CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Greedy Algorithms

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Announcements: HW1 Due

- Due: Mon 16 Sep @ 11:59PM
- Late Email submission to Instructor (kelinluo@buffalo.edu) and Head TAs Xiaoyu Zhang (zhang376@buffalo.edu) and Bahadir (ialtun@buffalo@edu): due 18 Sep @ 11:59PM
- Typed submission
- Potential Grading scheme will be released over the weekend.

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Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems

Common Paradigms for Algorithm Design

- Greedy Algorithms: shortest path problem
- Divide and Conquer: merge-sort, binary search
- Dynamic Programming: shortest path problem, Fibonacci number

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- They often run in polynomial time due to their simplicity: easy to come up with, easy to analyze running time.
- Hard to see correctness. Mostly, it is not correct. E.g. $\min f(x)$

Greedy Algorithm

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Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.

Outline

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: $60, 40, 25, 17, 12$
- Item sizes: $45, 41, 20, 19, 16$

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- Can put 4 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25, 16 \to 17$

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- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.

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- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:

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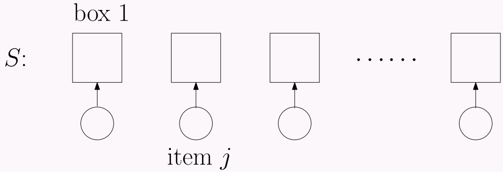

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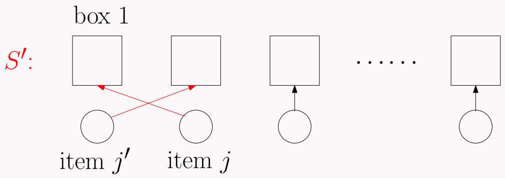

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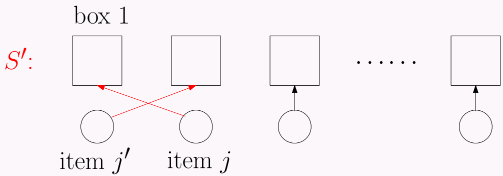

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- S' is also an optimum solution. In S', j is put into Box 1. $\square$

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- Trivial: we decided to put Item j into Box 1, and the remaining instance is obtained by removing Item j and Box 1.

Generic Greedy Algorithm

while the instance is non-trivial **do**
    make the choice using the greedy strategy
    reduce the instance

Greedy Algorithm for Box Packing

$T \leftarrow \{1, 2, 3, \cdots, m\}$
for $i \leftarrow 1$ to n **do**
    if some item in T can be put into box i **then**
        $j \leftarrow$ the largest item in T that can be put into box i
        print("put item j in box i")
        $T \leftarrow T \setminus \{j\}$

Running time

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- With sorted item-sizes and box-capacities, running time is $O(\max\{n, m\})$.