Homework 5

Computer Science Theory for the Information Age

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1. Let x_i, $1 \geq i \geq n$, be a set of indicator variables with identical probability distributions. Let $x = \sum_{i=1}^{n} x_i$ and suppose $E(x) \to \infty$. Show that if the x_i are statistically independent then $Prob(x = 0) \to 0$.

   Proof:

   The only way $E(x) \to \infty$ is $n \to \infty$.

   We can use the Chabyshev inequality

$$Var(x) = \sum_{i=1}^{n} Var(x_i) = nC_1$$

$$E(x) = \sum_{i=1}^{n} E(x_i) = nC_2$$

$$\begin{aligned} P(x = 0) &\leq P\big(|x - E(x)| \geq E(x)\big) \\ &\leq \frac{Var(x)}{E^2(x)} \\ &= \frac{nC_1}{n^2 C_2^2} \\ &= \frac{C_1}{nC_2^2} \end{aligned}$$

   As $E(x) \to \infty$, which means $n \to \infty$, $P(x = 0) \to 0$.

$\square$

2. Consider a model of random subset $N(n,p)$ of integers $\{1,2,\ldots,n\}$ where, $N(n,p)$ is the set obtained by independently at random including each of $\{1,2,\ldots,n\}$ into the set with probability p. Define what an "increasing property" of $N(n,p)$ means. Prove that every increasing peroperty of $N(n,p)$ has a threshold.

   Solution:

   We define an increasing property of $N(n,p)$ as follows:

   > Q is an increasing property of N if when a set N has the property any set obtained by adding numbers to N must also have the property.

   First we will proof the following lemma:

   > If Q is an increasing property and $0 \leq p \leq q \leq 1$, then the probability that $N(n,p)$ has property Q is less than or equal to the probability that $N(n,q)$ has property Q.

   Notice we could generate $N(n,q)$ in this way: First we generate $N(n,p)$, then we generate a set $N(n, \frac{q-p}{1-p})$ and take the union of them to get $N(n,q)$. If $N(n,p)$ has the property Q, $N(n,q)$ must has it too.

   Next we will prove

   > Every increasing property of $N(n,p)$ has a threshold at $p(n)$, where for each n, $p(n)$ is the minimum real number a for which the probability that $P(n,a)$ has the property Q is $\frac{1}{2}$.

   Suppose $p_0(n)$ is any function such that

   $$\lim_{n\to\infty} \frac{p_0(n)}{p(n)} = 0$$

   We will show that almost surely $N(n,p_0)$ does not have property Q. Suppose this is false. Then, the probability that $N(n,p_0)$ has the property Q does not converge to zero. By the definition of limit, there must be a positive real number ϵ for which the probability that $N(n,p_0)$ has property Q is at least ϵ on an inftyinite set I of n.

   Let $m = \lceil (1/\epsilon) \rceil$. Lset h be the m-fold replication of $N(n,p_0)$. Since from the m-fold method we have

   $$Prob\big(N(n,mp) \text{ does not have Q}\big) \leq \Big(Prob\big(N(n,p) \text{ does not have Q}\big)\Big)^m$$

Then the probability that H does not have Q is at most $(1-\epsilon)^m \le e^{-1} \le 1/2$ for all $n \in I$. So for these n, since $p(n)$ is the minimum real number a for which the probability that $N(n, a)$ has property Q is $1/2$, $mp_0 \ge p(n)$. This implies that $\frac{p_0(n)}{p(n)}$ is at leat $1/m$ inftyinitely often contradicting the hypothesis that $\lim\limits_{n\to\infty} \frac{p_0(n)}{p(n)} = 0$. A symmetric argument shows that for any $p_1(n)$ such that $\frac{p(n)}{p_1(n)} \to 0$, $N(n, p_1)$ almost surely has property Q.