CSE 431/531: Algorithm Analysis and Design (Fall 2024)

Connectivity, Graph Traversal and Bipartiteness

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University at Buffalo

Announcements: Quiz 3

- Posted on Ublearns
- Should take $<$ 30 minutes, 2 attempts
- Due Tue 10 Sep @ 11:59PM

Outline

Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)

two vertices $s, t \in V$

Output: whether there is a path connecting s to t in G

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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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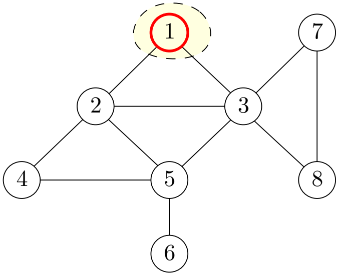

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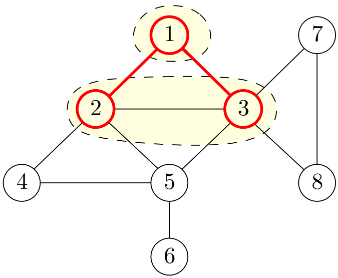

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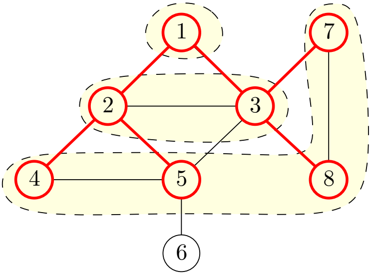

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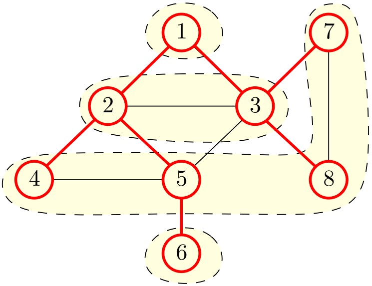


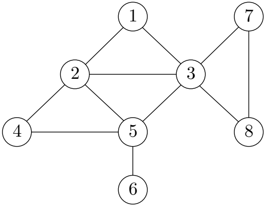
Implementing BFS using a Queue

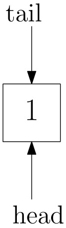
BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark s as "visited" and all other vertices as "unvisited"
while $head \leq tail$ **do**
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbors u of v **do**
        if u is "unvisited" **then**
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark u as "visited"

- Running time: $O(n + m)$.

Example of BFS via Queue





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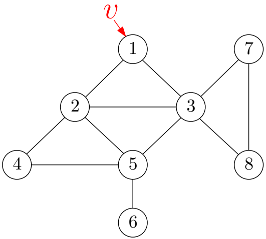

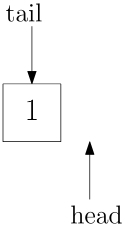

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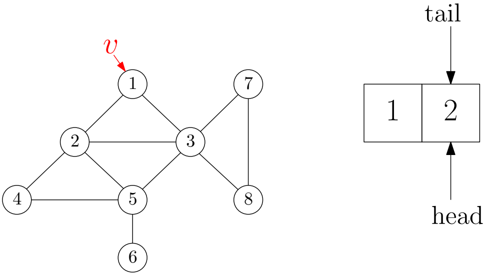

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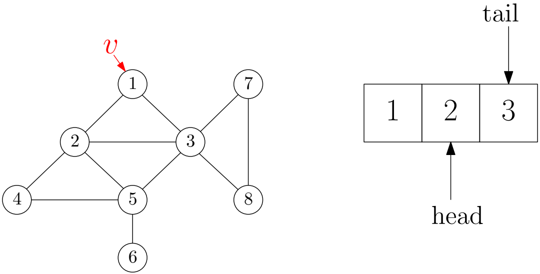

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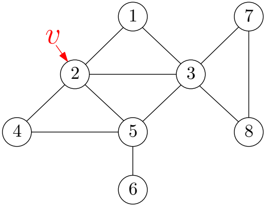

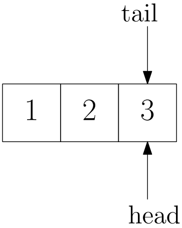

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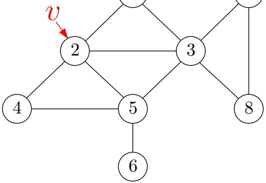

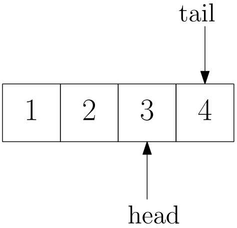

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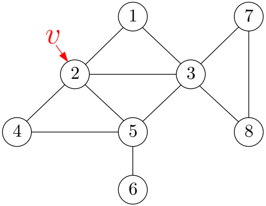

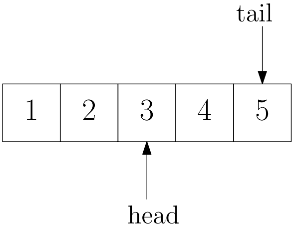

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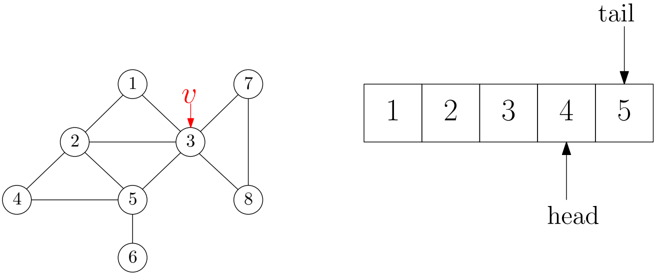

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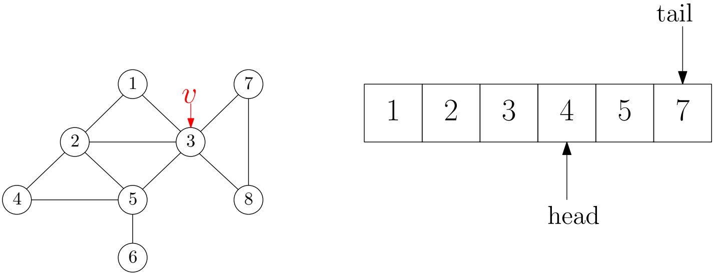

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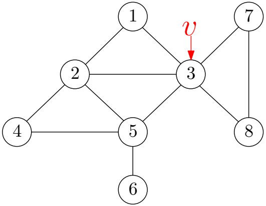

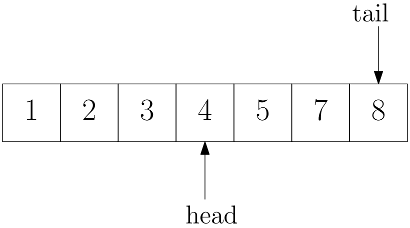

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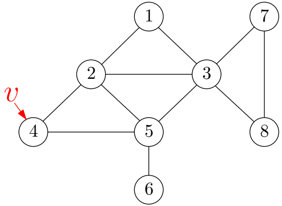

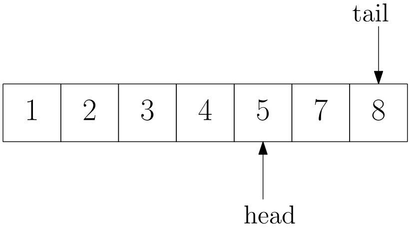

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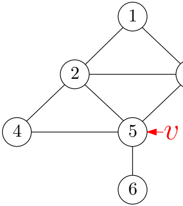

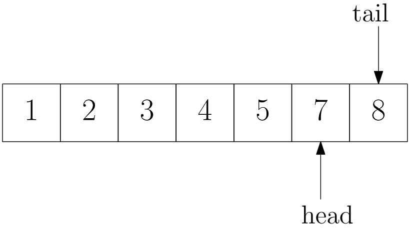

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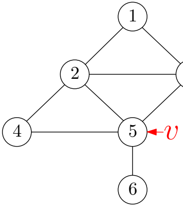

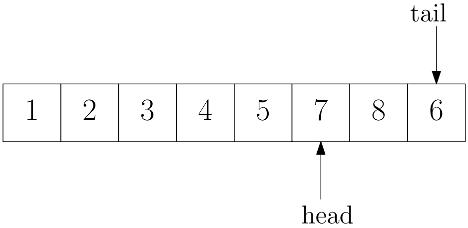

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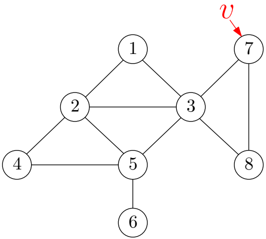

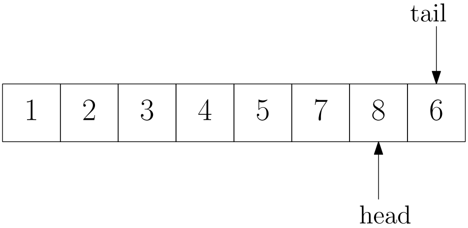

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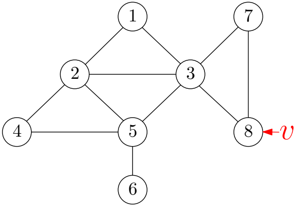

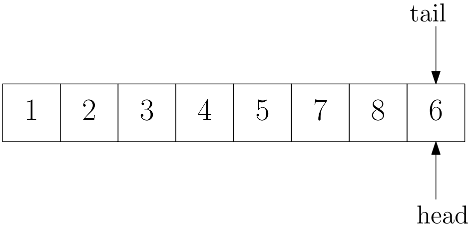

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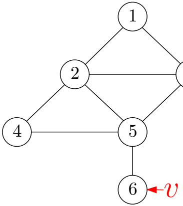

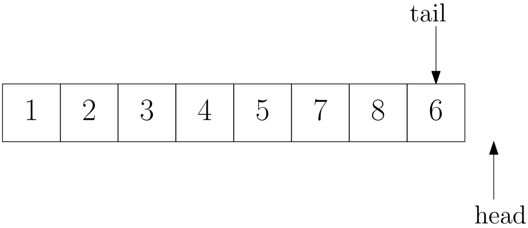

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Edges included in BFS algorithm starting with vertex 1: {1, 2}, {1, 3}, {2, 4}, {2, 5}, {3, 7}, {3, 8}, {5, 6}

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex ("dead-end"), go back
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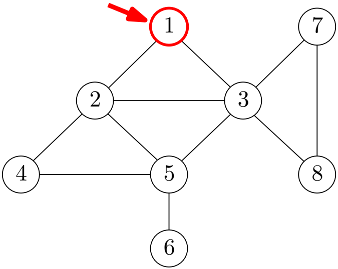

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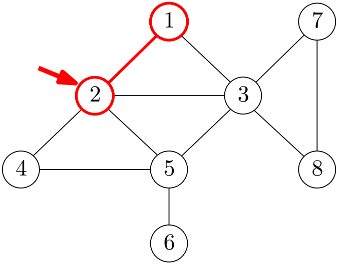

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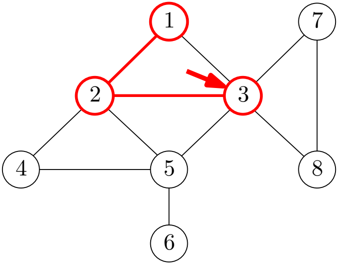

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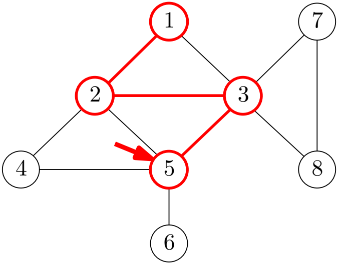

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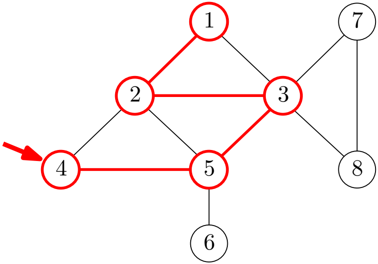

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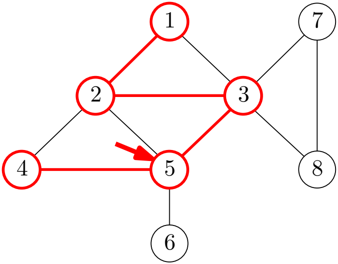

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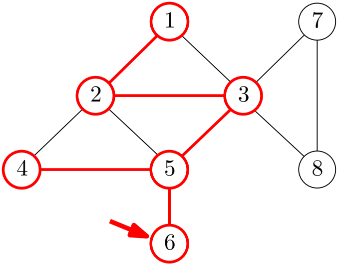

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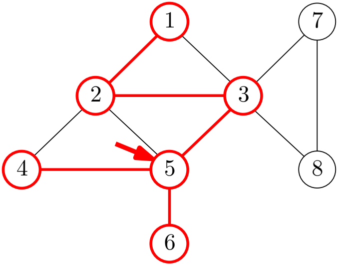

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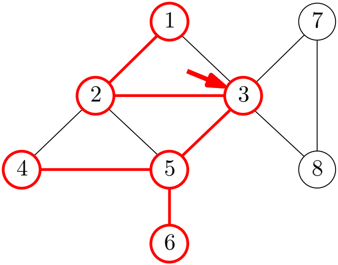

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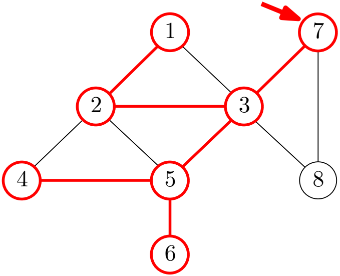

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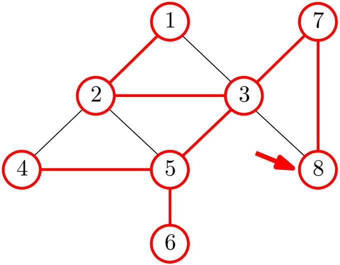

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Edges included in DFS algorithm starting with vertex 1: {1, 2}, {2, 3}, {3, 5}, {5, 4}, {5, 6}, {3, 7}, {7, 8}

Implementing DFS using Recurrsion

DFS(s)

```
mark all vertices as "unvisited"
recursive-DFS(s)
```

recursive-DFS(v)

```
mark v as "visited"
for all neighbors u of v do
    if u is unvisited then recursive-DFS(u)
```

Outline

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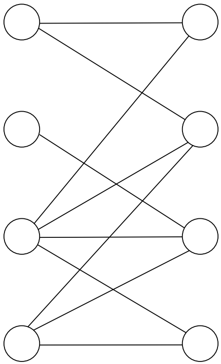
Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



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- Report "not a bipartite graph" if contradiction was found

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- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
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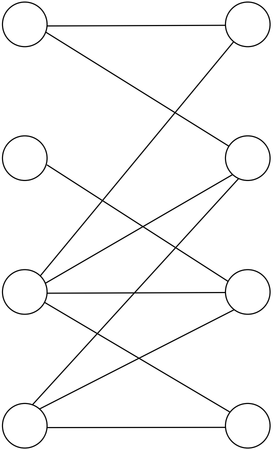
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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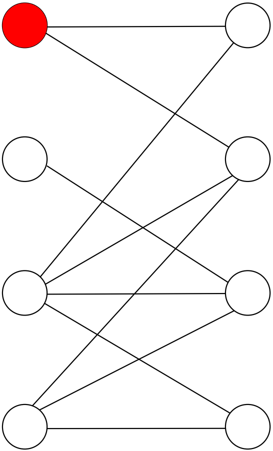

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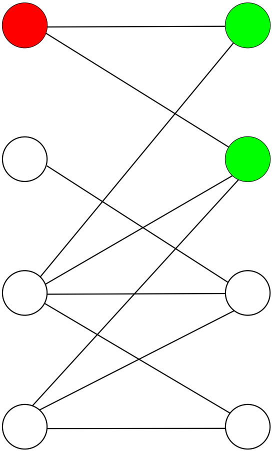

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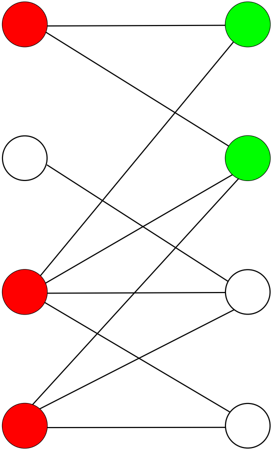

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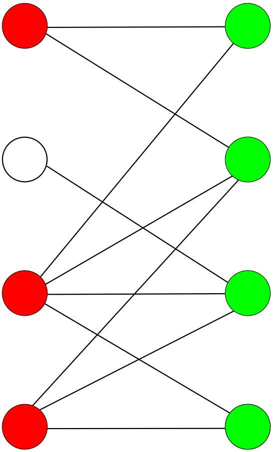

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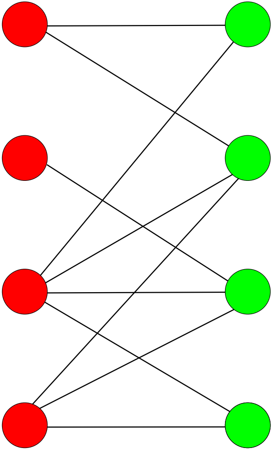
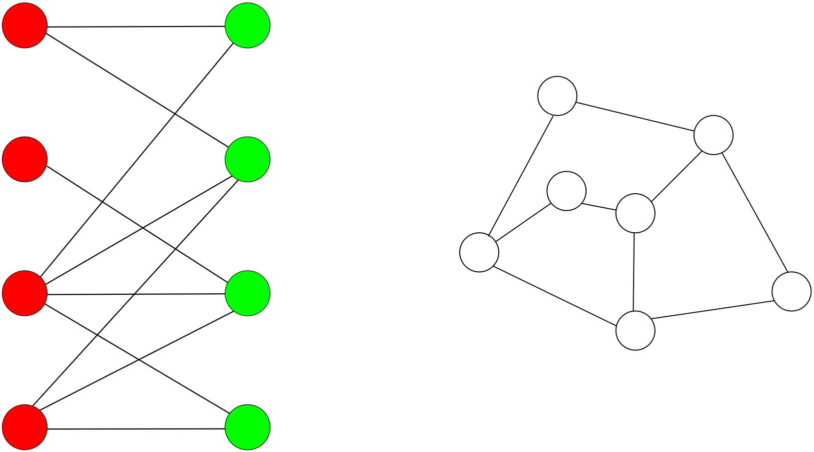


Test Bipartiteness



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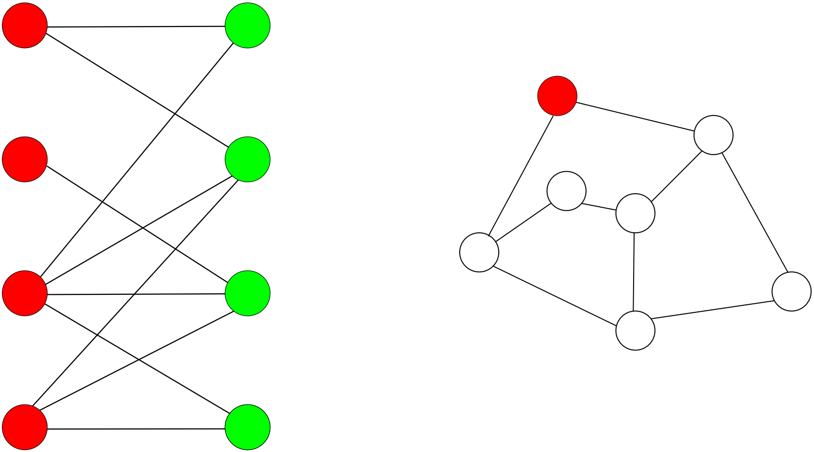

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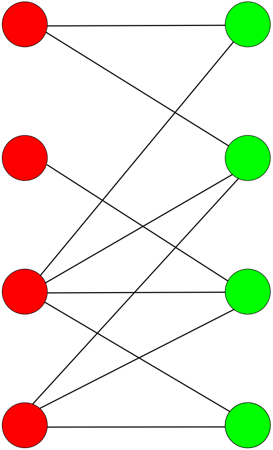

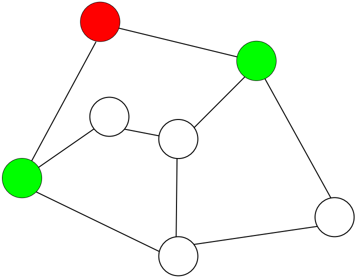

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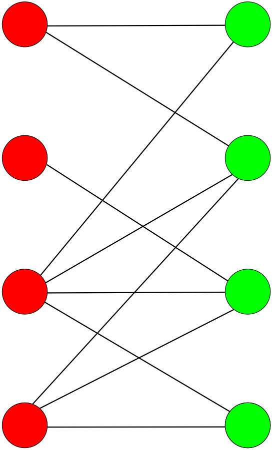

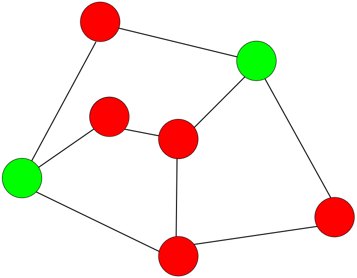

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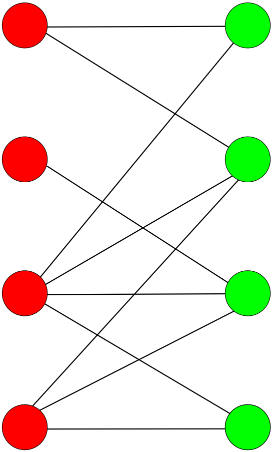

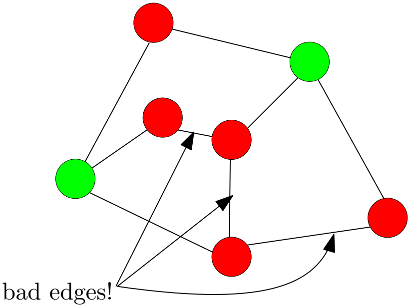

Testing Bipartiteness using BFS

BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark s as "visited" and all other vertices as "unvisited"
while $head \leq tail$ **do**
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbors u of v **do**
        if u is "unvisited" **then**
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark u as "visited"

Testing Bipartiteness using BFS

test-bipartiteness(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark s as "visited" and all other vertices as "unvisited"
$color[s] \leftarrow 0$
while $head \leq tail$ **do**
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbors u of v **do**
        if u is "unvisited" **then**
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark u as "visited"
            $color[u] \leftarrow 1 - color[v]$
        else if $color[u] = color[v]$ **then**
            print("G is not bipartite") and exit

Testing Bipartiteness using BFS

```
mark all vertices as "unvisited"
for each vertex v ∈ V do
    if v is "unvisited" then
        test-bipartiteness(v)
print("G is bipartite")
```

Testing Bipartiteness using BFS

```
mark all vertices as "unvisited"
for each vertex v ∈ V do
    if v is "unvisited" then
        test-bipartiteness(v)
print("G is bipartite")
```

Obs. Running time of algorithm $= O(n+m)$

Testing Bipartiteness using DFS

test-bipartiteness-DFS(s)

1: mark all vertices as "unvisited"
2: recursive-test-DFS(s)

recursive-test-DFS(v)

1: mark v as "visited"
2: **for** all neighbors u of v **do**
3:     **if** u is unvisited **then** , recursive-test-DFS(u)

Testing Bipartiteness using DFS

test-bipartiteness-DFS(s)

mark all vertices as "unvisited"
$color[s] \leftarrow 0$
recursive-test-DFS(s)

recursive-test-DFS(v)

mark v as "visited"
for all neighbors u of v **do**
    if u is unvisited **then**
        $color[u] \leftarrow 1 - color[v]$, recursive-test-DFS($u$)
    else if $color[u] = color[v]$ **then**
        print("G is not bipartite") and exit

Testing Bipartiteness using DFS

mark all vertices as "unvisited"
for each vertex $v \in V$ **do**
    if v is "unvisited" **then**
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Testing Bipartiteness using DFS

mark all vertices as "unvisited"
for each vertex $v \in V$ **do**
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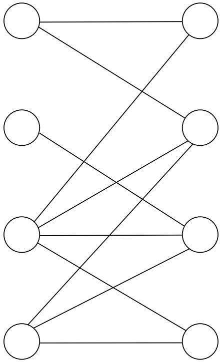
print("G is bipartite")

Obs. Running time of algorithm $= O(n+m)$

Bipartite Graph

Def. An undirected graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



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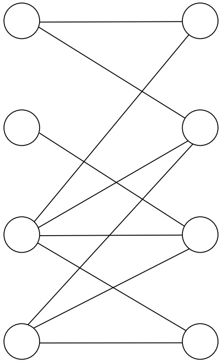
Obs. Bipartite graph may contain cycles.



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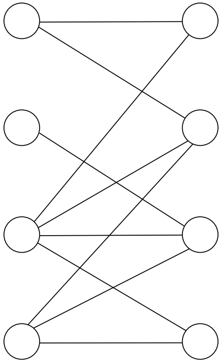
Obs. Bipartite graph may contain cycles.

Obs. If a graph is a tree, then it is also a bipartite graph.



BFS and DFS

Obs. BFS and DFS naturally induce a tree.

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