CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Greedy Algorithms

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Outline

1. Interval Scheduling
   - Interval Partitioning

2. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue

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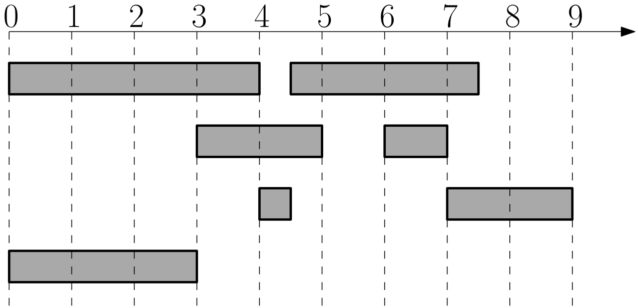
Interval Partitioning

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A minimum number of machines to schedule all jobs so that all jobs on a single machine are compatible.



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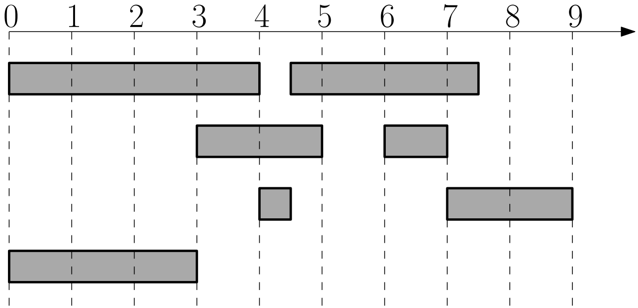

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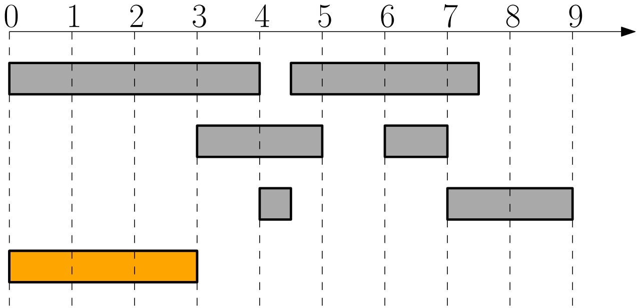

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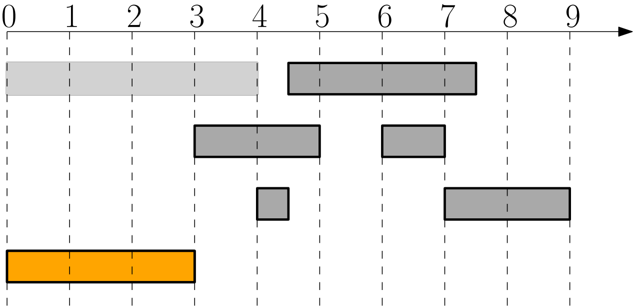

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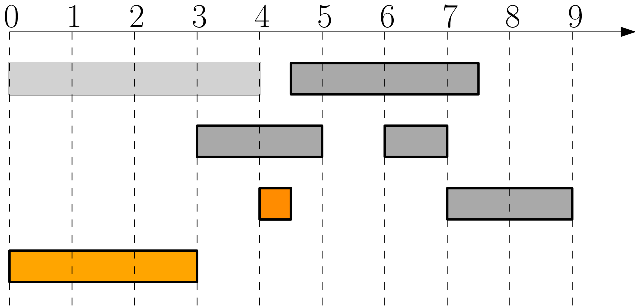

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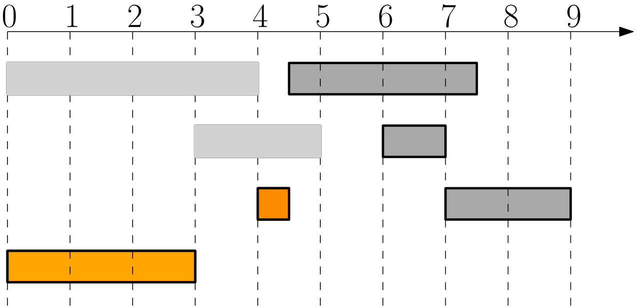

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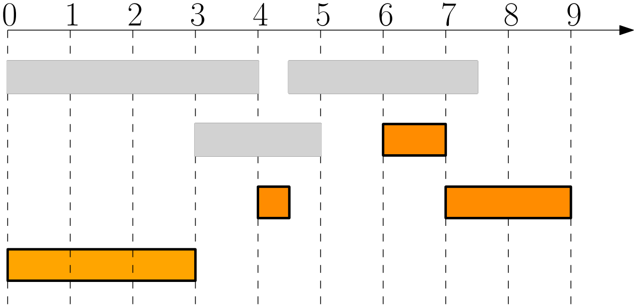

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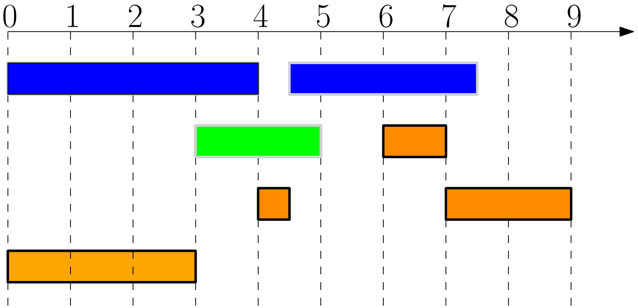

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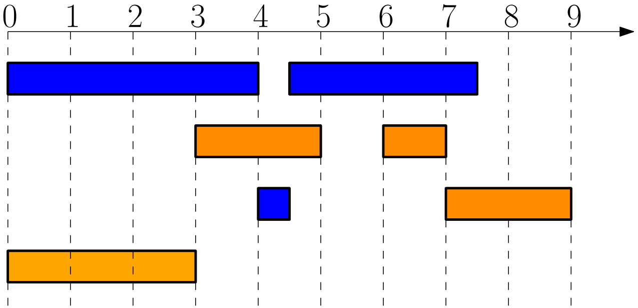

Greedy Algorithm for Interval Partitioning

Lemma It is safe to schedule the job j with the earliest starting time to a feasible machine: There exists an optimum solution where job j with the earliest starting time is scheduled first on a machine that is compatible with all jobs in that machine if applicable; otherwise, it can be scheduled by opening a new machine.

Proof.

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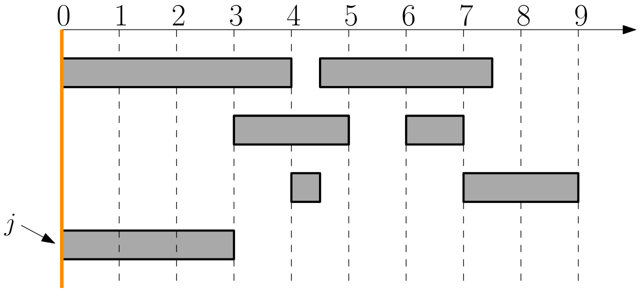
- Otherwise, replace all the jobs scheduled to the machine i in S with j and its subsequent jobs to obtain another optimum schedule S'. $\square$

Greedy Algorithm for Interval Partitioning

- What is the remaining task after we decided to schedule j?
- Is it another instance of interval partitioning problem?



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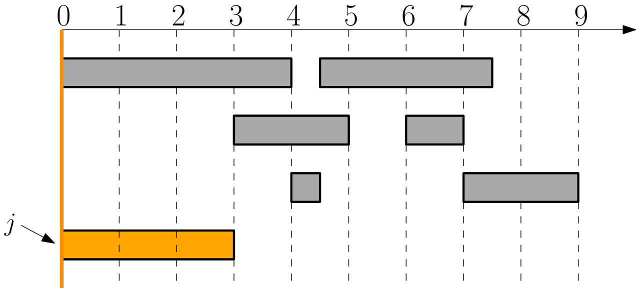

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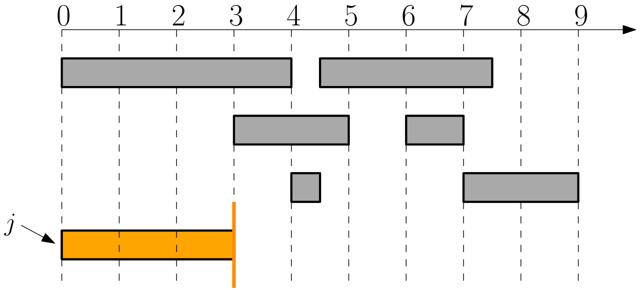

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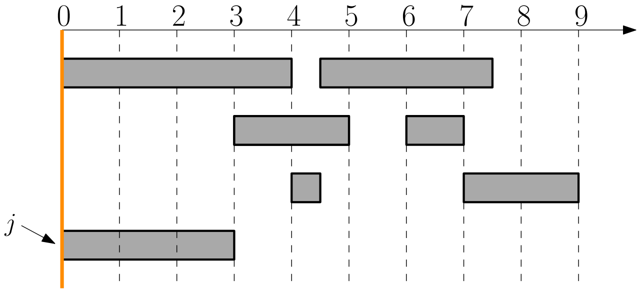

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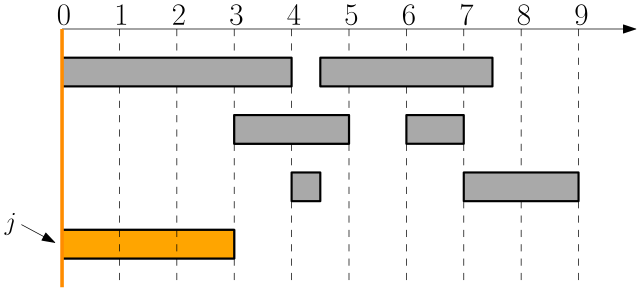

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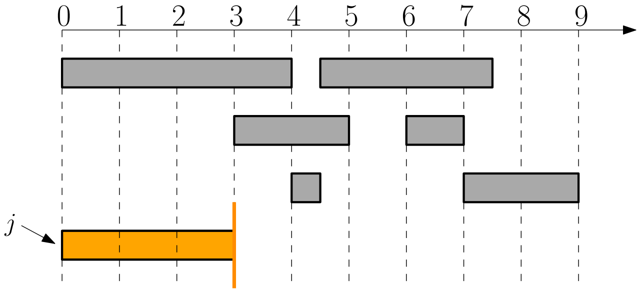

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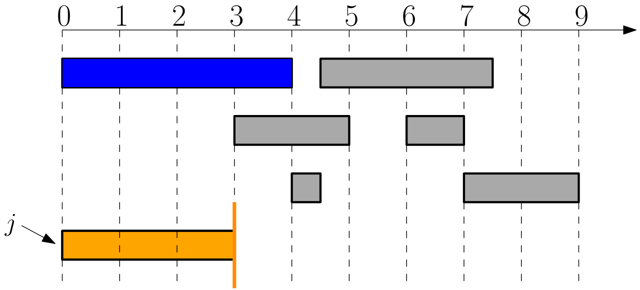

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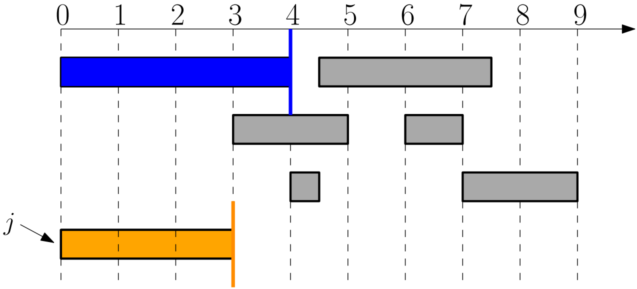

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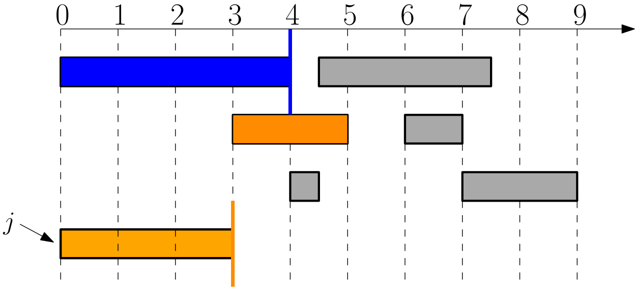

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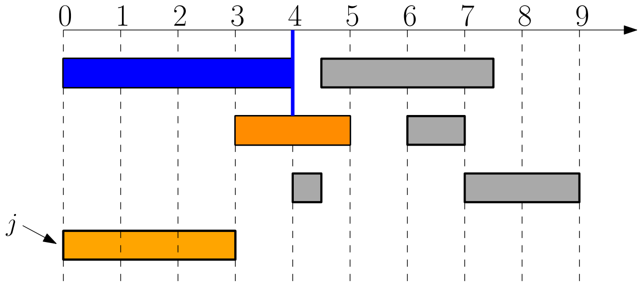

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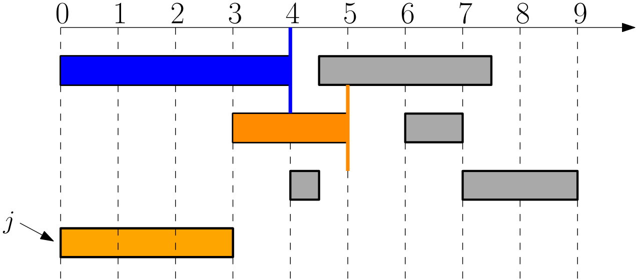

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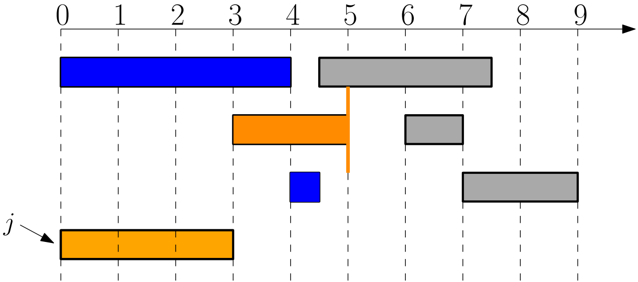

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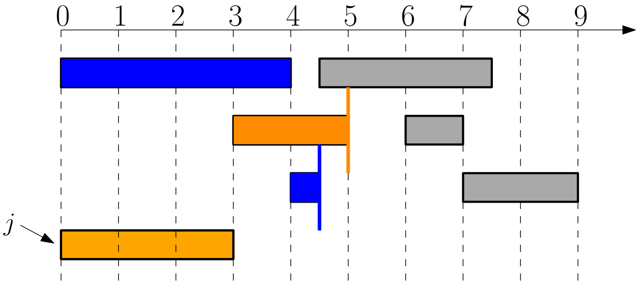

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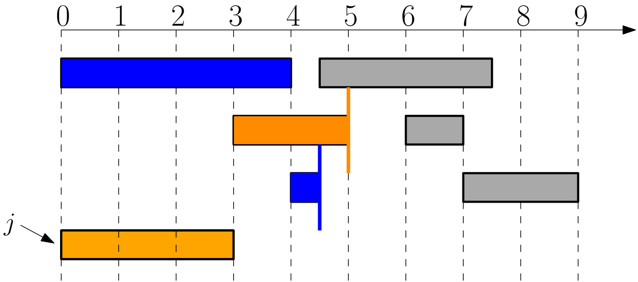

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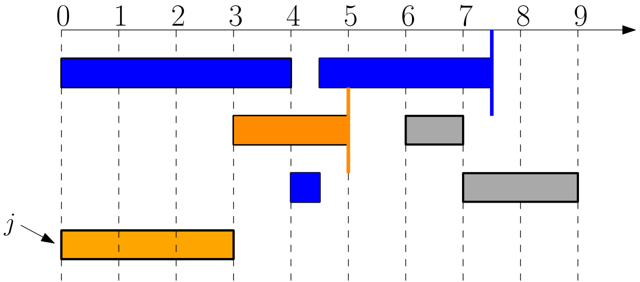

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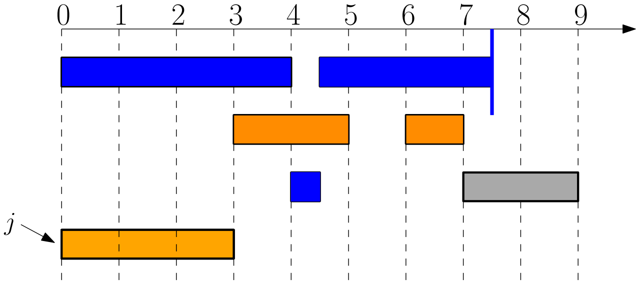

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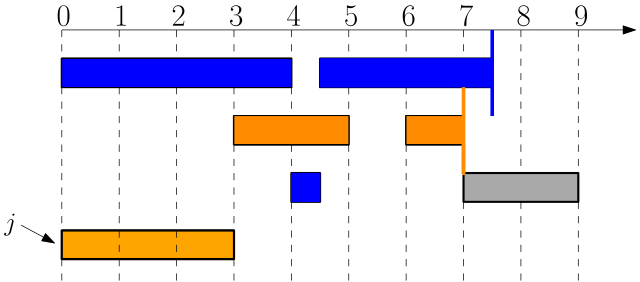

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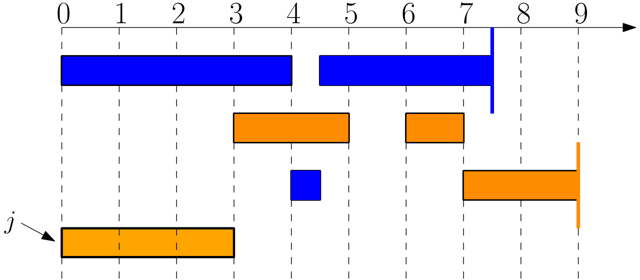

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Obs. Greedy algorithm never schedules two incompatible jobs in the same machine.

Why “Greedy algorithm” is optimal?

Theorem Greedy algorithm is optimal.

Proof.

- Let d be the number of machines that greedy algorithm used.

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- Observation: all these $d-1$ jobs starts earlier than s_j because we schedule the jobs in order of starting time. Thus, we have d jobs overlapping at time $s_j+\epsilon$. The jobs **depth** $\geq d$.

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- By the Observation in the previous slide, an optimal solution $\geq d$. Thus the greedy algorithm is optimal.

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Running time of algorithm?

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```
A ← {1, 2, ··· , n}, S ← {1}, t_1 = 0
while A ≠ ∅ do
    j ← arg min_{j'∈A} s_{j'}, S_j ← {i'}_{i'∈S, t_{i'} ≤ s_j}
    If S_j ≠ ∅, then schedule j to a machine i ∈ S_j and t_i = f_j
    Otherwise, schedule j to machine |S| + 1, S ← S ∪ {|S| + 1} and t_{|S|} = f_j
return S
```

Running time of algorithm?

- Naive implementation: $O(n^2)$ time

Greedy Algorithm for Interval Partitioning

Partition(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \{1\}$, $t_1 = 0$
2: **while** $A \neq \emptyset$ **do**
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5: Otherwise, schedule j to machine $|S| + 1$, $S \leftarrow S \cup \{|S| + 1\}$ and $t_{|S|} = f_j$
6: **return** S

Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time with Priority Queue.

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- Sequence of page requests

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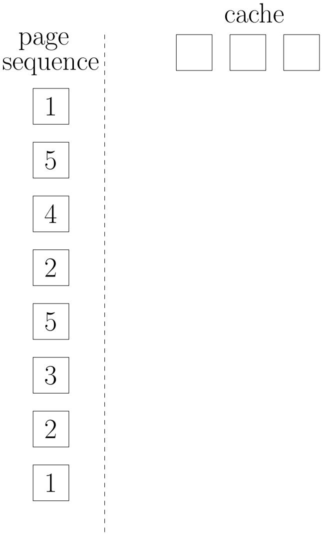

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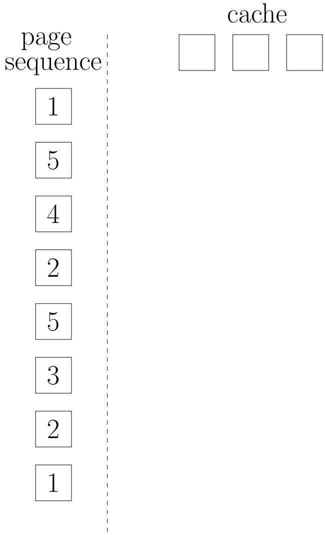

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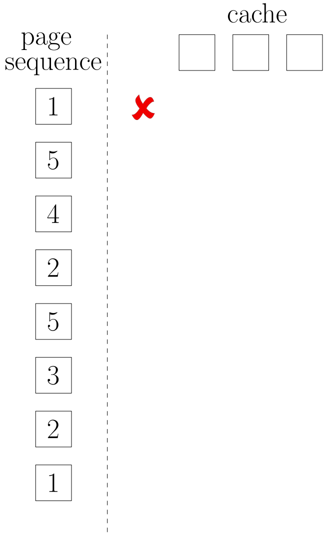

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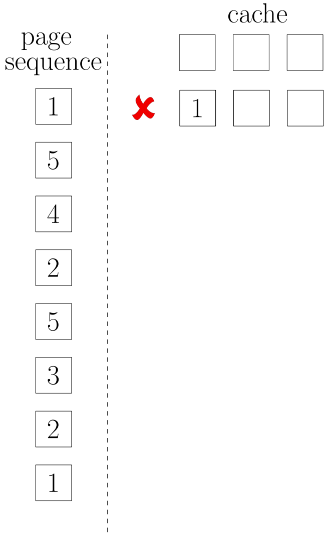

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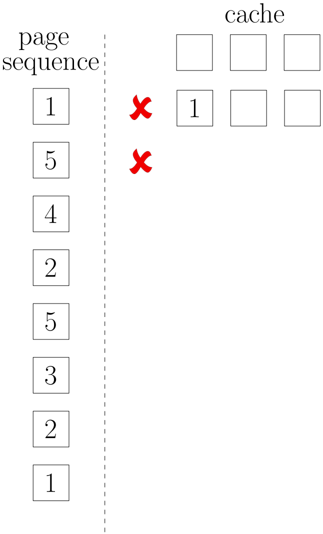

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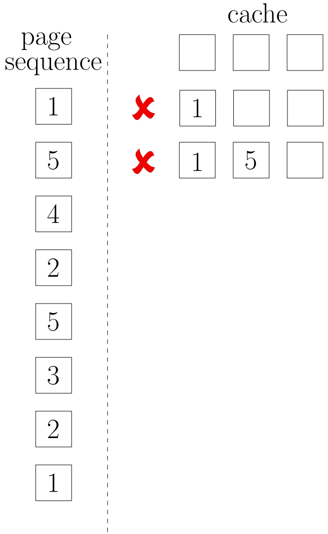

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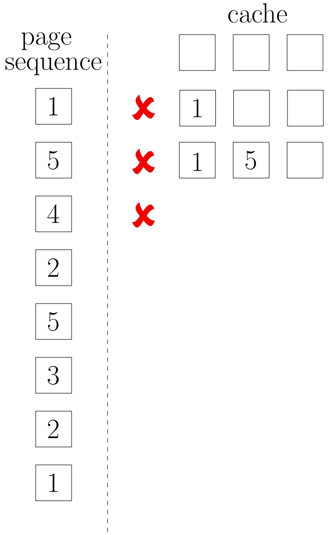

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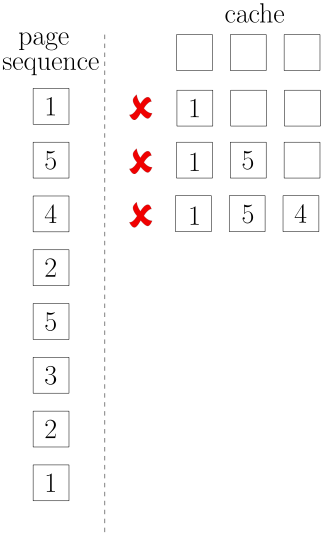

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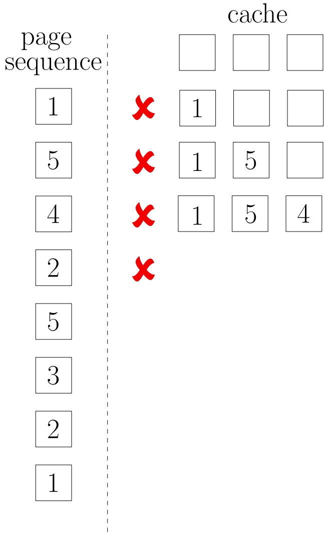

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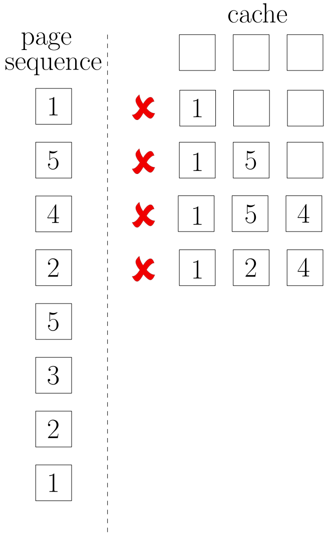

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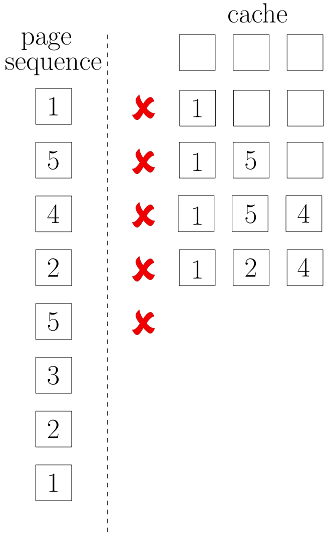

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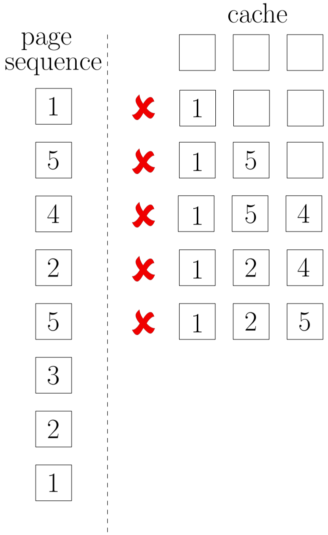

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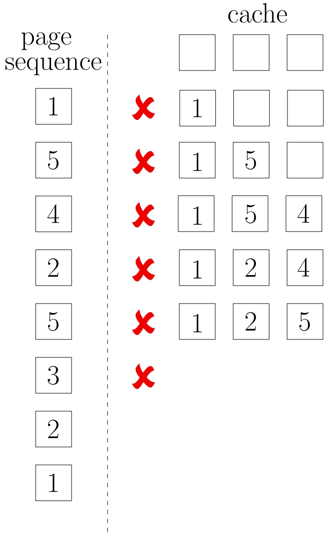

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page sequence		cache		
1	✘	1		
5	✘	1	5	
4	✘	1	5	4
2	✘	1	2	4
5	✘	1	2	5
3	✘	1	2	3
2				
1				

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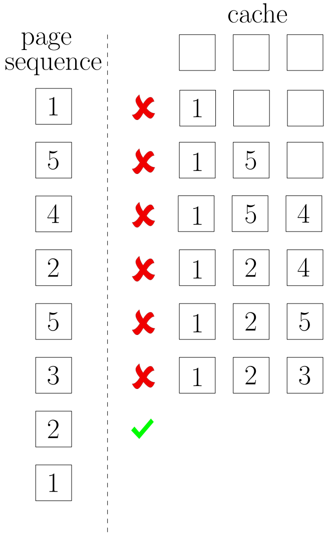
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.



Offline Caching

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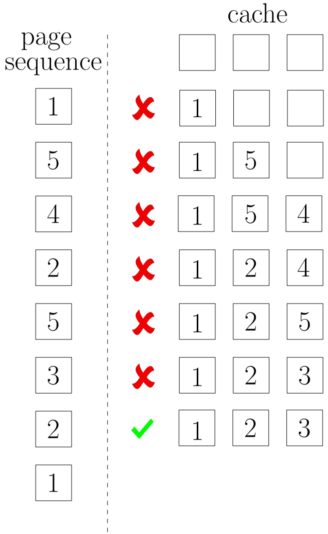

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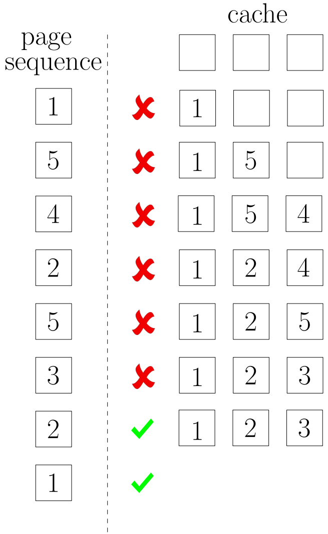

Offline Caching

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- Cache hit happens if requested page already in cache.

page sequence		cache		
1	✘	1		
5	✘	1	5	
4	✘	1	5	4
2	✘	1	2	4
5	✘	1	2	5
3	✘	1	2	3
2	✓	1	2	3
1	✓	1	2	3

Offline Caching

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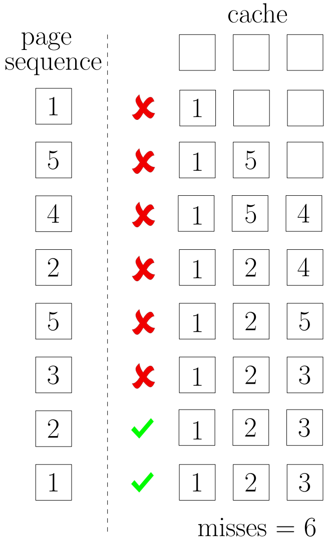

Offline Caching

- Cache that can store k pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.

page sequence		cache		
1	✗	1		
5	✗	1	5	
4	✗	1	5	4
2	✗	1	2	4
5	✗	1	2	5
3	✗	1	2	3
2	✓	1	2	3
1	✓	1	2	3

misses $= 6$

A Better Solution for Example

page sequence		cache				cache		
1	✗	1			✗	1		
5	✗	1	5		✗	1	5	
4	✗	1	5	4	✗	1	5	4
2	✗	1	2	4	✗	1	5	2
5	✗	1	2	5	✓	1	5	2
3	✗	1	2	3	✗	1	3	2
2	✓	1	2	3	✓	1	3	2
1	✓	1	2	3	✓	1	3	2
		misses = 6				misses = 5		

Offline Caching Problem

Input: k : the size of cache

n : number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\mathrm{hit}, \mathrm{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)