CSE 431/531: Algorithm Analysis and Design (Fall 2024)

NP-Completeness

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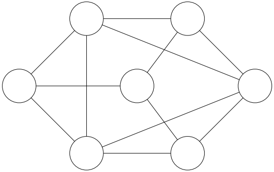
Outline

Maximum Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



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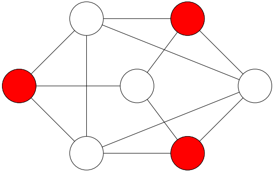

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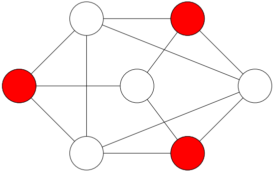

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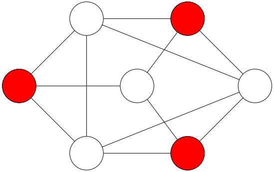

Maximum Independent Set Problem

Input: graph $G=(V,E)$

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- Maximum Independent Set is NP-hard

Formula Satisfiability

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Input: boolean formula with n variables, with $\vee, \wedge, \neg$ operators.

Output: whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
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Fact For each optimization problem X, there is a decision version X' of the problem. If we have a polynomial time algorithm for the decision version X', we can solve the original problem X in polynomial time.

Optimization to Decision

Shortest Path

Input: graph $G = (V, E)$, weight w, s, t and a bound L

Output: whether there is a path from s to t of length at most L

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Input: a graph G and a bound k

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- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)

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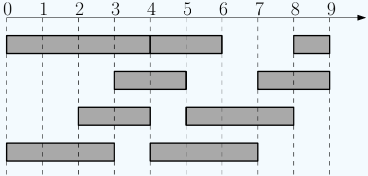
Example: Interval Scheduling Problem



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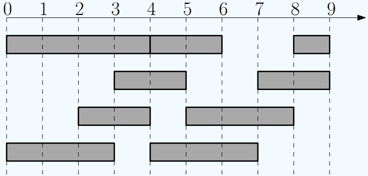

- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$

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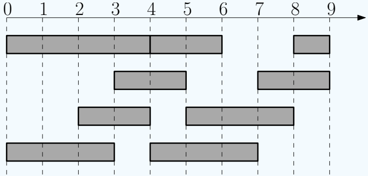

- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$
- Encode the sequence into a binary string as before

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A: No! As long as we are using a "natural" encoding. We only care whether the running time is polynomial or not

Define Problem as a Function $X : \{0,1\}^* \to \{0,1\}$

Def. A decision problem X is a function mapping $\{0,1\}^*$ to $\{0,1\}$ such that for any $s \in \{0,1\}^*$, $X(s)$ is the correct output for input s.

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Def. A has a polynomial running time if there is a polynomial function $p(\cdot)$ so that for every string s, the algorithm A terminates on s in at most $p(|s|)$ steps.

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- The decision versions of interval scheduling, shortest path and minimum spanning tree all in P.

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Def. The message Alice sends to Bob is called a certificate, and the algorithm Bob runs is called a certifier.

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- Certificate: a set of size k
- Certifier: check if the given set is really an independent set

The Complexity Class NP

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t, and outputs 0 or 1.
- there is a polynomial function p such that, $X(s) = 1$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s, t) = 1$.

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Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

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- HC(G) = 1 $\quad\Longleftrightarrow\quad$ $\exists S,\ B(G, S) = 1$

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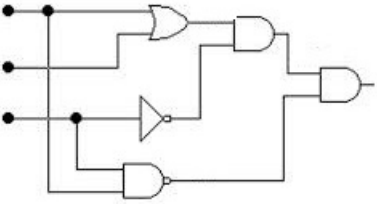
Circuit Satisfiablity (Circuit-Sat) Problem

Input: a circuit with and/or/not gates

Output: whether there is an assignment such that the output is 1?



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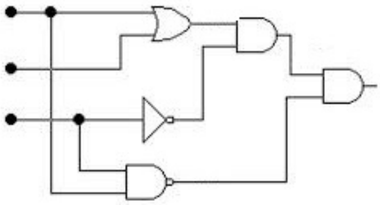

- Is Circuit-Sat $\in$ NP?

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- Unlikely
- Alice can only convince Bob that G is a no-instance
- $\overline{\text{HC}} \in$ Co-NP

The Complexity Class Co-NP

Def. For a problem X, the problem $\overline{X}$ is the problem such that $\overline{X}(s) = 1$ if and only if $X(s) = 0$.

Def. Co-NP is the set of decision problems X such that $\overline{X} \in \text{NP}$.