Homework 5

Computer Science Theory for the Information Age

致远 12 级 ACM 班　　刘爽　　5112409048

June 17, 2013

1. Let x_i, $1 \geq i \geq n$, be a set of indicator variables with identical probability distributions. Let $x = \sum_{i=1}^{n} x_i$ and suppose $E(x) \to \inf$. Show that if the x_i are statistically independent then $Prob(x = 0) \to 0$.

   Proof:

   We can use the Chabyshev inequality

   $$Var(x) = \sum_{i=1}^{n} Var(x_i) = C$$

   $$\begin{aligned} P(x = 0) &\leq P\big(|x - E(x)| \geq E(x)\big) \\ &\leq \frac{Var(x)}{E^2(x)} \\ &= \frac{C}{E^2(x)} \end{aligned}$$

   As $E(x) \to \inf$, $P(x = 0) \to 0$.

   $\square$

2. Consider a model of random subset $N(n,p)$ of integers $\{1, 2, \dots, n\}$ where, $N(n,p)$ is the set obtained by independently at random including each of $\{1, 2, \dots, n\}$ into the set with probability p. Define what an "increasing property" of $N(n,p)$ means. Prove that every increasing peroperty of $N(n,p)$ has a threshold.

Solution:

We define an increasing property of $N(n,p)$ as follows:

> Q is an increasing property of N if when a set N has the property any set obtained by adding numbers to N must also have the property.

First we will proof the following lemma:

> If Q is an increasing property and $0 \leq p \leq q \leq 1$, then the probability that $N(n,p)$ has property Q is less than or equal to the probability that $N(n,q)$ has property Q.

Notice we could generate $N(n,q)$ in this way: First we generate $N(n,p)$, then we generate a set $N(n, \frac{q-p}{1-p})$ and take the union of them to get $N(n,q)$. If $N(n,p)$ has the property Q, $N(n,q)$ must has it too.

Next we will prove

> Every increasing property of $N(n,p)$ has a threshold at $p(n)$, where for each n, $p(n)$ is the minimum real number a for which the probability that $P(n,a)$ has the property Q is $\frac{1}{2}$.

Suppose $p_0(n)$ is any function such that

$$\lim_{n \to \inf} \frac{p_0(n)}{p(n)} = 0$$

We will show that almost surely $N(n,p_0)$ does not have property Q. Suppose this is false. Then, the probability that $N(n,p_0)$ has the property Q does not converge to zero. By the definition of limit, there must be a positive real number ϵ for which the probability that $N(n,p_0)$ has property Q is at least ϵ on an infinite set I of n.

Let $m = \lceil (1/\epsilon) \rceil$. Lset h be the m-fold replication of $N(n,p_0)$. Since from the m-fold method we have

$$Prob\big(N(n,mp) \text{ does not have Q}\big) \leq \Big(Prob\big(N(n,p) \text{ does not have Q}\big)\Big)^m$$

Then the probability that H does not have Q is at most $(1-\epsilon)^m \leq e^{-1} \leq 1/2$ for all $n \in I$. So for thse n, since $p(n)$ is the minimum real number a for which the probability that $N(n, a)$ has property Q is $1/2$, $mp_0 \geq p(n)$. This implies that $\frac{p_0(n)}{p(n)}$ is at leat $1/m$ infinitely often contradicting the hypothesis that $\lim_{n \to \text{inf}} \frac{p_0(n)}{p(n)} = 0$. A symmetric argument shows that for any $p_1(n)$ such that $\frac{p(n)}{p_1(n)} \to 0$, $N(n, p_1)$ almost surely has property Q.