CSE 431/531: Algorithm Analysis and Design (Fall 2024)

Divide-and-Conquer

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University at Buffalo

Outline

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Quicksort Can Be Implemented as an “In-Place” Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses “small” extra space.

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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

$p \leftarrow$ random integer between ℓ and r, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while true **do**
    while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
    while $i < j$ and $A[i] < A[j]$ **do** $i \leftarrow i + 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
return i

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

if $\ell \geq r$ **then return**
$m \leftarrow$ patition(A, ℓ, r)
quicksort$(A, \ell, m-1)$
quicksort$(A, m+1, r)$

- To sort an array A of size n, call quicksort$(A, 1, n)$.

Note: We pass the array A by reference, instead of by copying.

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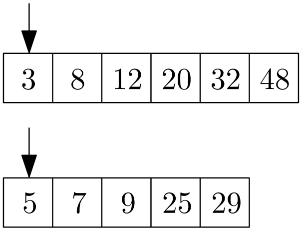

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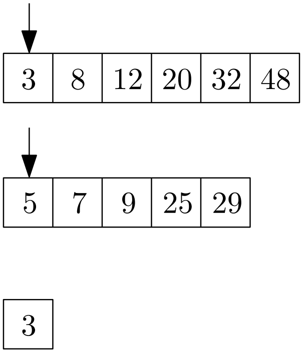

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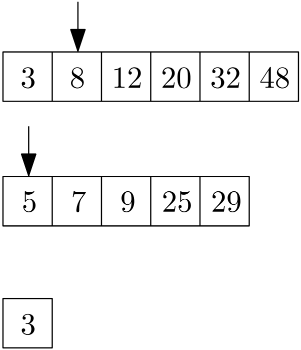

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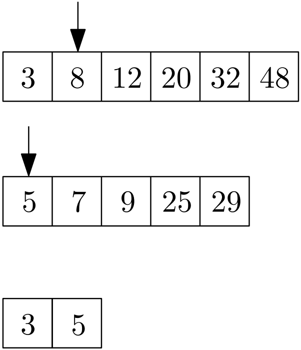

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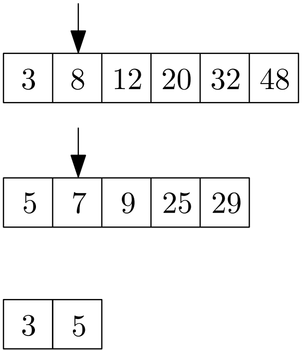

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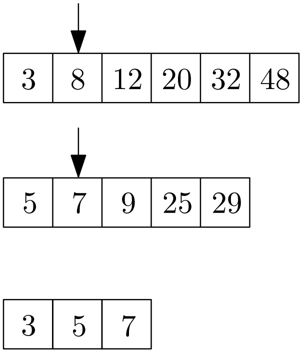

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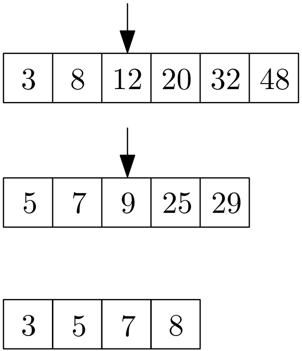

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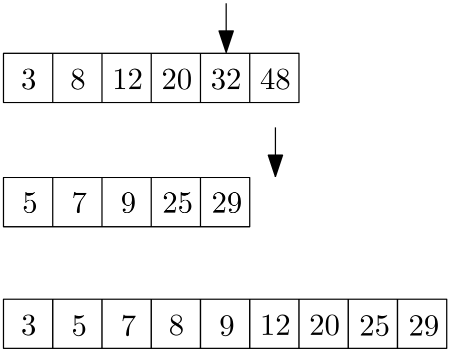

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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

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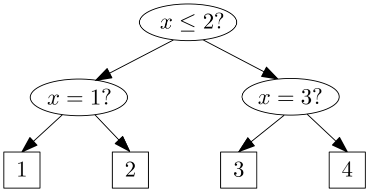

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \cdots, n\}$ in his hand.
- You can ask Bob "yes/no" questions about π.

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A: At least $\log_2 n! = \Theta(n \lg n)$

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Selection Problem

Input: a set A of n numbers, and $1 \leq i \leq n$

Output: the i-th smallest number in A

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- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

```
if n ≤ 1 then return A
x ← lower median of A
A_L ← elements in A that are less than x          ▷ Divide
A_R ← elements in A that are greater than x       ▷ Divide
B_L ← quicksort(A_L, A_L.size)                     ▷ Conquer
B_R ← quicksort(A_R, A_R.size)                     ▷ Conquer
t ← number of times x appear A
return the array obtained by concatenating B_L, the array
    containing t copies of x, and B_R
```

Selection Algorithm with Median Finder

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
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- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **thenreturn** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x $\triangleright$ Divide
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- expected running time $= O(n)$