Yet Another fast variant of Newton's method

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EUROPT, 24 August 2023, Budapest

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Problem motivation

Approximately solving the non convex problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

with a 'fast', adaptive convergent algorithm.

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Gradient descent (Since Cauchy Lemarechal [2012])

$$x_{k+1} = x_k - \gamma_k \nabla^1 f(x_k),$$

where γ_k can be chosen adaptively (LineSearch : Armijo [1966], Adaptive : Duchi, Hazan, and Singer [2011]; McMahan and Streeter [2010])

Advantages

- Cheap cost.
- Workhorse of modern ML.

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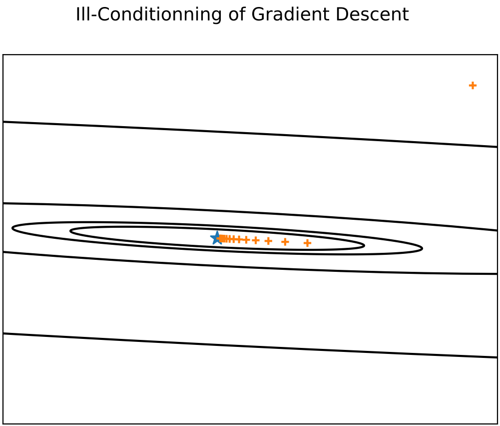
- Cheap cost.
- Workhorse of modern ML.

For deterministic optimization :

- Slow convergence rate (e.g : $\mathcal{O}(\epsilon^{-2})$ to $\|\nabla^1 f(x_\epsilon)\| \leq \epsilon$).
- Ill conditioning.

Illustrative example

Gradient Descent on a 2D quadratic



$\rightarrow$ Exploits second-order information

Newton's method

$$\text{Minimizes local quadratic : } s^\intercal \nabla^1 f(x_k) + \frac{1}{2} s^\intercal \nabla^2 f(x_k) s$$

$$x_{k+1} = x_k - (\nabla^2 f(x_k))^{-1} \nabla^1 f(x_k)$$

- Super-fast convergence locally [Bertsekas, 1995].
- Only need to solve a linear system.
- Effecient approximation of the Hessian (Quasi-Newton) [Nocedal and Wright, 2006].

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- No guaranteed decrease.

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Measure of first-order stationarity :

$$\boxed{\|\nabla f(x_\epsilon)\| \leq \epsilon.}$$

Vanilla Nonconvex Second-Order

Newton LineSearch :

$$x_{k+1} = x_k - \gamma_k (\nabla^2 f(x_k))^{-1} \nabla^1 f(x_k)$$

γ_k is adapted via a Line-Search.
Only need to solve a system.
$\mathcal{O}(\epsilon^{-2})$ rate [Cartis, Gould, and Toint, 2022b].

Trust-region : Optimize for $\|s\| \leq \delta_k$

$$s^\intercal \nabla^1 f(x_k) + \frac{1}{2} s^\intercal \nabla^2 f(x_k) s$$

where δ_k is updated to ensure convergence.
Efficient Algorithms [Conn et al., 2000, 6] (e.g : Steihaug-Toint).
For standard TR, $\mathcal{O}(\epsilon^{-2})$ rate as shown [Cartis et al., 2022b].

Cubic Regularization

Griewank [1981] and Nesterov and Polyak [2006] propose to minimize (exactly) the following cubic model :

$$m_k(s) = s^\intercal \nabla^1 f(x_k) + \frac{1}{2} s^\intercal \nabla^2 f(x_k) s + \frac{\sigma_k}{6} \|s\|^3$$

with σ_k bigger than a specific constant and set $x_{k+1} = x_k + s_k$.
Enjoys $\mathcal{O}\left(\epsilon^{-3/2}\right)$ [Nesterov and Polyak, 2006] complexity rate.

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- Adaptive (ARC) Cartis et al. [2011a,b], adaptive without function values [Gratton, J, and Toint, 2023a].
- Inexact Xu et al. [2019], on manifold Agarwal et al. [2020], probabilistic Bellavia et al. [2022]; Cartis and Scheinberg [2017], ...

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First-order stationarity condition on the model

$$\nabla^1 f(x_k) + \nabla^2 f(x_k) s_k + \frac{\sigma_k}{2} \|s_k\| s_k = 0$$

Implicit system:

$$s_k = -(\nabla^2 f(x_k) + \frac{\sigma_k}{2}\|s_k\| I_n)^{-1} \nabla^1 f(x_k).$$

Gradient Descent proposed in Carmon and Duchi [2019] or Lanczos process Cartis et al. [2011a].
Recently, Mishchenko [2023] and Doikov and Nesterov [2023] proposed for the **convex** case

$$s_k = -(\nabla^2 f(x_k) + \sqrt{\sigma_k \|\nabla^1 f(x_k)\|})^{-1} \nabla^1 f(x_k) \qquad (1),$$

with good initial numerical results Mishchenko [2023].

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Can we extend (1) to the non-convex case? Devise an algorithm

- Uses (1) when convex.
- Convergence rate close to $\mathcal{O}\left(\epsilon^{-3/2}\right)$.
- Scalable implementation.

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Denote by $g_k \stackrel{\text{def}}{=} \nabla^1 f(x_k)$ and $H_k \stackrel{\text{def}}{=} \nabla^2 f(x_k)$. Recall for convex:

$$s_k = -(H_k + \sqrt{\sigma_k \|g_k\|} I_n)^{-1} g_k.$$

AN2C

Denote by $g_k \stackrel{\text{def}}{=} \nabla^1 f(x_k)$ and $H_k \stackrel{\text{def}}{=} \nabla^2 f(x_k)$. Recall for convex:

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A natural extension for the nonconvex case,

$$s_k = -(H_k + (\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)]) I_n)^{-1} g_k$$

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Problem when $-\lambda_{\min}(H_k)$ is large $\rightarrow$ step too small.

$\rightarrow$ Use negative curvature instead.

Hence, **AN2C** (Adaptive Newton with Negative Curvature) denomination.

AN2C Algorithm presentation

Algorithm 1: AN2C Algorithm

Input: x_0, $\epsilon \in (0, 1]$, $\sigma_0, \sigma_{\min} > 0$, $0 < \eta_1 < \eta_2 < 1$, $0 < \gamma_1 < 1 < \gamma_2 \leq \gamma_3$, $\kappa_C > 0$ and $k = 0$.

Output: x_ϵ: an approximate first order point.

while $\|g_k\| > \epsilon$ **do**

Evaluate g_k and H_k.

Compute $\lambda_{\min}(H_k)$. If $\lambda_{\min}(H_k) \geq -\kappa_C \sqrt{\sigma_k \|g_k\|}$, compute s_k^{neig}

$$s_k^{neig} = -(H_k + (\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)]) I_n)^{-1} g_k.$$

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$$s_k^{neig} = -(H_k + (\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)]) I_n)^{-1} g_k.$$

Else $\lambda_{\min}(H_k) \leq -\kappa_C \sqrt{\sigma_k \|g_k\|}$,

$$g_k^\intercal u_k \leq 0, \|u_k\| = 1, H_k u_k = \lambda_{\min}(H_k) u_k \quad s_k^{curv} = \frac{\kappa_C \sqrt{\sigma_k \|g_k\|}}{\sigma_k} u_k.$$

Compute $\rho_k \leftarrow \frac{f(x_k) - f(x_k + s_k)}{-(g_k^\intercal s_k + \frac{1}{2} s_k^\intercal H_k s_k)}$ If $\rho_k \geq \eta_1$, $x_{k+1} \leftarrow x_k + s_k$. Else, $x_{k+1} \leftarrow x_k$.

Update σ_k with the values of η_1, η_2, γ_1, γ_2 and γ_3 as in [Cartis et al., 2022a, Algorithm 3.3.1].

$k \leftarrow k + 1$.

Comments on AN2C

- The $\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)]$ resembles the GQT method [Goldfeld, Quandt, and Trotter, 1966], $\mathcal{O}\left(\epsilon^{-2}\right)$ proven by Ueda and Yamashita [2014].
- Birgin and Martínez [2017] regularizes with $\max[0, -\lambda_{\min}(H_k)] + \mu$ and test multiple μ's to ensure $\mathcal{O}\left(\epsilon^{\frac{-3}{2}}\right)$ via 'cubic' descent.
- Negative curvature directions + gradient related ones. See Curtis and Robinson [2018]; Ferris et al. [1996]; Goldfarb [1980]; Gould et al. [2000].

Our algorithm enjoys a $\mathcal{O}\left(|\log \epsilon|\epsilon^{-\frac{3}{2}}\right)$ complexity rate.

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⚠ What about $\lambda_{\min}(H_k)$? Exact solve ?

Scalable variants will be discussed 😄.

Notations, bound on σ_k

We introduce

$$\mathcal{S}_k \stackrel{\text{def}}{=} \{0 \leq j \leq k |\, x_{j+1} = x_j + s_j, \}, \quad \mathcal{S}_k^{neig} \stackrel{\text{def}}{=} \left\{ j \in \mathcal{S}_k | s_j = s_j^{neig} \right\},$$

$$\mathcal{S}_k^{curv} \stackrel{\text{def}}{=} \left\{ j \in \mathcal{S}_k | s_j = s_j^{curv} \right\}.$$

As

$$\sigma_{k+1} \in \begin{cases} [\max(\sigma_{\min}, \gamma_1 \sigma_k), \sigma_k] & \text{if } \rho_k \geq \eta_2, \\ [\sigma_k, \gamma_2 \sigma_k] & \text{if } \rho_k \in [\eta_1, \eta_2), \\ [\gamma_2 \sigma_k, \gamma_3 \sigma_k] & \text{if } \rho_k < \eta_1. \end{cases}$$

From standard analysis [Cartis et al., 2022a, Lemma 2.4.1], only need a bound $\sigma_{\max}$ and $|\mathcal{S}_k|$.

AN2C

Assumptions

AS.1 f is a twice differentiable function.

AS.2 $\nabla^2 f$ is a Lipschitz function with constant L_H.

AS.3 f is lower bounded by f_{low}.

AS.4 $\lambda_{\min}(\nabla^2 f(x_k)) \geq -\kappa_H$.

Properties on when s_k^{neig}

$$\|s_k^{neig}\| \leq \sqrt{\frac{\|g_k\|}{\sigma_k}}, \quad -g_k^\intercal s_k^{neig} - \frac{1}{2} s_k^{neig} H_k s_k^{neig} \geq \sqrt{\sigma_k \|g_k\|} \|s_k^{neig}\|^2.$$

On s_k^{curv}

$$-g_k^\intercal s_k^{curv} - \frac{1}{2} s_k^{curv} H_k s_k^{curv} \geq \frac{1}{2} \sigma_k \|s_k^{curv}\|^3 = \frac{\eta_1 \kappa_C^3}{2\sqrt{\sigma_k}} \|g_k\|^{\frac{3}{2}}.$$

Decrease comes from step property, not from model as
in ARC Cartis et al. [2011a,b].

Bound on $\sigma +$ Elements

Bound on $\sigma_{\max}$.

From the local quadratic decrease $g_k^\intercal s_k + \frac{1}{2} s_k^\intercal H_k s_k$ in both cases + Lipschitz smoothness + σ_k update mechanism

$$\sigma_k \leq \sigma_{\max}.$$

Proof similar to ARp methods [Birgin, Gardenghi, Martínez, Santos, and Toint, 2016a].

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Proof similar to ARp methods [Birgin, Gardenghi, Martínez, Santos, and Toint, 2016a].
$k \in \mathcal{S}_k^{curv}$, guaranteed decrease

$$f(x_k) - f(x_{k+1}) \geq \eta_1(-g_k^\intercal s_k - \frac{1}{2} s_k^\intercal H_k s_k) \geq \frac{\eta_1 \kappa_C^3}{2\sqrt{\sigma_k}} \|g_k\|^{\frac{3}{2}} \geq \frac{\eta_1 \kappa_C^3}{2\sqrt{\sigma_{\min}}} \epsilon^{\frac{3}{2}}.$$

Decrease of required order. As $f(x) \geq f_{\text{low}}$, $|\mathcal{S}_k^{curv}| \leq \mathcal{O}\left(\epsilon^{-\frac{3}{2}}\right)$.
Lipschitz gradient error + negative curvature boundness and $\|s_k^{curv}\|$ bound

$$\|g_{k+1}\| \leq \left(\frac{L_H}{2\sigma_k} \kappa_C^2 + \frac{\kappa_H \kappa_C}{\sqrt{\epsilon \sigma_k}} + 1\right) \|g_k\|.$$

Newton decrease step

Lower bound required on $\|s_k^{neig}\|$. $k \in \mathcal{S}_k^{neig}$

$$
\begin{aligned}
\|g_{k+1}\| &\leq \|g_{k+1} - g_k - H_k s_k\| + \|g_k + H_k s_k\| \\
&= \|g_{k+1} - g_k - H_k s_k\| + \|(\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)]) s_k\| \\
&\leq \frac{L_H}{2}\|s_k\|^2 + (1 + \kappa_C)\sqrt{\sigma_k \|g_k\|}\|s_k\|,
\end{aligned}
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\end{aligned}$$

$$\|s_k\| \geq \frac{-(1+\kappa_C)\sqrt{\sigma_k\|g_k\|} + \sqrt{(1+\kappa_C)^2\sigma_k\|g_k\| + 2L_H\|g_{k+1}\|}}{L_H}.$$

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$$

When $\|g_{k+1}\| \ll \|g_k\|$, last bound becomes uninformative.

Newton steps division

Inspired by analysis of Mishchenko [2023], divide $\mathcal{S}_k^{neig}$ in two subsets.

$$\mathcal{S}_k^{divgrad} \stackrel{\text{def}}{=} \{k \in \mathcal{S}_k^{neig}, 2\kappa_m L_H \|g_{k+1}\| < \sigma_k \|g_k\|\}$$

where $\kappa_m = \frac{\sigma_{\max}}{L_H}$ depends only on problem constant.

$$\mathcal{S}_k^{decr} \stackrel{\text{def}}{=} \mathcal{S}_k^{neig} \setminus \mathcal{S}_k^{divgrad},$$

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$$\mathcal{S}_k^{decr} \stackrel{\text{def}}{=} \mathcal{S}_k^{neig} \setminus \mathcal{S}_k^{divgrad},$$

from $k \in \mathcal{S}_k^{decr}$ and $\|s_k^{neig}\|$ lower bound,

$$f(x_k) - f(x_{k+1}) \geq \kappa(\sigma_k \|g_k\|)^{\frac{3}{2}} \geq \mathcal{O}\left(\epsilon^{\frac{3}{2}}\right) \to |\mathcal{S}_k^{decr}| \leq \mathcal{O}\left(\epsilon^{\frac{-3}{2}}\right).$$

For $k \in \mathcal{S}_k^{divgrad}$,

$$\|g_{k+1}\| \leq \frac{\|g_k\|}{2}.$$

Bound on $|\mathcal{S}_k|$ and iterations number

$$|\mathcal{S}_k^{divgrad}| \leq \kappa_n |\mathcal{S}_k^{decr}| + \left(\frac{|\log \epsilon|}{2\log 2} + \kappa_{curv}\right)|\mathcal{S}_k^{curv}| + \frac{|\log(\epsilon)| + \log(\|g_0\|)}{\log 2} + 1$$

Idea : Since $\frac{\epsilon}{\|g_0\|} \leq \prod_{i\in\mathcal{S}_k} \frac{\|g_{i+1}\|}{\|g_i\|}$ with use bounds $\|g_{k+1}\|/\|g_k\|$ for three cases.

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Since $|\mathcal{S}_k| = |\mathcal{S}_k^{divgrad}| + |\mathcal{S}_k^{curv}| + |\mathcal{S}_k^{decr}|$

$$|\mathcal{S}_k| \leq \mathcal{O}\left(|\log \epsilon|\epsilon^{-3/2}\right)$$

and required iterations k

$$k \leq |\mathcal{S}_k|\left(1 + \frac{|\log \gamma_1|}{\log \gamma_2}\right) + \frac{1}{\log \gamma_2}\log\left(\frac{\sigma_{\max}}{\sigma_0}\right).$$

Comments

- Up to a $|\log \epsilon|$ from the optimal rate of second-order methods as shown Carmon et al. [2019].
- $|\log \epsilon|$ comes from negative curvature steps. Rare in practice.
- Retrieve same rate as the Truncated Newton-CG of Curtis et al. [2021].
- Better than past hybrid algorithms of [Curtis and Robinson, 2018], [Liu et al., 2018] and [Goldfeld et al., 1966].

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How to devise scalable variant ?

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Improvements

Goals : Avoid the computation of $\lambda_{\min}(H_k)$ and inexact resolution.

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Denote by

$$r_k^{neig} = (H_k + (\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(H_k)])I_n)s_k^{neig} + g_k$$

To compute s_k^{neig} only needs

$$\|r_k^{neig}\| \leq \min(\varsigma\sqrt{\sigma_k\|g_k\|}\|s_k^{neig}\|, \kappa_\theta\|g_k\|),$$

with $\varsigma < 1$.

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Comments

- $\kappa_\theta\|g_k\|$ standard for CG methods.
- $\varsigma\sqrt{\sigma_k\|g_k\|}\|s_k^{neig}\|$ mimics the condition on the cubic model gradient $\|\nabla_s^1 m_k(s_k)\| \leq \mathcal{O}\left(\|s_k\|^2\right)$, see [Curtis, Robinson, Royer, and Wright, 2021].

Definite step

Use only the gradient to "compensate" for the negative curvature. Consider $\kappa_a \geq 1$ and attempt to solve the following linear system

$$(H_k + \sqrt{\kappa_a \sigma_k \|g_k\|}) s_k^{def} = -g_k$$

to find a solution such that

$$(s_k^{def})^\intercal (H_k + \sqrt{\kappa_a \sigma_k \|g_k\|}) s_k^{def} > 0,$$

$$\|s_k^{def}\| \leq \kappa_\theta \sqrt{\frac{\|g_k\|}{\kappa_a \sigma_k}}.$$

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Comments

- Adds a new kind of step.
- First condition ensures that we have "sufficiently" regularized.
- Second condition ensures that the step is bounded w.r.t the gradient.
- Inexact resolution is possible.

Comments on the definite step

- Can use a factorization or iterative conjugate gradient to check the definiteness. See 'Capped-CG' subroutine of Royer, O'Neill, and Wright [2019]
- May slow numerical efficiency. For convex subregions, $\sqrt{\sigma_k \|g_k\|}$ is the "right" regularization term [Doikov and Nesterov, 2023; Mishchenko, 2023].
- Still needs to compute $\lambda_{\min}(H_k)$ if the step fails.

Propose subspace version of the exact AN2C.

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Subspaces Methods

Project the current derivatives into a "well-behaved" subspace. **We extend our AN2C to propose a AN2CK**

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Krylov methods.

Structured nested sequence of subspaces. See [Conn, Gould, and Toint, 2000; Nocedal and Wright, 2006] for trust-region variant and Newton-CG algorithm.

Condition on the subspace

Consider $p \in \{1, \ldots, n\}$ and

$$V_p \in \mathbb{R}^{n \times p}, \quad \widehat{g}_k \stackrel{\text{def}}{=} V_p^\intercal g_k \in \mathbb{R}^p, \quad \widehat{H}_k \stackrel{\text{def}}{=} V_p^\intercal H_k V_p \in \mathbb{R}^{p \times p}.$$

where

$$\|V_p\| \leq V_{\max} \text{ for all } p \in \{1, \ldots, n\}.$$

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$$\|V_p\| \leq V_{\max} \text{ for all } p \in \{1, \dots, n\}.$$

Let $\kappa_B \geq 1$. If $\lambda_{\min}(\widehat{H}_k) \geq -\kappa_C \sqrt{\sigma_k \|g_k\|}$,

$$(\widehat{H}_k + (\sqrt{\sigma_k \|g_k\|} + \max[0, -\lambda_{\min}(\widehat{H}_k)]) I_p) y_k^{neig} = -\widehat{g}_k.$$

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If

$$\|H_k V_p y_k^{neig} + g_k\|_n \leq \kappa_B \|\widehat{H}_k y_k^{neig} + \widehat{g}_k\|_p,$$

set $s_k^{neig} \stackrel{\text{def}}{=} V_p y_k^{neig}$.

Else, consider a new subspace V_p.

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Else $\lambda_{\min}(\widehat{H}_k) \le -\kappa_C\sqrt{\sigma_k\|g_k\|}$,

$$\widehat{g}_k^\intercal u_k \le 0, \|u_k\| = 1, \widehat{H_k} u_k = \lambda_{\min}(\widehat{H_k})u_k \quad s_k^{curv} = \frac{\kappa_C\sqrt{\sigma_k\|g_k\|}}{\sigma_k} V_p u_k.$$

Kryolov implementation details

Krylov via Lanczos

We use the Lanczos procedure to generate a sequence of orthonormal bases of the Krylov subspace. $V_{\max} = 1$ and $p = n$ gives a valid step.

Kryolov implementation details

Krylov via Lanczos

We use the Lanczos procedure to generate a sequence of orthonormal bases of the Krylov subspace. $V_{\max} = 1$ and $p = n$ gives a valid step. Exploit

the structure of the specific subproblem

- $\widehat{H}_k$ is a tridiagonal matrix.
- Computation of $\lambda_{\min}(\widehat{H}_k)$ can be performed easily Coakley and Rokhlin [2013].
- V_p can be regenerated at the end as in Gould et al. [2003].
- Preconditionner can be employed.

For more details on Krylov methods, See [Conn et al., 2000, Subsection 5.2]

Recap + Other Results

- Complexity rate $\mathcal{O}\left(|\log\epsilon|\epsilon^{-3/2}\right)$ still valid for all the introduced AN2C variants. See [Gratton, J, and Toint, 2023b].
- Second-order algorithm. $\mathcal{O}\left(|\log\epsilon_1|\max(\epsilon_1^{-3/2},\epsilon_2^{-3})\right)$ iterations to reach (ϵ_1,ϵ_2) second order point.
- Approximate negative curvature can be employed.

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Numerical experiments framework

- A first set of 117 small dimensional problems, second set of 74 medium problems and third set of 59 "largish" problems from CUTEst problems (OPM : Matlab library) Gratton and Toint [2021].
- Baselines : standard AR2 of Birgin et al. [2016b] and trust-region methods Conn et al. [2000] with factorized and iterative solver. with $\epsilon = 1\mathrm{e}{-6}$.
- Max number of iteration set out to 5000 or timeout if cputime > 1 hour.
- Results are reported using performance profiles Dolan et al. [2006].

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See [Gratton, J, and Toint, 2023b] for more details.

Variants of AN2C

- $\mathrm{AN2CE}$ for exact computation and $\mathrm{AN2CER}$ where the trial definite step is performed (regularization $\propto 10 * \sqrt{\sigma_k \|g_k\|}$).

- $\mathrm{AN2CKU}$ and $\mathrm{AN2CKYU}$ two variants of the Krylov subspace. First variant : aligned with $\lambda_{\min}(\widehat{H_k})$, second variant : uses past information and fraction of $\lambda_{\min}(\widehat{H_k})$.

Numerical performances small

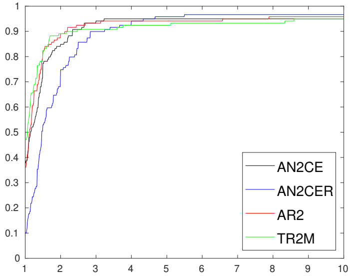

Figure: Exact and Factorized AN2C algorithms and Baselines

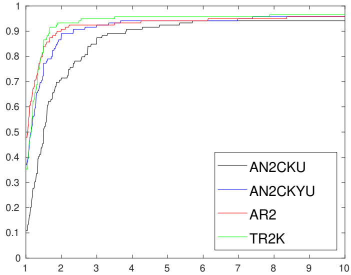

Figure: Two variants of Krylov AN2C and Baselines

Numerical performance medium

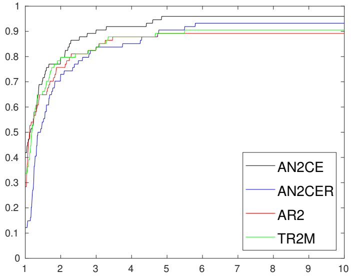

Figure: Exact and Factorized AN2C algorithms and Baselines

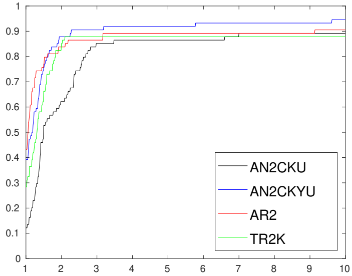

Figure: Two variants of Krylov AN2C and Baselines

Numerical performance large

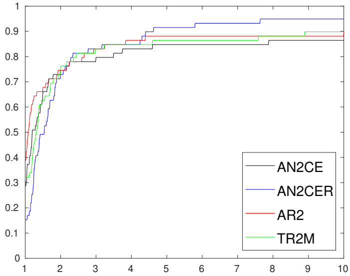

Figure: Exact and Factorized AN2C algorithms and Baselines

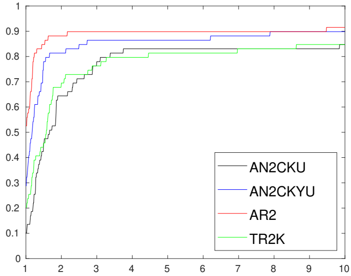

Figure: Two variants of Krylov AN2C and Baselines

Comments

- AN2C methods on par with standard methods. Best variant depends on the problems.
- AN2CE best for small datasets while AN2CER is the 'fastest' for large problems.
- Iterative methods: AN2CKU significantly trails the other variants $\rightarrow$ Does not exploit past informations

Comments

- AN2C methods on par with standard methods. Best variant depends on the problems.
- AN2CE best for small datasets while AN2CER is the 'fastest' for large problems.
- Iterative methods: AN2CKU significantly trails the other variants $\rightarrow$ Does not exploit past informations .
- Usage of negative curvature happens rarely (at most 0.25%). Basically a Newton method.
- When trial definite step is tested, used in at most 93% of case. one solution of a linear system required

More experiments are required to better asses these methods: cpu-time, preconditionner,...

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Conclusion

Recap

- Second order method alternating between Newton and negative curvature (In practice, mostly Newton).
- May require costly negative curvature information.
- Inexact solution + trial step that uses only gradient as a regularization + subspace implementation allow practical variants.
- Optimal complexity up to a log factor.
- On par with standard methods numerically.

Extensions

- β Hölder Hessian for $\beta \in (0, 1]$.

Perspectives

Other subspaces techniques : Random sketching methods [Mahoney, 2011; Woodruff, 2014] ?

Non-Euclidean norm [Doikov and Nesterov, 2023; Gratton and Toint, 2022] ?

For more details

Thank for your attention.

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