MTHH229 Fall 2006

The College of Staten Island
Department of Mathematics

Investigating Parametric Curves with MATLAB

1 Introduction

In this project we investigate curves in the plane. Specifically, we want to think about graphs of curves that are not necessarily given by simple functions $y = f(x)$. Such things fascinated ancient Greeks, confounded Enlightenment scholars, and often annoy students in Calculus 2 or 3. They are, however, very easy to visualize in Matlab.

1.1 New Math/MATLAB Topics

a. Defining parametric curves, $C : (x(t), y(t))$

b. Plotting parametric curves.

c. Exploring the calculus of curves

2 A parametric curve.

We are used to thinking about graphs of *functions* in the Cartesian plane (x, y) where $y = f(x)$. We all know that a function is a rule that assigns one and only one output value (y) for any allowable input value (x). While one can spend one's entire life studying such things, there is an whole different world of graceful, curious and important objects (curves) defined by collections of points (x, y) which do NOT fit the catagory of functions.

For example, let's consider the circle. This object is defined by the algebraic equation:

$$x^2 + y^2 = r^2$$

for some value of the constant r which is the radius of the circle.

How can we get Matlab to plot such a thing? Well, one way would be to solve for y as a function of x and use what we already know about plotting graphs. Try to do this and we immediately run into trouble!

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

What to do with the $\pm$? What to do with the limited domain of the 'function' defined above? If we wanted to plot the circle, we would need to carefully define the input values of x and then plot BOTH the y values of the positive square root and the negative square root. Try doing this in Matlab for some convenient value of r.

For example, for $r = 2$

```
>> x=linspace(-2,2);plot(x,sqrt(4 - x.^2))
>> hold on
>> plot(x,-sqrt(4-x.^2));
```

Yuck! That is an ugly looking circle and a lot of work. The reason the graph looks like an egg instead of a circle is because Matlab choses its own aspect ratio for the plot. (x is longer than y, for aesthetic reasons.) This is easily fixed with the `axis` command. Try typing:

```
>> axis('equal')
```

Better looking circle, but there is something unnatural about forcing our elegant curve which is NOT a function into being a function so we may plot it. The way around this is to define x and y in terms of a third variable, t. This is what we mean by *parameterizing* the curve (x, y) by t.

Knowing something about trigonemtry and the unit circle, we should recognize that a easy parameterization of the circle of radius r is given by:

$$\begin{aligned} x(t) &= r\cos t \\ y(t) &= r\sin t \\ 0 < \quad t &< 2\pi \end{aligned}$$

Now this is elegant and also convenient to enter into Matlab should we wish to look at the curve. For a circle of radius 2:

```
>> r =  2;
>> t=linspace(0,2*pi);
>> x = r*cos(t);
>> y = r*sin(t);
>> plot(x,y)
>> axis('equal')
```

Voila, the same circle. Note that the parameterization of the curve is not unique. What would happen if you redefined x and y but left t as it is? What would happen if you left x and y alone, but changed the definition of t?

Exercise 1:
Consider the curve defined by:

$$\begin{aligned} x(t) &= r\sin t \\ y(t) &= r\cos t \\ 0 < \quad t &< 2\pi \end{aligned}$$

a. Use Matlab to plot this curve for some value of r.

b. Where does the curve 'start'? In other words, what is the value of $(x(0), y(0))$? Compare this to the more standard parameterization given above.

Exercise 2:
Consider the curve defined by:

$$\begin{aligned} x(t) &= r\cos t \\ y(t) &= r\sin t \\ -\pi < & \; t < \pi \end{aligned}$$

a. Use Matlab to plot this curve for some value of r.

b. Compare this curve to the others. Are they different?

c. Where does the curve 'start'?

3 Some interesting curves

What happens if we let the radius of the circle grow (or shrink) depending upon the value of t? Try this out.

Consider the following curve:

$$\begin{aligned} x(t) &= t\cos t \\ y(t) &= t\sin t \\ 0 &< t \end{aligned}$$

What will the shape of this be? Thinking in terms of the circle, we see that now the radius is *increasing* linearly with time.

Exercise 3:
Use Matlab to plot the curve defined above.

a. What is this curve?

b. What happens to the curve as t increases? Look at different definitions of t, ie: `t = linspace(0,2*pi)`, `t = linspace(0,8*pi,1000)`

Exercise 4:
Consider the following curve:

$$\begin{aligned} x(t) &= t\cos t \\ y(t) &= r\sin t \\ 0 &< t \end{aligned}$$

where $r = $ *constant* What will the shape of this curve be? Think about it before asking Matlab.

a. Use Matlab to plot the curve for $0 \leq t \leq 10\pi$. Do not use the `>> axis('equal')` command.

b. Describe the curve

Exercise 5:
A *Lissajous Curve* (sometimes called a *Bowditch Curve*, if you are an Anglophile) is a parametric curve defined by:

$$\begin{aligned} x(t) &= a\sin(nt) \\ y(t) &= b\sin(t) \end{aligned}$$

for constants a, b, n.

a. For $n = 1$, predict what the curve will look like. In this case, it easy to solve for $y = f(x)$. What is $f(x)$? What effect do a and b have on the curve?

b. Use Matlab to plot the curve for $n = 2$. Start with $a = b = 1$. What happens when you change these?

c. If t starts at 0, at what value of t does the curve begin to repeat itself? (In other words, what is the period of the curve?)

d. What will change when $n = 1/2$?

e. Experiment with different values of n Try $n = 4, 6$ Try an odd value of n. What happens? What is the period of the curve for these integer values?

f. Try $n = 3/2, 2/3, 3/4$ etc. EXTRA CREDIT: Can you figure out a general formula for the period of the curve?

g. What happens when n is irrational? Try $n = \pi$ (or $n = e$, or $n = \sqrt{2}$) and use a very large range of t, say $0 < t < 200\pi$). What happens? Any idea why?

If we want to think of the parameter t as time, then we can visualize our curve as the line traced out by a point that *moves* along the position $(x(t), y(t))$. In Matlab, it is easy to make a movie of the moving point using the `movie` command.

For example, let us take the Lissajous curve defined above, with $a = b = 1$ and $n = 7/2$.

```
>> a =  1; b = 1;
>> t=linspace(0,4*pi,500);
>> x = a*sin(n*t);
>> y = b*sin(t);
>> plot(x,y)
>> axis('equal')
```

If we wanted to watch the curve evolve, we could now try:

```
>> M = moviein(50); % Set up movie in 'M'
>> for i =1:50     % Make 50 frames of a Movie
>> plot(x(1:10*i),y(1:10*i)); % Plot the first 10*i points of x,y
>> hold on
>> plot(x(10*i),y(10*i),'r*'); % plot the last point so far with a red star
>> axis([-a,a,-b,b])
>> M(:,i) = getframe;
>> hold off
>> end
```

Whew ... that may take a moment or ten, but now we have an animation of our curve. To view the animation, try:

```
>> movie(M,2)   %% View the movie two times.
```

or

```
>> movie(M,-2)   %% View the movie, forward and backward, two times.
```

Exercise 6:
Make a movie of one of your other curves. Or make up a curve and make a movie of it.

Back to calculus, given the parametric curve, we can now ask and answer reasonable questions about the speed and acceleration of our moving point. We can also easily ask, and answer, questions about the shape of the curve, such as the slope of the curve at any point.

For example, let us consider the simple paramterized form of the circle:

$$\begin{aligned} x(t) &= r\cos(t) \\ y(t) &= r\sin(t) \end{aligned}$$

for $0 \le t < 2 * \pi$. Where is the slope of the tangent to this curve identically equal to zero? Well, we know that the slope is given by the derivative, dy/dx. But the rate of change of y with respect to x is nothing but the ratio of the rate of change of y with respect to t to the rate of change of x with respect to t. In other words:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

In this example:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-r\cos(t)}{r\sin t} = -\frac{\cos(t)}{\sin t}$$

So, $dy/dx = 0$ whenever $\cos t = 0$, namely at $t = \pi/2$ and $t = 3\pi/2$. Does this make sense? Check it. $x(\pi/2) = 0, y(\pi/2) = r$. The slope of the circle is zero there and at $(0, -r)$. It works.

Exercise 7:
Consider *Talbot's* curve :

$$\begin{aligned} x(t) &= (\sin^2(t) + 1)\cos(t) \\ y(t) &= (\sin^2(t) - 1)\sin(t) \end{aligned}$$

for $0 \leq t < 2\pi$.

a. Analytically, find all the points where the slope of the tangent to the curve is equal to 0.

b. Graph the curve. On the curve, mark the point(s) where the slope of the tangent line is zero. Where are the points where the slope of the tangent line tends to ∞ or is not defined. How many such points are there? Mark these points with red *'s on the graph.

If any of this is of any interest, go check out the Web Repository of groovy curves located at:

`http://www-groups.dcs.st-and.ac.uk/ history/Curves`

Also, if you ever played with SPRIROGRAPH, you may be curious about *cycloids*, regular cycliods, epicycloids and hypocycloids. There is some simple Matlab code for investigating these creatures available at:

`www.math.csi.cuny.edu/ poje/cycloids`

Download the two Matlab functions `epicycloid.m` and `hypocycloid.m` to the folder where Matlab saves your m-files.

To run the demos, type

```
>> epicycloid(a,b,rev) % put in values for a b and rev!
```

or

```
>> hypocycloid(a,b,rev) % put in values for a b and rev!
```

where a is the radius of the wheel, b is the radius of the point of interest, rev is the number of revolutions you would like to see displayed.