Understanding Thin Accretion Disks: A Fluid Dynamical Perspective on Astrophysical Flows

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Abstract

Accretion disks are fundamental structures in astrophysics, governing the dynamics of mass transfer in systems ranging from young stellar objects to supermassive black holes. This project presents a focused study of thin, quasi-Keplerian accretion disks using the tools of classical fluid dynamics. We begin by outlining the key assumptions that allow a reduction of the full Navier–Stokes equations to a vertically integrated, axisymmetric form. Building on this framework, we examine the radial dependence of velocity, pressure, and temperature profiles, and derive scaling laws such as

$$T \propto R^{3/4}. \tag{1}$$

under steady-state conditions. Stress tensor components are analyzed in cylindrical coordinates to highlight the role of viscous shear in angular momentum transport.

1. Physical Formation of Accretion Disks

Accretion disks are commonly formed in binary systems consisting of a compact object and a companion star. Matter is transferred from the star via Roche lobe overflow, flowing through the L1 Lagrangian point towards the compact object. This transferred material already possesses intrinsic angular momentum due to the binary's orbital motion. In the simplest case, we assume that the incoming material's angular momentum axis aligns with that of the compact object's rotation; otherwise, torques would arise to reorient the system. [10]

As the gas flows inward under gravitational attraction, it tends to settle onto a plane defined by the net angular momentum vector. Any vertical (out-of-plane) motion is quickly damped on a free-fall timescale through shock interactions when gas streamers collide. Within this plane, a fluid minimizes its total energy for a given specific angular momentum by forming a circular orbit. Consequently, the material gradually arranges itself into quasi-circular orbits.

Vertically, the system achieves hydrostatic equilibrium: the vertical component of gravity is balanced by the pressure gradient within the disk. Radially, centripetal forces, arising from both gravitational pull and pressure gradients, govern the fluid motion. As matter spirals inward, it loses gravitational potential energy, converting it to radiation and, in the case of extremely massive objects, even releasing a significant fraction of the rest mass energy mc^2.

Thus, the infalling matter forms an accretion disk, gradually losing gravitational potential energy and spiraling inward with a small but finite radial velocity component. A fraction of the lost energy is converted into thermal energy through viscous dissipation. This thermal energy is eventually radiated away, powering the luminosity of accretion disks. [3]

Figure 1: Artist's Illustration of Accretion Disk Formation [NASA/CXC/M.Weiss]

2. Dynamics and Basic Assumptions

Within an accretion disk, fluid elements are in nearly Keplerian orbits, meaning their radial velocity (v_r) is much smaller than their azimuthal velocity (v_ϕ), but nonzero. A purely Keplerian orbit would not allow material to accrete inward; hence, a small but finite radial drift exists.

For gas to accrete, it must lose angular momentum. Several mechanisms have been proposed, including outflows, winds, and internal viscous stresses. In this work, we focus on accretion driven by internal viscous stresses.

We model the disk as geometrically thin ($H \ll r$) and use cylindrical polar coordinates. The key assumptions are: [11]

- Axisymmetry, $\partial/\partial\phi = 0$
- Hydrostatic equilibrium in the vertical direction, $v_z = 0$
- Radial pressure gradients negligible compared to gravitational forces,
- Radial velocity v_r much smaller than azimuthal velocity v_ϕ,
- The fluid behaves as an incompressible, Newtonian fluid with negligible bulk viscosity.

3. Mathematical Setup (General, Non-Steady State)

3.1 Continuity Equation (Mass Conservation)

Starting from the general continuity equation in cylindrical polar coordinates (r, ϕ, z): [8]

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho v_\phi) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Under the assumptions of:

- Axisymmetry ($\partial/\partial\phi = 0$),
- Vertical Hydrostatic Equilibrium ($v_z = 0$),

the continuity equation simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0$$

Defining the surface density Σ as:

$$\Sigma(r,t) = \int_{-\infty}^{\infty} \rho(r,z,t)\, dz$$

we assume that there is no vertical variation and integrate over z to obtain the vertically integrated continuity equation:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0$$

3.2 Navier-Stokes Equation

Incompressible Navier–Stokes equations with uniform viscosity (convective form):

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f} \tag{2}$$

[6]

3.2.1 Vertical (z) Component (Hydrostatic Equilibrium)

The vertical component of the Navier–Stokes equation, under the assumption of a thin disk and negligible vertical motions ($v_z = 0$), simplifies to a balance between the pressure gradient and the vertical component of gravity:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GMz}{r^3}$$

Rearranging:

$$\frac{\partial p}{\partial z} = -\rho\frac{GM}{r^3}z$$

This is the hydrostatic equilibrium equation in the vertical direction, indicating that the pressure gradient supports the disk material against the vertical pull of gravity.

Thus, if $H \ll R$, then $v_\phi \gg c_s$.

The azimuthal Mach number,

$$\mathcal{M}_\phi \equiv \frac{v_\phi}{c_s} \gg 1$$

meaning that the flow is supersonic.

3.2.2 Radial (r) Component (Centripetal Force Balance)

The radial component of Navier-Stokes is as follows:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} - \rho\frac{GM}{r^2}$$

Applying thin-disk approximations:

- Axisymmetry: $\partial/\partial\phi = 0$
- Vertical equilibrium: $v_z = 0$
- Small radial velocity: $v_r \ll v_\phi$

Now, $1/\rho \cdot \partial p/\partial r$ is of the order c_s^2/r which is $\ll GM/R^2$, that is, we can say that there is a negligible radial pressure gradient compared to gravity and centrifugal forces. We are now left with

$$\frac{v_\phi^2}{r} = \frac{GM}{r^2} \implies v_\phi = \sqrt{\frac{GM}{r}}$$

confirming Keplerian rotation at leading order.

Thus, from the Vertical and Radial components of the Navier-Stokes Equation, we get Hydrostatic Equilibrium and the omission of radial pressure gradients, that is the centripetal force is mostly Keplerian.

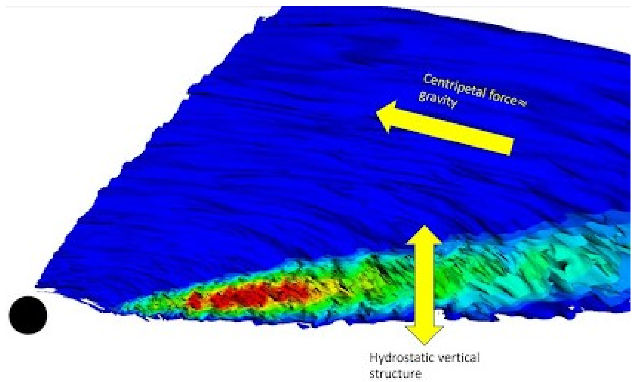

Figure 2: Vertical and Radial Accretion Disk Structure

3.3.3 Azimuthal (ϕ) Component (Angular Momentum Conservation)

The ϕ-component of the Navier–Stokes equation in cylindrical coordinates is:

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{\partial^2 v_\phi}{\partial z^2} - \frac{v_\phi}{r^2}\right] + \nu \frac{v_\phi}{r^2}\frac{\partial^2}{\partial \phi^2}$$

Applying

- Axisymmetry ($\partial/\partial\phi = 0$),
- Vertical Hydrostatic Equilibrium ($v_z = 0$),
- Neglecting radial pressure gradients in thin disks,

The equation simplifies to:

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = \nu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right) - \frac{v_\phi}{r^2}\right)$$

The viscous contribution to angular momentum transport (momentum flux) can be expressed as:

$$\text{Viscous Flux} = \nu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right) - \frac{v_\phi}{r^2}\right)$$

Simplifying, using $v_\phi = r\Omega(r)$ (where Ω is the angular velocity), we get the following:

$$= \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{d\Omega}{dr}\right)$$

Thus, the transport of viscous angular momentum depends on the radial gradient of the angular velocity.

3.3.4 Evolution of Angular Momentum (Vertically Integrated Form)

Multiplying by surface density Σ and integrating vertically, the equation for preserving angular momentum becomes:

$$\frac{\partial}{\partial t}(r\Sigma v_\phi) + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Sigma v_r v_\phi\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \frac{d\Omega}{dr}\right)$$

or equivalently:

$$\frac{\partial}{\partial t}(r\Sigma v_\phi) = -\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Sigma v_r v_\phi\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \frac{d\Omega}{dr}\right)$$

This is the fundamental evolution equation governing angular momentum transport in thin accretion disks. Applying mass conservation (continuity equation) and angular momentum conservation to this setup, we find that gradients in angular velocity lead to viscous shear within the disk. Fluid elements at smaller radii orbit faster than those at larger radii, resulting in the outward transport of angular momentum. Inner fluid elements, having lost angular momentum, drift inward, allowing accretion onto the central object.

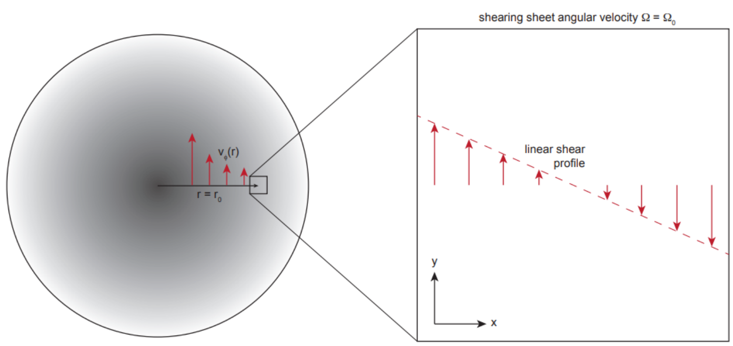

Figure 3: Geometry of the local or shearing sheet model of an accretion disk[2]

4. Surface Density Evolution

In the thin-disk approximation, we assume a Quasi-Keplerian scheme and use the vertically integrated angular momentum conservation equation and the continuity equation to get:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{1/2}\frac{\partial}{\partial r}\left(\nu \Sigma r^{1/2}\right)\right) \tag{3}$$

The equation governing the evolution of the disk's surface density (Σ) has a diffusion-like form. If the viscosity depends only on the radius, the equation is linear; if it depends on other properties (like temperature or surface density), it becomes non-linear. The diffusion-like nature means that an initially narrow ring of matter will spread out over time, with some material moving inward (accreting) and some moving outward.

This process explains how a stream of incoming gas from the companion star is transformed into a full accretion disk.

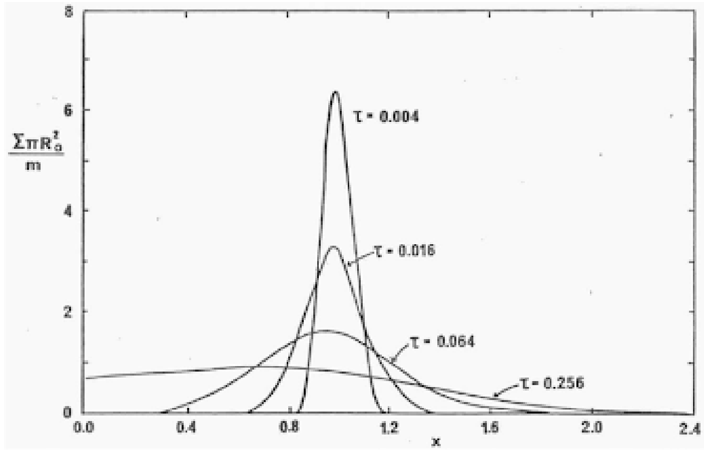

Figure 4: Viscous Evolution of an initially thin ring of mass (m) and radius (R_O) [9]

5. Nature of Viscosity in Disks

When estimating the viscosity based on molecular kinetic theory, we find unrealistically small values: the resulting accretion timescales exceed the age of the universe. Thus, the "viscosity" in accretion disks is not due to microscopic particle collisions but rather due to an anomalous turbulent viscosity.

This anomalous viscosity is now widely believed to originate from magnetohydrodynamic (MHD) turbulence, specifically via the magnetorotational instability (MRI). MRI can efficiently transport angular momentum outward even in weakly magnetized disks.

We now consider the azimuthal (ϕ) component of the steady-state Navier-Stokes (N-S) equation ($\partial/\partial t \to 0$): [13]

$$\rho \left(v_R \frac{\partial v_\phi}{\partial R} + \frac{v_R v_\phi}{R} \right) = \frac{\partial}{\partial R} \left(\eta \left(\frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R} \right) \right) + \frac{2\eta}{R} \left(\frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R} \right) \tag{4}$$

Given that

$$v_\phi \approx \sqrt{\frac{GM}{R}} \tag{5}$$

we find:

$$\frac{\partial v_\phi}{\partial R} = -\frac{1}{2}\sqrt{\frac{GM}{R^3}} = -\frac{1}{2}\frac{v_\phi}{R} \tag{6}$$

We now introduce the R–ϕ stress, which is the ϕ-directed stress on a surface whose outward normal vector points in the R direction:

$$t_{R\phi} = \eta \left(\frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R} \right) \tag{7}$$

(We can visualize this as acting on the side surface of a cylindrical shell (viewed from the top); shear viscosity acts to prevent adjacent azimuthal velocity streamlines from sliding past each other.)

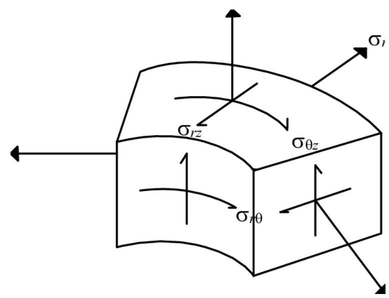

Figure 5: Stress Components in Cylindrical Polar Coordinate System[7]

The ϕ component of the steady-state N-S equation simplifies to:

$$\frac{1}{2}\rho\frac{v_R v_\phi}{R} = \frac{r}{2}\frac{t_{R\phi}}{R} \tag{8}$$

Now, the scalar pressure term that we usually know corresponds to the diagonal components of the viscous stress tensor. Meanwhile, terms like $t_{R\phi}$ are off-diagonal components of the viscous stress tensor, associated with shear stresses.

Since in the R–ϕ direction we are dealing with shearing motions, and because the viscous stress tensor and pressure have the same physical dimensions (force per unit area), it makes sense to treat them on the same footing.

However, the viscous stress is typically much smaller than the ambient pressure — it is just a small perturbation (or fraction) of it. Hence, it is reasonable to model the viscous stress component $t_{R\phi}$ as proportional to ambient pressure p.

Thus, Shakura and Sunyaev proposed:

$$t_{R\phi} = \alpha p \tag{9}$$

[12]

where:

- α is a dimensionless parameter,
- $0 < \alpha < 1$,
- α captures the relative strength of viscous stresses compared to the pressure.

We can think of this prescription by considering that in a reference frame rotating with the disk, the characteristic velocity scale is the sound speed c_s and the characteristic length scale is the thickness of the disk H. From dimensional analysis, the effective turbulent viscosity must be of the form:

$$\nu \sim c_s H \tag{10}$$

To capture the uncertain efficiency of this turbulence, Shakura and Sunyaev introduced the α-prescription:

$$\nu = \alpha c_s H \tag{11}$$

Alpha-disk models, based on this formulation, express the viscosity as a function of the disk temperature and vertical structure.

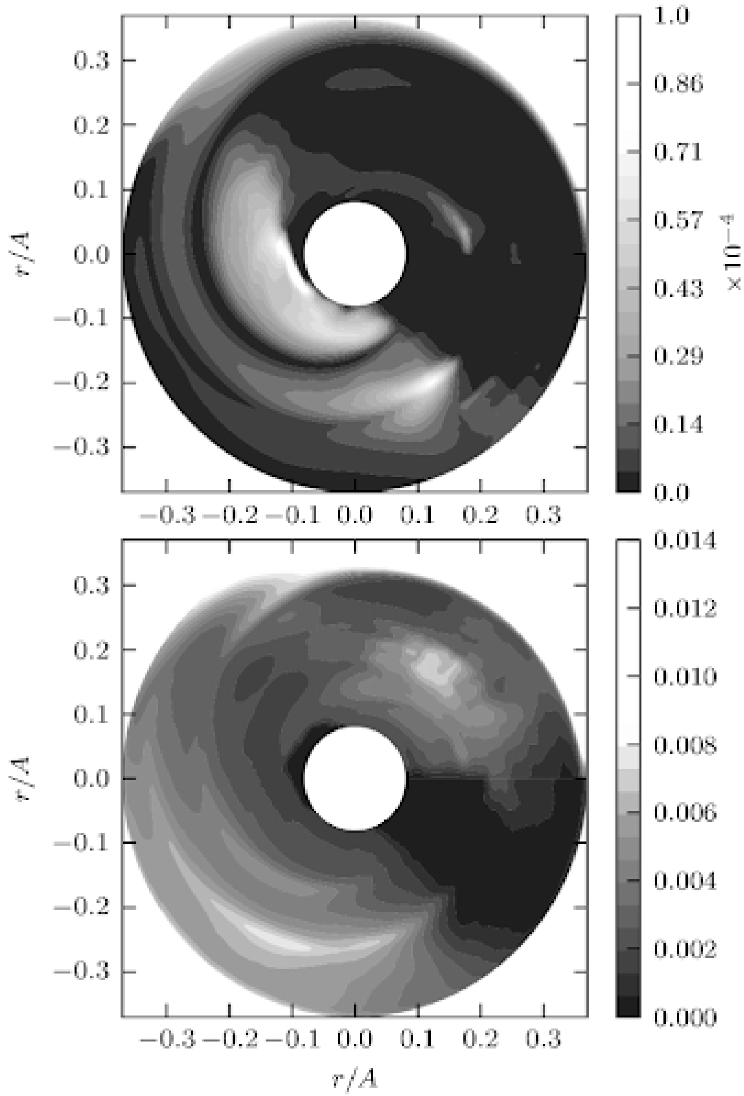

Figure 6: Perturbation amplitudes (upper panel) and the Shakura–Sunyaev parameter (bottom panel) in a disk[5]

6. Energy Dissipation

Assuming steady state, applying mass conservation and imposing zero torque at the boundary of the disk, we get [1]:

$$Q^{+}(R) = \frac{3GM\dot{M}}{8\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right) \tag{12}$$

Thus, far from the inner edge ($R \gg R_*$), the energy dissipation simplifies to:

$$Q^{+}(R) \approx \frac{3GM\dot{M}}{8\pi R^3} \tag{13}$$

Close to the inner boundary ($R \sim R_*$), the dissipation drops to zero as $R \to R_*$ because there is no viscous torque inside.

If the disk radiates locally as a blackbody, then:

$$\sigma_B T_{\text{eff}}^4 = Q^+ \tag{14}$$

where:

- σ_B is the Stefan–Boltzmann constant.
- $T_{\text{eff}}(R)$ is the effective temperature profile of the disk.

Substituting, we get

$$T_{\text{eff}}(R) = \left(\frac{3GM\dot{M}}{8\pi\sigma_B R^3} \left(1 - \sqrt{\frac{R_*}{R}}\right) \right)^{1/4} \tag{15}$$

Thus, for a stationary quasi-Keplerian accretion disk, we find

$$T \propto R^{3/4}. \tag{16}$$

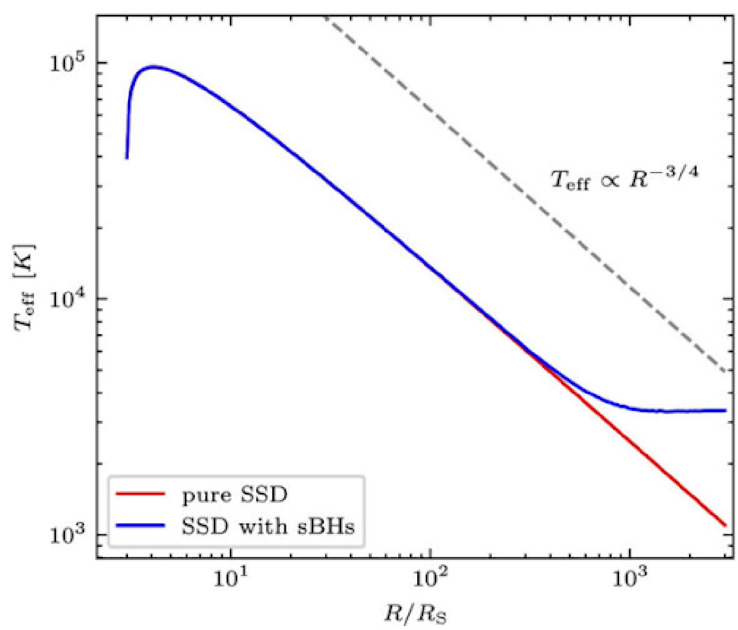

Figure 7: Modeled temperature of an accretion disk with (blue line) and without (red line) embedded stellar-mass black holes[4]

Conclusion

In summary, the structure and dynamics of steady-state accretion disks—particularly those described by the standard thin disk model—are governed by viscous angular momentum transport between adjacent Keplerian rings. While the bulk viscosity can be neglected, shear viscosity is essential to mediate the transfer of angular momentum outward, allowing mass to accrete inward. Despite this, in a steady-state configuration, none of the observable quantities—such as surface brightness, temperature profiles, or emitted spectra—depend explicitly on the microscopic viscosity. Instead, these depend on global quantities like the mass accretion rate and gravitational potential, making viscosity invisible to direct observational probes under steady-state conditions.

To extract information about the viscosity or its origin (e.g., turbulence or magnetorotational instability), one must study time-dependent behaviors—such as variability, instabilities, or transitions between accretion states—where the effects of viscosity become observable. This highlights the limitations of steady-state models and motivates the study of disk evolution, turbulence generation mechanisms, and the role of magnetic fields. Many open questions remain, particularly in understanding how microscopic viscosity emerges from macroscopic turbulence and how energy dissipation translates into the rich variety of emissions observed from astrophysical disks.

Ultimately, while the steady-state model provides a powerful framework for predicting disk structure and radiation, it only scratches the surface of the complex, dynamic processes shaping accretion in realistic astrophysical environments.

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