$F(n) = F(n-1) + F(n-2)$

For general linear homogeneous equations:

$$F(n) = \sum_{i=1}^{d} a_i F(n-i) \xrightarrow{F(n)=x^n} \xrightarrow{\div x^{n-d}}$$

$$\boxed{x^d = a_1 x^{d-1} + a_2 x^{d-2} + \cdots + a_d \quad \text{characteristic equation}}$$

Solve for x, $x = r$
→ a non-repeated root: r^n
→ a repeated root: nr^n, $n^2 r^n$, ...

→ $F(n) = c_1 r_1^n + c_2 r_2^n + \ldots$

→ derive constants from initial values.

⇒ Fibonacci: $F(n) = F(n-1) + F(n-2)$

$$\Rightarrow x^n = x^{n-1} + x^{n-2} \xrightarrow{\div x^{n-2}} x^2 - x - 1 = 0$$

$$x = \frac{1+\sqrt{5}}{2} = r_1$$

$$x = \frac{1-\sqrt{5}}{2} = r_2$$

$$F(n) = c_1 r_1^n + c_2 r_2^n$$

$$= c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F(0) = 1 \longrightarrow 1 = c_1 + c_2 \Rightarrow c_2 = 1 - c_1$$

$$F(1) = 1 \longrightarrow 1 = \left(\frac{1+\sqrt{5}}{2}\right) c_1 + \left(\frac{1-\sqrt{5}}{2}\right) c_2$$

$$\Rightarrow 1 = \frac{1+\sqrt{5}}{2} c_1 + \left(\frac{1-\sqrt{5}}{2}\right)(1 - c_1)$$

$$\Rightarrow 1 - \frac{1-\sqrt{5}}{2} = \sqrt{5}\, c_1 \Rightarrow \boxed{c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}}$$

$$\boxed{c_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}}$$

$$\Rightarrow F(n) = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

$\phi = 1.618$ golden number

$-\phi^{-1}$

$$\boxed{F(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - (-\phi^{-1})^{n+1}\right)}$$