curves

Higher degree approximations :—

① Explicit

$$y = f(x)$$

② Implicit

$$f(x,y) = 0$$

Defines curves implicitly as solution of equation system.

line : $ax + by + c = 0$

circle : $x^2 + y^2 - R^2 = 0$

③ Parametric

$$x = x(t), \; y = y(t)$$

Position on the curve is defined through a parameter.

Parametric curves form a rich variety of free form smooth curves.

"Splines"

Cubic Splines :—

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$= \sum_{i=1}^{4} B_i t^{i-1} \qquad t_1 \le t \le t_2$$

is defined as a cubic polynomial of the parameter t.

$$\Rightarrow \quad X(t) = \sum_{i=1}^{4} B_{x} t^{i-1}$$

$$\Rightarrow \quad x(t) = \sum_{i=1} B_x t$$

$$y(t) = \sum_{i=1}^{4} B_y t^{i-1}$$

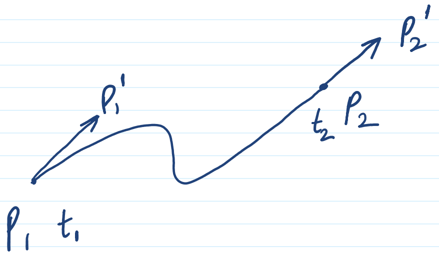
Vetors

$$P'(t) = B_2 + 2B_3 t' + 3B_4 t^2$$

Tangent vectors at some location.

<u>Example</u>:—

Given,



Two end points P_1 and P_2 of the cubic curve.
(position vectors)

P_1' and P_2' are the tangent vectors at point P_1 and P_2.

These points are defined for the parameter values t_1 and t_2.

Let $t_1 = 0$

$$P(0) = P_1 \qquad P(t_2) = P_2$$
$$P'(0) = P_1' \qquad P'(t_2) = P_2'$$

We know,

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$P'(t) = B_2 + 2B_3 t' + 3B_4 t^2$$

To obtain the generalized curve,

$P(0) = B_1 = P_1$

$P(t_2) = B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$

$P'(0) = B_2 = P_1'$

$P'(t_2) = B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$

We get,

$B_1 = P_1$ — ①

$B_2 = P_1'$ — ②

$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$ — ③

$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$ — ④

Using ① & ② in ③ & ④

$P_1 + P_1' t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$

$P_1' + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$

solving for B_3 and B_4,

Assuming $\left\{ \begin{array}{lll} P_1 \to a & P_1' \to b & t_2 \to c \\ P_2 \to d & P_2' \to e & \\ & B_3 \to x & B_4 \to y \end{array} \right\}$ for simplicity

$a + bc + \cancel{x c^2} + y c^3 = d$

$\frac{c}{2} \times [\; b + 2xc + 3y c^2 = e$

$\frac{bc}{2} + \cancel{x c^2} + \frac{3 y c^3}{2} = \frac{ec}{2}$

$-\qquad -\qquad -\qquad -$

$$a+bc-\frac{bc}{2}+y\,c^3-\frac{3y\,c^3}{2}=d-\frac{ec}{2}$$

$$a+\frac{bc}{2}-\frac{y\,c^3}{2}=d-\frac{ec}{2}$$

$$-\frac{y\,c^3}{2}=d-\frac{ec}{2}-a-\frac{bc}{2}$$

$$y=-\frac{2}{c^3}\left[d-\frac{ec}{2}-a-\frac{bc}{2}\right]$$

$$y=-\frac{2d}{c^3}+\frac{e}{c^2}+\frac{2a}{c^3}+\frac{b}{c^2}$$

$$b+2xc+3\left[-\frac{2d}{c^3}+\frac{e}{c^2}+\frac{2a}{c^3}+\frac{b}{c^2}\right]c^2=e$$

$$b+2xc-\frac{6d\cancel{c^2}}{c^{\cancel{3}}}+\frac{3e\cancel{c^2}}{\cancel{c^2}}+\frac{6a\cancel{c^2}}{c^{\cancel{3}}}+\frac{3b\cancel{c^2}}{\cancel{c^2}}=e$$

$$2xc-\frac{6d}{c}+2e+\frac{6a}{c}+4b=0$$

$$2xc=\frac{6d-6a}{c}-2e-4b$$

$$xc=\frac{3d-3a}{c}-e-2b$$

$$x=\frac{3d-3a}{c^2}-\frac{e}{c}-\frac{2b}{c}$$

$P_1\rightarrow a\quad P_1'\rightarrow b\quad t_2\rightarrow c$

$P_2\rightarrow d\quad P_2'\rightarrow e$

$B_3\rightarrow x\quad B_4\rightarrow y$

$$B_3=\frac{3P_2}{t_2^2}-\frac{3P_1}{t_2^2}-\frac{P_2'}{t_2}-\frac{2P_1'}{t_2}$$

$$\boxed{B_3=\frac{3P_2}{t_2^2}-\frac{3P_1}{t_2^2}-\frac{2P_1'}{t_2}-\frac{P_2'}{t_2}}$$

$$y = \frac{-2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}$$

$$B_4 = \frac{-2P_2}{t_2^3} + \frac{P_2'}{t_2^2} + \frac{2P_1}{t_2^3} + \frac{P_1'}{t_2^2}$$

$$= \frac{2P_1}{t_2^3} - \frac{2P_2}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$

$$\boxed{B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}}$$

Solving for B_1, B_2, B_3 and B_4

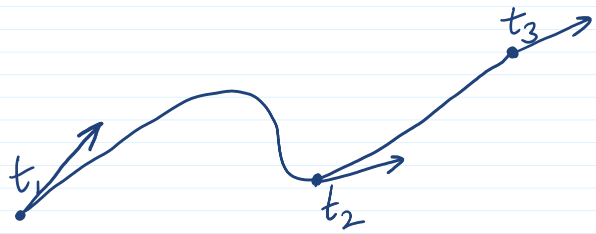
$$\left.\begin{aligned}
B_1 &= P_1 \\
B_2 &= P_2 \\
B_3 &= \frac{3(P_2 - P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\
B_4 &= \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}
\end{aligned}\right\} \quad A$$

where P_1 and P_2 gives the position of the endpoints.
and P_1' and P_2' gives the direction of the tangent vectors.

$$P(t) = P_1 + P_1' t + \left(\frac{3(P_2 - P_1)}{t_2} - \frac{2P'}{t_2} - \frac{P_2'}{t_2}\right) t^2$$
$$+ \left(\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right) t^3$$

$$+\left(\frac{2(P_1-P_2)}{t_2^3}+\frac{P_1'}{t_2^2}+\frac{P_2'}{t_2^2}\right)t^3$$

Extending this idea to set of n points.



Joining of segments

2 SEGMENTS: P_1 P_2 P_3 (Points)
P_1' P_2' P_3' (Tangents)

where P_2 and P_2' are the intermediate point and its tangent vector which is determined through some continuity constraint.

* Piecewise spline of degree k has continuity of order (k-1) at the internal joints.

Thus cubic splines have second order continuity i.e. $P_2''(t)$ is continuous over the joint.

$$P''(t) = \sum_{i=1}^{4}(i-1)(i-2)B_i t^{i-3} \qquad t_1 \le t \le t_2$$

at $t=t_2$

First Segment

$P'' = 6B_4 t_2 + 2B_3$

Second Segment

$P'' = 2B_3$

So, $(6B_4 t_2 + 2B_3)_{seg\,1} = (2B_3)_{seg\,2}$

Substitute the expressions for B_4 and B_3 and rearranging