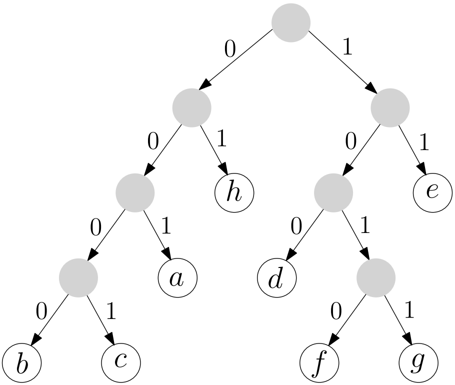
CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Greedy Algorithms

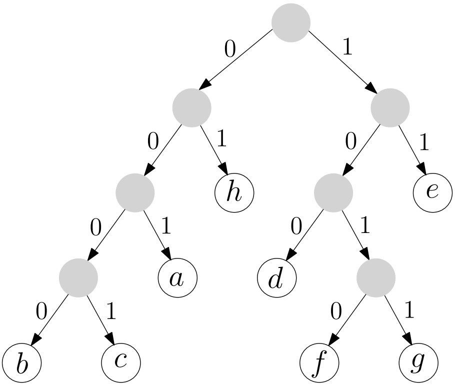
Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Outline



Properties of Encoding Tree



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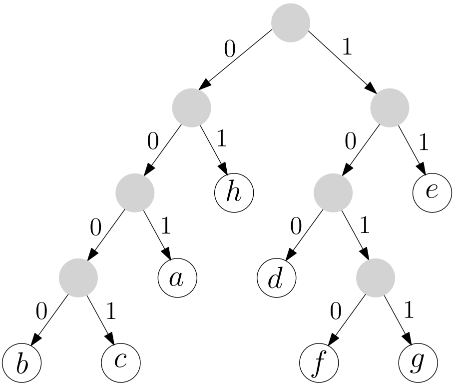
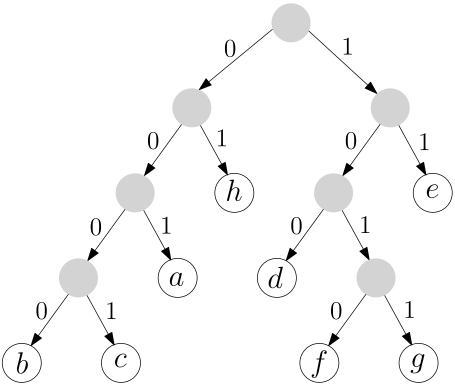
- Rooted binary tree



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1



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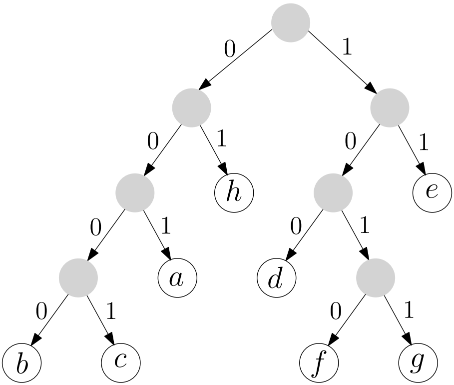

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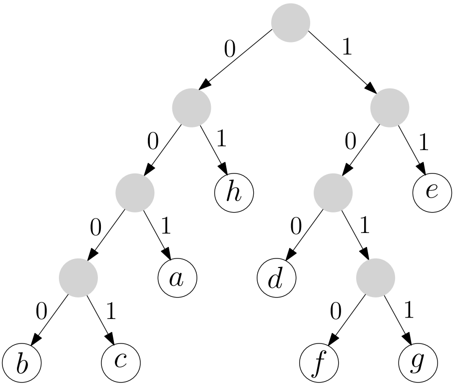

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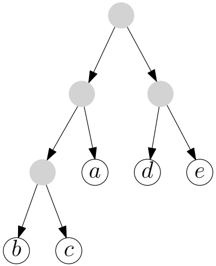
Best Prefix Codes

Input: frequencies of letters in a message

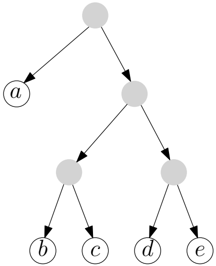
Output: prefix coding scheme with the shortest encoding for the message

example

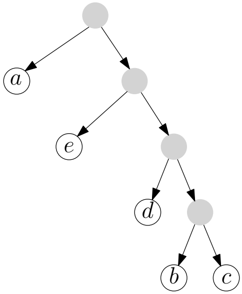
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



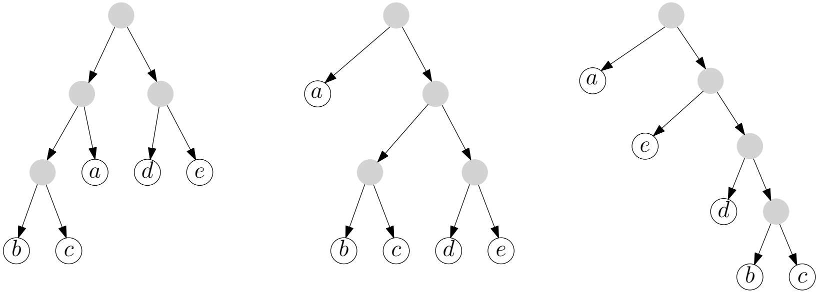
scheme 2



scheme 3

example

letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total $=$ 89
scheme 2 length	1	3	3	3	3	total $=$ 87
scheme 3 length	1	4	4	3	2	total $=$ 84



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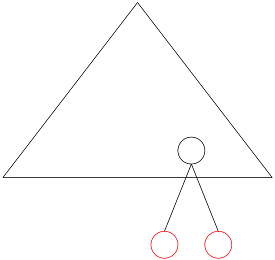
A: We can choose two letters and make them brothers in the tree.

Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree



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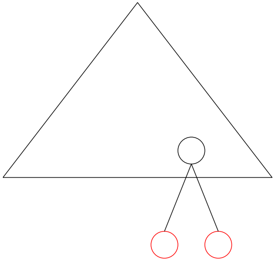
- There are two deepest leaves that are brothers



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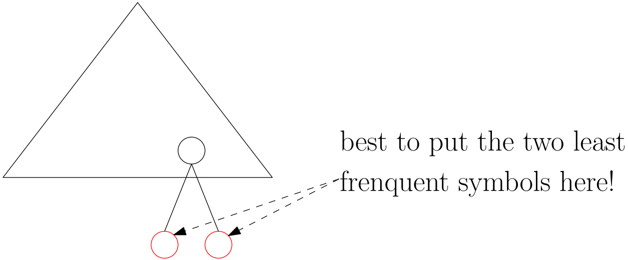

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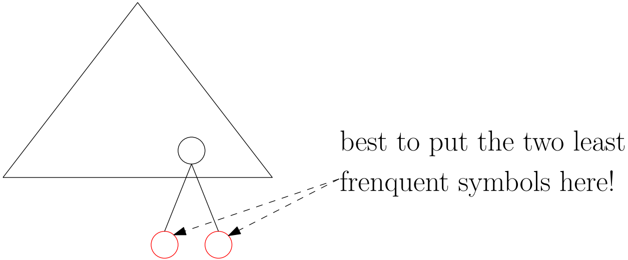

Lemma It is safe to make the two least frequent letters brothers.

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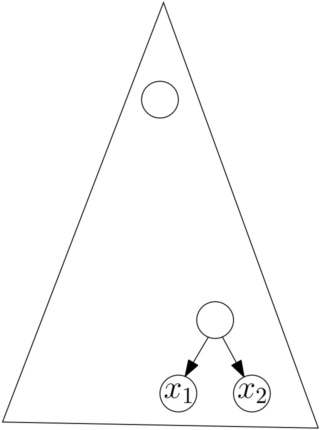
- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



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&\sum_{x \in S} f_x d_x \\
&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
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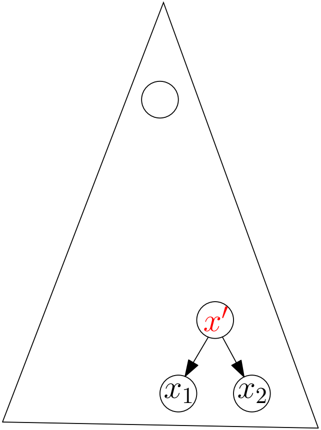

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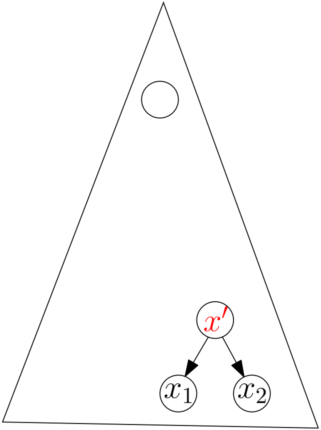

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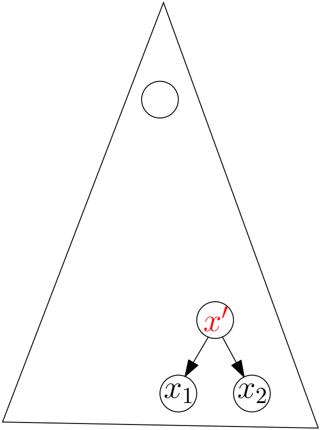

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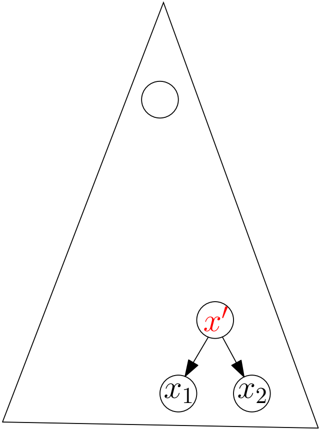

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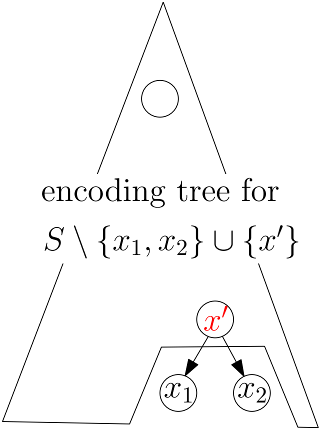

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In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

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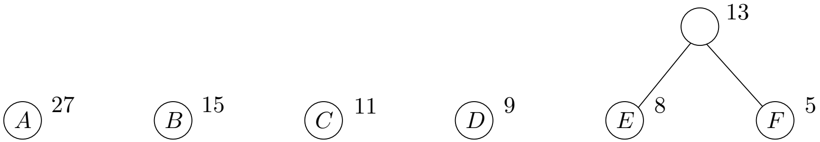

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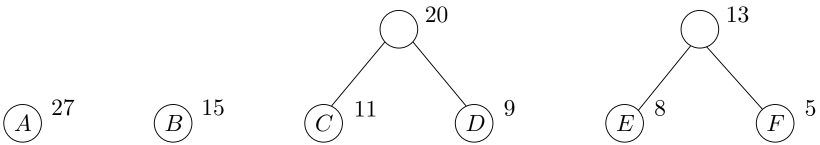

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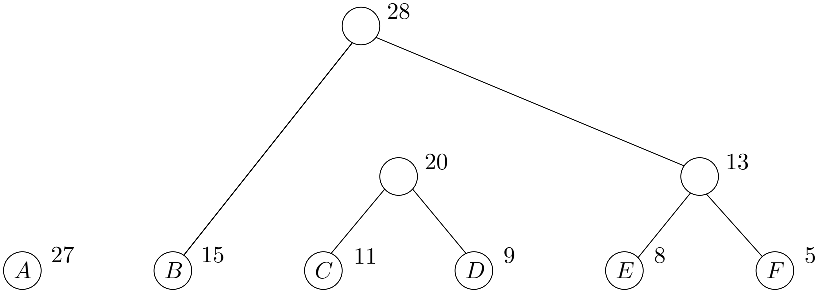

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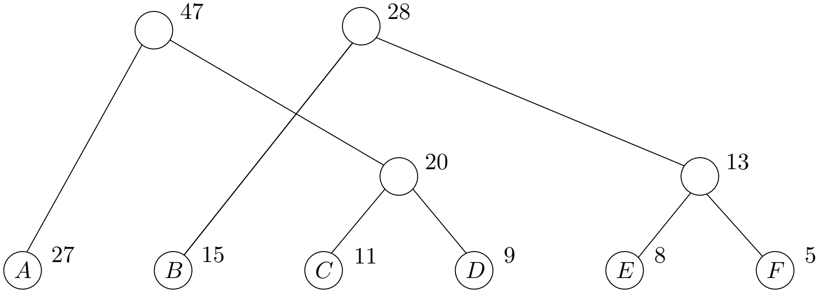

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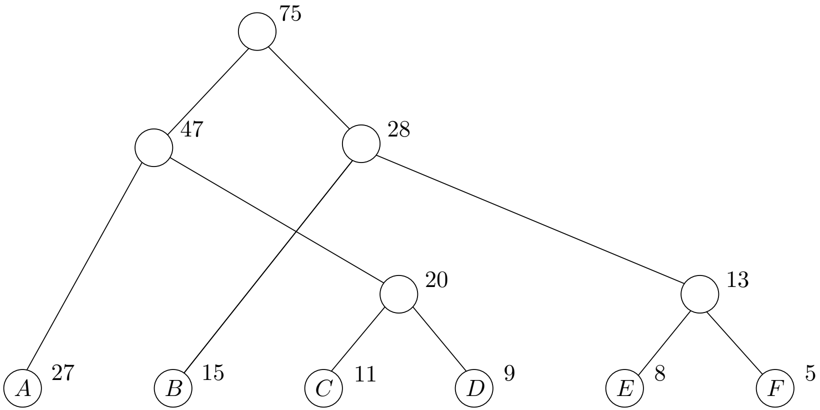

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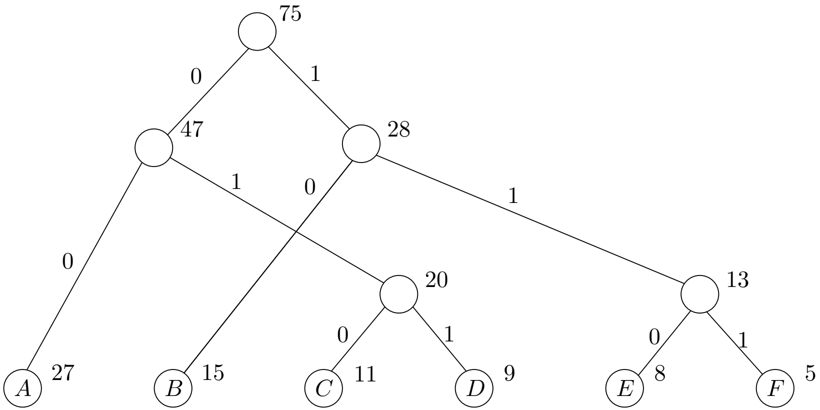

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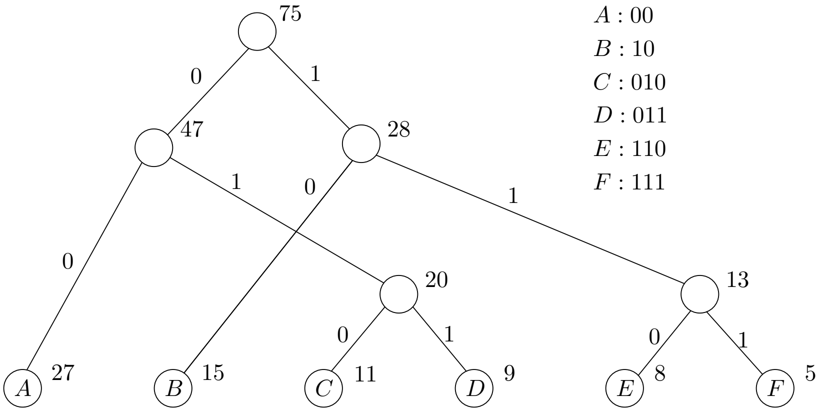

$A : 00$

$B : 10$

$C : 010$

$D : 011$

$E : 110$

$F : 111$

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Huffman(S, f)

while $|S| > 1$ **do**
    let x_1, x_2 be the two letters with the smallest f values
    introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
    let x_1 and x_2 be the two children of x'
    $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
return the tree constructed

Algorithm using Priority Queue

Huffman(S, f)

```
Q ← build-priority-queue(S)
while Q.size > 1 do
    x_1 ← Q.extract-min()
    x_2 ← Q.extract-min()
    introduce a new letter x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    Q.insert(x', f_{x'})
return the tree constructed
```

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- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

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- Safety: Prove that the reasonable strategy is "safe" (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

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- Interval scheduling problem: remove j^* and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one

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  - Topological Ordering problem: topological-sort algorithm (Queue or Stack)

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  - Exercise problems (Lecture: Monday, 30th September)