Calculus I Notes

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Thomas Calculus Ch 3. Notes

Definitions

Definition 1: Slope

The **slope** of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Definition 2: Derivative

The **derivative** of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

Definition 3: Alternative Derivative

$$f'(x_0) = \lim_{h\to 0} \frac{f(z) - f(x)}{z - x} \quad \text{(provided the limit exists).}$$

Definition 4: Theorem 1

Differentiable Implies Continuous If f has a derivative at $x = c$, then f is continuous at $x = c$.

Definition 5: Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Definition 6: Derivative of a positive integer power

If n is a positive integer then

$$\frac{dx^n}{dx} = nx^{n-1}.$$

Definition 7: Derivative constant multiple rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Definition 8: Derivative sum rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points:

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Definition 9: Derivative of e

$$\frac{d}{dx}(e^x) = e^x$$

Definition 10: Derivative product rule

If u and v are differentiable functions of x, then so is their product $u.v$ and

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Definition 11: Derivative quotient rule

If u and v are differentiable functions of x at $v(x) \neq 0$, then so is their quotient u/v and

$$\frac{d\frac{u}{v}}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Definition 12: Derivative sin rule

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(sinx) = cos(x)$$

Definition 13: Derivative cosine rule

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(cosx) = -sin(x)$$

Definition 14: Other important trig rules

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Definition 15: The chain rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Definition 16: Implicit differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Definition 17: Dervite rule for inverses

If f has an interval I as its domain and $f'(x)$ exists and is never zero on I , then f^{-1} is differentiable at every point in its domain (the range of f).
The value of $(f^{-1})'(b)$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Selected exercises 3.3

Q1. $y = -x^2 + 3$

A1. $dy/dx = -2x$, $d^2y/dx^2 = -2$

Q2. $y = x^2 + x + 8$

A2. $dy/dx = 2x + 1$, $d^2y/dx^2 = 2$

Q7. $w = 3z^{-2} - \frac{1}{z}$

A7. $dw/dz = -6z^{-3} + z^{-2}$, $d^2w/dz^2 = 18z^{-4} - 2z^{-3}$

Check with python

Function w(z):

$$-\frac{1}{z} + \frac{3}{z^2}$$

First derivative (dw/dz):

$$\frac{1}{z^2} - \frac{6}{z^3}$$

Second derivative (d$^{2w/dz}$2):

$$-\frac{2}{z^3} + \frac{18}{z^4}$$

Q.9 $y = 6x^2 - 10x - 5x^-2$

A.9 $dy/dx = 12x - 10 + 10x^{-3}$, $d^2w/dz^2 = 12 - 30x^{-4}$

Check with python

Function y(x):

$$6x^2 - 10x - \frac{5}{x^2}$$

First derivative (dy/dx):

$$12x - 10 + \frac{10}{x^3}$$

Second derivative (d$^{2y/dx}$2):

$$12 - \frac{30}{x^4}$$

Q.12 $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

A.12 $-\frac{12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5}$,

$\frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

Original Function:

$$\frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$$

First Derivative:

$$-\frac{12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5}$$

Second Derivative:

$$\frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$$