HW Week 1: Linear Machine Learning Models
(*due date: 05/22/2020*)

MATH 497 Summer 2020

Remark:

- You can choose any one format of Handwriting, LaTex, Microsoft Word, Mac Pages or Jupyter to finish this homework and then update to Canvas. (If you use Microsoft Word, Mac Pages or other softwares, please transfer your file to PDF version.)
- Please update the file with name "HWWeek1_YourName.pdf" or "HWWeek1_YourName.ipynb" (if you use Jupyter).

Problem 1 (3 pts) Consider $w, b \in \mathbb{R}$ and the function that

$$f(w,b) = e^{x_1 w + b},$$

for $x_1 \in \mathbb{R}$.

1. (**2 pts**) Consider the Hessian matrix of f defined by

$$H(w,b) = \nabla^2 f(w,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial w^2} & \frac{\partial^2 f}{\partial w \partial b} \\ \frac{\partial^2 f}{\partial b \partial w} & \frac{\partial^2 f}{\partial b^2} \end{pmatrix}.$$

   Verify that

$$H(w,b) = f(w,b)\boldsymbol{x}\boldsymbol{x}^T,$$

   where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}.$$

2. (**1 pts**) Prove that

$$v^T H(w,b) v \geq 0,$$

   for any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ and $(w,b) \in \mathbb{R}^2$.

Problem 2 (3 pts) Think about the next data set $A_1, A_2 \subset \mathbb{R}^2$ with

$$A_1 = \left\{(0,1), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (1,0), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-1,0), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (0,-1), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\right\},$$

and

$$A_2 = \left\{(0,2), (\sqrt{2}, \sqrt{2}), (2,0), (-\sqrt{2}, \sqrt{2}), (-2,0), (-\sqrt{2}, -\sqrt{2}), (0,-2), (\sqrt{2}, -\sqrt{2})\right\}.$$

1. (**1 pt**) Plot out A_1 and A_2 on $\mathbb{R}^2$.
2. (**2 pts**) Find a mapping
$$\varphi : \mathbb{R}^2 \mapsto \mathbb{R},$$
such that $\varphi(A_1)$ and $\varphi(A_2)$ are linearly separable where
$$\varphi(A_1) := \{\tilde{x} = \varphi(x) \mid x \in A_1\} \quad \text{and} \quad \varphi(A_2) := \{\tilde{x} = \varphi(x) \mid x \in A_2\}.$$

Problem 3 (2 pts) Prove that if A_1, A_2, A_3 are all-vs-one linearly separable, then they are linearly separable.

Problem 4 (2 pts) Give an example of $A_1, A_2, A_3 \subset \mathbb{R}^2$ such that they are linearly separable but not all-vs-one linearly separable.