CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Greedy Algorithms

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University at Buffalo

Announcements: Quiz 5

- Posted on Ublearns
- Should take $<$ 30 minutes, 2 attempts
- Due Mon 23 Sep @ 11:59PM

Outline

A Better Solution for Example

page sequence		cache				cache		
1	✗	1			✗	1		
5	✗	1	5		✗	1	5	
4	✗	1	5	4	✗	1	5	4
2	✗	1	2	4	✗	1	5	2
5	✗	1	2	5	✓	1	5	2
3	✗	1	2	3	✗	1	3	2
2	✓	1	2	3	✓	1	3	2
1	✓	1	2	3	✓	1	3	2
		misses $= 6$				misses $= 5$		

Offline Caching Problem

Input: k : the size of cache

n : number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\mathrm{hit}, \mathrm{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

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- Online Caching: we have to make decisions on the fly, before seeing future requests.

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A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the "competitive ratio" of online algorithms

Offline Caching: Potential Greedy Algorithms

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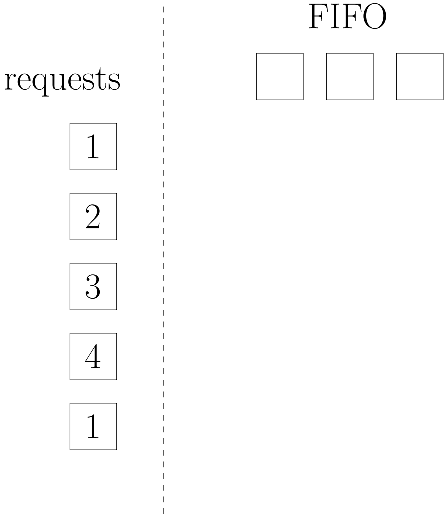
- All the above algorithms are not optimum!
- Indeed all the algorithms are "online", i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.

FIFO is not optimum



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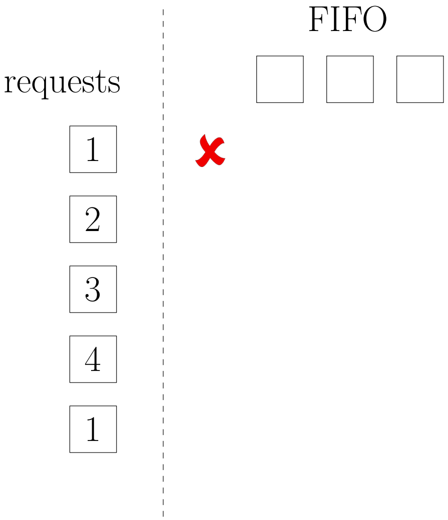

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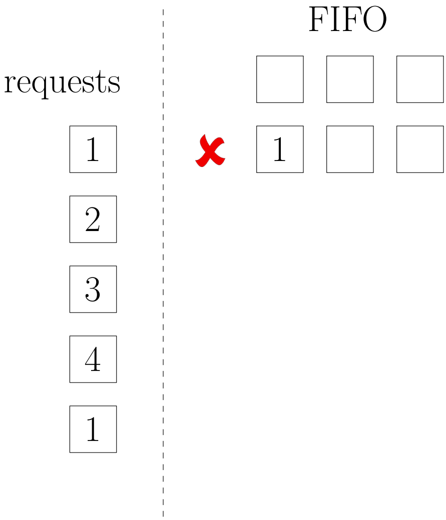

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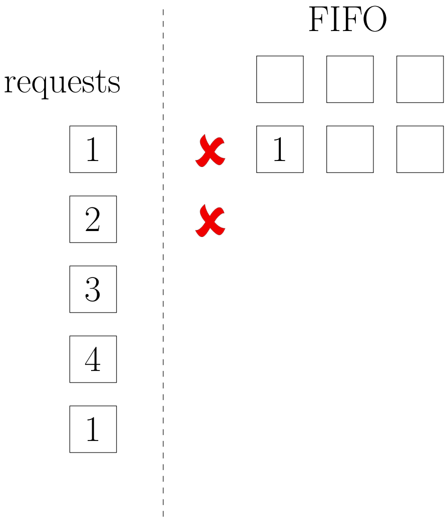

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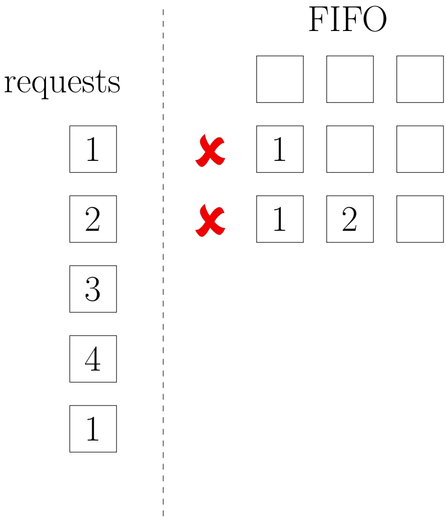

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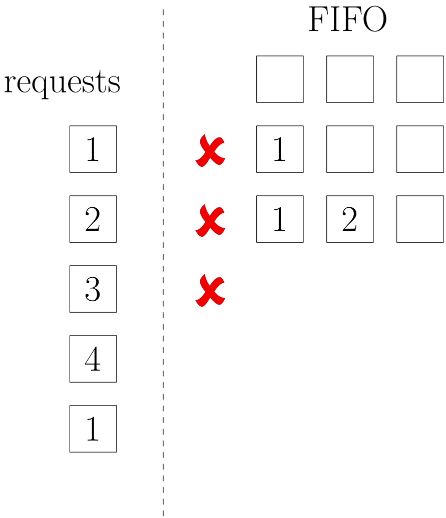

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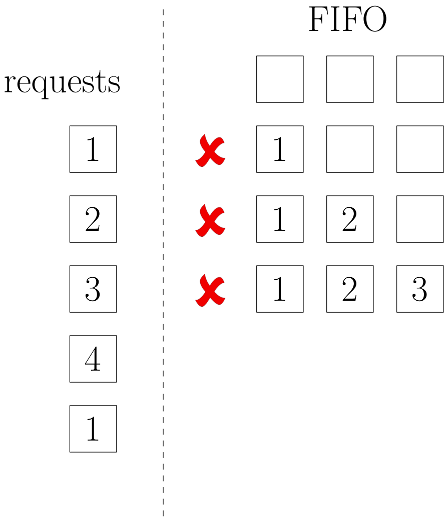

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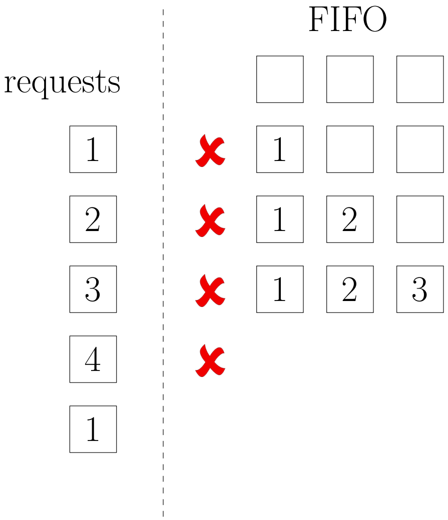

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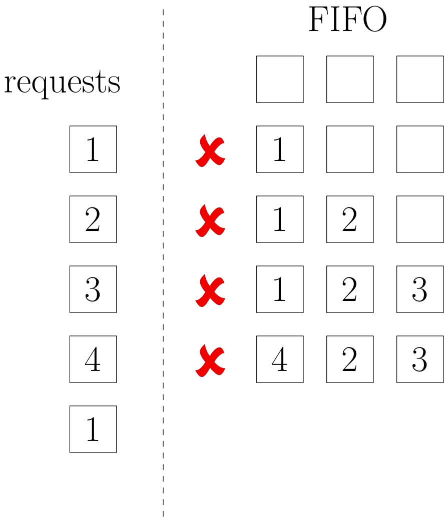

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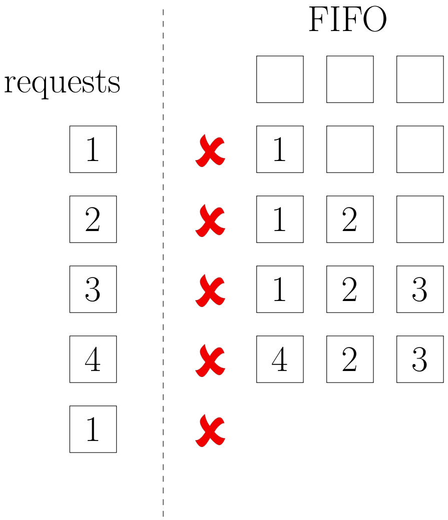

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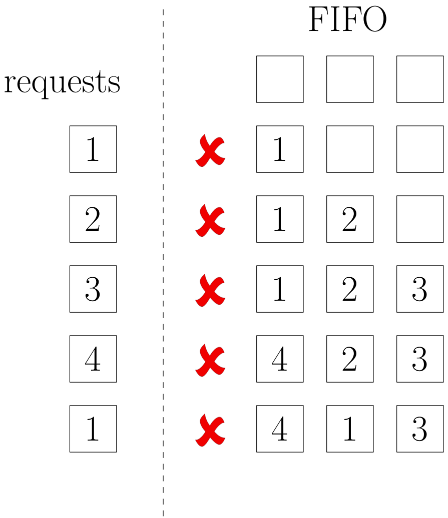


FIFO is not optimum

requests		FIFO		
1	✘	1		
2	✘	1	2	
3	✘	1	2	3
4	✘	4	2	3
1	✘	4	1	3

misses $= 5$

FIFO is not optimum

requests	FIFO				Furthest-in-Future			
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3
	misses $= 5$				misses $= 4$			

Optimum Offline Caching

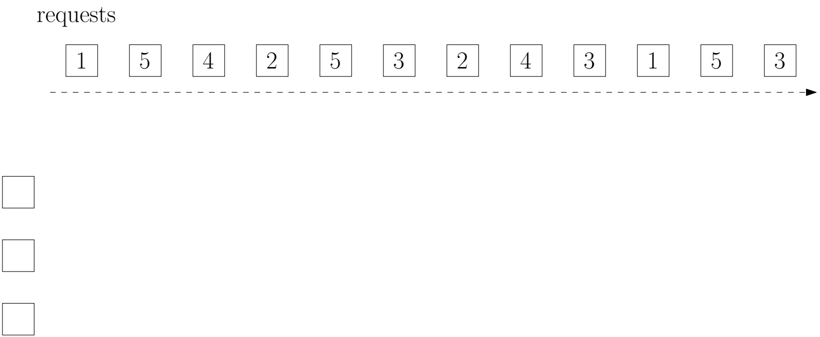
Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.

Furthest-in-Future (FF)

requests	FIFO				Furthest-in-Future			
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3
	misses = 5				misses = 4			

Example



Example

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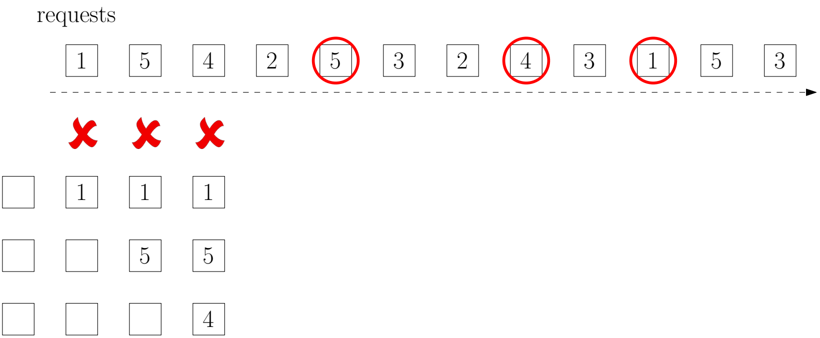

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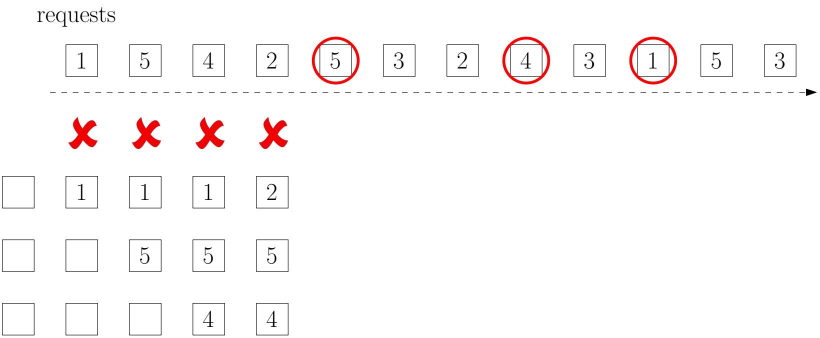

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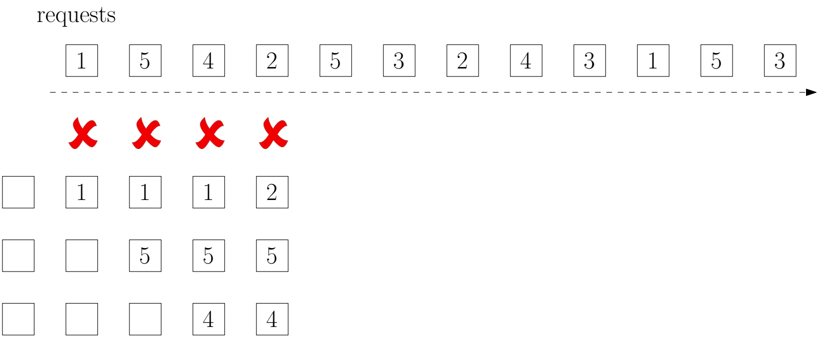

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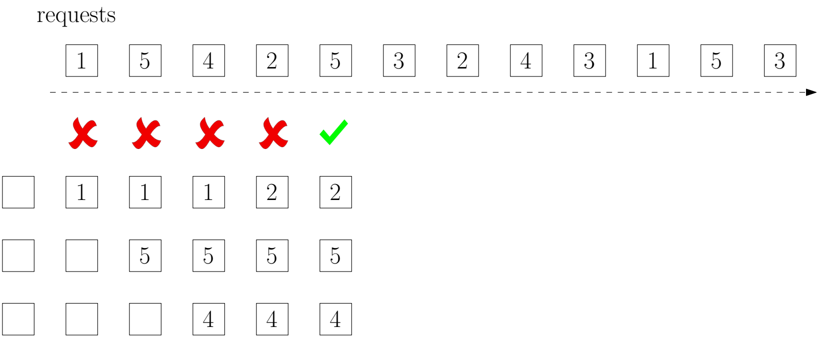

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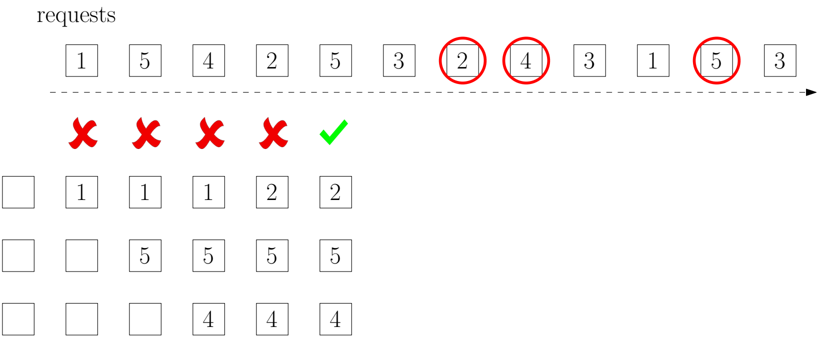

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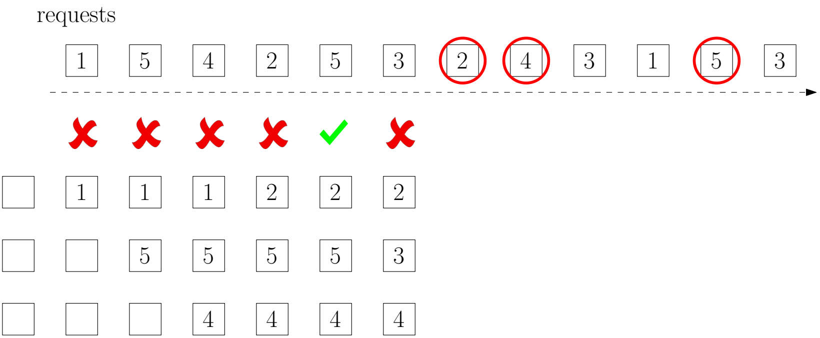

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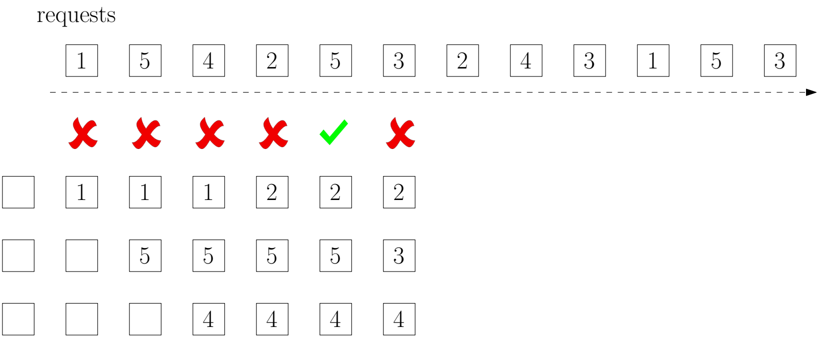

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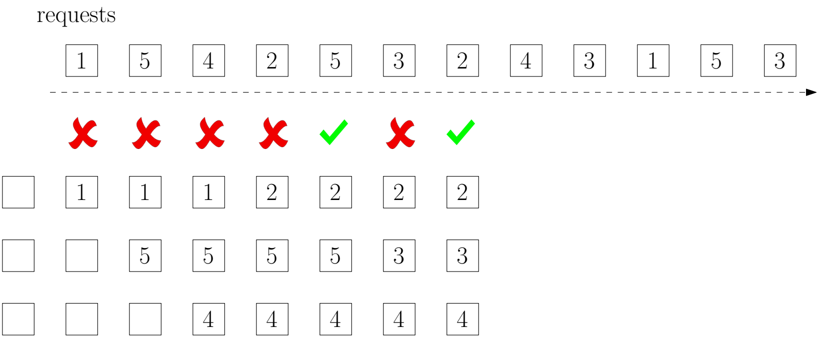

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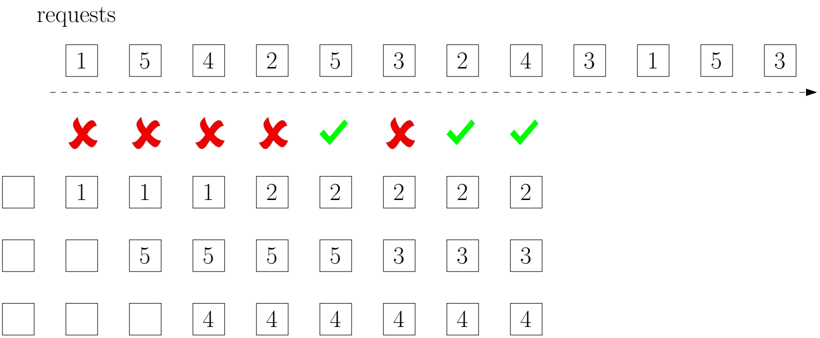

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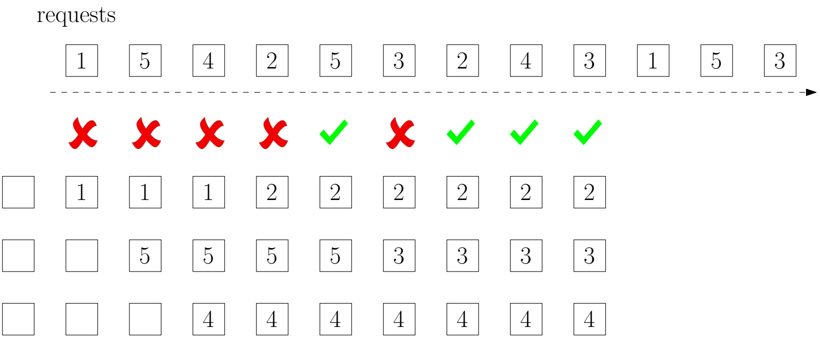

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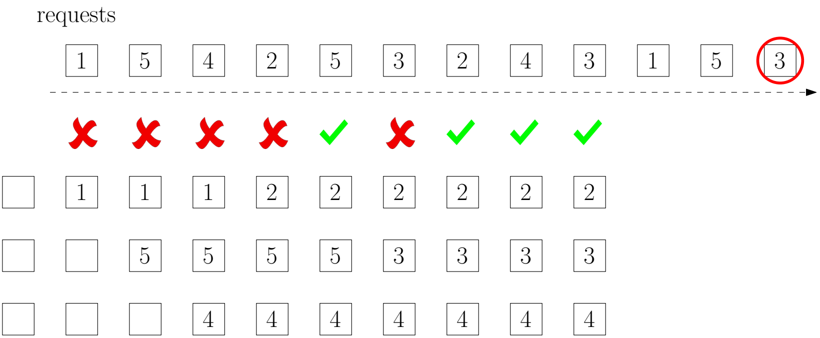

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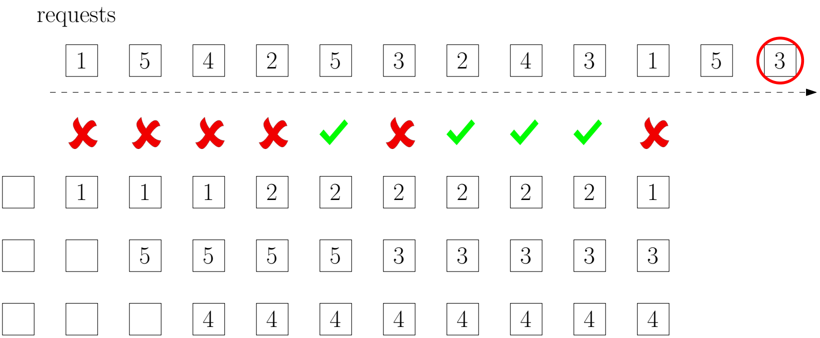

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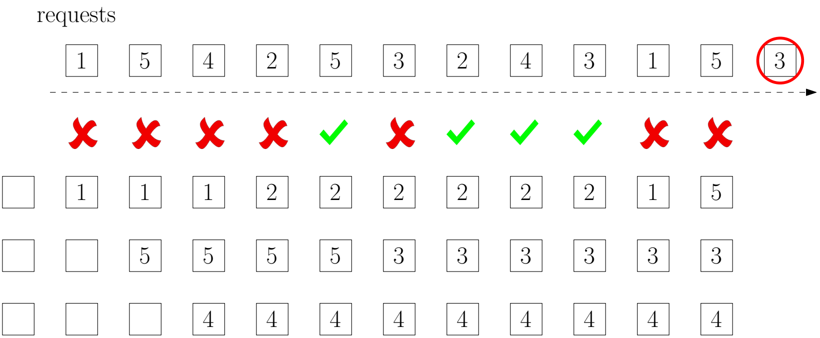
requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2 2 2 1 5
5 5 5 5 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4

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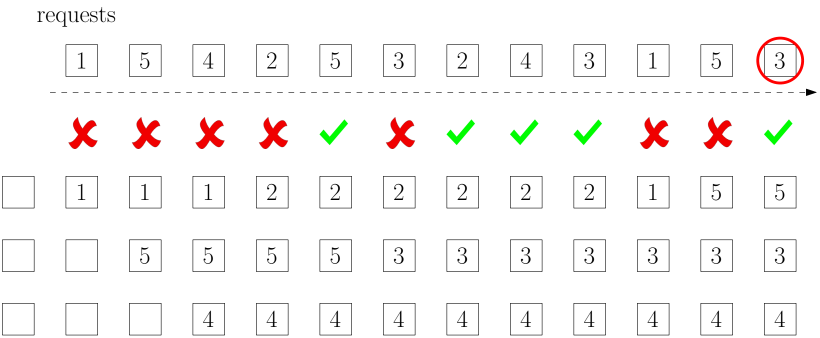
requests
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Recall: Designing and Analyzing Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is "safe" (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

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- empty stands for an empty page
- "hit" means evicting no pages

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$p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

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Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

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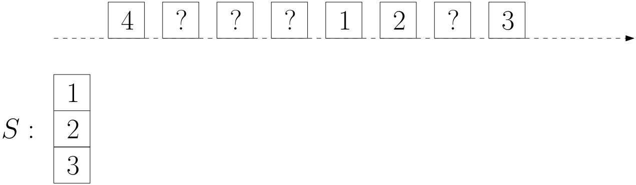
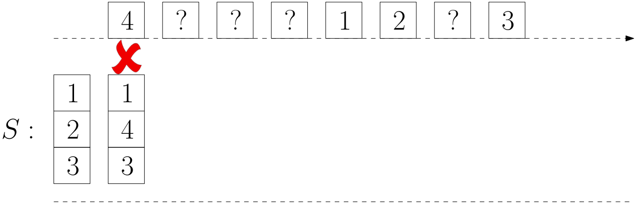
Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.



Proof.

1. S: any optimum solution
2. p^*: page in cache not requested until furthest in the future.
    - In the example, $p^* = 3$.



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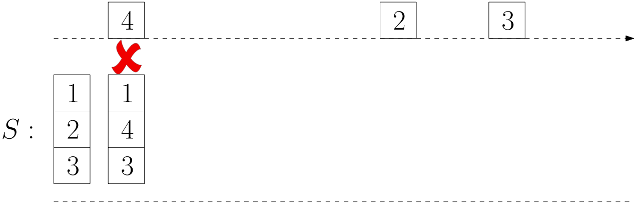

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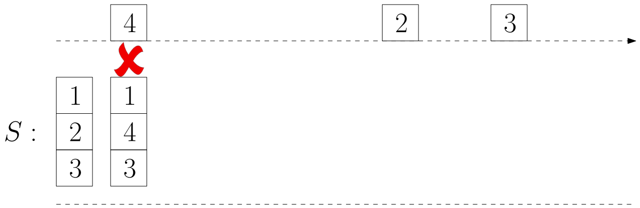

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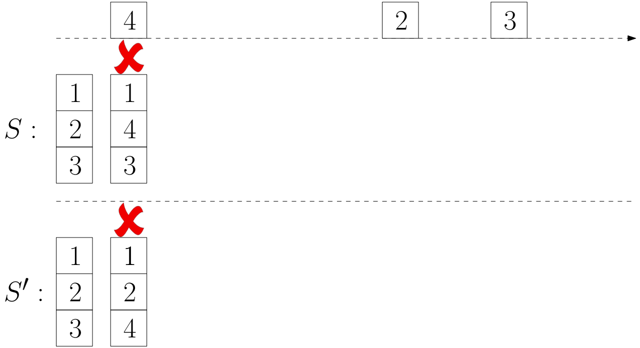

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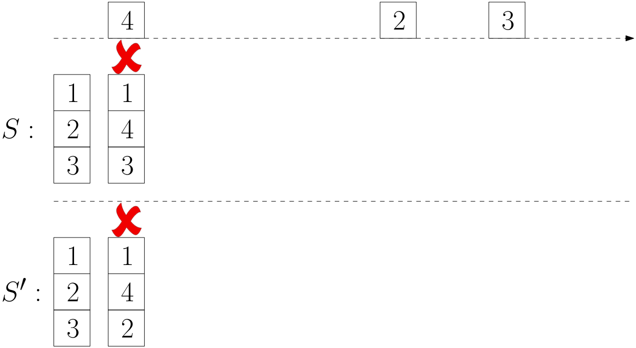
4. Create S'. S' evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.



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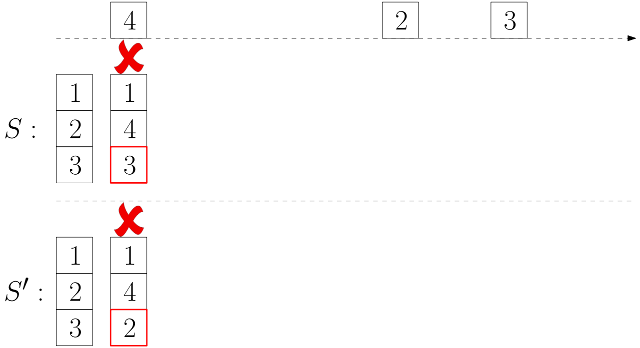

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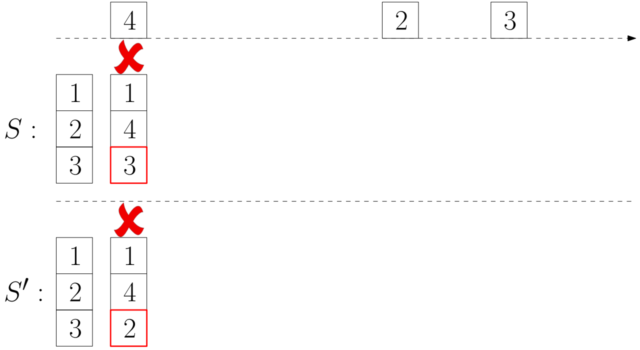

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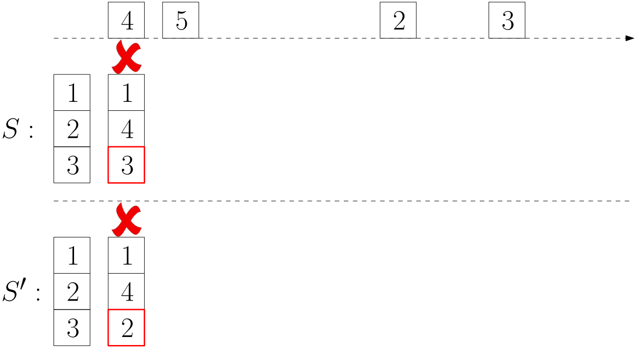
5. After time 1, cache status of S and that of S' differ by only 1 page. S' contains $p'(=2)$ and S contains $p^*(=3)$.
6. From now on, S' will "copy" S.



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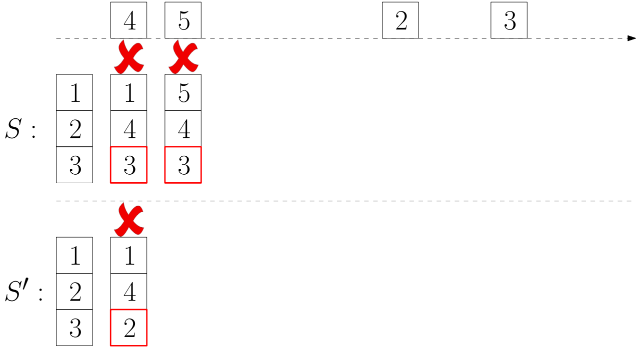

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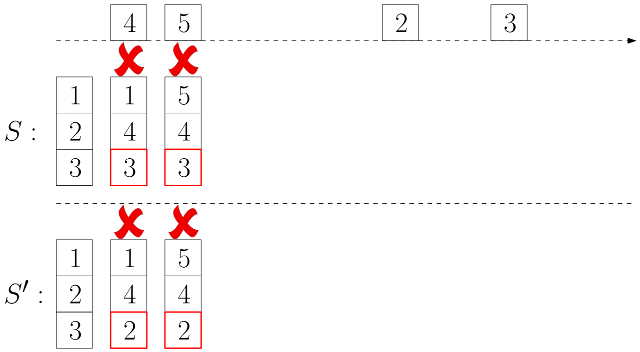

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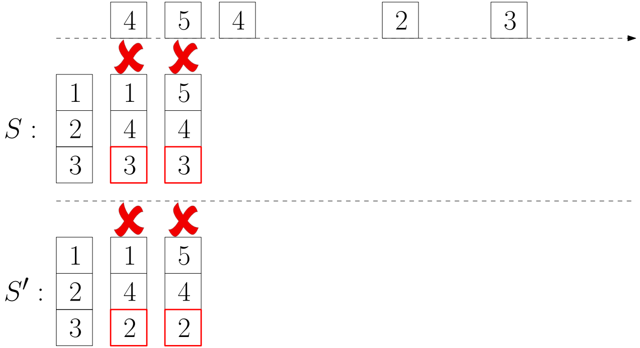

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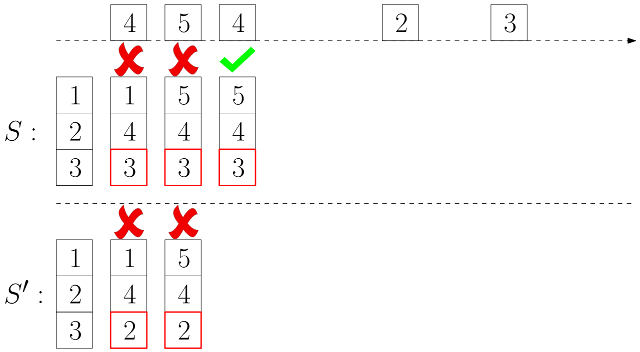

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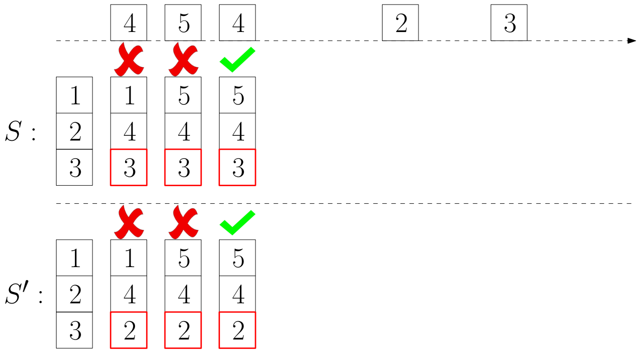

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5. After time 1, cache status of S and that of S' differ by only 1 page. S' contains $p'(=2)$ and S contains $p^*(=3)$.
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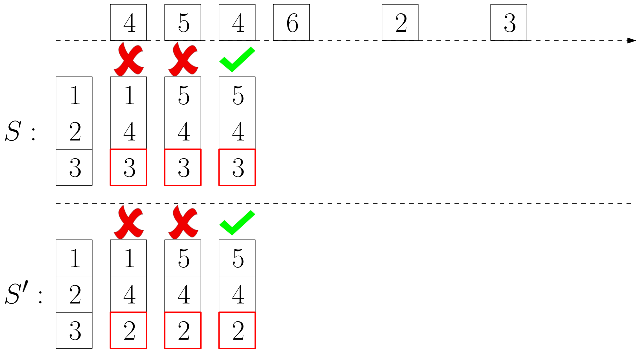

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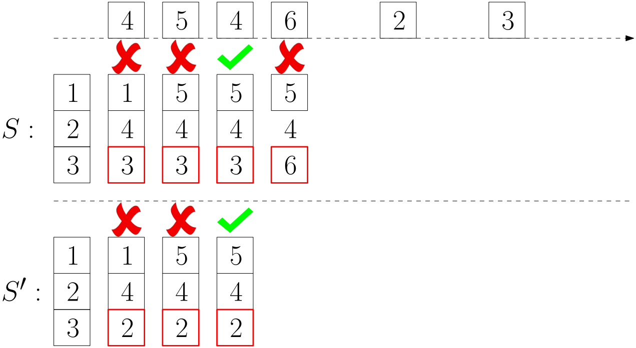

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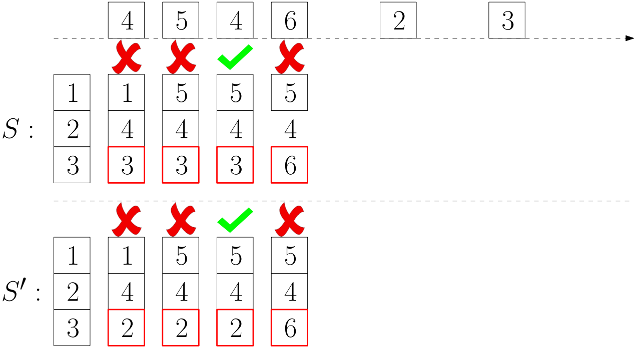

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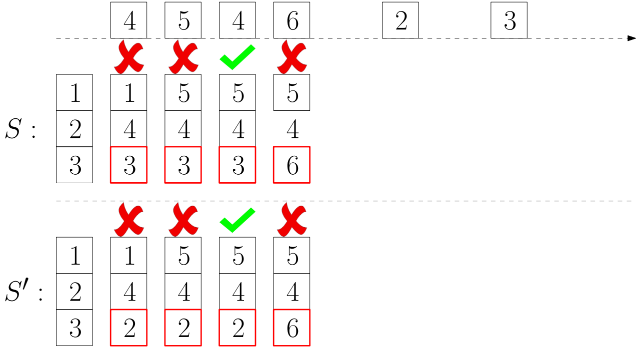
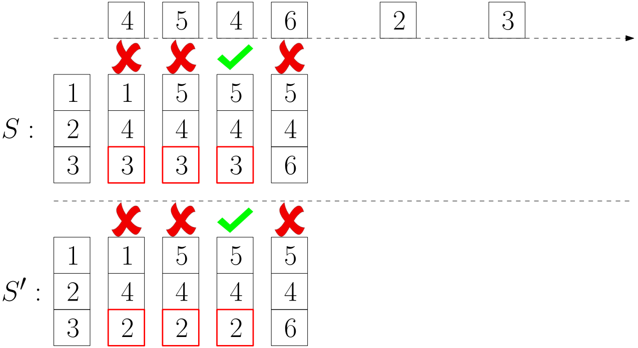


Proof.



Proof.

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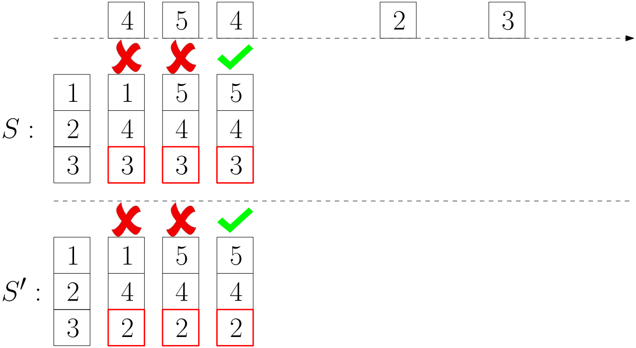

Proof.

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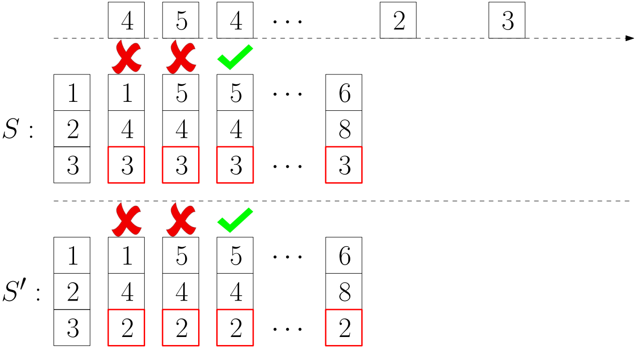
8. Assume S did not evict $p^*(=3)$ before we see $p'(=2)$.



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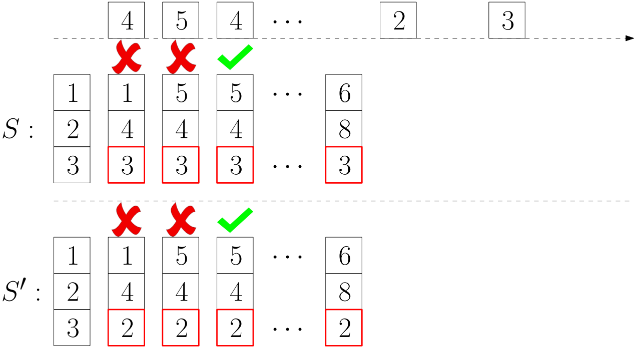

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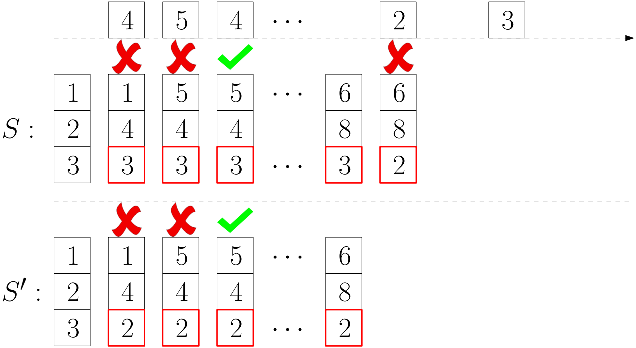

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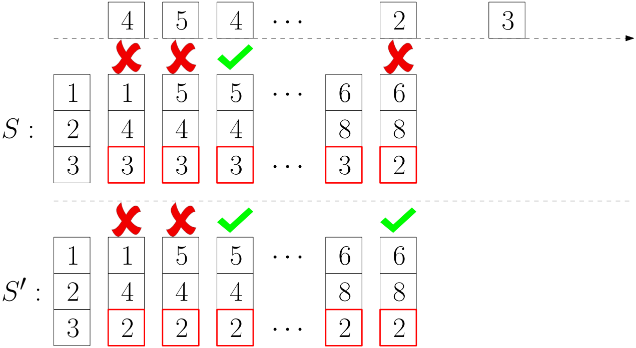
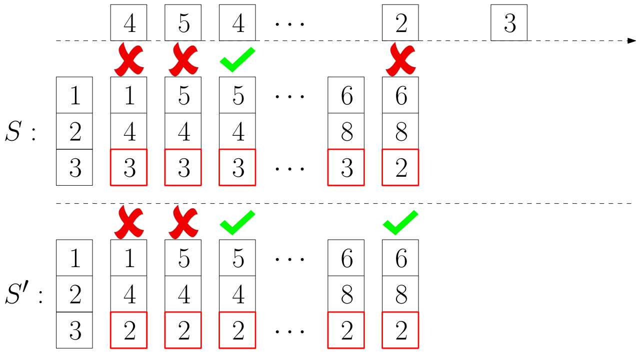


Proof.



Proof.

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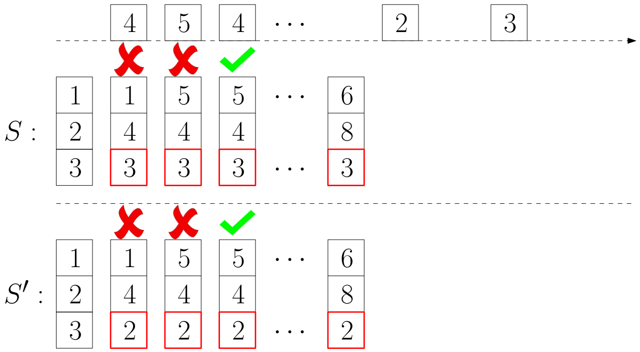

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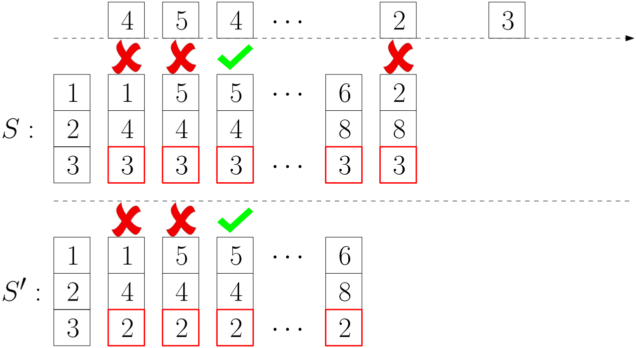

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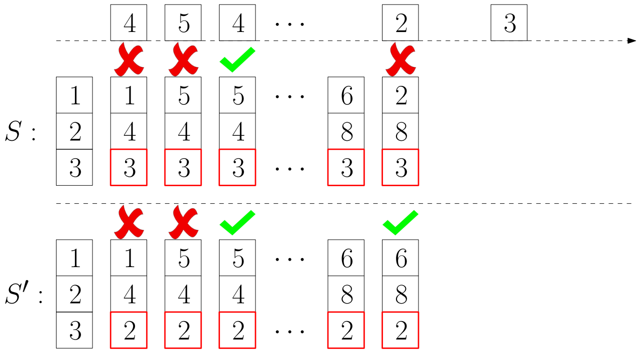

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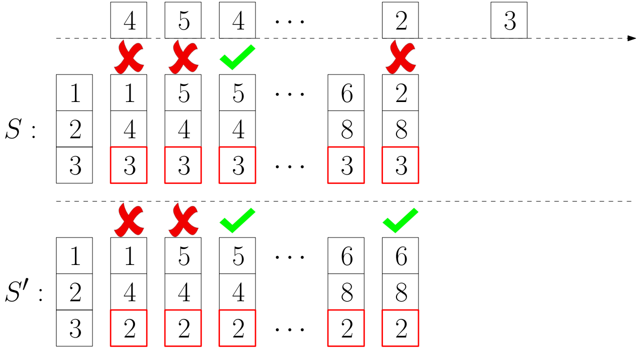
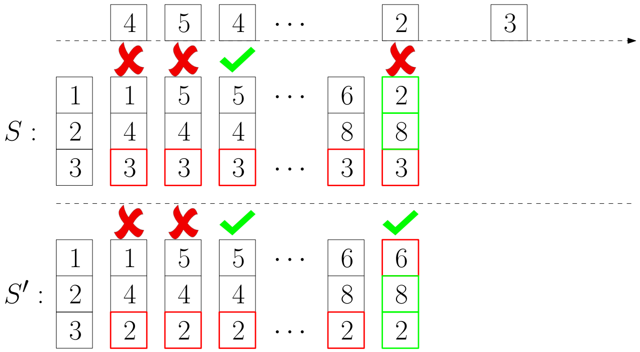


Proof.

9. If S evicts $p^*(=3)$ for $p'(=2)$, then S won't be optimum. Assume otherwise.
10. So far, S' has 1 less page-miss than S does.



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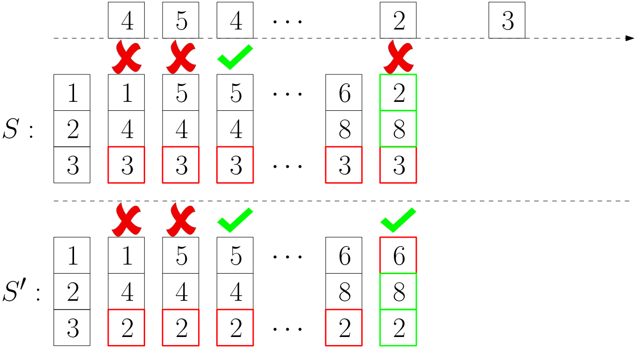
9. If S evicts $p^*(=3)$ for $p'(=2)$, then S won't be optimum. Assume otherwise.
10. So far, S' has 1 less page-miss than S does.
11. The status of S' and that of S only differ by 1 page.



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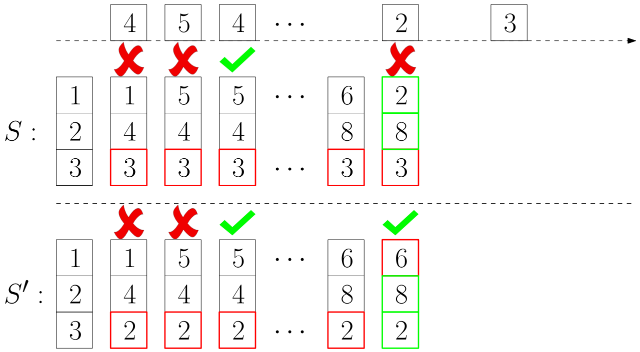

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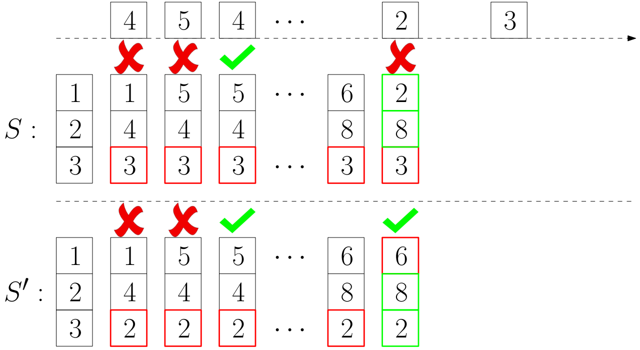
12. We can then guarantee that S' make at most the same number of page-misses as S does.



Proof.

12. We can then guarantee that S' make at most the same number of page-misses as S does.
    - Idea: if S has a page-hit and S' has a page-miss, we use the opportunity to make the status of S' the same as that of S. □

- Thus, we have shown how to create another solution S' with the same number of page-misses as that of the optimum solution S. Thus, we proved

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.

- Thus, we have shown how to create another solution S' with the same number of page-misses as that of the optimum solution S. Thus, we proved

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

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Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

Theorem The furthest-in-future strategy is optimum.

for $t \leftarrow 1$ to T **do**
  if ρ_t is in cache **then** do nothing
  else if there is an empty page in cache **then**
    evict the empty page and load ρ_t in cache
  else
    $p^* \leftarrow$ page in cache that is not used furthest in the future
    evict p^* and load ρ_t in cache

Q: How can we make the algorithm as fast as possible?

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- For each page p, use a linked list (or an array with dynamic size) to store the time steps in which p is requested.
  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
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priority queue

pages	priority values

time	0	1	2	3	4	5	6	7	8	9	10	11	12
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P1:	1	10	
P2:	4	7	
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priority queue

pages	priority values

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
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priority queue

pages	priority values
P1	10

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
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priority queue

pages	priority values
P1	10

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	$\boxed{10}$	
P2:	$\boxed{4}$	7	
P3:	$\boxed{6}$	9	12
P4:	$\boxed{3}$	8	
P5:	2	$\boxed{5}$	11

priority queue

pages	priority values
P1	10
P5	5

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1: 1 $\boxed{10}$

P2: $\boxed{4}$ 7

P3: $\boxed{6}$ 9 12

P4: $\boxed{3}$ 8

P5: 2 $\boxed{5}$ 11

priority queue

pages	priority values
P1	10
P5	5

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	$\boxed{10}$	
P2:	$\boxed{4}$	7	
P3:	$\boxed{6}$	9	12
P4:	3	$\boxed{8}$	
P5:	2	$\boxed{5}$	11

priority queue

pages	priority values
P1	10
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

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pages	priority values
P1	10
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P1:	1	10	
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P4:	3	8	
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priority queue

pages	priority values
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1: 1 $\boxed{10}$

P2: 4 $\boxed{7}$

P3: $\boxed{6}$ 9 12

P4: 3 $\boxed{8}$

P5: 2 $\boxed{5}$ 11

priority queue

pages	priority values
P2	7
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

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priority queue

pages	priority values
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P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓							

P1:	1	$\boxed{10}$	
P2:	4	$\boxed{7}$	
P3:	$\boxed{6}$	9	12
P4:	3	$\boxed{8}$	
P5:	2	5	$\boxed{11}$

priority queue

pages	priority values
P2	7
P5	11
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓							

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P2:	4	$\boxed{7}$	
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P4:	3	$\boxed{8}$	
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priority queue

pages	priority values
P2	7
P5	11
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

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time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1: 1 $\boxed{10}$

P2: 4 $\boxed{7}$

P3: 6 $\boxed{9}$ 12

P4: 3 $\boxed{8}$

P5: 2 5 $\boxed{11}$

priority queue

pages	priority values
P2	7
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗						

P1:	1	$\boxed{10}$	
P2:	4	$\boxed{7}$	
P3:	6	$\boxed{9}$	12
P4:	3	$\boxed{8}$	
P5:	2	5	$\boxed{11}$

priority queue

pages	priority values
P2	7
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓					

P1: 1 | $\boxed{10}$

P2: 4 | 7 | $\boxed{}$

P3: 6 | $\boxed{9}$ | 12

P4: 3 | $\boxed{8}$

P5: 2 | 5 | $\boxed{11}$

priority queue

pages	priority values
P2	∞
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓					

P1:	1	$\boxed{10}$	
P2:	4	7	$\boxed{}$
P3:	6	$\boxed{9}$	12
P4:	3	$\boxed{8}$	
P5:	2	5	$\boxed{11}$

priority queue

pages	priority values
P2	∞
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓				

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	∞
P3	9
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓				

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	∞
P3	9
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓			

P1:	1	$\boxed{10}$	
P2:	4	7	$\boxed{}$
P3:	6	9	$\boxed{12}$
P4:	3	8	$\boxed{}$
P5:	2	5	$\boxed{11}$

priority queue

pages	priority values
P2	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓			

P1:	1	$\boxed{10}$	
P2:	4	7	$\boxed{}$
P3:	6	9	$\boxed{12}$
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priority queue

pages	priority values
P2	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓			

P1:	1	$\boxed{10}$	
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P3:	6	9	$\boxed{12}$
P4:	3	8	$\boxed{}$
P5:	2	5	$\boxed{11}$

priority queue

pages	priority values
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗		

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P1	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗		

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P1	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗		

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	✓

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	∞
P4	∞

for every $p \leftarrow 1$ to n **do**
  $times[p] \leftarrow$ array of times in which p is requested, in increasing order ▷ put ∞ at the end of array
  $pointer[p] \leftarrow 1$
$Q \leftarrow$ empty priority queue
for every $t \leftarrow 1$ to T **do**
  $pointer[\rho_t] \leftarrow pointer[\rho_t] + 1$
  if $\rho_t \in Q$ **then**
    Q.increase-key$(\rho_t, times[\rho_t, pointer[\rho_t]])$, **print** "hit", **continue**
  if $Q.size() < k$ **then**
    print "load ρ_t to an empty page"
  else
    $p \leftarrow Q$.extract-max(), **print** "evict p and load ρ_t"
  Q.insert$(\rho_t, times[\rho_t, pointer[\rho_t]])$ ▷ add ρ_t to Q with key value $times[\rho_t, pointer[\rho_t]]$