10. Coordinate Transformation

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Source: First Principles of Computer Vision, Prof. Shree Nayyar

From <https://fpcv.cs.columbia.edu/>

Perspective Projection :-

$$\boxed{u = f_x \frac{x_c}{z_c} + O_x, \quad v = f_y \frac{y_c}{z_c} + O_y}$$

Using homogeneous representation,

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x x_c + z_c O_x \\ f_y y_c + z_c O_y \\ z_c \end{bmatrix} = \begin{bmatrix} f_x & 0 & O_x & 0 \\ 0 & f_y & O_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

"Linear Model for Perspective Projection"

Intrinsic Matrix :-

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & O_x & 0 \\ 0 & f_y & O_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

Calibration Matrix :

$$K = \begin{bmatrix} f_x & 0 & O_x \\ 0 & f_y & O_y \\ 0 & 0 & 1 \end{bmatrix}$$

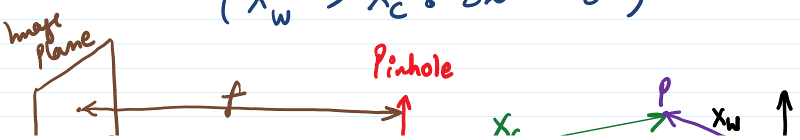
upper right triangular matrix

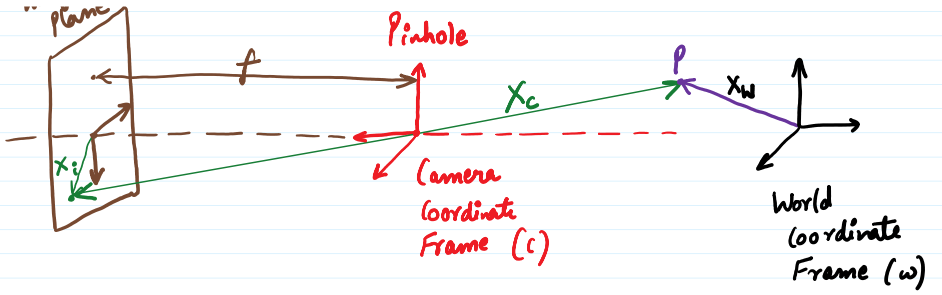
Intrinsic Matrix :

$$M_{int} = [K|0] = \begin{bmatrix} f_x & 0 & O_x & 0 \\ 0 & f_y & O_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Coordinate Transformation :-

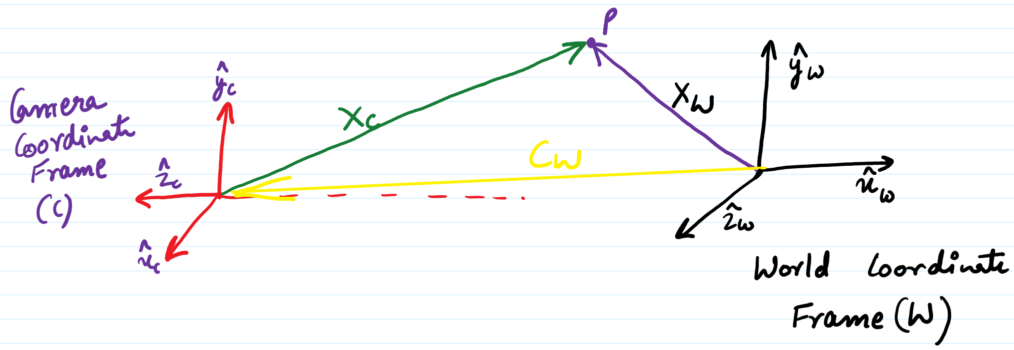
$$(X_w \rightarrow X_c : 3D \rightarrow 3D)$$





$$X_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \qquad X_W = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

Camera Coordinates — World Coordinates



Position C_W and orientation R of the camera in the world coordinate frame (W) are the camera's Extrinsic Parameters.

$$R = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{33} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\text{orthogonal matrix}} \begin{matrix} \rightarrow \text{Row 1: Direction of } \hat{x}_c \text{ in world coordinate frame.} \\ \rightarrow \text{Row 2: } \_\_\_ \; \hat{y}_c \; \_\_\_ . \\ \rightarrow \text{Row 3: } \_\_\_ \; \hat{z}_c \; \_\_\_ . \end{matrix}$$

Given (R, C_W) of the camera, the location of point P in the world coordinate frame (W).

$$X_c = R(X_W - C_W)$$

$$= R X_W - R C_W = R X_W + t \qquad \boxed{t = -RC_W}$$

$$= K x_w - K c_w = K x_w + t \qquad \boxed{t = -Kc_w}$$

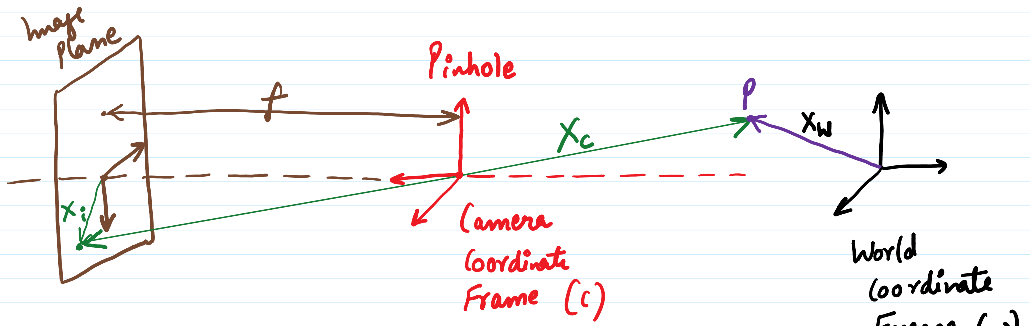
$$X_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Using homogeneous coordinates :-

$$X_c = \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Extrinsic Matrix (M_{ext})

$$= \begin{bmatrix} R_{3\times3} & t \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$X_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \xleftarrow{M_{int}} X_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \xleftarrow{M_{ext}} X_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

Image coordinates — Perspective Projection — Camera Coordinates — Coordinate Transformation — World coordinates

Camera to Pixel

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & o_x & 0 \\ 0 & f_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

World to Camera

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

$$\boxed{u = M_{int}\, x_c} \qquad \boxed{x_c = M_{ext}\, x_w}$$

Combining full projection matrix P:

$$\boxed{\begin{aligned} u &= M_{int} M_{ext}\, x_w \\ u &= P\, x_w \end{aligned}}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\underset{3\times 4}{\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}}}_{P \text{ (Projection Matrix)}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$