Source: "Fundamentals of Computer Graphics, 4th Edition"
by Steve Marschner and Peter Shirley, A K Peters/CRC Press, 2015.

Mathematics Recap –

Linear Algebra

Why Linear Algebra?

Graphics → Transformation/Change of points and vectors.

"Matrices"

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$$

Matrix Arithmetic →

① Matrix Multiplication

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ \boxed{a_{i1} \; \cdots \; a_{im}} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \boxed{\begin{matrix} b_{1j} \\ \vdots \\ b_{mj} \end{matrix}} & \cdots & b_{1c} \\ \vdots & & & & \vdots \\ b_{m1} & \cdots & & \cdots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \cdots & \boxed{p_{ij}} & \cdots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \cdots & p_{rj} & \cdots & p_{rc} \end{bmatrix}$$

Element $p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

Not commutative ⇒ $AB \neq BA$
$AB = AC \quad B \neq C$
} Not necessarily

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② Transpose A^T

$$a_{ij} = a_{ji}'$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

③ Determinants

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

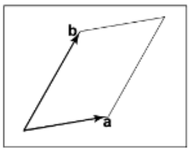
$$\det(A) = a * \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b * \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c * \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

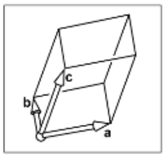
$$\det(A) = aei - afh - bdi + bfg + cdh - ceg$$

$|A|$

Typically thought as
solving the system of equations.

In computer graphics,
as multiplication of vectors





$|ab|$ = area of parallelogram

$|abc|$ = volume of parallelopipe

④ Inverse matrix A^{-1}

$$\boxed{AA^{-1} = I}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} & a^c_{41} \\ a^c_{12} & a^c_{22} & a^c_{32} & a^c_{42} \\ a^c_{13} & a^c_{23} & a^c_{33} & a^c_{43} \\ a^c_{14} & a^c_{24} & a^c_{34} & a^c_{44} \end{bmatrix}$$

Properties :-

→ $A^{-1}A = I$

→ $(AB)^{-1} = B^{-1}A^{-1}$

→ $(AB)^T = B^T A^T$

→ $|AB| = |A|\,|B|$

→ $|A^{-1}| = \dfrac{1}{|A|}$

→ $|A^T| = |A|$

→ Diagonal Matrix

All non-zero elements along diagonal.

→ **Symmetric Matrix**

$$A = A^T$$

→ **Orthogonal Matrix**

$$R R^T = I = R^T R$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

O ✓	O ✓	O ✗
S	S	S
D	D	D

Practice :- ① Compute the determinant of

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

② Inverse of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

⑤ Eigen Values and Eigen Vectors

Those non-zero vectors whose direction do not change when multiplied by the matrix.

$$Aa = \lambda a$$

Matrix

eigen value

eigen vector

$$(A - \lambda I)a = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{21})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Q:- Compute the eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

⑥ Eigen Value Decomposition

If A is symmetric matrix,

$$A = A^T$$

$$A = Q D Q^T$$

$Q \rightarrow$ Orthogonal Matrix
$D \rightarrow$ Diagonal Matrix

Eigen Vectors $\rightarrow$ Columns of Q.
Eigen Values $\rightarrow$ Diagonal elements of D.

Geometric Transformations:-

① 2D Linear Transformations

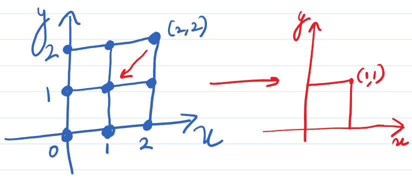
Changing a 2D vector using a 2x2 matrix

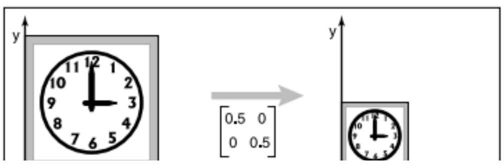
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

Scaling $\rightarrow$ scale along coordinate axes.

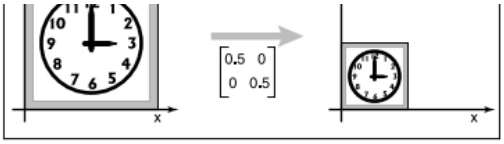
$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

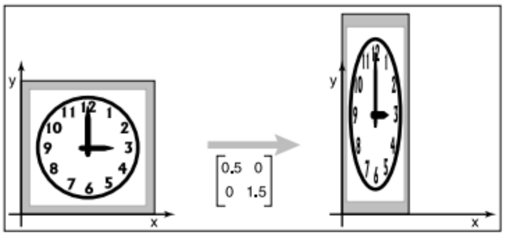




$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

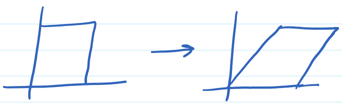


$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Shearing :-

"shear" pushing



shear matrices

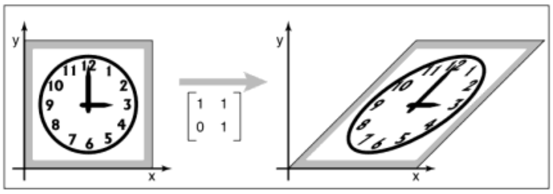
horizontal

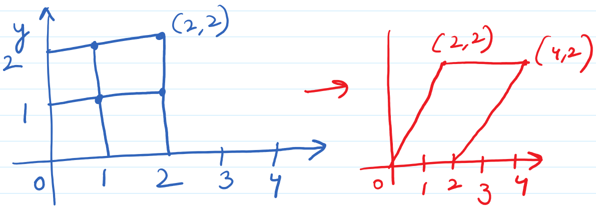
$$\text{shear-}x(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

vertical

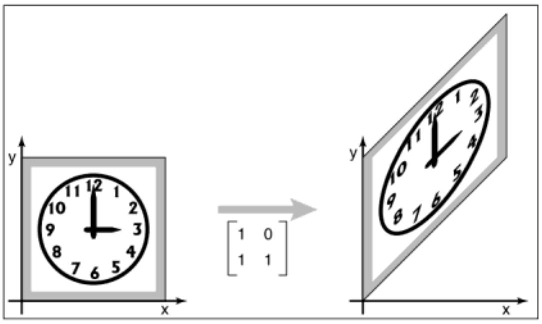
$$\text{shear-}y(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$s = \tan \phi$$





$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



$$\begin{bmatrix}1 & 1 & 2\\ 1 & 3 & 4\\ 0 & 2 & 5\end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} \begin{vmatrix}3 & 4\\ 2 & 5\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 2 & 5\end{vmatrix} & \begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} \\ -\begin{vmatrix}1 & 4\\ 0 & 5\end{vmatrix} & \begin{vmatrix}1 & 2\\ 0 & 5\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 1 & 4\end{vmatrix} \\ \begin{vmatrix}1 & 3\\ 0 & 2\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix}\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix}7 & -1 & -2\\ -5 & 5 & -2\\ 2 & -2 & 2\end{bmatrix}.$$

Given matrix,

$$A = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

$$A - \lambda I = 0$$

$$\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 0$$

$$\begin{bmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 1 = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

Solutions are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{\frac{3 \pm \sqrt{5}}{2} = \begin{bmatrix}2.618\\ 0.382\end{bmatrix} \quad \text{Exact eigen values}}$$

Associated Eigen Vectors :-

$$\begin{bmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$\lambda_1 = 2.618 \Rightarrow \begin{bmatrix}2-2.618 & 1\\ 1 & 1-2.618\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$$(x, y) = (0.8507, 0.5257)$$

$$\lambda_2 = 0.382 \Rightarrow \begin{bmatrix} 2-0.382 & 1 \\ 1 & 1-0.382 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(x,y) = (-0.5257, 0.5807)$$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is a symmetric matrix $(A = A^T)$ with eigen values $(2.618, 0.382)$ And one possible set of eigen vectors $\begin{bmatrix} 0.8507 \\ 0.5257 \end{bmatrix}$ $\begin{bmatrix} -0.5257 \\ 0.5807 \end{bmatrix}$

So it can be decomposed in the form,

$$A = Q D Q^T$$

where $Q \rightarrow$ orthogonal matrix
$D \rightarrow$ diagonal matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}}_{\substack{Q \\ \text{using eigen vectors}}} \underbrace{\begin{bmatrix} 2.618 & 0 \\ 0 & 0.382 \end{bmatrix}}_{\substack{\text{diagonal matrix } (D) \\ [\text{using eigen values}]}} \underbrace{\begin{bmatrix} 0.8507 & 0.5257 \\ -0.5257 & 0.8507 \end{bmatrix}}_{Q^T}$$

"Eigen Value Decomposition"

Singular Value Decomposition:-

Generalization of symmetric eigen value decomposition to non-symmetric matrices.

$$A = U S V^T$$

Both (U, V) are orthogonal but not same.

To solve for SVD,

$$M = AA^T$$
$$= (USV^T)(USV^T)^T$$
$$= (USV^T)(VS^TU^T)$$
$$\boxed{M = US^2U^T}$$

"M" is now a symmetric matrix.
So, applying method of eigen value decomposition.

$$N = A^TA$$
$$= (USV^T)^T(USV^T)$$
$$= (VS^TU^T)(USV^T)$$

$$= (USV^T)^T (USV^T)$$
$$= (VS^TU^T)(USV^T)$$
$$\boxed{N = VS^2V^T}$$

Q:- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ← Non symmetric matrix

$$M = AA^T$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Which was earlier computed.

$$\Rightarrow U = \begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}$$

$$N = A^TA$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Symmetric ?

$$V = \begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}}_{U} \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \underbrace{\begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}}_{V}$$