Differentiation
(Part – 2)
(Technique of Differentiation)

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Constant Rule:

If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

Example:

$$\frac{d}{dx}(1) = 0;$$

$$\frac{d}{dx}(\pi) = 0;$$

$$\frac{d}{dx}(1000) = 0;$$

$$\frac{d}{dx}(a) = 0.$$

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: Find $\frac{dy}{dx}$ for the function $y = \sqrt{x}$

Given that, $y = \sqrt{x}$

Taking derivative with respect to x on both sides,

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x})$$

$$= \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{1}{x^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Sum or Difference Rule:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

Example 1: Find $\frac{dy}{dx}$ for the function $y = \sqrt[3]{x} - \frac{1}{\sqrt{x}}$

Given that, $y = \sqrt[3]{x} - \frac{1}{\sqrt{x}}$

Taking derivative with respect to x on both sides,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x} - \frac{1}{\sqrt{x}}\right)$$

$$= \frac{d}{dx}\left(x^{\frac{1}{3}}\right) - \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{3}x^{\frac{1}{3}-1} - \left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{1}{3}\frac{1}{x^{\frac{2}{3}}} + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\frac{1}{\sqrt[3]{x^2}} + \frac{1}{2}\frac{1}{\sqrt{x^3}}$$

Sum or Difference Rule: (Cont...)

Example 2: Find $\frac{dy}{dx}$ for the function $y = e^x + 2\sin x - \frac{1}{2}\log x$

Given that, $y = e^x + 2\,sin\,x - \frac{1}{2}\,log\,x$

Taking derivative with respect to $x$on both sides,

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + 2\,sin\,x - \frac{1}{2}\,log\,x\right)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$= \frac{d}{dx}(e^x) + \frac{d}{dx}(2\,sin\,x) - \frac{d}{dx}\left(\frac{1}{2}\,log\,x\right)$$

$$\frac{d}{dx}(cx) = c\frac{d}{dx}(x)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$= e^x + 2\frac{d}{dx}(sin\,x) - \frac{1}{2}\frac{d}{dx}(log\,x)$$

$$\frac{d}{dx}(log\,x) = \frac{1}{x}$$

$$= e^x + 2\cos x - \frac{1}{2}\frac{1}{x}$$

$$\therefore \frac{dy}{dx} = e^x + 2\cos x - \frac{1}{2x}$$

Product Rule:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Example: Find $\frac{dy}{dx}$ for the function $y = x^2 \tan^{-1} x$

Given that, $y = x^2\, tan^{-1} x$

Taking derivative with respect to $x$on both sides,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2\, tan^{-1} x)$$

$$= x^2 \frac{d}{dx}(tan^{-1} x) + tan^{-1} x \frac{d}{dx}(x^2)$$

$$= x^2 \frac{1}{1+x^2} + tan^{-1} x\,.(2x)$$

$$\therefore \frac{dy}{dx} = \frac{x^2}{1+x^2} + 2x \tan^{-1} x$$

$$\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\therefore \frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

Quotient Rule:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Example: Find $\frac{dy}{dx}$ for the function $y = \frac{3x}{5-\tan x}$

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Given that, $y = \frac{3x}{5-\tan x}$

Taking derivative with respect to $x$on both sides,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x}{5-\tan x}\right)$$

$$= \frac{(5-\tan x)\frac{d}{dx}(3x) - 3x\frac{d}{dx}(5-\tan x)}{(5-\tan x)^2}$$

$$= \frac{(5-\tan x)3\frac{d}{dx}(x) - 3x\frac{d}{dx}(5-\tan x)}{(5-\tan x)^2}$$

$$\frac{dy}{dx} = \frac{(5-\tan x)3(1) - 3x(0-\sec^2 x)}{(5-\tan x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{15 - 3\tan x + 3x\sec^2 x}{(5-\tan x)^2}$$

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Chain Rule:

If y is a function of u, $(y = f(u))$ and u is a function of x, $(u = g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example: Find $\frac{dy}{dx}$ for the function $y = e^{2\cos^{-1} x}$

$$\boxed{\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}}$$

Given that, $y = e^{2\cos^{-1} x}$

Suppose $\quad u = 2\cos^{-1} x$

So $\quad y = e^u$

By Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(e^u) \cdot \frac{d}{dx}(2\cos^{-1} x)$$

$$= e^u \cdot 2\frac{d}{dx}(\cos^{-1} x)$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$$\frac{dy}{dx} = \left(e^{2\cos^{-1} x}\right) \cdot 2\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{-2e^{2\cos^{-1} x}}{\sqrt{1-x^2}}$$

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Find $\frac{dy}{dx}$ for the functions $x = a\cos\theta + b\sin\theta$ and $y = b\sin\theta$

Given parametric equations,

$$x = a\cos\theta + b\sin\theta$$

and $\quad y = b\sin\theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta + b\sin\theta)$$

$$= \frac{d}{d\theta}(a\cos\theta) + \frac{d}{d\theta}(b\sin\theta)$$

$$= a\frac{d}{d\theta}(\cos\theta) + b\frac{d}{d\theta}(\sin\theta)$$

$$= \mathrm{a}(-\sin\theta) + b\cos\theta$$

$$\therefore \frac{dx}{d\theta} = b\cos\theta - a\sin\theta$$

Again, $\quad y = b\sin\theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin\theta)$$

$$= b\frac{d}{d\theta}(\sin\theta)$$

$$\therefore \frac{dy}{d\theta} = b\cos\theta$$

By Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = b\cos\theta\cdot\frac{1}{b\cos\theta - a\sin\theta}$$

$$\therefore \frac{dy}{dx} = \frac{b\cos\theta}{b\cos\theta - a\sin\theta}$$

Function as Power of another Function:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then

$$\frac{d}{dx}(u^v) = u^v \frac{d}{dx}\left(v\, ln(u)\right)$$

Example: Find $\frac{dy}{dx}$ for the function $x^y + y^x = 1$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Given that, $x^y + y^x = 1$

Taking derivative with respect to $x$on both sides,

$$\frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = \frac{d}{dx}(1)$$

$$\Rightarrow x^y \;\frac{d}{dx}(y \ln x) + y^x \;\frac{d}{dx}(x \ln y) = 0$$

$$\Rightarrow x^y \left(y\frac{1}{x} + \ln x \frac{dy}{dx}\right) + y^x \left(x\frac{1}{y}\frac{dy}{dx} + \ln y\right) = 0$$

$$\Rightarrow x^y \; yx^{-1} + x^y \ln x \frac{dy}{dx} + y^x xy^{-1}\frac{dy}{dx} + y^x \ln y = 0$$

$$\Rightarrow x^{y-1}\; y + x^y \ln x \frac{dy}{dx} + y^{x-1}x\frac{dy}{dx} + y^x \ln y = 0$$

$$\frac{dy}{dx}(x^y \ln x + y^{x-1}x) = -x^{y-1}y - y^x \ln y$$

$$\frac{dy}{dx} = \frac{-x^{y-1}y - y^x \ln y}{x^y \ln x + y^{x-1}x}$$

$$\therefore \frac{dy}{dx} = \frac{-(x^{y-1}y + y^x\, ln\, y)}{x^y\, ln\, x + y^{x-1}x}$$

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Exercise

(a) Find the derivative of the following functions :

1. $y = (e^x + 1)\tan x$
2. $y = a^x x^3 \sin^{-1} x$
3. $y = \frac{\sin x}{x}$
4. $y = \frac{e^x}{1+x}$
5. $y = \ln(1 + e^x)$

(b) Find $\frac{dy}{dx}$ of the parametric equations:

$$x = e^{\tan^{-1} t} \text{ and } y = (1 + t^2)^5.$$