Mathematics Recap –

Linear Algebra

Why Linear Algebra?

Graphics → Transformation/Change of points and vectors.

"Matrices"

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$$

Matrix Arithmetic →

① Matrix Multiplication

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ \boxed{a_{i1} \ \cdots \ a_{im}} & & \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1c} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & \boxed{b_{mj}} & \cdots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \cdots & \boxed{p_{ij}} & \cdots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \cdots & p_{rj} & \cdots & p_{rc} \end{bmatrix}$$

Element $p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

Not commutative $\Rightarrow$ $AB \neq BA$

$AB = AC \quad B \neq C$ } Not necessarily

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② Transpose A^T

$$a_{ij} = a'_{ji}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

(3) Determinants

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

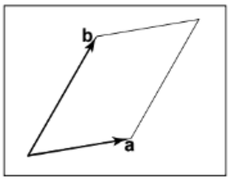
$$\det(A) = a * \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b * \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c * \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

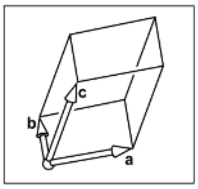
$$\det(A) = aei - afh - bdi + bfg + cdh - ceg$$

$|A|$

Typically thought as
solving the system of equations.

In computer graphics,
as multiplication of vectors





$|ab|$ = area of parallelogram

$|abc|$ = volume of parallelopipe

(4) Inverse matrix A^{-1}

$$\boxed{AA^{-1} = I}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} & a^c_{41} \\ a^c_{12} & a^c_{22} & a^c_{32} & a^c_{42} \\ a^c_{13} & a^c_{23} & a^c_{33} & a^c_{43} \\ a^c_{14} & a^c_{24} & a^c_{34} & a^c_{44} \end{bmatrix}$$

Properties :-

→ $A^{-1}A = I$

→ $(AB)^{-1} = B^{-1}A^{-1}$

→ $(AB)^T = B^T A^T$

→ $|AB| = |A|\,|B|$

→ $|A^{-1}| = \dfrac{1}{|A|}$

→ $|A^T| = |A|$

→ <u>Diagonal Matrix</u>

All non-zero elements along diagonal.

→ <u>Symmetric Matrix</u>

$$A = A^T$$

→ <u>Orthogonal Matrix</u>

$$RR^T = I = R^T R$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

O ✓	O ✓	O ×
S	S	S
D	D	D

Practice :- ① Compute the determinant of

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

② Inverse of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

⑤ Eigen Values and Eigen Vectors

Those non-zero vectors whose direction do not change when multiplied by the matrix.

$$Aa = \lambda a$$

Matrix

eigen value

eigen vector

$$(A - \lambda I)a = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{21})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Q :- Compute the eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

6 Eigen Value Decomposition

If A is symmetric matrix,

$$A = A^T$$

$$A = QDQ^T$$

$Q \rightarrow$ Orthogonal Matrix
$D \rightarrow$ Diagonal Matrix

Eigen Vectors $\rightarrow$ Columns of Q.

Eigen Values $\rightarrow$ Diagonal elements of D.

Geometric Transformations :-

1 2D Linear Transformations

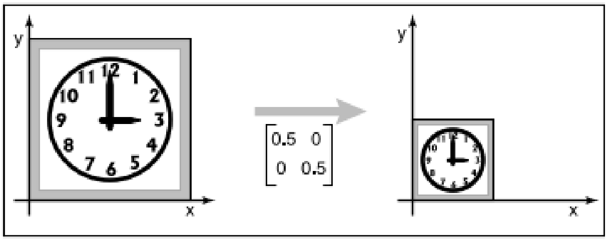
Changing a 2D vector using a 2x2 matrix

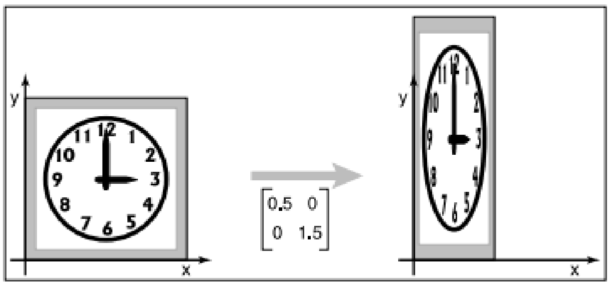
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

Scaling $\rightarrow$ scale along coordinate axes.

$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

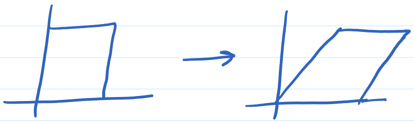
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$





Shearing :-

"shear" pushing



shear matrices

horizontal

$$\text{shear-}x(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

vertical

$$\text{shear-}y(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$s = \tan\phi$$