CSE 431/531: Algorithm Analysis and Design (Fall 2024)

Introduction III: Asymptotic Notation

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Outline

1. Introduction
   - What is an Algorithm?
   - More Computation Problems
   - Asymptotic Analysis

2. Asymptotic Notations

Outline

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Lecture Review

- Computational Problem: GCD, Sorting Problem
- Algorithms: Euclidean Alg, Insertion Sort Alg
- See correctness proof on Piazza-Resources-Lecture Notes
- Feel free to bring up more real-life problems on Piazza, and discuss potential algorithms.

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  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size n = worst running time over all possible arrays of length n

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Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

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Asymptotic Analysis: O-notation

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  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds

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- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
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  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

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- For Algorithm 2: if we increase n by a factor of 2, running time increases by a factor of 2

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ **do**
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- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$
  $= O(\frac{n(n+1)}{2} - 1) = O(n^2)$

Exercise: Runtime for Insertion Sort

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- The sequence A is given in ascending order.
  Total running time $= \sum_{j=2}^{n} O(1) = O(\sum_{j=2}^{n} 1) = O(n)$
- The sequence A is given in descending order.
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- What is the precision of real numbers?
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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

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- $100n - n^2/10 + 50$? No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

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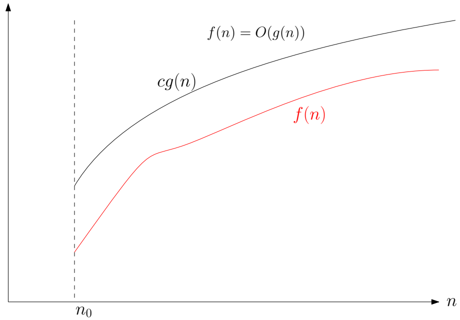

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Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 42n \le 0.$$
$$3n^2 + 2n \le c(n^2 - 10n)$$

$\square$

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Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\le$		

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"$=$" is asymmetric! Following equalities are wrong:

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- Analogy: Mike is a student. ~~A student is Mike.~~

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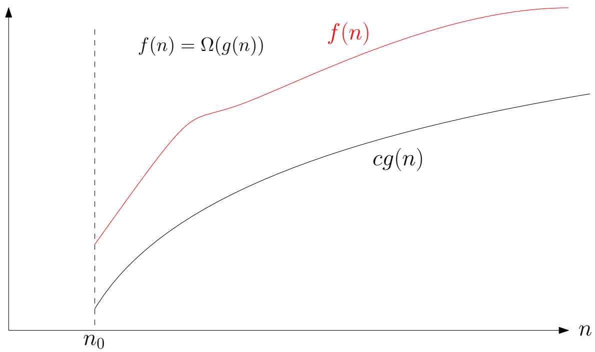

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  - $4n^2 = \Omega(n - 10)$
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Theorem $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$.

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- $f(n) = \Theta(g(n))$, then for large enough n, we have "$f(n) \approx g(n)$".

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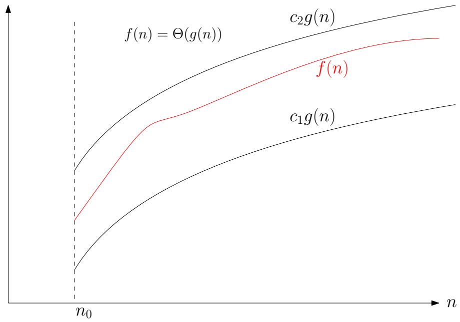

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Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

f	g	O	Ω	Θ
$n^3 - 100n$	$5n^2 + 3n$			
$3n - 50$	$n^2 - 7n$			
$n^2 - 100n$	$5n^2 + 30n$			
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