Standard RBC model

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1 Model

A standard Real Business Cycle (RBC) model with a representative agent including labour supply choice. The objective function of the agent is:

$$\max E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left\{ \frac{C_t^{1-\nu}}{1-\nu} - \chi \frac{H_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\}$$

where C_t is period t consumption, and H_t is period t labor supply. The real budget constraint is:

$$C_t + K_{t+1} = Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t \tag{1}$$

where K_t is the capital stock at the beginning of period t, and δ is the depreciation rate of capital. Total Factor Productivity (TFP) Z_t evolves by an exogenous process:

$$z_t = \rho_z z_{t-1} + \sigma_z \epsilon_t \tag{2}$$

where $z_t = \log(Z_t)$, ρ_z is the autocorrelation coefficient, and σ_z is the standard deviation of the shocks. The shocks ϵ_t are standard normally distributed ($\epsilon_t \sim \mathcal{N}(0,1)$).

The optimization problem can be written with an infinite Lagrangian:

$$\mathcal{L} = E \sum_{t=1}^{\infty} \beta^t \left\{ \frac{C_t^{1-\nu}}{1-\nu} - \chi \frac{H_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \lambda_t \left[Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t - C_t - K_{t+1} \right] \right\}$$

where λ_t is the shadow price of the budget constraint, and K_1 is given. Maximization of this Lagrangian with respect to hours H_t, consumption C_t and capital in next period K_{t+1} yields the following First Order Conditions:

$$\begin{aligned} C_t^{-\nu} &= \lambda_t \\ \chi H_t^{\frac{1}{\eta}} &= \lambda_t Z_t (1-\alpha) K_t^{\alpha} H_t^{-\alpha} \\ \lambda_t &= \beta \lambda_{t+1} \left[F_k (K_t, H_t) + 1 - \delta \right] \end{aligned}$$

Substituting out λ gives:

$$\chi H_t^{\frac{1}{\eta}} = C_t^{-\nu} Z_t \left(1-\alpha\right) K_t^{\alpha} H_t^{-\alpha} \tag{3}$$

$$C_t^{-\nu} = \beta E_t \left\{ C_{t+1}^{-\nu} \left[Z_{t+1} \alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + 1 - \delta \right] \right\} \tag{4}$$

We can derive a (log linear) analytical expression for labor supply, given capital, TFP and consumption using (3):

$$H_t = \left[\frac{1-\alpha}{\chi} C_t^{-\nu} Z_t K_t^{\alpha} \right]^{\frac{\eta}{1+\alpha\eta}} \tag{5}$$

2 Gauss-Hermite quadrature

The general rule for Gaussian-Hermite approximation is:

$$\int_{-\infty}^{\infty} \exp\left(-z^2\right) g\left(z\right) dz \approx \sum_{j=1}^{J} \omega_j g\left(\zeta_j\right) \tag{6}$$

with Gauss-Hermite nodes $j = 1, ..., J$, roots ζ_j and weights ω_j.

Assume we have a function $f\left(z_{t+1}, x\right)$ with exogenous variable z_{t+1}. This variable evolves according to (2) with standard normally distributed shocks $\epsilon_{t+1} \sim \mathcal{N}\left(0,1\right)$. The expected value of this function is:

$$E_t f\left(z_{t+1}, x\right) = \int_{-\infty}^{\infty} f\left(\rho_z z_t + \sigma_z \epsilon_{t+1}, x\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\epsilon_{t+1}^2/2\right) d\epsilon_{t+1} \tag{7}$$

To write (7) in the same form as (6) we need a change of variable $\phi = \frac{\epsilon_{t+1}}{\sqrt{2}}$, such that $\exp\left(-\epsilon_{t+1}^2/2\right) = \exp\left(-\phi^2\right)$. The approximation of the integral is:

$$\int_{-\infty}^{\infty} f\left(\rho_z z_t + \sigma_z \sqrt{2}\phi, x\right) \frac{1}{\sqrt{2\pi}} \exp\left(\phi\right) \sqrt{2} d\phi$$
$$\approx \sum_{j=1}^{J} \frac{\omega_j}{\sqrt{\pi}} f\left(\rho_z z_t + \sigma_z \sqrt{2}\zeta_j, x\right)$$

where the extra term$\sqrt{2}$ (before $d\phi$) follows from integration by substitution.

3 Steady state

To derive the analytical steady state we start with the Euler equation (4):

$$
\begin{aligned}
\overline{C}^{-\nu} &= \beta \left\{ \overline{C}^{-\nu} \left[\overline{Z} \alpha \overline{K}^{\alpha-1} H^{1-\alpha} + 1 - \delta \right] \right\} \\
\overline{H} &= \left[\frac{1-\beta\left(1-\delta\right)}{\overline{Z}\alpha\beta} \right]^{\frac{1}{1-\alpha}} \overline{K} = \Omega^{\frac{1}{1-\alpha}} \overline{K}
\end{aligned}
$$

with $\Omega = \frac{1-\beta(1-\delta)}{\alpha\beta\overline{Z}}$.

Substituting this into the resource constraint (1) yields:

$$
\begin{aligned}
\overline{C} + \overline{K} &= \overline{Z}\overline{K}^{\alpha} \overline{H}^{1-\alpha} + \left(1-\delta\right) \overline{K} \\
\overline{C} &= \overline{Z}\overline{K}^{\alpha} \left[\Omega^{\frac{1}{1-\alpha}} \overline{K} \right]^{1-\alpha} - \delta \overline{K} \\
&= \left(\overline{Z}\Omega - \delta \right) \overline{K}
\end{aligned}
$$

Substituting the expressions for $\overline{H}$ and $\overline{C}$ into the labor supply function (3) and solving for $\overline{K}$ yields:

$$
\overline{K} = \left[\left(\frac{1-\alpha}{\chi} \overline{Z} \left[\overline{Z}\Omega - \delta \right]^{-\nu} \right)^{\eta} \Omega^{\frac{\alpha\eta+1}{\alpha-1}} \right]^{\frac{1}{1+\eta\nu}}
$$