CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Divide-and-Conquer

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Outline

Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting n numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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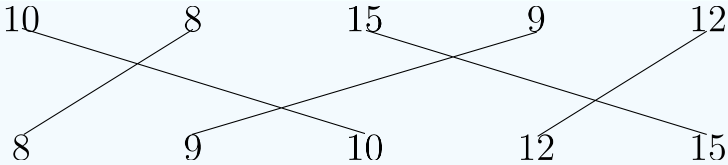

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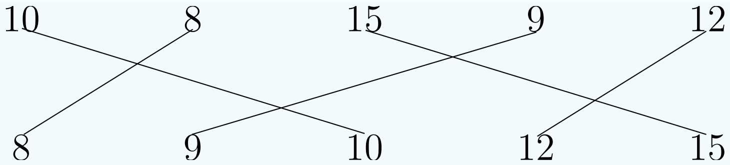


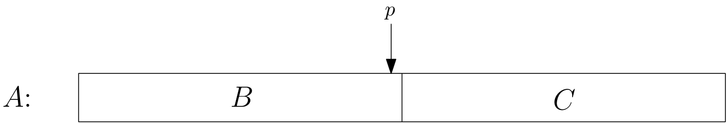
- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
c ← 0
for every i ← 1 to n − 1 do
    for every j ← i + 1 to n do
        if A[i] > A[j] then c ← c + 1
return c
```

Divide-and-Conquer



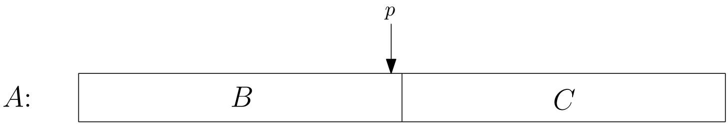
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \left|\left\{(i,j) : B[i] > C[j]\right\}\right|$$

Q: How fast can we compute m, via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer



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Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:	3	8	12	20	32	48

total$=$ 0

C:	5	7	9	25	29

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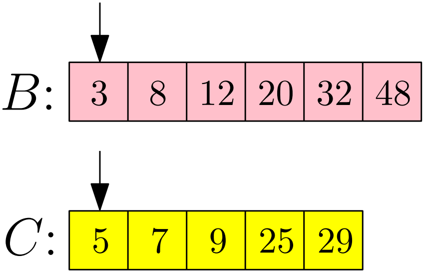

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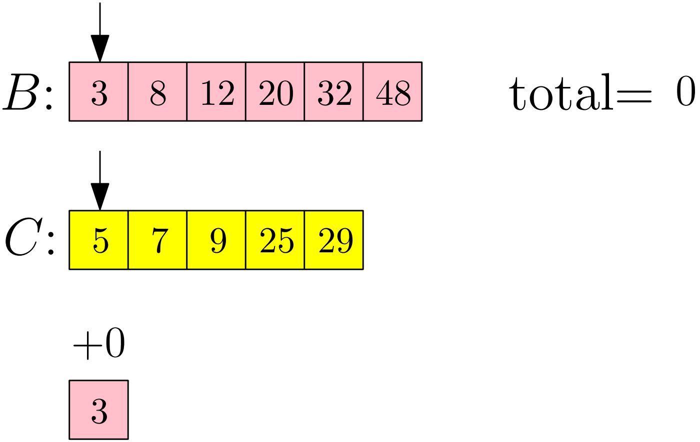

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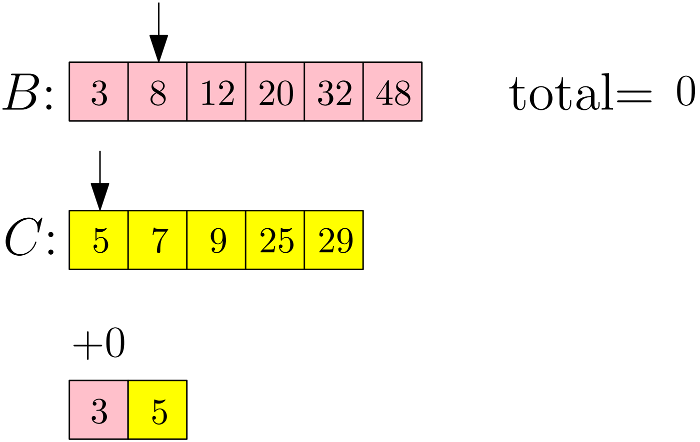

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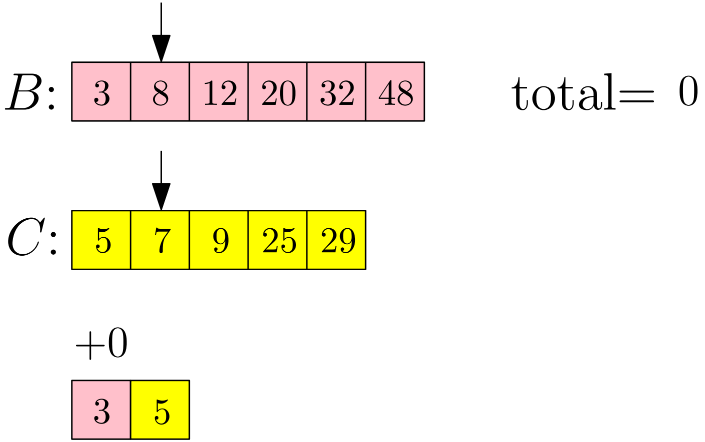

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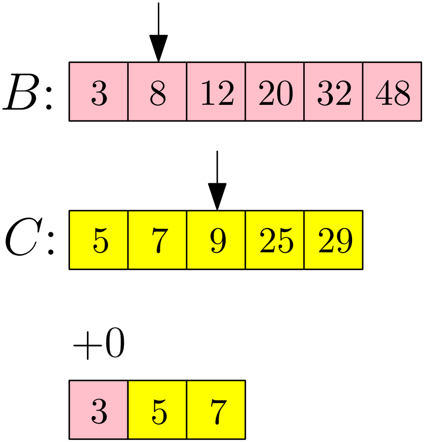

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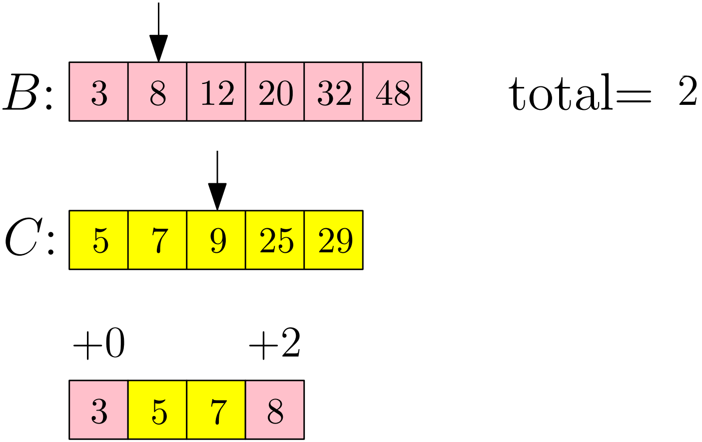

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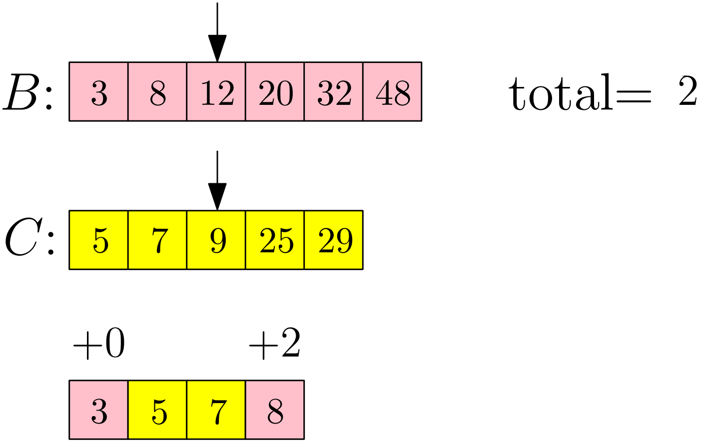

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B:

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total$=$ 2

C:

5	7	9	25	29

$+0$ $\quad$ $+2$

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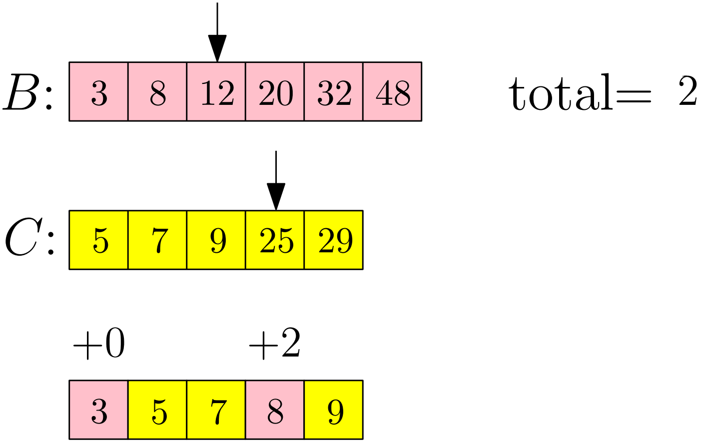

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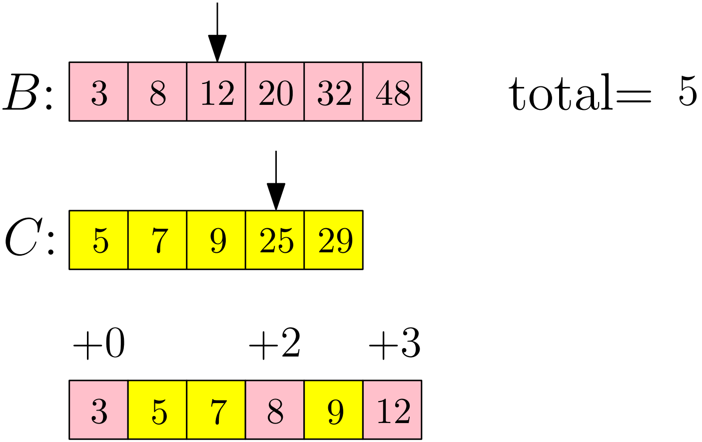

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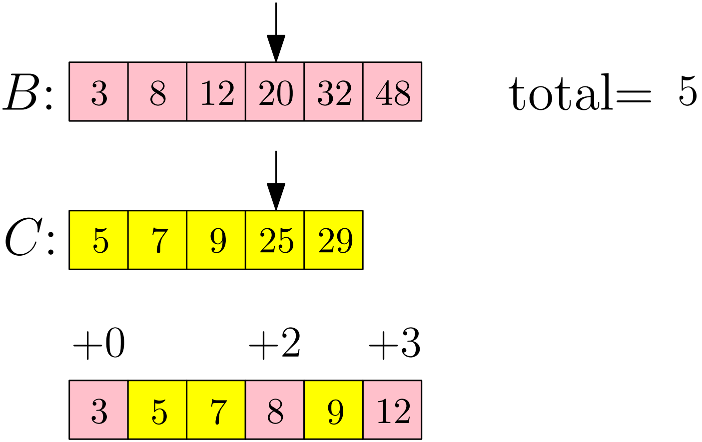

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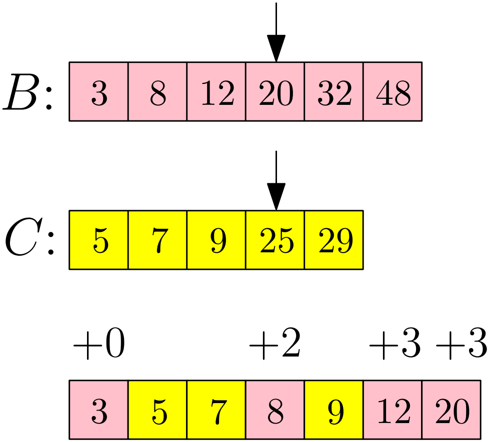
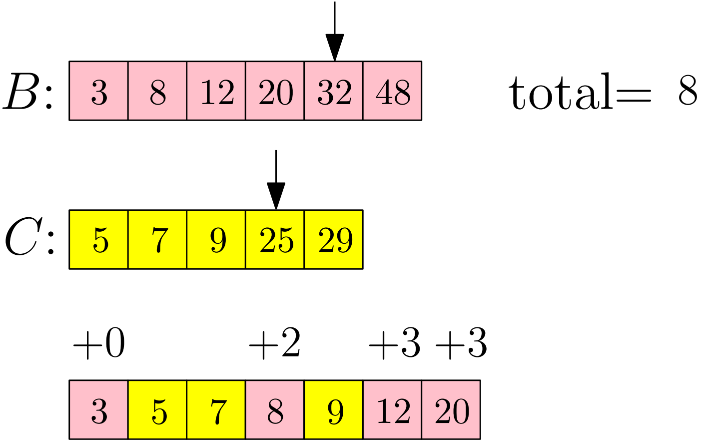


total= 8

Count pairs i, j such that $B[i] > C[j]$:



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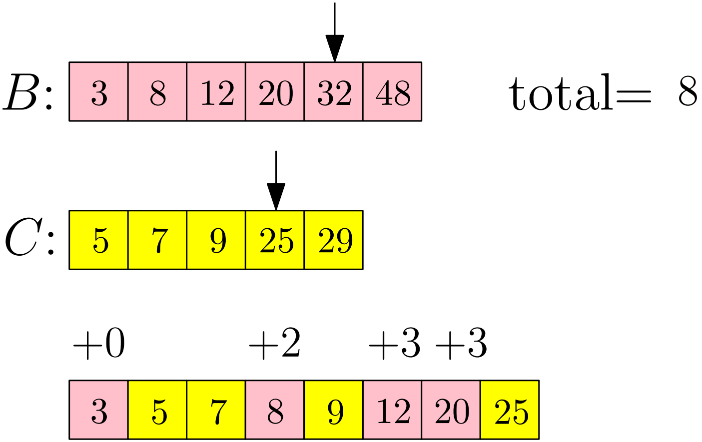

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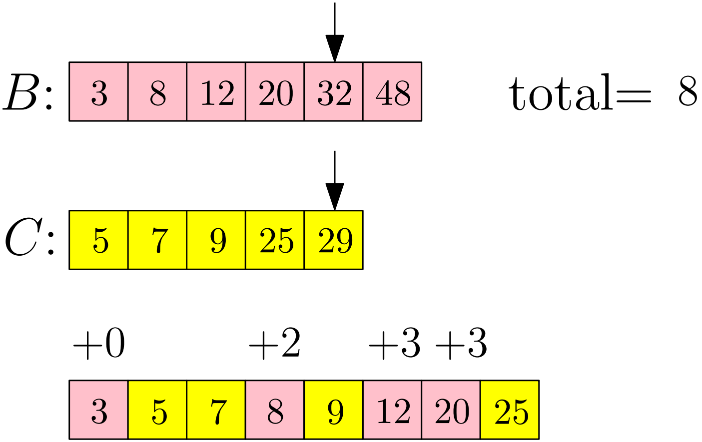

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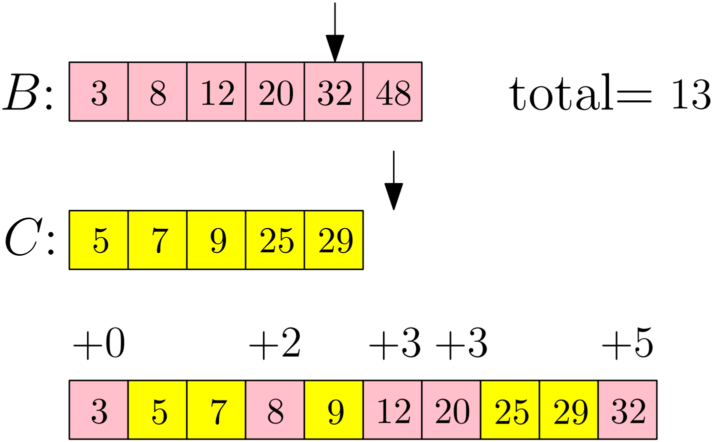

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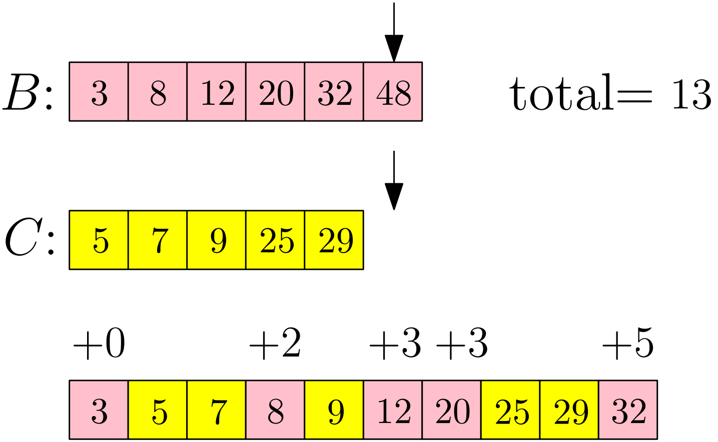

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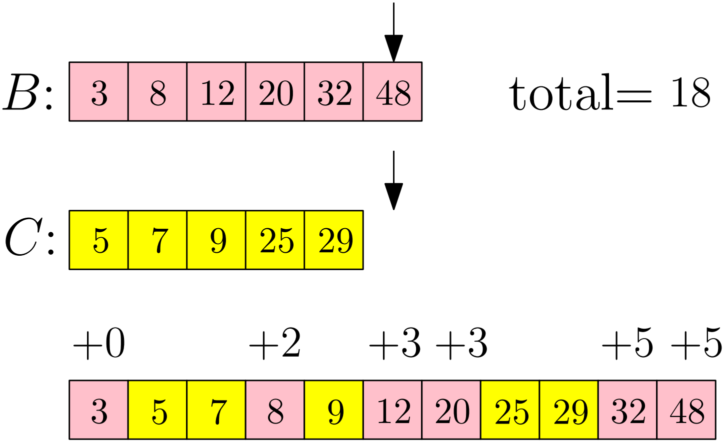

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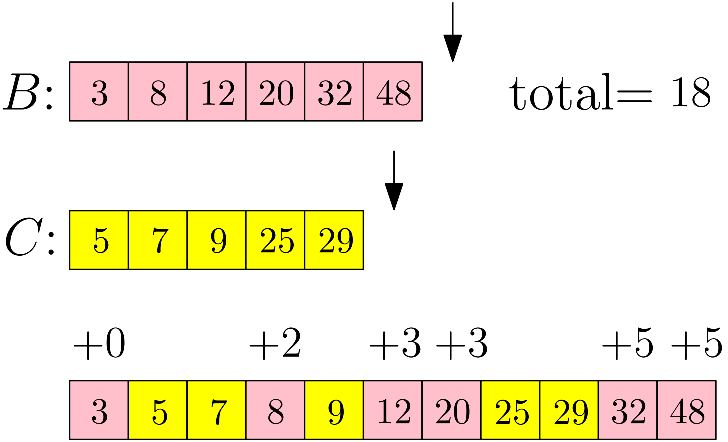

Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

$count \leftarrow 0$;
$A \leftarrow$ array of size $n_1 + n_2$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ or $j \leq n_2$ **do**
    if $j > n_2$ or $(i \leq n_1$ and $B[i] \leq C[j])$ **then**
        $A[i+j-1] \leftarrow B[i]$; $i \leftarrow i+1$
        $count \leftarrow count + (j-1)$
    else
        $A[i+j-1] \leftarrow C[j]$; $j \leftarrow j+1$
return $(A, count)$

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A:

sort-and-count(A, n)

```
if n = 1 then
    return (A, 0)
else
    (B, m_1) ← sort-and-count(A[1..⌊n/2⌋], ⌊n/2⌋)
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    (A, m_3) ← merge-and-count(B, C, ⌊n/2⌋, ⌈n/2⌉)
    return (A, m_1 + m_2 + m_3)
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- Divide: trivial
- Conquer: 4, 5
- Combine: 6, 7

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Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

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Quicksort

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ array of elements in A that are less than x \\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$ \\ Conquer
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Q: How to remove this assumption?

A:

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

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Q: Can computers really produce random numbers?

A: No! The execution of a computer programs is deterministic!

- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that "look like" random
- In theory: assume they can.

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Lemma The expected running time of the algorithm is $O(n \lg n)$.

Quicksort Can Be Implemented as an “In-Place” Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses “small” extra space.