CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Graph Algorithms

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Outline

1. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

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Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
    if $S_u \neq S_v$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

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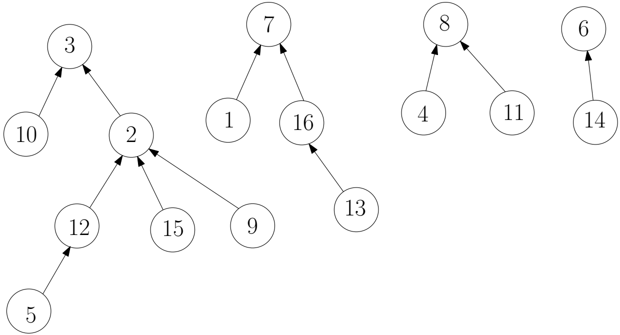
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

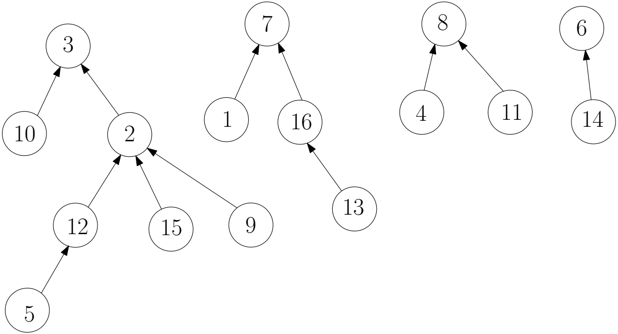
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure



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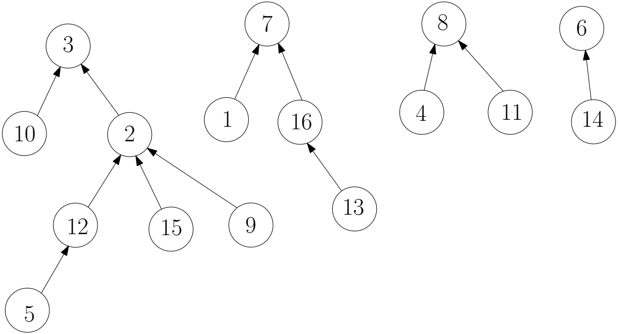

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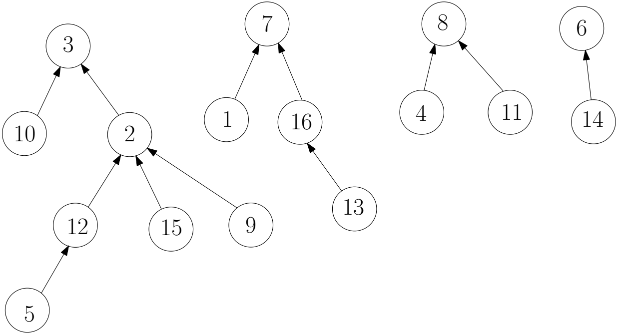

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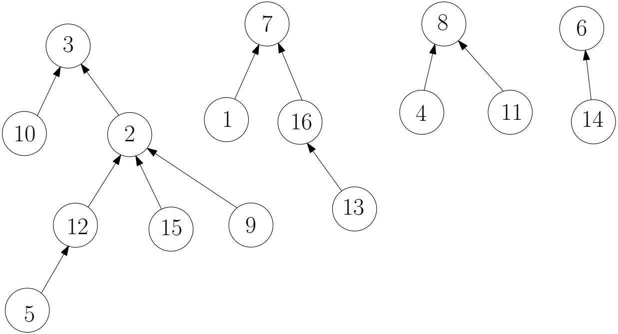

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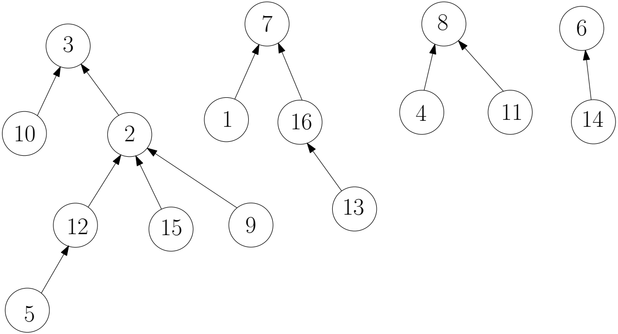

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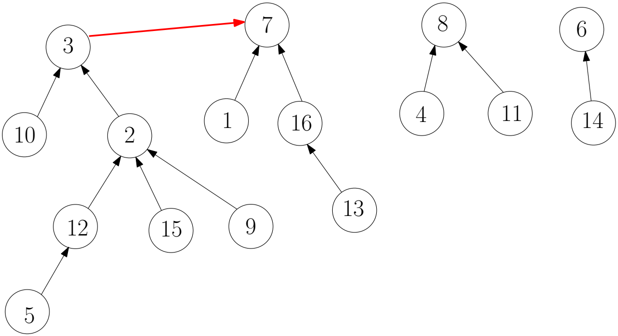

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if par[v] = ⊥ then
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Union-Find Data Structure

root(v)

1: **if** $par[v] = \perp$ **then**
2:     **return** v
3: **else**
4:     **return** root($par[v]$)

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- Improvement: all vertices in the path directly point to the root, saving time in the future.

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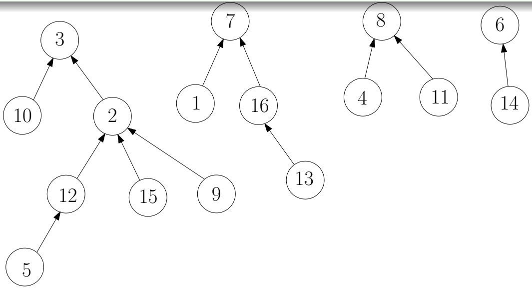

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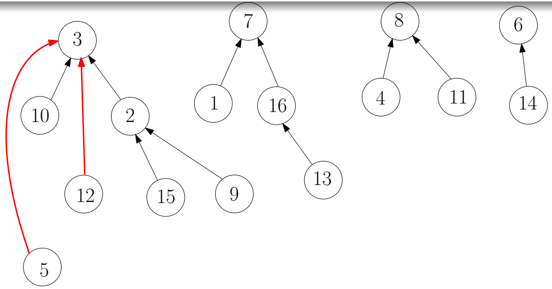

MST-Kruskal(G, w)

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    $u' \leftarrow \text{root}(u)$

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    if $u' \neq v'$ **then**

        $F \leftarrow F \cup \{(u, v)\}$

        $par[u'] \leftarrow v'$

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MST-Kruskal(G, w)

```
1: F ← ∅
2: for every v ∈ V do: par[v] ← ⊥
3: sort the edges of E in non-decreasing order of weights w
4: for each edge (u, v) ∈ E in the order do
5:     u' ← root(u)
6:     v' ← root(v)
7:     if u' ≠ v' then
8:         F ← F ∪ {(u, v)}
9:         par[u'] ← v'
10: return (V, F)
```

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

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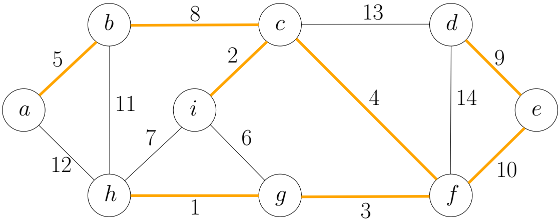
- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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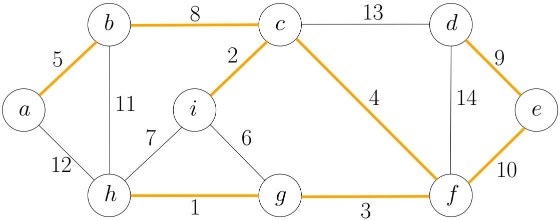

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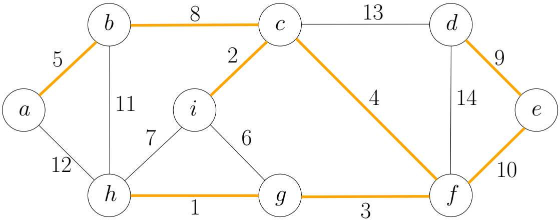

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

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Two Methods to Build a MST

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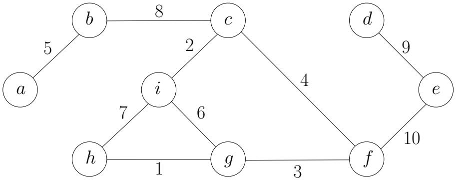

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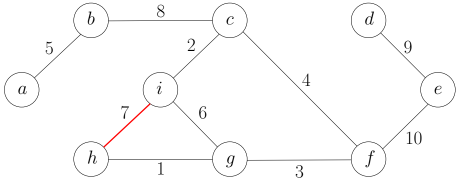

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A: The heaviest non-bridge edge.

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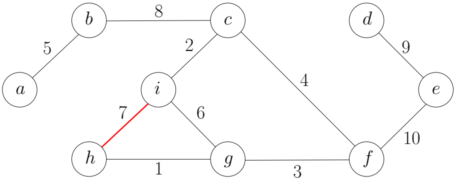


Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

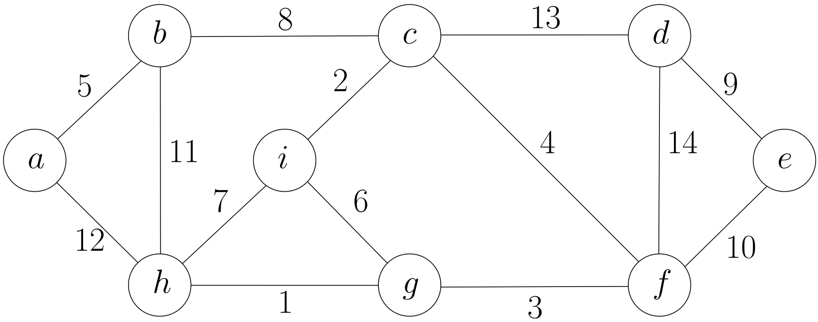
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

$F \leftarrow E$
sort E in non-increasing order of weights
for every e in this order **do**
    if $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
return (V, F)

Reverse Kruskal's Algorithm: Example



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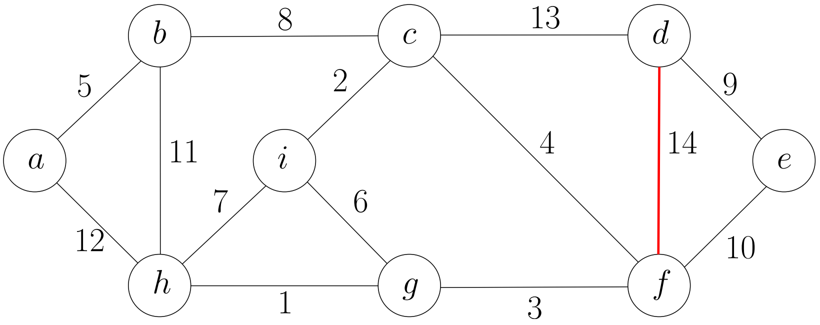

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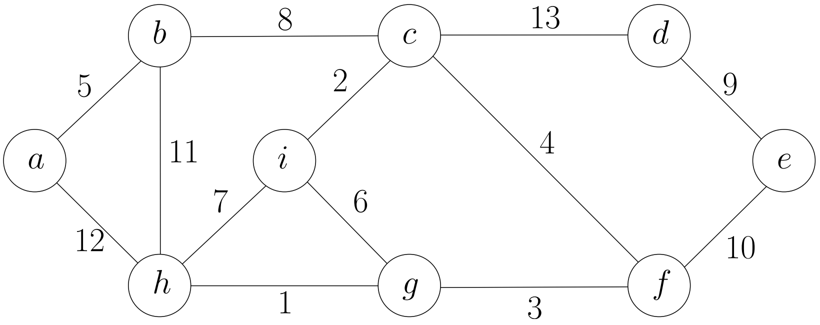

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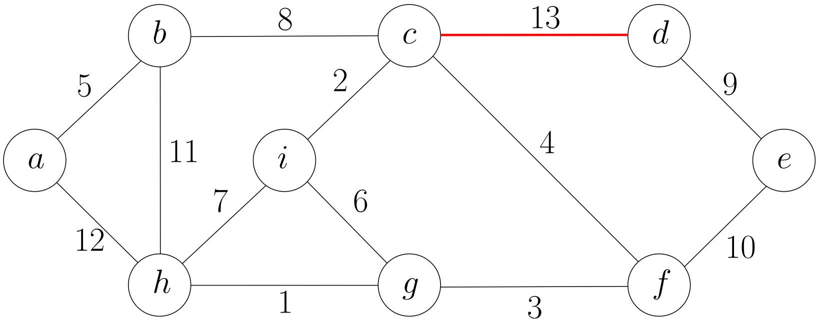

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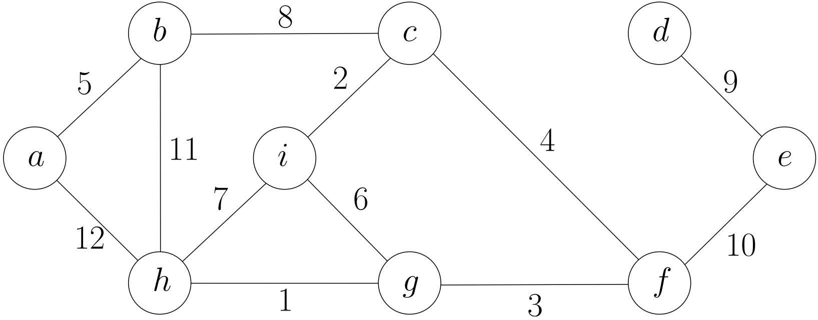

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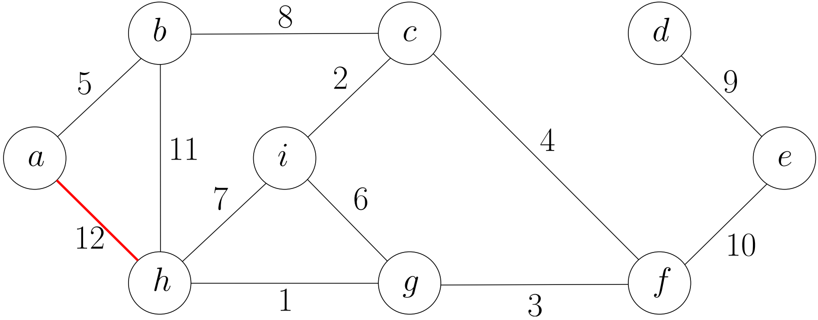

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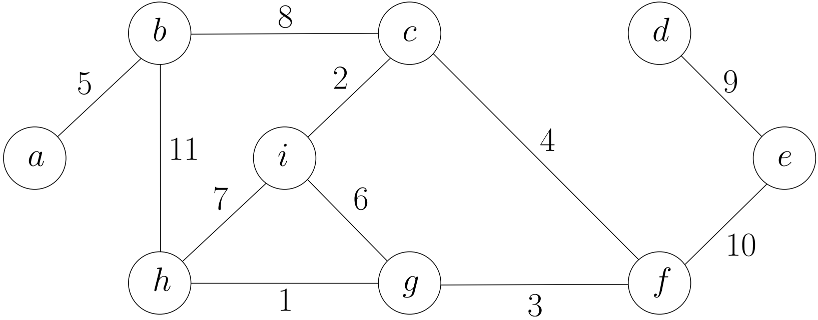

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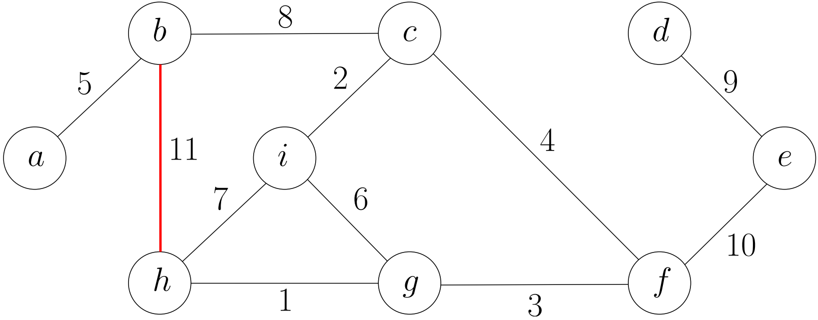

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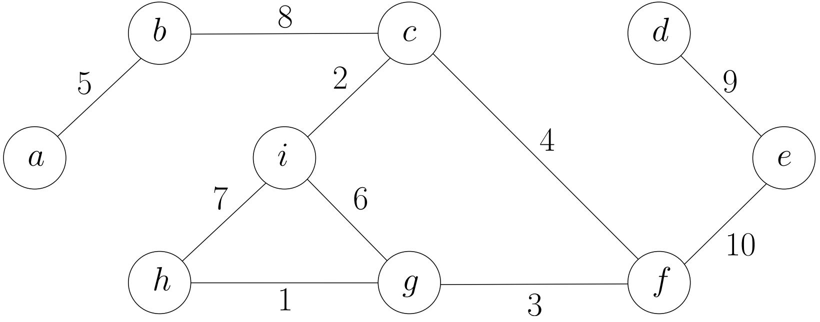

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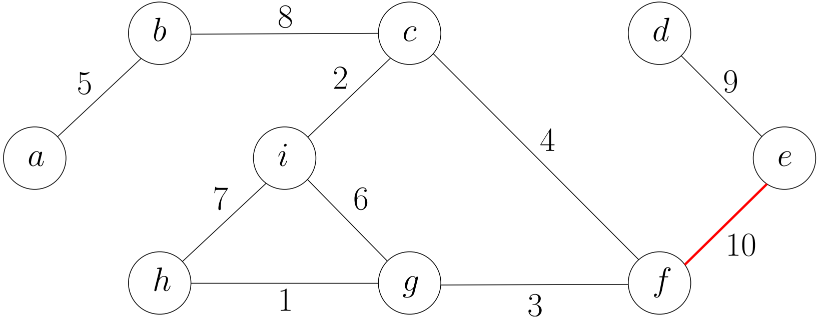

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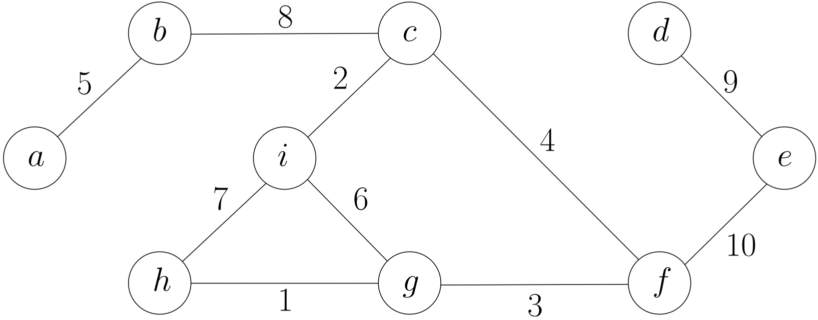

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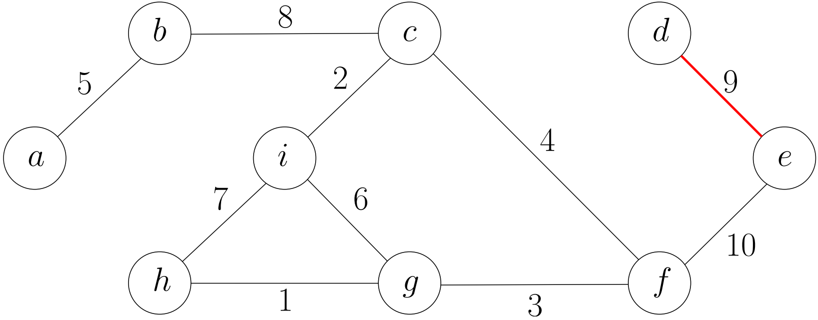

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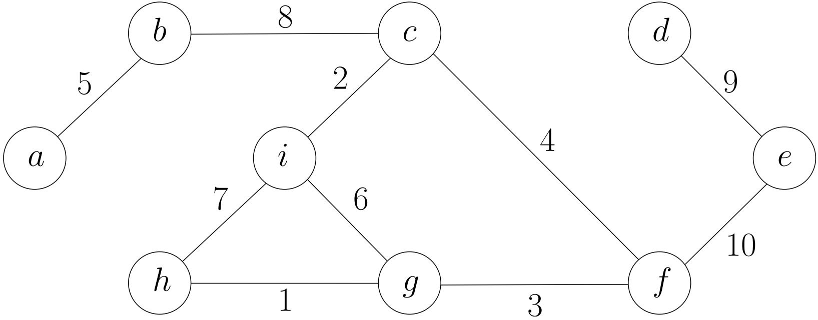

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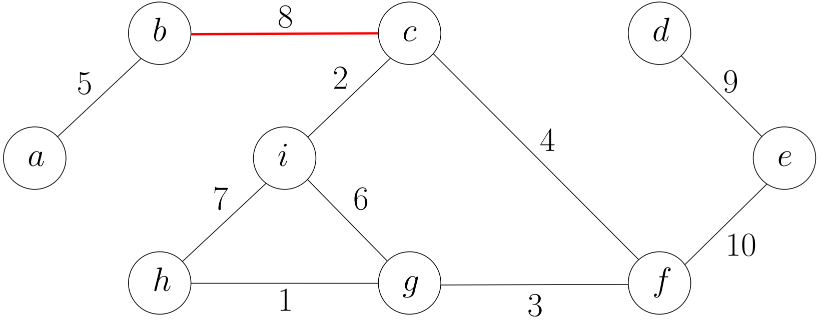

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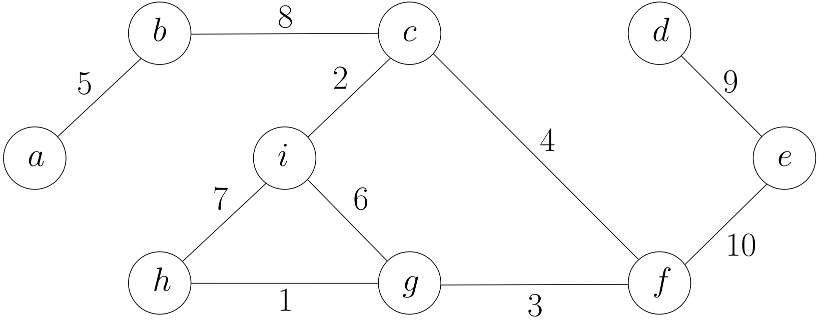

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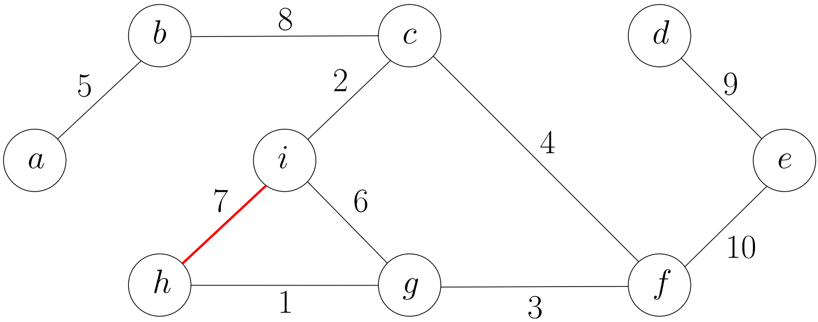

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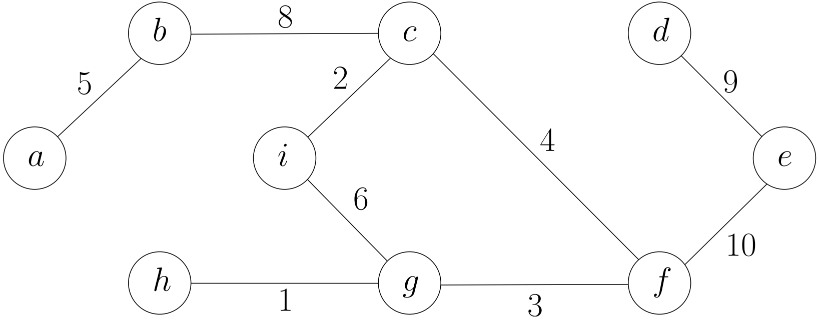

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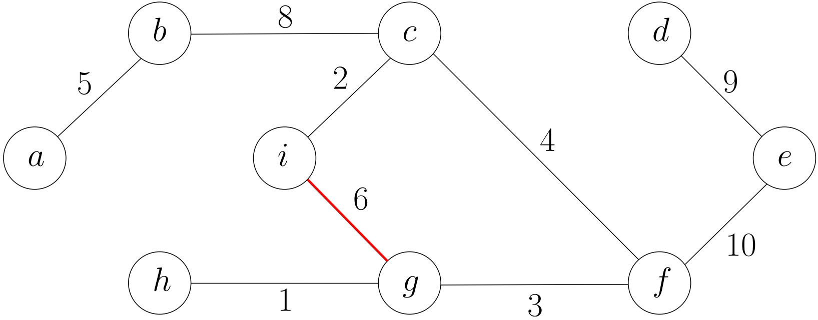

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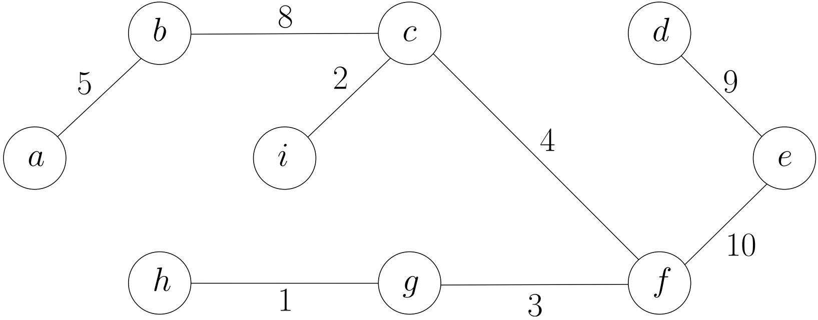

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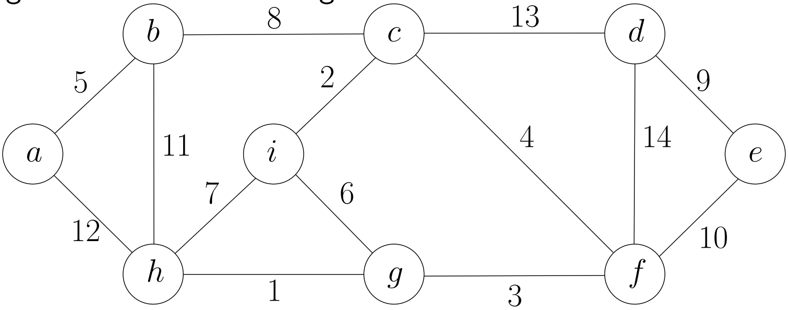
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



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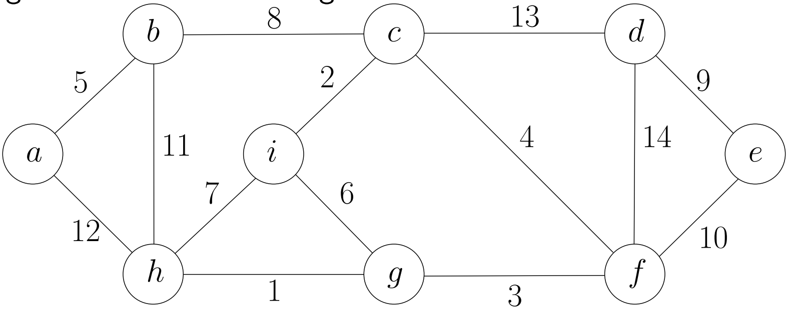

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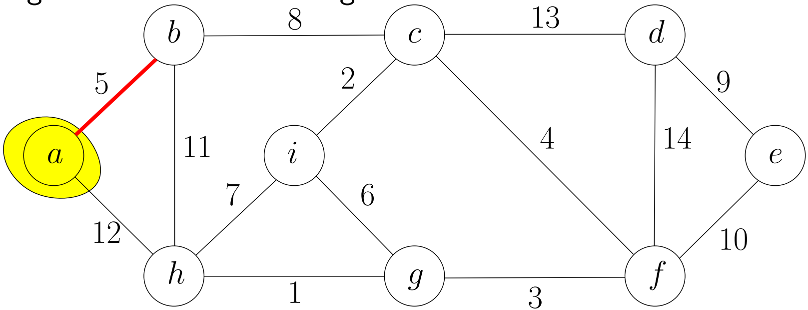

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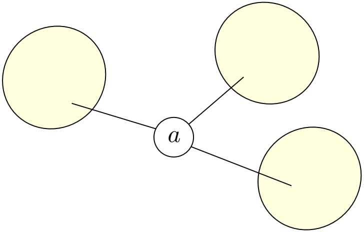

Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

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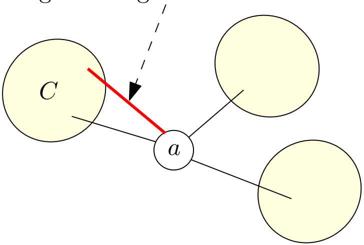

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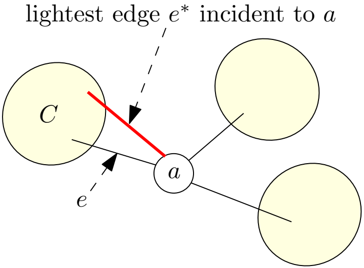

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- Let e be the edge in T connecting a to C

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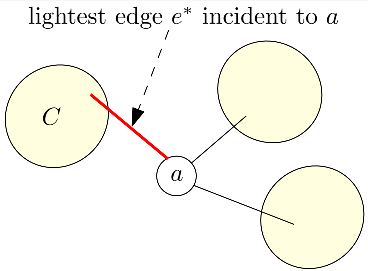

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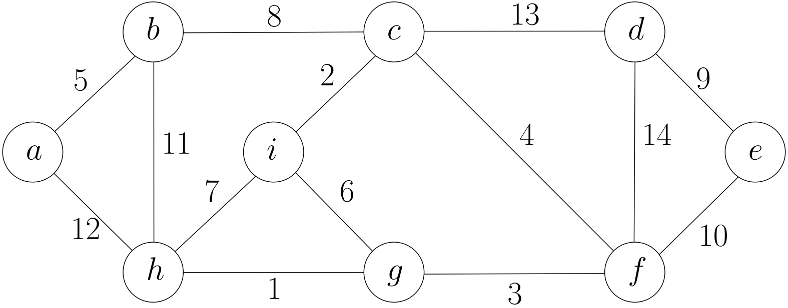
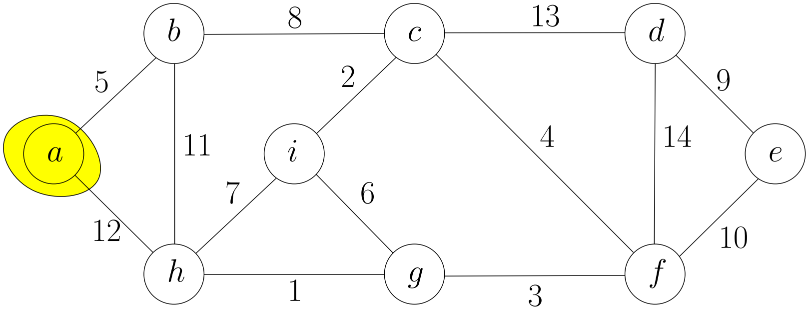
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ □

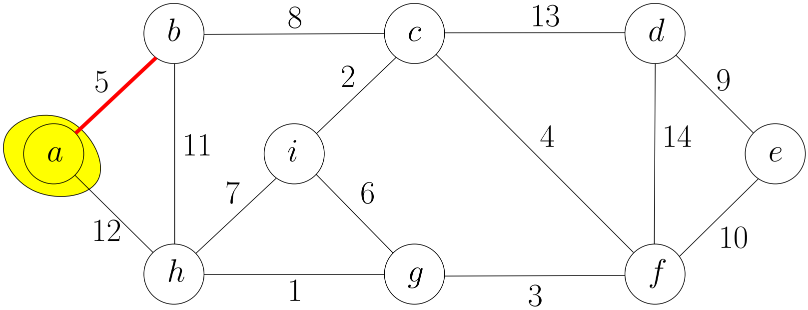
Prim's Algorithm: Example



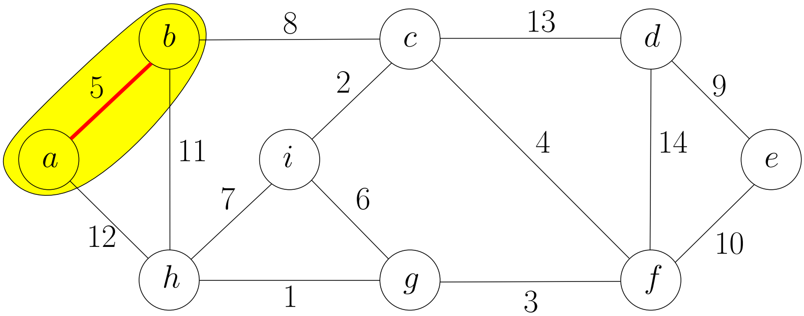
Prim's Algorithm: Example



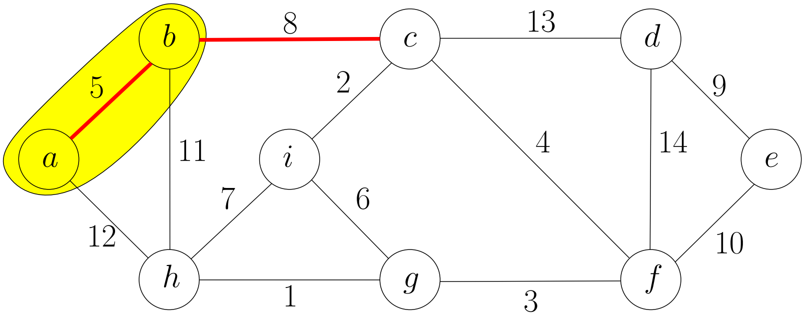
Prim's Algorithm: Example



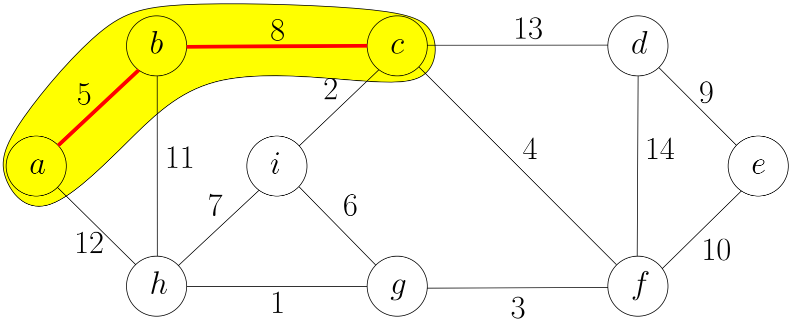
Prim's Algorithm: Example



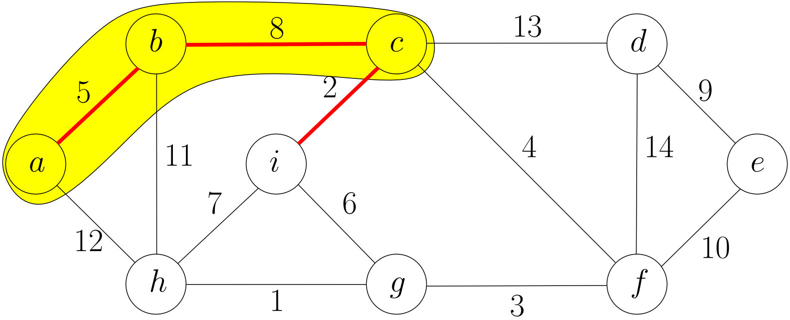
Prim's Algorithm: Example



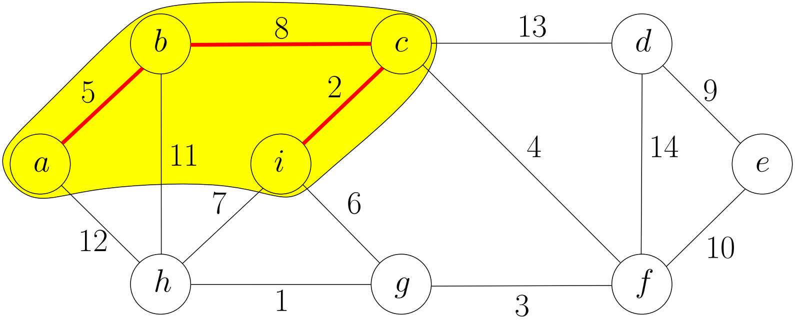
Prim's Algorithm: Example



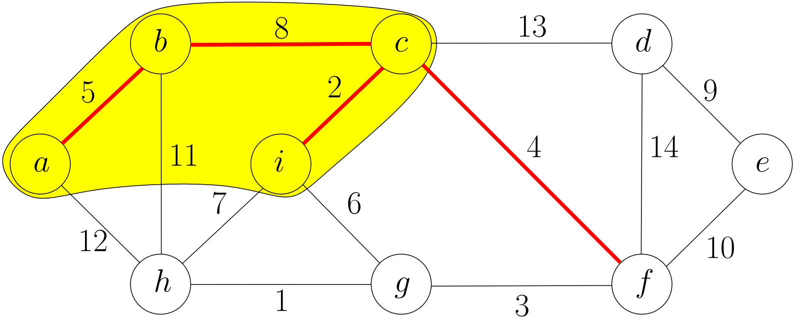
Prim's Algorithm: Example



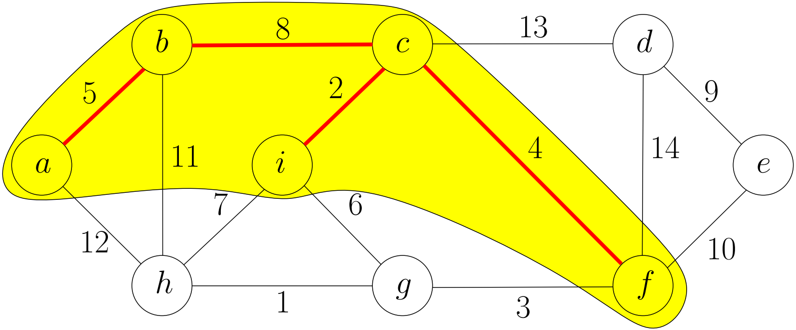
Prim's Algorithm: Example



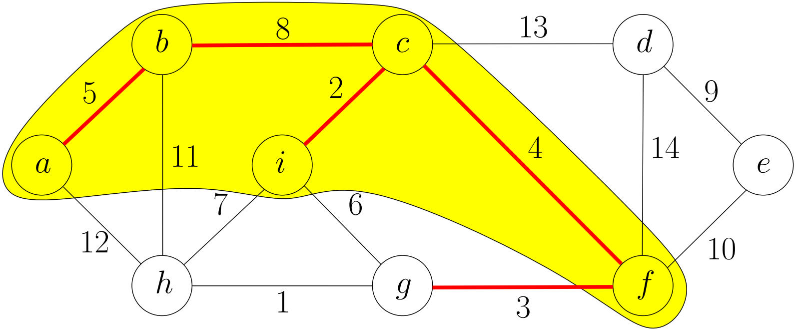
Prim's Algorithm: Example



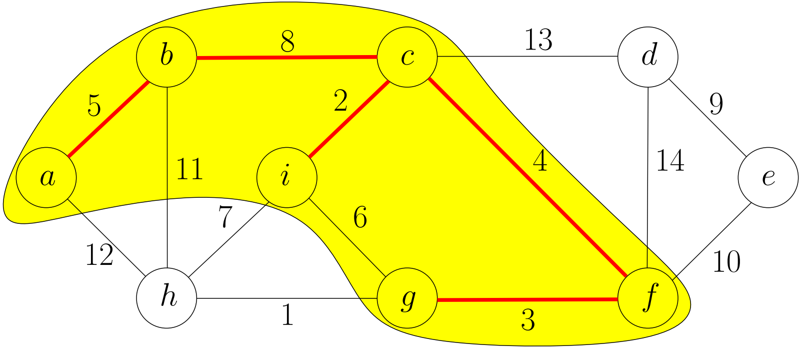
Prim's Algorithm: Example



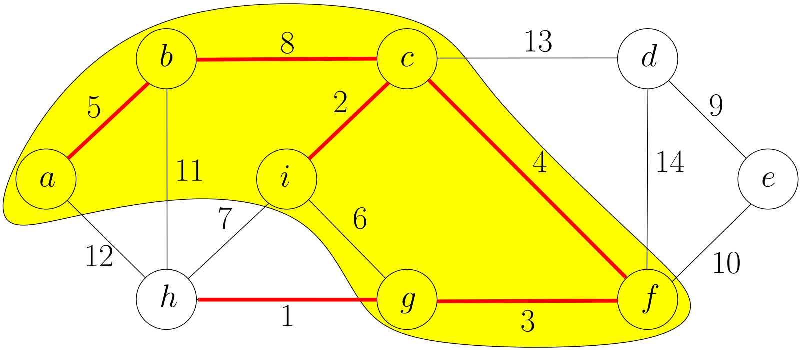
Prim's Algorithm: Example



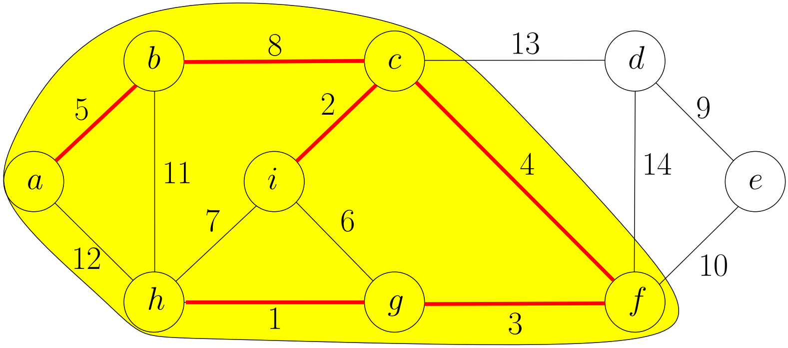
Prim's Algorithm: Example



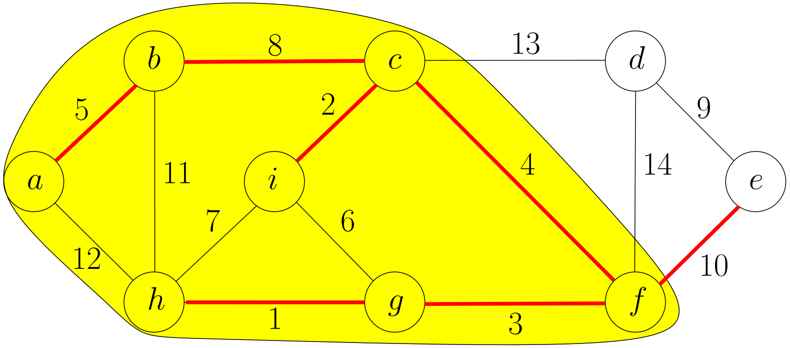
Prim's Algorithm: Example



Prim's Algorithm: Example



Prim's Algorithm: Example



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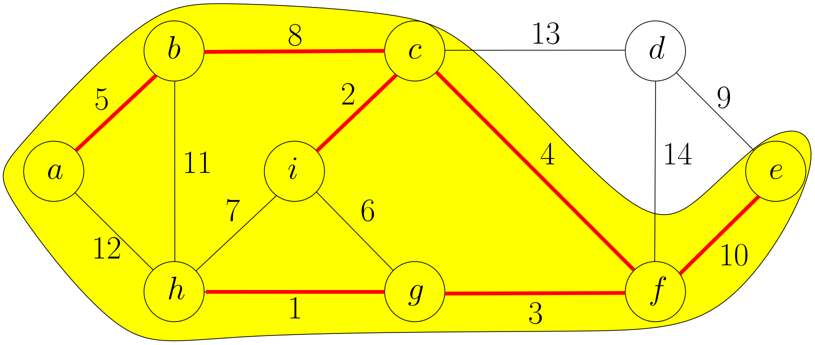
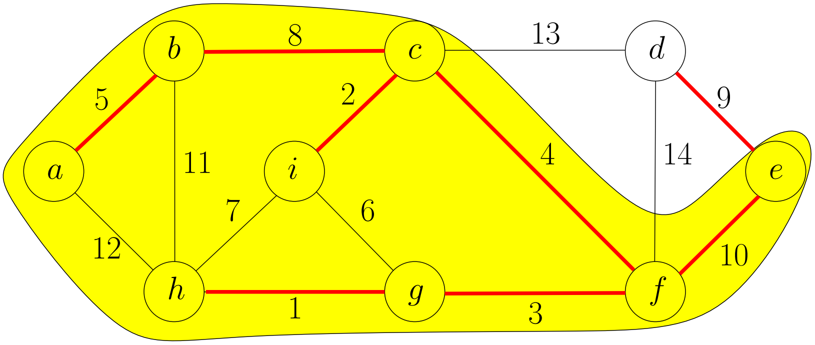


Prim's Algorithm: Example



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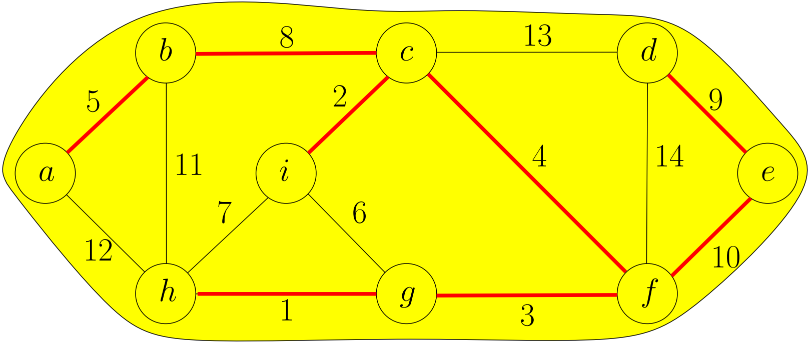

Greedy Algorithm

MST-Greedy1(G, w)

$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
    $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
        where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return (V, F)