CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Dynamic Programming

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Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
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5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
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Find Common Subsequence

$i \leftarrow n, j \leftarrow m, S \leftarrow ()$
while $i > 0$ and $j > 0$ **do**
    if $\pi[i, j] =$ "$\nwarrow$" **then**
        add $A[i]$ to beginning of S, $i \leftarrow i - 1, j \leftarrow j - 1$
    else if $\pi[i, j] =$ "$\uparrow$" **then**
        $i \leftarrow i - 1$
    else
        $j \leftarrow j - 1$
return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A and a string B

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

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Obs. $\#\text{OPs} = \text{length}(A) + \text{length}(B) - 2 \cdot \text{length}(\text{LCS}(A, B))$

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- Not related to LCS any more

Edit Distance with Replacing: Reduction to a Variant of LCS

- Need to match letters in A and B, every letter is matched at most once and there should be no crosses.
- However, we can match two different letters: Matching a same letter gives score 2, matching two different letters gives score 1.
- Need to maximize the score.
- DP recursion for the case $i > 0$ and $j > 0$:

$$opt[i,j] = \begin{cases} opt[i-1,j-1] + 2 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i-1,j] \\ opt[i,j-1] \\ opt[i-1,j-1] + 1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

- Relation : #OPs $=$ length(A) $+$ length(B) - max_score

Edit Distance (with Replacing): using DP directly

- $opt[i, j], 0 \le i \le n, 0 \le j \le m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.

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- if $i = 0$ then $opt[i, j] = j$; if $j = 0$ then $opt[i, j] = i$.

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$$opt[i, j] = \begin{cases} opt[i-1, j-1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i-1, j] + 1 \\ opt[i, j-1] + 1 \\ opt[i-1, j-1] + 1 \end{cases} & \text{if } A[i] \ne B[j] \end{cases}$$

Outline

- Longest Common Subsequence in Linear Space

Computing the Length of LCS

```
for j ← 0 to m do
    opt[0, j] ← 0
for i ← 1 to n do
    opt[i, 0] ← 0
    for j ← 1 to m do
        if A[i] = B[j] then
            opt[i, j] ← opt[i − 1, j − 1] + 1
        else if opt[i, j − 1] ≥ opt[i − 1, j] then
            opt[i, j] ← opt[i, j − 1]
        else
            opt[i, j] ← opt[i − 1, j]
```

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n + m)$

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Q: How to use this observation to reduce space?

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Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

for $j \leftarrow 0$ to m **do**
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i \text{ mod } 2, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m **do**
        if $A[i] = B[j]$ **then**
            $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j - 1] + 1$
        else if $opt[i \text{ mod } 2, j - 1] \geq opt[i - 1 \text{ mod } 2, j]$ **then**
            $opt[i \text{ mod } 2, j] \leftarrow opt[i \text{ mod } 2, j - 1]$
        else
            $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j]$
return $opt[n \text{ mod } 2, m]$

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  - Space: $O(m+n)$
  - Time: $O(nm)$