Homework 1

Computer Science Theory for the Information Age

致远 12 级 ACM 班　　刘爽　　5112409048

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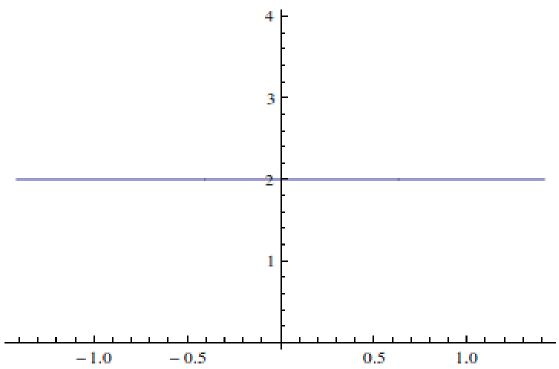
1. Project the surface area of a sphere of radius $\sqrt{d}$ in d-dimensions onto a line through the center. For d equal 2 and 3, derive an explicit formula for how the projected surface area changes as we move along the line. For large d, argue(intuitively)that the projected surface area should behave like a Gaussian.

   Solution: For any d greater than 1, the projected surface area of a d-dimensional along the line takes non-zero value for $x \in (-\sqrt{d}, \sqrt{d})$, has has the formula

$$(d - x^2)^{\frac{d-2}{2}} A(d-1)$$

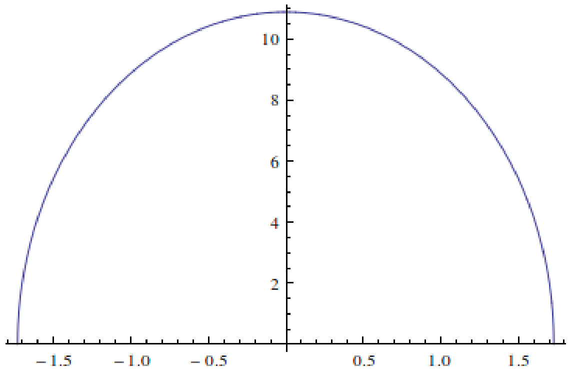
   for $d = 2$, the fomula is

$$x = 2$$



for $d = 3$, the fomula is

$$2\pi\sqrt{3 - x^2}$$

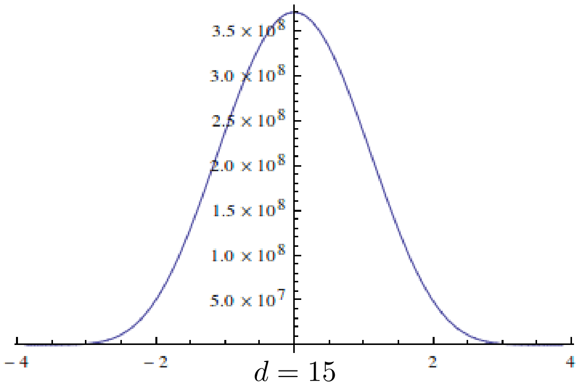


and for large d, first we can observe that it is symmetric bout the origin. Then we calculate the first and second derivatives of the function

$$\frac{dy}{dx} = C_1(-x)(d - x^2)^{\frac{d-4}{2}} \tag{1}$$

$$\frac{d^2y}{dx^2} = C_2(d - x^2)^{\frac{d-6}{2}}\left((d-3)x^2 - d\right) \tag{2}$$

it is strictly increasing when $x < 0$, and strictly decreasing when $x > 0$.In addition, it has inflection point at $x = \pm\sqrt{\frac{d}{d-3}}$, also it is concave down near origin and vice versa. Therefore it looks like a Gaussion for large d.



$d = 15$

2. For what value of d is the volume, $V(d)$, of a d-dimensional unit sphere maximum? (Hint: Consider the ratio $\frac{V(d)}{V(d-1)}$).

Solution: First,

$$\frac{V(d)}{V(d-2)} = \frac{(d-2)\sqrt{\pi}\Gamma(\frac{d-2}{2})}{d\Gamma(\frac{d}{2})}, \text{for } d \geq 1$$

for $d = 2k, k \geq 2$,

$$\frac{V(d)}{V(d-2)} = \frac{(k-1)\pi\Gamma(k-1)}{k\Gamma(k)} = \frac{\pi}{k}$$

then

$$\arg\max_k V(2k) = V(6) = \frac{\pi^3}{6}$$

for $d = 2k+1, k \geq 1$,

$$\frac{V(d)}{V(d-2)} = \frac{(2k-1)\pi\Gamma(\frac{2k-1}{2})}{(2k+1)\Gamma(\frac{2k+1}{2})} = \frac{2\pi}{2k+1}$$

then

$$\arg\max_k V(2k+1) = V(5) = \frac{8\pi^2}{15}$$

thus

$$\arg\max_d V(d) = \max(V(5), V(6)) = V(5)$$