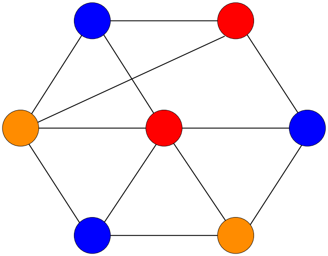
k-coloring problem

Def. A k-coloring of $G = (V, E)$ is a function $f : V \to \{1, 2, 3, \cdots, k\}$ so that for every edge $(u, v) \in E$, we have $f(u) \neq f(v)$. G is k-colorable if there is a k-coloring of G.



k-coloring problem

Input: a graph $G = (V, E)$

Output: whether G is k-colorable or not

2-Coloring Problem

Obs. A graph G is 2-colorable if and only if it is bipartite.

Q: How do we check if a graph G is 2-colorable?

2-Coloring Problem

Obs. A graph G is 2-colorable if and only if it is bipartite.

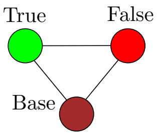
Q: How do we check if a graph G is 2-colorable?

A: We check if G is bipartite.

3-SAT $\leq_P$ 3-Coloring

- Construct the base graph

Base Graph



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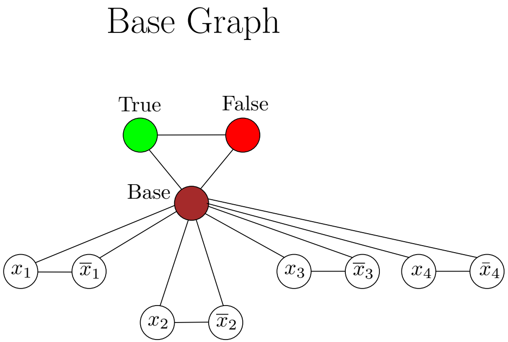

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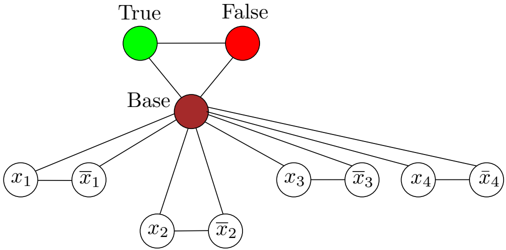
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

$$x_1 \vee \neg x_2 \vee x_3$$



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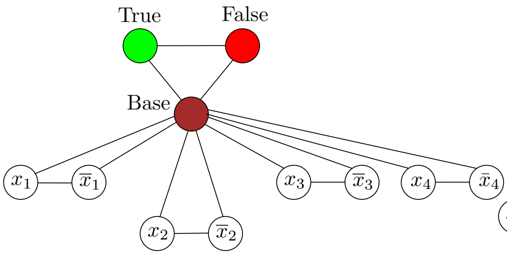

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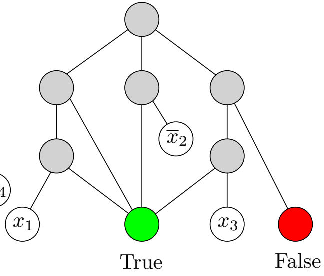

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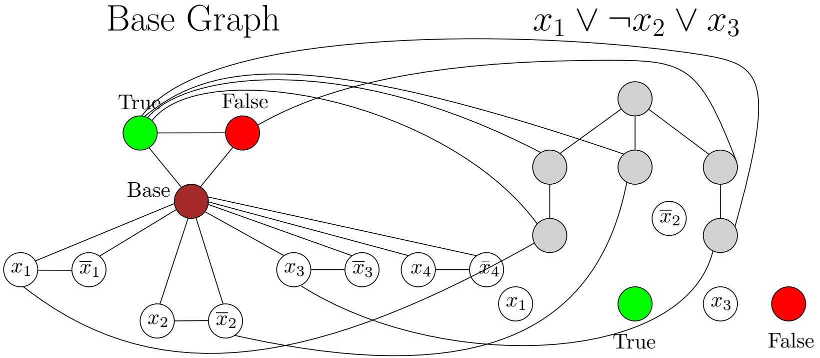

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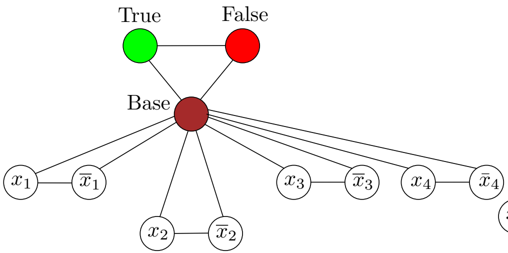

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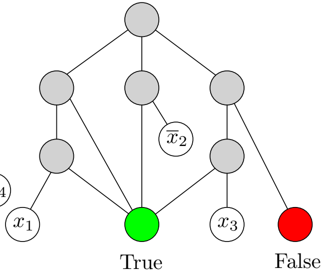

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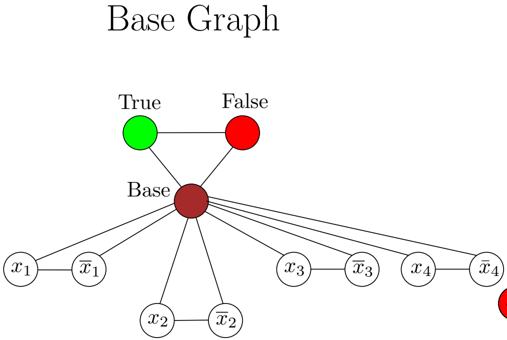

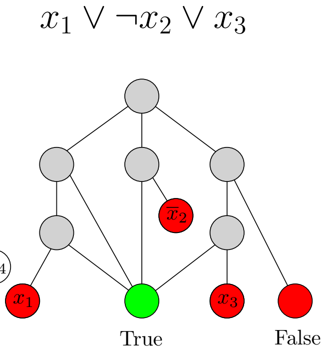

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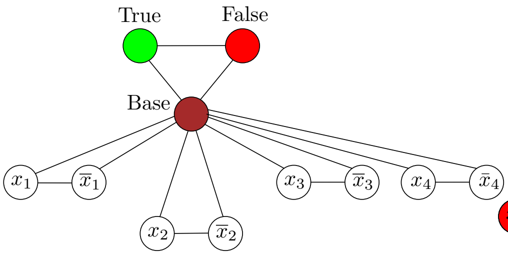

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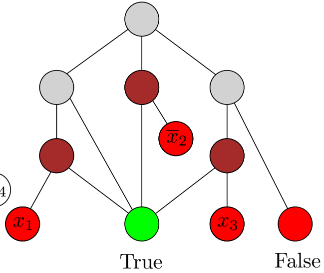

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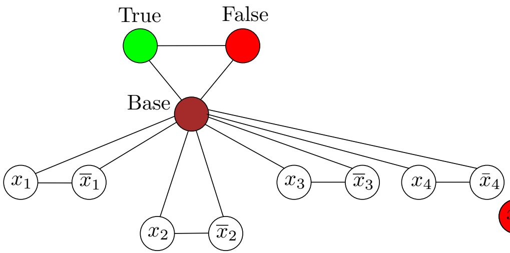

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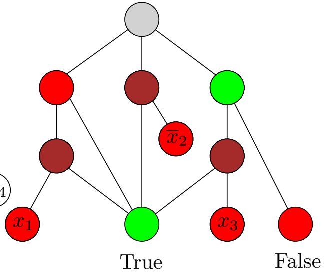

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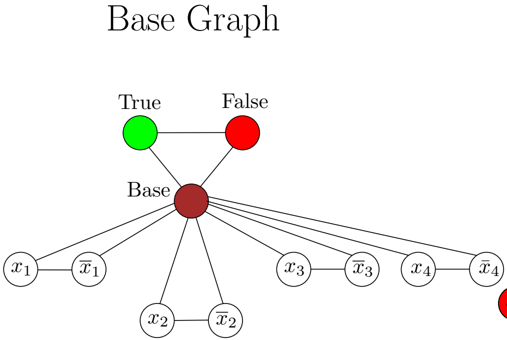

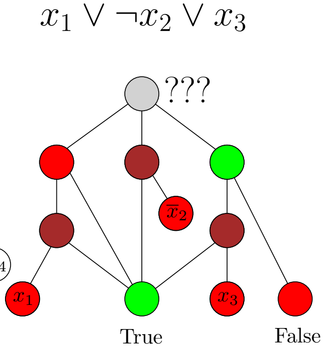

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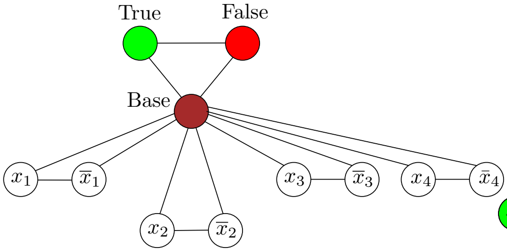

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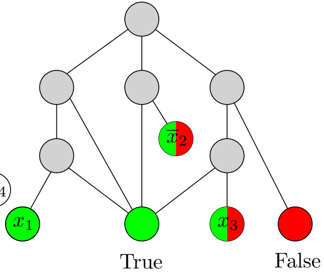

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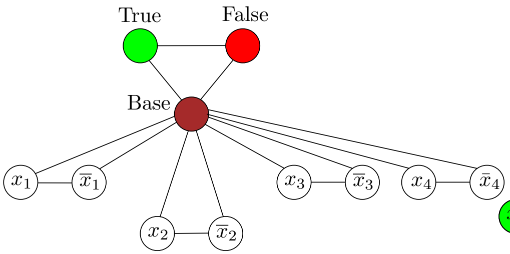

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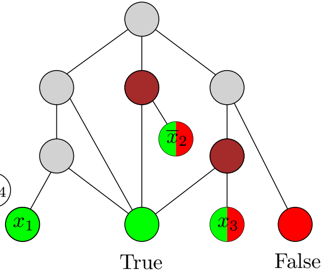

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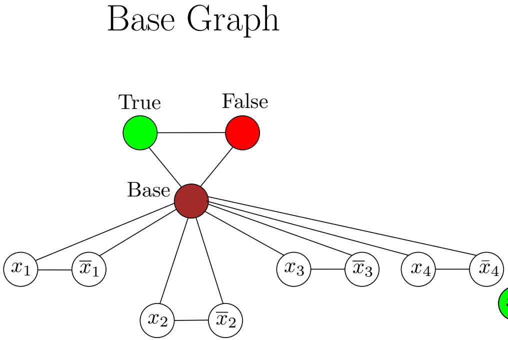

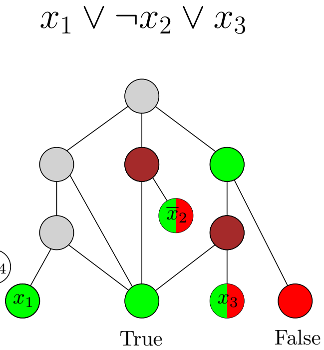

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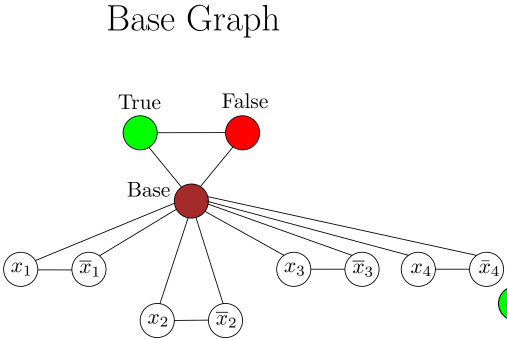

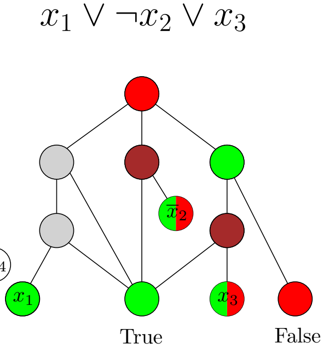

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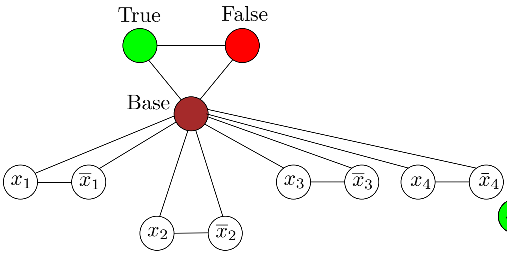

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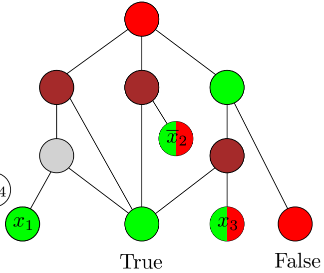

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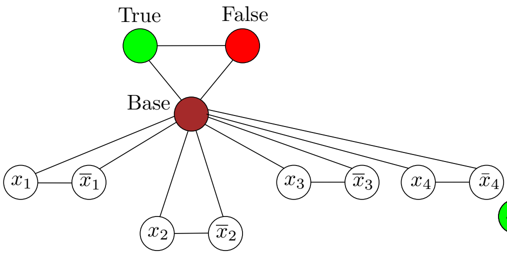

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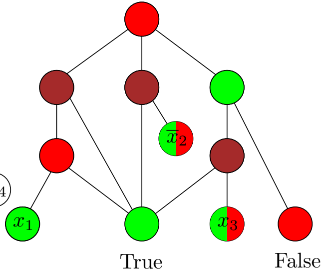

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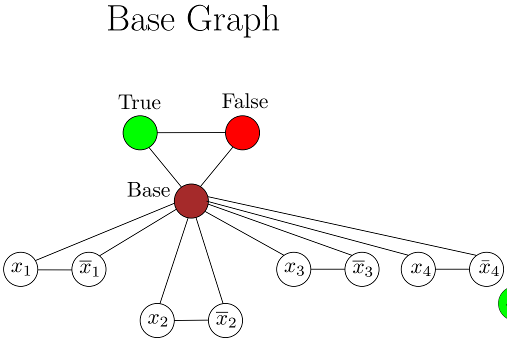

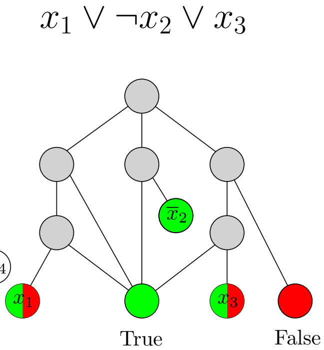

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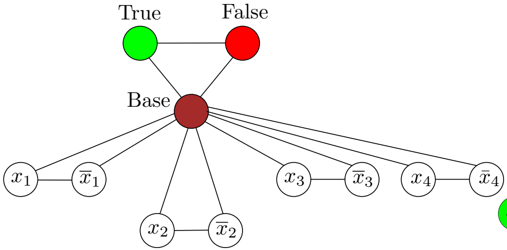

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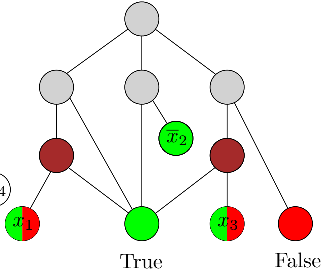

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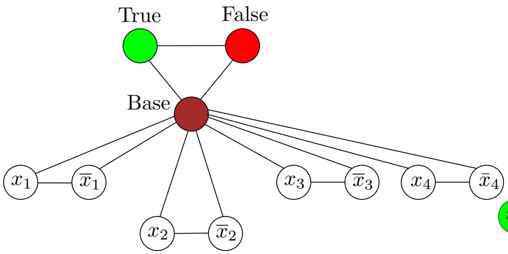

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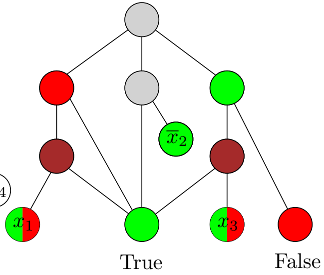

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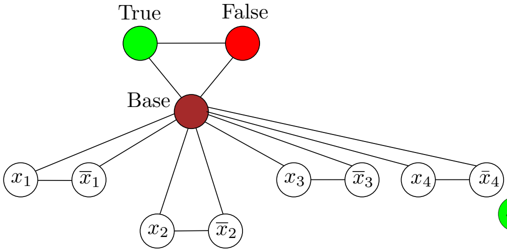

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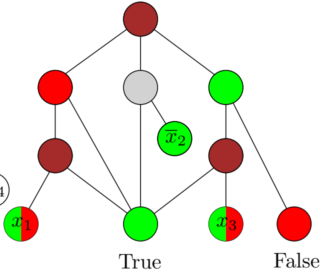

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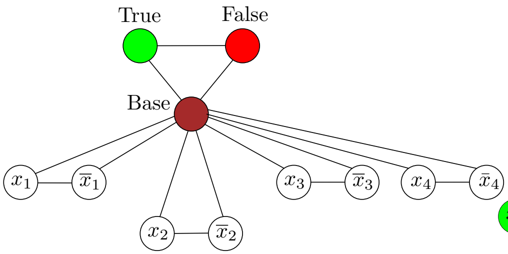

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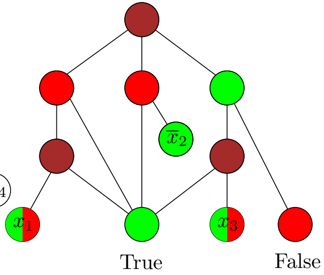

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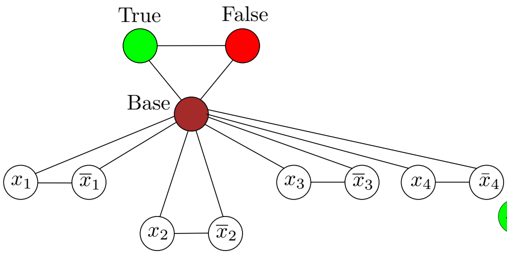

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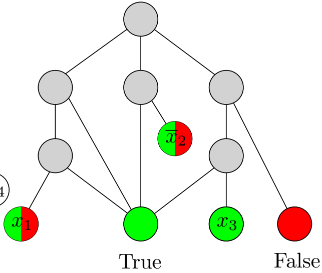

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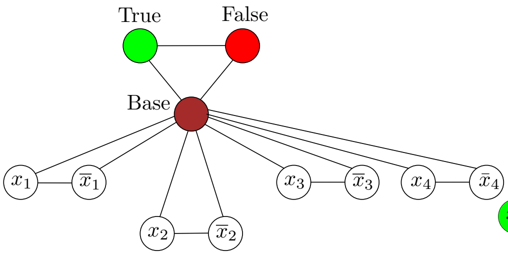

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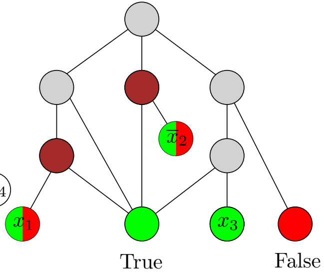

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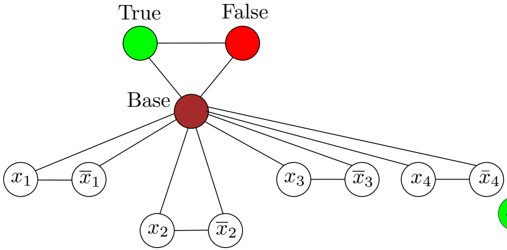

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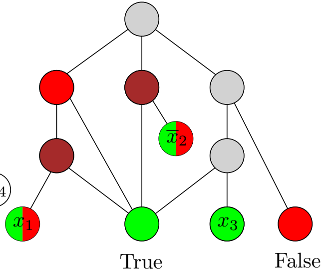

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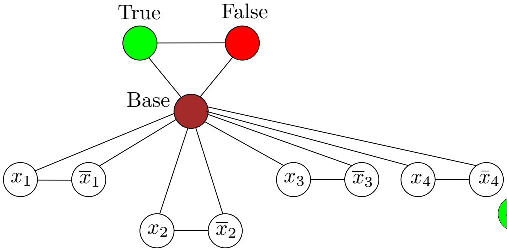

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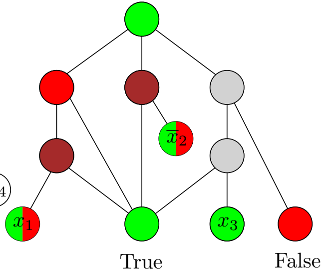

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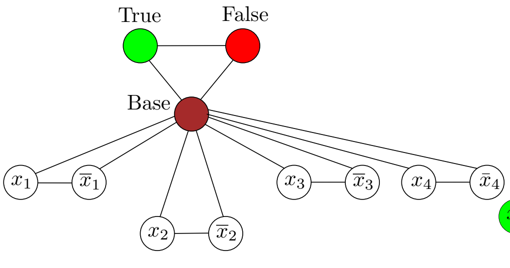

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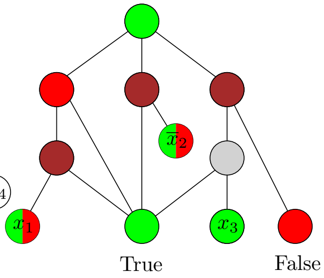

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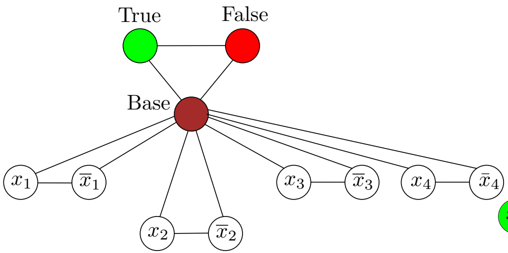

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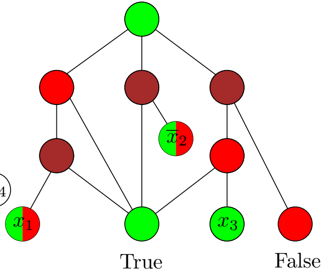

A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

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- In general, algorithm for Y can call the algorithm for X many times.
- However, for most reductions, we call algorithm for X only once
- That is, for a given instance s_Y for Y, we only construct one instance s_X for X

A Strategy of Polynomial Reduction

- Given an instance s_Y of problem Y, show how to construct in polynomial time an instance s_X of problem such that:
  - s_Y is a yes-instance of $Y \Rightarrow s_X$ is a yes-instance of X
  - s_X is a yes-instance of $X \Rightarrow s_Y$ is a yes-instance of Y

Outline

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  - Best algorithm runs in time $O(c^n)$ for some constant $c > 1$
  - Best lower bound is $\Omega(n)$

Q: How far away are we from proving or disproving P $=$ NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
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- Essentially we have no techniques for proving lower bound for running time

Dealing with NP-Hard Problems

- Faster exponential time algorithms
- Solving the problem for special cases
- Fixed parameter tractability
- Approximation algorithms

Faster Exponential Time Algorithms

3-SAT:

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Travelling Salesman Problem:

- Brute-force: $O(n! \cdot \text{poly}(n))$
- Better algorithm: $O(2^n \cdot \text{poly}(n))$
- In practice: TSP Solver can solve Euclidean TSP instances with more than 100,000 vertices

Solving the problem for special cases

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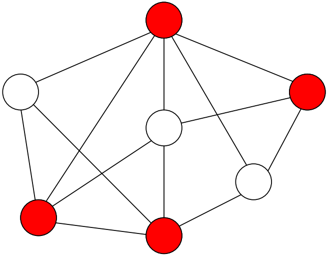
- Problem: whether there is a vertex cover of size k, for a small k (number of nodes is n, number of edges is $\Theta(n)$.)



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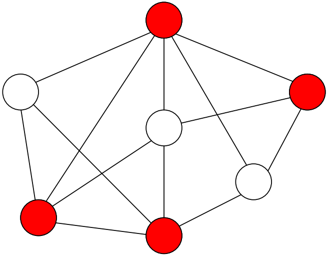

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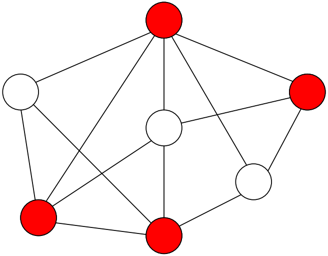

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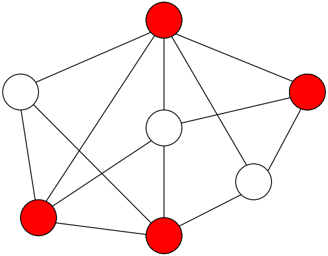

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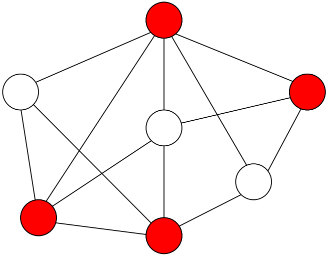
- Vertex-Cover is fixed-parameter tractable.



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- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time
- There is an 2-approximation for the vertex cover problem: we can efficiently find a vertex cover whose size is at most 2 times that of the optimal vertex cover

2-Approximation Algorithm for Vertex Cover

VertexCover(G)

1: $C \leftarrow \emptyset$
2: **while** $E \neq \emptyset$ **do**
3: select an edge $(u, v) \in E$, $C \leftarrow C \cup \{u, v\}$
4: Remove from E every edge incident on either u or v
5: **return** C

- Let the set C and C^* be the sets output by above algorithm and an optimal alg, respectively. Let S be the set of edges selected.
- Since no two edge in S are covered by the same vertex (Once an edge is picked in line 3, all other edges that are incident on its endpoints are removed from E in line 4), we have $|C^*| \geq |S|$;
- As we have added both vertices of edge (u, v), we get $|C| = 2|S|$ but C^* have to add one of the two, thus, $|C|/|C^*| \leq 2$.

Outline

Summary

- We consider decision problems
- Inputs are encoded as $\{0,1\}$-strings

Def. The complexity class P is the set of decision problems X that can be solved in polynomial time.

- Alice has a supercomputer, fast enough to run an exponential time algorithm
- Bob has a slow computer, which can only run a polynomial-time algorithm

Def. (Informal) The complexity class NP is the set of problems for which Alice can convince Bob a yes instance is a yes instance

Summary

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $X(s) = 1$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string t such that $B(s,t) = 1$ is called a certificate.

Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

Summary

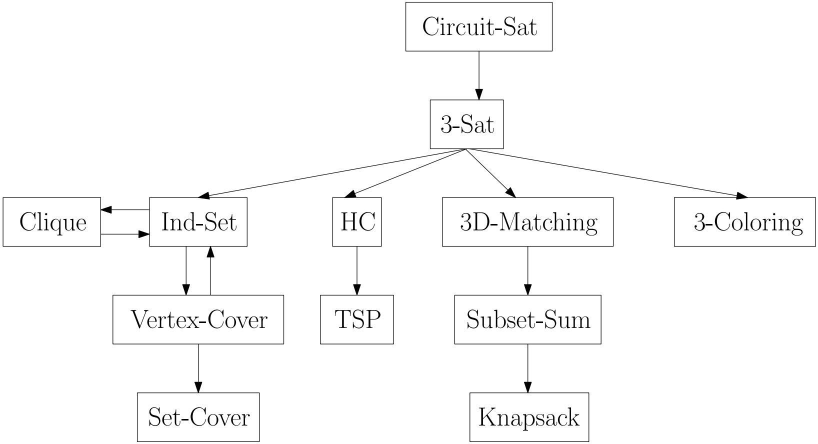
Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

Def. A problem X is called NP-complete if

1. $X \in$ NP, and
2. $Y \leq_\text{P} X$ for every $Y \in$ NP.

- If any NP-complete problem can be solved in polynomial time, then $P = NP$
- Unless $P = NP$, a NP-complete problem can not be solved in polynomial time

Summary


Circuit-Sat
3-Sat
Clique
Ind-Set
HC
3D-Matching
3-Coloring
Vertex-Cover
TSP
Subset-Sum
Set-Cover
Knapsack

Summary

Proof of NP-Completeness for Circuit-Sat

- Fact 1: a polynomial-time algorithm can be converted to a polynomial-size circuit
- Fact 2: for a problem in NP, there is a efficient certifier.
- Given a problem $X \in$ NP, let $B(s,t)$ be the certifier
- Convert $B(s,t)$ to a circuit and hard-wire s to the input gates
- s is a yes-instance if and only if the resulting circuit is satisfiable

- Proof of NP-Completeness for other problems by reductions