Project Parameter Estimation
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PPE Kriging	**Fall Semester 2020**	**Subject NR. 103-0787-00 L**

HOMEWORK 2: RANDOM FIELDS

1 Contents and Aim of Homework

- **Understanding stochastic processes as special cases of Random Fields**
- **Calculating Expectations and Covariances to evaluate if a Random Field is stationary**
- **Get a feeling for how Covariances typically look as a function of distance between the locations associated to Random Variables**
- **Understand Gaussian Processes and build your own ones**

Note: This homework is a group homework (2-3 people in each group).It is to be handed in till noon of 19th of November. E-mail: jemil.butt@geod.baug.ethz.ch

2 Tasks

2.1 Stationarity and Warm-up

a) The "Wiener Process" is just integrated white noise. Please formulate the equation for a discrete Wiener Process and explain if it is second order stationary or not.

2.2 Stochastic Process with dice (I)

Let a Stochastic Process $\{X_t\}_{t=1\dots n}$ be defined by *:*

$$\begin{aligned} X_1 &= 1.75 + 0.5D_1 \\ X_t &= 0.5D_{t-1} + 0.5D_t \end{aligned} \qquad \text{where } D_i \text{ is the i-th dice roll}$$

a) Calculate the expected value $\mathrm{E}[X_t]$ and the covariance $\sigma(X_t, X_{t+p})$ and draw the covariance as a function of p . Please explain!

b) Is this stochastic process second order stationary?

c) Derive the relationship on slide 7 between the covariance $\sigma(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$ and the semivariance $\gamma(X,Y) = \frac{1}{2}\mathrm{E}[(X-Y)^2]$ for second order stationary stochastic processes

d) Please grab a die and simulate $X_1 \dots X_{30}$. Then use the relationship from c) to estimate with Matlab or Excel $\sigma(X_t, X_{t+p})$ by using only $\hat{\gamma}(X_t, X_{t+p})$ and $\hat{\sigma}(X_t, X_t)$.

2.3 Stochastic Process with dice (II) (bonus points)

Let a Stochastic Process $\{X_t\}_{t=1\ldots n}$ be defined by *:*

$$X_1 = 1.75 + 0.5D_1$$

$$X_t = 0.5X_{t-1} + 0.5D_t \qquad \text{where } D_i \text{ is the i-th dice roll}$$

a) Give a first estimate of what you expect the covariance (as a function of distance p between two random variables) to be. Why do you think this behavior reasonable?

b) Calculate the expected Value $\mathrm{E}[X_t]$. **Hint:** Geometric series, see explanation on slides.

c) Calculate the covariance $\sigma(X_t, X_{t+p})$ and decide if X_t can be regarded as a second order stationary stochastic process. **Hint:** Geometric series, see explanation on slides.

d) After carrying out a simulation of this dice roll game, Bob realizes that he forgot to write down X_{50} . But he has all the other $X's: X_1 \ldots X_{100}$. Please help Bob and write down (schematically) the Kriging system for the best linear predictor for X_{50} .

2.4 Gaussian Stochastic Processes (SP)

A (discrete) Gaussian process is a collection of random variables $\{X_t\}_{t=1\ldots n}$; all together jointly Gaussian distributed.

a) Assume X_t to be a stationary Gaussian process. Is the expected value $\mathrm{E}[X_t]$ together with the covariance $\sigma(X_t, X_{t+1})$ enough to determine the whole process completely? Please explain (or give a counterexample).

b) Please define two second order stationary Gaussian processes $\{X_t\}_{t=1\ldots 300}$, $\{Y_t\}_{t=1\ldots 300}$ with:

$$\mathrm{E}[X_t] = 0 \qquad\qquad \mathrm{E}[Y_t] = 0$$

$$\sigma(X_t, X_{t+p}) = \sigma_0^2 e^{-\frac{|p|}{30}} \qquad\qquad \sigma(Y_t, Y_{t+p}) = \sigma_0^2 e^{-\left(\frac{|p|}{30}\right)^2} \qquad\qquad \sigma_0^2 = 1$$

(Hint: Think of what Information you need to fix to completely determine a Gaussian process -> a))

c) Write a Matlab program that simulates the two stochastic processes you just invented! You can use the pre-built Matlab function mvnrnd (generate multivariate Gaussian random numbers – see slides 18, 19). Please plot the values of your Gaussian processes and also the correlation coefficients as a function of the distance between the random variables.

d) Compare the two different stochastic processes by describing and explaining their different behavior by relating it to the two covariance functions.

2.5 Gaussian Random Fields (RF)

Let $\{X_s\}_{s\in\{1\ldots30\}\times\{1\ldots30\}}, \{Y_s\}_{s\in\{1\ldots30\}\times\{1\ldots30\}}$ be two 2D discrete Gaussian random fields. The goal is now to simulate drawing realizations from these random fields. This can be done in a straightforward way by considering the 2D matrix to be simulated as a single long vector with appropriate covariance matrix calculated from the respective 2D distances.

a) Write a Matlab program that simulates the two random fields $X_.$ and $Y_.$ with properties as sketched below.

$$\mathrm{E}[X_s]=0 \qquad\qquad \mathrm{E}[Y_s]=0$$

$$\sigma(X_s,X_{s+h})=\sigma_0^2 e^{-\frac{\|h\|}{5}} \qquad \sigma(Y_s,Y_{s+h})=\sigma_0^2 e^{-\left(\frac{\|h\|}{5}\right)^2} \qquad \sigma_0^2=1$$

where s and h are now vectors and $\|\cdot\|$ denotes the euclidean distance

b) Look at the covariance matrix of the random field you have just sampled from. Can you explain, why it looks how it looks?

c) Try to invent a random field with the weirdest behavior you can come up with.

2.6 Spectral Theory of Stochastic Processes (bonus points)

As is well known, Covariance matrices are positive definite and symmetric (see slide 18). Therefore, they have a unique eigenvalue-eigenvector decomposition. Many interesting properties like the mean energy of a stochastic processes and random fields or their smoothness properties can be traced back to these spectral properties.

a) Let $X_.$ be a discrete mean 0 SP and denote by $\langle X_.,X_.\rangle=\|X_.\|^2=E\left[\sum_{t=1}^{n}X_t^2\right]$ its squared norm.

Show that $\|X_.\|^2=\sum_{k=1}^{n}\lambda_k$ where λ_k are the eigenvalues of the covariance matrix.

Hint: $\mathrm{trace}(AB)=\mathrm{trace}(BA)$

b) Look at the sequence of eigenvalues of the covariance matrices you used for simulation in 2.4 b). Can you somehow derive some further info about these processes?

c) Let $X_.$ and $Y_.$ be two SP's and let $\langle X_.,Y_.\rangle=E\left[\sum_{t=1}^{n}X_tY_t\right]$ be their inner product. Show that $\langle\cdot,\cdot\rangle$ is bilinear, symmetric, positive definite; i.e. has the same properties as the standard scalar product $a\bullet b=\sum a_ib_i$ that you know from linear algebra and geometry. What implications does this have?

Good Luck!