NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

Theorem If X is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

Outline

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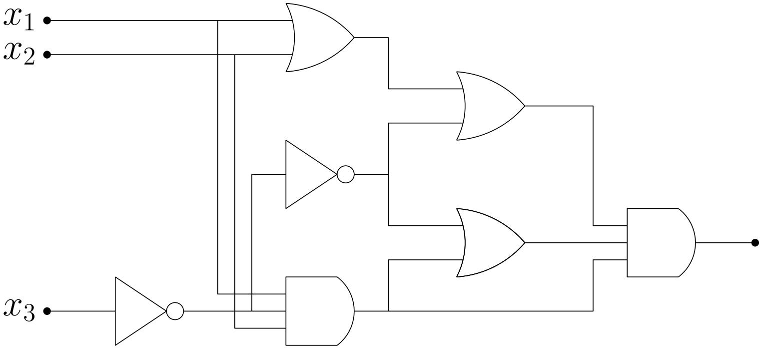
- How can we find a problem $X \in \text{NP}$ such that every problem $Y \in \text{NP}$ is polynomial time reducible to X? Are we asking for too much?
- No! There is indeed a large family of natural NP-complete problems

The First NP-Complete Problem: Circuit-Sat

Circuit Satisfiability (Circuit-Sat)

Input: a circuit

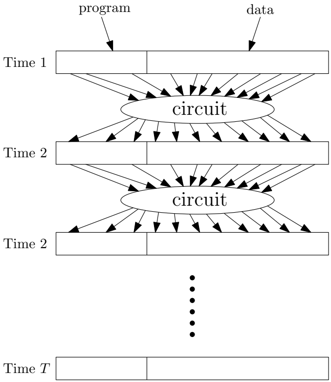
Output: whether the circuit is satisfiable



Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



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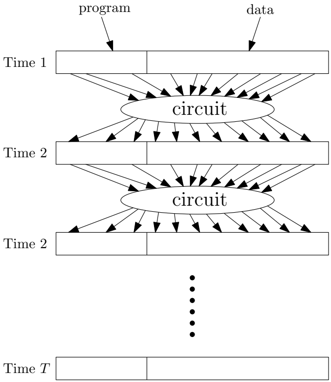

- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.
- We prove HC $\leq_P$ Circuit-Sat as an example.

HC $\leq_P$ Circuit-Sat

check-HC(G, S)

- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle in G and 0 otherwise.

HC $\leq_P$ Circuit-Sat

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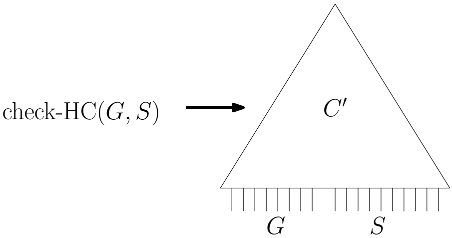
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1

HC $\leq_P$ Circuit-Sat



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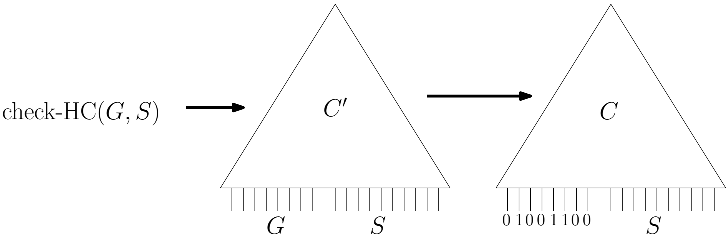
HC $\leq_P$ Circuit-Sat



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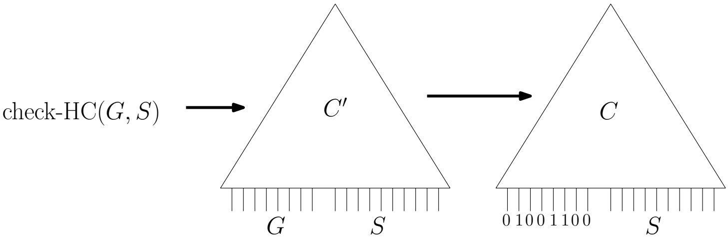
- hard-wire the instance G to the circuit C' to obtain the circuit C

HC $\leq_P$ Circuit-Sat



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- Construct a circuit C' for the algorithm check-HC
- hard-wire the instance G to the circuit C' to obtain the circuit C
- G is a yes-instance if and only if C is satisfiable

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s, t) be the certifier for problem Y: check-Y(s, t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s, t) returns 1
- Construct a circuit C' for the algorithm check-Y
- hard-wire the instance s to the circuit C' to obtain the circuit C
- s is a yes-instance if and only if C is satisfiable $\square$

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s,t) be the certifier for problem Y: check-Y(s,t) returns 1 if t is a valid certificate for s.
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- Construct a circuit C' for the algorithm check-Y
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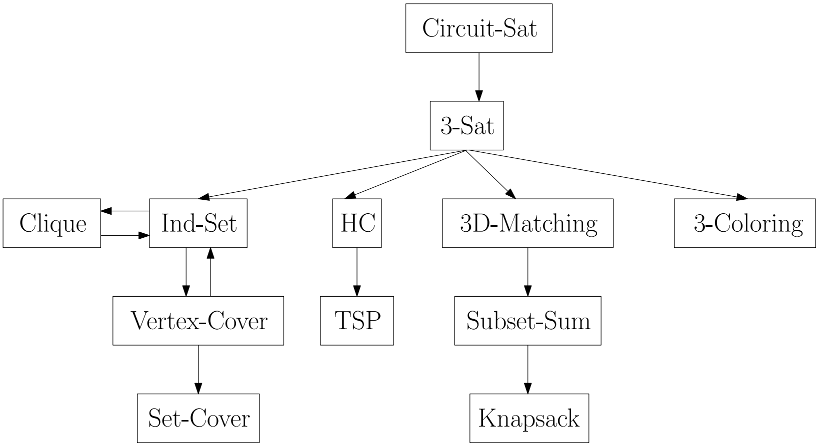
Theorem Circuit-Sat is NP-complete.

Reductions of NP-Complete Problems



3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

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- 3-CNF formula: conjunction ("and") of clauses: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

3-Sat

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Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

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- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal

3-Sat

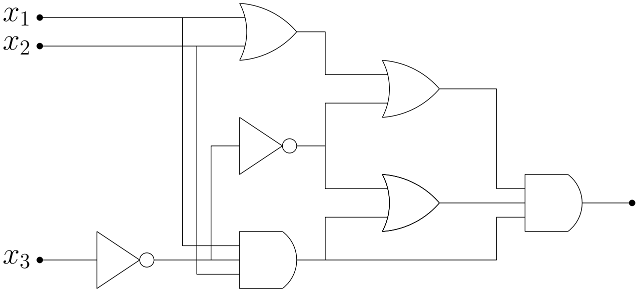
3-Sat

Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal
- Assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ satisfies
  $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

Circuit-Sat $\leq_P$ 3-Sat



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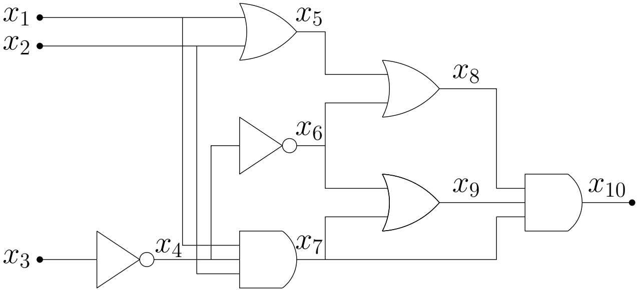

- Associate every wire with a new variable

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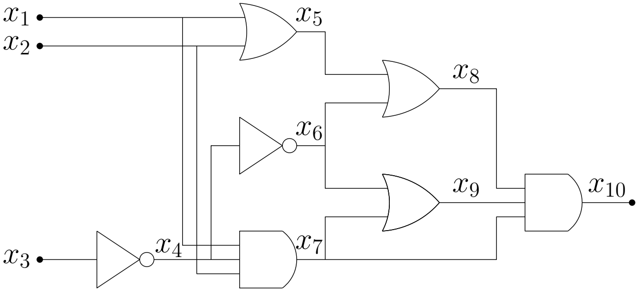

- Associate every wire with a new variable
- The circuit is equivalent to the following formula:

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
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\end{aligned}
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Convert each clause to a 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

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Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
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0	0	1	0
0	1	0	0
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Circuit-Sat $\leq_P$ 3-Sat

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Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

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- The circuit is satisfiable if and only if the 3-CNF is satisfiable

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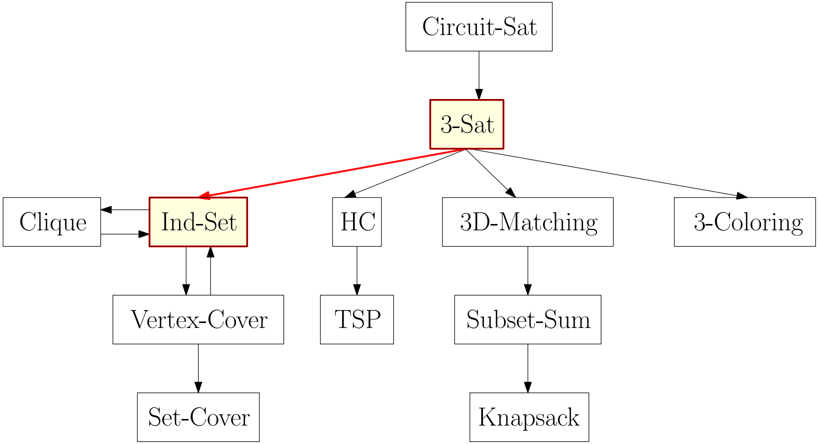
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit

Circuit-Sat $\leq_P$ 3-Sat

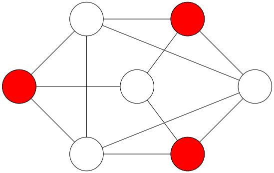
- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
- Thus, Circuit-Sat $\leq_P$ 3-Sat

Reductions of NP-Complete Problems



Recall: Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



Independent Set (Ind-Set) Problem

Input: $G = (V, E), k$

Output: whether there is an independent set of size k in G

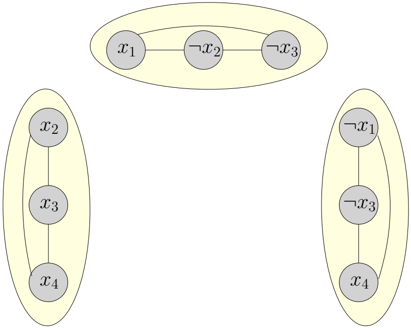
3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group



3-Sat $\leq_P$ Ind-Set

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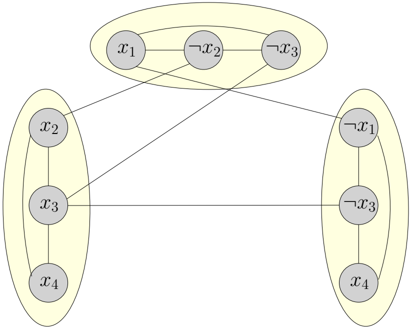
- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals



3-Sat $\leq_P$ Ind-Set

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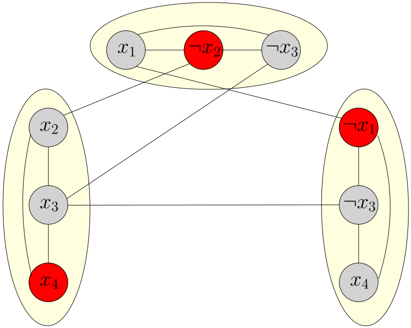
- Problem: whether there is an IS of size $k = \#$clauses



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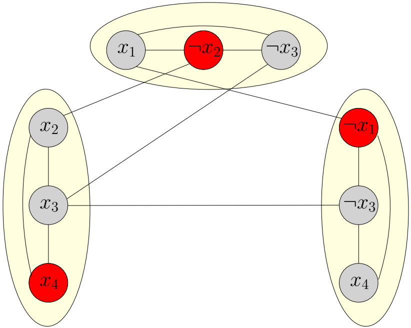
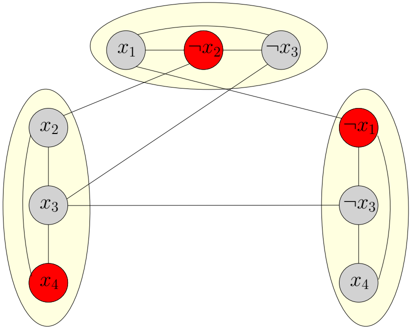


3-Sat instance is yes-instance $\Leftrightarrow$ Ind-Set instance is yes-instance:

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses

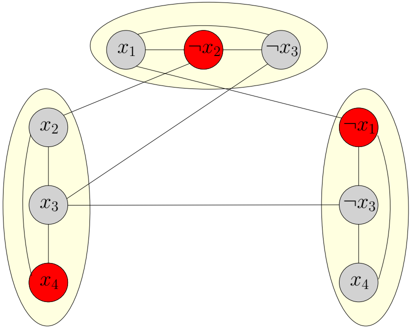


3-Sat instance is yes-instance $\Leftrightarrow$ Ind-Set instance is yes-instance:

- satisfying assignment $\Rightarrow$ independent set of size k
- independent set of size k $\Rightarrow$ satisfying assignment

Satisfying Assignment $\Rightarrow$ IS of Size k

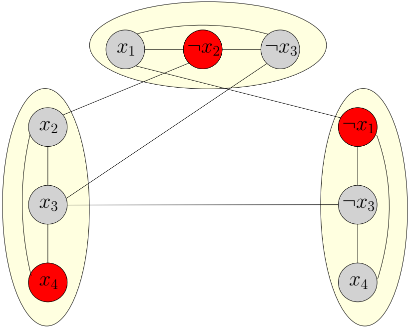
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

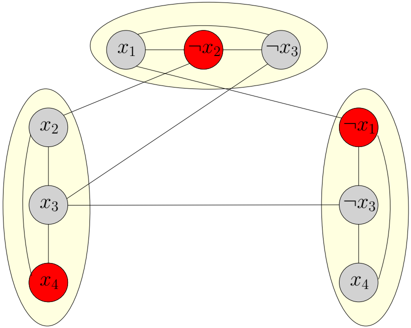
- For every clause, at least 1 literal is satisfied



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

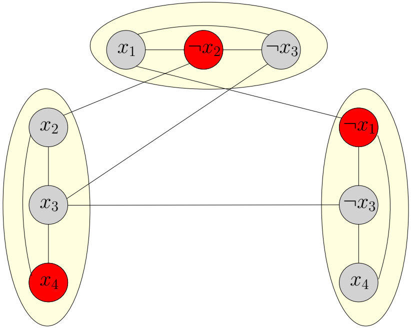
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

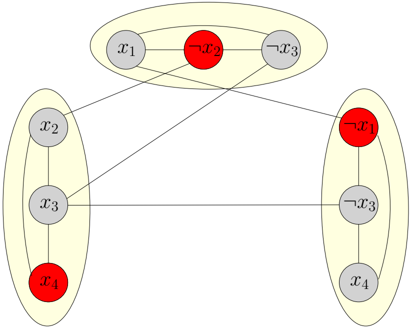
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

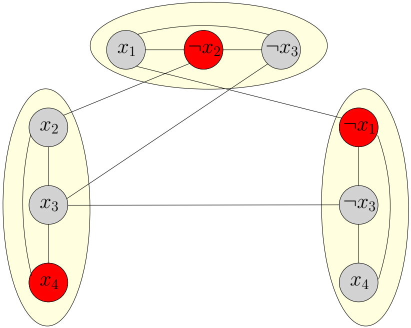
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals



Satisfying Assignment $\Rightarrow$ IS of Size k

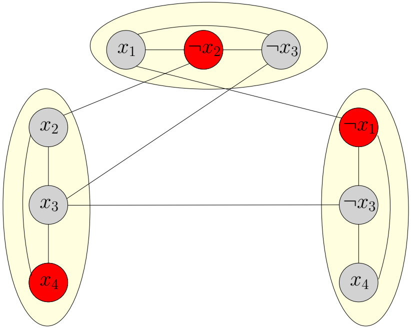
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals
- An IS of size k



IS of Size $k \Rightarrow$ Satisfying Assignment

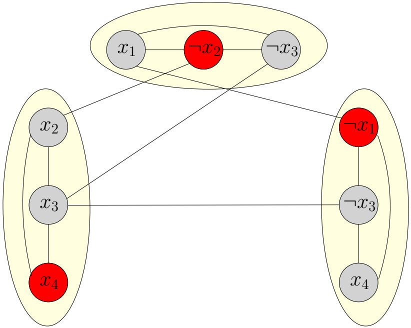
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

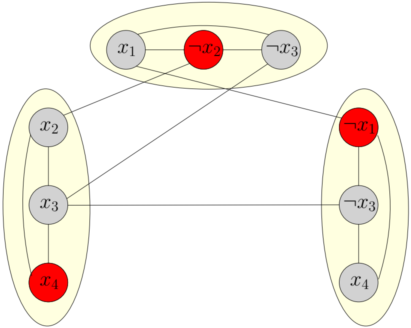
- For every group, exactly one literal is selected in IS



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

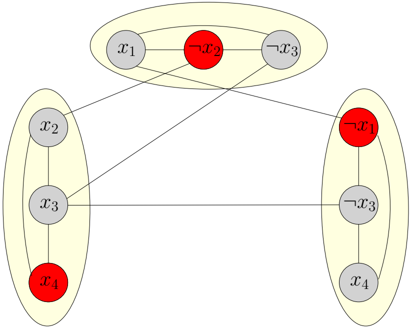
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

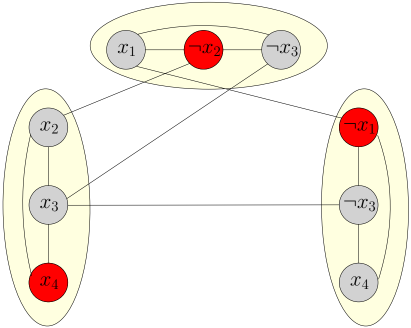
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If x_i is selected in IS, set $x_i = 1$



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If x_i is selected in IS, set $x_i = 1$
- If $\neg x_i$ is selected in IS, set $x_i = 0$



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If x_i is selected in IS, set $x_i = 1$
- If $\neg x_i$ is selected in IS, set $x_i = 0$
- Otherwise, set x_i arbitrarily