Primal problem

$$\begin{aligned} &\underset{x}{\text{minimize}} \ \|x\|_\infty \\ &\text{subject to } Ax = y. \end{aligned} \tag{1}$$

Basis of the kernel of the matrix $a^1, \dots, a^K$ is they are linearly independent and if $Aa^k = 0$ for all $k = 1, \dots, K$. Particular solution x^0 is any solution with $Ax^0 = y$. Then all solutions can be written by

$$\{x \mid Ax = y\} = \{x^0 + s^1 a^1 + \dots s^K a^K \mid s^1, \dots, s^K \in \mathbb{R}\}.$$

Idea

- Find $a_1, \dots, a_K$ offline.
- For given y find a single particular solution x^0
- Solve the equivalent problem

$$\underset{s}{\text{minimize}} \ \|x^0 + \sum_{k=1}^K s^k a^k\|_\infty. \tag{2}$$

1 Solve the previous using ADMM

Create matrix $B = [a^1, \dots, a^K]$ and vector $s = [s^1; \dots; s^K]$. Then

$$\sum_{k=1}^K s^k a^k = Bs.$$

Problem (2) is equivalent to

$$\begin{aligned} &\underset{s,z}{\text{minimize}} \ \|z\|_\infty \\ &\text{subject to } x^0 + Bs - z = 0. \end{aligned} \tag{3}$$

Scaled augmented Lagrangian

$$L(s, z; \mu) = \|z\|_\infty + \frac{\rho}{2}\|x^0 + Bs - z + \mu\|^2$$

ADMM [1, Equations 3.5-3.7] is the iterative procedure

$$\begin{aligned} s^{k+1} &\leftarrow \underset{s}{\operatorname{argmin}}\, L(\cdot, z^k; \mu^k) \\ z^{k+1} &\leftarrow \underset{z}{\operatorname{argmin}}\, L(s^{k+1}, \cdot; \mu^k) \\ \mu^{k+1} &\leftarrow \mu^k + x^0 + Bs^{k+1} - z^{k+1}. \end{aligned}$$

1.1 Update for s

Minimizing L with respect to s is equivalent to minimizing

$$L(s, z; \mu) = \frac{1}{2}\|x^0 + Bs - z^k + \mu^k\|^2,$$

which is the standard quadratic regression with the closed-form solution

$$s^{k+1} = (B^\top B)^{-1} B^\top (z^k - x^0 + \mu^k).$$

Matrix $C := (B^\top B)^{-1} B^\top$ can be precomputed online.

1.2 Update for z

Define

$$c := x^0 + Bs^{k+1} - \mu^k.$$

Then we need to solve

$$\text{minimize } \|z\|_\infty + \frac{\rho}{2}\|z^k - c\|^2 \tag{4}$$

Theorem 1.1. *Find the unique solution $z_{\max}$ of the equation*

$$\sum_{i=1}^{n} \max\{|c_i| - z_{\max}, 0\} = \frac{1}{\rho}.$$

Then the solution of (4) *equals to*

$$z_i^{k+1} = \begin{cases} \operatorname{sign}(c_i) z_{\max} & \textit{if } |c_i| > z_{\max}, \\ c_i & \textit{if } |c_i| \le z_{\max}. \end{cases}$$

Proof. It is quite complicated. $\square$

1.3 Update for μ

Already in a closed form.

References

[1] S. Boyd, N. Parikh, and E. Chu. *Distributed optimization and statistical learning via the alternating direction method of multipliers.* Now Publishers Inc, 2011.