CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Greedy Algorithms

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Outline

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for $t \leftarrow 1$ to T **do**
    if ρ_t is in cache **then** do nothing
    else if there is an empty page in cache **then**
        evict the empty page and load ρ_t in cache
    else
        $p^* \leftarrow$ page in cache that is not used furthest in the future
        evict p^* and load ρ_t in cache

for every $p \leftarrow 1$ to n **do**

  $times[p] \leftarrow$ array of times in which p is requested, in increasing order ▷ put ∞ at the end of array

  $pointer[p] \leftarrow 1$

$Q \leftarrow$ empty priority queue

for every $t \leftarrow 1$ to T **do**

  $pointer[\rho_t] \leftarrow pointer[\rho_t] + 1$

  if $\rho_t \in Q$ **then**

    Q.increase-key$(\rho_t, times[\rho_t, pointer[\rho_t]])$, **print** "hit", **continue**

  if $Q.size() < k$ **then**

    print "load ρ_t to an empty page"

  else

    $p \leftarrow Q$.extract-max(), **print** "evict p and load ρ_t"

  Q.insert$(\rho_t, times[\rho_t, pointer[\rho_t]])$ ▷ add ρ_t to Q with key value $times[\rho_t, pointer[\rho_t]]$

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- $\cdots$

Simple Implementations for Priority Queue

- n = size of ground set V

data structures	insert	extract_min	decrease_key
array			
sorted array			

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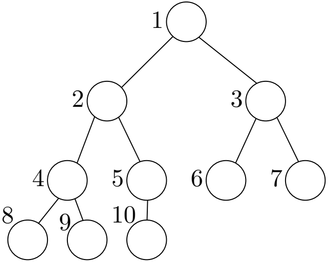
data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

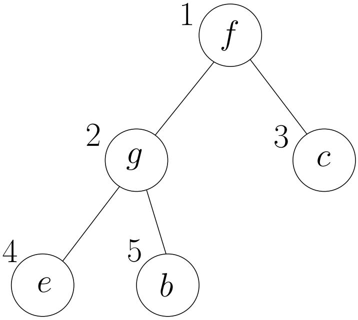


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p[v], v \in U$: the index of node containing v
- $key[v], v \in U$: the key value of element v

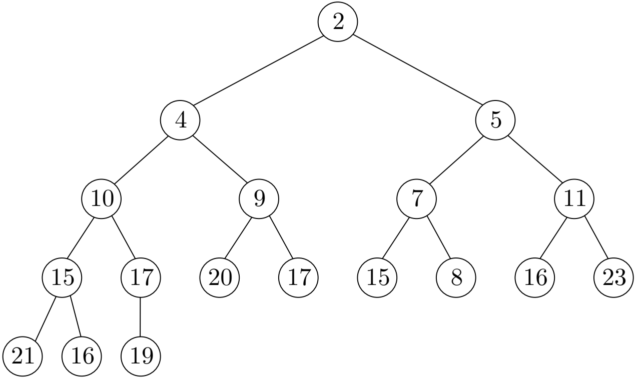


- $s = 5$
- $A = (`f\text{'}, `g\text{'}, `c\text{'}, `e\text{'}, `b\text{'})$
- $p[`f\text{'}] = 1, p[`g\text{'}] = 2, p[`c\text{'}] = 3,$
  $p[`e\text{'}] = 4, p[`b\text{'}] = 5$

Heap

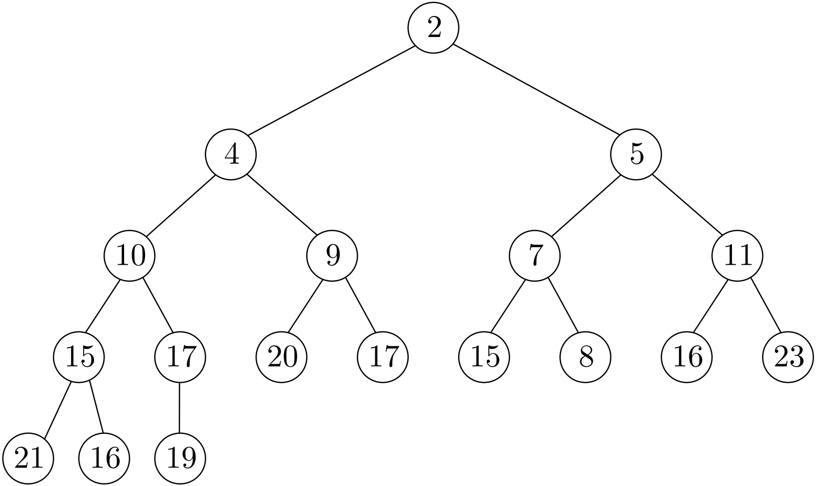
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key[A[i]] \leq key[A[j]]$.



A heap. Numbers in the circles denote key values of elements.

insert(v, key_value)



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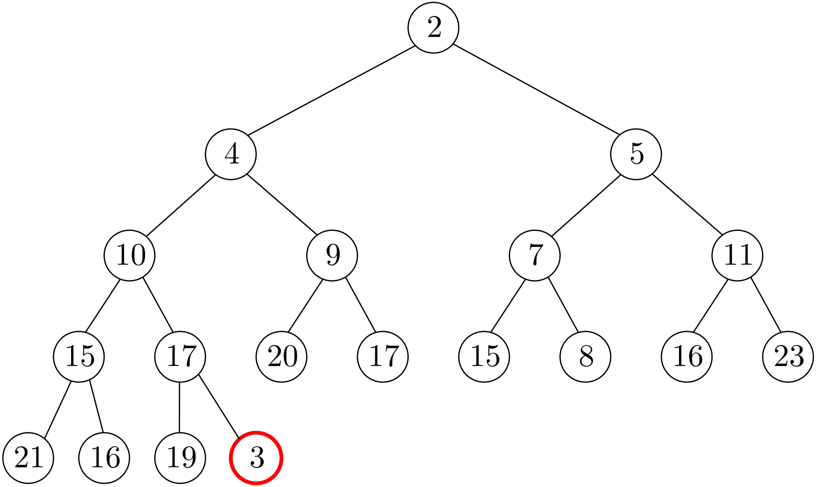

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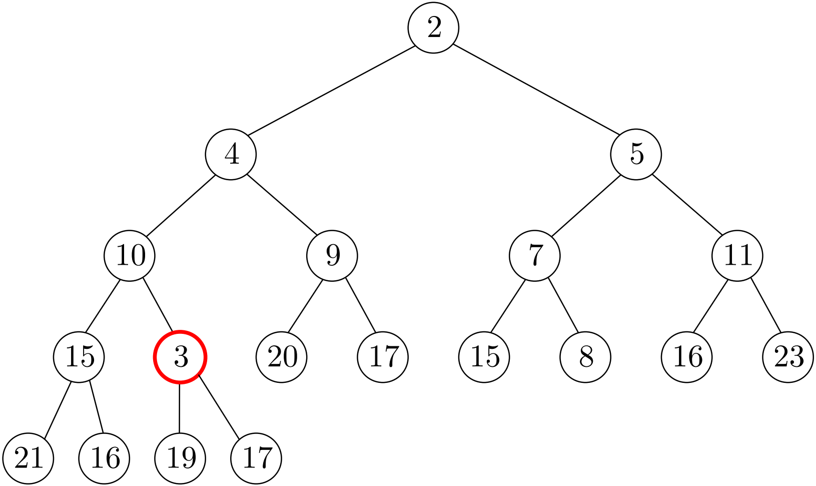

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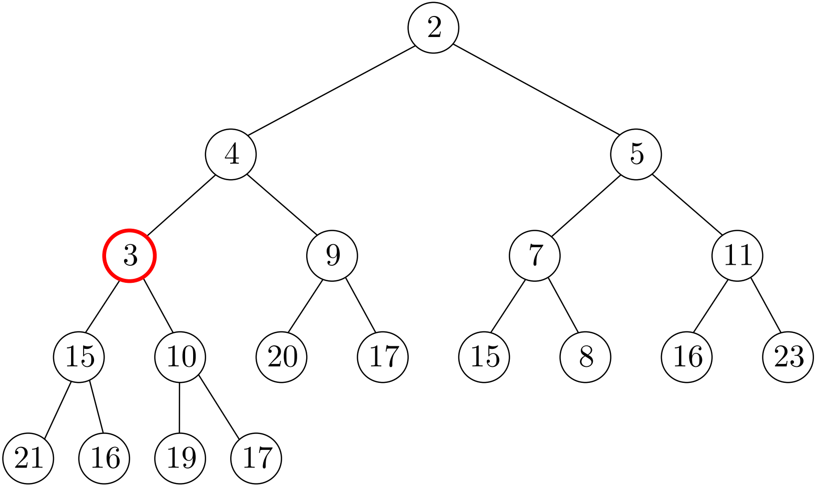

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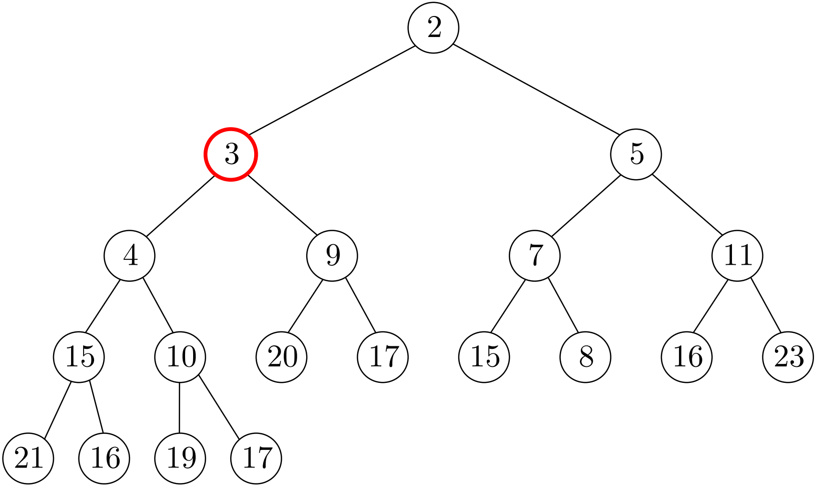


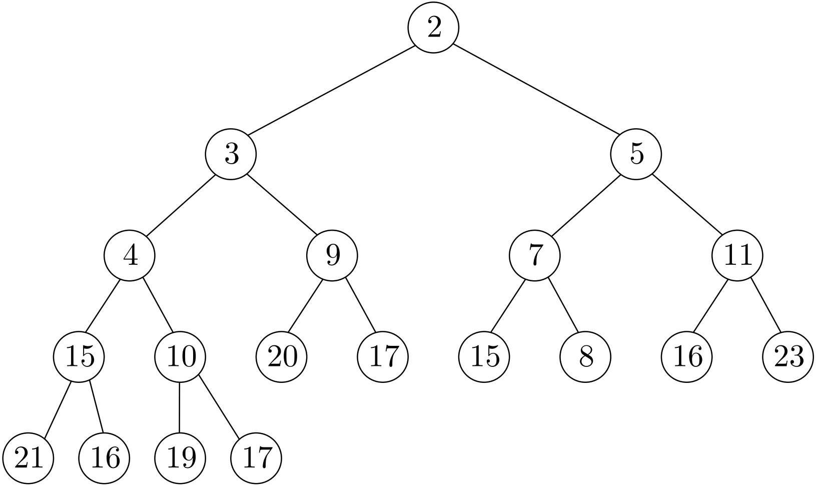
insert(v, key_value)

```
s ← s + 1
A[s] ← v
p[v] ← s
key[v] ← key_value
heapify_up(s)
```

heapify-up(i)

```
while i > 1 do
    j ← ⌊i/2⌋
    if key[A[i]] < key[A[j]] then
        swap A[i] and A[j]
        p[A[i]] ← i, p[A[j]] ← j
        i ← j
    else break
```

extract_min()



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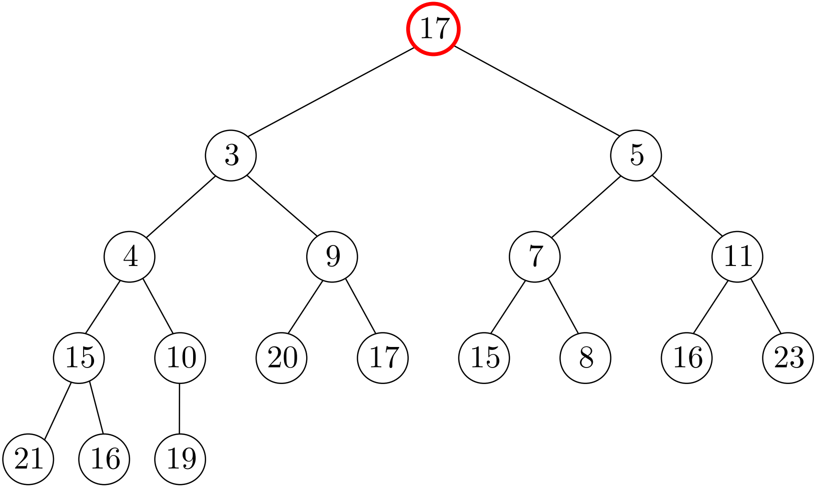

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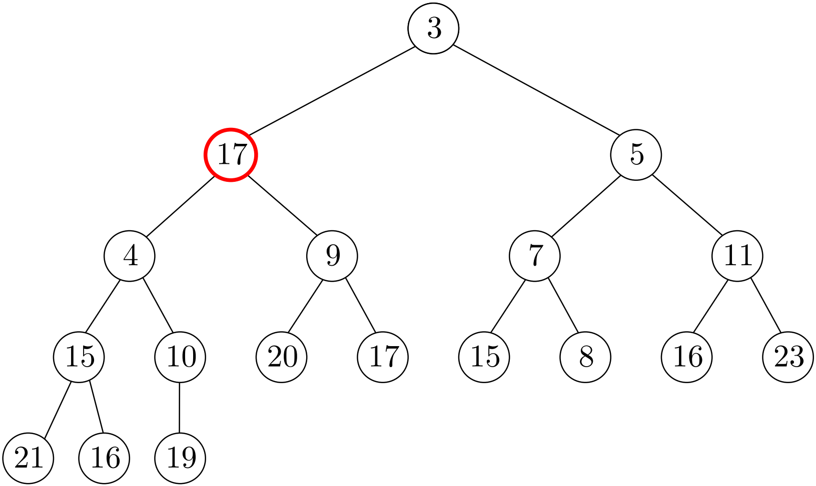

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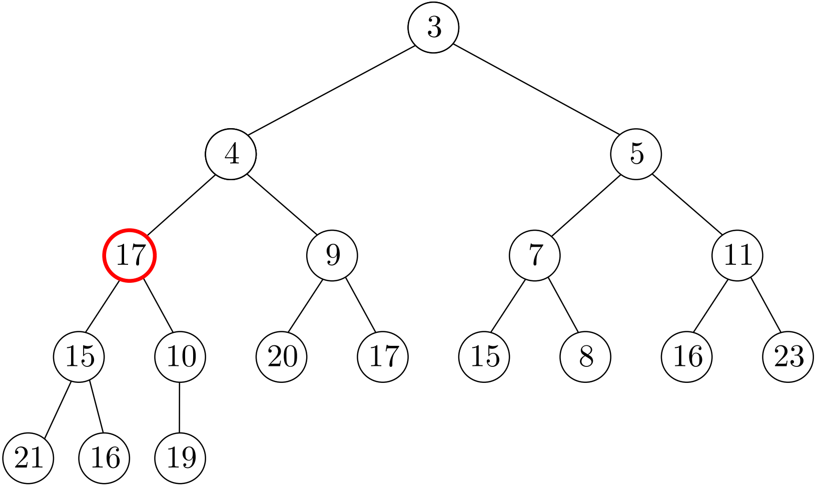

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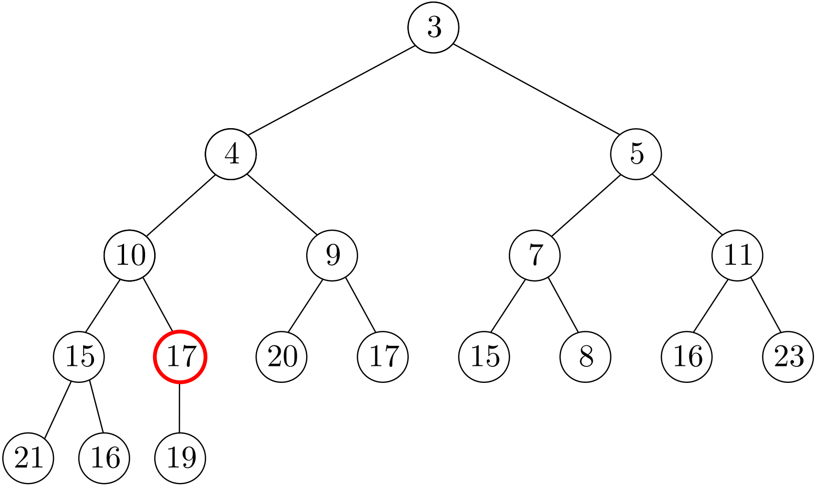

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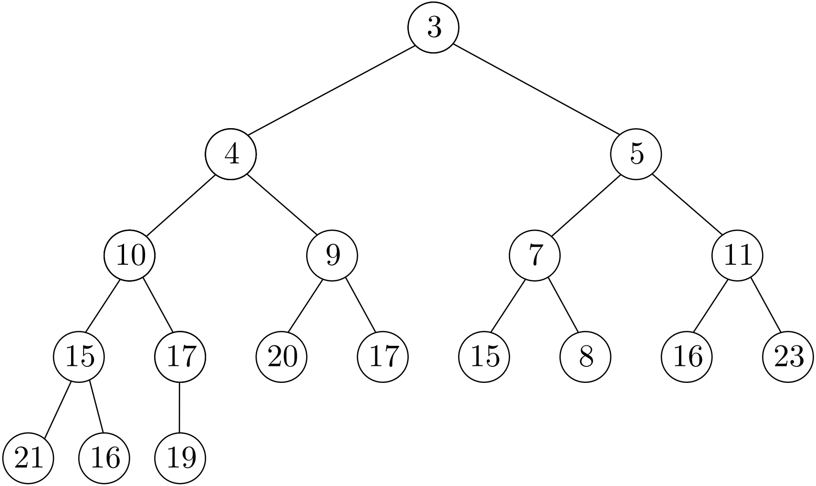

extract_min()

$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p[A[1]] \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ **then**
    heapify_down(1)
return ret

decrease_key(v, key_val

$key[v] \leftarrow key\_value$
heapify-up$(p[v])$

heapify-down(i)

while $2i \leq s$ **do**
    if $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ **then**
        $j \leftarrow 2i$
    else
        $j \leftarrow 2i + 1$
    if $key[A[j]] < key[A[i]]$ **then**
        swap $A[i]$ and $A[j]$
        $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
        $i \leftarrow j$
    else break

- Running time of heapify_up and heapify_down: $O(\lg n)$

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- Running time of insert, exact_min and decrease_key: $O(\lg n)$

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data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

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Outline

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

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A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

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Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

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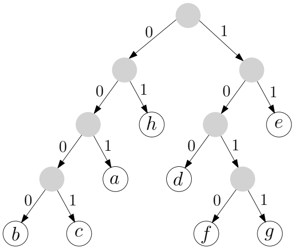
001	0000	0001	100
e	f	g	h
11	1010	1011	01



Prefix Codes Guarantee Unique Decoding

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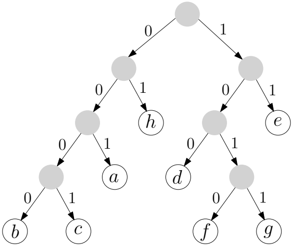

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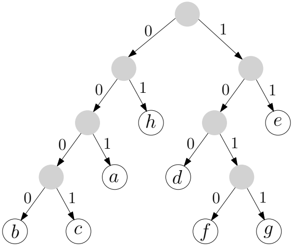

- 0001001100000001011110100001001

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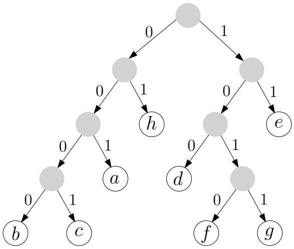

- 0001/0011000000101111010001001
- c

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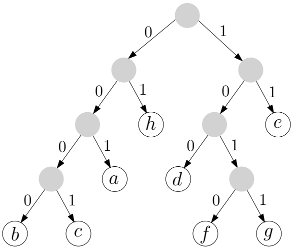

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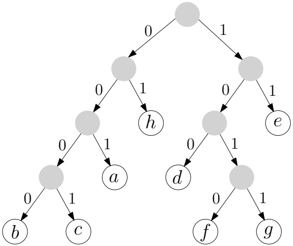

- 0001/001/100/00001011110100001001
- cad

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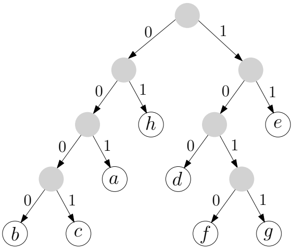

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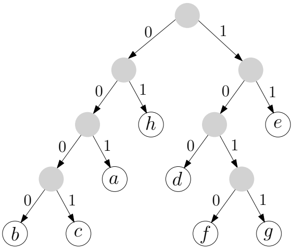

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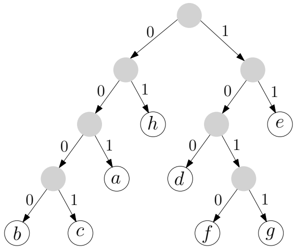

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- cadbhh

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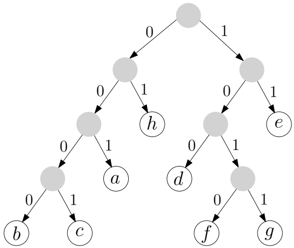

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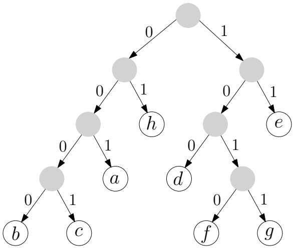

- 0001/001/100/0000/01/01/11/1010/0001001
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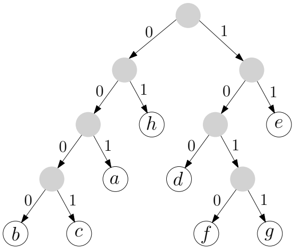

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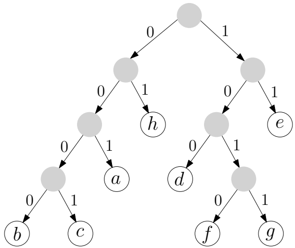
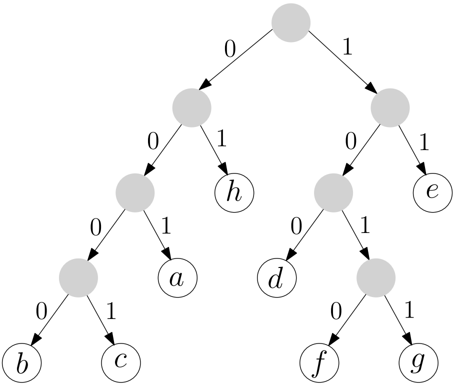
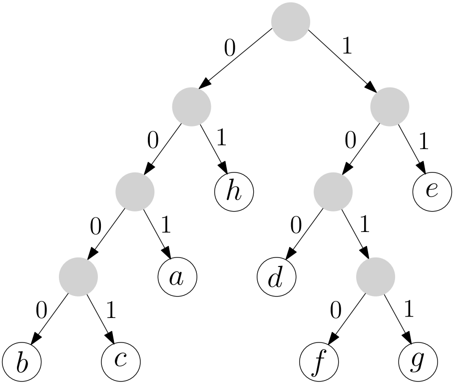


- 0001/001/100/0000/01/01/11/1010/0001/001/
- cadbhhefca

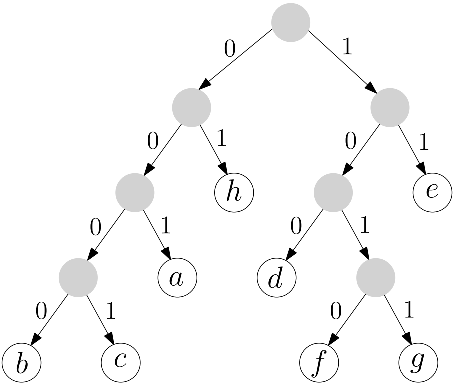


Properties of Encoding Tree



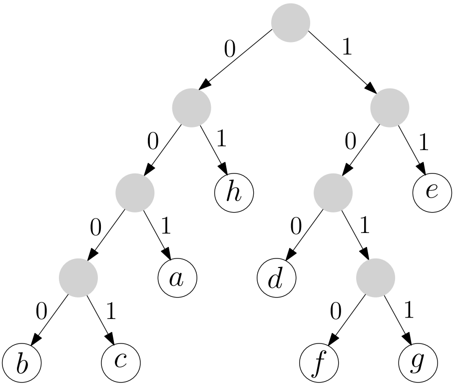
Properties of Encoding Tree

- Rooted binary tree



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1



Properties of Encoding Tree

- Rooted binary tree
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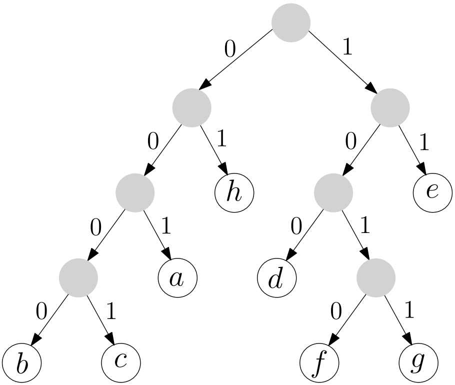
- A leaf corresponds to a code for some letter



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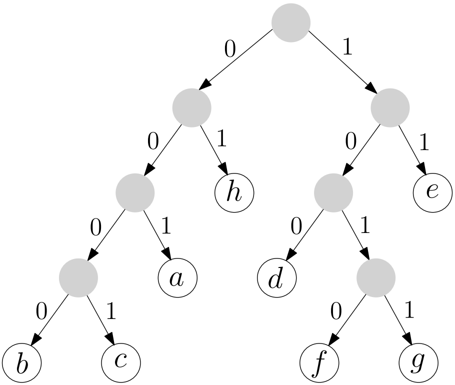
- If coding scheme is not wasteful: a non-leaf has exactly two children



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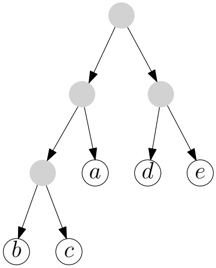
Best Prefix Codes

Input: frequencies of letters in a message

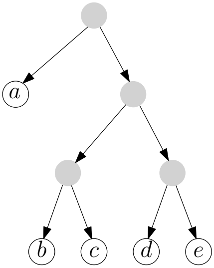
Output: prefix coding scheme with the shortest encoding for the message

example

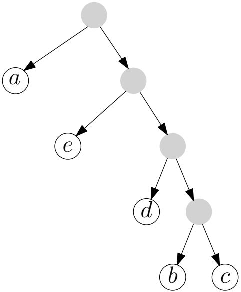
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



scheme 2



scheme 3