Source: 1. Computer Graphics by Donald Hearn and M. Pauline Baker

Introduction to Computer Graphics, NPTEL Course by Prof. Prem Kalra

2. **Computer Graphics** by Donald Hearn and M. Pauline Baker

Cubic Splines :-

$$P_k(u) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

$$0 \le u \le 1$$
$$1 \le k \le n-1$$

$$F_1(u) = 2u^3 - 3u^2 + 1$$
$$F_2(u) = -2u^3 + 3u^2$$
$$F_3(u) = u(u^2 - 2u + 1)\, t_{k+1}$$
$$F_4(u) = u(u^2 - u)\, t_{k+1}$$

where F_1, F_2, F_3, F_3 are called the Blending Functions.

$$\Rightarrow \quad P_k(u) = [F]\,[G]$$

parameter values (t_k)

given position and tangent vectors

where F is a blending function matrix and G is the geometric function.

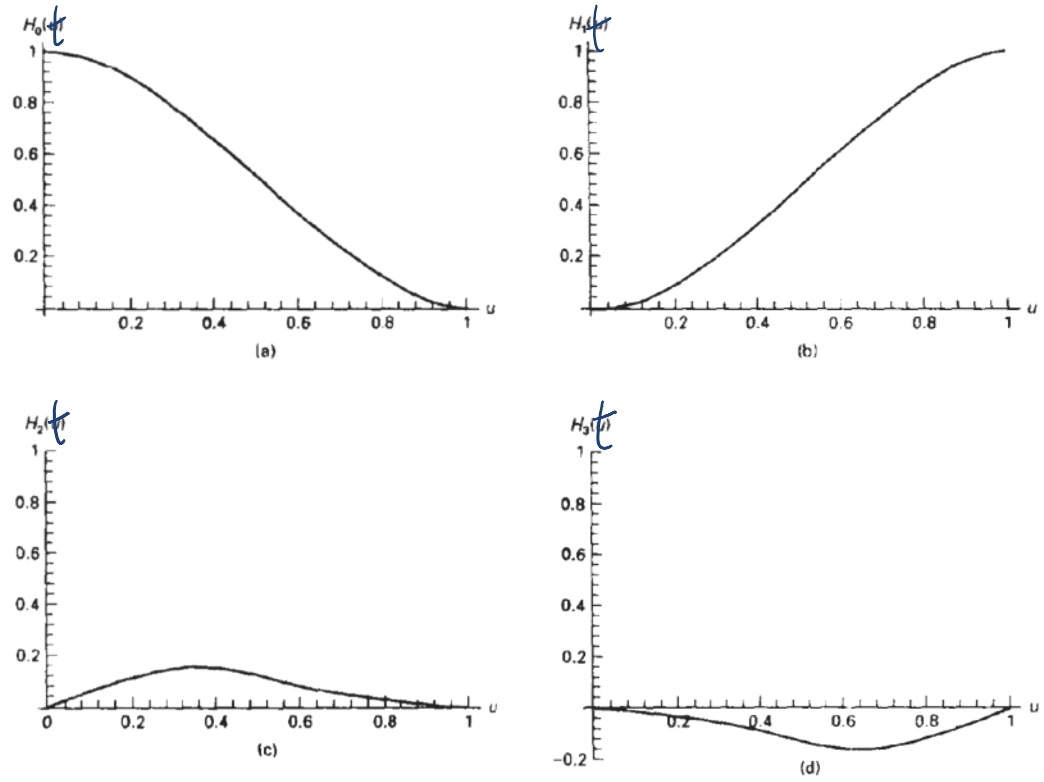


Figure 10-28
The Hermite blending functions.

If $\boxed{t_{k+1}=1}$ for all k then the spline is called

"Normalized Spline"

Blending functions becomes,

$H_0(t) = 2t^3 - 3t^2 + 1$

$H_1(t) = -2t^3 + 3t^2$

$H_2(t) = t^3 - 2t^2 + t$

$H_3(t) = t^3 - t^2$

"Hermite Polynomial Blending Functions.

"Hermite Polynomial Blending Functions.

$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix}^T = \underbrace{\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}}_{1\times4} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

A special case for cubic splines.

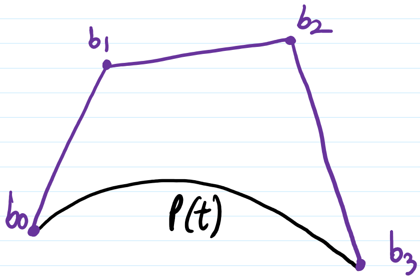
The matrix (B) for solving tangent vectors will become,

$$\underbrace{\begin{bmatrix} 1 & 0 & - & - & \cdots & \\ 1 & 4 & 1 & \cdots & & \\ & 1 & 4 & 1 & \cdots & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & \cdots & 0 & 1 \end{bmatrix}}_{\text{Constant Matrix}} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_{n-1}' \\ P_n' \end{bmatrix} = \begin{bmatrix} 3((P_3-P_2)+(P_2-P_1)) \\ \vdots \\ 3((P_n-P_{n+1})+(P_{n-1}-P_{n-2})) \end{bmatrix}$$

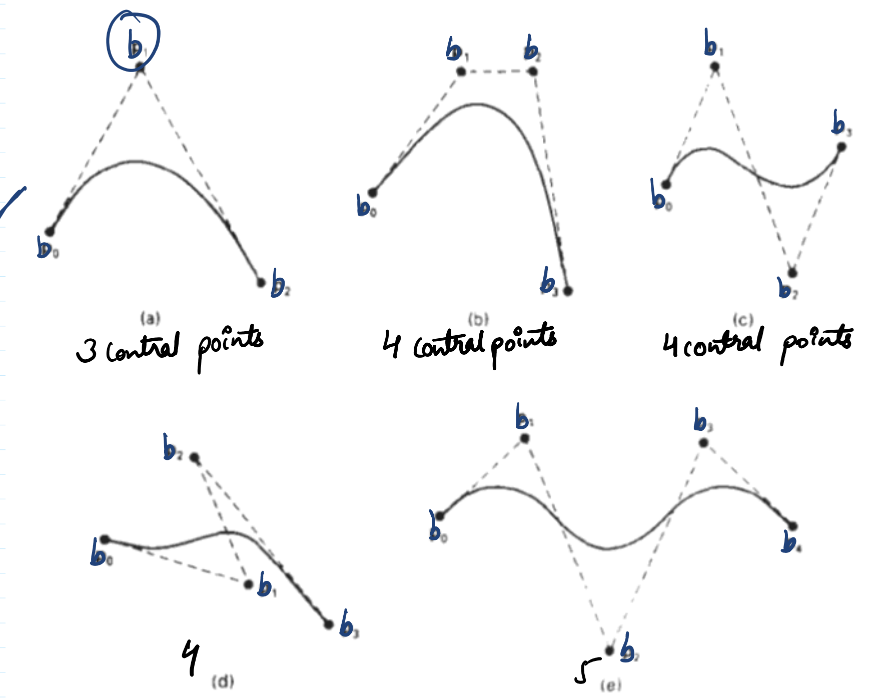
$\Rightarrow$ Less computations

Bezier Curves :-

User defined curves.



$b_0 b_1 b_2 b_3$ is the control polygon.



Mathematically,

$$P(t) = \sum_{i=0}^{n} b_i J_i^n(t) \quad 0 \le t \le 1$$

where J_i^n are called the Burnstein Blending Functions.

Burnstien Polynomials

$$\boxed{J_i^n(t) = {}^nC_i \, t^i (1-t)^{n-i}} = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

where, $J_0^0(t) = \underbrace{\frac{0!}{0!(0)!}}_{1} \times \underbrace{t^0}_{1} \times \underbrace{(1-t)^0}_{1}$

$= 1$

$$\boxed{\begin{array}{l} n^0 = 1 \\ 0! = 1 \\ 0^n = 0 \\ (-1)! = 0 \end{array}}$$

for $i \notin \{0, \ldots, n\}$

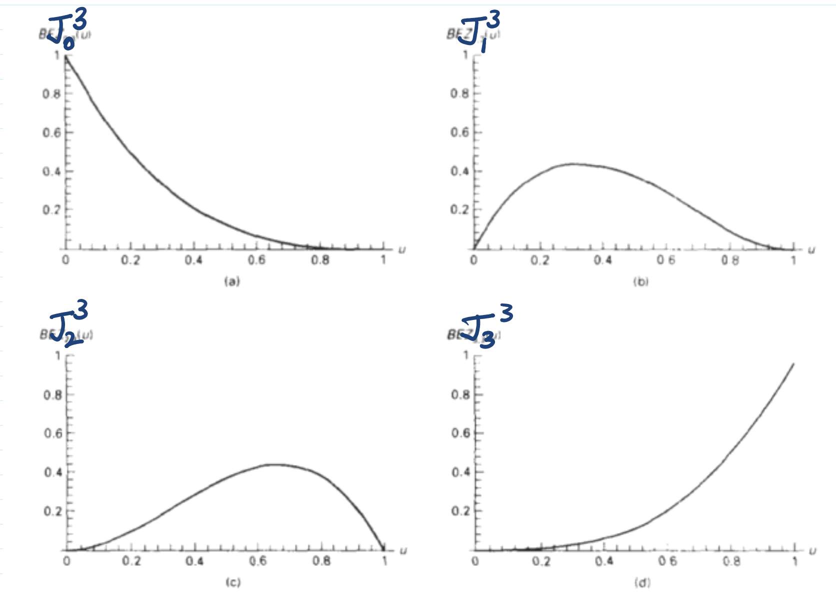
$J_i^n(t) = \frac{n!}{\ldots} \times t^{n+1} (1-t)^{n-(n+1)}$

$(n+1)!\,(n-(n+1))!$

$$= \frac{n!}{(n+1)!\,\underbrace{(-1)!}_{0}} \times t^{n+1}(1-t)^{-1} = 0$$

$$\sum_{i=0}^{n} J_i^n(t) = 1$$

For $n = 3$



$$J_0^3(t) = t^0(1-t)^3 = (1-t)^3$$
$$J_1^3(t) = 3t(1-t)^2$$
$$J_2^3(t) = 3t^2(1-t)$$
$$J_3^3(t) = t^3$$

$$P(t) = b_0 J_0^3 + b_1 J_1^3 + b_2 J_2^3 + b_3 J_3^3$$

$$P(t) = \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} b_0 \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Bezier Surfaces :-

Two sets of Bezier curves can be used to design an object surface by specifying an input mesh of control points.

$$P(u,v) = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{j,k} \, J_{i,m}(v) \, J_{k,n}(u)$$

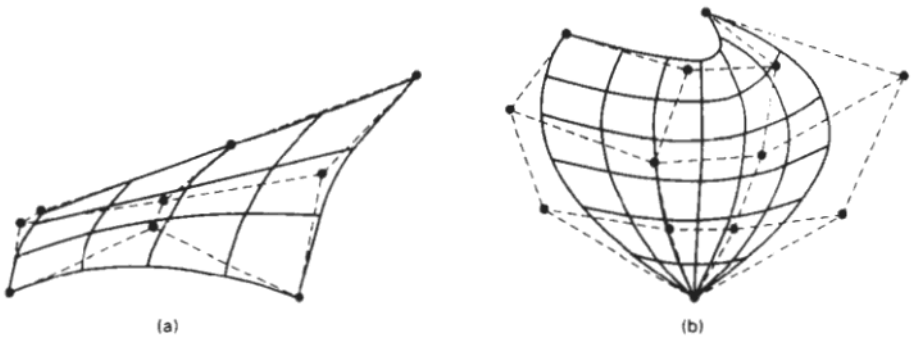
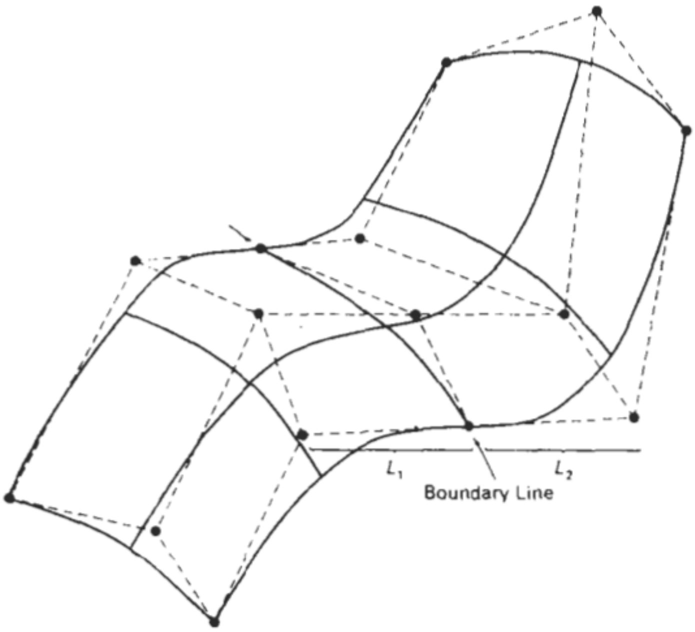


Figure 10-39
Bézier surfaces constructed for (a) $m = 3$, $n = 3$, and (b) $m = 4$, $n = 4$. Dashed lines connect the control points.


L_1
L_2
Boundary Line