CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Graph Algorithms

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Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm

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Greedy Algorithm

MST-Greedy1(G, w)

$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
    $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
        where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

Greedy Algorithm

MST-Greedy1(G, w)

```
S ← {s}, where s is arbitrary vertex in V
F ← ∅
while S ≠ V do
    (u, v) ← lightest edge between S and V \ S,
                        where u ∈ S and v ∈ V \ S
    S ← S ∪ {v}
    F ← F ∪ {(u, v)}
return (V, F)
```

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
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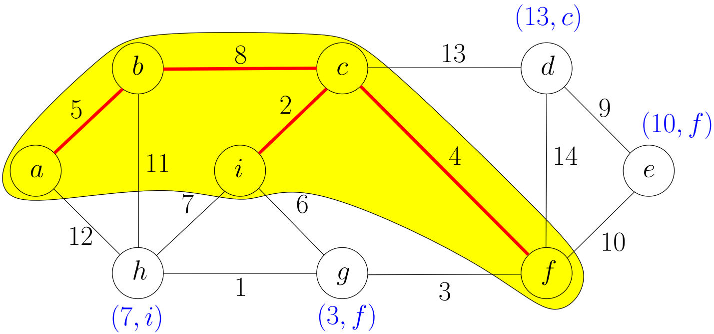

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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
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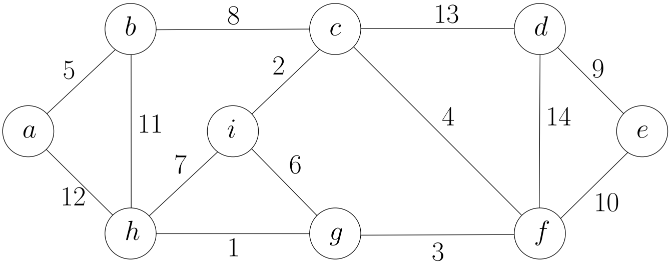
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
            $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Example



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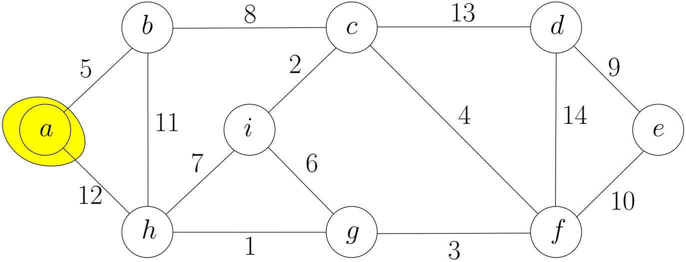

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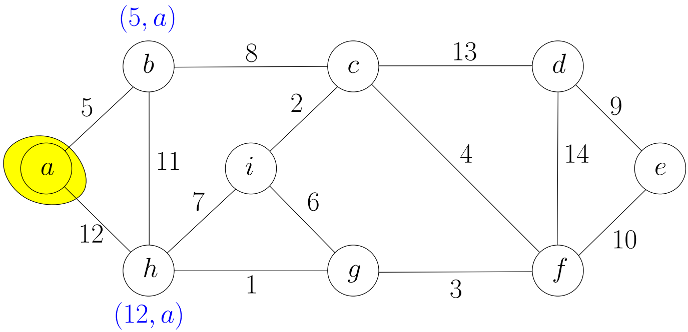

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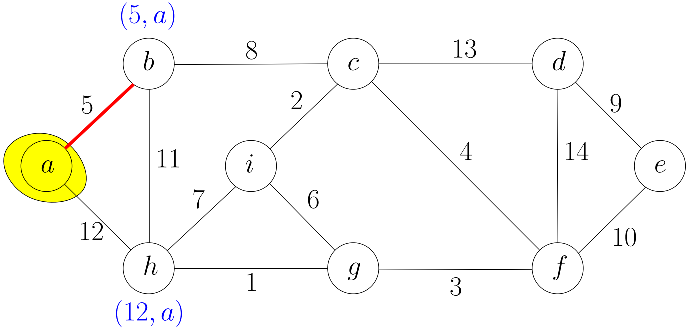

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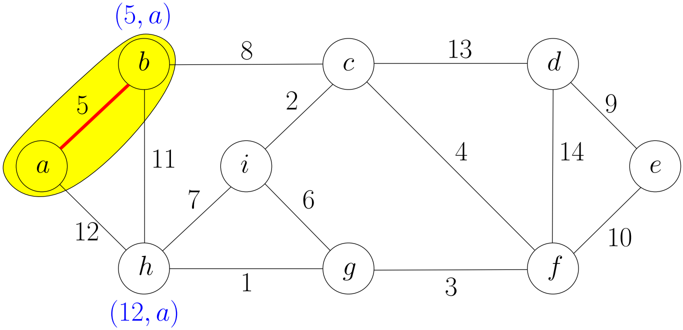

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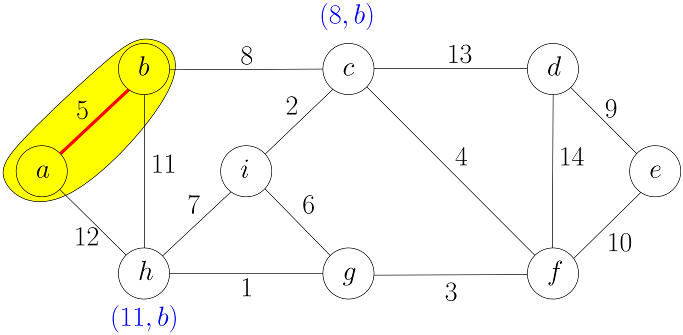

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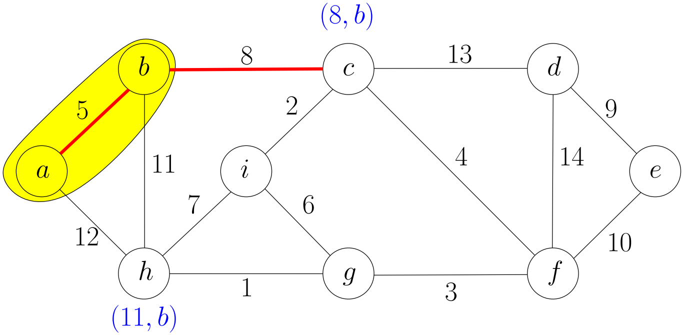

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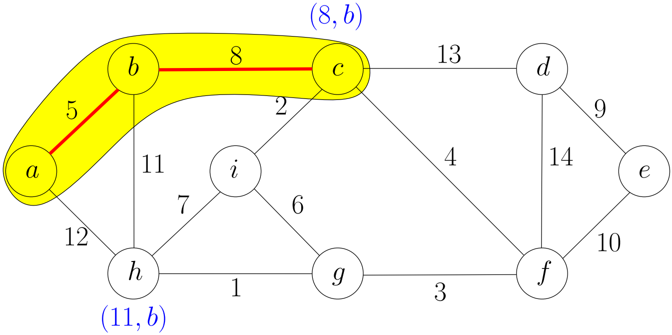

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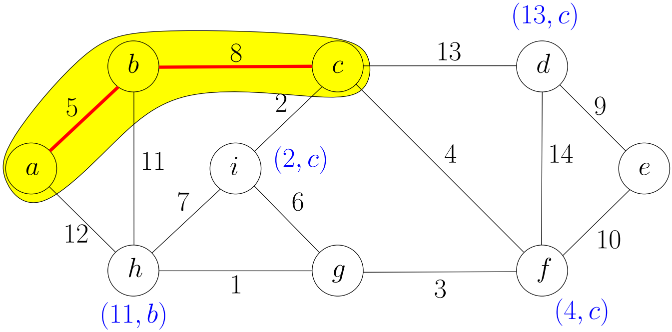

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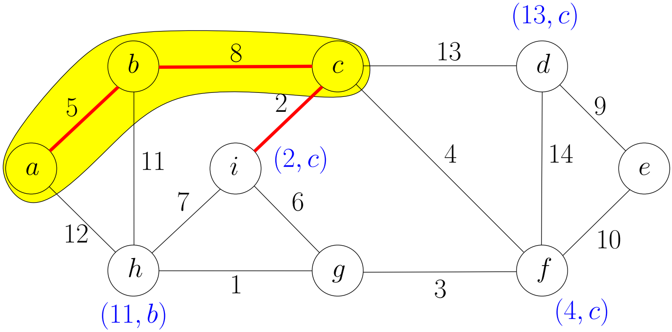

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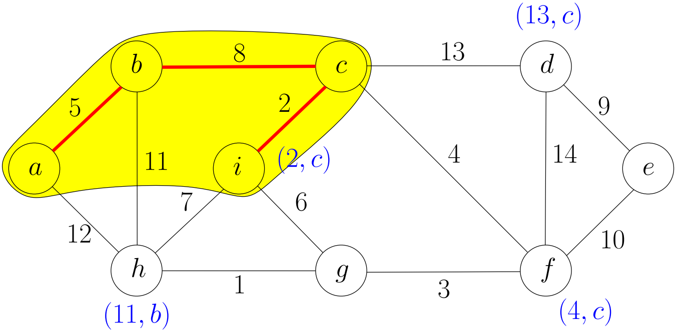

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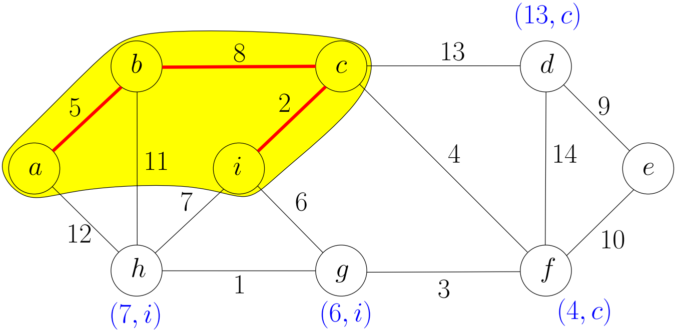

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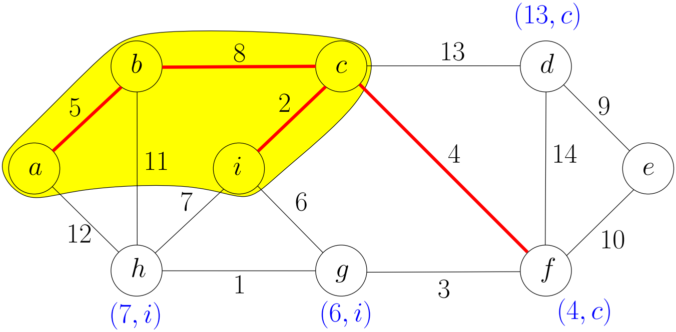

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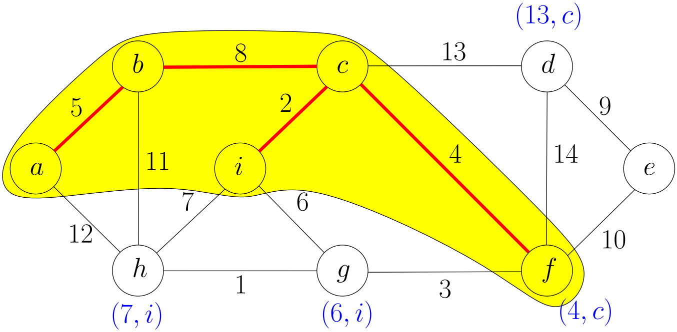

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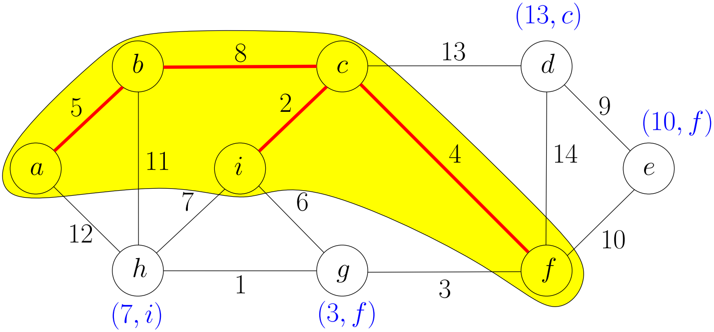

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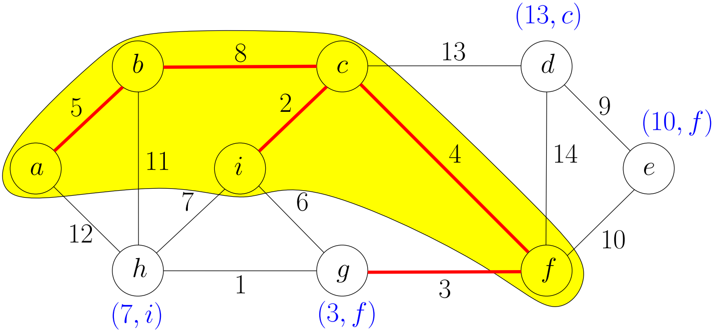

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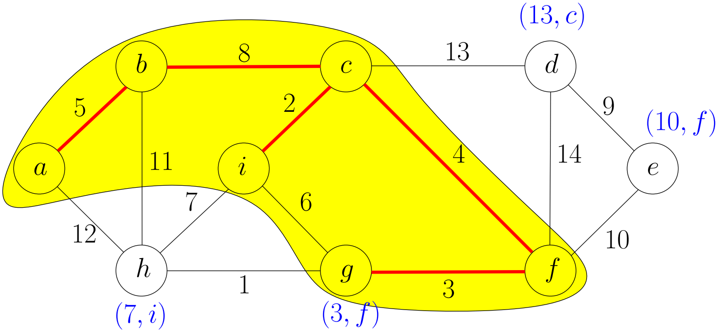

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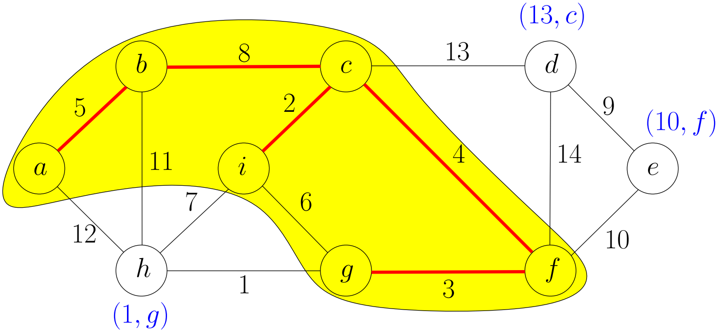

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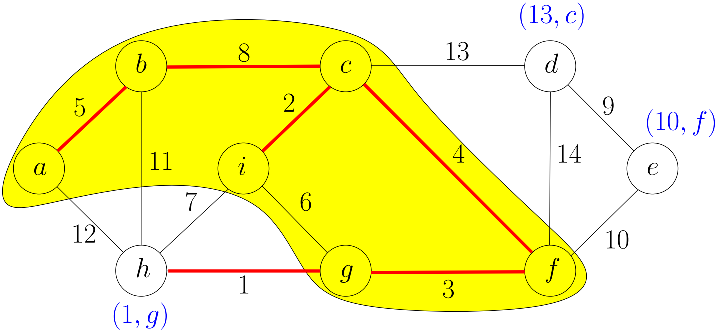

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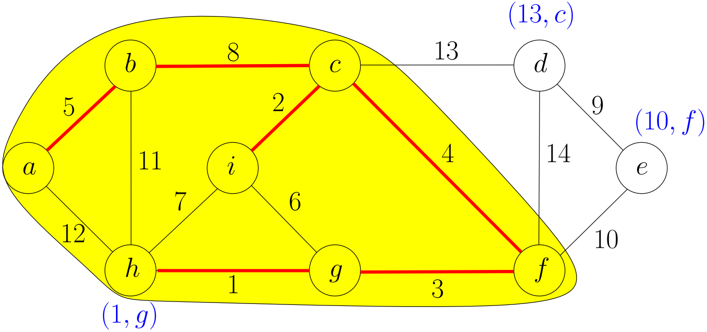

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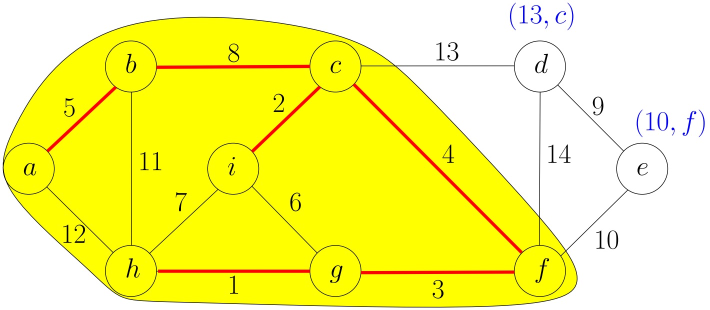

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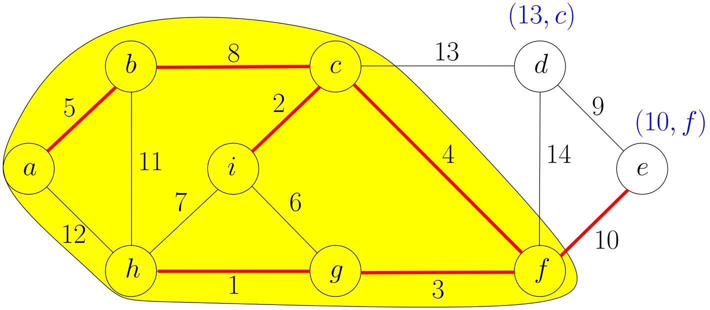

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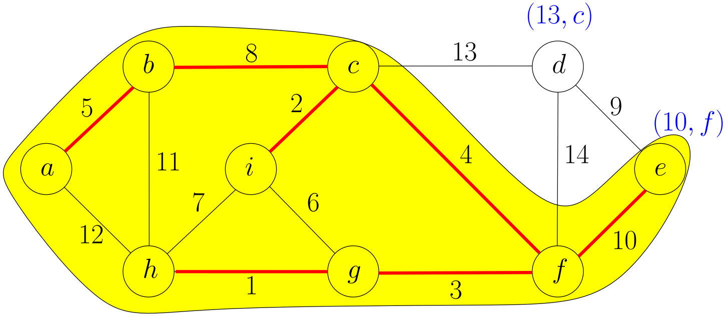

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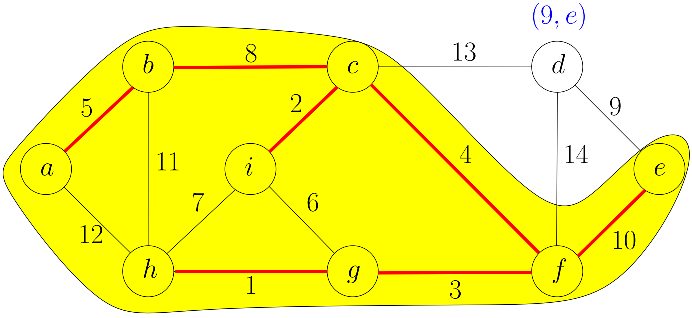

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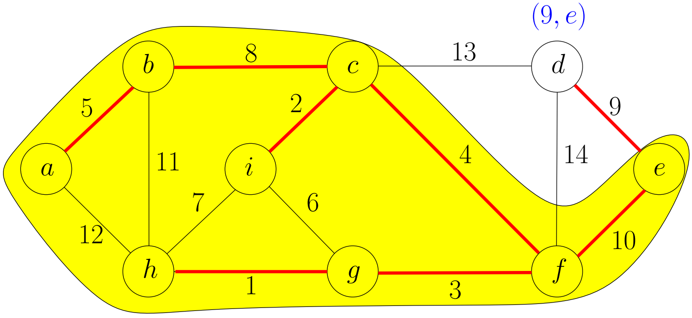

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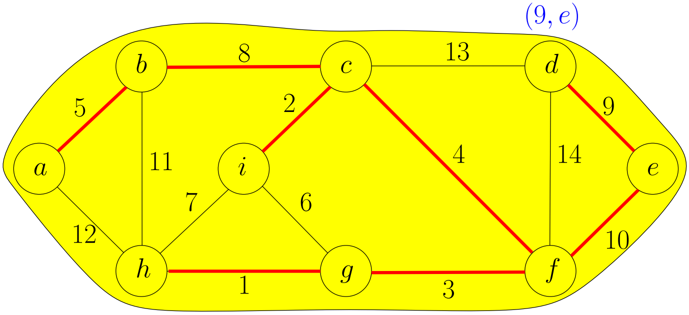

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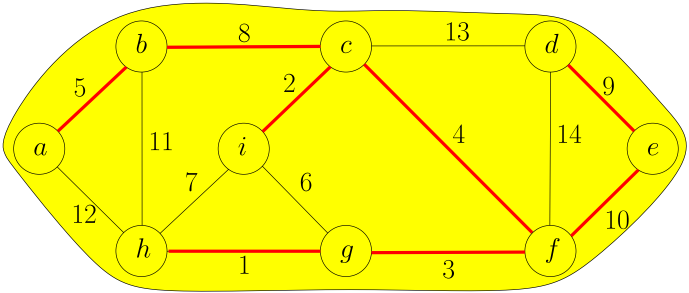

Prim's Algorithm

For every $v \in V \setminus S$ maintain

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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value — extract_min
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

MST-Prim(G, w)

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Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d[v])
while S ≠ V do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d[v] then
            d[v] ← w(u, v), Q.decrease_key(v, d[v])
            π[v] ← u
return {(u, π[u]) | u ∈ V \ {s}}
```

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+\ O(m)\times$ (time for decrease_key)

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concrete DS	extract_min	decrease_key	overall time
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Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.

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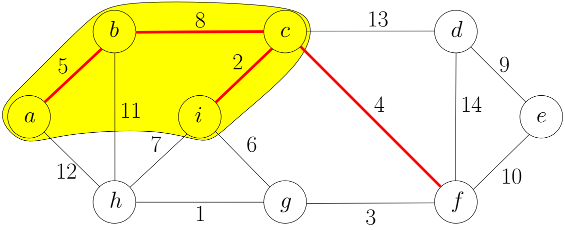

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$

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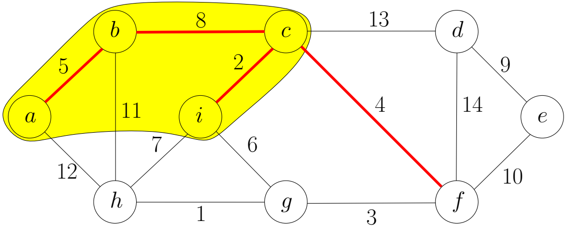

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

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Thus, the minimum spanning tree is unique with assumption.