6. Affine Transformations (30/01/24)

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Source: "Fundamentals of Computer Graphics, 4th Edition"
by Steve Marschner and Peter Shirley, A K Peters/CRC Press, 2015.

Linear Transformations :-

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = m_{11}x + m_{12}y$$

$$y' = m_{21}x + m_{22}y$$

In linear transforms, origin (0, 0) always remains fixed.
Cannot move the objects.

To translate / move a point

$$x' = x + x_t,$$

$$y' = y + y_t.$$

Use 2x2 matrix to perform this translation!

Representing the point (x, y) by a 3D vector $[x \; y \; 1]^T$

So, transformation matrix,

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

Using single matrix multiplication.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} m_{11}x + m_{12}y + x_t \\ m_{21}x + m_{22}y + y_t \\ 1 \end{bmatrix}$$

Affine Transformation :-

Implementation of linear transformation followed by a translation!

Addition of an extra dimension to implement the affine transformation is called as "Homogeneous Coordinates"

2D Translation Matrix :-

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + x_t \\ y + y_t \\ 1 \end{bmatrix}$$

3D Translation Matrix :-

$$\begin{bmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + x_t \\ y + y_t \\ z + z_t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \quad \begin{bmatrix} z + z_t \\ 1 \end{bmatrix}$$

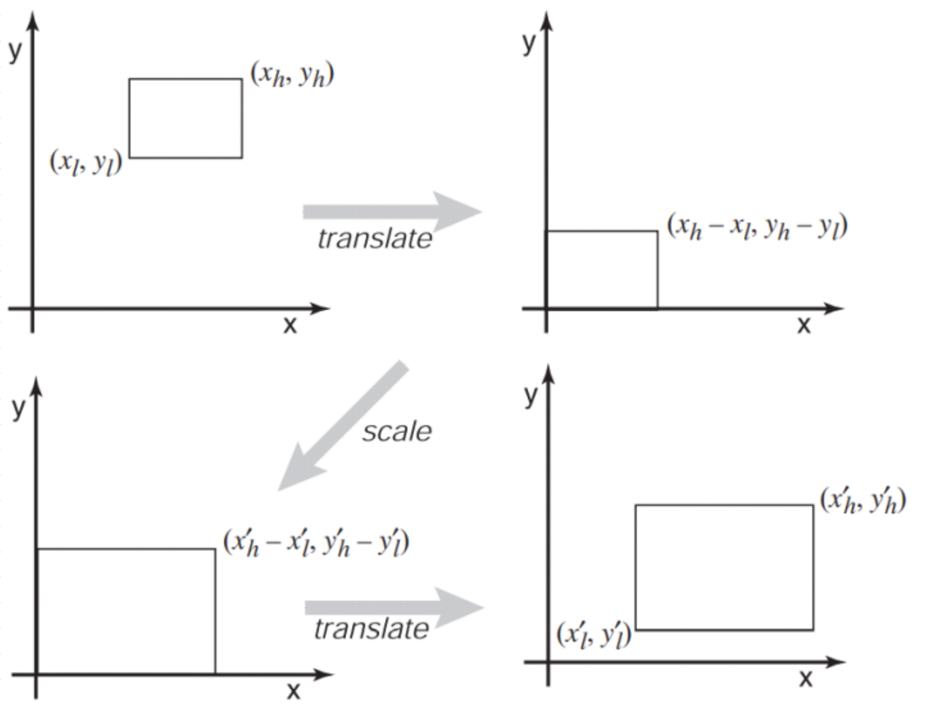
Example :-

Rotation (ϕ) + Translation (x_t, y_t)

$$M = \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation of rectangle

$$[x_l, x_h] \times [y_l, y_h] \longrightarrow [x_l', x_h'] \times [y_l', y_h']$$



① Move $(x_l, y_l) \rightarrow (0, 0)$

② Scale

③ Move $(0,0) \to (x'_e, y'_e)$

$$\Rightarrow \quad M = \text{translate}(x'_e, y'_e)\ \text{scale}\left(\frac{x'_h - x'_e}{x_h - x_e}, \frac{y'_h - y'_e}{y_h - y_e}\right)\ \text{translate}(-x_e, -y_e)$$

$$= \begin{bmatrix} 1 & 0 & x'_e \\ 0 & 1 & y'_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x'_h - x'_e}{x_h - x_e} & 0 & 0 \\ 0 & \frac{y'_h - y'_e}{y_h - y_e} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_e \\ 0 & 1 & -y_e \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & 1 \end{bmatrix}$$

In 3D,

$$[x_e, x_h] \times [y_e, y_h] \times [z_e, z_h] \longrightarrow [x'_e, x'_h] \times [y'_e, y'_h] \times [z'_e, z'_h]$$

$$\begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & 0 & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & \frac{z'_h - z'_l}{z_h - z_l} & \frac{z'_l z_h - z'_h z_l}{z_h - z_l} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general,

$$\begin{bmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & x_t \\ a_{21} & a_{22} & a_{23} & y_t \\ a_{31} & a_{32} & a_{33} & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & x_t \\ a_{21} & a_{22} & a_{23} & y_t \end{bmatrix}$$

Translation part

Rotation/scaling part

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} & y_t \\ a_{31} & a_{32} & a_{33} & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation part

Is also written in the form,

3x3 matrix

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

3-vector

Q:- Describe in words, what this 2D transform matrix does:

(1)
$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2)
$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverses of Transformation Matrices:-

$$M = M_1 M_2 M_3 \ldots M_n$$

$$M^{-1} = M_n^{-1} \ldots M_3^{-1} M_2^{-1} M_1^{-1}$$

Example:-

$$M = R_1 \, scale(\sigma_1, \sigma_2, \sigma_3) \, R_2$$

$$M^{-1} = R_2^T \, \text{scale}(1/\sigma_1, 1/\sigma_2, 1/\sigma_3) \, R_1^T .$$