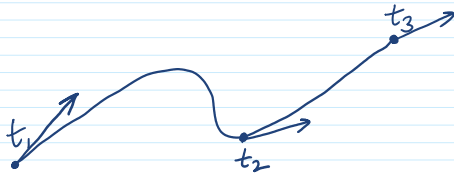


Source: 1. Computer Graphics by Donald Hearn and M. Pauline Baker

Introduction to Computer Graphics, NPTEL Course by Prof. Prem Kalra

2. Computer Graphics by Donald Hearn and M. Pauline Baker

Extending this idea to set of n points.



Joining of segments

2 SEGMENTS: P_1, P_2, P_3 (Points)

P'_1, P'_2, P'_3 (Tangents)

where P_2 and P'_2 are the intermediate point and its tangent vector which is determined through some continuity constraint.

* Piecewise spline of degree k has continuity of order $(k-1)$ at the internal joints.

Thus cubic splines have second order continuity i.e. $P''(t)$ is continuous over the joint.

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$= \sum_{i=1}^4 B_i t^{i-1} \quad t_1 \leq t \leq t_2$$

$$P'(t) = B_2 + 2B_3 t + 3B_4 t^2$$

$$= \sum_{i=1}^4 (i-1) B_i t^{i-2}$$

$$P''(t) = 2B_3 + 6B_4 t$$

$$= \sum_{i=1}^4 (i-1)(i-2) B_i t^{i-3} \quad t_1 \leq t \leq t_2$$

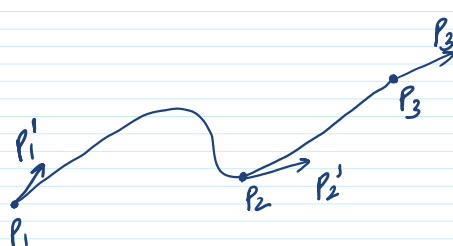
First Segment ($t=t_2$)

$$P'' = 6B_4 t_2 + 2B_3$$

Second Segment ($t=0$)

$$P'' = 2B_3$$

$$\text{So, } (6B_4 t_2 + 2B_3) = (2B_3) \dots \text{--- ①}$$



$$P' = 2B_3$$

$$S_0, (6B_4t_2 + 2B_3)_{\text{seg 1}} = (2B_3)_{\text{seg 2}} \quad \text{--- ①}$$

We know,

$$\left. \begin{aligned} b_1 &= P_1 \\ b_2 &= P_1' \\ b_3 &= \frac{3(P_2 - P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\ b_4 &= \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2} \end{aligned} \right\} \text{--- (A)}$$

$$(6B_4t_2 + 2B_3)_{\text{seg 1}} = (2B_3)_{\text{seg 2}} \quad \text{--- ①}$$

Substituting b_3 and b_4 from (A) in ①

$$3 \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2} \right] t_2 + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} \right]$$

$$= \left[\frac{3(P_3 - P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3} \right]$$

$$\frac{6(P_1 - P_2)}{t_2^2} + \frac{3P_1'}{t_2} + \frac{3P_2'}{t_2} + \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}$$

$$= \frac{3(P_3 - P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}$$

$$\frac{3P_1'}{t_2} + \frac{3P_2'}{t_2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3}$$

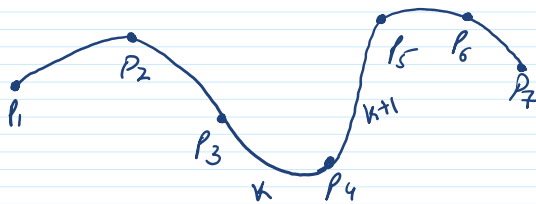
$$= \frac{3(P_3 - P_2)}{t_3^2} - \frac{6(P_1 - P_2)}{t_2^2} - \frac{3(P_2 - P_1)}{t_2^2}$$

$$t_2 t_3 \times \left[\frac{P_1'}{t_2} + \frac{2P_2'}{t_2} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3} \right] = \frac{3(P_3 - P_2)}{t_3^2} + \frac{3(P_2 - P_1)}{t_2^2}$$

$$t_3 P_1' + 2(t_3 + t_2) P_2' + t_2 P_3' = \frac{3t_2(P_3 - P_2)}{t_3} + \frac{3t_3(P_2 - P_1)}{t_2}$$

$$t_3 P_1' + 2(t_3 + t_2) P_2' + t_2 P_3' = \frac{3}{t_2 t_3} \left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1) \right)$$

$$\begin{bmatrix} t_3 & 2(t_3+t_2) & t_2 \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \\ p_3' \end{bmatrix} = \frac{3}{t_2 t_3} (t_2^2 (p_3 - p_2) + t_3^2 (p_2 - p_1))$$



In general, for the k^{th} and $(k+1)^{\text{th}}$ segment ($1 \leq k \leq n-2$)

$$\begin{bmatrix} t_{k+2} & 2(t_{k+1} + t_{k+2}) & t_{k+1} \end{bmatrix} \begin{bmatrix} p_k' \\ p_{k+1}' \\ p_{k+2}' \end{bmatrix} = \frac{3}{t_{k+1} t_{k+2}} (t_{k+1}^2 (p_{k+2} - p_{k+1}) + t_{k+2}^2 (p_{k+1} - p_k))$$

Set of $n-2$ equations form a linear system for the tangent vectors p_k'

$$\left\{ \begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & \dots \\ 0 & t_4 & 2(t_3+t_4) & t_3 & \vdots \\ \dots & \dots & t_m & 2(t_{m-1}+t_m) & t_{m-1} \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \end{bmatrix} = \begin{bmatrix} \frac{3}{t_2 t_3} (t_2^2 (p_3 - p_2) + t_3^2 (p_2 - p_1)) \\ \frac{3}{t_3 t_4} (t_3^2 (p_4 - p_3) + t_4^2 (p_3 - p_2)) \\ \vdots \\ \frac{3}{t_{m-1} t_m} (t_{m-1}^2 (p_m - p_{m-1}) + t_m^2 (p_{m-1} - p_{m-2})) \end{bmatrix} \right\} \quad \text{--- (8)}$$

This system of equations can be used to solve for the tangent vectors $p_1', p_2' \dots p_n'$

Solving for B_1, B_2, B_3 and B_4

$$B_{1k} = p_k$$

$$B_{2k} = p'_k$$

$$B_{3k} = \frac{3(p_{k+1} - p_k)}{t_{k+1}^2} - \frac{2p'_k}{t_{k+1}} - \frac{p'_{k+1}}{t_{k+1}}$$

$$B_{4k} = \frac{2(p_k - p_{k+1})}{t_{k+1}^3} + \frac{p'_k}{t_{k+1}^2} + \frac{p'_{k+1}}{t_{k+1}^2}$$

Rearranging,

$$\begin{bmatrix} B_{1k} \\ B_{2k} \\ B_{3k} \\ B_{4k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/3t_{k+1} & 1/2t_{k+1}^2 \end{bmatrix} \begin{bmatrix} p_k \\ p'_k \\ p_{k+1} \\ p'_{k+1} \end{bmatrix}$$

$$p_k(t) = \sum_{i=1}^4 B_{ik} t^{i-1} \quad 0 \leq t \leq t_{k+1}$$

$1 \leq k \leq n-1$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} B_{1k} & B_{2k} & B_{3k} & B_{4k} \end{bmatrix}^T$$

$$p_k(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{matrix} 1 \times 4 \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/3t_{k+1} & 1/2t_{k+1}^2 \end{bmatrix} \end{matrix} \begin{matrix} 4 \times 4 \\ \begin{bmatrix} p_k \\ p'_k \\ p_{k+1} \\ p'_{k+1} \end{bmatrix} \end{matrix} \quad \begin{matrix} 4 \times 1 \\ \end{matrix}$$

$$p_k(t) = \left[\left(1 - 3t^2/t_{k+1}^2 + 2t^3/t_{k+1}^3 \right) \quad \left(t - 2t^2/t_{k+1} + t^3/t_{k+1}^2 \right) \right]$$

$$\left[\left(\frac{3t^2}{t_{k+1}^2} - \frac{2t^3}{t_{k+1}^3} \right) \quad \left(-\frac{t^2}{t_{k+1}} + \frac{t^3}{t_{k+1}^2} \right) \right] \begin{bmatrix} p_k \\ p'_k \\ p_{k+1} \end{bmatrix}$$

1×4

$$1 \times 4 \begin{bmatrix} p_k \\ p_{k+1} \\ p_{k+1}' \\ p_{k+1}'' \end{bmatrix}_{4 \times 1}$$

Substituting $u = t/t_{k+1}$ rearranging

$$P_k(u) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ p_{k+1}' \\ p_{k+1}'' \end{bmatrix}$$

$$0 \leq u \leq 1$$

$$1 \leq k \leq n-1$$

$$F_1(u) = 2u^3 - 3u^2 + 1$$

$$F_2(u) = u(u^2 - 2u + 1) t_{k+1}$$

$$F_3(u) = -2u^3 + 3u^2$$

$$F_4(u) = u(u^2 - u) t_{k+1}$$

where F_1, F_2, F_3, F_4 are called the
Blending Functions.