QUBO formulation

$$\text{QUBO} = \min\left(\text{QO} + \sum_{1}^{7} P_i\right)$$

where the first term is the quadratic objective function:

$$\text{QO} = \sum_{l \in L} \sum_{j \in J} \rho_j \left(\sum_{c \in K_l} \beta_{c,j} x_c - K_{l,j}^{target} \right)^2$$

The second term represents the sum of all penalty terms that will be determined by constraints for the OneOpto optimization model using binary variables. There are four contraints:

1. Maximum number of bonds in basket (P_1)
2. Residual cash flow of porfolio (P_2 and P_3)
3. Guardrails on x_c (P_4 and P_5)
4. Guardrails on y_c (P_6 and P_7)

Before encoding constraints as penalty terms, it is worth to mention that the penaty terms can only be encoded from equality constraints by definition. However, in our case, all guardrails are inequality constraints, thus it is nessesary to convert those to equality ones by introducing slack variables (see [Constraints: Linear Inequality (Penalty Functions) from D-wave web page]). The QUBO formulation quickly grows in complexity, and using slack variables only makes things worse: they add extra variables—and thus require more qubits—an especially scarce resource on today's quantum hardware when running VQAs. There are several methods to formulate the constrained problem without slack variables. In this project, we will focus on unbalanced penalization (slack-free) [1] [2][3].

In the following part, we are showing how to encode these four constraints

Unbalanced penalization

Penaties

Maximum number of bonds in basket

The maximum number of bonds in basket is:

$$\sum_{c \in C} y_c \leq N$$

Applying the unbalanced penalization method (see Appendix), we obtain the first penalty term:

$$\boxed{P_1 = -\lambda_1^{(0)}\left(N - \sum_{c \in C} y_c\right) + \lambda_1^{(1)}\left(N - \sum_{c \in C} y_c\right)^2}$$

Residual cash flow constraint

The residual cash flow constraint is

$$\frac{max(RC)}{MV^b} \leq NC \leq \frac{min(RC)}{MV^b}$$

where:

- RC is the residual cash flow
- MV^b is the market value
- NC is the normalized cost of selected assets $c \in C$:

  $$NC = \sum_{c \in C} \frac{p_c \delta_c x_c}{100 MV^b} = \frac{NC'}{100 MV^b}$$

  with $NC' = \sum_{c \in C} p_c \delta_c x_c$

Similarly, we can determine the penalty terms, P_2 and P_3:

- Upper bound:

  $$\frac{NC'}{100MV^b} \leq \frac{min(RC)}{MV^b} \longrightarrow NC' \leq 100.min(RC)$$

  We obtain :

  $$\boxed{P_2 = -\lambda_2^{(0)}\left(100.min(RC) - NC'\right) + \lambda_2^{(1)}\left(100.min(RC) - NC'\right)^2}$$

- Lower bound:

  $$\frac{NC'}{100MV^b} \geq \frac{min(RC)}{MV^b} \longrightarrow NC' \geq 100.min(RC)$$

  We obtain :

  $$\boxed{P_3 = -\lambda_3^{(0)}\left(NC' - 100.min(RC)\right) + \lambda_3^{(1)}\left(NC' - 100.min(RC)\right)^2}$$

Guardrails on x_c

The min/max value of each characteristic j in each risk group l are:

$$b_{j,l}^{low} \leq \mathrm{MC}_{j,l} \leq b_{j,l}^{up}, \ \forall j \in J, l \in L$$

where:

$$\mathrm{MC}_{j,l} = \sum_{c \in K_l} \frac{p_c \delta_c}{100 MV^b} \beta_{c,j} x_c, \ \forall j \in J, l \in L$$

.

Similarly, the penaty terms, P_4 and P_5, are obtained:

- Upper bound:

$$\boxed{P_4 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_4^{(0)} \left(b_{j,l}^{up} - \mathrm{MC}_{j,l} \right) + \lambda_4^{(1)} \left(b_{j,l}^{up} - \mathrm{MC}_{j,l} \right)^2 \right]}$$

- Lower bound:

$$\boxed{P_5 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_5^{(0)} \left(\mathrm{MC}_{j,l} - b_{j,l}^{low} \right) + \lambda_5^{(1)} \left(\mathrm{MC}_{j,l} - b_{j,l}^{low} \right)^2 \right]}$$

Guardrails on y_c

The constraints on y_c is:

$$K_{j,l}^{low} \leq \sum_{c \in K_l} \beta_{c,j} y_c \leq K_{j,l}^{up}, \; \forall j \in J, l \in L$$

By doing the work, we obtain:

- Upper bound:

$$\boxed{P_6 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_6^{(0)} \left(K_{j,l}^{up} - \sum_{c \in K_l} \beta_{c,j} y_c \right) + \lambda_6^{(1)} \left(K_{j,l}^{up} - \sum_{c \in K_l} \beta_{c,j} y_c \right)^2 \right]}$$

- Lower bound:

$$\boxed{P_7 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_7^{(0)} \left(\sum_{c \in K_l} \beta_{c,j} y_c - K_{j,l}^{low} \right) + \lambda_7^{(1)} \left(\sum_{c \in K_l} \beta_{c,j} y_c - K_{j,l}^{low} \right)^2 \right]}$$

Converting QUBO formulation to Ising Hamiltonian.

Once the complete QUBO fomulation is found, we need to convert it into Ising Hamiltonian. It is worth to note that by using binary variables, x_c (how much of bond c is included in the basket) is no longer a variable, but is fixed to the average value it is allowed to have if c is included at all in the porfolio:

$$x_c = \frac{m_c + min\{M_c, i_c\}}{2\delta_c} y_c$$

where y_c is decision variable instead. $y_c \in \{0, 1\}$ is binary, represents whether bond c is included or not.

We can convert QUBO model to Ising model by mapping y_c into spin variable $z_i \in \{-1, 1\}$:

$$y_c = \frac{1 - z_c}{2}$$

Therefore:

$$x_c = \frac{1}{4\delta_c} (m_c + min\{M_c, i_c\})(1 - z_c)$$

In the following parts, we are showing the Ising models for each terms of QUBO formulation

The quadratic objective function QO

By using the new expression of x_c, we obtain:

$$\mathrm{QO} = \sum_{l\in L}\sum_{j\in J}\rho_j\left(\sum_{c\in K_l}\frac{\beta_{c,j}}{4\delta_c}(m_c + min\{M_c, i_c\})(1-z_c) - K_{l,j}^{target}\right)^2$$

Let:

$$O_{cjl} := \frac{\beta_{c,j}}{4\delta_c}(m_c + \min\{M_c, i_c\})$$

Then:

$$\mathrm{QO} = \sum_{l\in L}\sum_{j\in J}\rho_j\left(\sum_{c\in K_l} O_{cjl}(1-z_c) - K_{l,j}^{\text{target}}\right)^2$$

Let:

$$A_0 := \sum_{c\in K_l} O_{cjl} - K_{l,j}^{\text{target}}$$

Then expand:

$$\left(A_0 - \sum_{c\in K_l} O_{cjl}z_c\right)^2 = A_0^2 - 2A_0\sum_{c\in K_l} O_{cjl}z_c + \sum_{c,c'\in K_l} O_{cjl}O_{c'jl}z_c z_{c'}$$

Since $z_c^2 = 1$ and $\sum_{c\neq c'} z_c z_{c'} = 2\sum_{c<c'} z_c z_{c'}$ thus:

$$\left(A_0 - \sum_{c\in K_l} O_{cjl}z_c\right)^2 = A_0^2 - 2A_0\sum_{c\in K_l} O_{cjl}z_c + 2\sum_{c} O_{cjl}^2 + \sum_{c<c'} O_{cjl}O_{c'jl}z_c z_{c'}$$

Finally, we can write it as:

$$\mathrm{QO} = \mathrm{const}^{(0)} + \sum_{c\in K_l} h_c^{(0)} z_c + \sum_{c<c'} J_{cc'}^{(0)} z_c z_{c'}$$

Where:

- Constant term:

$$\mathrm{const}^{(0)} = \sum_{l\in L}\sum_{j\in J}\rho_j\left(A_0^2 + 2\sum_{c} O_{cjl}^2\right)$$

- Linear term:

$$h_c^{(0)} = \sum_{l\in L}\sum_{j\in J:\, c\in K_l}(-2\rho_j A_0 O_{cjl})$$

- Quadratic term:

$$J_{cc'}^{(0)} = \sum_{l \in L} \sum_{\substack{j \in J:\, c,c' \in K_l \\ c<c'}} \rho_j O_{cjl} O_{c'jl}$$

Penalty terms

Maximum number of bonds in basket

By using the new expression of y_c, we obtain:

$$P_1 = -\lambda_1^{(0)} \left(N - \sum_{c \in C} \frac{1 - z_c}{2} \right) + \lambda_1^{(1)} \left(N - \sum_{c \in C} \frac{1 - z_c}{2} \right)^2$$

We rewrite:

$$\sum_{c \in C} \frac{1 - z_c}{2} = \frac{|C|}{2} - \frac{1}{2} \sum_{c \in C} z_c$$

Now substitute into the penalty term:

$$P_1 = -\lambda_1^{(0)} \left(N - \frac{|C|}{2} + \frac{1}{2} \sum_{c \in C} z_c \right) + \lambda_1^{(1)} \left(N - \frac{|C|}{2} + \frac{1}{2} \sum_{c \in C} z_c \right)^2$$

We define: $A_1 := N - \frac{|C|}{2}$ then rewrite P_1 as

$$P_1 = -\lambda_1^{(0)} \left(A_1 + \frac{1}{2} \sum_{c \in C} z_c \right) + \lambda_1^{(1)} \left(A_1 + \frac{1}{2} \sum_{c \in C} z_c \right)^2$$

Now expand the square

$$\left(A_1 + \frac{1}{2} \sum_{c \in C} z_c \right)^2 = A_1^2 + A_1 \sum_{c \in C} z_c + \frac{1}{4} \sum_{c \neq c'} z_c z_{c'} + \frac{1}{4} \sum_{c \in C} z_c^2$$

but $z_c^2 = 1$ and $\sum_{c \neq c'} z_c z_{c'} = 2 \sum_{c<c'} z_c z_{c'}$ thus:

$$\left(A_1 + \frac{1}{2} \sum_{c \in C} z_c \right)^2 = A_1^2 + A_1 \sum_{c \in C} z_c + \frac{1}{2} \sum_{c<c'} z_c z_{c'} + \frac{|C|}{4}$$

The final Ising model for P_1 is:

$$P_1 = \mathrm{Const}^{(1)} + \sum_{c \in C} h_c^{(1)} z_c + \sum_{c<c'} J_{cc'}^{(1)} z_c z_{c'}$$

where :

- Constant term:

$$\mathrm{Const}^{(1)} = -\lambda_1^{(0)} A_1 + \lambda_1^{(1)} \left(A_1^2 + \frac{|C|}{4} \right)$$

- Linear term:

$$h_c^{(1)} = -\frac{\lambda_1^{(0)}}{2} + \lambda_1^{(1)} A_1$$

- Quadratic term:

$$J_{cc'}^{(1)} = \frac{\lambda_1^{(1)}}{2}$$

Residual cash flow of porfolio

Similarly, by using the new expression of x_c (for NC'), we have:

- $$P_2 = -\lambda_2^{(0)} \left(100.min(RC) - \sum_{c \in C} \frac{p_c}{4}(m_c + min\{M_c, i_c\})(1 - z_c)\right) \\ + \lambda_2^{(1)} \left(100.min(RC) - \sum_{c \in C} \frac{p_c}{4}(m_c + min\{M_c, i_c\})(1 - z_c)\right)^2$$
- $$P_3 = -\lambda_3^{(0)} \left(\sum_{c \in C} \frac{p_c}{4}(m_c + min\{M_c, i_c\})(1 - z_c) - 100.min(RC)\right) \\ + \lambda_3^{(1)} \left(\sum_{c \in C} \frac{p_c}{4}(m_c + min\{M_c, i_c\})(1 - z_c) - 100.min(RC)\right)^2$$

Let:

$$A_c = \frac{p_c}{4}(m_c + \min\{M_c, i_c\}) \quad \text{and} \quad B = 100 \cdot \min(RC)$$

Then the expression becomes:

$$P_2 = -\lambda_2^{(0)} \left(B - \sum_c A_c(1 - z_c)\right) + \lambda_2^{(1)} \left(B - \sum_c A_c(1 - z_c)\right)^2$$

We expand the inner sum:

$$P_2 = -\lambda_2^{(0)} \left(B - \sum_c A_c + \sum_c A_c z_c\right) + \lambda_2^{(1)} \left(B - \sum_c A_c + \sum_c A_c z_c\right)^2$$

Define:

$$\tilde{B} = B - \sum_c A_c$$

Then:

$$P_2 = -\lambda_2^{(0)} \left(\tilde{B} + \sum_c A_c z_c\right) + \lambda_2^{(1)} \left(\tilde{B} + \sum_c A_c z_c\right)^2$$

Now expand the square with using $z_c^2 = 1$:

$$\left(\tilde{B} + \sum_c A_c z_c\right)^2 = \tilde{B}^2 + 2\tilde{B}\sum_c A_c z_c + \sum_c A_c^2 + 2\sum_{c<c'} A_c A_{c'} z_c z_{c'}$$

Thus, we obtain:

$$P_2 = \mathrm{Const}^{(2)} + \sum_{c\in C} h_c^{(2)} z_c + \sum_{c<c'} J_{cc'}^{(2)} z_c z_{c'}$$

- Constant:

$$\mathrm{const}^{(2)} = -\lambda_2^{(0)}\tilde{B} + \lambda_2^{(1)}\left(\tilde{B}^2 + \sum_c A_c^2\right)$$

- Linear terms:

$$h_c^{(2)} = -\lambda_2^{(0)} A_c + 2\lambda_2^{(1)}\tilde{B}A_c$$

- Quadratic terms:

$$J_{cc'}^{(2)} = 2\lambda_2^{(1)} A_c A_{c'}$$

Similarly, we can obtain P_3 where $-\tilde{B}$ appears instead:

$$P_3 = -\lambda_3^{(0)}\left(-\tilde{B} - \sum_c A_c z_c\right) + \lambda_3^{(1)}\left(-\tilde{B} - \sum_c A_c z_c\right)^2$$

Simplifying:

$$P_3 = \mathrm{Const}^{(3)} + \sum_{c\in C} h_c^{(3)} z_c + \sum_{c<c'} J_{cc'}^{(3)} z_c z_{c'}$$

- Constant:

$$\mathrm{const}^{(3)} = \lambda_3^{(0)}\tilde{B} + \lambda_3^{(1)}\left(\tilde{B}^2 + \sum_c A_c^2\right)$$

- Linear terms:

$$h_c^{(3)} = \lambda_3^{(0)} A_c + 2\lambda_3^{(1)}\tilde{B}A_c$$

- Quadratic terms:

$$J_{cc'}^{(3)} = 2\lambda_3^{(1)} A_c A_{c'}$$

Guardrails on x_c

By using the new expression of x_c, the Ising models for P_4 and P_5 are:

- $$P_4 = \sum_{l \in L}$$

$$\sum_{j \in J} \left[-\lambda_4^{(0)} \left(b_j^{up} - \sum_{c \in K_l} \frac{p_c \beta_{c,j}}{400MV^b} (m_c + min\{M_c, i_c\})(1 - z_c) \right) + \lambda_4^{(1)} \left(b_j^{up} - \sum_{c \in} \right. \right.$$

- $$P_5 = \sum_{l \in L}$$

$$\sum_{j \in J} \left[-\lambda_5^{(0)} \left(\sum_{c \in K_l} \frac{p_c \beta_{c,j}}{400MV^b} (m_c + min\{M_c, i_c\})(1 - z_c) - b_j^{low} \right) + \lambda_5^{(1)} \left(\sum_{c \in K_l} \frac{p}{40} \right. \right.$$

Let:

- $$D_{cjl} = \frac{p_c \beta_{c,j}}{400MV^b} (m_c + \min\{M_c, i_c\})$$

Define:

- $$\tilde{E}_{jl} = b_j^{up} - \sum_{c \in K_l} D_{cjl}$$

Then:

$$P_4 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_4^{(0)} \left(\tilde{E}_{jl} + \sum_{c \in K_l} D_{cjl} z_c \right) + \lambda_4^{(1)} \left(\tilde{E}_{jl} + \sum_{c \in K_l} D_{cjl} z_c \right)^2 \right]$$

Let expand the square with using $z_c^2 = 1$:

$$\left(\tilde{E}_{jl} + \sum_{c \in K_l} D_{cjl} z_c \right)^2 = \tilde{E}_{jl}^2 + 2\tilde{E}_{jl} \sum_{c \in K_l} D_{cjl} z_c + \sum_{c \in K_l} D_{cjl}^2 + \sum_{c < c'} 2 D_{cjl} D_{c'jl} z_c z_{c'}$$

Finally, we obtain:

$$P_4 = \text{const}^{(4)} + \sum_c h_c^{(4)} z_c + \sum_{c < c'} J_{cc'}^{(4)} z_c z_{c'}$$

- Constant term:

$$\text{const}^{(4)} = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_4^{(0)} \tilde{E}_{jl} + \lambda_4^{(1)} \left(\tilde{E}_{jl}^2 + \sum_{c \in K_l} D_{cjl}^2 \right) \right]$$

- Linear term:

$$h_c^{(4)} = \sum_{l \in L} \sum_{j \in J : c \in K_l} D_{cjl} \left(-\lambda_4^{(0)} + 2\lambda_4^{(1)} \tilde{E}_{jl} \right)$$

- Quadratic term:

$$J_{cc'}^{(4)} = \sum_{l \in L} \sum_{j \in J : c, c' \in K_l} 2\lambda_4^{(1)} D_{cjl} D_{c'jl}$$

For P_5, we juste need to define:

- $$\tilde{F}_{jl} = \sum_{c \in K_l} D_{cjl} - b_j^{low}$$

Then:

$$P_5 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_5^{(0)} \left(\tilde{F}_{jl} - \sum_{c \in K_l} D_{cjl} z_c \right) + \lambda_5^{(1)} \left(\tilde{F}_{jl} - \sum_{c \in K_l} D_{cjl} z_c \right)^2 \right]$$

Let expand the square with using $z_c^2 = 1$:

$$\left(\tilde{F}_{jl} - \sum_{c \in K_l} D_{cjl} z_c \right)^2 = \tilde{F}_{jl}^2 - 2\tilde{F}_{jl} \sum_{c \in K_l} D_{cjl} z_c + \sum_{c \in K_l} D_{cjl}^2 + \sum_{c < c'} 2 D_{cjl} D_{c'jl} z_c z_{c'}$$

The final Ising model of P_5 is:

$$P_5 = \text{const}^{(5)} + \sum_c h_c^{(5)} z_c + \sum_{c < c'} J_{cc'}^{(5)} z_c z_{c'}$$

- Constant term:

$$\text{const}^{(5)} = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_5^{(0)} \tilde{F}_{jl} + \lambda_5^{(1)} \left(\tilde{F}_{jl}^2 + \sum_{c \in K_l} D_{cjl}^2 \right) \right]$$

- Linear term:

$$h_c^{(5)} = \sum_{l \in L} \sum_{j \in J : c \in K_l} D_{cjl} \left(\lambda_5^{(0)} - 2\lambda_5^{(1)} \tilde{F}_{jl} \right)$$

- Quadratic term:

$$J_{cc'}^{(5)} = \sum_{l \in L} \sum_{\substack{j \in J : c, c' \in K_l \\ c < c'}} 2\lambda_5^{(1)} D_{cjl} D_{c'jl}$$

Guardrails on y_c

By using the new expression of y_c, we obtain:

- $P_6 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_6^{(0)} \left(K_j^{up} - \sum_{c \in K_l} \frac{\beta_{c,j}}{2} (1 - z_c) \right) + \lambda_6^{(1)} \left(K_j^{up} - \sum_{c \in K_l} \frac{\beta_{c,j}}{2} (1 - z_c) \right.\right.$
- $P_7 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_7^{(0)} \left(\sum_{c \in K_l} \frac{\beta_{c,j}}{2} (1 - z_c) - K_j^{low} \right) + \lambda_7^{(1)} \left(\sum_{c \in K_l} \frac{\beta_{c,j}}{2} (1 - z_c) - K_j^{lc} \right.\right.$

Let's define:

- $$\tilde{K}_j^{up} = K_j^{up} - \sum_{c \in K_l} \frac{\beta_{c,j}}{2}$$

- $$\tilde{K}_j^{low} = \sum_{c \in K_l} \frac{\beta_{c,j}}{2} - K_j^{low}$$

Now, P_6 becomes:

$$P_6 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_6^{(0)} \left(\tilde{K}_j^{up} + \sum_{c \in K_l} \frac{\beta_{c,j}}{2} z_c \right) + \lambda_6^{(1)} \left(\tilde{K}_j^{up} + \sum_{c \in K_l} \frac{\beta_{c,j}}{2} z_c \right)^2 \right]$$

Let's expand the square with using $z^2 = 1$:

$$\left(\tilde{K}_j^{up} + \sum_{c \in K_l} \frac{\beta_{c,j}}{2} z_c \right)^2 = (\tilde{K}_j^{up})^2 + \tilde{K}_j^{up} \sum_{c \in K_l} \beta_{c,j} z_c + \sum_{c \in K_l} \frac{\beta_{c,j}^2}{4} + \sum_{c < c'} \frac{\beta_{c,j} \beta_{c',j}}{2} z_c z_{c'}$$

The final Ising model of P_6 is:

$$P_6 = \text{const}^{(6)} + \sum_c h_c^{(6)} z_c + \sum_{c < c'} J_{cc'}^{(6)} z_c z_{c'}$$

- Constant term:

$$\text{const}^{(6)} = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_6^{(0)} \tilde{K}_j^{up} + \lambda_6^{(1)} \left((\tilde{K}_j^{up})^2 + \sum_{c \in K_l} \frac{\beta_{c,j}^2}{4} \right) \right]$$

- Linear term:

$$h_c^{(6)} = \sum_{l \in L} \sum_{j \in J : c \in K_l} \frac{\beta_{c,j}}{2} \left(-\lambda_6^{(0)} + 2\lambda_6^{(1)} \tilde{K}_j^{up} \right)$$

- Quadratic term:

$$J_{cc'}^{(6)} = \sum_{l \in L} \sum_{\substack{j \in J : c, c' \in K_l \\ c < c'}} \lambda_6^{(1)} \frac{\beta_{c,j} \beta_{c',j}}{2}$$

Similarly, P_7 becomes:

$$P_7 = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_7^{(0)} \left(\tilde{K}_j^{low} - \sum_{c \in K_l} \frac{\beta_{c,j}}{2} z_c \right) + \lambda_7^{(1)} \left(\tilde{K}_j^{low} - \sum_{c \in K_l} \frac{\beta_{c,j}}{2} z_c \right)^2 \right]$$

Like P_5, we obtain:

$$P_7 = \text{const}^{(6)} + \sum_c h_c^{(7)} z_c + \sum_{c < c'} J_{cc'}^{(7)} z_c z_{c'}$$

- Constant term:

$$\text{const}^{(7)} = \sum_{l \in L} \sum_{j \in J} \left[-\lambda_7^{(0)} \tilde{K}_j^{low} + \lambda_7^{(1)} \left((\tilde{K}_j^{low})^2 + \sum_{c \in K_l} \frac{\beta_{c,j}^2}{4} \right) \right]$$

- Linear term:

$$h_c^{(7)} = \sum_{l \in L} \sum_{j \in J: c \in K_l} \frac{\beta_{c,j}}{2} \left(\lambda_7^{(0)} - 2\lambda_7^{(1)} \tilde{K}_j^{low} \right)$$

- Quadratic term:

$$J_{cc'}^{(7)} = \sum_{l \in L} \sum_{\substack{j \in J: c,c' \in K_l \\ c<c'}} \lambda_7^{(1)} \frac{\beta_{c,j}\beta_{c',j}}{2}$$

The total Ising Hamiltonian of QUBO model

$$H_{tot} = \text{const}^{tot} + \sum_c h_c^{tot} z_c + \sum_{c<c'} J_{cc'}^{tot} z_c z_{c'}$$

where:

- Total constant term:

$$\text{const}^{tot} = \sum_{i=0}^{7} \text{const}^{(i)}$$

$$= \Bigg[-\lambda_1^{(0)} A_1 + \lambda_1^{(1)} \left(A_1^2 + \frac{|C|}{4} \right) - \lambda_2^{(0)} \tilde{B} + \lambda_2^{(1)} \left(\tilde{B}^2 + \sum_c A_c^2 \right) + \lambda_3^{(0)} \tilde{B} + \lambda_3^{(1}$$

$$\sum_{j \in J} \Bigg[\Bigg[\rho_j \left(A_0^2 + 2 \sum_c O_{cjl}^2 \right) - \lambda_4^{(0)} \tilde{E}_{jl} + \lambda_4^{(1)} \left(\tilde{E}_{jl}^2 + \sum_{c \in K_l} D_{cjl}^2 \right) - \lambda_5^{(0)} \tilde{F}_{jl} + \lambda_5^{(1)} \big($$

$$+ \lambda_6^{(1)} \left((\tilde{K}_j^{up})^2 + \sum_{c \in K_l} \frac{\beta_{c,j}^2}{4} \right) - \lambda_7^{(0)} \tilde{K}_j^{low} + \lambda_7^{(1)} \left((\tilde{K}_j^{low})^2 + \sum_{c \in K} \right.$$

Simplifying:

$$\boxed{\begin{aligned} \text{const}^{tot} = &\Bigg[-\lambda_1^{(0)} A_1 + \lambda_1^{(1)} \left(A_1^2 + \frac{|C|}{4} \right) - (\lambda_2^{(0)} - \lambda_3^{(0)}) \tilde{B} + (\lambda_2^{(1)} + \lambda_3^{(1)}) \Bigg(\tilde{B}^2 + \sum_c \\ &\sum_{j \in J} \Bigg[\Bigg[\rho_j \left(A_0^2 + 2 \sum_c O_{cjl}^2 \right) - (\lambda_4^{(0)} \tilde{E}_{jl} + \lambda_5^{(0)} \tilde{F}_{jl} + \lambda_6^{(0)} \tilde{K}_j^{up} + \lambda_7^{(0)} \tilde{K}_j^{low}) + (\lambda_4^{(1)} \tilde{E}_{jl}^2 + \\ &+ \left(\lambda_4^{(1)} + \lambda_5^{(1)} \right) \sum_{c \in K_l} D_{cjl}^2 + + \left(\lambda_6^{(1)} + \lambda_7^{(1)} \right) \sum_{c \in K_l} \frac{\beta_{c,j}^2}{4} \Bigg] \Bigg] \end{aligned}}$$

- Total linear term:

$$h_c^{tot} = \sum_{i=0}^{7} h_c^{(i)} = \Bigg[-\frac{\lambda_1^{(0)}}{2} + \lambda_1^{(1)} A_1 - \lambda_2^{(0)} A_c + 2\lambda_2^{(1)} \tilde{B} A_c + \lambda_3^{(0)} A$$

$$\sum_{j \in J:\, c \in K_l} \Bigg[-2\rho_j A_0 O_{cjl} + D_{cjl} \left(-\lambda_4^{(0)} + 2\lambda_4^{(1)} \tilde{E}_{jl} \right) + D_{cjl} \left(\lambda_5^{(0)} - 2\lambda_5^{(1)} \tilde{F}_{jl} \right) + \frac{\beta_{c,j}}{2} \big(-\lambda_6^{($$

Simplifying:

$$\boxed{h_c^{tot} = \left[-\frac{\lambda_1^{(0)}}{2} + \lambda_1^{(1)} A_1 + \left(-\lambda_2^{(0)} + \lambda_3^{(0)} + 2\tilde{B}(\lambda_2^{(1)} + \lambda_3^{(1)})\right) A_c\right] + \sum_{l\in L} \sum_{j\in J:\, c\in K_l} \left[-2\rho_j A_0 O_{cjl} + D_{cjl}\left(-\lambda_4^{(0)} + \lambda_5^{(0)} + 2\lambda_4^{(1)}\tilde{E}_{jl} - 2\lambda_5^{(1)}\tilde{F}_{jl}\right) + \frac{\beta_{c,j}}{2}\left(-\lambda_6^{(0)} + \lambda_7^{(}\right.\right.}$$

- Total quadratic term:

$$J_{cc'}^{tot} = \sum_{i=0}^{7} J_{cc'}^{(i)} = \left[\frac{\lambda_1^{(1)}}{2} + 2\lambda_2^{(1)} A_c A_{c'} + 2\lambda_3^{(1)} A_c A_{c'}\right] + \sum_{l\in L} \sum_{\substack{j\in J:\, c,c'\in K_l \\ c<c'}} \left[\rho_j O_{cjl} O_{c'jl} + 2\lambda_4^{(1)} D_{cjl} D_{c'jl} + 2\lambda_5^{(1)} D_{cjl} D_{c'jl} + \lambda_6^{(1)} \frac{\beta_{c,j}\beta_{c',j}}{2} + \lambda_7^{(1)} \frac{\beta_{c,j}\beta_{c',j}}{2}\right]$$

Simplifying:

$$\boxed{J_{cc'}^{tot} = \sum_{i=0}^{7} J_{cc'}^{(i)} = \left[\frac{\lambda_1^{(1)}}{2} + 2A_c A_{c'}(\lambda_2^{(1)} + \lambda_3^{(1)})\right] + \sum_{l\in L} \sum_{\substack{j\in J:\, c,c'\in K_l \\ c<c'}} \left[\rho_j O_{cjl} O_{c'jl} + 2D_{cjl} D_{c'jl}(\lambda_4^{(1)} + \lambda_5^{(1)}) + \frac{\beta_{c,j}\beta_{c',j}}{2}(\lambda_6^{(1)} + \lambda_7^{(1)})\right]}$$

Double-click (or enter) to edit

Appendix

Input parameters

- A set of securities C with:
  - p_c market price,
  - m_c min trade,
  - M_c max trade,
  - i_c basket inventory,
  - δ_c minimum increment.
- A set L of risk buckets,
  - K_l are the bonds in bucket (not muatually exclusive)
- A set J of characteristic with

- $K_{l,j}^{target}$ target of characteristic j in risk bucket l,
- $b_{l,j}^{up}$ and $b_{l,j}^{low}$ guardrails for chracteristic j in risk bucket l,
- $\beta_{c,j}$ contributions of a unit of bond c to the target of characteristic j
- $K_{l,j}^{up}$ and $K_{l,j}^{low}$ guardrails for chracteristic j in risk bucket l (binary version)

- Global parameters:
  - N max number of bonds in porfolio
  - Min/max residual cash flow of porfolio

Unbalanced penalization

Upper bound

let's apply the **Unbalanced Penalization** method to the **upper bound constraint**:

$\sum y_i \leq b$

1. 🔸 Step 1: Define Constraint Violation

For the constraint:

$$\sum y_i \leq b \Rightarrow \text{Violation when } \sum y_i > b$$

So define:

$$h(y) = b - \sum y_i$$

- If $h(y) < 0$, the constraint is violated.
- If $h(y) \geq 0$, the constraint is satisfied.

2. 🔸 Step 2: Define Unbalanced Penalty Function

Montañez-Barrera et al. propose the following penalty function (Eq. 1 in their paper):

$$P(y) = -\lambda_1 \cdot h(y) + \lambda_2 \cdot h(y)^2$$

Note:

- When $h(y) \geq 0$, the function becomes **non-penalizing** if you tune λ_1 and λ_2 appropriately.
- When $h(y) < 0$, the function grows quadratically (like a parabola opening upward), creating a **steep energy increase** for violations.

Lower bound

Let's apply the **Unbalanced Penalization** method to the **lower-bound constraint**:

$\sum y_i \geq a$

1. 🔷 Step 1: Define Constraint Violation Function

For this constraint, a violation occurs when:

$$\sum y_i < a$$

So we define the violation function:

$$h(y) = \sum y_i - a$$

- $h(y) < 0$: **Violation** (too few items selected)
- $h(y) \geq 0$: **Feasible** (requirement met)

2. 🔷 Step 2: Define Unbalanced Penalty Function

As with the upper-bound case, the unbalanced penalty is:

$$P(y) = -\lambda_1 h(y) + \lambda_2 h(y)^2 = -\lambda_1 (\sum y_i - a) + \lambda_2 (\sum y_i - a)^2$$