

Differentiation (Part – 1)

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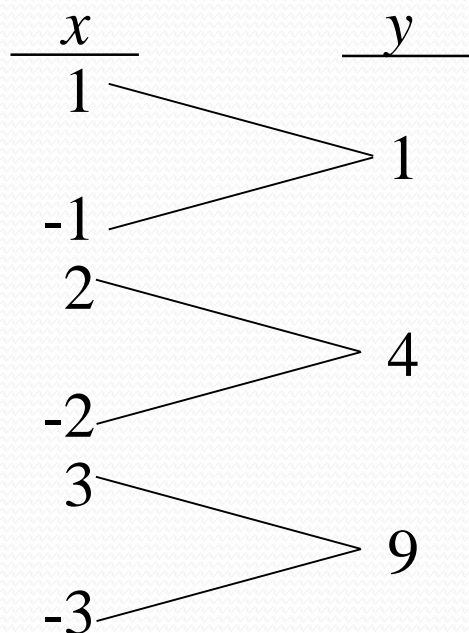
Function

If x and y are two variables related to one another in such a way that each value of x determines exactly one value of y , then we say that y is a function of x and it is simply written as $y = f(x)$, where x is an independent variable and y is a dependent variable. The value of y or $f(x)$ is called functional value.

$y = x^2$ is a function

For $x = \pm 1, \pm 2, \pm 3, \dots$

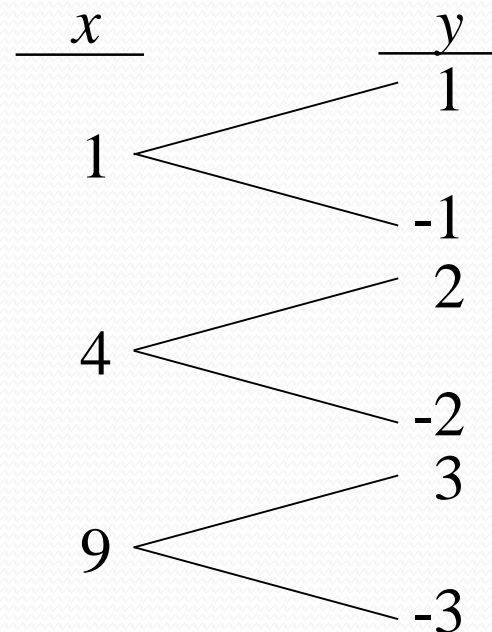
We get, $y = 1, 4, 9, \dots$



$y^2 = x$ or, $(y = \pm\sqrt{x})$ is not a function

For $x = 1, 4, 9, \dots$

We get, $y = \pm 1, \pm 2, \pm 3, \dots$

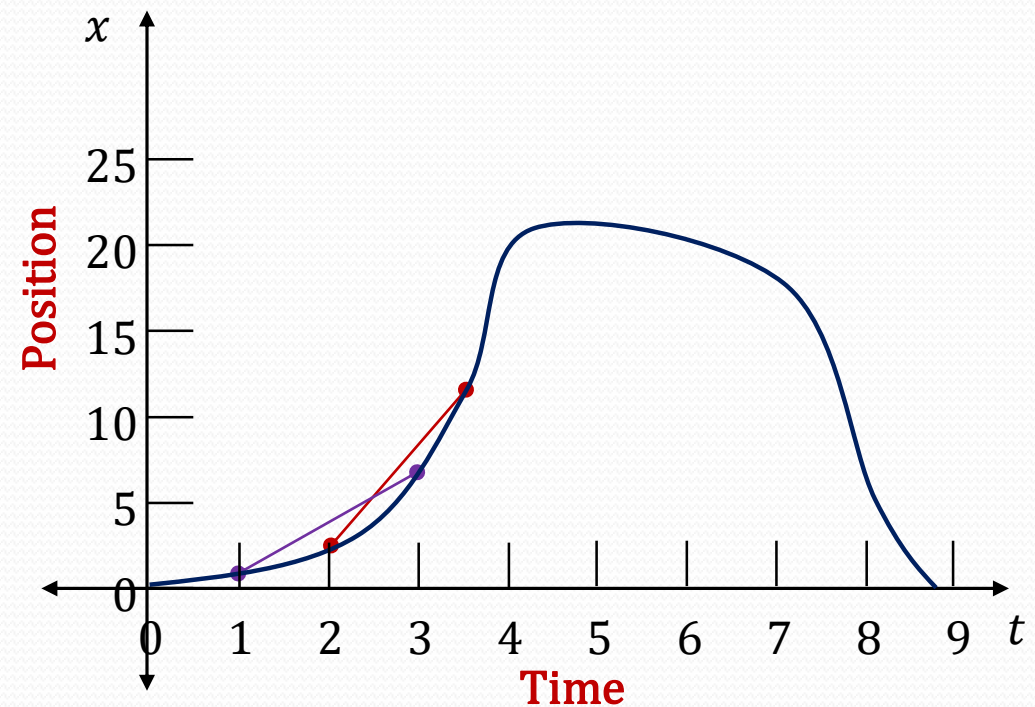
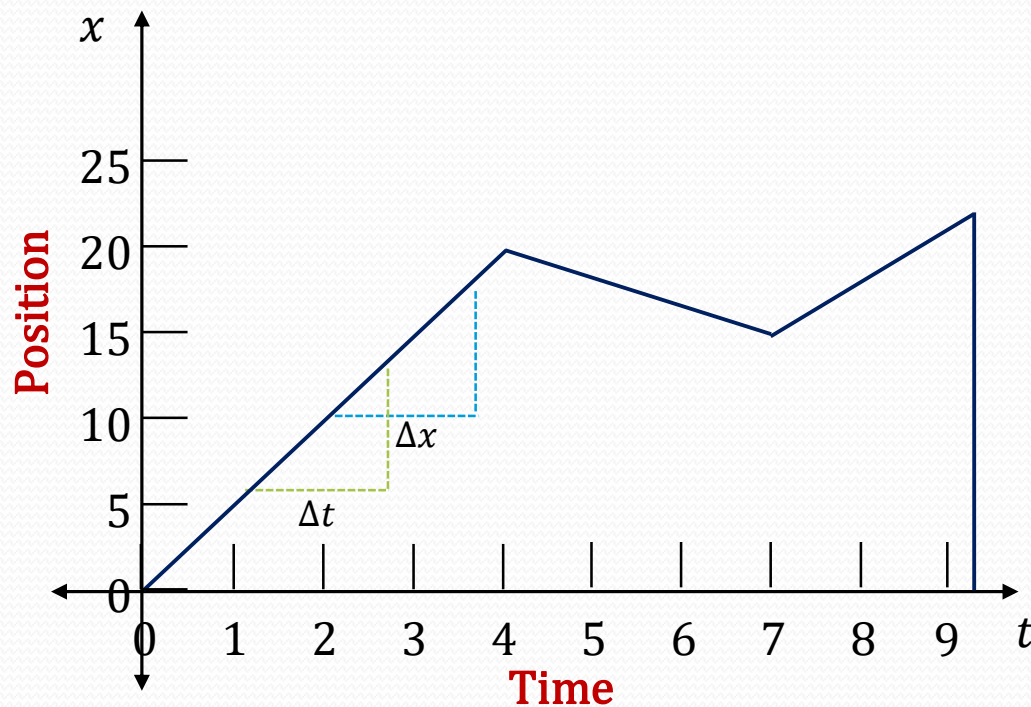


Concept of Derivative

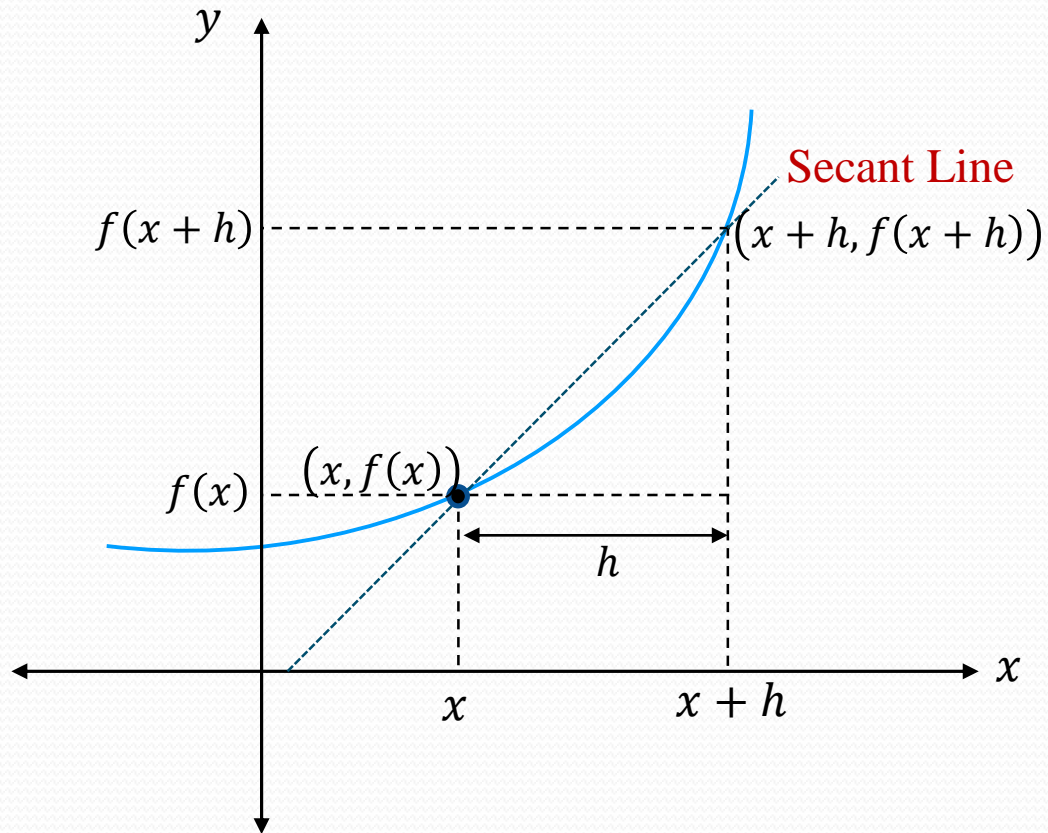
Differentiation → Instantaneous rate of change of a function with respect to one of its variables

≈ to find the slope of the tangent line to the function at a point.

$$\text{Speed} = \frac{\text{change of position}}{\text{change of time}} = \frac{\Delta x}{\Delta t}$$

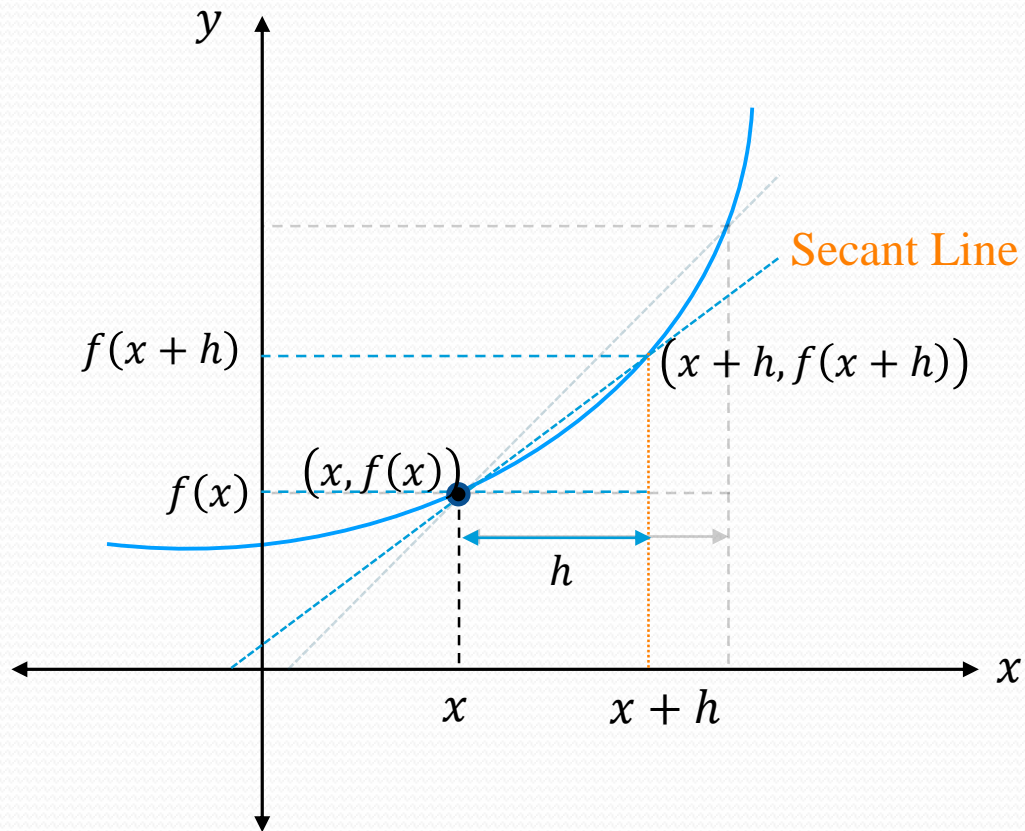


Geometrical Representation of Derivative



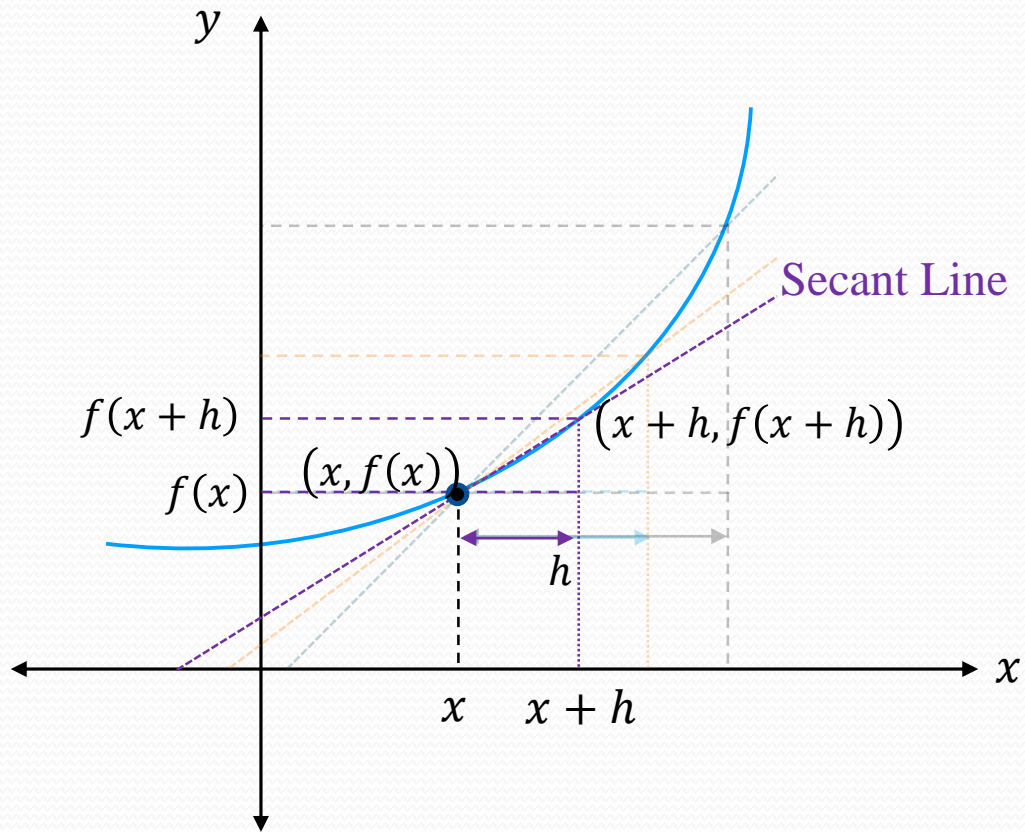
$$\begin{aligned}\text{Gradient/Slope} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{f(x+h) - f(x)}{x+h-x} \\ &= \frac{f(x+h) - f(x)}{h}\end{aligned}$$

Geometrical Representation of Derivative



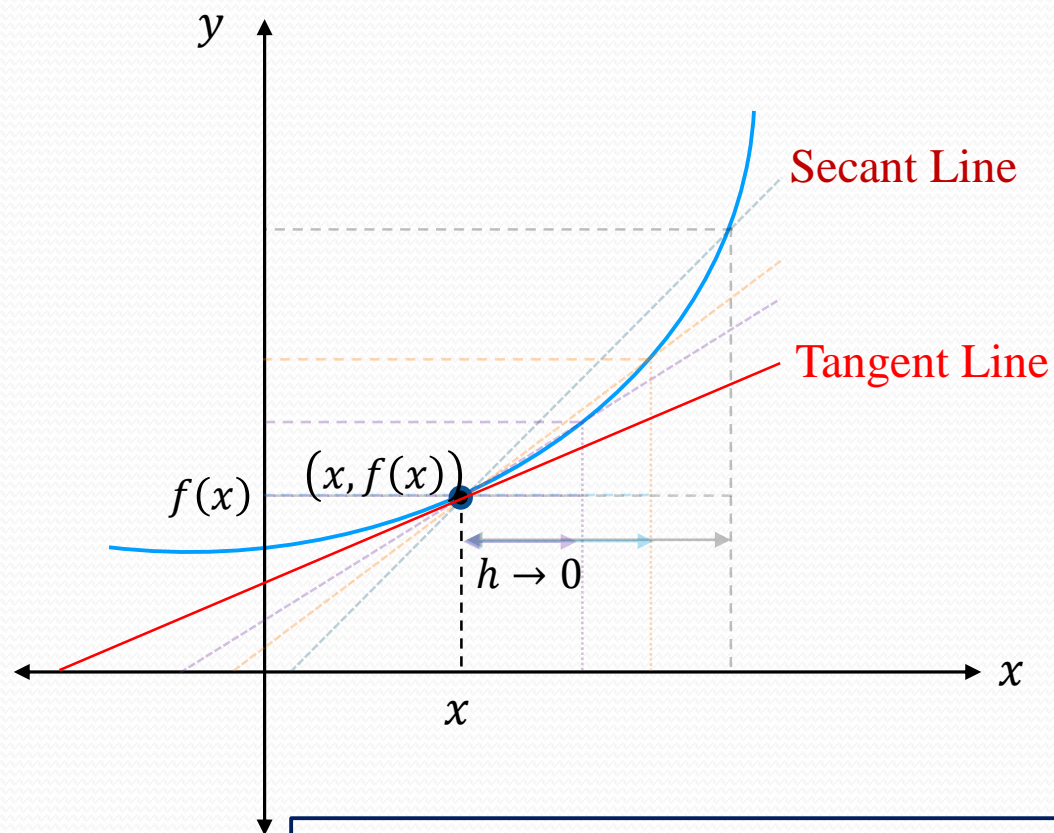
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Geometrical Representation of Derivative



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Geometrical Representation of Derivative



$$\text{Gradient/Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

First Principle: A function $f(x)$ is differentiable at a point x if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists (is finite).

Example 1: Find the derivative of $f(x) = c$ using first principle

Let $y = f(x) = c$

From **First Principle**,

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

$$\begin{aligned}f(x) &= c \\ f(x+h) &= c\end{aligned}$$

Example 2: Find the derivative of $f(x) = x$ using first principle

Let $y = f(x) = x$

From **First Principle**,

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \\ &= 1\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\begin{aligned}f(x) &= x \\ f(x+h) &= x+h\end{aligned}$$

Example 3: Find the derivative of $f(x) = x^n$ using first principle

Let $y = f(x) = x^n$

From **First Principle**,

$$\begin{aligned}f(x) &= x^n \\f(x+h) &= (x+h)^n\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= f'(x) \\&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\end{aligned}$$

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{\left\{x \left(1 + \frac{h}{x}\right)\right\}^n - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{x^n \left(1 + \frac{h}{x}\right)^n - x^n}{h} \\&= x^n \lim_{h \rightarrow 0} \frac{\left(1 + \frac{h}{x}\right)^n - 1}{h}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= x^n \lim_{h \rightarrow 0} \frac{1 + n\left(\frac{h}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3 + \dots - 1}{h} \\&= x^n \lim_{h \rightarrow 0} \frac{n\left(\frac{h}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3 + \dots}{h} \\&= x^n \lim_{h \rightarrow 0} \frac{h \left\{n\left(\frac{1}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right) + \dots\right\}}{h} \\&= x^n \lim_{h \rightarrow 0} \left\{n\left(\frac{1}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right) + \dots\right\} \\&= x^n n \left(\frac{1}{x}\right) = x^n n x^{-1} = n x^{n-1}\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^n) = n x^{n-1}$$

Example 4: Find the derivative of $f(x) = e^x$ using first principle

Let $y = f(x) = e^x$

From **First Principle**,

$$\frac{dy}{dx} = f'(x)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \dots - 1}{h}$$

$$\left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right]$$

$$\begin{aligned} f(x) &= e^x \\ f(x+h) &= e^{(x+h)} \end{aligned}$$

$$\frac{dy}{dx} = e^x \lim_{h \rightarrow 0} \frac{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \dots}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{h \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \dots \right)}{h}$$

$$= e^x \lim_{h \rightarrow 0} \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \dots \right)$$

$$= e^x \cdot 1$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x) = e^x$$

Example 5: Find the derivative of $f(x) = \ln(x)$ using first principle

Let $y = f(x) = \ln(x)$

From **First Principle**,

$$\frac{dy}{dx} = f'(x)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{h}{x} - \frac{1}{2} \frac{h^2}{x^2} + \frac{1}{3} \frac{h^3}{x^3} - \frac{1}{4} \frac{h^4}{x^4} + \dots}{h}$$

$$\left[\because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right]$$

$$\begin{aligned} f(x) &= \ln(x) \\ f(x+h) &= \ln(x+h) \end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{h \left(\frac{1}{x} - \frac{1}{2} \frac{h}{x^2} + \frac{1}{3} \frac{h^2}{x^3} - \frac{1}{4} \frac{h^3}{x^4} + \dots \right)}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{1}{x} - \frac{1}{2} \frac{h}{x^2} + \frac{1}{3} \frac{h^2}{x^3} - \frac{1}{4} \frac{h^3}{x^4} + \dots \right)$$

$$= \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\ln(x)) = \frac{1}{x}$$

Example 6: Find the derivative of $f(x) = \cos x$ using first principle

Let $y = f(x) = \cos x$

From **First Principle**,

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h}\end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = \lim_{h \rightarrow 0} \frac{h}{\sin h} = 1$$

$$\begin{aligned}f(x) &= \cos x \\ f(x+h) &= \cos(x+h)\end{aligned}$$

$$\cos a - \cos b = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$= - \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \times \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \quad \because h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0????$$

$$\begin{aligned}&= - \sin\left(\frac{2x+0}{2}\right) \times 1 \\ \therefore \frac{dy}{dx} &= \frac{d}{dx}(\cos x) = -\sin x\end{aligned}$$

$$\begin{aligned}h &\rightarrow 0 \\ \Rightarrow \frac{1}{2} \times h &\rightarrow \frac{1}{2} \times 0 \\ \Rightarrow \frac{h}{2} &\rightarrow 0\end{aligned}$$

Example 7: Find the derivative of $f(x) = \sin ax$ using first principle

Let $y = f(x) = \sin ax$

From **First Principle**,

$$\frac{dy}{dx} = f'(x)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(ax+ah) - \sin ax}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2ax+ah}{2}\right) \sin\left(\frac{ah}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \cos\left(\frac{2ax+ah}{2}\right) \times \lim_{h \rightarrow 0} \frac{\sin\left(\frac{ah}{2}\right)}{\frac{ah}{2}}$$

$$= \cos\left(\frac{2ax+a \cdot 0}{2}\right) \times \lim_{h \rightarrow 0} \frac{a \sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)}$$

$$\begin{aligned} f(x) &= \sin ax \\ f(x+h) &= \sin a(x+h) \\ &= \sin(ax+ah) \end{aligned}$$

$$\sin a - \sin b = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = \lim_{h \rightarrow 0} \frac{h}{\sin h} = 1$$

$$\begin{aligned} \frac{dy}{dx} &= (\cos ax) \times a \lim_{\frac{ah}{2} \rightarrow 0} \frac{\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)} \\ &\quad \left[\because h \rightarrow 0 \Rightarrow \frac{ah}{2} \rightarrow 0 \right] \\ &= (\cos ax) \times a(1) \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin ax) = a \cos ax$$

Example 8: Find the derivative of $f(x) = \tan x$ using first principle

Let $y = f(x) = \tan x$

$$\sin a \cos b - \cos a \sin b = \sin(a - b)$$

From **First Principle**,

$$\begin{aligned} f(x) &= \tan x \\ f(x + h) &= \tan(x + h) \end{aligned}$$

$$\frac{dy}{dx} = f'(x)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos(x+h) \cos x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x+h) \cos x - \cos(x+h) \sin x}{h \cos(x+h) \cos x}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x+h-x)}{h \cos(x+h) \cos x}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin h}{h \cos(x+h) \cos x} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \times \lim_{h \rightarrow 0} \frac{1}{\cos(x+h) \cos x} \\ &= 1 \times \frac{1}{\cos(x+0) \cos x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \\ \therefore \frac{dy}{dx} &= \frac{d}{dx} (\tan x) = \sec^2 x \end{aligned}$$

Example 9: Find the derivative of $f(x) = \tan^{-1} x$ using first principle

Let $y = f(x) = \tan^{-1} x$

From **First Principle**,

$$\begin{aligned} f(x) &= \tan^{-1} x \\ f(x+h) &= \tan^{-1}(x+h) \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= f'(x) \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \lim_{x \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \lim_{x \rightarrow 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} \dots (i) \end{aligned}$$

Suppose $\tan^{-1} x = y \Rightarrow x = \tan y$

and $\tan^{-1}(x+h) = y+k \Rightarrow x+h = \tan(y+k)$

$$\Rightarrow h = \tan(y+k) - x$$

$$\Rightarrow h = \tan(y+k) - \tan y$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{y+k-y}{h}$$

$$\frac{dy}{dx} = \lim_{k \rightarrow 0} \frac{k}{\tan(y+k) - \tan y} \quad [\because h \rightarrow 0 \Rightarrow k \rightarrow 0 ???]$$

$$x+h = \tan(y+k)$$

When $h \rightarrow 0$, $x = \tan(y+k)$

$$\Rightarrow \tan^{-1} x = y+k$$

$$\Rightarrow \tan^{-1} x = \tan^{-1} x + k$$

$$\Rightarrow k \rightarrow 0$$

Thus, $h \rightarrow 0 \Rightarrow k \rightarrow 0$

In concept of Limit

Example 9: Find the derivative of $f(x) = \tan^{-1} x$ using first principle (cont....)

$$\frac{dy}{dx} = \lim_{k \rightarrow 0} \frac{k}{\tan(y+k) - \tan y} \quad [\because h \rightarrow 0 \Rightarrow k \rightarrow 0]$$

$$= \lim_{k \rightarrow 0} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \lim_{k \rightarrow 0} \frac{k}{\frac{\sin(y+k) \cos y - \sin y \cos(y+k)}{\cos(y+k) \cos y}}$$

$$= \lim_{k \rightarrow 0} \frac{k \cos(y+k) \cos y}{\sin(y+k) \cos y - \cos(y+k) \sin y}$$

$$= \lim_{k \rightarrow 0} \frac{k \cos(y+k) \cos y}{\sin(y+k-y)}$$

$$= \lim_{k \rightarrow 0} \frac{k \cos(y+k) \cos y}{\sin k}$$

$$\sin a \cos b - \cos a \sin b = \sin(a - b)$$

$$\lim_{k \rightarrow 0} \frac{\sin k}{k} = \lim_{k \rightarrow 0} \frac{k}{\sin k} = 1$$

$$\frac{dy}{dx} = \lim_{k \rightarrow 0} \frac{k}{\sin k} \times \lim_{k \rightarrow 0} \cos(y+k) \cos y$$

$$= 1 \times \cos(y+0) \cos y$$

$$= \cos^2 y$$

$$= \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$[\because \tan y = x]$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}$$

Example 10: Find $\frac{dy}{dx}$ for $y = \sqrt{x}$ using first principle

Let $y = f(x) = \sqrt{x}$

From **First Principle**,

$$\frac{dy}{dx} = f'(x)$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \end{aligned}$$

$$\begin{aligned} f(x) &= \sqrt{x} \\ f(x+h) &= \sqrt{x+h} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Exercise

Find the derivative of the following functions using first principle:

1. $f(x) = a^x$

2. $f(x) = \cos ax$

3. $f(x) = \sin^{-1} x$

4. $f(x) = \cos^{-1} x$

5. $f(x) = \sin(x^2)$