Homework 7

Computer Science Theory for the Information Age

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1. Suppose A is an $n \times n$ matrix with block diagonal structure with k equal size blocks where all entries of the i^{th} block are a_i with $a_1 > a_2 > \cdots > a_k > 0$. Show that A has exactl k nonzero singular vectors $v_1, v_2, \ldots, v_k$ where v_i has the value $(\frac{k}{n})^{1/2}$ in the coordinates corresponding to the i^{th} block and 0 elsewhere. In other words, the singular vectors exactly indentify the blocks of the diagnal. What happens if $a_1 = a_2 = \cdots = a_k$? In the case where the a_i are equal, what is the structure of the set of all possible singular vectors?

   Hint: By symmetry, the top singular vector's components must be constant in each block.

   Solution:

   In fact, v_i are the normalized eigenvectors of $A^T A$, which is an matrix with block diagnal structure the same as A where all entries of the i^{th} block are $\frac{n}{k}a_i^2$. And the i^{th} block is a rank-1 matrix with all elements taking the same value, which has only one normalized $\frac{n}{k}$ dimension eigenvector $\left((\frac{k}{n})^{1/2}, (\frac{k}{n})^{1/2}, \ldots, (\frac{k}{n})^{1/2}\right)$, then $A^T A$ has k normalized eigenvectors v_i where v_i has the value $(\frac{k}{n})^{1/2}$ in the coordinates corresponding to the i^{th} block and 0 elsewhere.

   Then we notice that the k eigenvectors of $A^T A$ are linear independent. And on the other hand $A^T A$ is a rank-k matrix, it has at most k linear-independent eigenvectors. If all a_i are different, then $A^T A$ has k eigen-subspace. The k normalized eigenvectors are the singularvectors we want. If all a_i are the same, then $A^T A$ has a k-dimension eigen-subspace which is the span of v_i. Thus any orthonomal basis of this k-dimension subspace can be treat as the set of singular-vactors of A.

2. Computer the singular value decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution:

$$A^T A = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

First we can solve the eigenvalues of $A^T A$

$$\lambda_1 = 29.866, \quad \lambda_2 = 0.134$$

Then we can get the normalized eigenvectors

$$v_1 = (0.576, 0.817)^T$$
$$v_2 = (-0.817, 0.576)^T$$

and $\delta_1 = \sqrt{\lambda_1} = 5.465$, $\delta_2 = \sqrt{\lambda_2} = 0.366$

Finally

$$u_1 = \frac{Av_1}{\delta_1} = (0.405, 0.915)^T$$
$$u_2 = \frac{Av_2}{\delta_2} = (0.915, -0.405)^T$$

We have

$$A = UDV^T = \begin{pmatrix} 0.405 & 0.915 \\ 0.915 & -0.405 \end{pmatrix} \begin{pmatrix} 5.465 & 0 \\ 0 & 0.366 \end{pmatrix} \begin{pmatrix} 0.576 & 0.817 \\ -0.817 & 0.576 \end{pmatrix}$$