SICP 1-13 exercise

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1 Exercise

Prove that $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. Hint: let $\psi = \frac{1-\sqrt{5}}{2}$. Use induction and the defenition of the Fibonacci numbers to prove that $Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$.

2 Solutioin

So we need to prove that $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$ by proving that $Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$ using induction.

2.1 Base step

Let's check the formula for $n = 1$:

$$Fib(1) = \frac{\phi^1 - \psi^1}{\sqrt{5}}$$

$$1 = \frac{\phi^1 - \psi^1}{\sqrt{5}}$$

$$1 = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}}$$

$$1 = \frac{\frac{1+\sqrt{5}-1+\sqrt{5}}{2}}{\sqrt{5}}$$

$$1 = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}}$$

$$1 = \frac{\sqrt{5}}{\sqrt{5}}$$

$$1 = 1$$

2.2 Induction step

Assuming it works for $n = k$ and $n = k - 1$, it must work for $n = k + 1$ also:

$$Fib(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

By defenition of Fib: $Fib(k+1) = Fib(k-1) + Fib(k)$, so

$$Fib(k+1) = \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} + \frac{\phi^{k} - \psi^{k}}{\sqrt{5}}$$

Let's evaluate the right part of this equation and see what we got.

$$\frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} + \frac{\phi^{k} - \psi^{k}}{\sqrt{5}} =$$

$$\frac{\phi^{k-1} - \psi^{k-1} + \phi^{k} - \psi^{k}}{\sqrt{5}} =$$

$$\frac{\phi^{k-1} + \phi^{k} - \psi^{k} - \psi^{k-1}}{\sqrt{5}} = ...$$

Now let's stop for a sec and consider this property of ϕ and ψ :

$$\phi = \frac{1}{\phi} + 1$$

Same is true for ψ.

$$\frac{\phi^{k+1}(\phi^{-2} + \phi^{-1}) - \psi^{k+1}(\psi^{-2} + \psi^{-1}))}{\sqrt{5}} =$$

$$\frac{\phi^{k+1}\phi^{-1}(\phi^{-1} + 1) - \psi^{k+1}\psi^{-1}(\psi^{-1} + 1))}{\sqrt{5}} =$$

$$\frac{\phi^{k+1}\phi^{-1}(\phi^{-1} + 1) - \psi^{k+1}\psi^{-1}(\psi^{-1} + 1))}{\sqrt{5}} =$$

$$\frac{\phi^{k+1}\phi^{-1}(\frac{1}{\phi} + 1) - \psi^{k+1}\psi^{-1}(\frac{1}{\psi} + 1))}{\sqrt{5}} =$$

Here comes the fun part. Using the property mentioned above we can now reduce expressions in the brackets:

$$\frac{\phi^{k+1}\phi^{-1}(\phi) - \psi^{k+1}\psi^{-1}(\psi)}{\sqrt{5}} =$$

$$\frac{\phi^{k+1}\phi^{-1}(\phi) - \psi^{k+1}\psi^{-1}(\psi)}{\sqrt{5}} =$$

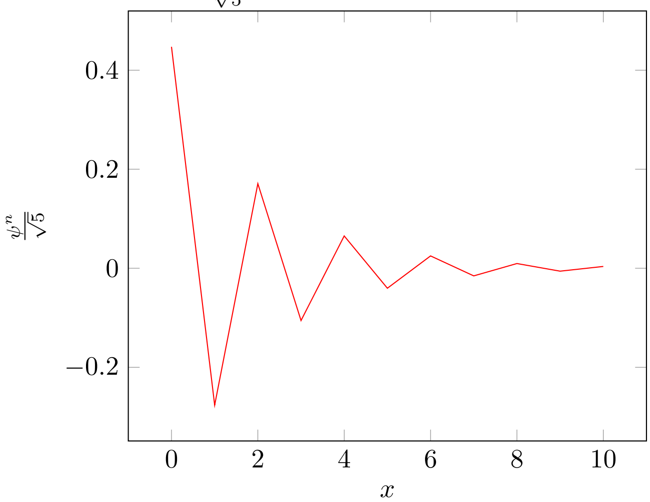
$$\frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

2.3 Original hypothesis proof

Now we can go back to the original hypothesis: $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$\frac{\phi^n}{\sqrt{5}}$ differs from $\frac{\phi^n - \psi^n}{\sqrt{5}}$ by $\frac{\psi^n}{\sqrt{5}}$ and always an integer for $n \in N$ so the prof is a matter of proving that $-1/2 < \frac{\psi^n}{\sqrt{5}} < 1/2$ where $n \in N$.

Let's just analyze $\frac{\psi^n}{\sqrt{5}}$:



It's a descending function and it's maximum value on N (including 0) is approximately 0.4472135955, at $x = 0$.

quod erat demonstrandum