

# Homework 5

## Computer Science Theory for the Information Age

致远 12 级 ACM 班

刘爽

5112409048

June 17, 2013

1. Let  $x_i, 1 \leq i \leq n$ , be a set of indicator variables with identical probability distributions. Let  $x = \sum_{i=1}^n x_i$  and suppose  $E(x) \rightarrow \infty$ . Show that if the  $x_i$  are statistically independent then  $\text{Prob}(x = 0) \rightarrow 0$ .

***Proof:***

We can use the Chabyshev inequality

$$\text{Var}(x) = \sum_{i=1}^n \text{Var}(x_i) = C$$

$$\begin{aligned} P(x = 0) &\leq P(|x - E(x)| \geq E(x)) \\ &\leq \frac{\text{Var}(x)}{E^2(x)} \\ &= \frac{C}{E^2(x)} \end{aligned}$$

As  $E(x) \rightarrow \infty$ ,  $P(x = 0) \rightarrow 0$ .

□

2. Consider a model of random subset  $N(n, p)$  of integers  $\{1, 2, \dots, n\}$  where,  $N(n, p)$  is the set obtained by independently at random including each of  $\{1, 2, \dots, n\}$  into the set with probability  $p$ . Define what an "increasing property" of  $N(n, p)$  means. Prove that every increasing property of  $N(n, p)$  has a threshold.

**Solution:**

We define an increasing property of  $N(n, p)$  as follows:

$Q$  is an increasing property of  $N$  if when a set  $N$  has the property any set obtained by adding numbers to  $N$  must also have the property.

First we will prove the following lemma:

If  $Q$  is an increasing property and  $0 \leq p \leq q \leq 1$ , then the probability that  $N(n, p)$  has property  $Q$  is less than or equal to the probability that  $N(n, q)$  has property  $Q$ .

Notice we could generate  $N(n, q)$  in this way: First we generate  $N(n, p)$ , then we generate a set  $N(n, \frac{q-p}{1-p})$  and take the union of them to get  $N(n, q)$ . If  $N(n, p)$  has the property  $Q$ ,  $N(n, q)$  must have it too.

Next we will prove

Every increasing property of  $N(n, p)$  has a threshold at  $p(n)$ , where for each  $n$ ,  $p(n)$  is the minimum real number  $a$  for which the probability that  $N(n, a)$  has the property  $Q$  is  $\frac{1}{2}$ .

Suppose  $p_0(n)$  is any function such that

$$\lim_{n \rightarrow \infty} \frac{p_0(n)}{p(n)} = 0$$

We will show that almost surely  $N(n, p_0)$  does not have property  $Q$ . Suppose this is false. Then, the probability that  $N(n, p_0)$  has the property  $Q$  does not converge to zero. By the definition of limit, there must be a positive real number  $\epsilon$  for which the probability that  $N(n, p_0)$  has property  $Q$  is at least  $\epsilon$  on an infinite set  $I$  of  $n$ .

Let  $m = \lceil (1/\epsilon) \rceil$ . Let  $h$  be the  $m$ -fold replication of  $N(n, p_0)$ . Since from the  $m$ -fold method we have

$$\text{Prob}(N(n, mp) \text{ does not have } Q) \leq \left( \text{Prob}(N(n, p) \text{ does not have } Q) \right)^m$$

Then the probability that  $h$  does not have  $Q$  is at most  $(1 - \epsilon)^m \leq e^{-1} \leq 1/2$  for all  $n \in I$ . So for these  $n$ , since  $p(n)$  is the minimum real number  $a$  for which the probability

that  $N(n, a)$  has property  $Q$  is  $1/2$ ,  $mp_0 \geq p(n)$ . This implies that  $\frac{p_0(n)}{p(n)}$  is at least  $1/m$  infinitely often contradicting the hypothesis that  $\lim_{n \rightarrow \infty} \frac{p_0(n)}{p(n)} = 0$ . A symmetric argument shows that for any  $p_1(n)$  such that  $\frac{p(n)}{p_1(n)} \rightarrow 0$ ,  $N(n, p_1)$  almost surely has property  $Q$ .