CSE 431/531: Algorithm Analysis and Design (Fall 2024)
Graph Basics

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Outline

Outline

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

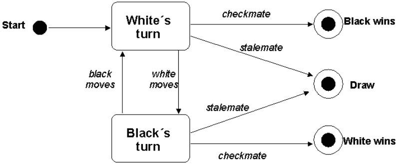
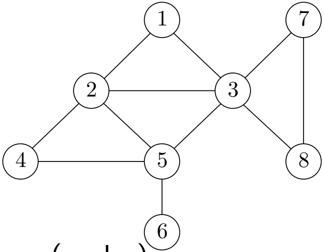


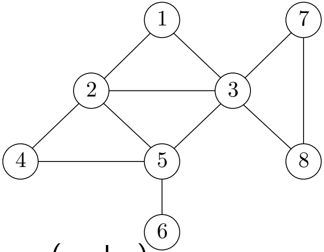
Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



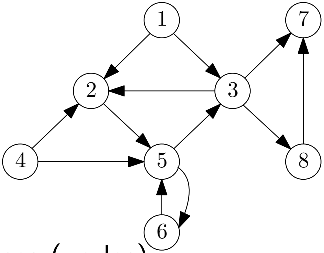
- V: set of vertices (nodes);
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

Directed Graph $G = (V, E)$



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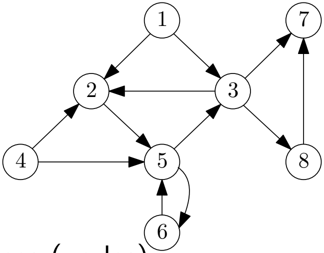

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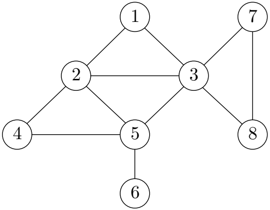
  - $E = \{(1,2), (1,3), (3,2), (4,2), (2,5), (5,3), (3,7), (3,8), (4,5), (5,6), (6,5), (8,7)\}$

Abuse of Notations

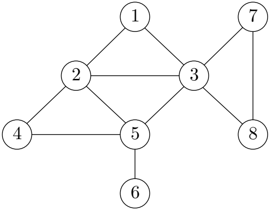
- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



- $E = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,5), (3,7), (3,8), (4,5), (5,6), (7,8)\}$

- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

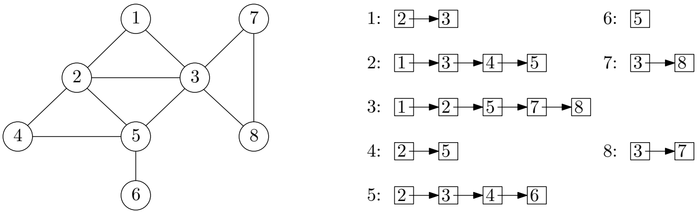
Representation of Graphs



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

Representation of Graphs



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- Linked lists
  - For every vertex v, there is a linked list containing all neighbors of v.

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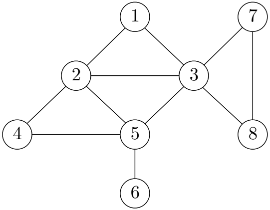

1: 2 3

2: 1 3 4 5

3: 1 2 5 7 8

4: 2 5

5: 2 3 4 6

6: 5

7: 3 8

8: 3 7

$d : (2, 4, 5, 2, 4, 1, 2, 2)$

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbors of v.
  - When graph is static, can use array of variant-length arrays.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbors of v		

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memory usage	$O(n^2)$	$O(m)$
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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	
time to list all neighbors of v		

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbors of v		

Comparison of Two Representations

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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u, v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbors of v	$O(n)$	

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

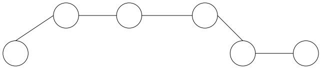
	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u, v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbors of v	$O(n)$	$O(d_v)$

Outline

1. Graph Basics
   - Graph Notations
   - Types of Graphs
2. Connectivity and Graph Traversal

Path Graph (or Linear Graph)

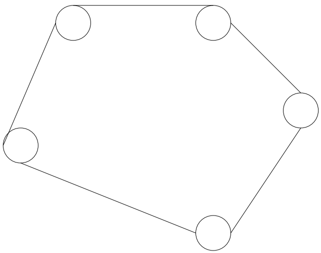
Def. An undirected graph $G = (V, E)$ is a path if the vertices can be listed in an order $\{v_1, v_2, ..., v_n\}$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, ..., n - 1$.



- Path graphs are connected graphs.

Cycle Graph (or Circular Graph)

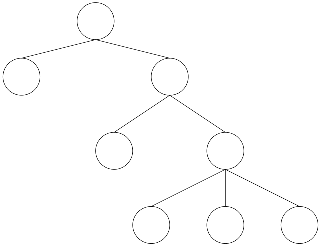
Def. An undirected graph $G = (V, E)$ is a cycle if its vertices can be listed in an order $v_1, v_2, ..., v_n$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, ..., n-1$, plus the edge $\{v_n, v_1\}$.



- The degree of all vertices is 2.

Tree

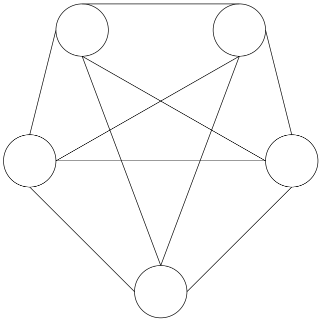
Def. An undirected graph $G = (V, E)$ is a tree if any two vertices are connected by exactly one path. Or the graph is a connected acyclic graph.



- Most important type of special graphs: most computational problems are easier to solve on trees or lines.

Complete Graph

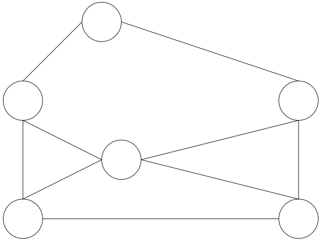
Def. An undirected graph $G = (V, E)$ is a complete graph if each pair of vertices is joined by an edge.



- A complete graph contains all possible edges.

Planar Graph

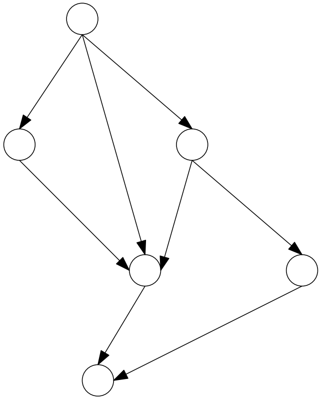
Def. An undirected graph $G = (V, E)$ is a planar graph if its vertices and edges can be drawn in a plane such that no two of the edges intersect.



- Most computational problems have good solutions in a planar graph.

Directed Acyclic Graph (DAG)

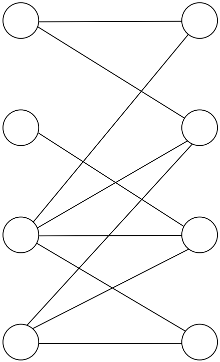
Def. A directed graph $G = (V, E)$ is a directed acyclic graph if it is a directed graph with no directed cycles



- DAG is equivalent to a partial ordering of nodes.

Bipartite Graph

Def. An undirected graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



Outline

Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

Output: whether there is a path connecting s to t in G