

EXERCISE 11.1

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Differentiate the following functions from the first principles:

1. e^{-x}

Solution:

We have to find the derivative of e^{-x} with the first principle method,

So let $f(x) = e^{-x}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{-(x+h)} - e^{-x}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{-x}(e^{-h} - 1)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{-x}(e^{-h} - 1)(-1)}{h(-1)}$$

$$[\text{By using } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1]$$

$$f'(x) = -e^{-x}$$

2. e^{3x}

Solution:

We have to find the derivative of e^{3x} with the first principle method,

So, let $f(x) = e^{3x}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

So, let $f(x) = e^{\cos x}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\cos(x+h)} - e^{\cos x}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\cos x} (e^{\cos(x+h) - \cos x} - 1)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\cos x} (e^{\cos(x+h) - \cos x} - 1)}{\cos(x+h) - \cos x} \cdot \frac{\cos(x+h) - \cos x}{h}$$

[By using $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$]

$$f'(x) = \lim_{h \rightarrow 0} e^{\cos x} \frac{\cos(x+h) - \cos x}{h}$$

Now by using $\cos(x+h) = \cos x \cos h - \sin x \sin h$

$$f'(x) = \lim_{h \rightarrow 0} e^{\cos x} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} e^{\cos x} \left[\frac{\cos x (\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right]$$

[By using $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\cos 2x = 1 - 2\sin^2 x$]

$$f'(x) = \lim_{h \rightarrow 0} e^{\cos x} \left[\frac{\cos x \left(-2\sin^2 \frac{h}{2} \right) \left(\frac{h}{4} \right)}{h \left(\frac{h}{4} \right)} - \sin x \right]$$

$$f'(x) = \lim_{h \rightarrow 0} e^{\cos x} \left[\frac{\cos x \left(-2\sin^2 \frac{h}{2} \right) \left(\frac{h}{4} \right)}{\frac{h^2}{2}} - \sin x \right]$$

$$f'(x) = -e^{\cos x} \sin x$$

5. $e^{\sqrt{2x}}$

Solution:

We have to find the derivative of $e^{\sqrt{2x}}$ with the first principle method,

So, let $f(x) = e^{\sqrt{2x}}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\sqrt{2(x+h)}} - e^{\sqrt{2x}}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)} - \sqrt{2x}} - 1)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)} - \sqrt{2x}} - 1)}{h} \times \frac{\sqrt{2(x+h)} - \sqrt{2x}}{\sqrt{2(x+h)} - \sqrt{2x}}$$

$$\left[\text{By using } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\sqrt{2x}}}{h} \times (\sqrt{2(x+h)} - \sqrt{2x}) \times \frac{\sqrt{2(x+h)} + \sqrt{2x}}{\sqrt{2(x+h)} + \sqrt{2x}}$$

[By rationalizing]

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{\sqrt{2x}}}{h} \times \frac{(2(x+h) - 2x)}{\sqrt{2(x+h)} + \sqrt{2x}}$$

$$f'(x) = \frac{e^{\sqrt{2x}}}{\sqrt{2x}}$$

EXERCISE 11.2

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Differentiate the following functions with respect to x:**1. Sin (3x + 5)****Solution:**

Given Sin (3x + 5)

Let $y = \sin (3x + 5)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx} [\sin(3x + 5)]$$

We know $\frac{d}{dx} (\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \cos(3x + 5) \frac{d}{dx} (3x + 5) \quad [\text{Using chain rule}]$$

$$\Rightarrow \frac{dy}{dx} = \cos(3x + 5) \left[\frac{d}{dx} (3x) + \frac{d}{dx} (5) \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos(3x + 5) \left[3 \frac{d}{dx} (x) + \frac{d}{dx} (5) \right]$$

However, $\frac{d}{dx} (x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos(3x + 5) [3 \times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 3 \cos(3x + 5)$$

$$\text{Thus, } \frac{d}{dx} [\sin(3x + 5)] = 3 \cos(3x + 5)$$

2. tan² x**Solution:**Given tan² x

$$\text{Let } y = \tan^2 x$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^2 x)$$

$$\text{We know } \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = 2 \tan^{2-1} x \frac{d}{dx}(\tan x) \quad [\text{Using chain rule}]$$

$$\Rightarrow \frac{dy}{dx} = 2 \tan x \frac{d}{dx}(\tan x)$$

$$\text{However, } \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = 2 \tan x (\sec^2 x)$$

$$\therefore \frac{dy}{dx} = 2 \tan x \sec^2 x$$

$$\text{Thus, } \frac{d}{dx}(\tan^2 x) = 2 \tan x \sec^2 x$$

3. $\tan(x^\circ + 45^\circ)$

Solution:

$$\text{Let } y = \tan(x^\circ + 45^\circ)$$

First, we will convert the angle from degrees to radians.

$$\text{Let } y = \tan(x^\circ + 45^\circ)$$

First, we will convert the angle from degrees to radians.

$$\text{We have } 1^\circ = \left(\frac{\pi}{180}\right)^c \Rightarrow (x + 45)^\circ = \left[\frac{(x+45)\pi}{180}\right]^c$$

$$\Rightarrow y = \tan\left[\frac{(x + 45)\pi}{180}\right]$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left\{ \tan \left[\frac{(x+45)\pi}{180} \right] \right\}$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \sec^2 \left[\frac{(x+45)\pi}{180} \right] \frac{d}{dx} \left[\frac{(x+45)\pi}{180} \right] \text{ [Using chain rule]}$$

$$\Rightarrow \frac{dy}{dx} = \sec^2(x^\circ + 45^\circ) \frac{\pi}{180} \frac{d}{dx}(x+45)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180} \sec^2(x^\circ + 45^\circ) \left[\frac{d}{dx}(x) + \frac{d}{dx}(45) \right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180} \sec^2(x^\circ + 45^\circ) [1 + 0]$$

$$\therefore \frac{dy}{dx} = \frac{\pi}{180} \sec^2(x^\circ + 45^\circ)$$

$$\text{Thus, } \frac{d}{dx} [\tan(x^\circ + 45^\circ)] = \frac{\pi}{180} \sec^2(x^\circ + 45^\circ)$$

4. Sin (log x)

Solution:

Given $\sin(\log x)$

Let $y = \sin(\log x)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} [\sin(\log x)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \cos(\log x) \frac{d}{dx}(\log x) \text{ [Using chain rule]}$$

$$\text{However, } \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \cos(\log x)$$

$$\text{Thus, } \frac{d}{dx}[\sin(\log x)] = \frac{1}{x} \cos(\log x)$$

5. $e^{\sin \sqrt{x}}$

Solution:

$$\text{Let } y = e^{\sin \sqrt{x}}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin \sqrt{x}})$$

$$\text{We know } \frac{d}{dx}(e^x) = e^x$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin \sqrt{x}} \frac{d}{dx}(\sin \sqrt{x}) \text{ [Using chain rule]}$$

$$\text{We have } \frac{d}{dx}(\sin x) = \cos x$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin \sqrt{x}} \cos \sqrt{x} \frac{d}{dx}(\sqrt{x}) \text{ [Using chain rule]}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin \sqrt{x}} \cos \sqrt{x} \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$\text{However, } \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin \sqrt{x}} \cos \sqrt{x} \left[\frac{1}{2} x^{\left(\frac{1}{2}-1\right)} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} e^{\sin \sqrt{x}} \cos \sqrt{x} x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} e^{\sin \sqrt{x}} \cos \sqrt{x}$$

$$\text{Thus, } \frac{d}{dx} (e^{\sin \sqrt{x}}) = \frac{1}{2\sqrt{x}} e^{\sin \sqrt{x}} \cos \sqrt{x}$$

6. $e^{\tan x}$

Solution:

$$\text{Let } y = e^{\tan x}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} (e^{\tan x})$$

$$\text{We know } \frac{d}{dx} (e^x) = e^x$$

$$\Rightarrow \frac{dy}{dx} = e^{\tan x} \frac{d}{dx} (\tan x) \quad [\text{Using chain rule}]$$

$$\text{We have } \frac{d}{dx} (\tan x) = \sec^2 x$$

$$\therefore \frac{dy}{dx} = e^{\tan x} \sec^2 x$$

$$\text{Thus, } \frac{d}{dx} (e^{\tan x}) = e^{\tan x} \sec^2 x$$

7. $\sin^2 (2x + 1)$

Solution:

$$\text{Let } y = \sin^2 (2x + 1)$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin^2(2x + 1)]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Using chain rule we get.

$$\Rightarrow \frac{dy}{dx} = 2 \sin^{2-1}(2x + 1) \frac{d}{dx}[\sin(2x + 1)]$$

$$\Rightarrow \frac{dy}{dx} = 2 \sin(2x + 1) \frac{d}{dx}[\sin(2x + 1)]$$

We have $\frac{d}{dx}(\sin x) = \cos x$

Now by using chain rule we have

$$\Rightarrow \frac{dy}{dx} = 2 \sin(2x + 1) \cos(2x + 1) \frac{d}{dx}(2x + 1)$$

$$\Rightarrow \frac{dy}{dx} = \sin[2(2x + 1)] \frac{d}{dx}(2x + 1) \quad [\because \sin(2\theta) = 2\sin\theta\cos\theta]$$

$$\Rightarrow \frac{dy}{dx} = \sin(4x + 2) \left[\frac{d}{dx}(2x) + \frac{d}{dx}(1) \right]$$

$$\Rightarrow \frac{dy}{dx} = \sin(4x + 2) \left[2 \frac{d}{dx}(x) + \frac{d}{dx}(1) \right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \sin(4x + 2) [2 \times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 2 \sin(4x + 2)$$

Thus, $\frac{d}{dx}[\sin^2(2x + 1)] = 2 \sin(4x + 2)$

8. $\log_7(2x - 3)$

Solution:

$$\text{Let } y = \log_7(2x - 3)$$

$$\text{We know that } \log_a b = \frac{\log b}{\log a}.$$

$$\Rightarrow \log_7(2x - 3) = \frac{\log(2x - 3)}{\log 7}$$

On differentiating y with respect to x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\frac{\log(2x - 3)}{\log 7} \right] \\ \Rightarrow \frac{dy}{dx} &= \left(\frac{1}{\log 7} \right) \frac{d}{dx} [\log(2x - 3)]\end{aligned}$$

$$\text{We know } \frac{d}{dx} (\log x) = \frac{1}{x}$$

Now by using chain rule we get

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\log 7} \right) \left(\frac{1}{2x-3} \right) \frac{d}{dx} (2x - 3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x - 3) \log 7} \left[\frac{d}{dx} (2x) - \frac{d}{dx} (3) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x - 3) \log 7} \left[2 \frac{d}{dx} (x) - \frac{d}{dx} (3) \right]$$

However, $\frac{d}{dx} (x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x - 3) \log 7} [2 \times 1 - 0]$$

$$\therefore \frac{dy}{dx} = \frac{2}{(2x - 3) \log 7}$$

$$\text{Thus, } \frac{d}{dx} [\log_7(2x - 3)] = \frac{2}{(2x - 3) \log 7}$$

9. $\tan 5x^\circ$

Solution:

Let $y = \tan(5x^\circ)$

First, we will convert the angle from degrees to radians. We have

$$1^\circ = \left(\frac{\pi}{180}\right)^c \Rightarrow 5x^\circ = 5x \times \frac{\pi}{180}^c$$

$$\Rightarrow y = \tan\left(5x \times \frac{\pi}{180}\right)$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan\left(5x \times \frac{\pi}{180}\right)\right]$$

$$\text{We know } \frac{d}{dx}(\tan x) = \sec^2 x$$

Now by using chain rule we have

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(5x \times \frac{\pi}{180}\right) \frac{d}{dx}\left(5x \times \frac{\pi}{180}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2(5x^\circ) \frac{\pi}{180} \frac{d}{dx}(5x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180} \sec^2(5x^\circ) \left[5 \frac{d}{dx}(x)\right]$$

$$\text{However, } \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180} \sec^2(5x^\circ) [5]$$

$$\therefore \frac{dy}{dx} = \frac{5\pi}{180} \sec^2(5x^\circ)$$

$$\text{Thus, } \frac{d}{dx}(\tan 5x^\circ) = \frac{5\pi}{180} \sec^2(5x^\circ)$$

10. 2^{x^3}

Solution:

Let $y = 2^{x^3}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2^{x^3})$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \frac{d}{dx}(x^3)$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \times 3x^{3-1}$$

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \times 3x^2$$

$$\therefore \frac{dy}{dx} = 2^{x^3} 3x^2 \log 2$$

Thus, $\frac{d}{dx}(2^{x^3}) = 2^{x^3} 3x^2 \log 2$

11. 3^{e^x}

Solution:

Let $y = 3^{e^x}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(3^{e^x})$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \frac{d}{dx}(e^x)$$

We have $\frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \times e^x$$

$$\therefore \frac{dy}{dx} = 3^{e^x} e^x \log 3$$

Thus, $\frac{d}{dx}(3^{e^x}) = 3^{e^x} e^x \log 3$

12. $\log_x 3$

Solution:

Let $y = \log_x 3$

We know that $\log_a b = \frac{\log b}{\log a}$.

$$\Rightarrow \log_x 3 = \frac{\log 3}{\log x}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\log 3}{\log x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx} \left(\frac{1}{\log x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx} (\log x)^{-1}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \log 3 [-1 \times (\log x)^{-1-1}] \frac{d}{dx} (\log x)$$

$$\Rightarrow \frac{dy}{dx} = -\log 3 (\log x)^{-2} \frac{d}{dx} (\log x)$$

We have $\frac{d}{dx} (\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = -\log 3 (\log x)^{-2} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1 \log 3}{x (\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1 \log 3}{x (\log x)^2} \times \frac{\log 3}{\log 3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1 (\log 3)^2}{x \log 3 (\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x \log 3} \left(\frac{\log 3}{\log x} \right)^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x \log 3 \times \left(\frac{\log x}{\log 3} \right)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x \log 3 (\log_3 x)^2}$$

Thus, $\frac{d}{dx} (\log_x 3) = -\frac{1}{x \log 3 (\log_3 x)^2}$

13. 3^{x^2+2x}

Solution:

Let $y = 3^{x^2+2x}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(3^{x^2+2x})$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \frac{d}{dx}(x^2 + 2x)$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \left[\frac{d}{dx}(x^2) + 2 \frac{d}{dx}(x) \right]$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 [2x + 2 \times 1]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 (2x + 2)$$

$$\therefore \frac{dy}{dx} = (2x + 2) 3^{x^2+2x} \log 3$$

Thus, $\frac{d}{dx}(3^{x^2+2x}) = (2x + 2) 3^{x^2+2x} \log 3$

14. $\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

Solution:

Let $y = \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{a^2 - x^2}{a^2 + x^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$

Using chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)$$

We know that $\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2) \frac{d}{dx} (a^2 - x^2) - (a^2 - x^2) \frac{d}{dx} (a^2 + x^2)}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2) \left(\frac{d}{dx} (a^2) - \frac{d}{dx} (x^2) \right) - (a^2 - x^2) \left(\frac{d}{dx} (a^2) + \frac{d}{dx} (x^2) \right)}{(a^2 + x^2)^2} \right]$$

However, $\frac{d}{dx} (x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2)(0 - 2x) - (a^2 - x^2)(0 + 2x)}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(a^2 + x^2) - 2x(a^2 - x^2)}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(a^2 + x^2 + a^2 - x^2)}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(2a^2)}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{a^2 - x^2}{a^2 + x^2} \right)^{-\frac{1}{2}} \left[\frac{-2xa^2}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}}} \left[\frac{-2xa^2}{(a^2 + x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}(a^2 - x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}\sqrt{a^2 - x^2}}$$

15. $3^{x \log x}$

Solution:

Let $y = 3^{x \log x}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(3^{x \log x})$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \frac{d}{dx}(x \log x)$$

We know that by product rule $(u v)' = v u' + u v'$

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \frac{d}{dx}(x \times \log x)$$

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \left[\log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x) \right]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \left[\log x \times 1 + x \times \frac{1}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 [\log x + 1]$$

$$\therefore \frac{dy}{dx} = (1 + \log x) 3^{x \log x} \log 3$$

Thus, $\frac{d}{dx}(3^{x \log x}) = (1 + \log x) 3^{x \log x} \log 3$

16. $\sqrt{\frac{1 + \sin x}{1 - \sin x}}$

Solution:

Let $y = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$

On differentiating y with respect to x, we get

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{1 + \sin x}{1 - \sin x}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\left(\frac{1 + \sin x}{1 - \sin x} \right)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{\frac{1}{2} - 1} \frac{d}{dx} \left(\frac{1 + \sin x}{1 - \sin x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1 + \sin x}{1 - \sin x} \right)$$

We know that

$$\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{(1 - \sin x) \frac{d}{dx}(1 + \sin x) - (1 + \sin x) \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{(1 - \sin x) \left(\frac{d}{dx}(1) + \frac{d}{dx}(\sin x) \right) - (1 + \sin x) \left(\frac{d}{dx}(1) - \frac{d}{dx}(\sin x) \right)}{(1 - \sin x)^2} \right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{(1 - \sin x)(0 + \cos x) - (1 + \sin x)(0 - \cos x)}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{(1 - \sin x) \cos x + (1 + \sin x) \cos x}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{(1 - \sin x + 1 + \sin x) \cos x}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{2 \cos x}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1 + \sin x}{1 - \sin x} \right)^{-\frac{1}{2}} \left[\frac{\cos x}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)^{-\frac{1}{2}}}{(1 - \sin x)^{-\frac{1}{2}}} \left[\frac{\cos x}{(1 - \sin x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)^{-\frac{1}{2}} \cos x}{(1 - \sin x)^{-\frac{1}{2} + 2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)^{-\frac{1}{2}} \cos x}{(1 - \sin x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x)^{1 + \frac{1}{2}} (1 + \sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x) (1 - \sin x)^{\frac{1}{2}} (1 + \sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x) \sqrt{(1 - \sin x)(1 + \sin x)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x) \sqrt{1 - \sin^2 x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x) \sqrt{\cos^2 x}} \quad (\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1 - \sin x) \cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - \sin x} \times \frac{1 + \sin x}{1 + \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \sin x}{1 - \sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \sin x}{\cos^2 x} \quad (\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x} \right)^2 + \left(\frac{1}{\cos x} \right) \left(\frac{\sin x}{\cos x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x + \sec x \tan x$$

$$\therefore \frac{dy}{dx} = \sec x (\sec x + \tan x)$$

Thus, $\frac{d}{dx} \left(\sqrt{\frac{1 + \sin x}{1 - \sin x}} \right) = \sec x (\sec x + \tan x)$

17. $\sqrt{\frac{1 + x^2}{1 - x^2}}$

Solution:

Let $y = \sqrt{\frac{1 - x^2}{1 + x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{1-x^2}{1+x^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\left(\frac{1-x^2}{1+x^2} \right)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$

Now by using chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{1-x^2}{1+x^2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1-x^2}{1+x^2} \right)$$

We know that $\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{(1+x^2) \frac{d}{dx} (1-x^2) - (1-x^2) \frac{d}{dx} (1+x^2)}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{(1+x^2) \left(\frac{d}{dx} (1) - \frac{d}{dx} (x^2) \right) - (1-x^2) \left(\frac{d}{dx} (1) + \frac{d}{dx} (x^2) \right)}{(1+x^2)^2} \right]$$

However, $\frac{d}{dx} (x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{(1+x^2)(0 - 2x) - (1-x^2)(0 + 2x)}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(1+x^2+1-x^2)}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x(2)}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-x^2}{1+x^2} \right)^{-\frac{1}{2}} \left[\frac{-2x}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}}} \left[\frac{-2x}{(1+x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$$

$$\text{Thus, } \frac{d}{dx} \left(\sqrt{\frac{1-x^2}{1+x^2}} \right) = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$$

18. $(\log \sin x)^2$

Solution:

Let $y = (\log \sin x)^2$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\log(\sin x))^2]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2(\log(\sin x))^{2-1} \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = 2 \log(\sin x) \frac{d}{dx}[\log(\sin x)]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2 \log(\sin x) \left[\frac{1}{\sin x} \frac{d}{dx}(\sin x) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x} \log(\sin x) \frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x} \log(\sin x) \cos x$$

$$\Rightarrow \frac{dy}{dx} = 2 \left(\frac{\cos x}{\sin x} \right) \log(\sin x)$$

$$\therefore \frac{dy}{dx} = 2 \cot x \log(\sin x)$$

Thus, $\frac{d}{dx}[(\log(\sin x))^2] = 2 \cot x \log(\sin x)$

19. $\sqrt{\frac{1+x}{1-x}}$

Solution:

$$\text{Let } y = \sqrt{\frac{1+x}{1-x}}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{1+x}{1-x}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} \right]$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1}$$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{1+x}{1-x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+x}{1-x} \right)$$

$$\text{We know that } \left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2} \text{ (quotient rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{(1-x) \frac{d}{dx} (1+x) - (1+x) \frac{d}{dx} (1-x)}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{(1-x) \left(\frac{d}{dx} (1) + \frac{d}{dx} (x) \right) - (1+x) \left(\frac{d}{dx} (1) - \frac{d}{dx} (x) \right)}{(1-x)^2} \right]$$

$$\text{However, } \frac{d}{dx} (x) = 1 \text{ and derivative of a constant is 0.}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{(1-x)(0+1) - (1+x)(0-1)}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{(1-x) + (1+x)}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{2}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left[\frac{1}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}} \left[\frac{1}{(1-x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$$

$$\text{Thus, } \frac{d}{dx} \left(\sqrt{\frac{1+x}{1-x}} \right) = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$$

20. $\sin \left(\frac{1+x^2}{1-x^2} \right)$

Solution:

Let $y = \sin\left(\frac{1+x^2}{1-x^2}\right)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[\sin\left(\frac{1+x^2}{1-x^2}\right) \right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

Now by using chain rule

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \frac{d}{dx} \left(\frac{1+x^2}{1-x^2} \right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{(1-x^2) \frac{d}{dx}(1+x^2) - (1+x^2) \frac{d}{dx}(1-x^2)}{(1-x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{(1-x^2) \left(\frac{d}{dx}(1) + \frac{d}{dx}(x^2) \right) - (1+x^2) \left(\frac{d}{dx}(1) - \frac{d}{dx}(x^2) \right)}{(1-x^2)^2} \right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{(1-x^2)(0+2x) - (1+x^2)(0-2x)}{(1-x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{2x(1-x^2) + 2x(1+x^2)}{(1-x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{2x(1-x^2+1+x^2)}{(1-x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{2x(2)}{(1-x^2)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right) \left[\frac{4x}{(1-x^2)^2} \right]$$

$$\therefore \frac{dy}{dx} = \frac{4x}{(1-x^2)^2} \cos\left(\frac{1+x^2}{1-x^2}\right)$$

$$\text{Thus, } \frac{d}{dx} \left[\sin\left(\frac{1+x^2}{1-x^2}\right) \right] = \frac{4x}{(1-x^2)^2} \cos\left(\frac{1+x^2}{1-x^2}\right)$$

21. $e^{3x} \cos 2x$

Solution:

$$\text{Let } y = e^{3x} \cos(2x)$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{3x} \cos 2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^{3x} \times \cos 2x)$$

We know that $(uv)' = vu' + uv'$ (product rule)

$$\Rightarrow \frac{dy}{dx} = \cos 2x \frac{d}{dx}(e^{3x}) + e^{3x} \frac{d}{dx}(\cos 2x)$$

$$\text{We know } \frac{d}{dx}(e^x) = e^x \text{ and } \frac{d}{dx}(\cos x) = -\sin x$$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \cos 2x \left[e^{3x} \frac{d}{dx}(3x) \right] + e^{3x} \left[-\sin 2x \frac{d}{dx}(2x) \right]$$

$$\Rightarrow \frac{dy}{dx} = e^{3x} \cos 2x \left[\frac{d}{dx}(3x) \right] - e^{3x} \sin 2x \left[\frac{d}{dx}(2x) \right]$$

$$\Rightarrow \frac{dy}{dx} = e^{3x} \cos 2x \left[3 \frac{d}{dx}(x) \right] - e^{3x} \sin 2x \left[2 \frac{d}{dx}(x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x} \cos 2x \left[\frac{d}{dx}(x) \right] - 2e^{3x} \sin 2x \left[\frac{d}{dx}(x) \right]$$

We have $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x} \cos 2x \times 1 - 2e^{3x} \sin 2x \times 1$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x} \cos 2x - 2e^{3x} \sin 2x$$

$$\therefore \frac{dy}{dx} = e^{3x}(3 \cos 2x - 2 \sin 2x)$$

Thus, $\frac{d}{dx}(e^{3x} \cos 2x) = e^{3x}(3 \cos 2x - 2 \sin 2x)$

22. Sin (log sin x)

Solution:

Let $y = \sin (\log \sin x)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log(\sin x))]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

By using chain rule,

$$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x)) \frac{d}{dx}[\log(\sin x)]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x)) \left[\frac{1}{\sin x} \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \cos(\log(\sin x)) \frac{d}{dx} (\sin x)$$

However, $\frac{d}{dx} (\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \cos(\log(\sin x)) \cos x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\cos x}{\sin x} \right) \cos(\log(\sin x))$$

$$\therefore \frac{dy}{dx} = \cot x \cos(\log(\sin x))$$

Thus, $\frac{d}{dx} [\sin(\log(\sin x))] = \cot x \cos(\log(\sin x))$

23. $e^{\tan 3x}$

Solution:

Let $y = e^{\tan 3x}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} (e^{\tan 3x})$$

We know $\frac{d}{dx} (e^x) = e^x$

By using chain rule,

$$\Rightarrow \frac{dy}{dx} = e^{\tan 3x} \frac{d}{dx} (\tan 3x)$$

We have $\frac{d}{dx} (\tan x) = \sec^2 x$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = e^{\tan 3x} \sec^2 3x \frac{d}{dx}(3x)$$

$$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x \frac{d}{dx}(x)$$

$$\text{However, } \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x \times 1$$

$$\therefore \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x$$

$$\text{Thus, } \frac{d}{dx}(e^{\tan 3x}) = 3e^{\tan 3x} \sec^2 3x$$

24. $e^{\sqrt{\cot x}}$

Solution:

$$\text{Let } y = e^{\sqrt{\cot x}}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{\cot x}})$$

$$\text{We know } \frac{d}{dx}(e^x) = e^x$$

By using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}} \frac{d}{dx}(\sqrt{\cot x})$$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}} \frac{d}{dx}[(\cot x)^{\frac{1}{2}}]$$

$$\text{We have } \frac{d}{dx}(x^n) = nx^{n-1}$$

By using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}} \left[\frac{1}{2} (\cot x)^{\frac{1}{2}-1} \frac{d}{dx} (\cot x) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} e^{\sqrt{\cot x}} (\cot x)^{-\frac{1}{2}} \frac{d}{dx} (\cot x)$$

However, $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} e^{\sqrt{\cot x}} (\cot x)^{-\frac{1}{2}} \operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}} \operatorname{cosec}^2 x}{2(\cot x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}} \operatorname{cosec}^2 x}{2\sqrt{\cot x}}$$

Thus, $\frac{d}{dx} (e^{\sqrt{\cot x}}) = -\frac{e^{\sqrt{\cot x}} \operatorname{cosec}^2 x}{2\sqrt{\cot x}}$

25. $\log \left(\frac{\sin x}{1 + \cos x} \right)$

Solution:

Let $y = \log \left(\frac{\sin x}{1 + \cos x} \right)$

$$\Rightarrow y = \log \left(\frac{\sin 2 \times \frac{x}{2}}{1 + \cos 2 \times \frac{x}{2}} \right)$$

We have $\sin 2\theta = 2\sin\theta\cos\theta$ and $1 + \cos 2\theta = 2\cos^2\theta$.

$$\Rightarrow y = \log \left(\frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right)$$

$$\Rightarrow y = \log\left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\right)$$

$$\Rightarrow y = \log\left(\tan \frac{x}{2}\right)$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\tan \frac{x}{2}\right)\right]$$

$$\text{We know } \frac{d}{dx}(\log x) = \frac{1}{x}$$

Now by using chain rule we have,

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\tan \frac{x}{2}}\right) \frac{d}{dx}\left(\tan \frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \cot \frac{x}{2} \frac{d}{dx}\left(\tan \frac{x}{2}\right)$$

$$\text{We have } \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \cot \frac{x}{2} \sec^2 \frac{x}{2} \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cot \frac{x}{2} \sec^2 \frac{x}{2} \frac{d}{dx}(x)$$

$$\text{However, } \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cot \frac{x}{2} \sec^2 \frac{x}{2} \times 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \times \frac{1}{\cos^2 \frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin 2 \times \frac{x}{2}} [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

$$\text{Thus, } \frac{d}{dx} \left[\log \left(\frac{\sin x}{1 + \cos x} \right) \right] = \operatorname{cosec} x$$

$$26. \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

Solution:

$$\text{Let } y = \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\log \sqrt{\frac{1 - \cos x}{1 + \cos x}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\log \left(\frac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}} \right]$$

$$\text{We know } \frac{d}{dx} (\log x) = \frac{1}{x}$$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}}} \frac{d}{dx} \left[\left(\frac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left[\left(\frac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Again by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{1}{2} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{1 - \cos x}{1 + \cos x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-\frac{1}{2}} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1 - \cos x}{1 + \cos x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-1} \frac{d}{dx} \left(\frac{1 - \cos x}{1 + \cos x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \frac{d}{dx} \left(\frac{1 - \cos x}{1 + \cos x} \right)$$

We know that $\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{(1 + \cos x) \left(\frac{d}{dx}(1) - \frac{d}{dx}(\cos x) \right) - (1 - \cos x) \left(\frac{d}{dx}(1) + \frac{d}{dx}(\cos x) \right)}{(1 + \cos x)^2} \right]$$

We know $\frac{d}{dx}(\cos x) = -\sin x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{(1 + \cos x)(0 + \sin x) - (1 - \cos x)(0 - \sin x)}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{(1 + \cos x) \sin x + (1 - \cos x) \sin x}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{(1 + \cos x + 1 - \cos x) \sin x}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right) \left[\frac{2 \sin x}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{(1 - \cos x)(1 + \cos x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{1 - \cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{\sin^2 x} (\because \sin^2 \theta + \cos^2 \theta = 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

$$\text{Thus, } \frac{d}{dx} \left(\log \sqrt{\frac{1 - \cos x}{1 + \cos x}} \right) = \operatorname{cosec} x$$

27. $\tan(e^{\sin x})$

Solution:

Let $y = \tan(e^{\sin x})$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} [\tan(e^{\sin x})]$$

We know $\frac{d}{dx} (\tan x) = \sec^2 x$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \sec^2(e^{\sin x}) \frac{d}{dx}(e^{\sin x})$$

We have $\frac{d}{dx}(e^x) = e^x$

Again by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \sec^2(e^{\sin x}) e^{\sin x} \frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \sec^2(e^{\sin x}) e^{\sin x} \cos x$$

$$\therefore \frac{dy}{dx} = e^{\sin x} \cos x \sec^2(e^{\sin x})$$

Thus, $\frac{d}{dx}[\tan(e^{\sin x})] = e^{\sin x} \cos x \sec^2(e^{\sin x})$

28. $\log(x + \sqrt{x^2 + 1})$

Solution:

Let $y = \log(x + \sqrt{x^2 + 1})$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(x + \sqrt{x^2 + 1})]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}(x + \sqrt{x^2 + 1})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{x^2 + 1}) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{d}{dx}(x) + \frac{d}{dx}(x^2 + 1)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

Again by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2}(x^2 + 1)^{\frac{1}{2}-1} \frac{d}{dx}(x^2 + 1) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \left(\frac{d}{dx}(x^2) + \frac{d}{dx}(1) \right) \right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x + 0) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \times 2x \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + x(x^2 + 1)^{-\frac{1}{2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{x}{\sqrt{x^2 + 1}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\text{Thus, } \frac{d}{dx} [\log(x + \sqrt{x^2 + 1})] = \frac{1}{\sqrt{x^2 + 1}}$$

29. $\frac{e^x \log x}{x^2}$

Solution:

$$\text{Let } y = \frac{e^x \log x}{x^2}$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{e^x \log x}{x^2} \right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2) \frac{d}{dx}(e^x \log x) - (e^x \log x) \frac{d}{dx}(x^2)}{(x^2)^2}$$

We have $(u v)' = vu' + u v'$ (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2) \left[\log x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\log x) \right] - (e^x \log x) \frac{d}{dx}(x^2)}{x^4}$$

We know $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^2) = 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2) \left[\log x \times e^x + e^x \times \frac{1}{x} \right] - (e^x \log x) \times 2x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2) \left[e^x \log x + \frac{e^x}{x} \right] - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x + xe^x - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x}{x^4} + \frac{xe^x}{x^4} - \frac{2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2} + \frac{e^x}{x^3} - \frac{2e^x \log x}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{x^2} \left(\log x + \frac{1}{x} - \frac{2 \log x}{x} \right)$$

$$\therefore \frac{dy}{dx} = e^x x^{-2} \left(\log x + \frac{1}{x} - \frac{2}{x} \log x \right)$$

$$\text{Thus, } \frac{d}{dx} \left(\frac{e^x \log x}{x^2} \right) = e^x x^{-2} \left(\log x + \frac{1}{x} - \frac{2}{x} \log x \right)$$

30. $\log (\operatorname{cosec} x - \cot x)$

Solution:

Let $y = \log (\operatorname{cosec} x - \cot x)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} [\log (\operatorname{cosec} x - \cot x)]$$

$$\text{We know } \frac{d}{dx} (\log x) = \frac{1}{x}$$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} \frac{d}{dx} (\operatorname{cosec} x - \cot x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} \left[\frac{d}{dx} (\operatorname{cosec} x) - \frac{d}{dx} (\cot x) \right]$$

$$\text{We know } \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \text{ and } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} [-\operatorname{cosec} x \cot x - (-\operatorname{cosec}^2 x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} [-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} [\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} [(\operatorname{cosec} x - \cot x) \operatorname{cosec} x]$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx} [\log(\operatorname{cosec} x - \cot x)] = \operatorname{cosec} x$

$$31. \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$$

Solution:

Let $y = \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}} \right)$$

We know that $\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x}) \frac{d}{dx} (e^{2x} + e^{-2x}) - (e^{2x} + e^{-2x}) \frac{d}{dx} (e^{2x} - e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\begin{aligned} &\Rightarrow \frac{dy}{dx} \\ &= \frac{(e^{2x} - e^{-2x}) \left[\frac{d}{dx} (e^{2x}) + \frac{d}{dx} (e^{-2x}) \right] - (e^{2x} + e^{-2x}) \left[\frac{d}{dx} (e^{2x}) - \frac{d}{dx} (e^{-2x}) \right]}{(e^{2x} - e^{-2x})^2} \end{aligned}$$

We know $\frac{d}{dx} (e^x) = e^x$

$$\begin{aligned} &\Rightarrow \frac{dy}{dx} \\ &= \frac{(e^{2x} - e^{-2x}) \left[e^{2x} \frac{d}{dx} (2x) + e^{-2x} \frac{d}{dx} (-2x) \right] - (e^{2x} + e^{-2x}) \left[e^{2x} \frac{d}{dx} (2x) - e^{-2x} \frac{d}{dx} (-2x) \right]}{(e^{2x} - e^{-2x})^2} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{dy}{dx} \\ &= \frac{(e^{2x} - e^{-2x}) \left[2e^{2x} \frac{d}{dx} (x) - 2e^{-2x} \frac{d}{dx} (x) \right] - (e^{2x} + e^{-2x}) \left[2e^{2x} \frac{d}{dx} (x) + 2e^{-2x} \frac{d}{dx} (x) \right]}{(e^{2x} - e^{-2x})^2} \end{aligned}$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} \times 1 - 2e^{-2x} \times 1] - (e^{2x} + e^{-2x})[2e^{2x} \times 1 + 2e^{-2x} \times 1]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} - 2e^{-2x}] - (e^{2x} + e^{-2x})[2e^{2x} + 2e^{-2x}]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x})(e^{2x} - e^{-2x}) - 2(e^{2x} + e^{-2x})(e^{2x} + e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2[(e^{2x} - e^{-2x})^2 - (e^{2x} + e^{-2x})^2]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x} + e^{2x} + e^{-2x})(e^{2x} - e^{-2x} - e^{2x} - e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(2e^{2x})(-2e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-8e^{2x+(-2x)}}{(e^{2x} - e^{-2x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{-8}{(e^{2x} - e^{-2x})^2}$$

$$\text{Thus, } \frac{d}{dx} \left(\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}} \right) = \frac{-8}{(e^{2x} - e^{-2x})^2}$$

32. $\log \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)$

Solution:

$$\text{Let } y = \log \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)$$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[\log \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right) \right]$$

We know $\frac{d}{dx} (\log x) = \frac{1}{x}$

By using chain rule, we have

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)} \frac{d}{dx} \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \frac{d}{dx} \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)$$

We know that $\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx}$$

$$= \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{(x^2 - x + 1) \frac{d}{dx} (x^2 + x + 1) - (x^2 + x + 1) \frac{d}{dx} (x^2 - x + 1)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{(x^2 - x + 1) \left(\frac{d}{dx} (x^2) + \frac{d}{dx} (x) + \frac{d}{dx} (1) \right) - (x^2 + x + 1) \left(\frac{d}{dx} (x^2) - \frac{d}{dx} (x) + \frac{d}{dx} (1) \right)}{(x^2 - x + 1)^2} \right]$$

We know $\frac{d}{dx} (x^2) = 2x$, $\frac{d}{dx} (x) = 1$ and derivative of constant is 0.

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{(x^2 - x + 1)(2x + 1 + 0) - (x^2 + x + 1)(2x - 1 + 0)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{(2x + 1)(x^2 - x + 1) - (2x - 1)(x^2 + x + 1)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{2x(x^2 - x + 1) + (x^2 - x + 1) - 2x(x^2 + x + 1) + (x^2 + x + 1)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{2x(x^2 - x + 1 - x^2 - x - 1) + (x^2 - x + 1 + x^2 + x + 1)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{2x(-2x) + (2x^2 + 2)}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{-4x^2 + 2x^2 + 2}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) \left[\frac{2 - 2x^2}{(x^2 - x + 1)^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 - 2x^2}{(x^2 + x + 1)(x^2 - x + 1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{(x^2 + 1)^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{(x^2 + 1)^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{x^4 + 2x^2 + 1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{(x^2 + 1)^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{x^4 + 2x^2 + 1 - x^2}$$

33. $\tan^{-1}(e^x)$

Solution:

Let $y = \tan^{-1}(e^x)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} e^x)$$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(e^x)^2} \frac{d}{dx}(e^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+e^{2x}} \frac{d}{dx}(e^x)$$

However, $\frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+e^{2x}} \times e^x$$

$$\therefore \frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$$

Thus, $\frac{d}{dx}(\tan^{-1} e^x) = \frac{e^x}{1+e^{2x}}$

34. $e^{\sin^{-1} 2x}$

Solution:

Let $y = e^{\sin^{-1} 2x}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin^{-1} 2x})$$

We know $\frac{d}{dx}(e^x) = e^x$

Using chain rule, we can write as

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{d}{dx} (\sin^{-1} 2x)$$

We have $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Using chain rule we get

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx} (2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 2 \frac{d}{dx} (x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times \frac{d}{dx} (x)$$

However, $\frac{d}{dx} (x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$$

Thus, $\frac{d}{dx} (e^{\sin^{-1} 2x}) = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$

35. $\sin (2 \sin^{-1} x)$

Solution:

Let $y = \sin (2 \sin^{-1} x)$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} [\sin(2 \sin^{-1} x)]$$

We know $\frac{d}{dx} (\sin x) = \cos x$

By using chain rule we get,

$$\Rightarrow \frac{dy}{dx} = \cos(2 \sin^{-1} x) \frac{d}{dx} (2 \sin^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = \cos(2 \sin^{-1} x) \times 2 \frac{d}{dx} (\sin^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = 2 \cos(2 \sin^{-1} x) \frac{d}{dx} (\sin^{-1} x)$$

We have $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{dy}{dx} = 2 \cos(2 \sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{2 \cos(2 \sin^{-1} x)}{\sqrt{1-x^2}}$$

Thus, $\frac{d}{dx} [\sin(2 \sin^{-1} x)] = \frac{2 \cos(2 \sin^{-1} x)}{\sqrt{1-x^2}}$

36. $e^{\tan^{-1} \sqrt{x}}$

Solution:

Let $y = e^{\tan^{-1} \sqrt{x}}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} (e^{\tan^{-1} \sqrt{x}})$$

We know $\frac{d}{dx} (e^x) = e^x$

Now by using chain rule, we can write as

$$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1} \sqrt{x}} \frac{d}{dx} (\tan^{-1} \sqrt{x})$$

We have $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$

Again by using chain rule we get,

$$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1} \sqrt{x}} \frac{1}{1+(\sqrt{x})^2} \frac{d}{dx} (\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1} \sqrt{x}}}{1+x} \frac{d}{dx} \left(x^{\frac{1}{2}} \right)$$

However, $\frac{d}{dx} (x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1} \sqrt{x}}}{1+x} \left(\frac{1}{2} x^{\frac{1}{2}-1} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1} \sqrt{x}}}{1+x} \left(\frac{1}{2} x^{-\frac{1}{2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1} \sqrt{x}}}{1+x} \left(\frac{1}{2\sqrt{x}} \right)$$

$$\therefore \frac{dy}{dx} = \frac{e^{\tan^{-1} \sqrt{x}}}{2\sqrt{x}(1+x)}$$

Thus, $\frac{d}{dx} (e^{\tan^{-1} \sqrt{x}}) = \frac{e^{\tan^{-1} \sqrt{x}}}{2\sqrt{x}(1+x)}$

37. $\sqrt{\tan^{-1} \left(\frac{x}{2} \right)}$

Solution:

Let $y = \sqrt{\tan^{-1} \frac{x}{2}}$

On differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\tan^{-1} \frac{x}{2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\left(\tan^{-1} \frac{x}{2} \right)^{\frac{1}{2}} \right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\tan^{-1} \frac{x}{2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\tan^{-1} \frac{x}{2} \right)$$

We have $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Again by using chain rule, we can write as

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{1}{1+\left(\frac{x}{2}\right)^2} \frac{d}{dx} \left(\frac{x}{2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{1}{1+\frac{x^2}{4}} \times \frac{1}{2} \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{4}{4+x^2} \times \frac{1}{2} \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{1}{4+x^2} \times \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{1}{4+x^2} \times 1$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1} \frac{x}{2} \right)^{-\frac{1}{2}} \frac{1}{4+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(4+x^2) \left(\tan^{-1} \frac{x}{2} \right)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(4 + x^2) \sqrt{\tan^{-1} \frac{x}{2}}}$$



EXERCISE 11.3

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Differentiate the following functions with respect to x:

1. $\cos^{-1} \left\{ 2x\sqrt{1-x^2} \right\}, \frac{1}{\sqrt{2}} < x < 1$

Solution:

$$\text{Let } y = \cos^{-1} \{ 2x\sqrt{1-x^2} \}$$

$$\text{let } x = \cos \theta$$

Now

$$y = \cos^{-1} \{ 2\cos \theta \sqrt{1-\cos^2 \theta} \}$$

$$= \cos^{-1} \{ 2\cos \theta \sqrt{\sin^2 \theta} \}$$

$$\text{Using } \sin^2 \theta + \cos^2 \theta = 1 \text{ and } 2 \sin \theta \cos \theta = \sin 2\theta$$

$$= \cos^{-1} (2 \cos \theta \sin \theta)$$

$$= \cos^{-1} (\sin 2\theta)$$

$$y = \cos^{-1} \left(\cos \left(\frac{\pi}{2} - 2\theta \right) \right)$$

Now by considering the limits,

$$\frac{1}{\sqrt{2}} < x < 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} < \cos \theta < 1$$

$$\Rightarrow 0 < \theta < \frac{\pi}{4}$$

$$\Rightarrow 0 < 2\theta < \frac{\pi}{2}$$

$$\Rightarrow 0 > -2\theta > -\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} > \frac{\pi}{2} - 2\theta > \frac{\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow 0 < \frac{\pi}{2} - 2\theta < \frac{\pi}{2}$$

Therefore,

$$y = \cos^{-1} \left(\cos \left(\frac{\pi}{2} - 2\theta \right) \right)$$

$$y = \cos^{-1} \left(\cos \left(\frac{\pi}{2} - 2\theta \right) \right)$$

$$y = \left(\frac{\pi}{2} - 2\theta \right)$$

$$y = \frac{\pi}{2} - 2 \cos^{-1} x$$

Differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - 2 \cos^{-1} x \right)$$

$$\Rightarrow \frac{dy}{dx} = 0 - 2 \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$2. \cos^{-1} \left\{ \sqrt{\frac{1+x}{2}} \right\}, -1 < x < 1$$

Solution:

Let

$$y = \cos^{-1} \left\{ \sqrt{\frac{1+x}{2}} \right\}$$

let $x = \cos 2\theta$

Now

$$y = \cos^{-1} \left\{ \sqrt{\frac{1 + \cos 2\theta}{2}} \right\}$$

$$y = \cos^{-1} \left\{ \sqrt{\frac{2 \cos^2 \theta}{2}} \right\}$$

Now by using $\cos 2\theta = 2\cos^2 \theta - 1$

$$y = \cos^{-1}(\cos \theta)$$

Considering the limits,

$$-1 < x < 1$$

$$-1 < \cos 2\theta < 1$$

$$0 < 2\theta < \pi$$

$$0 < \theta < \frac{\pi}{2}$$

$$\text{Now, } y = \cos^{-1}(\cos \theta)$$

$$y = \theta$$

$$y = \frac{1}{2} \cos^{-1} x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{1}{2} \left(-\frac{1}{\sqrt{1-x^2}} \right)$$

$$3. \sin^{-1} \left\{ \sqrt{\frac{1-x}{2}} \right\}, \quad 0 < x < 1$$

Solution:

$$y = \sin^{-1} \left\{ \sqrt{\frac{1-x}{2}} \right\}$$

Let,

$$\text{let } x = \cos 2\theta$$

Now

$$y = \sin^{-1} \left\{ \sqrt{\frac{1 - \cos 2\theta}{2}} \right\}$$

$$y = \sin^{-1} \left\{ \sqrt{\frac{2 \sin^2 \theta}{2}} \right\}$$

$$\text{Using } \cos 2\theta = 1 - 2\sin^2 \theta$$

$$y = \sin^{-1}(\sin \theta)$$

Considering the limits,

$$0 < x < 1$$

$$0 < \cos 2\theta < 1$$

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

$$\text{Now, } y = \sin^{-1}(\sin \theta)$$

$$y = \theta$$

$$y = \frac{1}{2} \cos^{-1} x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{1}{2} \left(-\frac{1}{\sqrt{1-x^2}} \right)$$

$$4. \sin^{-1} \left\{ \sqrt{1-x^2} \right\}, 0 < x < 1$$

Solution:

Let,

$$y = \sin^{-1} \left\{ \sqrt{1 - x^2} \right\}$$

$$\text{let } x = \cos \theta$$

Now

$$y = \sin^{-1} \left\{ \sqrt{1 - \cos^2 \theta} \right\}$$

$$\text{Using } \sin^2 \theta + \cos^2 \theta = 1$$

$$y = \sin^{-1}(\sin \theta)$$

Considering the limits,

$$0 < x < 1$$

$$0 < \cos \theta < 1$$

$$0 < \theta < \frac{\pi}{2}$$

$$\text{Now, } y = \sin^{-1}(\sin \theta)$$

$$y = \theta$$

$$y = \cos^{-1} x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

$$5. \tan^{-1} \left\{ \frac{x}{\sqrt{a^2 - x^2}} \right\}, \quad -a < x < a$$

Solution:

$$y = \tan^{-1} \left\{ \frac{x}{\sqrt{a^2 - x^2}} \right\}$$

$$\text{Let } x = a \sin \theta$$

Now

$$y = \tan^{-1} \left\{ \frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right\}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$

$$y = \tan^{-1} \left\{ \frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right\}$$

$$y = \tan^{-1} \left\{ \frac{\sin \theta}{\cos \theta} \right\}$$

$$y = \tan^{-1}(\tan \theta)$$

Considering the limits,

$$-a < x < a$$

$$-a < a \sin \theta < a$$

$$-1 < \sin \theta < 1$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Now, $y = \tan^{-1}(\tan \theta)$

$$y = \theta$$

$$y = \sin^{-1} \left(\frac{x}{a} \right)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1} \left(\frac{x}{a} \right) \right)$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

6. $\sin^{-1} \left\{ \frac{x}{\sqrt{x^2 + a^2}} \right\}$

Solution:

Let,

$$y = \sin^{-1} \left\{ \frac{x}{\sqrt{x^2 + a^2}} \right\}$$

Let $x = a \tan \theta$

Now

$$y = \sin^{-1} \left\{ \frac{a \tan \theta}{\sqrt{a^2 \tan^2 \theta + a^2}} \right\}$$

Using $1 + \tan^2 \theta = \sec^2 \theta$

$$y = \sin^{-1} \left\{ \frac{a \tan \theta}{a \sqrt{\tan^2 \theta + 1}} \right\}$$

$$y = \sin^{-1} \left\{ \frac{a \tan \theta}{a \sqrt{\sec^2 \theta}} \right\}$$

$$y = \sin^{-1} \left\{ \frac{\tan \theta}{\sec \theta} \right\}$$

$$y = \sin^{-1}(\sin \theta)$$

$$y = \theta$$

$$y = \tan^{-1} \left(\frac{x}{a} \right)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\tan^{-1} \left(\frac{x}{a} \right) \right)$$

$$\frac{dy}{dx} = \frac{a^2}{a^2 + x^2} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

7. $\sin^{-1} (2x^2 - 1)$, $0 < x < 1$

Solution:

Let,

$$y = \sin^{-1}\{2x^2 - 1\}$$

$$\text{let } x = \cos\theta$$

Now

$$y = \sin^{-1}\{\sqrt{2\cos^2\theta - 1}\}$$

$$\text{Using } 2\cos^2\theta - 1 = \cos 2\theta$$

$$y = \sin^{-1}(\cos 2\theta)$$

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

Considering the limits,

$$0 < x < 1$$

$$0 < \cos \theta < 1$$

$$0 < \theta < \frac{\pi}{2}$$

$$0 < 2\theta < \pi$$

$$0 > -2\theta > -\pi$$

$$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$$y = \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2\cos^{-1}x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - 2 \cos^{-1} x \right)$$

$$\frac{dy}{dx} = 0 - 2 \left(-\frac{1}{\sqrt{1-x^2}} \right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

8. $\sin^{-1}(1 - 2x^2)$, $0 < x < 1$

Solution:

Let,

$$y = \sin^{-1}\{1 - 2x^2\}$$

$$\text{let } x = \sin \theta$$

Now

$$y = \sin^{-1}\{\sqrt{1 - 2\sin^2 \theta}\}$$

$$\text{Using } 1 - 2\sin^2 \theta = \cos 2\theta$$

$$y = \sin^{-1}(\cos 2\theta)$$

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

Considering the limits,

$$0 < x < 1$$

$$0 < \sin \theta < 1$$

$$0 < \theta < \frac{\pi}{2}$$

$$0 < 2\theta < \pi$$

$$0 > -2\theta > -\pi$$

$$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$$

Now,

$$y = \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - 2\theta \right) \right\}$$

$$y = \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2 \sin^{-1} x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - 2 \cos^{-1} x \right)$$

$$\frac{dy}{dx} = 0 - 2 \left(\frac{1}{\sqrt{1-x^2}} \right)$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

9. $\cos^{-1} \left\{ \frac{x}{\sqrt{x^2 + a^2}} \right\}$

Solution:

Let,

$$y = \cos^{-1} \left\{ \frac{x}{\sqrt{x^2 + a^2}} \right\}$$

Let $x = a \cot \theta$

Now

$$y = \cos^{-1} \left\{ \frac{a \cot \theta}{\sqrt{a^2 \cot^2 \theta + a^2}} \right\}$$

Using $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

$$y = \cos^{-1} \left\{ \frac{a \cot \theta}{a \sqrt{\cot^2 \theta + 1}} \right\}$$

$$y = \cos^{-1} \left\{ \frac{a \cot \theta}{a \sqrt{\operatorname{cosec}^2 \theta}} \right\}$$

$$y = \cos^{-1} \left\{ \frac{\cot \theta}{\operatorname{cosec} \theta} \right\}$$

$$y = \cos^{-1}(\cos \theta)$$

$$y = \theta$$

$$y = \cot^{-1} \left(\frac{x}{a} \right)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\cot^{-1} \left(\frac{x}{a} \right) \right)$$

$$\frac{dy}{dx} = \frac{-a^2}{a^2 + x^2} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-a}{a^2 + x^2}$$

10. $\sin^{-1} \left\{ \frac{\sin x + \cos x}{\sqrt{2}} \right\}, -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Solution:

Let,

$$y = \sin^{-1} \left\{ \frac{\sin x + \cos x}{\sqrt{2}} \right\}$$

Now

$$y = \sin^{-1} \left\{ \sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}} \right\}$$

$$y = \sin^{-1} \left\{ \sin x \cos \left(\frac{\pi}{4} \right) + \cos x \sin \left(\frac{\pi}{4} \right) \right\}$$

Using $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$y = \sin^{-1} \left\{ \sin \left(x + \frac{\pi}{4} \right) \right\}$$

Considering the limits,

$$-\frac{3\pi}{4} < x < \frac{\pi}{4}$$

Differentiating it with respect to x ,

$$y = x + \frac{\pi}{4}$$

$$\frac{dy}{dx} = 1$$

$$11. \cos^{-1} \left\{ \frac{\cos x + \sin x}{\sqrt{2}} \right\}, -\frac{\pi}{4} < x < \frac{\pi}{4}$$

Solution:

Let,

$$y = \cos^{-1} \left\{ \frac{\cos x + \sin x}{\sqrt{2}} \right\}$$

Now

$$y = \cos^{-1} \left\{ \cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}} \right\}$$

$$y = \cos^{-1} \left\{ \cos x \cos \left(\frac{\pi}{4} \right) + \sin x \sin \left(\frac{\pi}{4} \right) \right\}$$

Using $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$y = \cos^{-1} \left\{ \cos \left(x - \frac{\pi}{4} \right) \right\}$$

Considering the limits,

$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < x - \frac{\pi}{4} < 0$$

Now,

$$y = -x + \frac{\pi}{4}$$

Differentiating it with respect to x ,

$$\frac{dy}{dx} = -1$$

12. $\tan^{-1} \left\{ \frac{x}{1 + \sqrt{1 - x^2}} \right\}, -1 < x < 1$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{x}{1 + \sqrt{1 - x^2}} \right\}$$

Let $x = \sin \theta$

Now

$$y = \tan^{-1} \left\{ \frac{\sin \theta}{1 + \sqrt{1 - \sin^2 \theta}} \right\}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$

$$y = \tan^{-1} \left\{ \frac{\sin \theta}{1 + \sqrt{\cos^2 \theta}} \right\}$$

$$y = \tan^{-1} \left\{ \frac{\sin \theta}{1 + \cos \theta} \right\}$$

Using $2 \cos^2 \theta = 1 + \cos 2\theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1} \left\{ \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} \right\}$$

$$y = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

Considering the limits,

$$-1 < x < 1$$

$$-1 < \sin \theta < 1$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2} \sin^{-1} x$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \sin^{-1} x \right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

$$13. \tan^{-1} \left\{ \frac{x}{a + \sqrt{a^2 - x^2}} \right\}, \quad -a < x < a$$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{x}{a + \sqrt{a^2 - x^2}} \right\}$$

Let $x = a \sin \theta$

Now

$$y = \tan^{-1} \left\{ \frac{a \sin \theta}{a + \sqrt{a^2 - a^2 \sin^2 \theta}} \right\}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$

$$y = \tan^{-1} \left\{ \frac{a \sin \theta}{a + a \sqrt{\cos^2 \theta}} \right\}$$

$$y = \tan^{-1} \left\{ \frac{\sin \theta}{1 + \cos \theta} \right\}$$

Using $2 \cos^2 \theta = 1 + \cos \theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1} \left\{ \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} \right\}$$

$$y = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

Considering the limits,

$$-a < x < a$$

$$-1 < \sin \theta < 1$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2} \sin^{-1} \frac{x}{a}$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \sin^{-1} \frac{x}{a} \right)$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}$$

14. $\sin^{-1} \left\{ \frac{x + \sqrt{1 - x^2}}{\sqrt{2}} \right\}, -1 < x < 1$

Solution:

Let,

$$y = \sin^{-1} \left\{ \frac{x + \sqrt{1 - x^2}}{\sqrt{2}} \right\}$$

Let $x = \sin \theta$

Now

$$y = \sin^{-1} \left\{ \frac{\sin \theta + \sqrt{1 - \sin^2 \theta}}{\sqrt{2}} \right\}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$

$$y = \sin^{-1} \left\{ \frac{\sin \theta + \cos \theta}{\sqrt{2}} \right\}$$

Now

$$y = \sin^{-1} \left\{ \sin \theta \frac{1}{\sqrt{2}} + \cos \theta \frac{1}{\sqrt{2}} \right\}$$

$$y = \sin^{-1} \left\{ \sin \theta \cos \left(\frac{\pi}{4} \right) + \cos \theta \sin \left(\frac{\pi}{4} \right) \right\}$$

Using $\sin (A + B) = \sin A \cos B + \cos A \sin B$

$$y = \sin^{-1} \left\{ \sin \left(\theta + \frac{\pi}{4} \right) \right\}$$

Considering the limits,

$$-1 < x < 1$$

$$-1 < \sin \theta < 1$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} + \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{\pi}{2} + \frac{\pi}{4}$$

$$-\frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4}$$

Now,

$$y = \sin^{-1} \left\{ \sin \left(\theta + \frac{\pi}{4} \right) \right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \sin^{-1} x + \frac{\pi}{4}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1} x + \frac{\pi}{4} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$15. \cos^{-1} \left\{ \frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right\}, \quad -1 < x < 1$$

Solution:

Let,

$$y = \cos^{-1} \left\{ \frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right\}$$

Let $x = \sin \theta$

Now

$$y = \cos^{-1} \left\{ \frac{\sin \theta + \sqrt{1 - \sin^2 \theta}}{\sqrt{2}} \right\}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$

$$y = \cos^{-1} \left\{ \frac{\sin \theta + \cos \theta}{\sqrt{2}} \right\}$$

Now

$$y = \cos^{-1} \left\{ \sin \theta \frac{1}{\sqrt{2}} + \cos \theta \frac{1}{\sqrt{2}} \right\}$$

$$y = \cos^{-1} \left\{ \sin \theta \sin \left(\frac{\pi}{4} \right) + \cos \theta \cos \left(\frac{\pi}{4} \right) \right\}$$

Using $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$y = \cos^{-1} \left\{ \cos \left(\theta - \frac{\pi}{4} \right) \right\}$$

Considering the limits,

$$-1 < x < 1$$

$$-1 < \sin \theta < 1$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} - \frac{\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{2} - \frac{\pi}{4}$$

$$-\frac{3\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{4}$$

Now,

$$y = \cos^{-1} \left\{ \cos \left(\theta - \frac{\pi}{4} \right) \right\}$$

$$y = - \left(\theta - \frac{\pi}{4} \right)$$

$$y = -\sin^{-1} x + \frac{\pi}{4}$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(-\sin^{-1} x + \frac{\pi}{4} \right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$16. \tan^{-1} \left\{ \frac{4x}{1-4x^2} \right\}, \quad -\frac{1}{2} < x < \frac{1}{2}$$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{4x}{1-4x^2} \right\}$$

$$\text{Let } 2x = \tan \theta$$

$$y = \tan^{-1} \left\{ \frac{2 \tan \theta}{1 - \tan^2 \theta} \right\}$$

$$\text{Using } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$$-\frac{1}{2} < x < \frac{1}{2}$$

$$-1 < 2x < 1$$

$$-1 < \tan \theta < 1$$

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

Now,

$$y = \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta$$

$$y = 2 \tan^{-1}(2x)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2 \tan^{-1} 2x)$$

$$\frac{dy}{dx} = 2 \times \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{4}{1 + 4x^2}$$

$$17. \tan^{-1} \left\{ \frac{2^{x+1}}{1 - 4^x} \right\}, \quad -\infty < x < 0$$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{2^{x+1}}{1 - 4^x} \right\}$$

$$\text{Let } 2^x = \tan \theta$$

$$y = \tan^{-1} \left\{ \frac{2 \times 2^x}{1 - (2^x)^2} \right\}$$

$$y = \tan^{-1} \left\{ \frac{2 \tan \theta}{1 - \tan^2 \theta} \right\}$$

$$\text{Using } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$$-\infty < x < 0$$

$$2^{-\infty} < 2^x < 2^0$$

$$0 < \tan \theta < 1$$

$$0 < \theta < \frac{\pi}{4}$$

$$0 < 2\theta < \frac{\pi}{2}$$

Now,

$$y = \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta$$

$$y = 2\tan^{-1}(2^x)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2^x)$$

$$\frac{dy}{dx} = 2 \times \frac{2^x \log 2}{1 + (2^x)^2}$$

$$\frac{dy}{dx} = \frac{2^{x+1} \log 2}{1 + 4^x}$$

18. $\tan^{-1} \left\{ \frac{2a^x}{1 - a^{2x}} \right\}, a > 1, -\infty < x < 0$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{2a^x}{1 - a^{2x}} \right\}$$

Let $a^x = \tan \theta$

$$y = \tan^{-1} \left\{ \frac{2 \tan \theta}{1 - \tan^2 \theta} \right\}$$

$$\text{Using } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$$-\infty < x < 0$$

$$a^{-\infty} < a^x < a^0$$

$$0 < \tan \theta < 1$$

$$0 < \theta < \frac{\pi}{4}$$

$$0 < 2\theta < \frac{\pi}{2}$$

$$\text{Now, } y = \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta$$

$$y = 2\tan^{-1}(a^x)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} a^x)$$

$$\frac{dy}{dx} = 2 \times \frac{a^x \log a}{1 + (a^x)^2}$$

$$\frac{dy}{dx} = \frac{2a^x \log a}{1 + a^{2x}}$$

$$19. \sin^{-1} \left\{ \frac{\sqrt{1+x} + \sqrt{1-x}}{2} \right\}, \quad 0 < x < 1$$

Solution:

$$\text{Let, } y = \sin^{-1} \left\{ \frac{\sqrt{1+x} + \sqrt{1-x}}{2} \right\}$$

Let $x = \cos 2\theta$

Now

$$y = \sin^{-1} \left\{ \frac{\sqrt{1 + \cos 2\theta} + \sqrt{1 - \cos 2\theta}}{2} \right\}$$

Using $1 - 2\sin^2\theta = \cos 2\theta$ and $2\cos^2\theta - 1 = \cos 2\theta$

$$y = \sin^{-1} \left\{ \frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{2} \right\}$$

Now

$$y = \sin^{-1} \left\{ \sin\theta \frac{1}{\sqrt{2}} + \cos\theta \frac{1}{\sqrt{2}} \right\}$$

$$y = \sin^{-1} \left\{ \sin\theta \cos\left(\frac{\pi}{4}\right) + \cos\theta \sin\left(\frac{\pi}{4}\right) \right\}$$

Using $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$y = \sin^{-1} \left\{ \sin\left(\theta + \frac{\pi}{4}\right) \right\}$$

Considering the limits,

$$0 < x < 1$$

$$0 < \cos 2\theta < 1$$

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

Now,

$$y = \sin^{-1} \left\{ \sin\left(\theta + \frac{\pi}{4}\right) \right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \frac{1}{2} \cos^{-1} x + \frac{\pi}{4}$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \cos^{-1} x + \frac{\pi}{4} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$$

$$20. \tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\}, \quad x \neq 0$$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\}$$

Let $ax = \tan \theta$

Now

$$y = \tan^{-1} \left\{ \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} \right\}$$

Using $\sec^2\theta = 1 + \tan^2\theta$

$$y = \tan^{-1} \left\{ \frac{\sqrt{\sec^2\theta}-1}{\tan\theta} \right\}$$

$$y = \tan^{-1} \left\{ \frac{\sec\theta-1}{\tan\theta} \right\}$$

$$y = \tan^{-1} \left\{ \frac{1-\cos\theta}{\sin\theta} \right\}$$

Using $2 \sin^2 \theta = 1 - \cos 2\theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1} \left\{ \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right\}$$

$$y = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2} \tan^{-1} ax$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} \tan^{-1} ax \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{a}{1 + (ax)^2}$$

$$\frac{dy}{dx} = \frac{a}{2(1 + a^2 x^2)}$$

21. $\tan^{-1} \left\{ \frac{\sin x}{1 + \cos x} \right\}, -\pi < x < \pi$

Solution:

Let,

$$y = \tan^{-1} \left\{ \frac{\sin x}{1 + \cos x} \right\}$$

Function y is defined for all real numbers where $\cos x \neq -1$

Using $2 \cos^2 \theta = 1 + \cos 2\theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1} \left\{ \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right\}$$

$$y = \tan^{-1} \left\{ \tan \frac{x}{2} \right\}$$

$$y = \frac{x}{2}$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{2} \right)$$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$22. \sin^{-1} \left\{ \frac{1}{\sqrt{1+x^2}} \right\}$$

Solution:

Let,

$$y = \sin^{-1} \left\{ \frac{1}{\sqrt{1+x^2}} \right\}$$

Let $x = \cot \theta$

Now

$$y = \sin^{-1} \left\{ \frac{1}{\sqrt{1+\cot^2 \theta}} \right\}$$

Using, $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

Now

$$y = \sin^{-1} \left\{ \frac{1}{\sqrt{\operatorname{cosec}^2 \theta}} \right\}$$

$$y = \sin^{-1} \left\{ \frac{1}{\operatorname{cosec} \theta} \right\}$$

$$y = \sin^{-1}(\sin \theta)$$

$$y = \theta$$

$$y = \cot^{-1}x$$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cot^{-1}x)$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

$$23. \cos^{-1} \left\{ \frac{1-x^{2n}}{1+x^{2n}} \right\}, 0 < x < \infty$$

Solution:

Let,

$$y = \cos^{-1} \left\{ \frac{1-x^{2n}}{1+x^{2n}} \right\}$$

$$\text{Let } x^n = \tan \theta$$

Now

$$y = \cos^{-1} \left\{ \frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right\}$$

$$\text{Using } \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta$$

$$y = \cos^{-1} \{\cos 2\theta\}$$

Considering the limits,

$$0 < x < \infty$$

$$0 < x^n < \infty$$

$$0 < \theta < \frac{\pi}{2}$$

$$\text{Now, } y = \cos^{-1}(\cos 2\theta)$$

$$y = 2\theta$$

$$y = \tan^{-1}(x^n)$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(x^n))$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1 + (x^n)^2}$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1 + x^{2n}}$$



EXERCISE 11.4

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Find dy/dx in each of the following:

1. $xy = c^2$

Solution:

Given $xy = c^2$;

Now we have to find $\frac{dy}{dx}$ of given equation, so by differentiating the equation on both sides with respect to x , we get,

By using the product rule on the left hand side,

$$\frac{d(xy)}{dx} = \frac{dc^2}{dx}$$

$$x \left(\frac{dy}{dx} \right) + y(1) = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

We can further solve it by putting the value of y ,

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

2. $y^3 - 3xy^2 = x^3 + 3x^2y$

Solution:

Given $y^3 - 3xy^2 = x^3 + 3x^2y$,

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\Rightarrow \frac{d}{dx}(y^3) - \frac{d}{dx}(3xy^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2y)$$

Now by using product rule we get,

$$\Rightarrow 3y^2 \frac{dy}{dx} - 3 \left[x \frac{d}{dx} (y^2) + y^2 \frac{d}{dx} (x) \right] = 3x^2 + 3 \left[x^2 \frac{d}{dx} (y) + y \frac{d}{dx} (x^2) \right]$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - 3 \left[x (2y) \frac{dy}{dx} + y^2 \right] = 3x^2 + 3 \left[x^2 \frac{dy}{dx} + y (2x) \right]$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - 6xy \frac{dy}{dx} - 3y^2 = 3x^2 + 3x^2 \frac{dy}{dx} + 6xy$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - 6xy \frac{dy}{dx} - 3x^2 \frac{dy}{dx} = 3x^2 + 6xy + 3y^2$$

$$\Rightarrow 3 \frac{dy}{dx} (y^2 - 2xy - x^2) = 3 (x^2 + 2xy + y^2)$$

Now by taking 3 as common we get,

$$\Rightarrow \frac{dy}{dx} = \frac{3(x + y)^2}{3(y^2 - 2xy - x^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x + y)^2}{y^2 - 2xy - x^2}$$

3. $x^{2/3} + y^{2/3} = a^{2/3}$

Solution:

Given $x^{2/3} + y^{2/3} = a^{2/3}$,

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{2}{3} \frac{1}{x^{1/3}} + \frac{2}{3} \frac{1}{y^{1/3}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

Now by substituting the value, we get

$$\frac{dy}{dx} = \frac{-\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}}$$

4. $4x + 3y = \log(4x - 3y)$

Solution:

Given $4x + 3y = \log(4x - 3y)$,

Now we have to find dy/dx of it, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{d}{dx}(4x) + \frac{d}{dx}(3y) = \frac{d}{dx}\{\log(4x - 3y)\}$$

$$\Rightarrow 4 + 3\frac{dy}{dx} = \frac{1}{(4x - 3y)} \frac{d}{dx}(4x - 3y)$$

$$\Rightarrow 4 + 3\frac{dy}{dx} = \frac{1}{(4x - 3y)} \left(4 - 3\frac{dy}{dx}\right)$$

$$\Rightarrow 3\frac{dy}{dx} + \frac{3}{(4x - 3y)} \frac{dy}{dx} = \frac{4}{(4x - 3y)} - 4$$

$$\Rightarrow 3\frac{dy}{dx} \left\{1 + \frac{1}{(4x - 3y)}\right\} = 4 \left\{\frac{1}{(4x - 3y)} - 1\right\}$$

$$\Rightarrow 3\frac{dy}{dx} \left\{\frac{4x - 3y + 1}{(4x - 3y)}\right\} = 4 \left\{\frac{1 - 4x + 3y}{(4x - 3y)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{3} \left\{\frac{1 - 4x + 3y}{(4x - 3y)}\right\} \left(\frac{4x - 3y}{4x - 3y + 1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{3} \left(\frac{1 - 4x + 3y}{4x - 3y + 1}\right)$$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-xb^2}{ya^2}$$

6. $x^5 + y^5 = 5xy$

Solution:

Given $x^5 + y^5 = 5xy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(y^5) = \frac{d}{dx}(5xy)$$

Now by using product rule, we get

$$\Rightarrow 5x^4 + 5y^4 \frac{dy}{dx} = 5 \left[x \frac{dy}{dx} + y \frac{d}{dx}(x) \right]$$

$$\Rightarrow 5x^4 + 5y^4 \frac{dy}{dx} = 5 \left[x \frac{dy}{dx} + y(1) \right]$$

$$\Rightarrow 5x^4 + 5y^4 \frac{dy}{dx} = 5x \frac{dy}{dx} + 5y$$

$$\Rightarrow 5y^4 \frac{dy}{dx} - 5x \frac{dy}{dx} = 5y - 5x^4$$

$$\Rightarrow 5 \frac{dy}{dx} (y^4 - x) = 5 (y - x^4)$$

$$\Rightarrow \frac{dy}{dx} = \frac{5(y - x^4)}{5(y^4 - x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x^4}{y^4 - x}$$

7. $(x + y)^2 = 2axy$

Solution:

Given $(x + y)^2 = 2axy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\Rightarrow \frac{d}{dx}(x + y)^2 = \frac{d}{dx}(2axy)$$

Now by using product rule, we get

$$\Rightarrow 2(x + y) \frac{d}{dx}(x + y) = 2a \left[x \frac{dy}{dx} + y \frac{d}{dx}(x) \right]$$

$$\Rightarrow 2(x + y) \left[1 + \frac{dy}{dx} \right] = 2a \left[x \frac{dy}{dx} + y(1) \right]$$

$$\Rightarrow 2(x + y) + 2(x + y) \frac{dy}{dx} = 2ax \frac{dy}{dx} + 2ay$$

$$\Rightarrow \frac{dy}{dx} [2(x + y) - 2ax] = 2ay - 2(x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2[ay - x - y]}{2[x + y - ax]}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{ay - x - y}{x + y - ax} \right)$$

8. $(x^2 + y^2)^2 = xy$

Solution:

Given $(x + y)^2 = 2axy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x , we get,

$$\Rightarrow \frac{d}{dx} \left[(x^2 + y^2)^2 \right] = \frac{d}{dx} (xy)$$

Now by applying product rule we get,

$$\Rightarrow 2(x^2 + y^2) \frac{d}{dx} (x^2 + y^2) = x \frac{dy}{dx} + y \frac{d}{dx} (x)$$

$$\Rightarrow 2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) = x \frac{dy}{dx} + y(1)$$

$$\Rightarrow 4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = x \frac{dy}{dx} + y$$

$$\Rightarrow 4y(x^2 + y^2) \frac{dy}{dx} - x \frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} [4y(x^2 + y^2) - x] = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x(x^2 + y^2) - y}{x - 4y(x^2 + y^2)}$$

9. $\tan^{-1}(x^2 + y^2)$

Solution:

Given $\tan^{-1}(x^2 + y^2) = a$,

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$10. e^{x-y} = \log \left(\frac{x}{y} \right)$$

Solution:

$$e^{x-y} = \log \left(\frac{x}{y} \right)$$

Given

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{d}{dx} (e^{x-y}) = \frac{d}{dx} \left\{ \log \left(\frac{x}{y} \right) \right\}$$

$$\Rightarrow e^{(x-y)} \frac{d}{dx} (x-y) = \frac{1}{\left(\frac{x}{y} \right)} \times \frac{d}{dx} \left(\frac{x}{y} \right)$$

Now by applying quotient rule we get

$$\Rightarrow e^{(x-y)} \left(1 - \frac{dy}{dx} \right) = \frac{y}{x} \left[\frac{y \frac{d}{dx}(x) - x \frac{dy}{dx}}{y^2} \right]$$

$$\Rightarrow e^{(x-y)} - e^{(x-y)} \frac{dy}{dx} = \frac{1}{xy} \left[y(1) - x \frac{dy}{dx} \right]$$

$$\Rightarrow e^{(x-y)} - e^{(x-y)} \frac{dy}{dx} = \frac{1}{x} - \frac{1}{y} \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} - e^{(x-y)} \frac{dy}{dx} = \frac{1}{x} - e^{(x-y)}$$

$$\Rightarrow \frac{dy}{dx} \left[\frac{1}{y} - \frac{e^{(x-y)}}{1} \right] = \frac{1}{x} - \frac{e^{(x-y)}}{1}$$

$$\Rightarrow \frac{dy}{dx} \left[\frac{1 - ye^{(x-y)}}{y} \right] = \frac{1 - xe^{(x-y)}}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \left[\frac{1 - xe^{(x-y)}}{1 - ye^{(x-y)}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{-x} \left[\frac{xe^{(x-y)} - 1}{ye^{(x-y)} - 1} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \left[\frac{xe^{(x-y)} - 1}{ye^{(x-y)} - 1} \right]$$

11. $\sin xy + \cos(x + y) = 1$

Solution:

Given $\sin xy + \cos(x + y) = 1$

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x , we get,

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\cos(x + y) = \frac{d}{dx}(1)$$

$$\Rightarrow \cos xy \frac{d}{dx}(xy) - \sin(x + y) \frac{d}{dx}(x + y) = 0$$

$$\Rightarrow \cos xy \left[x \frac{dy}{dx} + y \frac{d}{dx}(x) \right] - \sin(x + y) \left[1 + \frac{dy}{dx} \right] = 0$$

$$\Rightarrow \cos xy \left[x \frac{dy}{dx} + y(1) \right] - \sin(x + y) - \sin(x + y) \frac{dy}{dx} = 0$$

$$\Rightarrow x \cos xy \frac{dy}{dx} + y \cos xy - \sin(x + y) - \sin(x + y) \frac{dy}{dx} = 0$$

$$\Rightarrow [x \cos xy - \sin(x + y)] \frac{dy}{dx} = [\sin(x + y) - y \cos xy]$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{\sin(x + y) - y \cos xy}{x \cos xy - \sin(x + y)} \right]$$

12. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Solution:

Given $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Let $x = \sin A$ and $y = \sin B$

Then given equation becomes,

$$\Rightarrow \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \cos A + \cos B = a(\sin A - \sin B)$$

$$\Rightarrow a = \frac{\cos A + \cos B}{\sin A - \sin B}$$

Now by applying the formula we get,

$$\Rightarrow a = \frac{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}$$

$$\Rightarrow a = \cot \left(\frac{A-B}{2} \right)$$

$$\Rightarrow \cot^{-1} a = \frac{A-B}{2}$$

$$\Rightarrow 2 \cot^{-1} a = A - B$$

$$\Rightarrow 2 \cot^{-1} a = \sin^{-1} x - \sin^{-1} y.$$

Now by differentiating with respect to x we get,

$$\frac{d}{dx} (2 \cot^{-1} a) = \frac{d}{dx} (\sin^{-1} x) - \frac{d}{dx} (\sin^{-1} y)$$

$$\Rightarrow 0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

13. If $y = \sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Solution:

Given, $y = \sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

Let $x = \sin A$ and $y = \sin B$

Then given equation becomes,

$$\Rightarrow \sin B \sqrt{1-\sin^2 A} + \sin A \sqrt{1-\sin^2 B} = 1$$

Now by applying the identity, we get

$$\Rightarrow \sin B \cos A + \sin A \cos B = 1$$

$$\Rightarrow \sin(A+B) = 1$$

$$\Rightarrow A+B = \sin^{-1}(1)$$

Now by substituting the values of A and B, we get

$$\Rightarrow \sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$$

Now by differentiating with respect to x , we get

$$\Rightarrow \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\sin^{-1} y) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

14. If $xy = 1$, prove that $\frac{dy}{dx} + y^2 = 0$.

Solution:

Given $xy = 1$

Differentiating with respect to x , we get

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

By using product rule,

$$\Rightarrow x \frac{dy}{dx} + y \frac{d}{dx}(x) = 0.$$

$$\Rightarrow x \frac{dy}{dx} + y(1) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

We have $xy = 1$, therefore $x = 1/y$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{\frac{1}{y}}$$

$$\Rightarrow \frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} + y^2 = 0$$

15. If $xy^2 = 1$, prove that $2\frac{dy}{dx} + y^3 = 0$.

Solution:

Given $xy^2 = 1$

Now differentiating given equation with respect to x, we get

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(1)$$

$$\Rightarrow x \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x) = 0$$

$$\Rightarrow x(2y) \frac{dy}{dx} + y^2(1) = 0$$

$$\Rightarrow 2xy \frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{2xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{2x}$$

Now by substituting $x = 1/y^2$ in above equation we get

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{2\left(\frac{1}{y^2}\right)}$$

$$\Rightarrow 2\frac{dy}{dx} = -y^3$$

$$\Rightarrow 2\frac{dy}{dx} + y^3 = 0$$



EXERCISE 11.5

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Differentiate the following functions with respect to x:

1. $x^{1/x}$

Solution:

Let $y = x^{\frac{1}{x}}$

Taking log both the sides:

$$\Rightarrow \log y = \log x^{\frac{1}{x}}$$

We know that $\log x^a = a \log x$, substituting this in above equation we get

$$\Rightarrow \log y = \frac{1}{x} \log x$$

Differentiating with respect to x, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x} \log x\right)}{dx}$$

Now by using the product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^{-1})}{dx}$$

$$\text{We have } \left\{ \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} ; \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx} \right\}, \text{ by using this we get,}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \times \frac{1}{x} \frac{dx}{dx} + \log x \left(\frac{-1}{x^2} \right)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2} \log x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1 - \log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{1 - \log x}{x^2} \right)$$

Put the value of $y = x^{\frac{1}{x}}$

$$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}} \left(\frac{1 - \log x}{x^2} \right)$$

2. $x^{\sin x}$

Solution:

Let $y = x^{\sin x}$

Taking log both the sides

$$\log y = \log (x^{\sin x})$$

$$\log y = \sin x \log x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x , we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin x \log x)}{dx}$$

Now by using product rule, we can write as

$$\Rightarrow \frac{d(\log y)}{dx} = \sin x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x)}{dx}$$

Again we have, $\left\{ \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} \text{ \& } \frac{d(\sin x)}{dx} = \cos x \right\}$, by using this we can write as

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin x \times \frac{1}{x} \frac{dx}{dx} + \log x (\cos x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\sin x}{x} + \log x \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{\sin x}{x} + \log x \cos x \right)$$

Put the value of $y = x^{\sin x}$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cos x \right)$$

3. $(1 + \cos x)^x$

Solution:

$$\text{Let } y = (1 + \cos x)^x$$

Taking log on both the sides

$$\Rightarrow \log y = \log (1 + \cos x)^x$$

$$\Rightarrow \log y = x \log (1 + \cos x) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d[x \log (1 + \cos x)]}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d[\log(1 + \cos x)]}{dx} + \log(1 + \cos x) \times \frac{dx}{dx}$$

$$\text{Again we have, } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{(1 + \cos x)} \frac{d(1 + \cos x)}{dx} + \log(1 + \cos x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{(1 + \cos x)} (-\sin x) + \log(1 + \cos x)$$

$$\left\{ \frac{d(1 + \cos x)}{dx} = \frac{d(1)}{dx} + \frac{d(\cos x)}{dx} = 0 + (-\sin x) \frac{dx}{dx} = -\sin x \right\}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{-x \sin x}{1 + \cos x} + \log(1 + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{-x \sin x}{1 + \cos x} + \log(1 + \cos x) \right\}$$

Put the value of $y = (1 + \cos x)^x$

$$\Rightarrow \frac{dy}{dx} = (1 + \cos x)^x \left\{ \frac{-x \sin x}{1 + \cos x} + \log(1 + \cos x) \right\}$$

4. $x^{\cos^{-1} x}$

Solution:

Let $y = x^{\cos^{-1} x}$

Taking log both the sides

$$\Rightarrow \log y = \log x^{\cos^{-1} x}$$

$$\Rightarrow \log y = \cos^{-1} x \log x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos^{-1} x \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\cos^{-1} x)}{dx}$$

Again we have, $\left\{ \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} \text{ \& } \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}} \right\}$, from this we can write as

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\cos^{-1} x}{x} + \log x \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right\}$$

Put the value of $y = x^{\cos^{-1} x}$

$$\Rightarrow \frac{dy}{dx} = x^{\cos^{-1} x} \left\{ \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right\}$$

5. $(\log x)^x$

Solution:

$$\text{Let } y = (\log x)^x$$

Taking log both the sides

$$\Rightarrow \log y = \log (\log x)^x$$

$$\Rightarrow \log y = x \log (\log x) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log \log x)}{dx}$$

By product rule, we have

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log \log x)}{dx} + \log \log x \times \frac{dx}{dx}$$

$$\text{We know that } \left\{ \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{\log x} \frac{d(\log x)}{dx} + \log \log x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x}{\log x} \times \frac{1}{x} + \log \log x$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{1}{\log x} + \log \log x \right\}$$

Put the value of $y = (\log x)^x$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log \log x \right\}$$

6. $(\log x)^{\cos x}$ **Solution:**

$$\text{Let } y = (\log x)^{\cos x}$$

Taking log both the sides, we get

$$\Rightarrow \log y = \log (\log x)^{\cos x}$$

$$\Rightarrow \log y = \cos x \log (\log x) \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log \log x)}{dx} + \log \log x \times \frac{d(\cos x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$ & $\frac{d(\cos x)}{dx} = -\sin x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \cos x \times \frac{1}{\log x} \frac{d(\log x)}{dx} + \log \log x (-\sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\cos x}{\log x} \times \frac{1}{x} - \sin x \log \log x$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\cos x}{x \log x} - \sin x \log \log x \right\}$$

Put the value of $y = (\log x)^{\cos x}$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x} \left\{ \frac{\cos x}{x \log x} - \sin x \log \log x \right\}$$

7. $(\sin x)^{\cos x}$

Solution:

Let $y = (\sin x)^{\cos x}$

Taking log both the sides

$$\Rightarrow \log y = \log (\sin x)^{\cos x}$$

$$\Rightarrow \log y = \cos x \log \sin x \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log \sin x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\cos x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$; $\frac{d(\cos x)}{dx} = -\sin x$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \cos x \times \frac{1}{\sin x} \frac{d(\sin x)}{dx} + \log \sin x (-\sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \cot x (\cos x) - \sin x \log \sin x$$

$$\Rightarrow \frac{dy}{dx} = y \{ \cos x \cot x - \sin x \log \sin x \}$$

Put the value of $y = (\sin x)^{\cos x}$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x} \{ \cos x \cot x - \sin x \log \sin x \}$$

8. $e^{x \log x}$

Solution:

Let $y = e^{x \log x}$

Taking log both the sides, we get

$$\Rightarrow \log y = \log (e)^{x \log x}$$

$$\Rightarrow \log y = x \log x \log e \quad \{ \log x^a = a \log x \}$$

$$\Rightarrow \log y = x \log x \quad \{ \log e = 1 \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{x} \frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y \{ 1 + \log x \}$$

Put the value of $y = e^{x \log x}$

$$\Rightarrow \frac{dy}{dx} = e^{x \log x} \{ 1 + \log x \}$$

$$\Rightarrow \frac{dy}{dx} = e^{\log x^x} \{ 1 + \log x \} \{ e^{\log a} = a; a \log x = x^a \}$$

$$\Rightarrow \frac{dy}{dx} = x^x \{ 1 + \log x \}$$

9. $(\sin x)^{\log x}$

Solution:

Let $y = (\sin x)^{\log x}$

Taking log both the sides

$$\Rightarrow \log y = \log (\sin x)^{\log x}$$

$$\Rightarrow \log y = \log x \log \sin x \{ \log x^a = a \log x \}$$

Differentiating with respect to x , then we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log \sin x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\log x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log x \times \frac{1}{\sin x} \frac{d(\sin x)}{dx} + \log \sin x \left(\frac{1}{x} \frac{dx}{dx} \right)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\log x}{\sin x} (\cos x) + \frac{\log \sin x}{x}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \log x \cot x + \frac{\log \sin x}{x} \right\}$$

Put the value of $y = (\sin x)^{\log x}$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \left\{ \log x \cot x + \frac{\log \sin x}{x} \right\}$$

10. $10^{\log \sin x}$

Solution:

Let $y = 10^{\log \sin x}$

Taking log both the sides

$$\Rightarrow \log y = \log 10^{\log \sin x}$$

$$\Rightarrow \log y = \log \sin x \log 10 \{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log 10 \log \sin x)}{dx}$$

Now by using chain rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log 10 \times \frac{d(\log \sin x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log 10 \times \frac{1}{\sin x} \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\log 10}{\sin x} (\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y \{\log 10 \cot x\}$$

Put the value of $y = 10^{\log \sin x}$

$$\Rightarrow \frac{dy}{dx} = 10^{\log \sin x} \{\log 10 \cot x\}$$

11. $(\log x)^{\log x}$

Solution:

Let $y = (\log x)^{\log x}$

Taking log both the sides

$$\Rightarrow \log y = \log (\log x)^{\log x}$$

$$\Rightarrow \log y = \log x \log (\log x) \{\log x^a = a \log x\}$$

Differentiating with respect to x , then we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log (\log x))}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log (\log x))}{dx} + \log (\log x) \times \frac{d(\log x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log x \times \frac{1}{\log x} \frac{d(\log x)}{dx} + \log \log x \left(\frac{1}{x} \frac{dx}{dx} \right)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\log x}{\log x} \left(\frac{1}{x} \frac{dx}{dx} \right) + \frac{\log(\log x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{1}{x} + \frac{\log(\log x)}{x} \right\}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{1 + \log(\log x)}{x} \right\}$$

Put the value of $y = (\log x)^{\log x}$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\log x} \left\{ \frac{1 + \log(\log x)}{x} \right\}$$

12. $10^{(10^x)}$

Solution:

Let $y = 10^{(10^x)}$

Taking log both the sides

$$\Rightarrow \log y = \log 10^{(10^x)}$$

$$\Rightarrow \log y = 10^x \log 10 \{ \log x^a = a \log x \}$$

$$\Rightarrow \log y = (10 \log 10) x$$

Differentiating with respect to x ,

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\{(10 \log 10)x\}}{dx}$$

Here $10 \log 10$ is a constant term, therefore by using chain rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = 10 \times \log(10) \times \frac{d(x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} ; \frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 10 \log(10)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 10 \log(10)$$

$$\Rightarrow \frac{dy}{dx} = y\{10 \log(10)\}$$

Put the value of $y = 10^{(10^x)}$

$$\Rightarrow \frac{dy}{dx} = 10^{10x} \{10 \log(10)\}$$

13. $\sin(x^x)$

Solution:

Let $y = \sin(x^x)$

Take sin inverse both sides

$$\Rightarrow \sin^{-1} y = \sin^{-1}(\sin x^x)$$

$$\Rightarrow \sin^{-1} y = x^x$$

Taking log both the sides

$$\Rightarrow \log(\sin^{-1} y) = \log x^x$$

$$\Rightarrow \log(\sin^{-1} y) = x \log x \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = \frac{d(x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\text{We know that } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{\sin^{-1} y} \frac{d(\sin^{-1} y)}{dx} = x \times \frac{1}{x} \frac{dx}{dx} + \log x$$

Again we have, $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$ by using this result we get

$$\Rightarrow \frac{1}{\sin^{-1} y} \times \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{1}{\sin^{-1} y (\sqrt{1-y^2})} \frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} y (\sqrt{1-y^2}) (1 + \log x)$$

Put the value of $y = \sin(x^x)$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} (\sin x^x) (\sqrt{1 - \sin^2(x^x)}) (1 + \log x)$$

From $\sin^2 x + \cos^2 x = 1$, we can write as

$$\Rightarrow \frac{dy}{dx} = x^x (\sqrt{\cos^2(x^x)}) (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x \cos x^x (1 + \log x)$$

14. $(\sin^{-1} x)^x$

Solution:

Let $y = (\sin^{-1} x)^x$

Taking log both the sides

$$\Rightarrow \log y = \log (\sin^{-1} x)^x$$

$$\Rightarrow \log y = x \log (\sin^{-1} x) \{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log (\sin^{-1} x))}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log (\sin^{-1} x))}{dx} + \log(\sin^{-1} x) \times \frac{dx}{dx}$$

$$\text{We know that } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{\sin^{-1} x} \frac{d(\sin^{-1} x)}{dx} + \log(\sin^{-1} x)$$

$$\text{Again we have, } \frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \text{ by using this result we get}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x}{\sin^{-1} x} \times \frac{1}{\sqrt{1-x^2}} \frac{dx}{dx} + \log(\sin^{-1} x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x}{\sin^{-1} x \sqrt{1-x^2}} + \log(\sin^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{x}{\sin^{-1} x \sqrt{1-x^2}} + \log(\sin^{-1} x) \right\}$$

Put the value of $y = (\sin^{-1} x)^x$

$$\Rightarrow \frac{dy}{dx} = (\sin^{-1} x)^x \left\{ \frac{x}{\sin^{-1} x \sqrt{1-x^2}} + \log(\sin^{-1} x) \right\}$$

15. $x^{\sin^{-1} x}$

Solution:

$$\text{Let } y = x^{\sin^{-1} x}$$

Taking log both the sides

$$\Rightarrow \log y = \log x^{\sin^{-1} x}$$

$$\Rightarrow \log y = \sin^{-1} x \log x \{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin^{-1} x \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \sin^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin^{-1} x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$; $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin^{-1} x \times \frac{1}{x} \frac{dx}{dx} + \log x \times \frac{1}{\sqrt{1-x^2}} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}} \right\}$$

Put the value of $y = x^{\sin^{-1} x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1} x} \left\{ \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}} \right\}$$

16. $(\tan x)^{1/x}$

Solution:

Let $y = (\tan x)^{\frac{1}{x}}$

Taking log both the sides, we get

$$\Rightarrow \log y = \log(\tan x)^{\frac{1}{x}}$$

$$\Rightarrow \log y = \frac{1}{x} \log \tan x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x} \log \tan x\right)}{dx}$$

By using product rule, we can write as

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log \tan x)}{dx} + \log \tan x \times \frac{d(x^{-1})}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\tan x} \frac{d(\tan x)}{dx} + \log \tan x (-x^{-2})$$

Again we have $\frac{d(\tan x)}{dx} = \sec^2 x$ by using this result in the above expression

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x \tan x} (\sec^2 x) - \frac{\log \tan x}{x^2}$$

$$\frac{dy}{dx} = y \left\{ \frac{\sec^2 x}{x \tan x} - \frac{\log \tan x}{x^2} \right\}$$

Put the value of $y = (\tan x)^{\frac{1}{x}}$

$$\frac{dy}{dx} = (\tan x)^{\frac{1}{x}} \left\{ \frac{\sec^2 x}{x \tan x} - \frac{\log \tan x}{x^2} \right\}$$

17. $x^{\tan^{-1} x}$

Solution:

Let $y = x^{\tan^{-1} x}$

Taking log both the sides

$$\Rightarrow \log y = \log x^{\tan^{-1} x}$$

$$\Rightarrow \log y = \tan^{-1} x \log x \{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\tan^{-1} x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \tan^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\tan^{-1} x)}{dx}$$

Again we know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$; $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{u^2 + 1} \frac{du}{dx}$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \tan^{-1} x \times \frac{1}{x} \frac{dx}{dx} + \log x \times \frac{1}{x^2 + 1} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{\tan^{-1} x}{x} + \frac{\log x}{x^2 + 1}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\tan^{-1} x}{x} + \frac{\log x}{x^2 + 1} \right\}$$

Put the value of $y = x^{\tan^{-1} x}$

$$\Rightarrow \frac{dy}{dx} = x^{\tan^{-1} x} \left\{ \frac{\tan^{-1} x}{x} + \frac{\log x}{x^2 + 1} \right\}$$

18. (i) $(x^x)^{\sqrt{x}}$

Solution:

$$\text{Let } y = (x^x)^{\sqrt{x}}$$

Taking log both the sides

$$\Rightarrow \log y = \log(x^x)^{\sqrt{x}}$$

$$\Rightarrow \log y = \log(x)^x + \log \sqrt{x} \quad \{\text{Log}(ab) = \log a + \log b\}$$

$$\Rightarrow \log y = \log(x)^x + \log x^{\frac{1}{2}}$$

$$\Rightarrow \log y = x \log x + \frac{1}{2} \log x \quad \{\text{Log } x^a = a \log x\}$$

$$\Rightarrow \log y = \left(x + \frac{1}{2}\right) \log x$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\left(x + \frac{1}{2}\right) \log x\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \left(x + \frac{1}{2}\right) \times \frac{d(\log x)}{dx} + \log x \times \frac{d\left(x + \frac{1}{2}\right)}{dx}$$

Again we have to use chain rule for the above expression,

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \left(x + \frac{1}{2}\right) \times \frac{1}{x} \frac{dx}{dx} + \log x \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{(2x + 1)}{2} \times \frac{1}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{(2x + 1)}{2x} + \log x \right\}$$

Put the value of $y = (x)^x \sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = (x)^x \sqrt{x} \left\{ \frac{(2x + 1)}{2x} + \log x \right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x \sqrt{x} \left\{ \frac{2x}{2x} + \frac{1}{2x} + \log x \right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x \sqrt{x} \left\{ 1 + \frac{1}{2x} + \log x \right\}$$

$$18.(ii) \ x^{(\sin x - \cos x)} + \frac{x^2 - 1}{x^2 + 1}$$

Solution:

$$\text{Let } y = x^{(\sin x - \cos x)} + \frac{x^2 - 1}{x^2 + 1}$$

$$\Rightarrow y = a + b$$

$$\text{where } a = x^{(\sin x - \cos x)}; b = \frac{x^2 - 1}{x^2 + 1}$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = x^{(\sin x - \cos x)}$$

Taking log both the sides to the above expressions we get

$$\Rightarrow \log a = \log x^{(\sin x - \cos x)}$$

$$\Rightarrow \log a = (\sin x - \cos x) \log x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((\sin x - \cos x) \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = (\sin x - \cos x) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x - \cos x)}{dx}$$

To the above expression we have to use chain rule,

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = (\sin x - \cos x) \times \frac{1}{x} \frac{dx}{dx} + \log x \left(\frac{d(\sin x)}{dx} - \frac{d(\cos x)}{dx} \right)$$

$$\text{We know that } \frac{d(\cos x)}{dx} = -\sin x; \frac{d(\sin x)}{dx} = \cos x$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x (\cos x - (-\sin x))$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x (\cos x + \sin x)$$

$$\Rightarrow \frac{da}{dx} = a \left\{ \frac{\sin x - \cos x}{x} + \log x (\cos x + \sin x) \right\}$$

Put the value of $a = x^{(\sin x - \cos x)}$

$$\Rightarrow \frac{da}{dx} = x^{(\sin x - \cos x)} \left\{ \frac{\sin x - \cos x}{x} + \log x (\cos x + \sin x) \right\}$$

$$b = \frac{x^2 - 1}{x^2 + 1}$$

To differentiate above expression with respect to x we have to use quotient rule,

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 + 1) \frac{d(x^2 - 1)}{dx} - (x^2 - 1) \frac{d(x^2 + 1)}{dx}}{(x^2 + 1)^2}$$

Now by using chain rule, we get

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 + 2x) - (2x^3 - 2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 + 2x - 2x^3 + 2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{4x}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting all the values in above expressions we get

$$\Rightarrow \frac{dy}{dx} = x^{(\sin x - \cos x)} \left\{ \frac{\sin x - \cos x}{x} + \log x (\cos x + \sin x) \right\} + \frac{4x}{(x^2 + 1)^2}$$

18. (iii) $x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Solution:

$$\text{Let } y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

$$\Rightarrow y = a + b$$

$$\text{where } a = x^{x \cos x}; b = \frac{x^2 + 1}{x^2 - 1}$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = x^{x \cos x}$$

Taking log both the sides to the above equation we get

$$\Rightarrow \log a = \log x^{x \cos x}$$

$$\Rightarrow \log a = x \cos x \log x$$

$$\{\log x^a = a \log x\}$$

Differentiating with respect to x ,

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x \cos x \log x)}{dx}$$

Now by using product rule, we can write as

$$\Rightarrow \frac{d(\log a)}{dx} = x \cos x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x \cos x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x \cos x \times \frac{d(\log x)}{dx} + \log x \left\{ x \frac{d(\cos x)}{dx} + \cos x \right\}$$

Again we have, $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$ by using this result in the above expressions we get

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = x \cos x \times \frac{1}{x} \frac{dx}{dx} + \log x \{ x (-\sin x) + \cos x \}$$

We know that $\frac{d(\cos x)}{dx} = -\sin x$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{x \cos x}{x} + \log x (\cos x - x \sin x)$$

$$\Rightarrow \frac{da}{dx} = a \{ \cos x + \log x (\cos x - x \sin x) \}$$

Put the value of $a = x^{x \cos x}$:

$$\Rightarrow \frac{da}{dx} = x^{x \cos x} \{ \cos x + \log x (\cos x - x \sin x) \}$$

$$\Rightarrow \frac{da}{dx} = x^{x \cos x} \{ \cos x + \log x \cos x - x \sin x \log x \}$$

$$\Rightarrow \frac{da}{dx} = x^{x \cos x} \{ \cos x (1 + \log x) - x \sin x \log x \}$$

$$b = \frac{x^2 + 1}{x^2 - 1}$$

Now we have to differentiate above expression using quotient rule, then we get

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 - 1) \frac{d(x^2 + 1)}{dx} - (x^2 + 1) \frac{d(x^2 - 1)}{dx}}{(x^2 - 1)^2}$$

Now apply chain rule for the above equation,

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 - 2x) - (2x^3 + 2x)}{(x^2 - 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 - 2x - 2x^3 - 2x)}{(x^2 - 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{-4x}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

By substituting all values in the above expression we get

$$\Rightarrow \frac{dy}{dx} = x^{\cos x} \{ \cos x (1 + \log x) - x \sin x \log x \} - \frac{4x}{(x^2 + 1)^2}$$

18.(iv) $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Solution:

$$\text{Let } y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow y = a + b$$

$$\text{where } a = (x \cos x)^x; b = (x \sin x)^{\frac{1}{x}}$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = (x \cos x)^x$$

Taking log both the sides, we get

$$\Rightarrow \log a = \log(x \cos x)^x$$

$$\Rightarrow \log a = x \log(x \cos x)$$

$$\{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x \log(x \cos x))}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(x \cos x))}{dx} + \log(x \cos x) \times \frac{dx}{dx}$$

$$\text{We know that } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = x \times \frac{1}{x \cos x} \frac{d(x \cos x)}{dx} + \log(x \cos x)$$

Again by using product rule, we can write as

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{x}{x \cos x} \left\{ x \frac{d(\cos x)}{dx} + \cos x \right\} + \log(x \cos x)$$

$$\text{We have } \frac{d(\cos x)}{dx} = -\sin x \text{ using this result we can write as}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{1}{\cos x} \{ x(-\sin x) + \cos x \} + \log(x \cos x)$$

$$\Rightarrow \frac{da}{dx} = a \left\{ \frac{\cos x - x \sin x}{\cos x} + \log(x \cos x) \right\}$$

Put the value of $a = (x \cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (x \cos x)^x \left\{ \frac{\cos x - x \sin x}{\cos x} + \log(x \cos x) \right\}$$

$$\Rightarrow \frac{da}{dx} = (x \cos x)^x \{ 1 - x \tan x + \log(x \cos x) \}$$

$$b = (x \sin x)^{\frac{1}{x}}$$

Taking log both the sides

$$\Rightarrow \log b = \log(x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log b = \frac{1}{x} \log(x \sin x) \quad \{ \log x^a = a \log x \}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x} \log(x \sin x)\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x} \times \frac{d(\log(x \sin x))}{dx} + \log(x \sin x) \times \frac{d(x^{-1})}{dx}$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{1}{x} \times \frac{1}{x \sin x} \frac{d(x \sin x)}{dx} + \log(x \sin x) (-x^{-2})$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{1}{x^2 \sin x} \left(x \frac{d(\sin x)}{dx} + \sin x \frac{dx}{dx} \right) - \frac{\log(x \sin x)}{x^2}$$

$$\text{We know that } \frac{d(\sin x)}{dx} = \cos x$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{x \cos x + \sin x}{x^2 \sin x} - \frac{\log(x \sin x)}{x^2} \right\}$$

Put the value of $b = (x \sin x)^{\frac{1}{x}}$:

$$\Rightarrow \frac{db}{dx} = (x \sin x)^{\frac{1}{x}} \left\{ \frac{x \cos x + \sin x}{x^2 \sin x} - \frac{\log(x \sin x)}{x^2} \right\}$$

$$\Rightarrow \frac{db}{dx} = (x \sin x)^{\frac{1}{x}} \left\{ \frac{x \cot x + 1}{x^2} - \frac{\log(x \sin x)}{x^2} \right\}$$

$$\Rightarrow \frac{db}{dx} = (x \sin x)^{\frac{1}{x}} \left\{ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting all the values in above expression we get

$$\Rightarrow \frac{dy}{dx} = (x \cos x)^x \{1 - x \tan x + \log(x \cos x)\} \\ + (x \sin x)^{\frac{1}{x}} \left\{ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right\}$$

$$18.(v) \left(x + \frac{1}{x} \right)^x + x^{(1+\frac{1}{x})}$$

Solution:

$$\text{Let } y = \left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$$

$$\Rightarrow y = a + b$$

$$\text{where } a = \left(x + \frac{1}{x}\right)^x; b = x^{\left(1 + \frac{1}{x}\right)}$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = \left(x + \frac{1}{x}\right)^x$$

Taking log both the sides, we get

$$\Rightarrow \log a = \log \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log a = x \log \left(x + \frac{1}{x}\right) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d \left(x \log \left(x + \frac{1}{x}\right)\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d \left(\log \left(x + \frac{1}{x}\right)\right)}{dx} + \log \left(x + \frac{1}{x}\right) \times \frac{dx}{dx}$$

$$\text{Again we know that } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = x \times \frac{1}{x + \frac{1}{x}} \frac{d \left(x + \frac{1}{x}\right)}{dx} + \log \left(x + \frac{1}{x}\right)$$

Again by using chain rule in the above expression we get

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{x}{x^2 + 1} \left\{ \frac{dx}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx} \right\} + \log\left(x + \frac{1}{x}\right)$$

By using $\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}$,

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{x^2}{x^2 + 1} \left\{ 1 + \left(-\frac{1}{x^2}\right) \right\} + \log\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{da}{dx} = a \left\{ \frac{x^2}{x^2 + 1} \left\{ 1 - \frac{1}{x^2} \right\} + \log\left(x + \frac{1}{x}\right) \right\}$$

Put the value of $a = \left(x + \frac{1}{x}\right)^x$:

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x \left\{ \frac{x^2}{x^2 + 1} \left\{ 1 - \frac{1}{x^2} \right\} + \log\left(x + \frac{1}{x}\right) \right\}$$

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x \left\{ \frac{x^2}{x^2 + 1} - \frac{1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right) \right\}$$

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x \left\{ \frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right) \right\}$$

$$b = x^{\left(1 + \frac{1}{x}\right)}$$

Taking log both the sides

$$\Rightarrow \log b = \log x^{\left(1 + \frac{1}{x}\right)}$$

$$\Rightarrow \log b = \left(1 + \frac{1}{x}\right) \log x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\left(1 + \frac{1}{x}\right) \log x\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \left(1 + \frac{1}{x}\right) \times \frac{d(\log x)}{dx} + \log x \times \frac{d\left(1 + \frac{1}{x}\right)}{dx}$$

Again for the above expression we have to apply chain rule,

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x+1}{x} \times \frac{1}{x} \frac{dx}{dx} + \log x \left(\frac{d(1)}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx} \right)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x+1}{x^2} + \log x \left(-\frac{1}{x^2} \right)$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{x+1}{x^2} - \frac{\log x}{x^2} \right\}$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{x+1-\log x}{x^2} \right\}$$

Put the value of $b = x^{(1+\frac{1}{x})}$:

$$\Rightarrow \frac{db}{dx} = x^{(1+\frac{1}{x})} \left\{ \frac{x+1-\log x}{x^2} \right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting the all the values in above expression we get

$$\Rightarrow \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left\{ \frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right) \right\} + x^{(1+\frac{1}{x})} \left\{ \frac{x+1-\log x}{x^2} \right\}$$

18. (vi) $e^{\sin x} + (\tan x)^x$

Solution:

$$\text{Let } y = e^{\sin x} + (\tan x)^x$$

$$\Rightarrow y = a + b$$

$$\text{Where } a = e^{\sin x}; b = (\tan x)^x$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = e^{\sin x}$$

Taking log both the sides, we get

$$\Rightarrow \log a = \log e^{\sin x}$$

$$\Rightarrow \log a = \sin x \log e \quad \{\log x^a = a \log x\}$$

$$\Rightarrow \log a = \sin x \{\log e = 1\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\sin x)}{dx}$$

Again we have $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx} ; \frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \cos x$$

$$\Rightarrow \frac{da}{dx} = a (\cos x)$$

Put the value of $a = e^{\sin x}$

$$\Rightarrow \frac{da}{dx} = e^{\sin x} \cos x$$

$$b = (\tan x)^x$$

Taking log both the sides:

$$\Rightarrow \log b = \log (\tan x)^x$$

$$\Rightarrow \log b = x \log (\tan x) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log (\tan x))}{dx}$$

Again by using product rule,

$$\Rightarrow \frac{d(\log b)}{dx} = x \times \frac{d(\log(\tan x))}{dx} + \log(\tan x) \times \frac{dx}{dx}$$

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

We know that

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = x \times \frac{1}{\tan x} \frac{d(\tan x)}{dx} + \log(\tan x)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x}{\tan x} (\sec^2 x) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x \cos x}{\sin x} \left(\frac{1}{\cos^2 x} \right) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x}{\sin x} \left(\frac{1}{\cos x} \right) + \log(\tan x)$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{x}{\sin x \cos x} + \log(\tan x) \right\}$$

Put the value of $b = (\tan x)^x$

$$\Rightarrow \frac{db}{dx} = (\tan x)^x \left\{ \frac{x}{\sin x \cos x} + \log(\tan x) \right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x} \cos x + (\tan x)^x \left\{ \frac{x}{\sin x \cos x} + \log(\tan x) \right\}$$

18. (vii) $(\cos x)^x + (\sin x)^{1/x}$

Solution:

$$\text{Let } y = (\cos x)^x + (\sin x)^{\frac{1}{x}}$$

$$\Rightarrow y = a + b$$

$$\text{where } a = (\cos x)^x; b = (\sin x)^{\frac{1}{x}}$$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = (\cos x)^x$$

Taking log both the sides

$$\Rightarrow \log a = \log(\cos x)^x$$

$$\Rightarrow \log a = x \log(\cos x) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x \log(\cos x))}{dx}$$

Now by using product rule, we have

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(\cos x))}{dx} + \log(\cos x) \times \frac{dx}{dx}$$

$$\text{Again we have } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = x \times \frac{1}{\cos x} \frac{d(\cos x)}{dx} + \log(\cos x)$$

$$\text{We know that } \frac{d(\cos x)}{dx} = -\sin x$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{x}{\cos x} (-\sin x) + \log(\cos x)$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{-x \sin x}{\cos x} + \log(\cos x)$$

$$\Rightarrow \frac{da}{dx} = a \{-x \tan x + \log(\cos x)\}$$

Put the value of $a = (\cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (\cos x)^x \{-x \tan x + \log(\cos x)\}$$

$$b = (\sin x)^{\frac{1}{x}}$$

Taking log both the sides

$$\Rightarrow \log b = \log(\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log b = \frac{1}{x} \log(\sin x) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x} \log(\sin x)\right)}{dx}$$

Again by product rule we have

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x} \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{d(x^{-1})}{dx}$$

$$\text{We know that } \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{1}{x} \times \frac{1}{\sin x} \frac{d(\sin x)}{dx} + \log(\sin x) (-x^{-2})$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{1}{x \sin x} (\cos x) - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{\cos x}{x \sin x} - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2} \right\}$$

Put the value of $b = (\sin x)^{\frac{1}{x}}$:

$$\Rightarrow \frac{db}{dx} = (\sin x)^{\frac{1}{x}} \left\{ \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2} \right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x \{-x \tan x + \log(\cos x)\} + (\sin x)^{\frac{1}{x}} \left\{ \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2} \right\}$$

18.(viii) $x^{x^2-3} + (x-3)^{x^2}$

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

$$\Rightarrow y = a + b$$

where $a = x^{x^2-3}$; $b = (x-3)^{x^2}$

Now we have to differentiate $y = a + b$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = x^{x^2-3}$$

Taking log both the sides

$$\Rightarrow \log a = \log x^{x^2-3}$$

$$\Rightarrow \log a = (x^2 - 3) \log x \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((x^2 - 3) \log x)}{dx}$$

Now by using product rule,

$$\Rightarrow \frac{d(\log a)}{dx} = (x^2 - 3) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^2 - 3)}{dx}$$

Again by using chain rule we get

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = (x^2 - 3) \times \frac{1}{x} \frac{dx}{dx} + \log x \times (2x)$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{(x^2 - 3)}{x} + 2x \log x$$

$$\Rightarrow \frac{da}{dx} = a \left\{ \frac{(x^2 - 3)}{x} + 2x \log x \right\}$$

Put the value of $a = x^{x^2-3}$:

$$\Rightarrow \frac{da}{dx} = x^{x^2-3} \left\{ \frac{(x^2 - 3)}{x} + 2x \log x \right\}$$

$$b = (x - 3)^{x^2}$$

Taking log both the sides:

$$\Rightarrow \log b = (x - 3)^{x^2}$$

$$\Rightarrow \log b = x^2 \log(x - 3) \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x :

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x^2 \log(x - 3))}{dx}$$

Again by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = x^2 \times \frac{d(\log(x - 3))}{dx} + \log(x - 3) \times \frac{d(x^2)}{dx}$$

For the above expression now we have to use chain rule,

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = x^2 \times \frac{1}{(x - 3)} \frac{d(x - 3)}{dx} + \log(x - 3) \times (2x)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x^2}{(x - 3)} \left(\frac{dx}{dx} - \frac{d(3)}{dx} \right) + 2x \log(x - 3)$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = \frac{x^2}{(x - 3)} (1) + 2x \log(x - 3)$$

$$\Rightarrow \frac{db}{dx} = b \left\{ \frac{x^2}{(x-3)} + 2x \log(x-3) \right\}$$

Put the value of $b = (x-3)^{x^2}$:

$$\Rightarrow \frac{db}{dx} = (x-3)^{x^2} \left\{ \frac{x^2}{(x-3)} + 2x \log(x-3) \right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2-3} \left\{ \frac{(x^2-3)}{x} + 2x \log x \right\} + (x-3)^{x^2} \left\{ \frac{x^2}{(x-3)} + 2x \log(x-3) \right\}$$

19. $y = e^x + 10^x + x^x$

Solution:

Let $y = e^x + 10^x + x^x$

$$\Rightarrow y = a + b + c$$

Where $a = e^x$; $b = 10^x$; $c = x^x$

Now we have to differentiate $y = a + b + c$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$$a = e^x$$

Taking log both the sides

$$\Rightarrow \log a = \log e^x$$

$$\Rightarrow \log a = x \log e$$

$$\{\log x^a = a \log x\}$$

$$\Rightarrow \log a = x \{\log e = 1\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = 1$$

$$\Rightarrow \frac{da}{dx} = a$$

Put the value of $a = e^x$

$$\Rightarrow \frac{da}{dx} = e^x$$

$$b = 10^x$$

Taking log both the sides:

$$\Rightarrow \log b = \log 10^x$$

$$\Rightarrow \log b = x \log 10$$

$$\{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log 10)}{dx}$$

Now by using chain rule,

$$\Rightarrow \frac{d(\log b)}{dx} = \log 10 \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = b(\log 10)$$

$$\Rightarrow \frac{db}{dx} = b(\log 10)$$

Put the value of $b = 10^x$

$$\Rightarrow \frac{db}{dx} = 10^x(\log 10)$$

$$c = x^x$$

Taking log both the sides

$$\Rightarrow \log c = \log x^x$$

$$\Rightarrow \log c = x \log x$$

$$\{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{c} \frac{dc}{dx} = x \times \frac{1}{x} \frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{c} \frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of $c = x^x$

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x + 10^x(\log 10) + x^x\{1 + \log x\}$$

20. $y = x^n + n^x + x^x + n^n$

Solution:

$$\text{Let } y = x^n + n^x + x^x + n^n$$

$$\Rightarrow y = a + b + c + m$$

$$\text{Where } a = x^n; b = n^x; c = x^x; m = n^n$$

Now we have to differentiate $y = a + b + c + m$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$$a = x^n$$

Taking log both the sides

$$\Rightarrow \log a = \log x^n$$

$$\Rightarrow \log a = n \log x$$

$$\{\log x^a = a \log x\}$$

$$\Rightarrow \log a = n \log x \{\log e = 1\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(n \log x)}{dx}$$

Again by chain rule, we can write as

$$\Rightarrow \frac{d(\log a)}{dx} = n \frac{d(\log x)}{dx}$$

$$\text{We know that } \frac{d(\log u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = n \times \frac{1}{x} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{a} \frac{da}{dx} = \frac{n}{x}$$

$$\Rightarrow \frac{da}{dx} = \frac{an}{x}$$

Put the value of $a = x^n$

$$\frac{da}{dx} = \frac{nx^n}{x}$$

$$\frac{da}{dx} = nx^{n-1}$$

$$b = n^x$$

Taking log both the sides

$$\Rightarrow \log b = \log n^x$$

$$\Rightarrow \log b = x \log n \quad \{\log x^a = a \log x\}$$

Differentiating with respect to x using chain rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \log n \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{b} \frac{db}{dx} = b(\log n)$$

$$\Rightarrow \frac{db}{dx} = b(\log n)$$

Put the value of $b = n^x$

$$\Rightarrow \frac{db}{dx} = n^x(\log n)$$

$$c = x^x$$

Taking log both the sides

$$\Rightarrow \log c = \log x^x$$

$$\Rightarrow \log c = x \log x$$

$$\{\log x^a = a \log x\}$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{c} \frac{dc}{dx} = x \times \frac{1}{x} \frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{c} \frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of $c = x^x$

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

$$m = n^n$$

$$\Rightarrow \frac{dm}{dx} = \frac{d(n^n)}{dx}$$

$$\Rightarrow \frac{dm}{dx} = 0$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\} + 0$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\}$$

EXERCISE 11.6

PAGE NO: 11.98

1. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \text{to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{2y - 1}$.

Solution:

Given,

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \text{to } \infty}}}$$

$$y = \sqrt{x + y}$$

$$\text{Where } y = \sqrt{x + \sqrt{x + \dots \text{to } \infty}}$$

On squaring both sides,

$$y^2 = x + y$$

Differentiating both sides with respect to x ,

$$2y \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} (2y - 1) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y - 1}$$

Hence proved.

2. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \text{to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$.

Solution:

Given,

$$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \text{to } \infty}}}$$

$$y = \sqrt{\cos x + y}$$

Where $y = \sqrt{\cos x + \sqrt{\cos x + \dots \text{to } \infty}}$

Squaring on both sides,

$$y^2 = \cos x + y$$

Differentiating both sides with respect to x ,

$$2y \frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\frac{dy}{dx} (2y - 1) = -\sin x$$

$$\frac{dy}{dx} = -\frac{\sin x}{2y - 1}$$

$$\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$$

Hence proved.

3. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \text{to } \infty}}}$, prove that $(2y - 1) \frac{dy}{dx} = \frac{1}{x}$.

Solution:

Given

$$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \text{to } \infty}}}$$

$$y = \sqrt{\log x + y}$$

Where $y = \sqrt{\log x + \sqrt{\log x + \dots \text{to } \infty}}$

Squaring on both sides,

$$y^2 = \log x + y$$

Differentiating both sides with respect to x ,

$$2y \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x(2y - 1)}$$

Hence proved.

4. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \text{to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{2y - 1}$.

Solution:

Given,

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \text{to } \infty}}}$$

$$y = \sqrt{\tan x + y}$$

On squaring both sides,

$$y^2 = \tan x + y$$

Differentiating both sides with respect to x ,

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \sec^2 x$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{(2y - 1)}$$

Hence proved.

EXERCISE 11.7

PAGE NO: 11.103

Find dy/dx , when

1. $x = at^2$ and $y = 2at$

Solution:Given that $x = at^2$, $y = 2at$ Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating $y = 2at$ with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

2. $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

Solution:Given that $x = at^2$, $y = 2at$ Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating $y = 2at$ with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$x = a (\theta + \sin \theta)$$

Differentiating it with respect to θ ,

$$\frac{dx}{d\theta} = a(1 + \cos \theta) \dots\dots (1)$$

And,

$$y = a (1 - \cos \theta)$$

Differentiating it with respect to θ ,

$$\frac{dy}{d\theta} = a(0 + \sin \theta)$$

$$\frac{dy}{d\theta} = a \sin \theta \dots\dots (2)$$

Using equation (1) and (2),

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{a \sin \theta}{a(1 - \cos \theta)} \\ &= \frac{\frac{2 \sin \theta}{2} \frac{(\cos \theta)}{2}}{\frac{2 \sin^2 \theta}{2}}, \end{aligned}$$

$$\left\{ \text{Since, } 1 - \cos \theta = \frac{2 \sin^2 \theta}{2} \right\}$$

$$= \frac{dy}{dx} = \frac{\tan \theta}{2}$$

3. $x = a \cos \theta$ and $y = b \sin \theta$

Solution:

Given $x = a \cos \theta$ and $y = b \sin \theta$

Now by differentiating x with respect to θ we get,

$$\frac{dx}{d\theta} = \frac{d(a\cos\theta)}{d\theta} = -a\sin\theta$$

Again by differentiating y with respect to θ we get,

$$\frac{dy}{d\theta} = \frac{d(b\sin\theta)}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

4. $x = a e^{\theta} (\sin \theta - \cos \theta)$, $y = a e^{\theta} (\sin \theta + \cos \theta)$

Solution:

Given that $x = a e^{\theta} (\sin \theta - \cos \theta)$

Differentiating it with respect to θ

$$\frac{dx}{d\theta} = a \left[e^{\theta} \frac{d(\sin\theta - \cos\theta)}{d\theta} + (\sin\theta - \cos\theta) \frac{d(e^{\theta})}{d\theta} \right]$$

$$= a [e^{\theta} (\cos \theta + \sin \theta) + (\sin \theta - \cos \theta) e^{\theta}]$$

$$\frac{dx}{d\theta} = a[2e^{\theta}\sin\theta] \dots\dots (1)$$

And also given that, $y = a e^{\theta} (\sin \theta + \cos \theta)$

Differentiating it with respect to θ ,

$$\frac{dy}{d\theta} = a \left[e^{\theta} \frac{d(\sin\theta + \cos\theta)}{d\theta} + (\sin\theta + \cos\theta) \frac{d(e^{\theta})}{d\theta} \right]$$

$$= a [e^{\theta} (\cos \theta - \sin \theta) + (\sin \theta + \cos \theta) e^{\theta}]$$

$$\frac{dy}{d\theta} = a[2e^{\theta}\cos\theta] \dots\dots (2)$$

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{a(2e^{\theta}\cos\theta)}{a(2e^{\theta}\sin\theta)}$$

$$\frac{dy}{dx} = \cot\theta$$

5. $x = b \sin^2 \theta$ and $y = a \cos^2 \theta$

Solution:

Given that $x = b \sin^2 \theta$

Now by differentiating above equation with respect to θ , we get

$$\frac{dx}{d\theta} = \frac{d(b\sin^2 \theta)}{d\theta} = 2b\sin\theta\cos\theta$$

And also given that $y = a \cos^2 \theta$

Now by differentiating above equation with respect to θ , we get

$$\frac{dy}{d\theta} = d(a\cos^2 \theta) = -2a\cos\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2a\cos\theta\sin\theta}{2b\sin\theta\cos\theta} = -\frac{a}{b}$$

6. $x = a(1 - \cos \theta)$ and $y = a(\theta + \sin \theta)$ at $\theta = \pi/2$

Solution:

Given $x = a(1 - \cos \theta)$

Differentiate x with respect to θ , we get

$$\frac{dx}{d\theta} = \frac{d[a(1 - \cos\theta)]}{d\theta} = a(\sin\theta)$$

And also given that $y = a(\theta + \sin \theta)$

Differentiate x with respect to θ , we get

$$\frac{dy}{d\theta} = \frac{d(\theta + \sin\theta)}{d\theta} = a(1 + \cos\theta)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos\theta)}{a(\sin\theta)} \bigg|_{\left(\theta = \frac{\pi}{2}\right)} \\ &= \frac{a(1 + 0)}{a} = 1 \end{aligned}$$

$$7. x = \frac{e^t + e^{-t}}{2} \text{ and } y = \frac{e^t - e^{-t}}{2}$$

Solution:

$$\text{Given } x = \frac{e^t + e^{-t}}{2}$$

Differentiating above equation with respect to t

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2} \left[\frac{d(e^t)}{dt} + \frac{d(e^{-t})}{dt} \right] \\ &= \frac{1}{2} \left[e^t + e^{-t} \frac{d(-t)}{dt} \right] \end{aligned}$$

$$\frac{dx}{dt} = \frac{1}{2} (e^t - e^{-t}) = y \dots\dots (1)$$

$$\text{And also given that } y = \frac{e^t - e^{-t}}{2}$$

Differentiating above equation with respect to t,

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{2} \left[\frac{d(e^t)}{dt} - \frac{d(e^{-t})}{dt} \right] \\ &= \frac{1}{2} \left[e^t - e^{-t} \frac{d(-t)}{dt} \right] \end{aligned}$$

$$= \frac{1}{2}(e^t - e^{-t}(-1))$$

$$\frac{dy}{dt} = \frac{e^t + e^{-t}}{2} = x \dots\dots (2)$$

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$8. \ x = \frac{3at}{1+t^2} \text{ and } y = \frac{3at^2}{1+t^2}$$

Solution:

$$\text{Given } x = \frac{3at}{1+t^2}$$

Differentiating above equation with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{\left((1+t^2) \frac{d(3at)}{dt} - 3at \frac{d(1+t^2)}{dt} \right)}{(1+t^2)^2} \right]$$

$$= \left[\frac{(1+t^2)(3a) - 3at(2t)}{(1+t^2)^2} \right]$$

$$= \left[\frac{(3a) + 3at^2 - 6at^2}{(1+t^2)^2} \right]$$

$$= \left[\frac{3a - 3at^2}{(1+t^2)^2} \right]$$

$$\frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2} \dots\dots (1)$$

And also given that $y = \frac{3at^2}{1+t^2}$

Differentiating above equation with respect to t using quotient rule

$$\frac{dy}{dx} = \left[\frac{(1+t^2) \frac{d(3at^2)}{dt} - 3at^2 \frac{d(1+t^2)}{dt}}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2)(6at) - (3at^2)(2t)}{(1+t^2)^2} \right]$$

$$= \left[\frac{6at + 6at^3 - 6at^3}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = \frac{6at}{(1+t^2)^2} \dots\dots (2)$$

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{(1+t^2)^2} \times \frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{2t}{1-t^2}$$

9. $x = a (\cos \theta + \theta \sin \theta)$ and $y = a (\sin \theta - \theta \cos \theta)$

Solution:

Given $x = a (\cos \theta + \theta \sin \theta)$

Now differentiating x with respect to θ

$$\frac{dx}{d\theta} = a \left[\frac{d}{d\theta} \cos \theta + \frac{d}{d\theta} (\theta \sin \theta) \right]$$

$$= a \left[-\sin \theta + \frac{\theta d}{d\theta} (\sin \theta) + \sin \theta \frac{d}{d\theta} (\theta) \right]$$

$$= a [-\sin \theta + \theta \cos \theta + \sin \theta] = a \theta \cos \theta$$

And also given $y = a (\sin \theta - \theta \cos \theta)$,

Now differentiating x with respect to θ

$$\begin{aligned}\frac{dy}{d\theta} &= a \left[\frac{d}{d\theta} (\sin\theta) - \frac{d}{d\theta} (\theta \cos\theta) \right] \\ &= a \left[\cos\theta - \left\{ \frac{\theta d}{d\theta} (\cos\theta) + \cos\theta \frac{d}{d\theta} (\theta) \right\} \right] \\ &= a [\cos\theta + \theta \sin\theta - \cos\theta] \\ &= a \theta \sin \theta\end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta \sin\theta}{a\theta \cos\theta} = \tan\theta$$

10. $x = e^{\theta} \left(\theta + \frac{1}{\theta} \right)$ and $y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$

Solution:

Given $x = e^{\theta} \left(\theta + \frac{1}{\theta} \right)$

Differentiating x with respect to θ using the product rule,

$$\begin{aligned}\frac{dx}{d\theta} &= e^{\theta} \frac{d}{d\theta} \left(\theta + \frac{1}{\theta} \right) + \left(\theta + \frac{1}{\theta} \right) \frac{d}{d\theta} (e^{\theta}) \\ &= e^{\theta} \left(1 - \frac{1}{\theta^2} \right) + \frac{\theta^2 + 1}{\theta} (e^{\theta}) \\ &= e^{\theta} \left(1 - \frac{1}{\theta^2} + \frac{\theta^2 + 1}{\theta} \right) \\ &= e^{\theta} \left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2} \right)\end{aligned}$$

$$\frac{dx}{d\theta} = e^{\theta} \left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2} \right) \dots\dots (1)$$

And also given that, $y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$

Differentiating y with respect to θ using the product rule,

$$\frac{dy}{d\theta} = e^{-\theta} \frac{d}{d\theta} \left(\theta - \frac{1}{\theta} \right) + \left(\theta - \frac{1}{\theta} \right) \frac{d}{d\theta} (e^{-\theta})$$

$$= e^{-\theta} \left(1 + \frac{1}{\theta^2} \right) + \left(\theta - \frac{1}{\theta} \right) e^{-\theta} \frac{d}{d\theta} (-\theta)$$

$$= e^{-\theta} \left(1 + \frac{1}{\theta^2} \right) + \left(\theta - \frac{1}{\theta} \right) e^{-\theta} (-1)$$

$$\frac{dy}{d\theta} = e^{-\theta} \left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta} \right)$$

$$= e^{-\theta} \left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2} \right)$$

$$\frac{dy}{d\theta} = e^{-\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2} \right) \dots\dots (2)$$

Divide equation (2) by (1)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = e^{-\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2} \right) \times \frac{1}{e^{\theta} \left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2} \right)}$$

$$= e^{-2\theta} \left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1} \right)$$

11. $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$

Solution:

Given, $x = \frac{2t}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\begin{aligned}\frac{dx}{dt} &= \left[\frac{(1+t^2) \frac{d}{dt}(2t) - 2t \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right] \\ &= \left[\frac{(1+t^2)(2) - 2t(2t)}{(1+t^2)^2} \right] \\ &= \left[\frac{2 + 2t^2 - 4t^2}{(1+t^2)^2} \right] \\ &= \left[\frac{2 - 2t^2}{(1+t^2)^2} \right] \\ \frac{dx}{dt} &= \left[\frac{2-2t^2}{(1+t^2)^2} \right] \dots\dots (1)\end{aligned}$$

And also given that, $y = \frac{1-t^2}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\begin{aligned}\frac{dy}{dt} &= \left[\frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right] \\ &= \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \right] \\ &= \left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2} \right] \\ \frac{dy}{dt} &= \left[\frac{-4t}{(1+t^2)^2} \right] \dots\dots (2)\end{aligned}$$

Dividing equation (2) by (1),

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left[\frac{-4t}{(1+t^2)^2} \right] \times \frac{1}{\left[\frac{2-2t^2}{(1+t^2)^2} \right]} \\ &= -\frac{2t}{1-t^2}\end{aligned}$$

$$\frac{dy}{dx} = -\frac{x}{y} \left[\text{since, } \frac{x}{y} = \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} = \frac{2t}{1-t^2} \right]$$

12. $x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1} \frac{1}{\sqrt{1+t^2}}$, $t \in R$

Solution:

Given $x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}$

Differentiating x with respect to t using chain rule,

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{\sqrt{1 - \left(\frac{1}{\sqrt{1+t^2}}\right)^2}} \frac{d}{dt} \left(\frac{1}{\sqrt{1+t^2}} \right) \\ &= -\frac{1}{\sqrt{1 - \frac{1}{1+t^2}}} \left\{ -\frac{1}{2(1+t^2)^{\frac{3}{2}}} \right\} \frac{d}{dt} (1+t^2) \\ &= -\frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}} \left\{ -\frac{1}{2(1+t^2)^{\frac{3}{2}}} \right\} (2t) \\ &= -\frac{t}{\sqrt{t^2} \times (1+t^2)} \\ \frac{dx}{dt} &= -\frac{1}{1+t^2} \dots\dots (1) \end{aligned}$$

Also given that, $y = \sin^{-1} \frac{1}{\sqrt{1+t^2}}$

Differentiating y with respect to t using chain rule,

$$\frac{dy}{dt} = \frac{1}{\sqrt{1 - \left(\frac{1}{\sqrt{1+t^2}}\right)^2}} \frac{d}{dt} \left(\frac{1}{\sqrt{1+t^2}} \right)$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{1 - \frac{1}{1+t^2}}} \left\{ -\frac{1}{2(1+t^2)^{\frac{3}{2}}} \right\} \frac{d}{dt} (1+t^2) \\
 &= \frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}} \left\{ -\frac{1}{2(1+t^2)^{\frac{3}{2}}} \right\} (2t) \\
 &= \frac{t}{\sqrt{t^2} \times (1+t^2)} \\
 \frac{dy}{dt} &= -\frac{1}{1+t^2} \dots\dots (2)
 \end{aligned}$$

Dividing equation (2) by (1),

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{1+t^2} \times -\frac{1+t^2}{1} \\
 \frac{dy}{dx} &= 1
 \end{aligned}$$

13. $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Solution:

Given $x = \frac{1-t^2}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\begin{aligned}
 \frac{dx}{dt} &= \left[\frac{(1+t^2) \frac{d}{dt} (1-t^2) - (1-t^2) \frac{d}{dt} (1+t^2)}{(1+t^2)^2} \right] \\
 &= \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \right]
 \end{aligned}$$

$$= \left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2} \right]$$

$$\frac{dx}{dt} = \left[\frac{-4t}{(1+t^2)^2} \right] \dots\dots (1)$$

And also given that, $y = \frac{2t}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\frac{dy}{dt} = \left[\frac{(1+t^2) \frac{d}{dt}(2t) - (2t) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right]$$

$$= \left[\frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2} \right]$$

$$= \left[\frac{2 + 2t^2 - 4t^2}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \dots\dots (2)$$

Divide equation (2) by (1) so,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\frac{-4t}{(1+t^2)^2}}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{-4t}$$

14. If $x = 2 \cos \theta - \cos 2\theta$ and $y = 2 \sin \theta - \sin 2\theta$, prove that $\frac{dy}{dx} = \tan \left(\frac{3\theta}{2} \right)$.

Solution:

Given $x = 2 \cos \theta - \cos 2\theta$

Differentiating x with respect to θ using chain rule,

$$\frac{dx}{d\theta} = 2(-\sin\theta) - (-\sin 2\theta) \frac{d}{d\theta}(2\theta)$$

$$= -2\sin\theta + 2\sin 2\theta$$

$$\frac{dx}{d\theta} = 2(\sin 2\theta - \sin\theta) \dots\dots (1)$$

And also given that, $y = 2\sin \theta - \sin 2\theta$

Differentiating y with respect to θ using chain rule,

$$\frac{dy}{d\theta} = 2\cos\theta - \cos 2\theta \frac{d}{d\theta}(2\theta)$$

$$= 2\cos\theta - \cos 2\theta(2)$$

$$= 2\cos\theta - 2\cos 2\theta$$

$$\frac{dy}{d\theta} = 2(\cos\theta - \cos 2\theta) \dots\dots (2)$$

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2(\cos\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)}$$

$$= \frac{(\cos\theta - \cos 2\theta)}{(\sin 2\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{-2 \sin\left(\frac{\theta + 2\theta}{2}\right) \sin\left(\frac{\theta - 2\theta}{2}\right)}{2 \cos\left(\frac{\theta + 2\theta}{2}\right) \sin\left(\frac{2\theta - \theta}{2}\right)}$$

$$\left[\cos a - \cos b = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \right]$$

$$= - \frac{\sin\left(\frac{3\theta}{2}\right) \left(\sin\left(-\frac{\theta}{2}\right)\right)}{\cos\left(\frac{3\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)}$$

$$= - \frac{\sin\left(\frac{3\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)}$$

$$\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$$



EXERCISE 11.8

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1. Differentiate x^2 with respect to x^3 .**Solution:**Let $u = x^2$ and $v = x^3$.We have to differentiate u with respect to v that is find $\frac{du}{dv}$.On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{du}{dx} = 2x^{2-1}$$

$$\therefore \frac{du}{dx} = 2x$$

Now, on differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(x^3)$$

$$\Rightarrow \frac{dv}{dx} = 3x^{3-1}$$

$$\therefore \frac{dv}{dx} = 3x^2$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{3x^2}$$

$$\therefore \frac{du}{dv} = \frac{2}{3x}$$

$$\text{Thus, } \frac{du}{dv} = \frac{2}{3x}$$

2. Differentiate $\log(1+x^2)$ with respect to $\tan^{-1}x$.**Solution:**

Let $u = \log(1+x^2)$ and $v = \tan^{-1}x$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx} [\log(1+x^2)]$$

$$\text{We know } \frac{d}{dx} (\log x) = \frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \frac{d}{dx} (1+x^2)$$

Now by using chain rule, we get

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \left[\frac{d}{dx} (1) + \frac{d}{dx} (x^2) \right]$$

However, $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} [0 + 2x^{2-1}]$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} [2x]$$

$$\therefore \frac{du}{dx} = \frac{2x}{1+x^2}$$

Now, on differentiating v with respect to x , we get

$$\therefore \frac{dv}{dx} = \frac{1}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{\frac{1}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{1+x^2} \times (1+x^2)$$

$$\therefore \frac{du}{dv} = 2x$$

Thus, $\frac{du}{dv} = 2x$

3. Differentiate $(\log x)^x$ with respect to $\log x$.

Solution:

Let $u = (\log x)^x$ and $v = \log x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = (\log x)^x$

Taking log on both sides, we get

$$\log u = \log (\log x)^x$$

$$\Rightarrow \log u = x \times \log (\log x) \quad [\because \log a^m = m \times \log a]$$

On differentiating both the sides with respect to x , we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[x \times \log(\log x)]$$

We know that $(u v)' = v u' + u v'$

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\log x) \frac{d}{dx}(x) + x \frac{d}{dx}[\log(\log x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\log x) \times 1 + x \left[\frac{1}{\log x} \frac{d}{dx}(\log x) \right]$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \log(\log x) + \frac{x}{\log x} \frac{d}{dx}(\log x)$$

But, $u = (\log x)^x$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{(\log x)^x} \frac{du}{dx} = \log(\log x) + \frac{x}{\log x} \times \frac{1}{x}$$

$$\Rightarrow \frac{1}{(\log x)^x} \frac{du}{dx} = \log(\log x) + \frac{1}{\log x}$$

$$\therefore \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$$

Now, on differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\log x)$$

$$\therefore \frac{dv}{dx} = \frac{1}{x}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]}{\frac{1}{x}}$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x \left[\frac{\log(\log x) \log x + 1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dv} = \frac{x(\log x)^x}{\log x} [\log(\log x) \log x + 1]$$

$$\therefore \frac{du}{dv} = x(\log x)^{x-1} [1 + \log x \log(\log x)]$$

$$\text{Thus, } \frac{du}{dv} = x(\log x)^{x-1} [1 + \log x \log(\log x)]$$

4. Differentiate $\sin^{-1} \sqrt{1-x^2}$ with respect to $\cos^{-1}x$, if

(i) $x \in (0, 1)$

(ii) $x \in (-1, 0)$

Solution:

(i) Given $\sin^{-1} \sqrt{1-x^2}$

Let $u = \sin^{-1} \sqrt{1-x^2}$ and $v = \cos^{-1}x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1} \sqrt{1-x^2}$

By substituting $x = \cos \theta$, we have

$$u = \sin^{-1} \sqrt{1 - (\cos \theta)^2}$$

$$\Rightarrow u = \sin^{-1} \sqrt{1 - \cos^2 \theta}$$

$$\Rightarrow u = \sin^{-1} \sqrt{\sin^2 \theta} \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(\sin \theta)$$

Given $x \in (0, 1)$

However, $x = \cos \theta$.

$$\Rightarrow \cos \theta \in (0, 1)$$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin \theta) = \theta$.

$$\Rightarrow u = \cos^{-1}x$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\text{We know } \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{\sqrt{1-x^2}} \times (-\sqrt{1-x^2})$$

$$\therefore \frac{du}{dv} = 1$$

$$\text{Thus, } \frac{du}{dv} = 1$$

(ii) Given $\sin^{-1} \sqrt{1-x^2}$

Let $u = \sin^{-1} \sqrt{1-x^2}$ and $v = \cos^{-1}x$.

Now we have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1} \sqrt{1 - x^2}$

By substituting $x = \cos \theta$, we get

$$u = \sin^{-1} \sqrt{1 - (\cos \theta)^2}$$

$$\Rightarrow u = \sin^{-1} \sqrt{1 - \cos^2 \theta}$$

$$\Rightarrow u = \sin^{-1} \sqrt{\sin^2 \theta} \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(\sin \theta)$$

Given $x \in (-1, 0)$

However, $x = \cos \theta$.

$$\Rightarrow \cos \theta \in (-1, 0)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{2}, \pi\right)$$

Hence, $u = \sin^{-1}(\sin \theta) = \pi - \theta$.

$$\Rightarrow u = \pi - \cos^{-1} x$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - \cos^{-1} x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(\cos^{-1} x)$$

We know $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - \left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1} x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\text{We have } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times (-\sqrt{1-x^2})$$

$$\therefore \frac{du}{dv} = -1$$

$$\text{Thus, } \frac{du}{dv} = -1$$

5. Differentiate $\sin^{-1}(4x\sqrt{1-4x^2})$ with respect to $\sqrt{1-4x^2}$ if,

(i) $x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

(ii) $x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

(iii) $x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

Solution:

(i) Let

$$u = \sin^{-1}(4x\sqrt{1-4x^2}) \text{ And } v = \sqrt{1-4x^2}.$$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}(4x\sqrt{1-4x^2})$

$$\Rightarrow u = \sin^{-1}(4x\sqrt{1-(2x)^2})$$

By substituting $2x = \cos \theta$, we have

$$u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{\sin^2 \theta}) \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sin \theta)$$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

Given $x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

However, $2x = \cos \theta \Rightarrow x = \frac{\cos \theta}{2}$

$$\Rightarrow \frac{\cos \theta}{2} \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \cos \theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$$\Rightarrow u = \pi - 2\cos^{-1}(2x)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}[\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}[2 \cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2 \frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2 \left[-\frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx}(2x) \right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}} \left[\frac{d}{dx}(2x) \right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}} \left[2 \frac{d}{dx}(x) \right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

Now, we have $v = \sqrt{1-4x^2}$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sqrt{1-4x^2})$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1-4x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2} (1 - 4x^2)^{\frac{1}{2}-1} \frac{d}{dx} (1 - 4x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2} (1 - 4x^2)^{-\frac{1}{2}} \left[\frac{d}{dx} (1) - \frac{d}{dx} (4x^2) \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}} \left[\frac{d}{dx} (1) - 4 \frac{d}{dx} (x^2) \right]$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}} [0 - 4(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}} [-8x]$$

$$\therefore \frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x} \right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

(ii) Let

$$u = \sin^{-1}(4x\sqrt{1-4x^2}) \text{ and } v = \sqrt{1-4x^2}.$$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

$$\text{We have } u = \sin^{-1}(4x\sqrt{1-4x^2})$$

$$\Rightarrow u = \sin^{-1}(4x\sqrt{1-(2x)^2})$$

By substituting $2x = \cos \theta$, we have

$$u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{\sin^2 \theta}) \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sin \theta)$$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

$$\text{Given } x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$$

$$\text{However, } 2x = \cos \theta \Rightarrow x = \frac{\cos \theta}{2}$$

$$\Rightarrow \frac{\cos \theta}{2} \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$$

$$\Rightarrow \cos \theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

$$\text{Hence, } u = \sin^{-1}(\sin 2\theta) = 2\theta.$$

$$\Rightarrow u = 2\cos^{-1}(2x)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx} [2 \cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2 \frac{d}{dx} [\cos^{-1}(2x)]$$

We know $\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 2 \left[-\frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx} (2x) \right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}} \left[\frac{d}{dx} (2x) \right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}} \left[2 \frac{d}{dx} (x) \right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}} \frac{d}{dx} (x)$$

However, $\frac{d}{dx} (x) = 1$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = \frac{1}{x}$$

$$\text{Thus, } \frac{du}{dv} = \frac{1}{x}$$

(iii) Let

$$u = \sin^{-1}(4x\sqrt{1-4x^2}) \text{ And } v = \sqrt{1-4x^2}.$$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

$$\text{We have } u = \sin^{-1}(4x\sqrt{1-4x^2})$$

$$\Rightarrow u = \sin^{-1}(4x\sqrt{1-(2x)^2})$$

By substituting $2x = \cos \theta$, we have

$$u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{1-(\cos \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sqrt{\sin^2 \theta}) \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(2 \cos \theta \sin \theta)$$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

$$\text{Given } x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$$\text{However, } 2x = \cos \theta \Rightarrow x = \frac{\cos \theta}{2}$$

$$\Rightarrow \frac{\cos \theta}{2} \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \cos \theta \in \left(-1, -\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{4}, \pi\right)$$

$$\Rightarrow 2\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\pi - 2\theta$.

$$\Rightarrow u = 2\pi - 2\cos^{-1}(2x)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}[2\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(2\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-4x^2}}{4x} \right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

6. Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ with respect to $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$, if $-1 < x < 1$, $x \neq 0$.

Solution:

Let $u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ and $v = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$

By substituting $x = \tan \theta$, we have

$$u = \tan^{-1} \left(\frac{\sqrt{1+(\tan \theta)^2}-1}{\tan \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\sqrt{1 + \tan^2 \theta} - 1}{\tan \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\sqrt{\sec^2 \theta - 1}}{\tan \theta} \right) [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{1 - \cos \left(2 \times \frac{\theta}{2} \right)}{\sin \left(2 \times \frac{\theta}{2} \right)} \right)$$

But, $\cos 2\theta = 1 - 2\sin^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$.

$$\Rightarrow u = \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\tan \frac{\theta}{2} \right)$$

Given $-1 < x < 1 \Rightarrow x \in (-1, 1)$

However, $x = \tan \theta$

$\Rightarrow \tan \theta \in (-1, 1)$

$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right)$

$$\Rightarrow \frac{\theta}{2} \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$$

Hence, $u = \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2}$

$$\Rightarrow u = \frac{1}{2} \tan^{-1} x$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{1}{2} \tan^{-1} x \right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} \frac{d}{dx} (\tan^{-1} x)$$

We know $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{1}{2(1+x^2)}$$

Now, we have $v = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$

By substituting $x = \tan \theta$, we have

$$v = \sin^{-1} \left(\frac{2 \tan \theta}{1 + (\tan \theta)^2} \right)$$

$$\Rightarrow v = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right)$$

$$\Rightarrow v = \sin^{-1} \left(\frac{2 \tan \theta}{\sec^2 \theta} \right) \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow v = \sin^{-1} \left(\frac{2 \times \frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos^2 \theta}} \right)$$

$$\Rightarrow v = \sin^{-1} \left(2 \times \frac{\sin \theta}{\cos \theta} \times \cos^2 \theta \right)$$

$$\Rightarrow v = \sin^{-1}(2\sin\theta\cos\theta)$$

But, $\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow v = \sin^{-1}(\sin 2\theta)$$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $v = \sin^{-1}(\sin 2\theta) = 2\theta$

$$\Rightarrow v = 2\tan^{-1}x$$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2 \frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = \frac{1}{4}$$

$$\text{Thus, } \frac{du}{dv} = \frac{1}{4}$$

7. Differentiate $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, if,

(i) $x \in (0, 1/\sqrt{2})$

(ii) $x \in (1/\sqrt{2}, 1)$

Solution:

(i) Let

$$u = \sin^{-1}(2x\sqrt{1-x^2}) \text{ And } v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right).$$

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

$$\text{We have } u = \sin^{-1}(2x\sqrt{1-x^2})$$

By substituting $x = \sin \theta$, we have

$$u = \sin^{-1}(2 \sin \theta \sqrt{1 - (\sin \theta)^2})$$

$$\Rightarrow u = \sin^{-1}(2 \sin \theta \sqrt{1 - \sin^2 \theta})$$

$$\Rightarrow u = \sin^{-1}(2 \sin \theta \sqrt{\cos^2 \theta}) \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}(2 \sin \theta \cos \theta)$$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

$$\text{Now, we have } v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

By substituting $x = \sin \theta$, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1 - (\sin \theta)^2}}\right)$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\sqrt{1 - \sin^2 \theta}} \right)$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\sqrt{\cos^2 \theta}} \right) [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\cos \theta} \right)$$

$$\Rightarrow v = \sec^{-1}(\sec \theta)$$

Given $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

However, $x = \sin \theta$

$$\Rightarrow \sin \theta \in \left(0, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$$\Rightarrow u = 2\sin^{-1}(x)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1} x)$$

$$\Rightarrow \frac{du}{dx} = 2 \frac{d}{dx}(\sin^{-1} x)$$

We know $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}}$$

We have $\theta \in \left(0, \frac{\pi}{4}\right)$

Hence, $v = \sec^{-1}(\sec \theta) = \theta$

$$\Rightarrow v = \sin^{-1}x$$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

$$\text{We know } \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = 2$$

$$\text{Thus, } \frac{du}{dv} = 2$$

(ii) Let

$$u = \sin^{-1}(2x\sqrt{1-x^2}) \text{ And } v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right).$$

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

$$\text{We have } u = \sin^{-1}(2x\sqrt{1-x^2})$$

By substituting $x = \sin \theta$, we have

$$u = \sin^{-1} \left(2 \sin \theta \sqrt{1 - (\sin \theta)^2} \right)$$

$$\Rightarrow u = \sin^{-1} \left(2 \sin \theta \sqrt{1 - \sin^2 \theta} \right)$$

$$\Rightarrow u = \sin^{-1} \left(2 \sin \theta \sqrt{\cos^2 \theta} \right) [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1} (2 \sin \theta \cos \theta)$$

$$\Rightarrow u = \sin^{-1} (\sin 2\theta)$$

Now, we have $v = \sec^{-1} \left(\frac{1}{\sqrt{1-x^2}} \right)$

By substituting $x = \sin \theta$, we have

$$v = \sec^{-1} \left(\frac{1}{\sqrt{1 - (\sin \theta)^2}} \right)$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\sqrt{1 - \sin^2 \theta}} \right)$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\sqrt{\cos^2 \theta}} \right) [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow v = \sec^{-1} \left(\frac{1}{\cos \theta} \right)$$

$$\Rightarrow v = \sec^{-1} (\sec \theta)$$

Given $x \in \left(\frac{1}{\sqrt{2}}, 1 \right)$

However, $x = \sin \theta$

$$\Rightarrow \sin \theta \in \left(\frac{1}{\sqrt{2}}, 1 \right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right)$$

$$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \pi \right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$$\Rightarrow u = \pi - 2\sin^{-1}(x)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - 2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

We have $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Hence, $v = \sec^{-1}(\sec \theta) = \theta$

$$\Rightarrow v = \sin^{-1}x$$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

We have

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = -2$$

$$\text{Thus, } \frac{du}{dv} = -2$$

8. Differentiate $(\cos x)^{\sin x}$ with respect to $(\sin x)^{\cos x}$.

Solution:

Let $u = (\cos x)^{\sin x}$ and $v = (\sin x)^{\cos x}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = (\cos x)^{\sin x}$

Taking log on both sides, we get

$$\log u = \log (\cos x)^{\sin x}$$

$$\Rightarrow \log u = (\sin x) \times \log (\cos x) \quad [\because \log a^m = m \times \log a]$$

On differentiating both the sides with respect to x , we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[\sin x \times \log(\cos x)]$$

We know that $(uv)' = vu' + uv'$

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\cos x) \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}[\log(\cos x)]$$

$$\text{We know } \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(\sin x) = \cos x$$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\cos x) \times \cos x + \sin x \left[\frac{1}{\cos x} \frac{d}{dx} (\cos x) \right]$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \cos x \log(\cos x) + \frac{\sin x}{\cos x} \frac{d}{dx} (\cos x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \cos x \log(\cos x) + \tan x \frac{d}{dx} (\cos x)$$

We know $\frac{d}{dx} (\cos x) = -\sin x$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \cos x \log(\cos x) + \tan x (-\sin x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

But, $u = (\cos x)^{\sin x}$

$$\Rightarrow \frac{1}{(\cos x)^{\sin x}} \frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

$$\therefore \frac{du}{dx} = (\cos x)^{\sin x} [\cos x \log(\cos x) - \tan x \sin x]$$

Now, we have $v = (\sin x)^{\cos x}$

Taking log on both sides, we get

$$\log v = \log (\sin x)^{\cos x}$$

$$\Rightarrow \log v = (\cos x) \times \log (\sin x) \quad [\because \log a^m = m \times \log a]$$

On differentiating both the sides with respect to x , we get

$$\frac{d}{dv} (\log v) \times \frac{dv}{dx} = \frac{d}{dx} [\cos x \times \log(\sin x)]$$

We know that $(uv)' = vu' + uv'$ (product rule)

$$\Rightarrow \frac{d}{du} (\log u) \times \frac{dv}{dx} = \log(\sin x) \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} [\log(\sin x)]$$

We know $\frac{d}{dx} (\log x) = \frac{1}{x}$ and $\frac{d}{dx} (\cos x) = -\sin x$

$$\Rightarrow \frac{1}{v} \times \frac{dv}{dx} = \log(\sin x) \times (-\sin x) + \cos x \left[\frac{1}{\sin x} \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\sin x \log(\sin x) + \frac{\cos x}{\sin x} \frac{d}{dx} (\sin x)$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \frac{d}{dx} (\sin x)$$

We know $\frac{d}{dx} (\sin x) = \cos x$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \times (\cos x)$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

But, $v = (\sin x)^{\cos x}$

$$\Rightarrow \frac{1}{(\sin x)^{\cos x}} \frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

$$\therefore \frac{dv}{dx} = (\sin x)^{\cos x} [-\sin x \log(\sin x) + \cot x \cos x]$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\cos x)^{\sin x} [\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x} [-\sin x \log(\sin x) + \cot x \cos x]}$$

$$\therefore \frac{du}{dv} = \frac{(\cos x)^{\sin x} [\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x} [\cot x \cos x - \sin x \log(\sin x)]}$$

Thus, $\frac{du}{dv} = \frac{(\cos x)^{\sin x} [\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x} [\cot x \cos x - \sin x \log(\sin x)]}$

9. Differentiate $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$ with respect to $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$, if $0 < x < 1$.

Solution:

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting $x = \tan \theta$, we have

$$u = \sin^{-1}\left(\frac{2 \tan \theta}{1 + (\tan \theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2 \tan \theta}{\sec^2 \theta}\right) [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2 \times \frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos^2 \theta}}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2 \times \frac{\sin \theta}{\cos \theta} \times \cos^2 \theta\right)$$

$$\Rightarrow u = \sin^{-1}(2 \sin \theta \cos \theta)$$

But, $\sin 2\theta = 2 \sin \theta \cos \theta$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

Given $0 < x < 1 \Rightarrow x \in (0, 1)$

However, $x = \tan \theta$

$$\Rightarrow \tan \theta \in (0, 1)$$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$

$$\Rightarrow u = 2\tan^{-1}x$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = 2 \frac{d}{dx}(\tan^{-1}x)$$

$$\text{We know } \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{2}{1+x^2}$$

$$\text{Now, we have } v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

By substituting $x = \tan \theta$, we have

$$v = \cos^{-1}\left(\frac{1 - (\tan \theta)^2}{1 + (\tan \theta)^2}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1 - \tan^2 \theta}{\sec^2 \theta}\right) [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\sec^2 \theta} - \frac{\tan^2 \theta}{\sec^2 \theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{\frac{1}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta}} - \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta}}\right)$$

$$\Rightarrow v = \cos^{-1}(\cos^2 \theta - \sin^2 \theta)$$

$$\text{But, } \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\Rightarrow v = \cos^{-1}(\cos 2\theta)$$

$$\text{However, } \theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

$$\text{Hence, } v = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\Rightarrow v = 2\tan^{-1}x$$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2 \frac{d}{dx}(\tan^{-1}x)$$

$$\text{We know } \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\text{We have } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = 1$$

$$\text{Thus, } \frac{du}{dv} = 1$$

10. Differentiate $\tan^{-1} \left(\frac{1+ax}{1-ax} \right)$ with respect to $\sqrt{1+a^2x^2}$.

Solution:

Let $u = \tan^{-1} \left(\frac{1+ax}{1-ax} \right)$ and $v = \sqrt{1+a^2x^2}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1} \left(\frac{1+ax}{1-ax} \right)$

By substituting $ax = \tan \theta$, we have

$$u = \tan^{-1} \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right)$$

$$\Rightarrow u = \tan^{-1} \left(\tan \left(\frac{\pi}{4} + \theta \right) \right) \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \right]$$

$$\Rightarrow u = \frac{\pi}{4} + \theta$$

$$\Rightarrow u = \frac{\pi}{4} + \tan^{-1}(ax)$$

On differentiating u with respect to x , we get

$$\frac{du}{dx} = \frac{d}{dx} \left[\frac{\pi}{4} + \tan^{-1}(ax) \right]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx} \left(\frac{\pi}{4} \right) + \frac{d}{dx} [\tan^{-1}(ax)]$$

We know $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{1+(ax)^2} \frac{d}{dx} (ax)$$

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{1 + (ax)^2} \frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + a^2x^2} \left[a \frac{d}{dx}(x) \right]$$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1 + a^2x^2} \frac{d}{dx}(x)$$

We know $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1 + a^2x^2} \times 1$$

$$\therefore \frac{du}{dx} = \frac{a}{1 + a^2x^2}$$

Now, we have $v = \sqrt{1 + a^2x^2}$

On differentiating v with respect to x , we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sqrt{1 + a^2x^2})$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1 + a^2x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 + a^2x^2)^{\frac{1}{2}-1} \frac{d}{dx}(1 + a^2x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 + a^2x^2)^{-\frac{1}{2}} \left[\frac{d}{dx}(1) + \frac{d}{dx}(a^2x^2) \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 + a^2x^2}} \left[\frac{d}{dx}(1) + a^2 \frac{d}{dx}(x^2) \right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}} [0 + a^2(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}} [2a^2x]$$

$$\therefore \frac{dv}{dx} = \frac{a^2x}{\sqrt{1+a^2x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{a}{1+a^2x^2}}{\frac{a^2x}{\sqrt{1+a^2x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{a}{1+a^2x^2} \times \frac{\sqrt{1+a^2x^2}}{a^2x}$$

$$\therefore \frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$$

Thus, $\frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$