Source: 1. Computer Graphics by Donald Hearn and M. Pauline Baker

Introduction to Computer Graphics, NPTEL Course by Prof. Prem Kalra

2. **Computer Graphics** by Donald Hearn and M. Pauline Baker

Higher degree approximations :—

① Explicit

$$y = f(x)$$

② Implicit

$$f(x,y) = 0$$

Defines curves implicitly as solution of equation system.

line : $ax + by + c = 0$

circle : $x^2 + y^2 - R^2 = 0$

③ Parametric

$$x = x(t),\ y = y(t)$$

Position on the curve is defined through a parameter.

Parametric curves form a rich variety of free form smooth curves.

"Splines"

Cubic Splines :—

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$= \sum_{i=1}^{4} B_i t^{i-1} \qquad t_1 \le t \le t_2$$

is defined as a cubic polynomial of the parameter t.

$$\Rightarrow \quad x(t) = \sum_{i=1}^{4} B_x t^{i-1}$$

$$y(t) = \sum_{i=1}^{4} B_y t^{i-1}$$

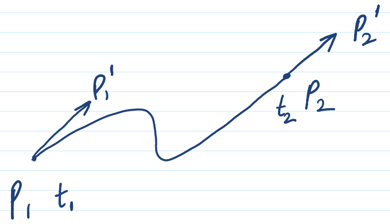
Vetors

$P'(t) = B_2 + 2B_3t' + 3B_4t^2$

Tangent vectors at some location.

Example:—

Given,



Two end points P_1 and P_2 of the cubic curve.
(position vectors)

P_1' and P_2' are the tangent vectors at point P_1 and P_2.

These points are defined for the parameter values t_1 and t_2.

Let $t_1 = 0$

$P(0) = P_1 \qquad P(t_2) = P_2$

$P'(0) = P_1' \qquad P'(t_2) = P_2'$

We know,

$P(t) = B_1 + B_2t + B_3t^2 + B_4t^3$

$P'(t) = B_2 + 2B_3t' + 3B_4t^2$

To obtain the generalized curve,

$P(0) = B_1 = P_1$

$P(t_2) = B_1 + B_2t_2 + B_3t_2^2 + B_4t_2^3 = P_2$

$P'(0) = B_2 = P_1'$

$P'(t_2) = B_2 + 2B_3t_2 + 3B_4t_2^2 = P_2'$

We get,

$$B_1 = P_1 \quad \text{—} \quad (1)$$

$$B_2 = P_1' \quad \text{—} \quad (2)$$

$$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2 \quad \text{—} \quad (3)$$

$$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P_2' \quad \text{—} \quad (4)$$

Using (1) & (2) in (3) & (4)

$$P_1 + P_1' t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$$

$$P_1' + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$$

solving for B_3 and B_4,

$$\text{Assuming} \left\{ \begin{array}{ccc} P_1 \to a & P_1' \to b & t_2 \to c \\ P_2 \to d & P_2' \to e & \\ & B_3 \to x & B_4 \to y \end{array} \right\} \text{for simplicity}$$

$$a + bc + \cancel{x c^2} + y c^3 = d$$

$$\frac{c}{2} \times \left[b + 2xc + 3y c^2 = e \right.$$

$$-\frac{bc}{2} - \cancel{xc^2} - \frac{3yc^3}{2} = -\frac{ec}{2}$$

$$a + bc - \frac{bc}{2} + y c^3 - \frac{3y c^3}{2} = d - \frac{ec}{2}$$

$$a + \frac{bc}{2} - \frac{y c^3}{2} = d - \frac{ec}{2}$$

$$-\frac{y c^3}{2} = d - \frac{ec}{2} - a - \frac{bc}{2}$$

$$y = -\frac{2}{c^3}\left[d - \frac{ec}{2} - a - \frac{bc}{2}\right]$$

$$y = -\frac{2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}$$

$$b + 2xc + 3\left[-\frac{2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}\right] c^2 = e$$

$$b + 2xc + 3\left[\frac{-2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}\right]c^2 = e$$

$$b + 2xc - \frac{6d\cancel{c^2}}{\cancel{c^3}} + \frac{3e\cancel{c^2}}{\cancel{c^2}} + \frac{6a\cancel{c^2}}{\cancel{c^3}} + \frac{3b\cancel{c^2}}{\cancel{c^2}} = e$$

$$2xc - \frac{6d}{c} + 2e + \frac{6a}{c} + 4b = 0$$

$$2xc = \frac{6d - 6a}{c} - 2e - 4b$$

$$xc = \frac{3d - 3a}{c} - e - 2b$$

$$x = \frac{3d - 3a}{c^2} - \frac{e}{c} - \frac{2b}{c}$$

$P_1 \to a$ $\quad P_1' \to b$ $\quad t_2 \to c$

$P_2 \to d$ $\quad P_2' \to e$

$B_3 \to x$ $\quad B_4 \to y$

$$B_3 = \frac{3P_2}{t_2^2} - \frac{3P_1}{t_2^2} - \frac{P_2'}{t_2} - \frac{2P_1'}{t_2}$$

$$\boxed{B_3 = \frac{3P_2}{t_2^2} - \frac{3P_1}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}}$$

$$y = \frac{-2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}$$

$$B_4 = \frac{-2P_2}{t_2^3} + \frac{P_2'}{t_2^2} + \frac{2P_1}{t_2^3} + \frac{P_1'}{t_2^2}$$

$$= \frac{2P_1}{t_2^3} - \frac{2P_2}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$

$$\boxed{B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}}$$

Solving for B_1, B_2, B_3 and B_4

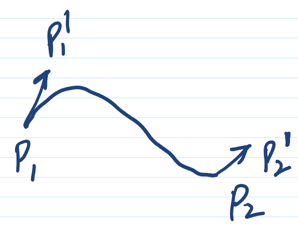
$B_1 = P_1$

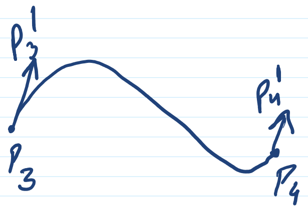
$$\left.\begin{aligned} B_1 &= P_1 \\ B_2 &= P_1' \\ B_3 &= \frac{3(P_2-P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\ B_4 &= \frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2} \end{aligned}\right\} \text{(A)}$$

where P_1 and P_2 gives the position of the endpoints.
and P_1' and P_2' gives the direction of the tangent vectors.

$$P(t) = P_1 + P_1't + \left(\frac{3(P_2-P_1)}{t_2} - \frac{2P'}{t_2} - \frac{P_2'}{t_2}\right)t^2 + \left(\frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right)t^3$$

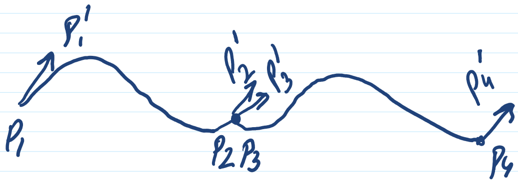
Extending this idea to set of n points.





Joining of segments

2 SEGMENTS: P_1 P_2 P_3 (Points)
P_1' P_2' P_3' (Tangents)



where P_2 and P_2' are the intermediate point and its tangent vector which is determined through some continuity contraint.

* Piecewise spline of degree k has continuity of order (k-1) at the internal joints.

Thus cubic splines have second order continuity i.e. $P_2''(t)$ is continuous over the joint.

$$P''(t) = \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3} \quad t_1 \le t \le t_2$$

at $t = t_2$

First Segment

$$P'' = 6 B_4 t_2 + 2 B_3$$

Second Segment

$$P'' = 2 B_3$$

So, $(6 B_4 t_2 + 2 B_3)_{seg\,1} = (2 B_3)_{seg\,2}$

Substitute the expressions for B_4 and B_3 and rearranging