Ordinary Differential Equations: Euler Method

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Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

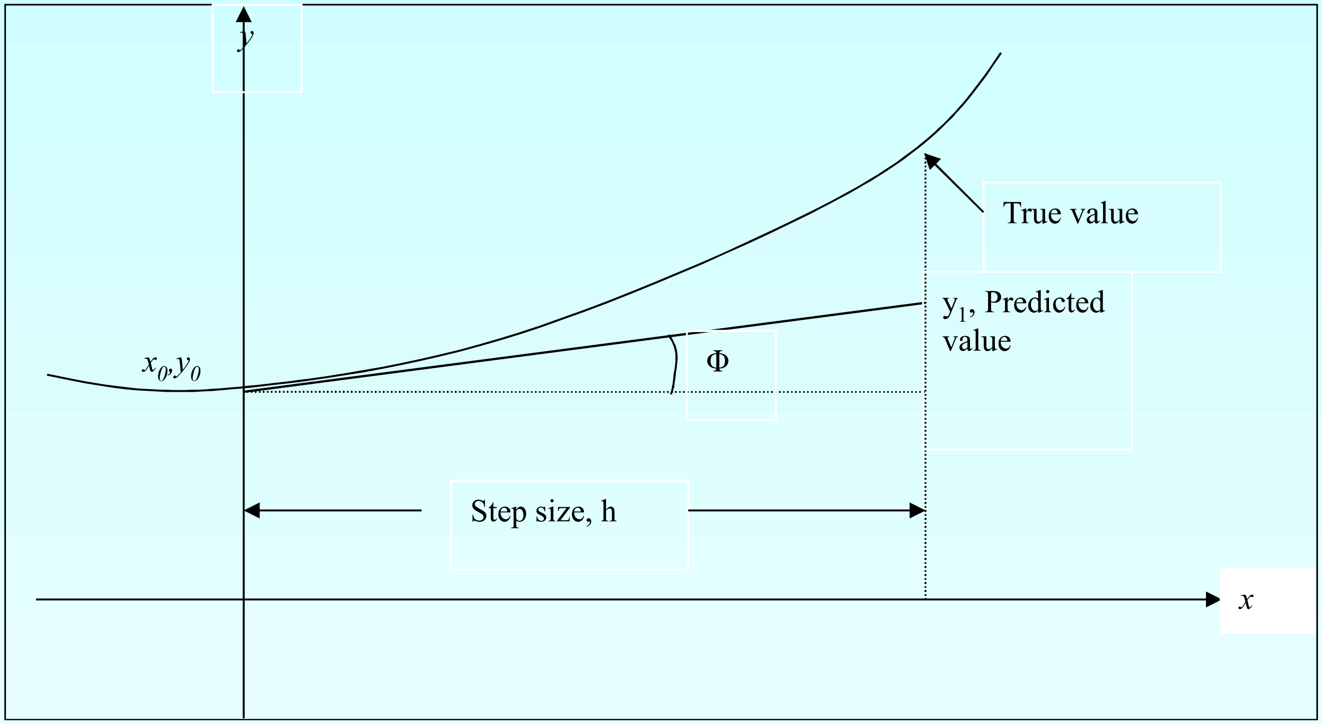

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

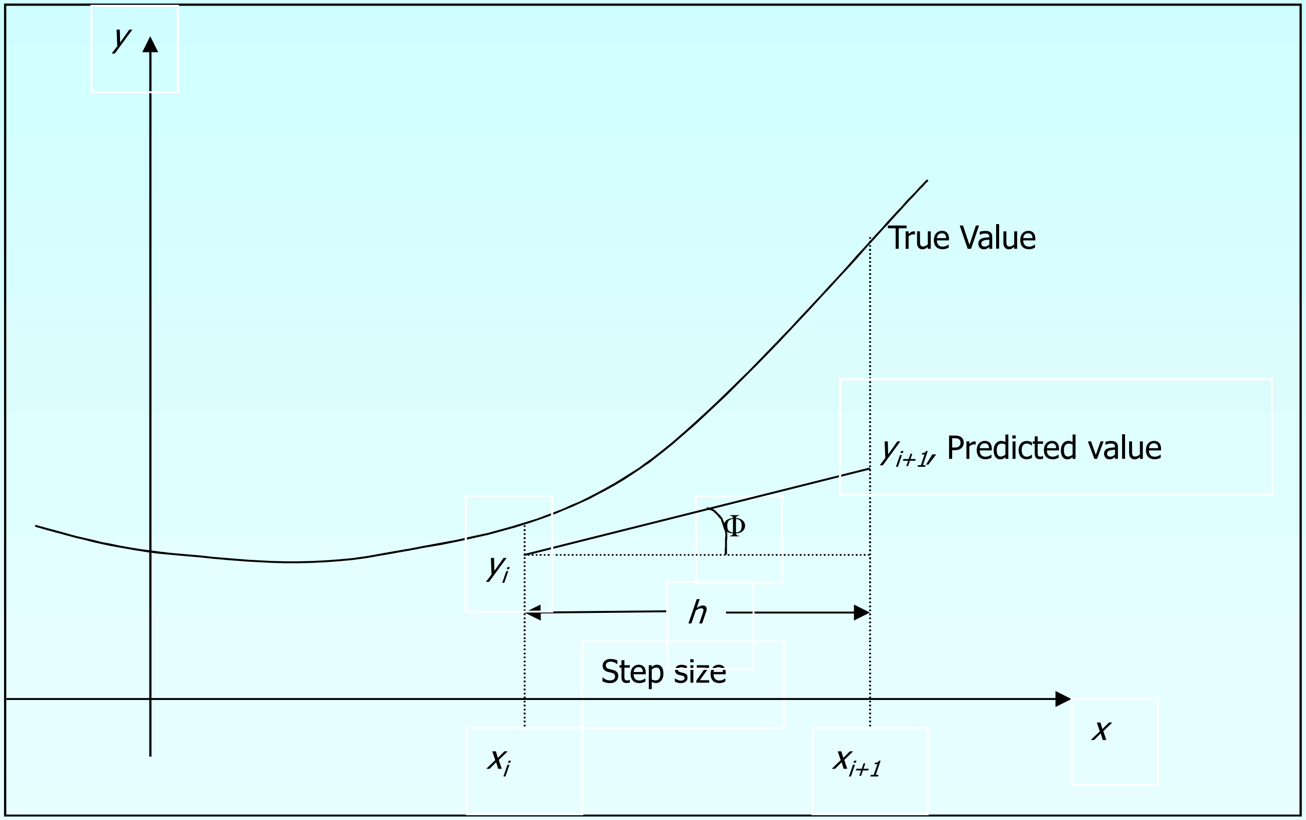

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \theta(0) = 1200\,K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0,\theta_0)h\\ &= 1200 + f(0,1200)240\\ &= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240\\ &= 1200 + (-4.5579)240\\ &= 106.09K\end{aligned}$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution Cont

Step 2: For $i = 1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$
\begin{aligned}
\theta_2 &= \theta_1 + f\left(t_1, \theta_1\right)h \\
&= 106.09 + f\left(240, 106.09\right)240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right)240 \\
&= 106.09 + \left(0.017595\right)240 \\
&= 110.32K
\end{aligned}
$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta\left(480\right) \approx \theta_2 = 110.32K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison of Exact and Numerical Solutions

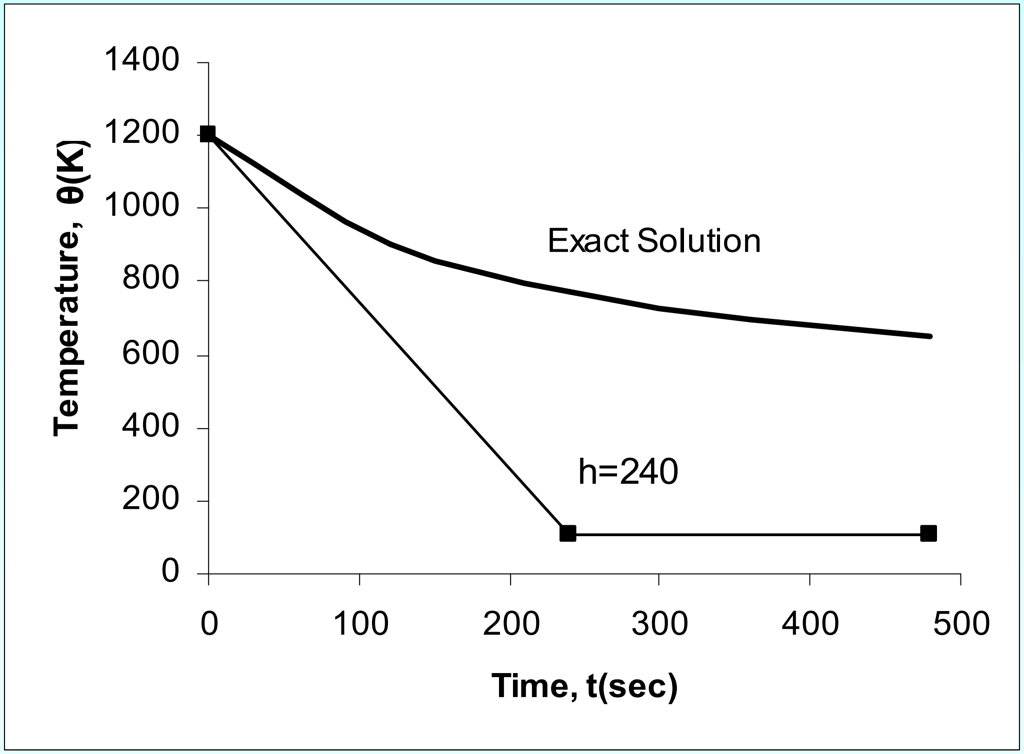

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

$\theta(480) = 647.57K$ (exact)

Comparison with exact results

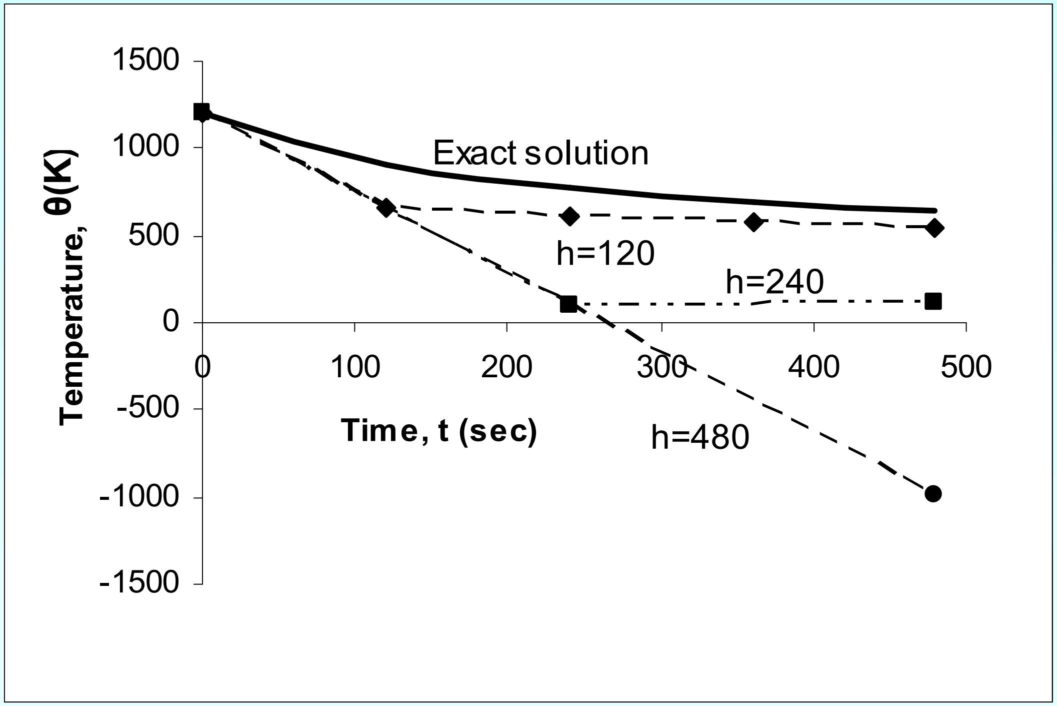

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

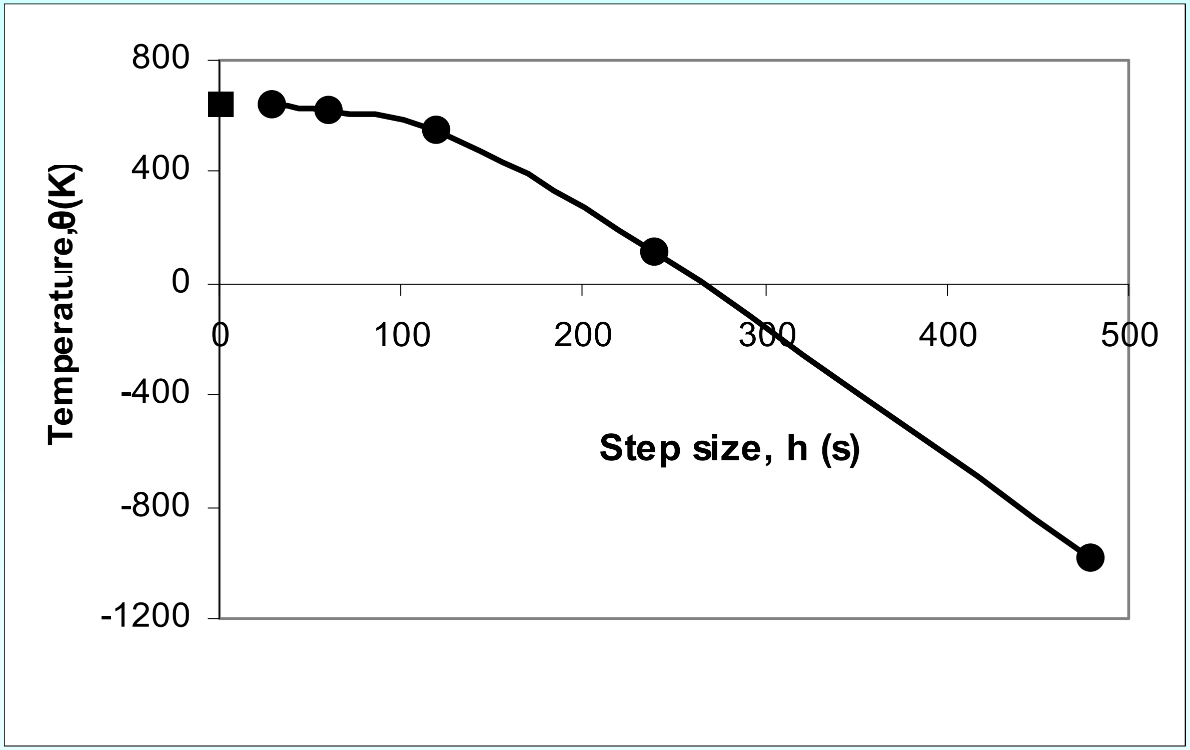

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2 y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3 y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method- Runge Kutta 1st Order Method

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$

Ordinary differential Equations: Runge-Kutta 2nd Order Method

Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

$a_1 + a_2 = 1$

$a_2 p_1 = 1/2$

$a_2 q_{11} = 1/2$

3 equations 4 unknowns- assume 1 and other 3 then can be determined

Generally a_2 is chosen→1/2, 1 and 2/3

Heun's Method

Heun's method

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

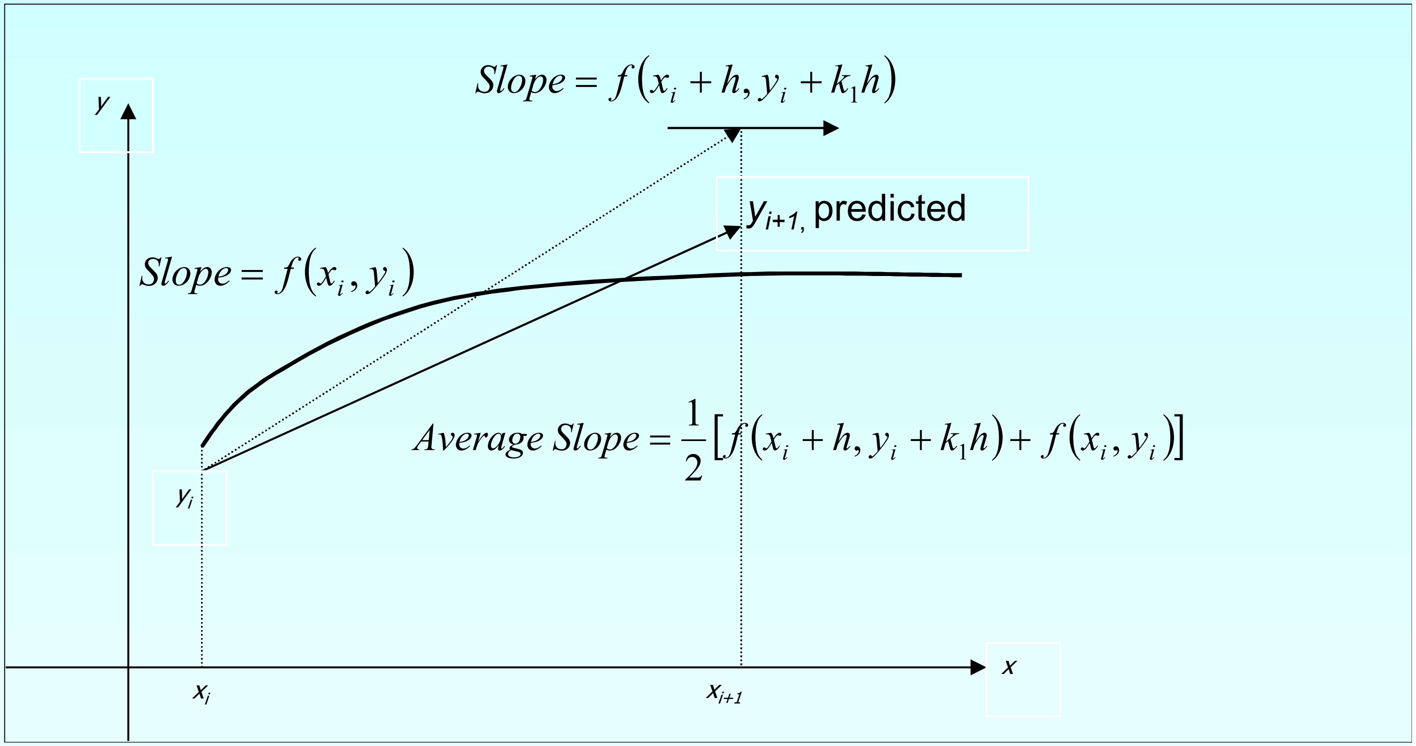

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f(t,\theta) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$

$$
\begin{aligned}
k_1 &= f(t_0, \theta_o) \\
&= f(0,1200) \\
&= -2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right) \\
&= -4.5579
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\
&= f(0 + 240, 1200 + (-4.5579)240) \\
&= f(240, 106.09) \\
&= -2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right) \\
&= 0.017595
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\
&= 1200 + (-2.2702)240 \\
&= 655.16K
\end{aligned}
$$

Solution Cont

Step 2: $i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$\begin{aligned} k_1 &= f(t_1, \theta_1) \\ &= f(240, 655.16) \\ &= -2.2067 \times 10^{-12}\left(655.16^4 - 81 \times 10^8\right) \\ &= -0.38869 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\ &= f(240 + 240, 655.16 + (-0.38869)240) \\ &= f(480, 561.87) \\ &= -2.2067 \times 10^{-12}\left(561.87^4 - 81 \times 10^8\right) \\ &= -0.20206 \end{aligned}$$

$$\begin{aligned} \theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\ &= 655.16 + (-0.29538)240 \\ &= 584.27K \end{aligned}$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison with exact results

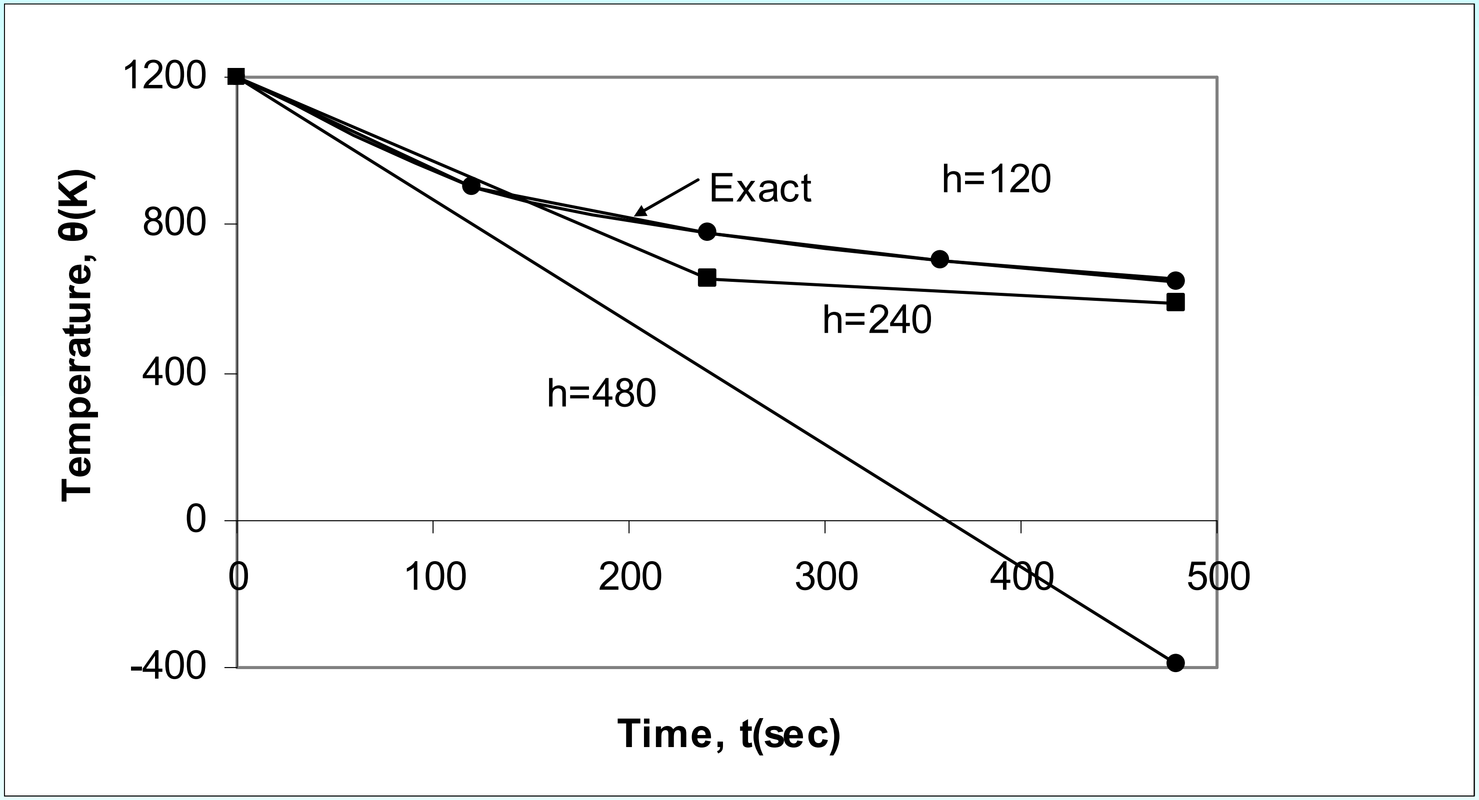

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145
30	648.21	−0.63219	0.097625

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Effects of step size on Heun's Method

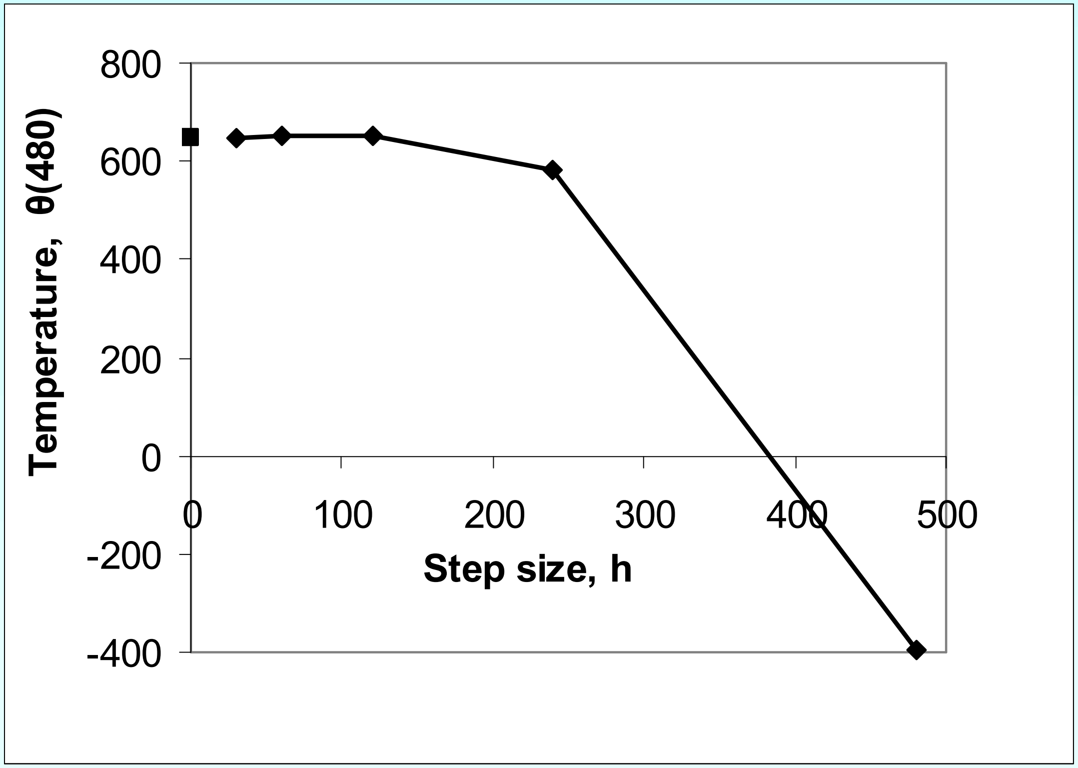

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left\|\in_t\right\|\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$$\theta(480) = 647.57K$$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

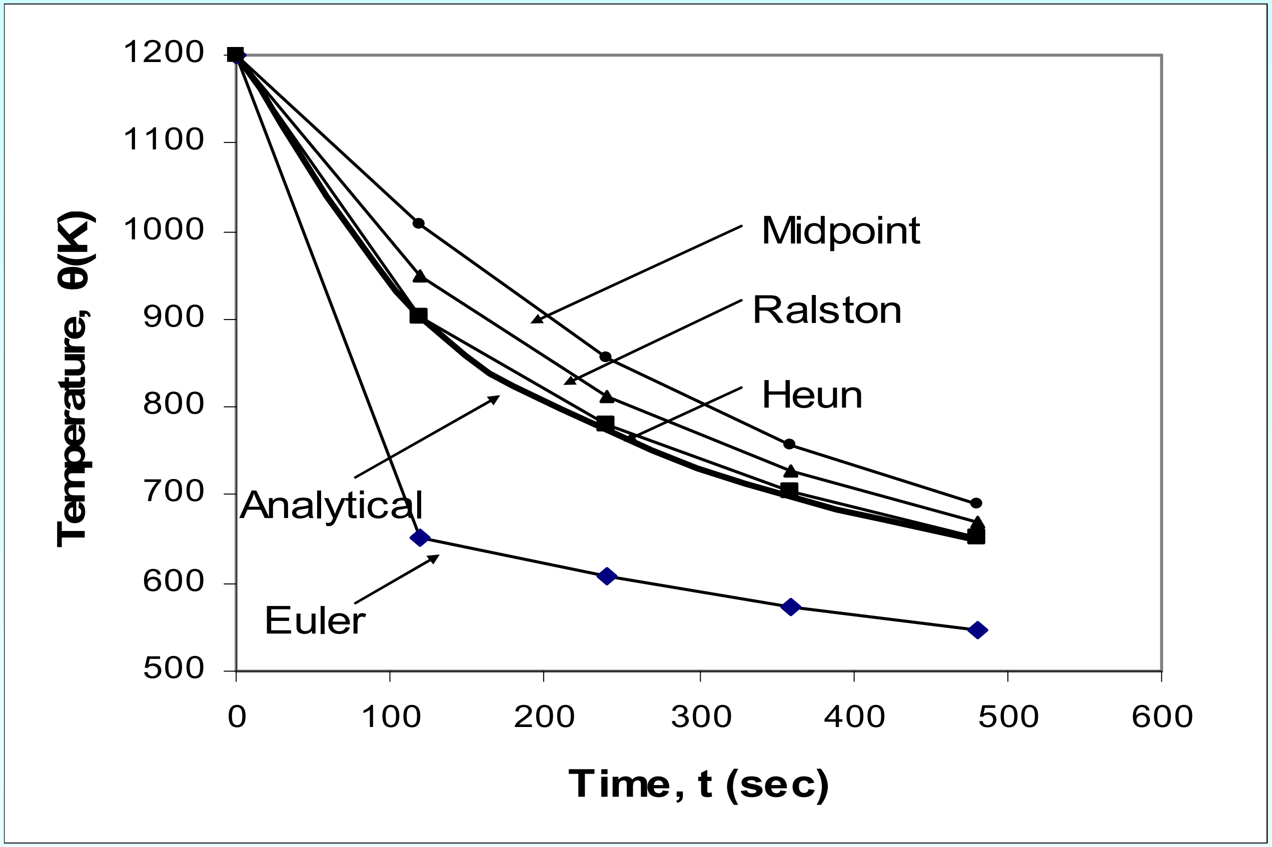

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Ordinary differential Equations: Runge-Kutta 4^{th} Order Method

Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method.

Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$f(t,\theta) = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f(t_0, \theta_o) = f(0,1200) = -2.2067\times 10^{-12}\left(1200^4 - 81\times 10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$

$$= f(120, 653.05) = -2.2067\times 10^{-12}\left(653.05^4 - 81\times 10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$

$$= f(120, 1154.0) = 2.2067\times 10^{-12}\left(1154.0^4 - 81\times 10^8\right) = -3.8954$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h) = f(0 + (240), 1200 + (-3.984)240)$$

$$= f(240, 265.10) = 2.2067\times 10^{-12}\left(265.10^4 - 81\times 10^8\right) = 0.0069750$$

Solution Cont

$$\theta_1 = \theta_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 1200 + \frac{1}{6}\left(-4.5579 + 2\left(-0.38347\right) + 2\left(-3.8954\right) + \left(0.069750\right)\right)240$$

$$= 1200 + \frac{1}{6}\left(-2.1848\right)240$$

$$= 675.65K$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta(240) \approx \theta_1 = 675.65K$$

Solution Cont

Step 2: $i = 1, t_1 = 240, \theta_1 = 675.65K$

$$k_1 = f(t_1, \theta_1) = f(240,675.65) = -2.2067\times10^{-12}\left(675.65^4 - 81\times10^8\right) = -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right)$$

$$= f(360,622.61) = -2.2067\times10^{-12}\left(622.61^4 - 81\times10^8\right) = -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372)240\right)$$

$$= f(360, 638.00) = 2.2067\times10^{-12}\left(638.00^4 - 81\times10^8\right) = -0.34775$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h) = f(240 + (240), 675.65 + (-0.34775)240)$$

$$= f(480,592.19) = 2.2067\times10^{-12}\left(592.19^4 - 81\times10^8\right) = -0.25351$$

Solution Cont

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 675.65 + \frac{1}{6}\left(-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351)\right)240$$

$$= 675.65 + \frac{1}{6}\left(-2.0184\right)240$$

$$= 594.91K$$

θ_2 is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta(480) \approx \theta_2 = 594.91K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}\left(0.003333\theta\right) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

Comparison with exact results

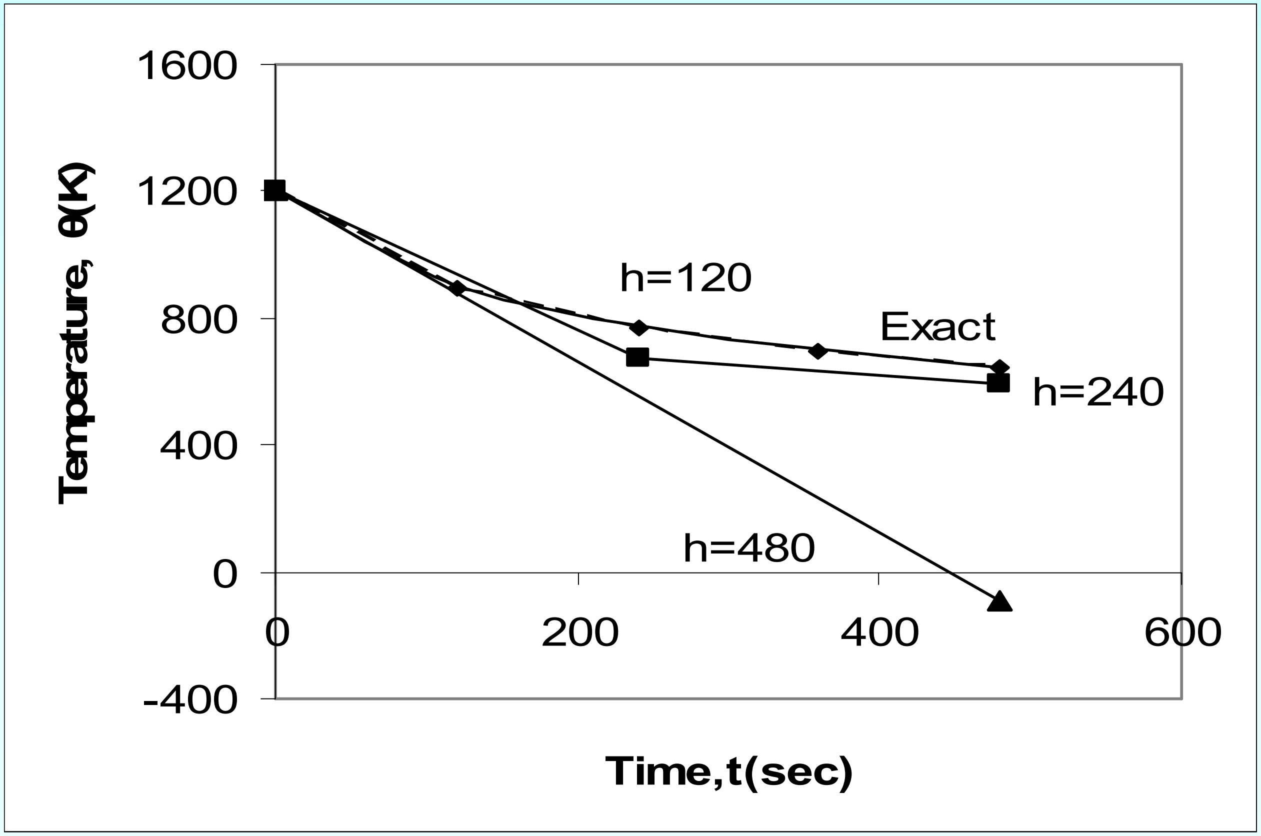

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, *h*	θ (480)	E_t	$\lvert \epsilon_t \rvert$%
480	−90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

$\theta(480) = 647.57K$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

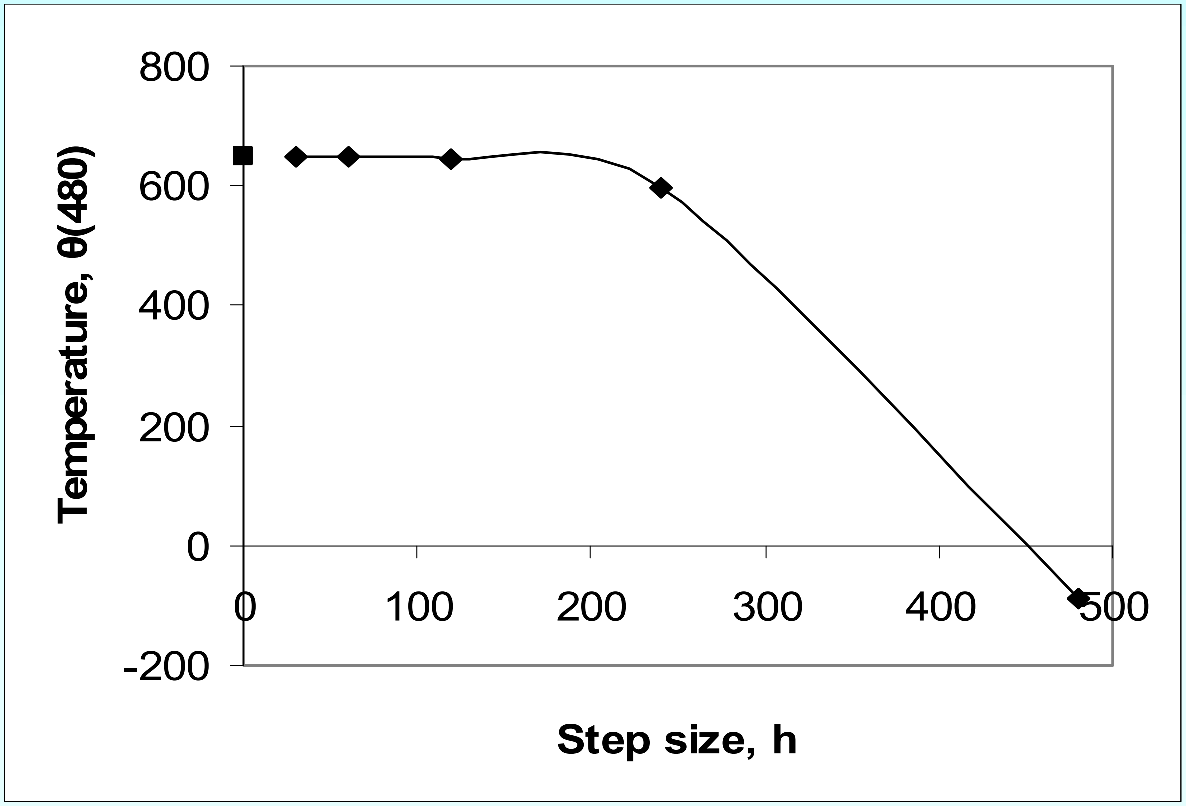

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

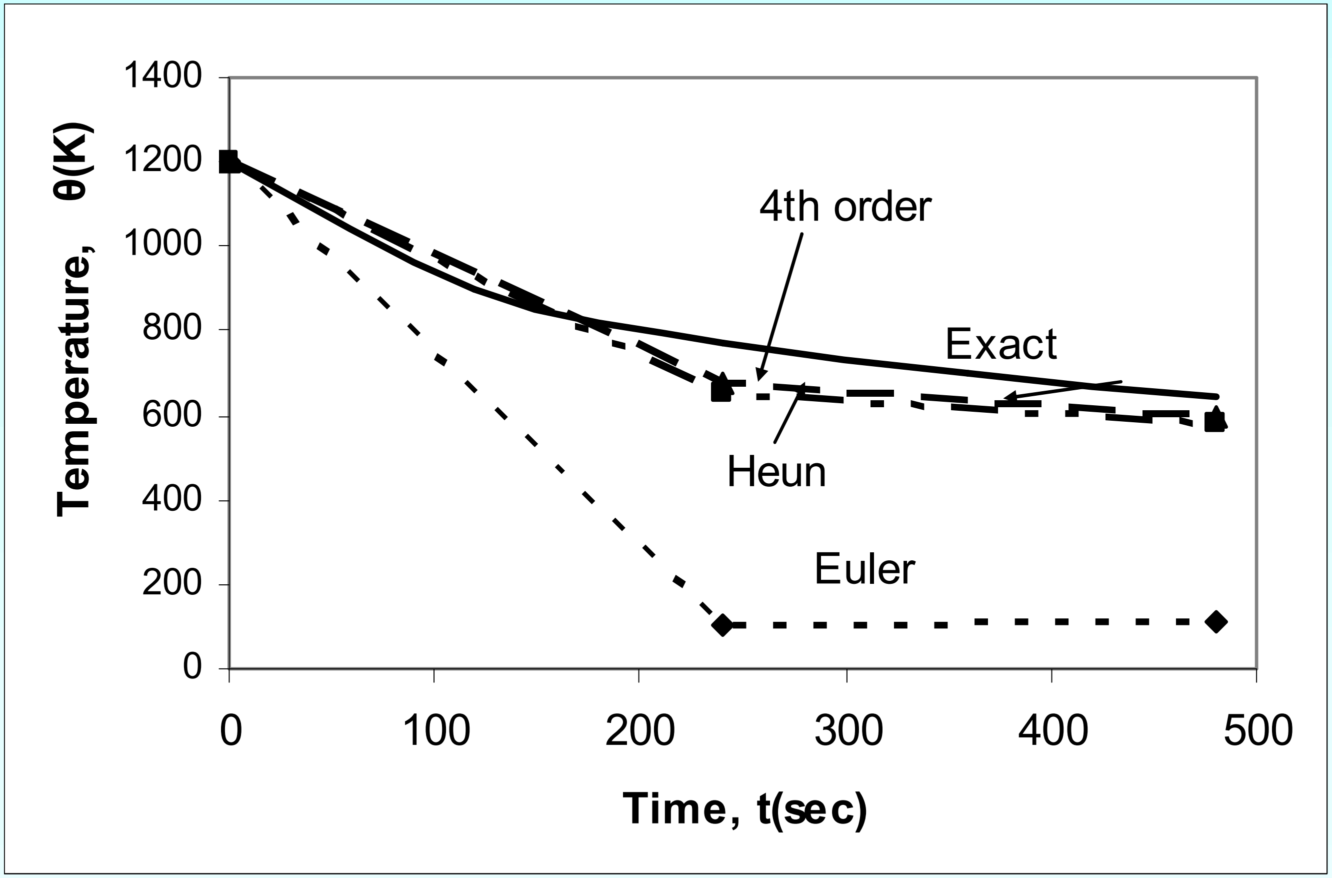

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.