CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Testing Bipartiteness and Topological Ordering

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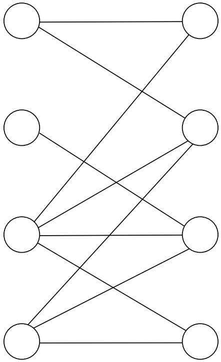
Department of Computer Science and Engineering
University at Buffalo

Outline

1. Testing Bipartiteness
2. Topological Ordering
   - Applications: Word Ladder

Bipartite Graph

Def. An undirected graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



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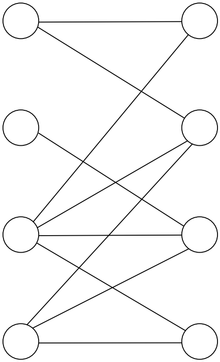
Obs. Bipartite graph may contain cycles.



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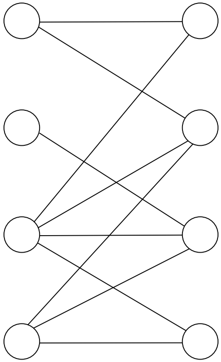
Obs. Bipartite graph may contain cycles.

Obs. If a graph is a tree, then it is also a bipartite graph.



BFS and DFS

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- True: simple, undirected graph, starting at the same vertex
- Not True: starting at different vertices or an directed graph

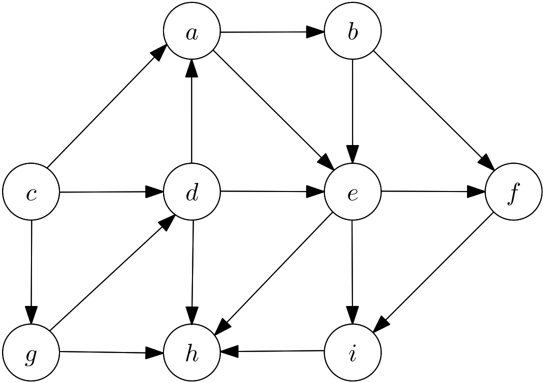
Outline

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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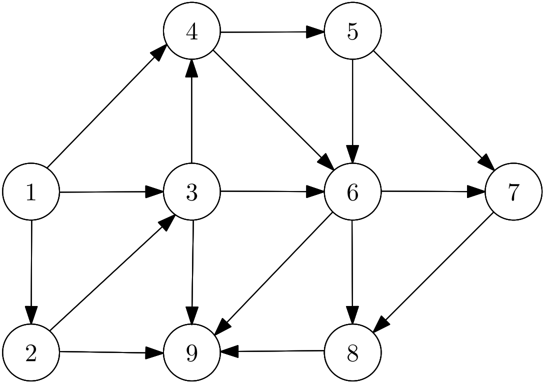
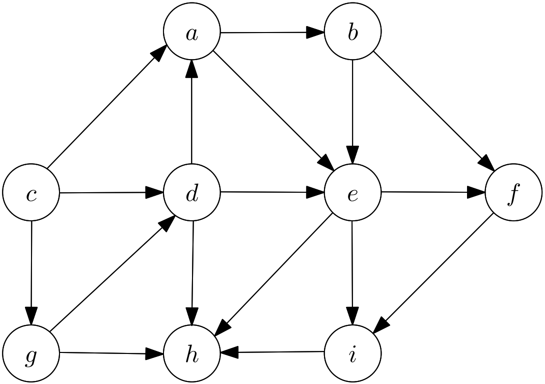


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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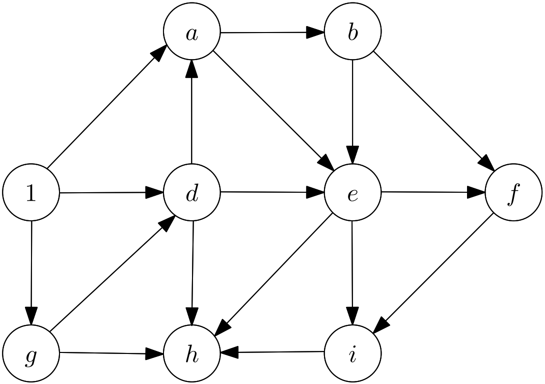

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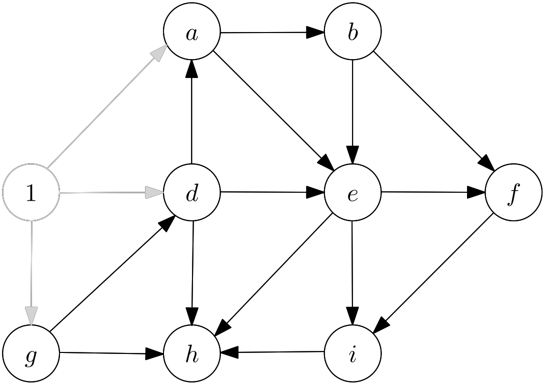

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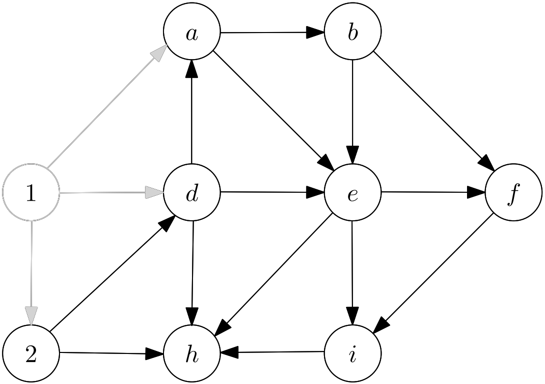

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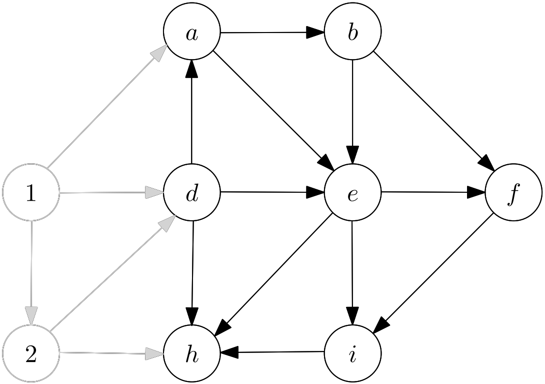

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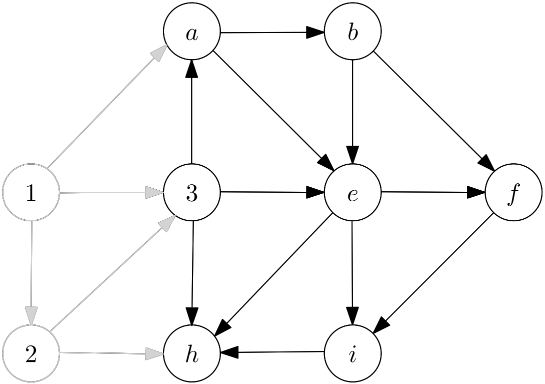

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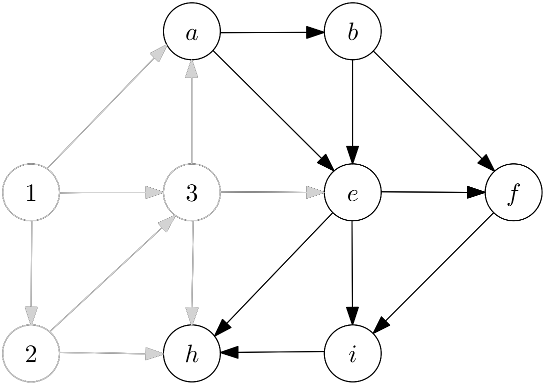

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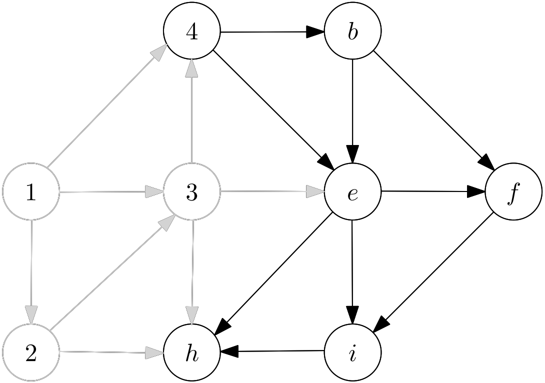

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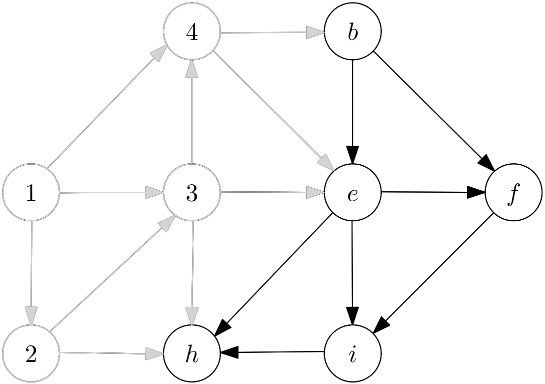

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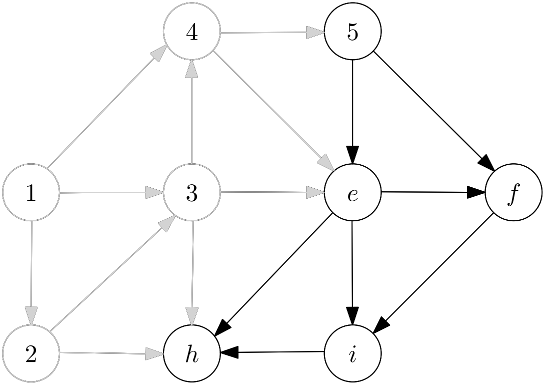

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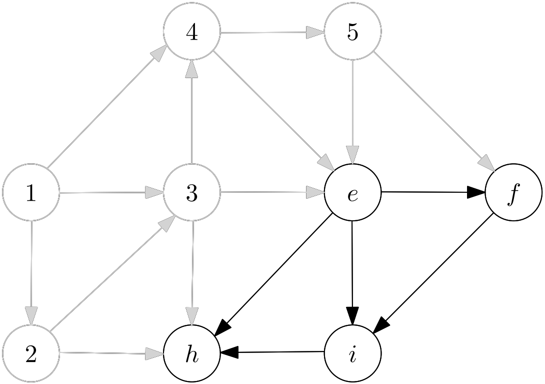

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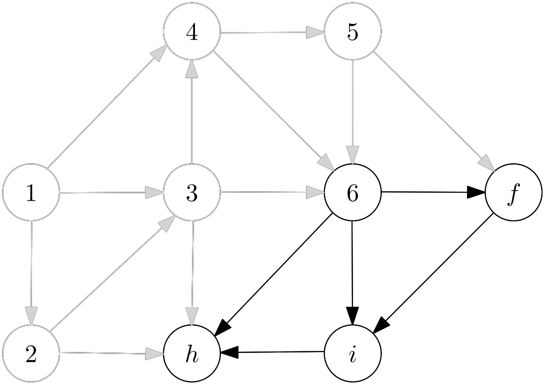

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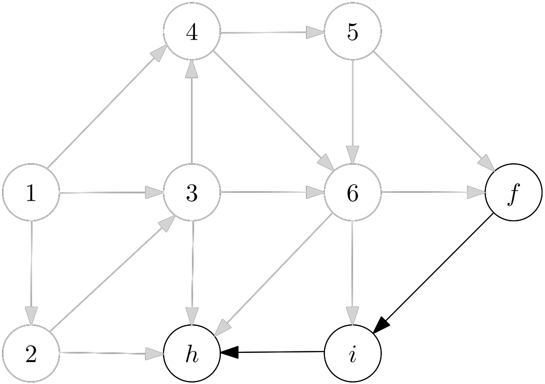

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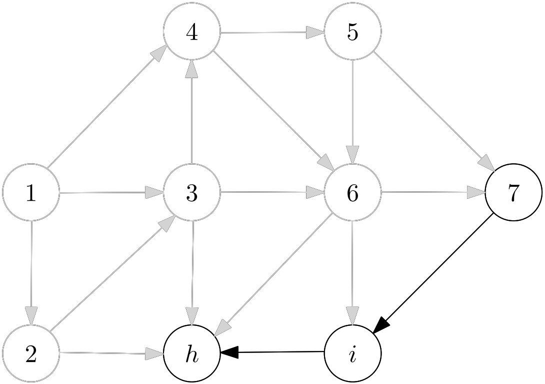

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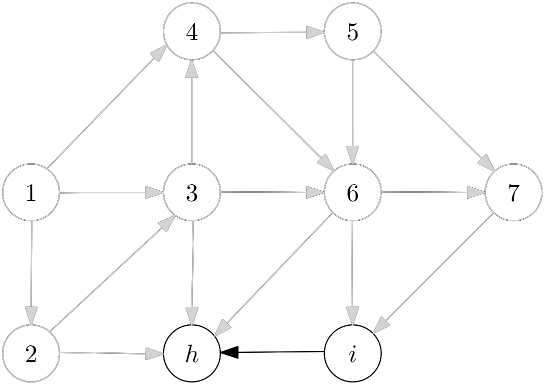

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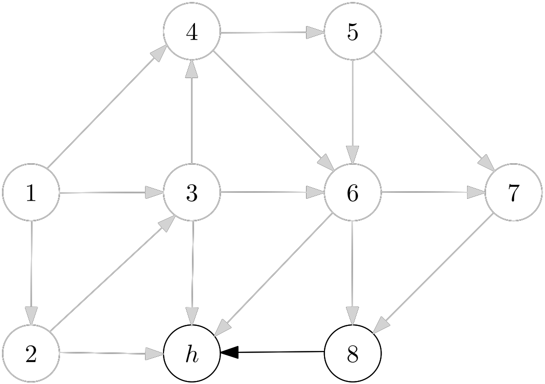

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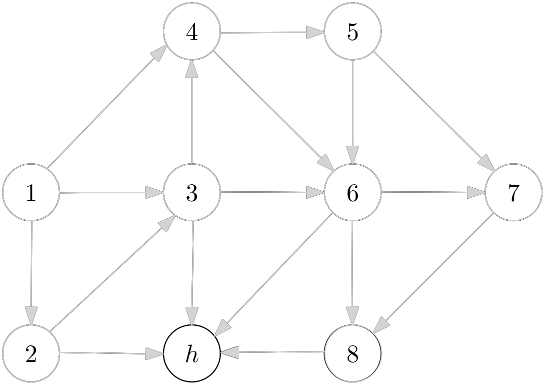

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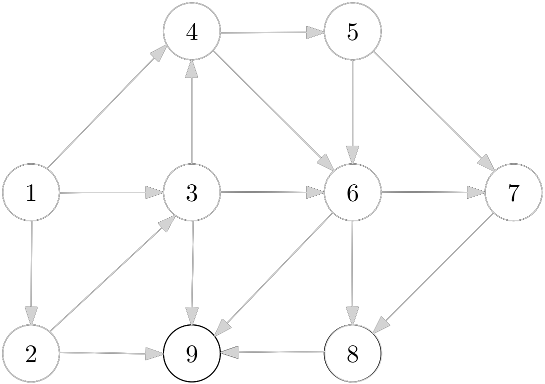

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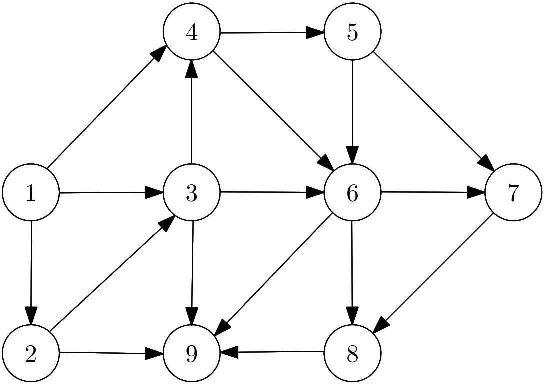

Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.

topological-sort(G)

$i \leftarrow 0$
for $V \neq \emptyset$ **do**
  $v \leftarrow$ vertex in V without incoming edges
  $i \leftarrow i + 1$, $\pi(v) \leftarrow i$ and $V \leftarrow V \setminus \{v\}$
  while $V \neq \emptyset$ **do**
    for every $u \in V$ such that $(v, u) \in E$ **do**
      $E \leftarrow E \setminus \{(v, u)\}$

- Running time $= O(n + m)$

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Topological Ordering

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

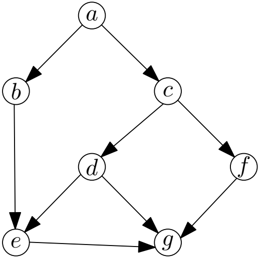
let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$ **do**
    for every u such that $(v, u) \in E$ **do**
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$ **do**
    $v \leftarrow$ arbitrary vertex in S, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every u such that $(v, u) \in E$ **do**
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ **then** add u to S

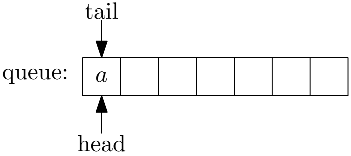
- S can be represented using a queue or a stack
- Running time $= O(n + m)$

S as a Queue or a Stack

DS	Queue	Stack
Initialization	$head \leftarrow 1,\ tail \leftarrow 0$	$top \leftarrow 0$
Non-Empty?	$head \leq tail$	$top > 0$
Add(v)	$tail \leftarrow tail + 1$ $S[tail] \leftarrow v$	$top \leftarrow top + 1$ $S[top] \leftarrow v$
Retrieve v	$v \leftarrow S[head]$ $head \leftarrow head + 1$	$v \leftarrow S[top]$ $top \leftarrow top - 1$

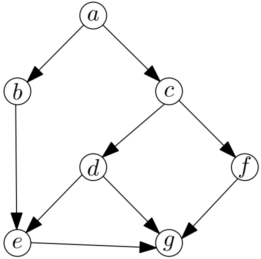
Example

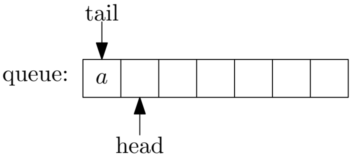




	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

Example





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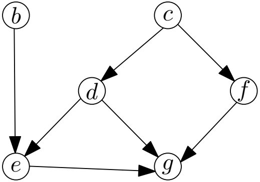

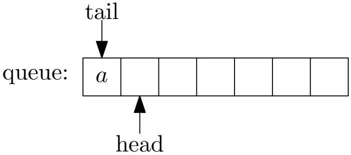

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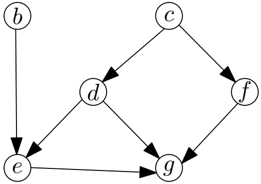

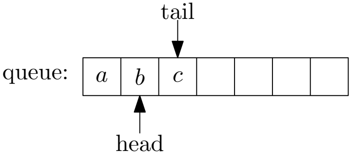

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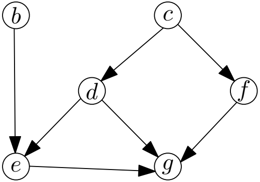

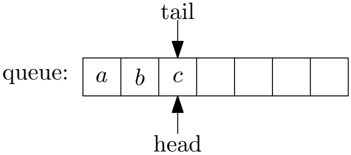

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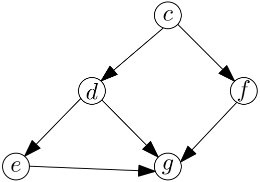

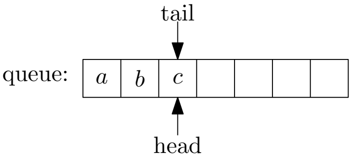

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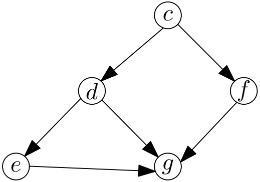

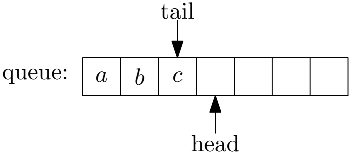

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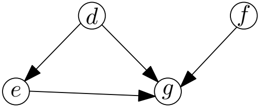

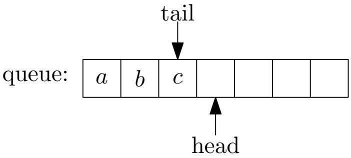

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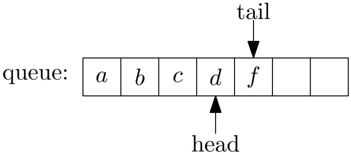

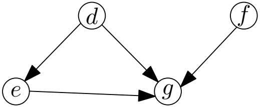

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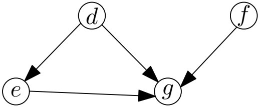


tail

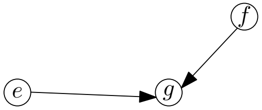
queue:

a	b	c	d	f		

head

	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

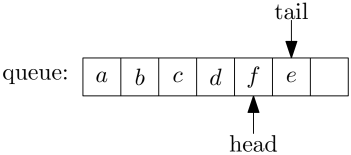
Example

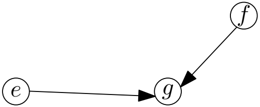


tail				↓		
queue: a	b	c	d	f		
				↑ head		

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example





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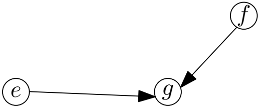

tail

queue:	a	b	c	d	f	e	

head

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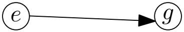
degree	0	0	0	0	0	0	2

Example

tail

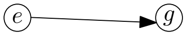
queue:							
	a	b	c	d	f	e	

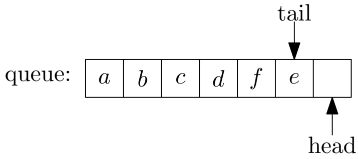
head



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example





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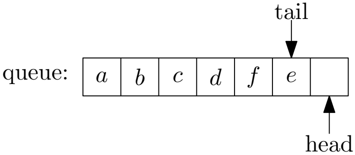

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example

tail

queue:	a	b	c	d	f	e	g

head

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degree	0	0	0	0	0	0	0

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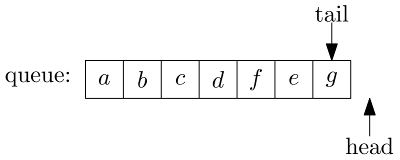

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Outline

Def. Word: A string formed by letters.

Def. Adjacency words: Word A and B are adjacent if they differ in exactly one letter.

e.g. wor$d$ and wor$k$; t$e$ll and t$a$ll; ask$b$e and ask$e$e.

Def. Word Ladder: Players start with one word, and in a series of steps, change or transform that word into another word.

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- The objective is to make the change in the smallest number of steps, with each step involving changing a **single letter** of the word to create a new valid word.

Word Ladder Problem

Input: Two words S and T, a list of words $A = \{W_1, W_2, ..., W_k\}$.

Output: " The smallest word ladder" if we can change S to T by moving between adjacency words in $A \cup \{S, T\}$; Otherwise, "No word ladder".

Example:

- S="a e f g h", T = "d l m i h"
- W_1="a e f i h", W_2 = "a e m g h", W_3="d l f i h"
  W_4 = "s e f i h", W_5="a d f g h", W_6 = "d e m i h"
  W_7="d e f i h", W_8 = "d e m g h", W_9 = "s e m i h"

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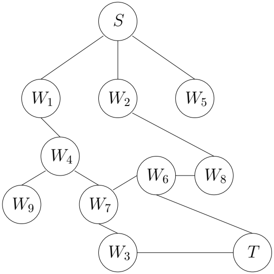
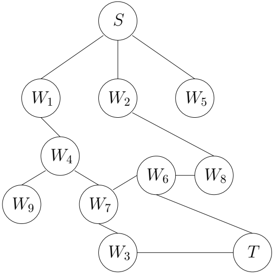


- Each vertex corresponds to a word.
- Two vertices are adjacent if the corresponding words are adjacent.



- Each vertex corresponds to a word.
- Two vertices are adjacent if the corresponding words are adjacent.
- Hints: Given vertex v, check its nearest neighbor.