Problem one:

$$f(w,b) = e^{x_1 w + b}, \quad x_1 \in \mathbb{R}$$

1. $\frac{\partial f}{\partial w} = x_1 e^{x_1 w + b}$ $\quad \frac{\partial^2 f}{\partial w^2} = x_1^2 e^{x_1 w + b}$

$\frac{\partial^2 f}{\partial w \partial b} = x_1 e^{x_1 w + b}$ $\quad \frac{\partial f}{\partial b} = e^{x_1 w + b}$

$\frac{\partial^2 f}{\partial b^2} = e^{x_1 w + b}$ $\quad \frac{\partial f}{\partial b \partial w} = x_1 e^{x_1 w + b}$.

Then we have $\quad$ Let $y = e^{x_1 w + b}$

$$H(w,b) = \nabla^2 f(w,b) = \begin{pmatrix} x_1^2 e^{x_1 w + b} & x_1 e^{x_1 w + b} \\ x_1 e^{x_1 w + b} & e^{x_1 w + b} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^2 y & x_1 y \\ x_1 y & y \end{pmatrix} = y \begin{pmatrix} x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix}$$

for $x = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$, $\quad f(w,b)\, x x^T$

$$= y \begin{pmatrix} x_1 \\ 1 \end{pmatrix} (x_1 \;\; 1) \qquad 2\times1 \times 1\times 2 \Rightarrow 2\times 2$$

$$= y \begin{pmatrix} x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix} = H(w,b).$$

2. Let $v \in \mathbb{R}^2$ i.e. $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

consider

$$v^T H(w,b) v = (v_1 \;\; v_2) \begin{pmatrix} x_1^2 y & x_1 y \\ x_1 y & y \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= (v_1 \;\; v_2) \begin{pmatrix} v_1 x_1^2 y + v_2 x_1 y \\ v_1 x_1 y + v_2 y \end{pmatrix}$$

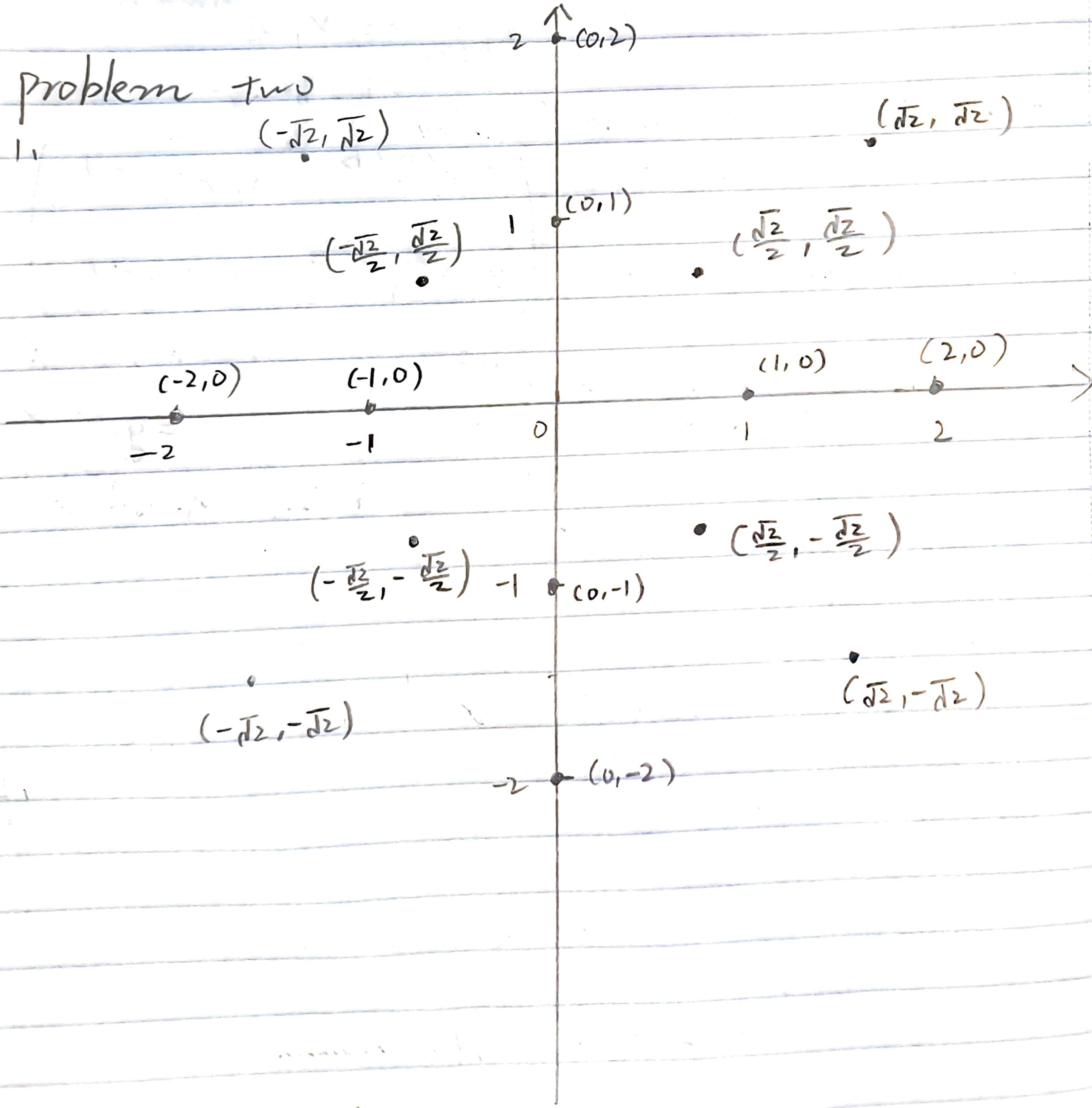
$$v^T H(w,b) v = v_1^2 x_1^2 y + 2 v_1 v_2 x_1 y + v_2^2 y$$
$$= y(v_1^2 x_1^2 + v_2^2 + 2 v_1 v_2 x_1)$$
$$= y(v_1 x_1 + v_2)^2$$

since for any $(w,b) \in \mathbb{R}^2$, $y = e^{x_1 w + b} > 0$,

$$v^T H(w,b) v \geq 0 \quad \forall (w,b) \in \mathbb{R}^2, v \in \mathbb{R}^2$$

problem two

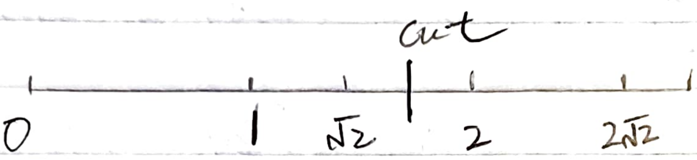
1.



2. want| A map $\varphi: \mathbb{R}^2 \to \mathbb{R}$ s.t. $\varphi(A_1)$ and $\varphi(A_2)$ are linearly separable.

define φ as $\varphi(x,y) = |x|+|y| \quad \forall (x,y) \in \mathbb{R}^2$.

Then $\varphi(A_1) = \{1, \sqrt{2}\}$ $\quad \varphi(A_2) = \{2, 2\sqrt{2}\}$.



from the graph, it is easy to see that $\varphi(A_1)$ and $\varphi(A_2)$ are linearly separable by a plane (in $\mathbb{R}$, hyperplane is a cut (or a pt)).

or we claim $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ separates $\varphi(A_1), \varphi(A_2)$;)

Let $w' = 1-3 = -2$ $\quad b' = 4-1 = 3$.

then $h = xw' + b' > 0 \quad \forall x \in \varphi(A_1)$ ($x=1, h=1$;

$h=\sqrt{2}, h = 3-2\sqrt{2} > 0$)

$h = xw' + b' < 0 \quad \forall x \in \varphi(A_2)$ ($x=2, h=-1$;

$x = 2\sqrt{2}$, $3-4\sqrt{2} < 0$)

$\Rightarrow (w,b)$ separates $\varphi(A_1), \varphi(A_2)$.

$\Rightarrow \varphi(A_1)\, \varphi(A_2)$ are linearly separable. QED//

Problem 3.

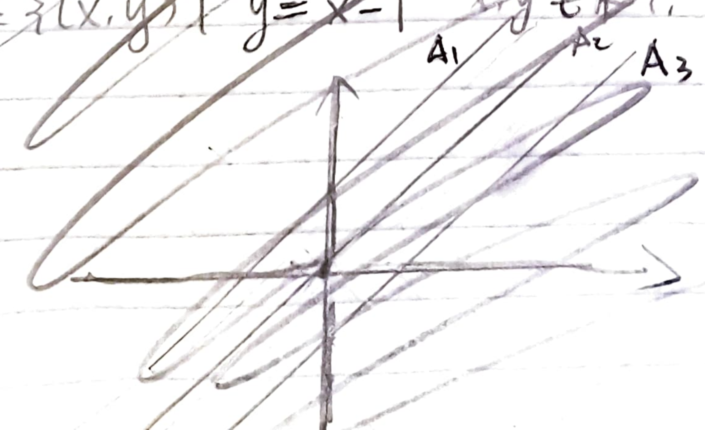
Given set A_1, A_2, A_3. They are all-vs-one linearly separable.

want| They are linearly separable.

All-vs-one separable $\Rightarrow \exists\ w_1, w_2, w_3, b_1, b_2, b_3 \ldots$
s.t. $w_i x + b_i$ separates A_i from $\bigcup_{j \neq i} A_j$.
WLOG, we assume $i=1$, then we have
$w_1 x + b_1$ separates A_1 from $A_2 \cup A_3$.
By definition of linearly separable for $k=2$, we have
$w_1 x + b_1 > 0$ for $x \in A_1$ and $w_1 x + b_1 < 0$ for $x \in A_i$
where $i = 2$ or 3.
By repeating the same process above for $i=2$ and
$i=3$, we get
$w_i x + b_i > 0$ for $x \in A_i$ and $w_i x + b_i < 0$ for $x \in A_j$
where $j \neq i$.
Let $W = (w_1^T, w_2^T, w_3^T)^T \quad b = (b_1, b_2, b_3)^T$
Then we have $(Wx+b)_i > 0$ and $(Wx+b)_j < 0$
for $x \in A_i$ and $j \neq i$.
i.e. $(Wx+b)_i > 0 > (Wx+b)_j \quad \forall x \in A_i,\ i \neq j$.
$\Rightarrow A_1, A_2, A_3$ are linearly separable QED

~~Problem 4~~

~~Let $A_1 = \{(x,y) \mid y = x,\ x,y \in \mathbb{R}\}$~~
~~$A_2 = \{(x,y) \mid y = x+1,\ x,y \in \mathbb{R}\}$~~
~~$A_3 = \{(x,y) \mid y = x-1,\ x,y \in \mathbb{R}\}$~~

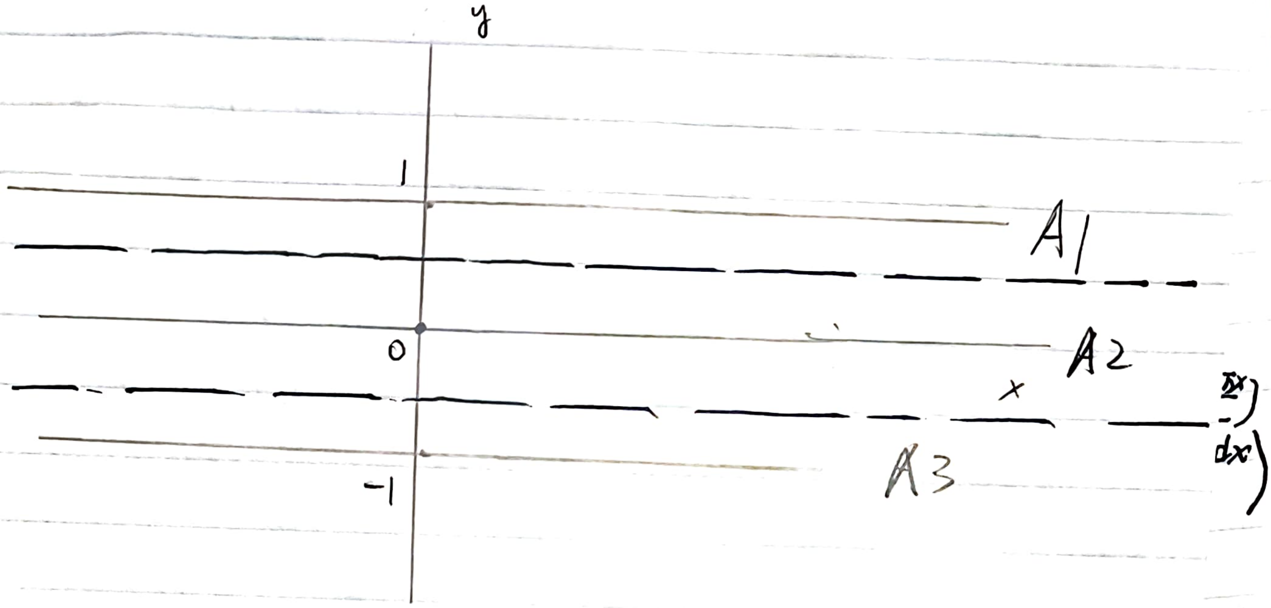


Problem 4

$A_1 = \{(x,y) \mid y=1, x\in\mathbb{R}\}$

$A_2 = \{(x,y) \mid y=0, x\in\mathbb{R}\}$

$A_3 = \{(x,y) \mid y=-1, x\in\mathbb{R}\}$



see the graph it is obivious that $\nexists$ a single plane can separate A_2 from $A_1 \cup A_3$.

Formally, if $\exists$ $w=(w_1, w_2)$, $b\in\mathbb{R}$ (fixed) separates A_2 from $A_1\cup A_3$, then we have

$w_1x + w_2y + b > 0 \quad \forall (x,y)\in A_2$

$w_1x + w_2y + b < 0 \quad \forall (x,y)\in A_1\cup A_3$.

Since all pts in A_2 have 0 y-coordinate, i.e. we have

$w_1 x + b > 0 \quad \forall x\in\mathbb{R}$, this implies $w_1 = 0$.

if $w_1 > 0$, find $x<0$ $x\in\mathbb{N}$ s.t. $w_1|x| > b$ (we can can always find it by the archemidean property)

then $w_1x + b < 0$ $(\rightarrow\leftarrow)$

Similar proof for $w_1<0$.

Also $b>0$. to satisfy ~~the~~ condition $w_1x+b>0$.

Jinnan Li

3.4.3 (b) $\int f_{odd}(x) = \sum_{n=1}^{\infty} B_n <x, \sin\frac{n\pi x}{}$

Now consider $(x,1) \in A_1$, we have
$w_2 + b < 0$. i.e. $w_2 < 0$
But then for $(x,-1) \in A_3$, we will have
$-w_2 + b > 0$ since both $-w_2 > 0$ and $b > 0$
$(\Rightarrow\Leftarrow)$
$\Rightarrow \nexists (w, b)$ separates A_2 from $A_1 \cup A_2$
$\Rightarrow$ not All-vs-one linearly separable,
But A_1, A_2, A_3 are linearly separable,
see the graph (two lines in btw A_1 and A_2, and A_2 and A_3 parallel to A_2.)
or we can claim $W = \begin{pmatrix} 0 & 3 \\ 0 & 1 \\ 0 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ separate
A_1, A_2, A_3 linearly.

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w.