Asymptotic Analysis Proof

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1 Problem in Lecture Note 3

1.1 $2^{n/3+\sqrt{n}+100} = \Omega(2^{n/3})$

Proof. Proof by Ω Definition.

There exists a constant $c_1 = 1$ and $n_0 = 10$, for any number $n \geq n_0$, we have $2^{n/3+\sqrt{n}+100} \geq 2^{n/3}$, thus $2^{n/3+\sqrt{n}+100}$ is in $\Omega(2^{n/3})$. $\square$

1.2 $2^{n/3+\sqrt{n}+100} \notin O(2^{n/3})$

Proof. Proof by Contradiction.

Assume $2^{n/3+\sqrt{n}+100} \in O(2^{n/3})$. By definition, we know that there exists a constant $c_1 > 0$ and $n_0 > 0$, for any number $n \geq n_0$, we have $2^{n/3+\sqrt{n}+100} \leq c_1 * 2^{n/3}$.

However, for any constant $c_1 > 0$ and $n_0 > 0$, there exists a number $n' > \max\{\log_2^2 c_1, n_0\}$, $2^{n'/3+\sqrt{n'}+100} > 2^{n'/3} * 2^{\log_2 c_1} * 2^{100} > c_1 * 2^{n'/3}$ holds, which derives a contradiction. Thus $2^{n/3+\sqrt{n}+100}$ is not in $O(2^{n/3})$. $\square$

1.3 $2^{n/3+\sqrt{n}+100} \notin \Theta(2^{n/3})$

Proof. Since $2^{n/3+\sqrt{n}+100} \notin O(2^{n/3})$, we know that $2^{n/3+\sqrt{n}+100} \notin \Theta(2^{n/3})$. $\square$

2 Mote examples

Example 1: $f(n) = \log_2 n$**,** $g(n) = \log_8 n$

Proof Approach (by definition):

There exists a constant $c_1 = 4$ and $n_0 = 8$, for any number $n \geq n_0$, we have $f(n) = \log_2 n < 4 \log_8 n = c_1 * g(n)$, thus $f(n)$ is in $O(g(n))$.

There exists a constant $c_1 = 1$ and $n_0 = 8$, for any number $n \geq n_0$, we have $f(n) = \log_2 n > \log_8 n = c_1 * g(n)$, thus $f(n)$ is in $\Omega(g(n))$.

Since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, $f(n)$ is in $\Theta(g(n))$.

Exercise: Please try prove that $\log_a n = \Theta(\log_b n)$ **for any constant** $a, b > 0$**.**

Example 2: $f(n) = n^2$**,** $g(n) = 1$ **if** n **is odd and** $g(n) = n^3$ **if** n **is even.**

Proof Approach (by definition): We prove the "not in " relationship by contradiction.

For any constant $c_1 > 0$ and $n_0 > 0$, there exists an odd number $n > \max\{\sqrt{c_1}, n_0\}$, $f(n) = n^2 > c_1 = c_1 * g(n)$ holds, thus $f(n)$ is not in $O(g(n))$.

For any constant $c_2 > 0$ and $n_0 > 0$, there exists an even number $n > \max\{1/c_2, n_0\}$, $f(n) = n^2 < c_2 * g(n)$ holds, thus $f(n)$ is not in $\Omega(g(n))$.

Since $f(n)$ is not in $O(g(n))$, $f(n)$ is not in $\Theta(g(n))$.

Example 3: $f(n) = 2n$**,** $g(n) = n$ **if** n **is odd and** $g(n) = n^2$ **if** n **is even.**

Proof Approach (by definition):

There exists a constant $c_1 = 3$ and $n_0 = 1$, for any number $n \geq n_0$, we have $f(n) \leq c_1 * g(n)$, thus $f(n)$ is in $O(g(n))$.

For any constant $c_2 > 0$ and $n_0 > 0$, there exists even number $n > \max\{2/c_2, n_0\} > n_0$, we have $f(n) < c_2 * g(n)$, thus f(n) is not in $\Omega(g(n))$.

Since $f(n)$ is not in $\Omega(g(n))$, $f(n)$ is not in $\Theta(g(n))$.

Hints for the homework: Firstly try to simplify both $f(n)$ **and** $g(n)$**, and then analyse the asymptotic relationships.**