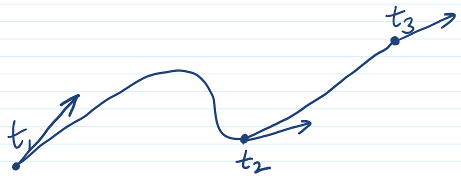
Source: 1. Computer Graphics by Donald Hearn and M. Pauline Baker

Introduction to Computer Graphics, NPTEL Course by Prof. Prem Kalra

2. **Computer Graphics** by Donald Hearn and M. Pauline Baker

Extending this idea to set of n points.



Joining of segments

2 SEGMENTS: P_1 P_2 P_3 (Points)

P_1' P_2' P_3' (Tangents)

where P_2 and P_2' are the intermediate point and its tangent vector which is determined through some continuity contraint.

* Piecewise spline of degree k has continuity of order $(k-1)$ at the internal joints.

Thus cubic splines have second order continuity i.e. $P_2''(t)$ is continuous over the joint.

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$= \sum_{i=1}^{4} B_i t^{i-1} \qquad t_1 \le t \le t_2$$

$$P'(t) = B_2 + 2B_3 t + 3B_4 t^2$$

$$= \sum_{i=1}^{4} (i-1) B_i t^{i-2}$$

$$P''(t) = 2B_3 + 6B_4 t$$

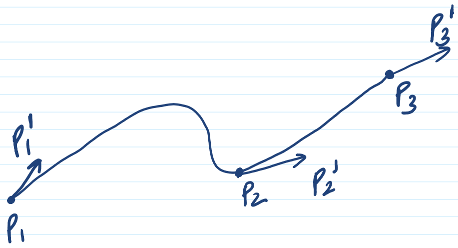
$$= \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3} \qquad t_1 \le t \le t_2$$

First Segment $(t = t_2)$

$P'' = 6B_4 t_2 + 2B_3$

Second Segment $(t = 0)$

$P'' = 2B_3$



So. $(6 B_4 t_2 + 2B_3) = (2B_3)$ —— ①

$$P'' = 2B_3$$

So, $(6B_4 t_2 + 2B_3)_{seg\,1} = (2B_3)_{seg\,2}$ —— (1)

We know,

$$\left.\begin{aligned} B_1 &= P_1 \\ B_2 &= P_1' \\ B_3 &= \frac{3(P_2-P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\ B_4 &= \frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2} \end{aligned}\right\} (A)$$

$$(6B_4 t_2 + 2B_3)_{seg\,1} = (2B_3)_{seg\,2} \text{ —— (1)}$$

Substituting B_3 and B_4 from (A) in (1)

$$3\left[\frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right]t_2 + \left[\frac{3(P_2-P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] = \left[\frac{3(P_3-P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}\right]$$

$$\frac{6(P_1-P_2)}{t_2^2} + \frac{3P_1'}{t_2} + \frac{3P_2'}{t_2} + \frac{3(P_2-P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} = \frac{3(P_3-P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}$$

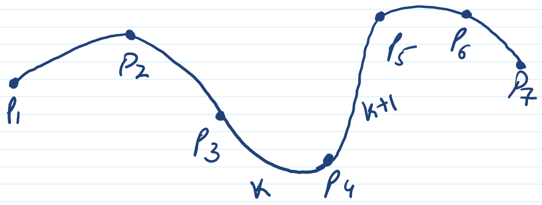
$$\frac{3P_1'}{t_2} + \frac{3P_2'}{t_2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3} = \frac{3(P_3-P_2)}{t_3^2} - \frac{6(P_1-P_2)}{t_2^2} - \frac{3(P_2-P_1)}{t_2^2}$$

$$t_2 t_3 \times \left[\frac{P_1'}{t_2} + \frac{2P_2'}{t_2} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3}\right] = \frac{3(P_3-P_2)}{t_3^2} + \frac{3(P_2-P_1)}{t_2^2}$$

$$t_3 P_1' + 2(t_3+t_2)P_2' + t_2 P_3' = \frac{3t_2(P_3-P_2)}{t_3} + \frac{3t_3(P_2-P_1)}{t_2}$$

$$t_3 P_1' + 2(t_3+t_2)P_2' + t_2 P_3' = \frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2) + t_3^2(P_2-P_1)\right)$$

$$\begin{bmatrix} t_3 & 2(t_3+t_2) & t_2 \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ P_3' \end{bmatrix} = \frac{3}{t_2 t_3} \left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1) \right)$$



In general, for the k^{th} and $(k+1)^{th}$ segment $(1 \le k \le n-2)$

$$\begin{bmatrix} t_{k+2} & 2(t_{k+1} + t_{k+2}) & t_{k+1} \end{bmatrix} \begin{bmatrix} P_k' \\ P_{k+1}' \\ P_{k+2}' \end{bmatrix}$$

$$= \frac{3}{t_{k+1}\, t_{k+2}} \left(t_{k+1}^2 (P_{k+2} - P_{k+1}) + t_{k+2}^2 (P_{k+1} - P_k) \right)$$

Set of $n-2$ equations form a linear system for the tangent vectors P_k'

$$\left.\begin{aligned} &\begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & \cdots \\ 0 & t_4 & 2(t_3+t_4) & t_3 & \vdots \\ \cdots & \cdots & t_m & 2(t_m+t_{m-1}) & t_{m-1} \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{t_2 t_3} \left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1) \right) \\ \frac{3}{t_3 t_4} \left(t_3^2 (P_4 - P_3) + t_4^2 (P_3 - P_2) \right) \\ \vdots \\ \frac{3}{t_{m-1} t_m} \left(t_{m-1}^2 (P_m - P_{m-1}) + t_m^2 (P_{m-1} - P_{m-2}) \right) \end{bmatrix} \end{aligned}\right\} \text{(B)}$$

This system of equations can be used to solve for the tangent vectors P_1', P_2' --- P_n'

Solving for B_1 B_2 B_3 and B_4

$$B_{1K} = P_{K}$$
$$B_{2K} = P_K'$$
$$B_{3K} = \frac{3(P_{K+1} - P_K)}{t_{K+1}^2} - \frac{2P_K'}{t_{K+1}} - \frac{P_{K+1}'}{t_{K+1}}$$
$$B_{4K} = \frac{2(P_K - P_{K+1})}{t_{K+1}^3} + \frac{P_K'}{t_{K+1}^2} + \frac{P_{K+1}'}{t_{K+1}^2}$$

Rearranging,

$$\begin{bmatrix} B_{1K} \\ B_{2K} \\ B_{3K} \\ B_{4K} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{K+1}^2 & -2/t_{K+1} & 3/t_{K+1}^2 & -1/t_{K+1} \\ 2/t_{K+1}^3 & 1/t_{K+1}^2 & -2/t_{K+1}^3 & 1/t_{K+1}^2 \end{bmatrix} \begin{bmatrix} P_K \\ P_K' \\ P_{K+1} \\ P_{K+1}' \end{bmatrix}$$

$$P_K(t) = \sum_{i=1}^{4} B_{iK}\, t^{i-1} \qquad 0 \le t \le t_{K+1}$$

$$1 \le K \le n-1$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} B_{1K} & B_{2K} & B_{3K} & B_{4K} \end{bmatrix}^T$$

$$P_K(t) = \underset{1\times4}{\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}} \underset{4\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{K+1}^2 & -2/t_{K+1} & 3/t_{K+1}^2 & -1/t_{K+1} \\ 2/t_{K+1}^3 & 1/t_{K+1}^2 & -2/t_{K+1}^3 & 1/t_{K+1}^2 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} P_K \\ P_K' \\ P_{K+1} \\ P_{K+1}' \end{bmatrix}}$$

$$P_K(t) = \underset{1\times4}{\left[\left(1 - \frac{3t^2}{t_{K+1}^2} + \frac{2t^3}{t_{K+1}^3}\right) \quad \left(t - \frac{2t^2}{t_{K+1}} + \frac{t^3}{t_{K+1}^2}\right) \quad \left(\frac{3t^2}{t_{K+1}^2} - \frac{2t^3}{t_{K+1}^3}\right) \quad \left(-\frac{t^2}{t_{K+1}} + \frac{t^3}{t_{K+1}^2}\right) \right]} \begin{bmatrix} P_K \\ P_K' \\ P_{K+} \end{bmatrix}$$

$$\ldots_{1\times 4}\begin{bmatrix} \vdots \\ P_{k+1} \\ P'_{k+1} \end{bmatrix}_{4\times 1}$$

Substituting $u = t/t_{k+1}$ rearranging

$$P_k(u) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P'_k \\ P'_{k+1} \end{bmatrix}$$

$$0 \le u \le 1$$
$$1 \le k \le n-1$$

$$F_1(u) = 2u^3 - 3u^2 + 1$$
$$F_2(u) = -2u^3 + 3u^2$$
$$F_3(u) = u(u^2 - 2u + 1)\, t_{k+1}$$
$$F_4(u) = u(u^2 - u)\, t_{k+1}$$

where F_1, F_2, F_3, F_3 are called the Blending Functions.