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Conformal Field Theory and the SYK Model

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Abstract

This project focuses on conformal field theory and its application to the Sachdev-Ye-Kitaev model in the context of the AdS/CFT correspondence. On this matter, the general properties of CFTs in $d \geq 3$ dimensions are derived and discussed. These features are then used to study the 1d SYK model, which is nearly conformal in the infrared. In particular, two-point functions, effective and Schwarzian actions are computed and used in a qualitative analysis of the four-point functions and their role in the leading-order corrections to the (out-of-)time-ordered correlation functions. The link to an incomplete AdS dual is finally made through these (O)TOCs by comparing them to those of Jackiw-Teitelboim gravity, and a comment is made on the speculated complete dual of the SYK model.

Contents

1 Introduction

One of the most important developments in the understanding of quantum gravity is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, which connects certain spaces used most notably in theories of quantum gravity and string theories (AdS) with certain types of quantum field theories (CFT). It was conjectured by Maldacena back in 1997 [1], when he noticed that the large N limit of some CFTs included in their Hilbert space a sector describing gravity on the product of AdS spacetimes and some other compact manifolds.
This project is a literature review aiming to introduce the reader to the SYK model as an example of AdS/CFT correspondence, focusing on the CFT side, by building the foundation needed from basic quantum mechanics and special relativity knowledge. To achieve this goal, we first introduce the reader to some concepts of classical field theory and Lie groups and algebras, give a crash course on quantum field theory and use the new knowledge built up to this point to present conformal field theory in $d \geq 3$ dimensions and some of its most striking properties. We then introduce the SYK model in detail, filling as best as possible the gaps left by the two main papers reviewed [2, 3] and detailing computations when needed. We wrap up our discussion with a comparison of correlation functions in the SYK model and in JT gravity, and some comments on the possible complete AdS dual of the model.
Note that if the reader is already familiar with classical and quantum field theory, they may skip Sections 2 and 3 of the present work.

2 Basics

Before setting out on our journey to the realm of the infinitely small and discussing quantum fields and the way they behave under certain transformations, in particular conformal ones, we must define what a field is. In addition, we'll rigorously introduce Noether's theorem, which links the symmetries of a system and its conserved quantities, and define the energy-momentum tensor. We'll wrap up our discussion of classical field theory by writing a general formula of the action for a field living on a general manifold endowed with a metric tensor $g_{\mu\nu}$. To prepare for

2.1 Classical Field Theory

Some of the most common examples of fields encountered in classical physics are the electric and magnetic fields $\vec{E}$ and $\vec{B}$, which are known to obey Maxwell's equations. Much like the equation of motion for a particle can be derived from a lagrangian using the principle of stationary action, it is possible to encapsulate all the information about the behaviour of a field in a similar formalism. [4]
However, a key difference between particles and fields lies in the number of degrees of freedom they have. The former can only have a finite number of them, while the latter has infinitely many. In fact, while the determination of a particle's trajectory at any time t relies only on the knowledge of its coordinate functions $x_i(t)$, its velocities $\dot{x}_i(t)$ and the constraints they are subject to, a field is a collection of functions defined at all space-time points. Following this definition, we denote a field by $\Phi(x) = (\phi_1, ..., \phi_n)$, where $x^\alpha = (ct, \mathbf{x})$ is a d-dimensional spacetime point, with d-1 spatial dimensions and the ϕ_j's are functions. Seen differently, a field generalises the quantities $t, x_i(t)$ and $\dot{x}_i(t)$ by replacing them respectively with $x^\alpha, \Phi(x)$ and $\partial^\alpha\Phi(x) = (\partial^\alpha\phi_1, ..., \partial^\alpha\phi_n)$. The last quantity is the d-gradient of the fields, which takes the place of the velocities of the particle. We have not yet mentioned what the fields actually were. Mathematically, a field is a function that assigns a scalar, a vector or a tensor to each point of its domain. It could be argued that scalars and vectors are just rank 0 and rank 1 tensors, but fields are traditionally divided in these three categories. In physics, the domain is generally a region of 3-dimensional space or 4-dimensional space-time. Since tensor fields will not appear in any of our calculations, we only define vector fields. We further restrict ourselves to vector fields whose domains are regions of $\mathbb{R}^n$ or the pseudo-euclidian Lorentz-Minkowski space $\mathbb{E}^{1,d}$ since more general spaces will not appear. A vector field V is then (may need to change something here, since I believe vector fields don't necessarily have to yield vectors that live in the same space as their input)

$$V : F \longrightarrow F, \tag{2.1.1}$$

$$x \longmapsto V(x_1, ..., x_n) = \begin{pmatrix} V_1(x_1, ..., x_n) \\ \vdots \\ V_n(x_1, ..., x_n) \end{pmatrix}, \tag{2.1.2}$$

where F is either $\mathbb{R}^n$ or $\mathbb{E}^{1,d}$. A scalar field on any of those spaces is then a function $S : F \longrightarrow \mathbb{K}$, where $\mathbb{K}$ is a field (the algebraic structure, not what we're defining right now), usually $\mathbb{R}$ or $\mathbb{C}$. To provide a little bit of context, let us give an example. Recall the d'Alembert equation describing a wave equation with speed c

$$\Box\phi \equiv -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = 0. \tag{2.1.3}$$

In vacuum, c is usually the speed of light, and if ϕ is e.g. the electric field $\vec{E}$, this equation (along with a similar one for $\vec{B}$) describes the motion of an electromagnetic wave. Since our

goal was to generalize the lagrangian formalism to fields, we want to find an action which, once minimized, gives the d'Alembert equation. If we take ϕ to be a scalar field for simplicity, such an action is

$$S[\phi] = \frac{1}{2}\int d^4x \left[-\left(\frac{\partial\phi}{\partial x^0}\right)^2 + \nabla\phi\cdot\nabla\phi \right]. \tag{2.1.4}$$

Using the notation defined earlier, the action can be nicely written as

$$S[\phi] = \frac{1}{2}\int d^4x \left(\partial^\mu\phi\eta_{\mu\nu}\partial^\nu\phi\right) = \frac{1}{2}\int d^4x\partial^\mu\phi\partial_\mu\phi, \tag{2.1.5}$$

where $\eta_{\mu\nu}$ is the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1.6}$$

which allows to lower indices in the following way

$$\eta_{\mu\nu}\partial^\nu = \partial_\mu. \tag{2.1.7}$$

Note that the Einstein summation convention is used, which means that repeated indices are summed over. Also note that the Minkowski metric can be defined in a similar way with both indices up, with the same matrix form, to allow for the raising instead of the lowering of indices. The integrand in the action is an object known as a lagrangian density $\mathcal{L}$, since integration over the three spatial dimensions would lead us back to an action of the form $S = \int dtL$. By abuse of language, we will also call the lagrangian density a lagrangian. A slightly modified version of the above density is

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.1.8}$$

where m is the "mass" of the field. This lagrangian leads to the Klein-Gordon equation of motion for the field

$$(-\partial^2 + m^2)\phi = 0 \tag{2.1.9}$$

which we will come back to when we introduce quantum field theory. Here, $\partial^2 = \partial^\mu\partial_\mu = \partial_0^2 - \nabla^2$. Also note that $\phi = \phi(x)$, where $x = (t, \mathbf{x})$ is a four-vector.

2.1.1 Symmetries, Noether's Theorem and Conserved Quantities

Symmetries play an extremely important role throughout all of physics. In fact, this whole project revolves around a certain type of symmetry, conformal symmetry, and its consequences on certain systems. The question we might ask is then: "What are the effects of a symmetry in a physical system?" To answer this question, we take a look at what the action is. The action is a scalar, which means that it should be the same regardless of the choice of coordinates. However, certain transformations that act on the fields directly are also symmetries. We therefore define transformations acting on both the coordinates and the fields. Let Φ be a collection of fields and $\partial_\mu\Phi$ its derivatives. Then a transformation will be of the form

$$\begin{gathered} x \longrightarrow x' \\ \Phi(x) \longrightarrow \Phi'(x') = \mathcal{F}(\Phi(x)), \end{gathered} \tag{2.1.10}$$

with $\mathcal{F}$ a functional (i.e. a function of functions). This way of viewing the transformation is called active, whereas if we had $\Phi'(x) = \Phi(Ax)$ instead of $\Phi'(x) = \Phi(A^{-1}x)$, the transformation would be considered passive. We now take a look at how the action changes under such a transformation[5].

$$\begin{aligned} S' &= \int d^d x' \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x'), g'_{\mu\nu}(x')) \\ &= \int d^d x' \mathcal{L} \left(\mathcal{F}(\Phi(x)), \partial'_\mu \mathcal{F}(\Phi(x)), \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \right) \\ &= \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L} \left(\mathcal{F}(\Phi(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \partial_\nu \mathcal{F}(\Phi(x)), \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \right) \end{aligned} \tag{2.1.11}$$

Where we used the definitions in eq. (2.1.10) to go from the first to the second line, and finished the calculation by changing variables $x \longrightarrow x$ and writing the derivative ∂' in terms of x. We have made the lagrangian depend on the metric explicitly, since a non-rigid (i.e. that is a function of position) transformation changes the metric.
To derive Noether's theorem, we introduce infinitesimal transformations, i.e. transformations of the form

$$\begin{aligned} x'^\mu &= x^\mu + \omega_a(x) \frac{\delta x^\mu}{\delta \omega_a(x)}, \\ \Phi'(x') &= \Phi(x) + \omega_a(x) \frac{\delta \mathcal{F}(x)}{\delta \omega_a(x)}, \end{aligned} \tag{2.1.12}$$

where $\frac{\delta}{\delta \omega_a(x)}$ is the functional derivative with respect to $\omega_a(x)$, a set of infinitesimal parameters labeled by a. We keep everything to first order, and calculate the approximations of the terms appearing in eq. (2.1.11) for infinitesimal transformations. The jacobian matrix terms become

$$\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu \left(\omega_a(x) \frac{\delta x^\nu}{\delta \omega_a(x)} \right) \tag{2.1.13}$$

which, once inverted, are

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu \left(\omega_a(x) \frac{\delta x^\nu}{\delta \omega_a(x)} \right). \tag{2.1.14}$$

The jacobian is given by the determinant, which we calculate using the approximation

$$\det(1 + E) \approx 1 + \mathrm{Tr}(E), \tag{2.1.15}$$

giving

$$\left| \frac{\partial x'}{\partial x} \right| \approx 1 + \partial_\mu \left(\omega_a(x) \frac{\delta x^\mu}{\delta \omega_a(x)} \right). \tag{2.1.16}$$

The change of the metric is given by

$$\begin{aligned} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) &= \left[\delta^\alpha_\mu - \partial_\mu \left(\omega_a(x) \frac{\delta x^\alpha}{\delta \omega_a(x)} \right) \right] \left[\delta^\beta_\nu - \partial_\nu \left(\omega_a(x) \frac{\delta x^\beta}{\delta \omega_a(x)} \right) \right] g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) - \delta^\alpha_\mu \partial_\nu \left(\omega_a(x) \frac{\delta x^\beta}{\delta \omega_a(x)} \right) g_{\alpha\beta}(x) - \delta^\beta_\nu \partial_\mu \left(\omega_a(x) \frac{\delta x^\alpha}{\delta \omega_a(x)} \right) g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) - \partial_\nu \left(\omega_a(x) \frac{\delta x^\beta}{\delta \omega_a(x)} \right) g_{\mu\beta}(x) - \partial_\mu \left(\omega_a(x) \frac{\delta x^\alpha}{\delta \omega_a(x)} \right) g_{\alpha\nu}(x) \\ &= g_{\mu\nu}(x) - \partial_\nu \left(\omega_a(x) \frac{\delta x_\mu}{\delta \omega_a(x)} \right) - \partial_\mu \left(\omega_a(x) \frac{\delta x_\nu}{\delta \omega_a(x)} \right) \\ &= g_{\mu\nu}(x) - \partial_\nu \epsilon_\mu(x) - \partial_\mu \epsilon_\nu(x), \end{aligned} \tag{2.1.17}$$

where we have written $\epsilon_\mu(x) = \omega_a(x)\frac{\delta x_\mu}{\delta\omega_a(x)}$ for the sake of clarity. Plugging all of this back in the transformed action and not noting the dependence on x to lighten the notation, we get

$$\begin{aligned}
S' &= \int d^dx\,(1+\partial_\mu\epsilon^\mu)\,\mathcal{L}\left(\Phi+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a},\left(\delta^\nu_\mu-\partial_\mu\epsilon^\nu\right)\partial_\nu\left(\Phi+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right),g_{\mu\nu}-\partial_\nu\epsilon_\mu-\partial_\mu\epsilon_\nu\right)\\
&= \int d^dx\,(1+\partial_\mu\epsilon^\mu)\,\mathcal{L}(\Phi,\partial_\mu\Phi,g_{\mu\nu})\\
&\quad+\left[\frac{\partial\mathcal{L}}{\partial\Phi}\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\left(\partial_\mu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)-\partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\Phi\right)-\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}(\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu)\right]\\
&= S\\
&\quad+\int d^dx\left[\partial_\mu\delta^\mu_\nu\epsilon^\nu\mathcal{L}+\frac{\partial\mathcal{L}}{\partial\Phi}\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\left(\partial_\mu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)-\partial_\mu\epsilon^\nu\partial_\nu\Phi\right)-\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}(\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu)\right].
\end{aligned}\tag{2.1.18}$$

Disregarding expansion in the metric, the second term of the last equality is the change in the action, $\delta S = S' - S$, which should be 0 under a symmetry transformation, since it leaves the action invariant. The last part of the RHS should also be 0 for the same reason. Rewriting the cumbersome expression in (2.1.18) in terms of the derivatives of ω_a gives

$$\delta S = -\int d^dx(\partial_\mu\omega_a)j^\mu_a,\tag{2.1.19}$$

where

$$j^\mu_a = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi-\delta^\mu_\nu\mathcal{L}\right]\frac{\delta x^\nu}{\delta\omega_a}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\mathcal{F}}{\delta\omega_a}\tag{2.1.20}$$

is the conserved current associated with the infinitesimal transformation (2.1.12). Integrating eq. (2.1.19) by parts gives (assuming there is no current at the boundary)

$$\delta S = \int d^dx\omega_a\partial_\mu j^\mu_a.\tag{2.1.21}$$

Since $\delta S = 0$ for any parameter $\omega_a(x)$, we find that the current should be conserved

$$\partial_\mu j^\mu_a = 0\tag{2.1.22}$$

Since the current is ambiguous, it is common to define call the real conserved quantity the charge Q_a, given by

$$Q_a = \int d^{d-1}x j^0_a,\tag{2.1.23}$$

where $d^{d-1}x$ is the purely spatial integration measure, and j^0_a the time-component of the current. This charge is indeed conserved, since its time-derivative is zero

$$\begin{aligned}
\frac{dQ_a}{dt} &= \int d^{d-1}x\frac{\partial}{\partial t}j^0_a\\
&= -\int d^{d-1}x\partial_i j^i_a = 0.
\end{aligned}\tag{2.1.24}$$

We used Gauss's law in the last equality, assuming that there is no flow of current that crosses the boundary at infinity. Note that the Latin alphabet indices represent the spatial dimensions only, while the zero-th index is usually time.

Let us come back to the claim that the current is ambiguously defined. As can be seen from eq. (2.1.22), the current can be freely rescaled by a constant or modified by the addition of the divergence of an antisymmetric tensor

$$j^{'\mu}_a \longrightarrow j^\mu_a + \partial_\nu B^{\nu\mu}_a \quad , \quad \text{with} B^{\nu\mu}_a = -B^{\mu\nu}_a.\tag{2.1.25}$$

It is important to remember that Noether's theorem does not say much about the symmetries of the quantum system, although it puts constraints on the correlation functions we will introduce in the next section. In certain settings, it can even happen that the symmetry is broken at the quantum level, making it an anomaly. Before moving on to Quantum Field Theory, we take a look at the quantity we put aside earlier in our derivation of Noether's theorem, namely the lagrangian's first-order expansion in its metric dependence

$$\delta S = -\int d^d x \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} (\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu) \stackrel{!}{=} 0. \tag{2.1.26}$$

Using integration by part, we get

$$\delta S = -\int d^d x - (\epsilon_\nu \partial_\mu + \epsilon_\mu \partial_\nu) \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = -\int d^d x - 2\epsilon_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \right) \stackrel{!}{=} 0, \tag{2.1.27}$$

where we assume symmetry under exchange of indices in the penultimate step. Since the last equality must be true for any choice of ϵ, we find that

$$\partial_\nu T^{\mu\nu} = 0, \text{with } T^{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = -2\frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.1.28}$$

This quantity is called the energy-momentum tensor, which in this case is by definition symmetric. Note that it generally isn't, but can be made symmetric through the addition of the divergence of another tensor since it is a conserved current. In particular, this is possible in theories with rotation or Lorentz invariance. The resulting tensor is generally called the Belinfante energy-momentum tensor. See [5] for more details (maybe footnote?).

There's still an issue we have swept under the rug in the discussion where we defined the lagrangian density (2.1.8). In the action integrand, $d^d x \mathcal{L}$ is a scalar, but neither of the two components is. Under a coordinate transformation, $d^d x$ picks up a jacobian [6]. This means that $\mathcal{L}$ is not a scalar either. To go back to a formulation of the action that is not explicitly dependent on coordinates, we use the fact that the jacobian is equal to the square root of the determinant of the metric tensor

$$|J| = \sqrt{\det(g_{\mu\nu})} \equiv \sqrt{g}, \tag{2.1.29}$$

and therefore we can write the action

$$S = \int d^d x \sqrt{g} \mathcal{L}, \tag{2.1.30}$$

because $d^d x \sqrt{g}$ is constant regardless of the coordinate system in use. Note that this factor is equal to 1 for both the Euclidian and Minkowski metrics. In the Minkowski case, we take the absolute value of the determinant to avoid an imaginary action.

2.2 Lie Groups and Algebras

When introducing symmetries earlier, we omitted the fact that they are actually described by groups. Just as there are countable and uncountable sets, there are finite and continuous groups, that respectively realize discrete and continuous symmetries. Note that only the latter ones are the subject of Noether's theorem, although the former can lead to restrictions on the system such as selection rules. Continuous groups have an uncountable infinity of elements, and are generally Lie groups (although there are some exceptions that are not important to

us). A Lie group G is a group where the group elements $g = g(\vec{\theta})$ depend on n real parameters collected in $\vec{\theta} = (\theta_1, ..., \theta_n)$. We take the parameters to be real, because we can separate a complex number into its imaginary and real parts, and make it two parameters. The $\vec{\theta_k}$ are defined such that $g(\vec{0}) = e$, the identity element of the group. The group axioms for continuous groups are formulated as follows

- **Closure**. We require that $g(\vec{\theta})g(\vec{\phi}) = g(\vec{\xi})$, meaning that there must exist a mapping of the parameters $\vec{f}(\vec{\theta}, \vec{\phi}) = \vec{\xi}$, such that $g(\vec{\xi}) \in G$.
- **Associativity**: $\vec{f}$ must also satisfy $\vec{f}(\vec{\theta}, \vec{f}(\vec{\phi}, \vec{\xi})) = \vec{f}(\vec{f}(\vec{\theta}, \vec{\phi}), \vec{\xi})$
- **Identity**: Since we chose to take the identity element to be $g(\vec{0})$, we need to have $\vec{f}(\vec{0}, \vec{\theta}) = \vec{f}(\vec{\theta}, \vec{0})$
- **Inverse**: The inverse of $g(\vec{\theta})$ is given by some parameters $\vec{\theta'}$ such that $g(\vec{\theta'}) = g^{-1}(\vec{\theta})$. Then we need to have $\vec{f}(\vec{\theta}, \vec{\theta'}) = \vec{0}$

If the parameters are all independent and if $\vec{f}$ is a smooth function, then G is a Lie group of dimension n.
Now, elements of a group are technically abstract objects that have certain properties when acting on other objects, such as states in quantum mechanics. However, for us to make computations with those elements, we need a way to apply them to the states. To evaluate $g(\vec{\theta})|\psi\rangle$, we need to "translate" the group element to an operator in the Hilbert space $|\psi\rangle$ lives in. To do this, we use representations. Formally, a representation is a homomorphism between G and a set of what we would want the elements to be, for example invertible matrices. In this case a representation is a mapping

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(n, \mathbb{C}) \\ g(\vec{\theta}) &\longmapsto D(g(\vec{\theta})), \end{aligned} \tag{2.2.1}$$

where $D(g(\vec{\theta})) \in \mathrm{GL}(n, \mathbb{C})$ the group of invertible, complex $n \times n$ matrices. This representation assigns a matrix to every element of the group. The mapping D needs to preserve the multiplication law of the group

$$D(g(\vec{\theta}))D(g(\vec{\phi})) = D(g(\vec{\theta})g(\vec{\phi})) = D(g(\vec{\xi})). \tag{2.2.2}$$

Representation theory is a very interesting topic of much importance in physics, but we will stop there since it doesn't really concern this project. We needed to introduce representations to be able to introduce the generators of a Lie group, that allow to deduce the important properties of the group by looking at a finite number of operators.
Consider a representation $D(g(\vec{\theta}))$, that we'll denote $D(\vec{\theta})$ for simplicity. Then $D(\vec{0}) = \mathbb{1}$, the unit matrix that represents the identity element. Since we assumed that the mapping $\vec{f}$ was smooth, we can move to an element infinitesimally close to the identity, with $\theta_k \ll 1$ and Taylor expand the representation to first order:

$$D(\vec{\theta}) = \mathbb{1} + \theta_k \left.\frac{\partial}{\partial \theta_k} D(\vec{\theta})\right|_{\vec{\theta}=0} + ... \tag{2.2.3}$$

Recall that we are using Einstein's summation convention, and that therefore all θ_k's appear. We can rewrite the previous expression as

$$D(\vec{\theta}) = \mathbb{1} - i\theta_k X_k + ..., \tag{2.2.4}$$

where we have introduced the infinitesimal generators X_k

$$X_k = i \left. \frac{\partial}{\partial \theta_k} D(\vec{\theta}) \right|_{\vec{\theta}=0}. \tag{2.2.5}$$

The generators act on the same vector space as the matrices D, and are themselves also matrices (or operators if we take a different representation).
We now have a representation of group elements infinitesimally close to the identity given by $\mathbb{1} - i\theta_k X_k$. For larger values of θ_k, we can raise an infinitesimal element to a large power to generate a finite element with finite parameter θ_k. Taking the limit, we have

$$D(\theta_k) = \lim_{n\to\infty} \left(\mathbb{1} - \frac{i\theta_k X_k}{n} \right)^n = \exp(-i\theta_k X_k). \tag{2.2.6}$$

Now, this means that the group multiplication law (2.2.2) can be rewritten

$$e^{-i\vec{\theta}\cdot\vec{X}} e^{-i\vec{\phi}\cdot\vec{X}} = e^{-i\vec{\xi}\cdot\vec{X}} \tag{2.2.7}$$

where in general

$$e^{-i\vec{\xi}\cdot\vec{X}} \neq e^{-i(\vec{\theta}+\vec{\phi})\cdot\vec{X}} \tag{2.2.8}$$

because of the Baker-Campbell-Hausdorff formula (3.2.5). We will see this formula in a little more detail when we introduce path integrals. If the commutator $[X_j, X_k] = 0$, then the extra terms vanish and we have $\vec{\xi} = \vec{\theta} + \vec{\phi}$. Putting the group multiplication law together with the formula, we can have $e^{-i\vec{\xi}\cdot\vec{X}}$ on one side and an expansion in terms of $[X_j, X_k]$ (at first order in θ_j, ϕ_k). This means that it is possible to express the commutator as a linear combination of the generators :

$$[X_j, X_k] = i c^l_{jk} X_l. \tag{2.2.9}$$

This equation defines the Lie algebra of the group. Note that the generators of an algebra that are also generators of an infinitesimal transformation are in fact conserved charges. We can therefore interchangeably refer to them as one or the other. A proof of this statement can be found in [5]. Now, we can move on to discussing the conformal group.

3 Quantum Field Theory

In the previous section, we briefly mentioned Lorentz invariance and special relativity, as they are closely related to the most well-explored area of classical field theory: the theory of electromagnetism. Historically, Einstein's theory of special relativity and later of general relativity was only one of the two major discoveries of physics of the early twentieth century. The other, of course, is quantum mechanics, which describes the behaviour of matter and light with effects typically observed at subatomic scales.
Having both special relativity and quantum mechanics on hand, one might ask if it is possible to combine them and give a quantum description of a relativistic particle. It turns out that it is, and that the most widely used framework to do so is called quantum field theory (QFT), which we'll review in this section. It is an extremely vast subject and we'll focus on introducing the aspects of importance for our needs.

3.1 The Klein-Gordon Equation

Attempting to write the the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\phi,t\right\rangle = H\left|\phi,t\right\rangle \tag{3.1.1}$$

with the hamiltonian associated with a free particle moving at relativistic speeds

$$H = \sqrt{\mathbf{P}^2c^2 + m^2c^4} \tag{3.1.2}$$

yields, in the position basis [7],

$$i\hbar\frac{\partial}{\partial t}\phi(\mathbf{x},t) = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\phi(\mathbf{x},t). \tag{3.1.3}$$

This unfortunately doesn't look very promising: the difference in treatment of the two "variables" (position in 4 dimensional spacetime really is 3 variables hidden under $\mathbf{x}$) isn't what one should find in a relativistic theory. Since both act the same way under Lorentz transformations, we expect some kind of symmetry between them. We can do a bit of damage control and square the the operators on both sides, yielding

$$-\hbar^2\frac{\partial^2}{\partial t^2}\phi(\mathbf{x},t) = \left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\phi(\mathbf{x},t). \tag{3.1.4}$$

This is, albeit in a slightly different form than eq. (2.1.9), the Klein-Gordon equation. A bit of notational work (see [7]), a quick introduction to the covariant formulation of special relativity and the choice to set $c = \hbar = 1$ gives us the form we introduced in the previous section. It is important to note that the Klein-Gordon equation is relativistic, meaning that it is the same in every inertial frame.
Unfortunately, another issue arises as soon as we're done dealing with relativity: solutions of the Klein-Gordon equation cannot be interpreted as as single-particle probability amplitude. A way to see this is to consider the general form of a local conservation law

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}, \tag{3.1.5}$$

where we would interpret ρ as a probability density and $\mathbf{J}$ its associated current in the context of quantum mechanics. In the case of the Schrödinger equation, $\rho = \phi^\star\phi = |\phi|^2$, where $\phi^\star$ is

the complex conjugate of ϕ [8]. Being equal to a divergence, we have that the spatial integral of ρ is a conserved quantity, here the probability amplitude. For the Klein-Gordon equation, however, we find $\rho = \phi^\star \partial_t \phi - \phi \partial_t \phi^\star$. Since this density is not always positive, it cannot be treated as a probability (as postulated by the Kolmogorov axioms).
There is no real way to solve this issue while still treating ϕ as a single-particle wave function. In fact, there exists no quantum description of a single relativistic particle. This is due to the fact that mass is not necessarily conserved in special relativity, as opposed to energy (at least in the same inertial frame of reference). From the quantum point of view, we can see another issue with our approach: we're attempting to mix time and space, even though they do not have the same status in quantum mechanics. As a matter of fact, the position $\mathbf{x}$ is (the eigenvalue of) an observable represented by a hermitian operator while time is introduced in the theory through a simple parameter t that is nothing more than that.
The way of patching this up that leads us to quantum field theory is to demote position from its status as an operator, and treat it simply as another parameter. But we still need operators, right? Of course, but of a different kind. We will, as the previous chapter may have hinted, treat the ϕ appearing in (3.1.4) as a an operator field. What this means is that to each spacetime point $(\mathbf{x}, t)$ we will assign an operator. We will now write φ instead of ϕ. Note that, since $\varphi(\mathbf{x}\ , t)$ is an operator (field), its time-dependence can be written in the Heisenberg picture as

$$\varphi(\mathbf{x}, t) = e^{iHt/\hbar} \varphi(\mathbf{x}, 0) e^{-iHt/\hbar}, \tag{3.1.6}$$

where H is the hamiltonian of the system. Going back to the Klein-Gordon lagrangian in eq. (2.1.8), this means we have to take our scalar field (which assigns a scalar to each spacetime point) and turn it into an operator field. This is called canonical quantization.
To familiarize ourselves with QFT and this procedure, let us first rewrite quantum mechanics with a fixed number n of particles as a quantum field theory. We start with the Schrödinger equation in the position basis for n particles of same mass m, under the influence of an external potential U($\mathbf{x}$) and an interparticle potential $V(\mathbf{x}_j - \mathbf{x}_k)$

$$i\hbar \frac{\partial}{\partial t} \psi = \left[\sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) + \sum_{j=1}^{n} \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \right], \tag{3.1.7}$$

where $\psi = \psi(\mathbf{x}_1, ..., \mathbf{x}_n, t)$ is the wave function in the position basis. It can be shown that this hamiltonian can be rewritten in terms of creation $a(\mathbf{x})$ and annihilation $a^\dagger(\mathbf{x})$ operators in the following way (see [9], Chapter 2)

$$H = \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) + \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}). \tag{3.1.8}$$

Note that here both a and $a^\dagger$ are quantum fields, and therefore H is the hamiltonian operator of the quantum field theory. They have the commutation relations

$$\begin{aligned} [a(\mathbf{x}), a(\mathbf{x}')] &= 0 \\ [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] &= 0 \\ [a(\mathbf{x}), a^\dagger(\mathbf{x}')] &= \delta^3(\mathbf{x} - \mathbf{x}'), \end{aligned} \tag{3.1.9}$$

where $\delta^3(\mathbf{x})$ is the three-dimensional Dirac delta function, which can be expressed by

$$\delta^3(\mathbf{x} - \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x} - \mathbf{x}')}. \tag{3.1.10}$$

$a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ are thus akin to harmonic oscillator creation and annihilation operators. However, we will see that instead of moving up and down energy states, they "add" or "remove" a

particle to the state. Consider a general time-dependent state

$$|\psi, t\rangle = \int d^3x_1 ... d^3x_n \psi(\mathbf{x}_1, ..., \mathbf{x}_n) a^\dagger(\mathbf{x}_1) ... a^\dagger(\mathbf{x}_n)|0\rangle, \tag{3.1.11}$$

where $|0\rangle$ is the vacuum state, the state annihilated by all the a's

$$a(\mathbf{x}) = 0. \tag{3.1.12}$$

We can show that, using our new hamiltonian defined in eq. (3.1.8), the equations (3.1.1) and (3.1.7) are equivalent, i.e. if ψ satisfies one then it verifies the other and vice-versa.
This means that we can interpet the vacuum state $|0\rangle$ as the state with "zero particles", $a^\dagger(\mathbf{x}_1)|0\rangle$ as the state with one particle located at $\mathbf{x}_1$, $a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)|0\rangle$ as the state with two particles, one located at $\mathbf{x}_1$ and the other at $\mathbf{x}_2$, etcaetera. Using these ladder operators, we can define the local density operator $\rho(\mathbf{x})$ which measures the particle density at $\mathbf{x}$

$$\rho(\mathbf{x}) = a^\dagger(\mathbf{x})a(\mathbf{x}). \tag{3.1.13}$$

Integrating this over all possible positions naturally yields the number operator which counts the total number of particles

$$N = \int d^3x a^\dagger(\mathbf{x})a(\mathbf{x}). \tag{3.1.14}$$

It commutes with the Hamiltonian, which means that the total number of particles is fixed in this theory. In general, however, a slightly different hamiltonian could have interacting or different terms that do not commute with N, and would imply a time-dependent number of particles. This should be expected. There are multiple phenomena where particles are "created" or "destroyed", such as when an atom emits or absorbs a photon. It is therefore a good thing to be able to describe such systems with our formalism.
During all the above discussion, we have not mentioned the kind of particles we were dealing with. Since it's a quantum theory, they are indistinguishable from one another, but they could be either bosons or fermions (at least in three+-dimensional space, anyons will not be discussed in this project). The Klein-Gordon equation deals with scalar fields, and is therefore related only to spinless bosons. This is due to a result known as the spin-statistics theorem, which we will not cover because it has little importance for the rest of our work. Higher spin bosons and fermions are described by different equations and different types of fields. In particular, there are two types of fermions, regular (Dirac) ones and Majorana fermions. The SYK model deals with the latter ones, whose main feature is their hermicity, i.e. the operators verify $\chi_i = \chi_i^\dagger$. In other words, they are their own antiparticle, which generally means that they are neutrally charged fermions. In addition, they follow the anticommutation relation $\{\chi_i, \chi_j\} = 2\delta_{ij}$.
Let us forget about Majorana fermions for a while, and come back to quantum field theory. For the sake of brevity, and because it is not necessary to understand what follows, we will not delve into the details of how one goes from the Klein-Gordon "classical" lagrangian to the quantized Hamiltonian. It is given by

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega} \omega a^\dagger(\mathbf{k})a(\mathbf{k}), \tag{3.1.15}$$

where $a^\dagger(\mathbf{k}) = -i \int d^3x e^{ikx} \overset{\leftrightarrow}{\partial_o} \varphi(x)$ and $\omega = (\mathbf{k}^2 + m^2)^{1/2}$. Notice that we are working in momentum space as opposed to position space earlier on, which means that the creation operator creates a particle with a certain momentum instead of at a certain position. The commutation relations for the momentum-space creation and annihilation operators are similar to those in eq. (3.1.9), except that we have

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega \delta^3(\mathbf{k} - \mathbf{k}'). \tag{3.1.16}$$

Now, we have to ask ourselves how to extract meaningful and testable data from this formalism that we have just created. In the same way as in quantum mechanics, we're interested in the probability amplitude of going from an initial state $|i\rangle$ to a final state $|f\rangle$. In the free theory (i.e. free relativistic particles, the Klein-Gordon equation), whatever these states are, they can be created by acting on the vacuum state $|0\rangle$ with a creation operator of the form given earlier. We assume that it will the same in an interacting theory, although the operators will become time-dependent. Now, $\langle f|i\rangle$ is the scattering amplitude, the probability of a scattering reaction happening. We'll skip over some not very enlightening computations and simply introduce the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula for a real scalar field

$$\begin{aligned}\langle f|i\rangle = i^{n+n'} \int & d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)...d^4x_n e^{ik_nx_n}(-\partial_n^2+m^2) \\ & d^4x'_1 e^{-ik'_1x'_1}(-\partial_{1'}^2+m^2)...d^4x'_n e^{-ik'_nx'_n}(-\partial_{n'}^2+m^2) \\ & \times \langle 0|\mathrm{T}\varphi(x_1)...\ \varphi(x'_1)...|0\rangle.\end{aligned} \tag{3.1.17}$$

There exists other reduction formulas of the same type, and they are all related to the last term of the equation, namely

$$\langle 0|\mathrm{T}\varphi(x_1)...\ \varphi(x'_1)...|0\rangle. \tag{3.1.18}$$

Here T is the time-ordering operator, which sorts the following operators in chronological order

$$\mathrm{T}(\varphi(x_1)...\varphi(x_n)) = \varphi(x_1)...\varphi(x_n) \quad \text{if} \quad x_1^0 > x_2^0 > ... > x_n^0. \tag{3.1.19}$$

The term in eq. (3.1.18) is called a correlation function, and we will see how to compute it shortly. Before that, let us mention that the LSZ formula (3.1.17) is valid only if the field $\varphi(x)$ obeys

$$\langle 0|\varphi(x)|0\rangle \quad \text{and} \quad \langle k|\varphi(x)|0\rangle = e^{-ikx}, \tag{3.1.20}$$

where $|k\rangle$ is a single-particle state of momentum k. The Lagrangian therefore needs to be renormalized in general, but we will not concern ourselves with that.
Let us now move on to developping the tools needed to evaluate eq. (3.1.18).

3.2 Path Integrals

Multiple different but equivalent formalisms can be used to treat quantum mechanics. The most common and the one usually taught in the first courses on the subject is known as canonical quantization, where classical quantities or observables such as energy, position, etc become operators acting on a Hilbert space where the state of the system, represented by vectors, lives. A different and more intuitive formalism is that of path integration, which was first developed by Feynman. To get used to it, let us begin with a simple system of a single particle first, then look at how it works for fields.

3.2.1 One Degree of Freedom

The hamiltonian of a single nonrelativistic quantum particle in one dimension is

$$H(P,Q) = \frac{P^2}{2m} + V(Q), \tag{3.2.1}$$

with $[Q,P] = i$. Notice $\hbar$ does not appear because we work in natural units. Suppose we're interested in the probability amplitude of a particle going from a point (q',t') in space-time to

another point (q'', t''). We know that probability amplitudes are evaluated at the same time, and therefore we wish to calculate

$$\langle q''|e^{-iH(t''-t')}|q'\rangle. \tag{3.2.2}$$

Here, $|q'\rangle$ and $|q''\rangle$ are both eingenstates of Q. In the Heisenberg picture, the state of the system is time-independent and the operators aren't anymore. In that case, we have $Q(t) = e^{iHt}Q(0)e^{-iHt}$. We then have instantaneous eigenstates $Q(t)|q,t\rangle = q(t)|q,t\rangle$. However, a problem may arise : we don't know the value when $Q(t)$ is applied to a state $|q;\tilde{t}\rangle$

$$Q(t)|q,\tilde{t}\rangle = \begin{cases} \neq q(t)|q;\tilde{t}\rangle \\ = q(\tilde{t})|q;\tilde{t}\rangle \end{cases}. \tag{3.2.3}$$

We would therefore rather use the Schrödinger picture. Let us then come back to eq. (3.2.2). Plugging in the explicit expression of H (3.2.1), we get

$$\langle q''|e^{-i\left[\frac{P^2}{2m}+V(Q)\right](t''-t')}|q'\rangle, \tag{3.2.4}$$

which is much less nice than it looks, for two reasons. The first one is that, although $e^{-iV(Q)(t''-t')}|q'\rangle = e^{-iV(q')(t''-t')}|q'\rangle$ is easy to compute, this is not the case for $e^{-i\frac{P^2}{2m}(t''-t')}|q'\rangle$. In addition, since P and Q are operators that do not commute, the exponential of their sum follows the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\dots}. \tag{3.2.5}$$

The trick to work around this annoying expansion is to divide the time interval $t''-t'$ in $N+1$ small steps, with $\delta t := \frac{t''-t'}{N+1}$. That way, we will be able to separate the two terms of the hamiltonian since the remaining terms will be of order δt^2. We can write

$$\langle q'';t''|q';t'\rangle = \langle q''|\underbrace{e^{-iH\delta t}\dots e^{-iH\delta t}}_{\text{N+1 factors}}|q''\rangle. \tag{3.2.6}$$

Then, inserting the completeness relation $\mathbb{1} = \int dq_j|q_j\rangle\langle q_j|$ in between each exponential yields

$$\begin{aligned}
\langle q'';t''|q';t'\rangle &= \int \prod_{j=1}^{N} dq_j \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle\dots\langle q_1|e^{-iH\delta t}|q_1\rangle \\
&\approx \int \prod_{j=1}^{N} dq_j \langle q''|e^{-i\frac{P^2}{2m}\delta t}e^{-iV(Q)\delta t}|q_N\rangle\dots\langle q_1|e^{-i\frac{P^2}{2m}\delta t}e^{-iV(Q)\delta t}|q'\rangle \\
&= \int \prod_{j=1}^{N} dq_j \langle q''|e^{-i\frac{P^2}{2m}\delta t}|q_N\rangle e^{-iV(q_N)\delta t}\dots.
\end{aligned} \tag{3.2.7}$$

We once again insert a completeness relation, but on the p's $\mathbb{1} = \int dp_j|p_j\rangle\langle p_j|$ this time, leading to

$$\langle q'';t''|q';t'\rangle = \int \prod_{j=1}^{N} dq_j dp_j dp_0 \langle q''|p_N\rangle e^{-i\frac{p_N^2}{2m}\delta t}e^{-iV(q_N)\delta t}\langle p_N|q_N\rangle\dots. \tag{3.2.8}$$

The q_j's are always paired with a p_{j-1}, hence the dp_o appearing. We know that $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\exp(ipq)$, i.e. $\langle q|p\rangle$ is a plane wave, and therefore that $\langle p|q\rangle = \frac{1}{\sqrt{2\pi}}\exp(-ipq)$ is its adjoint. Plugging this back in eq. (3.2.8), we end up with

$$\langle q'';t''|q';t'\rangle = \int \underbrace{\prod_{k=1}^{N} dq_k}_{\mathcal{D}q} \underbrace{\prod_{j=0}^{N} dp_j}_{\mathcal{D}p} \exp\left[\sum_{l=0}^{N} ip_l \overbrace{(q_{l+1}-q_l)}^{\approx \dot{q}\delta t} - iH(p_l,q_l)\delta t\right], \tag{3.2.9}$$

where we relabeled $q_0 := q', q_{N+1} = q''$. Taking the limit $\delta t \longrightarrow 0$, the sum formally becomes an integral from t' to t'', and the amplitude takes the final form

$$\begin{aligned}\langle q''; t''|q'; t'\rangle &= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt(p(t)\dot{q}(t) - H(p(t), q(t)))\right] \\ &= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt L(q(t), \dot{q}(t))\right].\end{aligned} \tag{3.2.10}$$

To go to the last line, we noticed that the terms under the time integral are the Legendre transformation of the hamiltonian, which means they actually are the lagrangian of the system. In turn, it follows that the exponential is nothing more that e^{iS}, with S the action. The idea behind this formula is the following: to compute the probability amplitude, we "try" all possible paths between the two points in space-time, and weight them with the exponential of the associated action, which will oscillate since it's a complex exponential. Most of the possible trajectories are irregular and would not usually be observed, but they contribute almost nothing to the total amplitude because the oscillating exponential tends to make them cancel each other. Now, as we've seen earlier, what we want to compute are correlation functions. In this quantum mechanical formalism with one degree of freedom, such functions would be of the form

$$\langle q''; t''|Q(t_1)|q'; t'\rangle = \int \mathcal{D}q q(t_1) e^{iS}. \tag{3.2.11}$$

However, we quickly see an issue arise if we start to include more than one operator $Q(t_i)$. Indeed, what happens if we try to compute $\langle q''; t''|Q(t_1)Q(t_2)|q'; t'\rangle$, but $t_1 < t_2$? The answer is that we should swap them, since the trajectory can't go forward in time to t_2, then back to t_1, and then finally to t''. Therefore we should introduce time-ordering. This is a good thing, because it appears in the LSZ formula (3.1.17). In general, it allows us to write

$$\begin{aligned}\langle q''; t''|Q(t_1)...Q(t_n)P(\tilde{t}_1)...P(\tilde{t}_n)|q'; t'\rangle &= \int \mathcal{D}q\mathcal{D}p q(t_1)...p(\tilde{t}_1)...e^{i\int_{t'}^{t''} dt(p\dot{q}-H)} \\ &= (\star),\end{aligned} \tag{3.2.12}$$

where we added P operators acting at different times. To go further however, we need to bring back a tool we introduced when discussing Noether's theorem: the functional derivative. Its most interesting feature in our case it that

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2), \tag{3.2.13}$$

with $\delta(t)$ the Dirac delta. Functional derivatives function like normal derivatives, except they act on the functions themselves. Now, let's modify our free theory by adding arbitrary functions $h(t)$ and $f(t)$ acting on the particle in the following way

$$H(p, q) \to H(p, q) - f(t)q(t) - h(t)p(t), \tag{3.2.14}$$

where $f(t)$ and $h(t)$ are functions that depend on the theory. We then write

$$\langle q''; t''|q'; t'\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q \exp\left[i\int_{t'}^{t''} dt\,(p\dot{q} - H + fq + hp)\right]. \tag{3.2.15}$$

This function is called the generating functional, a name that will become clear in the next few lines. Indeed, using the functional derivative to bring down factors of $q(t_i)$ and $p(t_j)$, we get

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\langle q''; t''|q'; t'\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q q(t_1) \exp\left[i\int_{t'}^{t''} dt\,(p\dot{q} - H + fq + hp)\right], \tag{3.2.16}$$

for a single operator, and we are able to rewrite eq. (3.2.12) as

$$(\star) = \frac{1}{i}\frac{\delta}{\delta f(t_1)}...\frac{1}{i}\frac{\delta}{\delta f(t_n)}\frac{1}{i}\frac{\delta}{\delta h(\tilde{t}_1)}...\frac{1}{i}\frac{\delta}{\delta h(\tilde{t}_n)}\langle q'';t''|q';t'\rangle_{f,h}\Big|_{f=h=0}. \tag{3.2.17}$$

The reason we put both f and h to zero after all calculations have been done is because we want to recover the original Hamiltonian. Now, let's take a look at how this would work for the ground state as both initial and final state after an infinite time, before moving on to path integration for fields. With the help of the identity trick we used when developping path integrals, we have

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty}\lim_{t''\to+\infty}\int dq''dq' \underbrace{\langle 0|q'';t''\rangle}_{\psi_0^*(q'')}\langle q'';t''|q';t'\rangle_{f,h}\underbrace{\langle q';t'|0\rangle}_{=\psi_0(q')}. \tag{3.2.18}$$

We wish to simplify this formula, as it is not easy to use. To do so, we let $|n\rangle$ denote an eigenstate of H with eigenvalue E_n. We assume that $E_0 = 0$, and if it is not the case we shift the hamiltonian so that it is. Then we can write

$$\begin{aligned}|q';t'\rangle &= e^{iHt'}|q'\rangle \\ &= \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle \\ &= \sum_{n=0}^{\infty}\psi_n^*(q')e^{iE_nt'}|n\rangle.\end{aligned} \tag{3.2.19}$$

Now, taking the prescription $H \to (1-i\epsilon)H$ with $\epsilon > 0$ fixed and the limit $t' \to -\infty$ yields

$$\begin{aligned}\lim_{t'\to-\infty}\sum_{n=0}^{\infty}\psi_n^*(q')e^{i(1-i\epsilon)Ht'}|n\rangle &= \lim_{t'\to-\infty}\sum_{n=0}^{\infty}\psi_n^*(q')\underbrace{e^{i(1-i\epsilon)E_nt'}}_{\to 0\forall n>0}|n\rangle \\ &= \psi_0^*(q')|0\rangle.\end{aligned} \tag{3.2.20}$$

We then multiply eq. (3.2.18) by an arbitrary $\chi(q')$ such that $\langle 0|\chi\rangle \neq 0$ and integrate over q' which gives a constant multiplying $|0\rangle$. This constant can be taken care of by the normalization of the path integral, and therefore is ignored. A similar approach yields the same result for $\langle q'';t''| = \langle q''|e^{-iHt}$ and $t \to +\infty$.
This discussion allows us to be rather free with the boundary conditions we set at the beginning and the end of the path, as long as they are reasonable. Therefore we have

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{+\infty}dt(p\dot{q} - (1-i\epsilon)H + fq + hp)\right]. \tag{3.2.21}$$

For future reference, let's also look at what happens when $H = H_0 + H_1$, where H_0 is diagonalisable and H_1 is a perturbation. In other words, H_0 leads to a "doable" path integral and H_1 is "small". Then we have

$$\begin{aligned}\langle 0|0\rangle_{f,h} &= \int \mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{+\infty}dt(p\dot{q} - (1-i\epsilon)(H_0+H_1) + fq + hp)\right] \\ &= \int \mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{+\infty}dt(p\dot{q} - (1-i\epsilon)H_0 - H_1 + fq + hp)\right].\end{aligned} \tag{3.2.22}$$

The ϵ prescription disappeared against H_1, because we consider both to be small and therefore their product to vanish. We can rewrite this expression as

$$\begin{aligned}\langle 0|0\rangle_{f,h} =& \exp\left[-i\int_{-\infty}^{+\infty}dtH_1\left(\frac{1}{i}\frac{\delta}{\delta h(t)},\frac{1}{i}\frac{\delta}{\delta f(t)}\right)\right] \\ &\times\int\mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{+\infty}dt(p\dot{q} - (1-i\epsilon)H_0 + fq + hp)\right].\end{aligned} \tag{3.2.23}$$

To compute the amplitude, we then expand the exponential in powers of H_1. In practice, stealing a joke from my QFT professor, we only keep a few powers until the accuracy is better than what experimentalists can test, and wait for them to catch up before adding more powers. Now that we have all those tools working for a single degree of freedom, let's move on to our objects of interest: fields.

3.2.2 Quantum Fields

To begin with, let us go back to the hamiltonian density of a free scalar field

$$\mathcal{H}_0 = \frac{1}{2}\left(\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right), \tag{3.2.24}$$

where we "translated" (in the linguistic meaning of translation)

$$\begin{aligned} Q(t) &\longrightarrow \varphi(t,\mathbf{x}) \\ P(t) &\longrightarrow \Pi(t,\mathbf{x}) \\ \omega &\longrightarrow m \end{aligned} \tag{3.2.25}$$

in the harmonic oscillator hamiltonian

$$H = \frac{1}{2}(P^2 + \omega^2 Q^2). \tag{3.2.26}$$

Note that eq. (3.2.24) and eq. (3.2.25) are not on the same footing, since $\mathcal{H}_0$ is a density. Therefore, one cannot directly translate from one to the other and vice-versa.
Anyways, let us look at our generating functional. Since our lagrangian only depends on $\varphi(x)$, we will only need a single source function. However, we have moved up in dimensions and thus need to translate once again $f(t)$ to $J(t,\mathbf{x})$. The generating functional of the free field theory can then be written as a path integral with a measure over φ instead of q and p. It looks like

$$Z_0(J) = \langle 0|0\rangle_J = \int \mathcal{D}\varphi \exp\left[\int d^4x \left(\mathcal{L}_0(\varphi, \partial_\mu\varphi) + J(x)\varphi(x)\right)\right], \tag{3.2.27}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\phi^2 \tag{3.2.28}$$

is the lagrangian density of the free field theory and

$$\mathcal{D}\varphi \propto \prod_x d\varphi(x) \tag{3.2.29}$$

is the functional measure. Here, the product runs over all possible x. The path we're considering is thus a path in the space of field configurations (commonly called Fock space). To compute (3.2.27), we start with the four-dimensional Fourier transforms

$$\begin{aligned} \tilde{\varphi}(k) &= \int d^4x e^{-ikx}\varphi(x) \quad \text{and} \quad \varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\varphi}(k) \\ \tilde{J}(k) &= \int d^4x e^{-ikx}J(x) \quad \text{and} \quad J(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{J}(k), \end{aligned} \tag{3.2.30}$$

with $kx = -k^0t + \mathbf{k}\cdot\mathbf{x}$, and k^0 is an integration variable. Denoting $S_0 = \int d^4x\,[\mathcal{L}_0 + J\varphi]$, we can rewrite (full steps in annex?)

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\tilde{\varphi}(k)(k^2+m^2)\tilde{\varphi}(-k) + \tilde{J}(k)\tilde{\varphi}(-k) + \tilde{J}(-k)\tilde{\varphi}(k)\right], \tag{3.2.31}$$

where $k^2 = \mathbf{k}^2 - (k^0)^2$. To turn this into a slightly more tractable expression, we complete the square by writing

$$\tilde{\chi}(k) := \tilde{\varphi}(k) - \frac{\tilde{J}(k)}{k^2 + m^2}, \tag{3.2.32}$$

which leads us to

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2} - \tilde{\chi}(k)(k^2+m^2)\tilde{\chi}(-k)\right]. \tag{3.2.33}$$

Notice that the since χ is just $\varphi + c$, the path integration measure does not change $\mathcal{D}\varphi = \mathcal{D}\chi$, and therefore

$$Z_0(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\epsilon}\right]\cdot\int \mathcal{D}\chi \exp\left[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{\chi}(k)(k^2+m^2)\tilde{\chi}(-k)\right] \tag{3.2.34}$$

We could take the first part out of the path integral since it doesn't contain any χ. We reintroduced the epsilon prescription for later use, because we don't want a divergence when $k^2 = -m^2$ and complex integration allows us to avoid that. Now, coming back to our expression, we notice a very important feature: the second term is equivalent to $\langle 0|0\rangle_{f=0}$. This expression is the probability of the ground state staying in the ground state in a free theory. Since it has a mass gap, there is no way for the ground state to magically transition to another state, and thus $\langle 0|0\rangle_{f=0} = 1$. We end up with

$$\begin{aligned} Z_0(J) &= \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\epsilon}\right] \\ &= \exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right], \end{aligned} \tag{3.2.35}$$

where we have introduced the Feynman propagator

$$\Delta(x-x') := \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{3.2.36}$$

which is a Green's function for the Klein-Gordon equation, i.e

$$(-\partial_x^2 + m^2)\Delta(x-x') = \delta^4(x-x'). \tag{3.2.37}$$

Now, let's return to our correlation functions and use our newly developed tools to evaluate a few examples. Remember that

$$\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}...\frac{1}{i}\frac{\delta}{\delta J(x_n)}Z_0(J). \tag{3.2.38}$$

Note that we use $\langle 0|0\rangle_J = Z_0(J)$ because both the initial and final states are the ground state in our case. Now let's move on to the examples. We will start with the simple case $\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle$ and then progress to the slightly more enlightening case $\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$.

$$
\begin{aligned}
\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]\Big|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left\{Z_0(J)\frac{1}{2}\frac{\delta}{\delta J(x_2)}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right\}\Big|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left\{Z_0(J)\int d^4x \Delta(x_2-x)J(x)\right\}\Big|_{J=0} \\
&= \frac{1}{i}\left[\underbrace{\frac{\delta Z_0(J)}{\delta J(x_1)}\left\{\int d^4x\Delta(x_2-x)J(x)\right\}\Big|_{J=0}}_{=0 \text{ since we set } J=0} + Z_0(J)\int d^4x\Delta(x_2-x)\underbrace{\frac{\delta J(x)}{\delta J(x_1)}}_{=\delta(x-x_1)}\Big|_{J=0}\right] \\
&= \frac{1}{i}Z_0(J)\Delta(x_2-x_1)\Big|_{J=0} = \frac{1}{i}\Delta(x_2-x_1)
\end{aligned}
\tag{3.2.39}
$$

The result we end up with is the Feynman propagator between two x_1 and x_2. Notice that if we have an odd number of φ's, there is always one J left that will end up making everything vanish. This is not valid in general. This means that we need to pair functional derivatives together in order to get rid of the J's. The next example is evaluated in a similar way, although we will skip the steps to avoid tedious equation writing.

$$
\begin{aligned}
\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle =& \frac{1}{i^2}[\Delta(x_1-x_2)\Delta(x_3-x_4) \\
&+ \Delta(x_1-x_3)\Delta(x_2-x_4) \\
&+ \Delta(x_1-x_4)\Delta(x_2-x_3)].
\end{aligned}
\tag{3.2.40}
$$

Generalising this result to $2n$ operators gives a version of Wick's theorem

$$
\langle 0|\mathrm{T}(\varphi(x_1)...\varphi(x_{2n}))|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}}\Delta(x_{i_1}-x_{i_2})...\Delta(x_{i_{2n-1}}-x_{i_{2n}}). \tag{3.2.41}
$$

The full theorem involves contraction of operators and their normal ordering, see next section for more details.
The propagators that appear in eq. (3.2.40) can be translated into (very simple) Feynman diagrams, as seen in 1.

Figure 1: Feynman diagrams for the free theory

For now, just remembering that a line signifies a propagator from one point to another is enough. Note that this is the only time when we will write the x_i's. We'll see more interesting diagrams very shortly when we look at interacting theories and some slightly complicated ones when we study the SYK model.
Recall the lagrangian of our free theory

$$
\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2, \tag{3.2.42}
$$

and let us now turn on interactions. There are multiple ways to do so while still requiring Lorentz invariance. The easiest one is adding Lorentz scalars. In our case, any simple function of the field φ is a Lorentz scalar, and therefore we can add, for example, a term proportional to φ^3. Other theories are possible and have different properties, but this one makes for relatively easy diagrams that allow us to understand Feynman rules. The lagrangian density reads as

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{6}g\varphi^3, \tag{3.2.43}$$

where g is a coupling constant. However, remember that to use the LSZ formula, we need to enforce $\langle 0|\varphi(x)|0\rangle = 0$ and $\langle k|\varphi(x)|0\rangle = e^{-ikx}$, with $\langle 0|0\rangle = 1$ and $|k\rangle$ has four-momentum k^μ such that $k^2 = k^\mu k_\mu = -m^2$ and $\langle k'|k\rangle = (2\pi)^3 2k^0\delta^3(\mathbf{k}' - \mathbf{k})$. Therefore, we would shift and rescale the parameters to have

$$\mathcal{L} = -\frac{1}{2}Z_\varphi\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{6}g\varphi^3 + Y\varphi, \tag{3.2.44}$$

with Z_i and Y unknown constants ensuring the conditions of the LSZ formula. The parameters m and g are set by respectively requiring m to correspond to the mass of the particle represented by the fields and g to have a specific relationship with a particular cross-section.
To use the tools of perturbation theory developed in eq.(3.2.23), we want to split up our lagrangian in the following way

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \text{where} \begin{cases} \mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \\ \mathcal{L}_1 = \frac{1}{6}Z_g g\varphi^3 \underbrace{-\frac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi}_{\mathcal{L}_{ct}}. \end{cases} \tag{3.2.45}$$

The second term of $\mathcal{L}_1$ is the counterterm lagrangian $\mathcal{L}_{ct}$. According to eq.(3.2.23), the path integral for the full theory can be written

$$Z(J) = \exp\left[i\int d^4x\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]\int\mathcal{D}\varphi\exp\left[i\int d^4x\left(\mathcal{L}_0 + J\varphi\right)\right]. \tag{3.2.46}$$

Notice that when $g = Y = 0$ and $Z_m = Z_\varphi = 1$, we recover the free theory and therefore

$$Z_0(J) := Z(J)\big|_{\substack{y\,=\,Z\,=\,0 \\ Z_\varphi = Z_m = 1}} = \exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x - x')J(x')\right]. \tag{3.2.47}$$

This is expected, and it allows us write

$$Z(J) \propto \exp\left[i\int d^4x\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]Z_0(J), \tag{3.2.48}$$

with $Z(J)$ proportional and not equal to the rhs because the ϵ prescription doesn't give the correct overall normalization [7]. Since we are expecting to recover the free theory when we remove the φ^3 term, we assume some kind of "continuity" and expect that as $g \to 0$, $Y \to 0$ and $Z_i \to 1$, so that $\mathcal{L}_{ct}$ vanishes. To make our lives easier, let us ignore counterterms and write

$$Z_1(J) :\propto \exp\left[\frac{i}{6}Z_g g\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]Z_0(J). \tag{3.2.49}$$

Normalisation is enforced by $Z_1(0) = 1$, which gives the proportionality factor. As mentioned in the previous section, since we cannot directly compute both exponentials, we expand them

respectively in powers of g for the interaction part and J for the free part. We end up with

$$\begin{aligned} Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i}{6} Z_g g \int d^4x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right]^V \\ \times \sum_{P=0}^{\infty} \frac{1}{P!} \left[\frac{i}{2} \int d^4y d^4z J(y) \Delta(y-z) J(z) \right]^P . \end{aligned} \tag{3.2.50}$$

Following a convention used in [7], the letters V and P stand for "vertex" and "propagator". To better understand the diagrammatics that will follow, it is enlightening to try and focus on certain values of V and P. To make our life easier, we'll restrain ourselves to the first few powers of both expansions. Let's take $V = 0, P = 1$ to begin with. This factor of the expansion will simply look like

$$\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta(x_1 - x_2) J(x_2). \tag{3.2.51}$$

We already know that we represent propagators (in fact, the term $\frac{1}{i}\Delta(x_1 - x_2)$ by a line between x_1 and x_2. How should we represent the sources $J(x_i)$? Feynman's answer was by filled circles or "blobs", which correspond to the term $i \int d^4x_1 J(x_1)$. Drawn, this factor corresponds to the diagram in 2

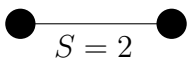


Figure 2: Feynman diagram with $P = 1, V = 0$

Before proceeding to the next set of values, notice that we have introduced the symmetry factor S. This counts how many times we obtain the same diagram by rearranging derivatives or rearranging sources. This factor is best computed by looking at the diagrams and considering what can or cannot be swapped. In addition, we will only concern ourselves with connected diagrams that have sources, due to a result that says that $Z_1(J)$ is the exponential of the sum of these type of diagrams only [7, 10]

$$Z_1(J) = \exp[iW_1(J)], \tag{3.2.52}$$

where $iW_1(J)$ is the sum we mentioned above. A diagram is connected if it is possible to go from any point to any other with the available edges of the graph. In φ^3 theory, an easy way to calculate the number of surviving (or external) sources is the formula $E = 2P - 3V$, where E stands for external. Now that we have 2, we see that $E = 2, P = 1$ and therefore we have two sources linked by a propagator.
To understand the "vertex" terminology for V let us look at $V = 1, P = 2 \implies E = 1$. This term has the form

$$\begin{aligned} (\star\star) = \left[\frac{i}{6} Z_g g \int d^4x_1 \left(\frac{1}{i} \frac{\delta}{\delta J(x_1)} \right)^3 \right] \\ \times \frac{1}{2} \left[\frac{i}{2} \int d^4y_1 d^4z_1 J(y_1) \Delta(y_1 - z_1) J(z_1) \right] \left[\frac{i}{2} \int d^4y_2 d^4z_2 J(y_2) \Delta(y_2 - z_2) J(z_2) \right]. \end{aligned} \tag{3.2.53}$$

The indices on the variables do not mean that the variables are related. Recalling how we computed the n-point functions earlier, we know that the first term will act on the second one by removing sources and turning them into Dirac deltas

$$(\star\star) = \frac{Z_g g}{24} \int d^4x_1 d^4y_1 d^4y_2 d^4z_1 d^4z_2 \left(\frac{\delta}{\delta J(x_1)} \right)^3 J(y_1) \Delta(y_1 - z_1) J(z_1) J(y_2) \Delta(y_2 - z_2) J(z_2) \tag{3.2.54}$$

Computing everything is rather tedious, but we can see that the terms will look like

$$\Delta(0)\Delta(x_1 - r_i)J(r_i), \tag{3.2.55}$$

where $r = y, z$ and $i = 1, 2$. The vertex in this theory is where three lines join, and it is associated with $iZ_g g \int d^4x$. Since we only have two propagators but $V = 1$, we may think that there is a problem. The $\Delta(0)$ factor fixes this issue because it is a loop, a propagator that starts somewhere and ends at the same point. The associated diagram is drawn in 3

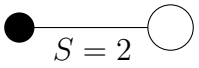


Figure 3: Connected diagram with $E = 1, V = 1$

A more "normal" vertex would simply look like three lines joining together, but we wanted to introduce loops. We avoided values of V and E that yielded multiple different diagrams. For a more complete discussion on Feynman diagrams, see [7, 10].
We briefly introduced a very important equation earlier, namely eq. (3.2.52). It allows us to compute the generating functional simply by drawing diagrams, and stopping at the order of g that we are interested in.
There is little left to introduce about quantum field theory that is of interest to us, other than the the link between classical symmetries and quantum fields.

3.3 Correlation Functions

Throughout the previous section, a few important results related to n-point functions have been scattered and barely brushed over. This section therefore collects them, summarizes them and gives a bit more information about the Wick theorem. In addition, we will study how correlation functions behave under transformations. This will lead us to the Ward identities which connect the classically conserved Noether current and the quantum fields.

3.3.1 Summary

We have seen that correlation or n-point functions are time-ordered strings of field operators sandwiched between two ground states which look like

$$\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle \tag{3.3.1}$$

and can be computed through the use of path integrals of the form

$$Z_0(J) = \langle 0|0\rangle_J = \int \mathcal{D}\varphi \exp\left[\int d^4x \left(\mathcal{L}_0(\varphi, \partial_\mu\varphi) + J(x)\varphi(x)\right)\right], \tag{3.3.2}$$

where we have defined a generating functional $Z_0(J)$. Note that as was done in eq. (3.2.21), it is possible to add more sources J to the generating functional. In the case of a generic complex scalar field that is not necessarily hermitian, we would for example introduce both J and $J^\dagger$, since both φ and $\varphi^\dagger$ appear in the lagrangian density. The general rule is that we have one source per field appearing in the lagrangian. It is also important to understand that in the case of interacting theories, the method relies on the coupling constants being relatively small. We have not really discussed (re)normalization (apart from the brief mention of the

Z_i, Y parameters in eq. (3.2.43)), but it is common to need the use of extra tricks to keep infinities in check because path integrals are quick to diverge [7, 10].
We then saw how to compute generating functionals in interacting theories based on sums of Feynman diagrams such as the ones pictured in 2 and 3, using eq. (3.2.52). This finally allows us to evaluate correlation functions the following relationship

$$\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}...\frac{1}{i}\frac{\delta}{\delta J(x_n)}Z_0(J). \tag{3.3.3}$$

As mentioned before, if more sources were present, we would have to follow a similar structure as in eq. (3.2.17)
We also came across an interesting result called Wick's theorem. Let's investigate it a little further.

3.3.2 Wick's Theorem

This theorem is ultimately about the relationship between an operation called contraction, time-ordering and normal-ordering of creation and annihilation operators. Before stating the theorem, let us first define contraction of two operators A and B as the difference of AB and their normal ordering

$$\overbrace{AB} := AB - :AB: . \tag{3.3.4}$$

The definition of normal ordering varies depending on what the operators are in the context of the theory. The only general feature is that the expectation value of a normal ordered product should be zero. In the case of quantum field theory, the specific expectation value is the vacuum expectation value (VEV), where we surround the product with ground states. This is why the Wick theorem we'll outline here and the one written in eq. (3.2.41) look slightly different, although they are the same.
Now, the general statement of the theorem is that a string of operators $ABCDE...$ can be rewritten as its normal ordered product, plus the sums of the normal ordered products with one, two, etc contractions. Mathematically,

$$\begin{aligned} ABCDE... = &: ABDCE... : \\ &+ \sum_{\text{one contraction}} : \overbrace{AB}CDE... : \\ &+ \sum_{\text{two}} : \overbrace{AB}\overbrace{C}\overbrace{D}E... : \\ &+ .... \end{aligned} \tag{3.3.5}$$

Applied to our case of quantum field theory, the operators are $\varphi(x_i)$, and the subject of interest is the time-ordered product

$$\mathrm{T}\varphi(x_1)\varphi(x_2) = \langle\varphi(x_1)\varphi(x_2)\rangle + :\mathrm{T}\varphi(x_1)\varphi(x_2): \tag{3.3.6}$$

where we denote by $\langle\varphi(x_1)\varphi(x_2)\rangle$ the 2-point correlation function. We can see that this relation holds because when we sandwich the whole expression in ground states, the normal-ordered product is zero due to its properties, and we recover the expression we have for correlation functions. Applied to

$$\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle, \tag{3.3.7}$$

we recover the result found in eq. (3.2.40).

3.3.3 Transformations and Ward Identities

Noether's theorem tells us that a continuous symmetry of the system, corresponding to the invariance of the action under a specific transformation, signifies that there exists an associated conserved current. This is only valid at the classical level. In the quantum world, continuous symmetries require that correlation functions follow certain equations: the Ward identities. Before deriving them, let us start by considering a collection of fields that we denote by Φ and the action of the theory $S[\Phi]$. A general n-point function of this theory is

$$\langle 0|\mathrm{T}\Phi(x_1)...\Phi(x_n)|0\rangle = \int \mathcal{D}\Phi\Phi(x_1)...\Phi(x_n)e^{iS[\Phi]}. \tag{3.3.8}$$

Recall the infinitesimal transformations as defined in eq. (2.1.10). If we assume that the jacobian of the transformation $\Phi \longrightarrow \mathcal{F}(\Phi)$ is trivial, i.e. it does not depend on the fields in Φ themselves, we can prove that

$$\langle 0|\mathrm{T}\Phi(x_1')...\Phi(x_n')|0\rangle = \langle 0|\mathrm{T}\mathcal{F}(\Phi(x_1))...\mathcal{F}(\Phi(x_n))|0\rangle. \tag{3.3.9}$$

This is important, because we see where Noether's theorem starts to break down. The action might very well be invariant under the transformation but if the path integral measure isn't, the symmetry disappears. Such a symmetry is then called an anomaly. Now, with the assumption, the proof is as follows [5]

$$\begin{aligned}
\langle 0|\mathrm{T}\Phi(x_1')...\Phi(x_n')|0\rangle &= \int \mathcal{D}\Phi\mathrm{T}\Phi(x_1')...\Phi(x_n')e^{iS[\Phi]} \\
&= \int \mathcal{D}\Phi'\mathrm{T}\Phi'(x_1')...\Phi'(x_n')e^{iS[\Phi']} \\
&= \int \mathcal{D}\Phi\mathrm{T}\mathcal{F}(\Phi(x_1))...\mathcal{F}(\Phi(x_n))e^{iS[\Phi]} \\
&= \langle 0|\mathrm{T}\mathcal{F}(\Phi(x_1))...\mathcal{F}(\Phi(x_n))|0\rangle.
\end{aligned} \tag{3.3.10}$$

Going from the first line to the second is simply a matter of relabelling, with no actual change of variables. We then use the definition of a transformation to jump to the next line, as well as the assumptions of trivial jacobian and invariance of the action. The trivial jacobian assumption does not come entirely from nowhere, since for example translations and Lorentz transformations of scalar fields have $\mathcal{F}(\Phi) = \Phi$. Conformal invariance, however, suffers from this issue. It is necessary for a theory to be scale invariant for it to be conformally invariant, but

$$\langle 0|\mathrm{T}\varphi(\lambda x_1)...\varphi(\lambda x_n)|0\rangle = \lambda^{-\Delta_1}...\lambda^{-\Delta_n}\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle, \tag{3.3.11}$$

where the Δ_i's are the scaling dimensions of the scale transformation

$$\begin{aligned}
x' &= \lambda x \\
\Phi'(\lambda x) &= \lambda^{-\Delta}\Phi(x).
\end{aligned} \tag{3.3.12}$$

We will come back to these transformations in our discussion of conformal invariance.
Now, let us look at a general infinitesimal transformation such as the one in eq. (2.1.12), i.e. of the form

$$\Phi'(x) = \Phi(x) + \omega_a(x)\frac{\delta\mathcal{F}(x)}{\delta\omega_a(x)}. \tag{3.3.13}$$

To facilitate the reading of the equations that will follow, let us denote by X the string $\Phi(x_1)...\Phi(x_n)$, and by δX its variation under the transformation. Without the assumption

that the action is invariant, we know that it will change by a δS given by eq. (2.1.21). The change in the correlation function is then

$$\langle 0|\mathrm{T}X|0\rangle = \int \mathcal{D}\Phi' \mathrm{T}(X+\delta X)\exp\left[iS[\Phi] + \int d^d x \omega_a(x)\partial_\mu j_a^\mu(x)\right]. \tag{3.3.14}$$

Assuming that the measure is invariant $\mathcal{D}\Phi' = \mathcal{D}\Phi$, we can expand the above to first order in ω_a and get

$$-i\langle 0|\mathrm{T}\delta X|0\rangle = \int d^d x \partial_\mu \langle 0|\mathrm{T}j_a^\mu(x)X|0\rangle \omega_a(x). \tag{3.3.15}$$

The $-i$ factor comes from the factorisation in the exponential. The explicit expression of δX is

$$\begin{aligned} \mathrm{T}\delta X &= \sum_{i=1}^{n}\left(\mathrm{T}\Phi(x_1)....\frac{\delta\mathcal{F}(x)}{\delta\omega_a}\Phi(x_i)...\Phi(x_n)\right)\omega_a(x_i) \\ &= \int d^d x \omega_a(x)\sum_{i=1}^{n}\delta(x-x_i)\left(\mathrm{T}\Phi(x_1)....\frac{\delta\mathcal{F}(x)}{\delta\omega_a}\Phi(x_i)...\Phi(x_n)\right). \end{aligned} \tag{3.3.16}$$

These are, modulo taking the VEV of the expression, the Schwinger-Dyson equations. These relations hold for any infinitesimal ω_a, allowing us to write the Ward identity for j_a^μ by combining the local form of δX in eq. (3.3.16) and eq. (3.3.15):

$$\partial_\mu\langle 0|\mathrm{T}j_a^\mu\Phi(x_1)...\Phi(x_n)|0\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle 0|\mathrm{T}\Phi(x_1)...\frac{\delta\mathcal{F}(x)}{\delta\omega_a}\Phi(x_i)...\Phi(x_n)|0\rangle. \tag{3.3.17}$$

In a similar way as before, the current is ambiguous up to the addition of a divergenceless quantity. Note that if $x \neq x_{1,...,n}$, the correlation function is zero. In other words, the classical equation of motion holds for a quantum field inside the n-point function if its argument is different from those of the other fields. If not, we get contact terms whose form depend on the transformation.

4 Conformal Field Theory

The end of our introduction of the concepts necessary to understand the SYK model is coming. We will briefly introduce notions related to Lie groups and algebras so that we can study conformal symmetry, the associated transformations, their generators and commutation relations. This knowledge will enable us to look at how a conformal transformation acts on a classical field, construct conformal invariants and subsequently look at the constraints put on correlation functions at the quantum level.
We will the define the notion of primary and descendant fields in $d \geq 3$ conformal field theories, introduce radial quantization and the operator product expansion for primary fields, and finish our tour by briefly glancing over conformal blocks.

4.1 Conformal Group

The adjective conformal means that angles between curves do not change if they are both acted on by the same transformation. In general, as first seen in eq. (2.1.11), a change of coordinates $x \longrightarrow x'$ results in a change in the metric given by:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \ . \tag{4.1.1}$$

In the case of a conformal transformation, since we expect angles to be conserved, we could guess that it should locally behave as a rotation and a dilation. Mathematically, this translates to leaving the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{4.1.2}$$

To study this transformation, we want to determine its generators. As seen in the previous section, we want to look at an infinitesimal transformation, and determine how the above equation constrains it. For a generic infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$, the metric changes by (cf. eq. (2.1.17)):

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_\nu \epsilon_\mu(x) - \partial_\mu \epsilon_\nu(x). \tag{4.1.3}$$

Since we want the transformation to be conformal, we need the terms added to the metric to be proportional to it, i.e.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \tag{4.1.4}$$

Taking the trace on both sides of the equation yields:

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho. \tag{4.1.5}$$

Assuming we live in flat Euclidean space, we can manipulate eq. (4.1.4) so it becomes

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f, \tag{4.1.6}$$

which, once contracted with $\eta^{\mu\nu}$ gives:

$$2\partial^2 \epsilon_\mu = (2-d)\partial_\mu f. \tag{4.1.7}$$

We can further manipulate the last expression by applying ∂_ν to it, and equate it with (4.1.4) to which we have applied ∂^2 to end up with

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f. \tag{4.1.8}$$

Lastly, contracting again with $\eta^{\mu\nu}$ yields:

$$(d-1)\partial^2 f = 0. \tag{4.1.9}$$

Using eqs. (4.1.4)-(4.1.9), we can derive the infinitesimal forms of the transformations. For $d = 2$, the solutions are relatively easy. For $d \geq 3$, we find the following general expression [5]:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} + c_{\mu\nu\rho}x^\nu x^\rho \tag{4.1.10}$$

where

$$\begin{aligned} b_{\mu\nu} &= \lambda\eta_{\mu\nu} + m_{\mu\nu} \qquad m_{\mu\nu} = -m_{\nu\mu} \\ c_{\mu\nu\rho} &= c_{\mu\rho\nu} \\ c_{\mu\nu\rho} &= \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\nu \quad \text{where } b_\mu \equiv \frac{c^\sigma_{\sigma\nu}}{d} \end{aligned} \tag{4.1.11}$$

The λ part of $b_{\mu\nu}$ is an infinitesimal scale transformation, while its antisymmetric part represents an infinitesimal rigid rotation. The transformation associated to $c_{\mu\nu\rho}$ is named the special conformal transformation:

$$x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2 \tag{4.1.12}$$

The associated finite transformations are:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \quad \text{(translation)} \\ x'^\mu &= M^\mu_\nu x^\nu \quad \text{(lorentz transformation)} \\ x'^\mu &= \lambda x^\mu \quad \text{(dilation)} \\ x'^\mu &= b^\mu x^2 - 2x^\mu b\cdot x \quad \text{(special conformal transformation).} \end{aligned} \tag{4.1.13}$$

The special conformal transformation can also be expressed as

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu \tag{4.1.14}$$

i.e. an inversion $x^\mu \to x^\mu/x^2$, then a translation and finally another translation. Assuming that the fields are not affected by the transformations, the generators of the conformal group are

$$\begin{aligned} P_\mu &= -i\partial_\mu && \text{(translation)} \\ L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && \text{(rotations)} \\ D &= -ix^\mu\partial_\mu && \text{(dilation)} \\ K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) && \text{(SCT).} \end{aligned} \tag{4.1.15}$$

Note that the rotations are actually Lorentz transformations, as the infinitesimal parameter of the transformation is ω^μ_ν and not $\omega_{\mu\nu}$ [11]. The conformal algebra is defined by the following commutation relations:

$$\begin{aligned} [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\ [K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\ [P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \end{aligned} \tag{4.1.16}$$

It is often easier to define different generators based on the previous ones to have easier commutation relations:

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \\ J_{-1,0} &= D \\ J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu), \end{aligned} \tag{4.1.17}$$

with $J_{ab} = -J_{ba}$ antisymmetric, for $a, b \in \{-1, 0, 1, ..., d\}$. These generators then have the same commutation relations as the ones of the $SO(d+1,1)$ algebra [5]:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{4.1.18}$$

where $\eta_{ab} = \text{diag}(-1, 1, ..., 1, -1)$ for Minkowski spacetime (only the first -1 appears in Euclidean spacetime). This relationship shows the existence of an isomorphism between the conformal group in d dimensions and $SO(d,2)$ (or $SO(d+1,1)$ in Euclidean spacetime). Note that the group composed of the Poincaré group (Lorentz transformations + translations) put together with dilations is a subgroup of the conformal group. This means that invariance to special conformal transformations is not implied by invariance under translations, rotations and dilations [5].
Now that we know what conformal transformations look like, let us construct conformal invariants, i.e. functions of N points x_i that are unaffected by them. We first notice that invariance under translations and rotations is only possible for distances $|x_i - x_j|$ between pairs of points. In addition, scale invariance that we should have no notion of length scales, and therefore that invariants should be ratios of distances:

$$\frac{|x_i - x_j|}{|x_k - x_l|}. \tag{4.1.19}$$

To see what constraints are put in place by a special conformal transformation, let us look at how it affects the distance between two points:

$$|x'_i - x'_j| = \frac{|x_i - x_j|}{(1 - 2b \cdot x_i + b^2 x_i^2)^{\frac{1}{2}}(1 - 2b \cdot x_j + b^2 x_j^2)^{\frac{1}{2}}}. \tag{4.1.20}$$

We see that for these factors to cancel, we want each x_i to appear twice in an invariant, once in the numerator and once in the denominator. For this to happen with distances, we therefore need at least four distinct points. These expressions are called cross ratios or anharmonic ratios. Some simple examples that are not 0 or 1 are:

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \text{ and } \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}. \tag{4.1.21}$$

With N distinct points, there is a total of $\frac{N(N-3)}{2}$ independent cross-ratios that can be constructed. A proof for this statement can be found in [12].

4.2 Aspects of Conformal Field Theories

Previously, we vaguely defined a conformal field theory as a field theory that was left invariant under conformal transformations. Now, we add a few more conditions [11]:

1. There is a collection of fields Φ that contains in particular all derivatives of each field.

2. There is a subset of fields $\{\phi\} \subseteq \Phi$ that transform like

$$\phi(x) \longmapsto \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x) \tag{4.2.1}$$

under a conformal transformation $x \to x'$. These fields are called primary (if $d = 2$, they are quasi-primary, but this distinction does not exist for $d \geq 3$). The Δ factor is the scaling dimension we mentioned in eq. (3.3.12). In this setting, we call it either the conformal weight or the dimension of the field itself, while d is the dimension of the spacetime. $|\partial x'/\partial x|$ is the jacobian of the conformal transformation of the coordinates, related to the $\Omega(x)$ by [5]

$$\left|\frac{\partial x'}{\partial x}\right| = \Omega(x)^{-\frac{d}{2}}. \tag{4.2.2}$$

3. The set of primary fields $\{\phi\}$ forms a generating family that allows us to construct the rest of Φ by linear combination.

4. There exists a ground state $|0\rangle$ that is invariant under the conformal group.

Since every field can be constructed by a linear combination of primary fields, they are our object of interest. Let us therefore study their correlation functions, in particular 2- and 3-point ones. Using eq. (3.3.9), we see that an n-point function of primary fields behaves like

$$\langle 0|\mathrm{T}\phi_1(x_1)...\phi_n(x_n)|0\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} ... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\frac{\Delta_n}{d}} \langle 0|\mathrm{T}\phi_1(x_1')...\phi(x_n')|0\rangle, \tag{4.2.3}$$

where i is not necessarily different from j. In other words, the same primary field can appear twice. Remembering the method we used to construct invariants earlier, we look at how conformal transformations constraint the functions. As before, rotational and translational invariance imply that the correlation function must be a function of distance:

$$\langle 0|\mathrm{T}\phi_1(x_1)\phi_2(x_2)|0\rangle = f(|x_1 - x_2|). \tag{4.2.4}$$

Using eq. (4.2.3) with a scale transformation gives us

$$\langle 0|\mathrm{T}\phi_1(x_1)\phi_2(x_2)|0\rangle = \lambda^{\Delta_1+\Delta_2}\langle 0|\mathrm{T}\phi_1(\lambda x_1)\phi_2(\lambda x_2)|0\rangle. \tag{4.2.5}$$

Then $f(x) \overset{!}{=} \lambda^{\Delta_1+\Delta_2} f(\lambda x)$, and therefore

$$\langle 0|\mathrm{T}\phi_1(x_1)\phi_2(x_2)|0\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{4.2.6}$$

with C_{12} a constant. Lastly, adding invariance under special conformal transformations that have jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b \cdot x + b^2x^2)^d}, \tag{4.2.7}$$

we end up with

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \quad , \gamma_i = (1 - 2b \cdot x_i + b^2x_i^2). \tag{4.2.8}$$

This only holds if $\Delta_1 = \Delta_2$, which means that two primary fields are correlated if and only if they have the same conformal weight:

$$\langle 0\mathrm{T}|\phi_1(x_1)\phi_2(x_2)|0\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \text{if} \quad \Delta_1 = \Delta_2 \\ 0 & \text{otherwise} \end{cases} \tag{4.2.9}$$

The same method can be used to fix the form of the three-point function [5]:

$$\langle 0|\mathrm{T}\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)|0\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{13}^{\Delta_1+\Delta_3-\Delta_2}}, \tag{4.2.10}$$

where $r_{ij} = |x_i - x_j|$. Our lucky streak unfortunately ends here: it is not possible fix $N \geq 4$-point functions up to a constant like we've done for 2- and 3-point ones. This is easily seen with the general form of a 4-point function:

$$\langle 0|\mathrm{T}\phi_1(x_1)...\phi_4(x_4)|0\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{14}r_{23}}\right) \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\Delta/3}, \tag{4.2.11}$$

with $\Delta = \sum_{i=1}^4 \Delta_i$ and F arbitrary. Note that the above is true for spinless (scalar) primary fields. For spinning fields, see [13].
Let us come back to the statement that primary fields allow to construct any other field in the theory. Recall the commutation relations (4.1.16). It can be proven that, for a generic field $\Phi(x)$ [5, 13]:

$$\begin{aligned} [L_{\mu\nu}, \Phi^a(0)] &= (S_{\mu\nu})^a_b \Phi^b(0) \\ [D, \Phi(0)] &= \Delta\Phi(0), \end{aligned} \tag{4.2.12}$$

where $S_{\mu\nu}$ are matrices that satisfy the same algebra as $L_{\mu\nu}$, a, b are indices for the $SO(d)$ representation of Φ (i.e. spin indices, which disappear if Φ is scalar for example or because it is easier to write equations without) and Δ is the dimension or conformal weight of the field. Notice that these relations are taken at the origin, and can be translated to any point x by $\Phi(x) = e^{x\cdot P}\Phi(0)e^{-x\cdot P}$ with $P = (P^\mu)$ the generator of translations. In fact, doing allows us the write the action of the generators [5]:

$$\begin{aligned} L_{\mu\nu}\Phi(x) &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x), \\ D\Phi(x) &= -i(x^\nu\partial_\nu + \Delta)\Phi(x). \end{aligned} \tag{4.2.13}$$

From these, we find out that K_μ is a lowering operator for dimension,

$$\begin{aligned} DK_\mu\Phi(0) &= ([D, K_\mu] + K_\mu D)\Phi(0) \\ &= (\Delta - 1)K_\mu\Phi(0). \end{aligned} \tag{4.2.14}$$

and that we could technically lower dimension arbitrarily by applying n K_{μ_i} operators to $\Phi(0)$. However, "dimensions are bounded from below in physically sensible theories" [13] and there must therefore exist fields that commute with K_μ

$$[K_\mu, \Phi(0)] = 0. \tag{4.2.15}$$

These are the primary fields we defined earlier, from which we can construct fields of higher dimensions called descendants. This is done using the momentum generators P_μ because they act like raising operators for dimension. In particular, $\Phi(x) = e^{x\cdot P}\Phi(0)e^{-x\cdot P}$ is an infinite linear combination of descendant fields. We then find

$$[K_\mu, \Phi(x)] = (k_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu})\Phi(x), \tag{4.2.16}$$

with $k_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu$.
Another aspect that we have yet to discuss is the tracelessness of the energy-momentum tensor in theories invariant under conformal transformations. Recall that under an infinitesimal transformation of the coordinates $x^\mu \to x^\mu + \epsilon^\mu$, the variation of the action is

$$\delta S = \frac{1}{2}\int d^d x T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu), \tag{4.2.17}$$

with $T^{\mu\nu}$ symmetric. This symmetry follows from the discussion on the Belinfante tensor in the section on classical field theory. Now, using the definition of an infinitesimal conformal transformation (4.1.4) and (4.1.5), we end up with

$$\delta S = \frac{1}{d}\int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho. \tag{4.2.18}$$

The tracelessness of the tensor thus implies conformal invariance of the action. The converse is not true, although we have not found an example of a conformal field theory that has a energy-momentum tensor with non-vanishing trace. Other arguments seem to claim that in general, the stress tensor vanishes at fixed points because it behaves like the β function in perturbation theory [14].

4.3 Radial quantization

We will assume without proof that every result we have derived until now does not depend on how one quantizes the theory [13]. This allows to choose a quantization that keeps the symmetry alive at the quantum level. In the case of a scale-invariant theory, it makes sense to "cut" spacetime with spheres of growing radii centered at the origin, and evolve states from smaller to larger spheres with the dilation operator, since we choose time to be orthogonal to the spatial surfaces where the states live. To each sphere we associate a Hilbert space $\mathcal{H}$, on which we can act by inserting fields defined on the sphere. In practice, this is done by inserting an operator defined on the surface in the path integral.
When the theory is quantized in that way, a correlation function is a radially ordered product

$$\langle\phi_1(x_1)...\phi_n(x_n)\rangle = \langle 0|\mathrm{R}\phi_1(x_1)...\phi_n(x_n)|0\rangle, \tag{4.3.1}$$

where R acts like

$$\mathrm{R}(\phi(x_1)...\phi(x_n)) = \phi(x_1)...\phi(x_n) \quad \text{if} \quad x_1 > x_2 > ... > x_n. \tag{4.3.2}$$

It is possible to choose to quantize around another point than the origin, amounting to a kind of change of frame of reference. Radial ordering is consistent with time-ordering in a canonically quantized theory, because fields at the same radius but different angles on the sphere commute the same way space-like separated fields do usually.
Radial quantization allows for an interesting correspondence between states and local operators. Local operators are defined as eingenstates of the dilation operator. A specific construction (see e.g. [5, 11, 13]) allows to write

$$\Phi(0) \quad \longleftrightarrow \quad \Phi(0)|0\rangle =: |\Phi\rangle. \tag{4.3.3}$$

The idea of this construction is that to prepare a state on the boundary of the spheres, one computes a path integral over their interiors and inserting fields. On the other hand, eigenstates of the dilation operator can be used as fields by "gluing" them on the boundary of holes cut out of the path integral around positions x_i, which gives a quantity that behaves like a correlator of

local operators. Alternatively, we can also define the state as the limit of $\Phi(x)|0\rangle$ when $x \to 0$. The conformal group then acts on states in the following way:

$$\begin{aligned} [K_\mu, \Phi(0)] = 0 \quad &\longleftrightarrow \quad K_\mu|\Phi\rangle = 0, \\ [D, \Phi(0)] = \Delta\Phi(0) \quad &\longleftrightarrow \quad D|\Phi\rangle = \Delta|\Phi\rangle, \\ [L_{\mu\nu}, \Phi(0)] = S_{\mu\nu}\Phi(0) \quad &\longleftrightarrow \quad L_{\mu\nu}|\Phi\rangle = S_{\mu\nu}|\Phi\rangle. \end{aligned} \tag{4.3.4}$$

This is easily seen by remembering the fact that K, D and $L_{\mu\nu}$ all follow $\mathcal{O}|0\rangle = 0$, and applying the left hand side of the equivalence to the ground state.
In radial quantization, we can define a conformal multiplet, i.e. a string of a primary state and higher-dimensional descendants created by acting with momentum generators on the state

$$|\Phi\rangle, P_\mu|\Phi\rangle, P_\mu P_\nu|\Phi\rangle, ..., \tag{4.3.5}$$

which is equivalent to acting with derivatives of $\Phi(x)$ at the origin, e.g. [13]

$$\partial_\mu \Phi(x)|_{x=0}|0\rangle = [P_\mu, \Phi(0)]|0\rangle = P_\mu|\Phi\rangle. \tag{4.3.6}$$

As mentioned earlier, $\Phi(x) = e^{x\cdot P}\Phi(0)$ creates an infinite linear combination of descendant states:

$$\Phi(x)|0\rangle = e^{x\cdot P}\Phi(0)e^{-x\cdot P}|0\rangle = e^{x\cdot P}|\Phi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(x\cdot P)^n|\Phi\rangle. \tag{4.3.7}$$

4.4 Operator Product Expansion

The insertion of two fields $\phi_i(x)\phi_j(0)$ inside a ball and subsequent computation of the path integral over its interior yields a state on the boundary. Every state being a linear combination of primaries and descendants allows us to write

$$\phi_i(x)\phi_j(0)|0\rangle = \sum_k C_{ijk}(x, P)\phi_k(0)|0\rangle, \tag{4.4.1}$$

where k is an index running over primary fields and $C_{ijk}(x, P)$ is an operator that puts together primaries and descendents in the k-th conformal multiplet. In other words, it's more or less a sum of increasingly long strings of momentum operators. This equation can be used inside the path integral as long as the other fields of the theory lie outside the sphere of radius $|x|$.
The state-operator transformation allows us to rewrite the above as:

$$\phi_i(x_1)\phi_j(x_2) = \sum_k C_{ijk}(r_{12}, \partial_2)\phi_k(x_2), \tag{4.4.2}$$

which is valid in correlation functions as long as the other fields $\phi_n(x_n)$ are further away from x_2 than $\phi_i(x_1)$, i.e. $r_{2n} \geq r_{12}$. This is called the Operator Product Expansion (OPE) (and not fields because the fields located at a point are technically operators). The expansion need not be centered around one of the fields. We could choose to perform radial quantization around x_3, yielding

$$\phi_i(x_1)\phi_j(x_2) = \sum_k C'_{ijk}(r_{13}, r_{23}, \partial_3)\phi_k(x_3), \tag{4.4.3}$$

with C'_{ijk} a distinct differential operator. This means that it is possible to do an OPE between two fields as long as it is possible to draw a sphere that isolates them from the other fields.

4.4.1 Constraints of Conformal Invariance

Conformal invariance puts restrictions on the form the OPE can take. For the sake of simplicity, we'll omit calculations in what follows and simply state results, taken from [13]. Note that all fields are scalars.
The operator C_{ijk} is completely fixed by conformal invariance. Taking the correlation function of both sides of the OPE:

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_k)\rangle = \sum_{k'} C_{ijk'}(r_{12},\partial_2)\langle \phi'_k(x_2),\phi_k(x_3)\rangle. \tag{4.4.4}$$

The left-hand side is given by eq. (4.2.10), while the right-hand side can be computed by choosing an orthogonal basis of primary fields so that $\langle \phi_k(x_2)\phi'_k(x_3)\rangle = \delta_{kk'} r_{23}^{-2\Delta_k}$. We end up with

$$\frac{f_{ijk}}{r_{12}^{\Delta_i+\Delta_j-\Delta_k} r_{23}^{\Delta_j+\Delta_k-\Delta_i} r_{13}^{\Delta_i+\Delta_k-\Delta_j}} = C_{ijk}(r_{12},\partial_2) r_{23}^{-2\Delta_k} \tag{4.4.5}$$

Notice how important the f_{ijk} coefficient is. Knowing it, along with the scaling dimensions Δ and the $SO(d)$ irreducible representations of the primary fields (when they have spin) allows to determine all correlation functions of local fields. This set of constants is called the "CFT data". Eq. (4.4.4) hints at a recursive relation that will help us to compute correlation functions.

4.4.2 Using the OPE: Evaluating Correlators

It is in general possible to use the operator product expansion to reduce n-point functions to $(n-1)$-point functions as follows:

$$\langle \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle = \sum_k C_{12k}(r_{12},\partial_2)\langle \phi_k(x_2)...\phi_n(x_n)\rangle. \tag{4.4.6}$$

Iterating this relation, we get a long sum of one-point functions given by

$$\langle \Phi(x)\rangle = \begin{cases} 1 & \text{if } \Phi \text{ is the unit field,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.7}$$

We now have an algorithm that allows us to compute flat-space correlation functions!

4.5 Conformal Blocks

Suppose that we want to compute a four-point correlation function of identical scalar fields $\phi(x)$. Using eq. (4.2.11) with identical fields gives

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{14}r_{23}}\right)\prod_{i<j} r_{ij}^{-2\Delta/3} \tag{4.5.1}$$

The OPE is [13]

$$\phi(x_1)\phi(x_2) = \sum_\Phi f_{\phi\phi\Phi} C_a(r_{12},\partial_2)\Phi^a(x_2), \tag{4.5.2}$$

with Φ^a can have non-zero spin, if it transforms in a spin-l traceless symmetric tensor representation of $SO(d)$ [13]. Similarly as in the OPE, as long as we can draw a sphere that contains x_1, x_2 and not the other points, we can pair up the fields and perform the OPE:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \prod_{i<j} r_{ij}^{-2\Delta/3}\left[\sum_\Phi f^2_{\phi\phi\Phi} g_{\Delta_\Phi,l_\Phi}(x_i)\right], \tag{4.5.3}$$

with $g_{\Delta,l}(x_i)$ the conformal blocks, where Δ_Φ is the dimension of Φ and l_Φ its spin. Although this is not obvious from their definition, they are functions of the anharmonic ratios that decompose F:

$$F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{14}r_{23}}\right) = \sum_\Phi f^2_{\phi\phi\Phi} g_{\Delta_\Phi, l_\Phi}\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{14}r_{23}}\right). \tag{4.5.4}$$

We will study more examples of these blocks when we discuss the SYK model. For now, it is enough to know that they decompose the function of anharmonic rations in the four-point correlation function.

To conclude this section on Conformal Field Theory, we come back to the fact that any correlation function of local operators can be determined using the CFT data. That is true, although a random set of data doesn't necessarily define a consistent CFT. It needs to satisfy additional conditions one of which is OPE associativity, which gives a crossing symmetry and leads to the conformal bootstrap [13, 14, 15].

5 The Sachdev-Ye-Kitaev Model

The SYK model is a toy model for a lot of interesting phenomena in physics such as quantum chaos, traversable wormholes and strange metals (although one has to generalize the model a little for that last application), to name a few. In this project, we consider the original model which has a four-fermion interaction vertex ($q = 4$). It is the simplest non-trivial and non-degenerate case.
We will discuss the model's basic properties, i.e. its behavior and the look of its diagrams for a large number of fermions (large N diagrammatics), the emergence of conformal symmetry in the IR limit, its effective and its Schwarzian action. We will wrap up this section with the statement of its four-point function and the computation of time-ordered-correlators (TOC) and out-of-time-ordered correlators (OTOC), avoiding the calculations due to their bulkiness. A complete description of them can be found in [2].

5.1 Basic Definitions

This model is a quantum mechanical system of $N \gg 1$ Majorana fermions with all-to-all random couplings. In other words, each particle interacts with one another and the coupling J_{ijkl} follows a probability distribution. The action is written as

$$I_{SYK} = \int d\tau \left[\frac{1}{2} \sum_{i_1}^{N} \chi_i(\tau)\dot{\chi}_i(\tau) - \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau) \right], \tag{5.1.1}$$

with $\dot{\chi}_i = \frac{d\chi_i}{d\tau}$. Here, τ is the euclidean time, related to the Lorentzian time t by a Wick rotation, $\tau = it$. As opposed to the rest of the project up until now, we will work in euclidean time, unless we explicitly state otherwise. The Majorana operators are hermitian, $\chi_i = \chi_i^\dagger$ and obey the anticommutation relations

$$\{\chi_i, \chi_j\} = \chi_i\chi_j - \chi_j\chi_i = \delta_{ij}, \quad i, j = 1, ..., N, \tag{5.1.2}$$

which define the Clifford algebra. In one dimension, Majorana fermions are dimensionless and scalar fields. The couplings are all independent from one another and distributed randomly following a gaussian distribution with the probability density function

$$P(J_{ijkl}) = \exp\left(-\frac{N^3 J_{ijkl}^2}{12J^2} \right) \quad \text{for every} \quad J_{ijkl}. \tag{5.1.3}$$

Note that there is no implicit summation over the indices i, j, k and l. We can derive important properties from the distribution (5.1.3). For one, it tells us (as is customary for gaussian distributions) the average and average square of the couplings

$$\overline{J_{ijkl}} = 0, \quad \overline{J_{ijkl}^2} = \frac{3!J^2}{N^3}, \tag{5.1.4}$$

with J a constant with dimensions of mass. In addition, we can write a Wick-style decomposition of even moments into a sum of all possible products of second moments (average squares), similar to (3.2.40). For the fourth moment, we have

$$\begin{aligned} \overline{J_{i_1i_2i_3i_4}J_{j_1j_2j_3j_4}J_{k_1k_2k_3k_4}J_{l_1l_2l_3l_4}} =& \overline{J_{i_1i_2i_3i_4}J_{j_1j_2j_3j_4}}\,\overline{J_{k_1k_2k_3k_4}J_{l_1l_2l_3l_4}} \\ &+ \overline{J_{i_1i_2i_3i_4}J_{k_1k_2k_3k_4}}\,\overline{J_{j_1j_2j_3j_4}J_{l_1l_2l_3l_4}} \\ &+ \overline{J_{i_1i_2i_3i_4}J_{l_1l_2l_3l_4}}\,\overline{J_{j_1j_2j_3j_4}J_{k_1k_2k_3k_4}}. \end{aligned} \tag{5.1.5}$$

This also implies that the odd moments are zero, because a Wick-style decomposition would contain averages of single couplings in each term, which are trivial as derived in (5.1.4). We also have that the anticommutation relations outlined in eq. (5.1.2) imply antisymmetry of the couplings:

$$J_{ijkl} = \text{sgn}\sigma J_{\sigma(i)\sigma(j),\sigma(k)\sigma(l)}, \quad \sigma : i \to \sigma(i), \quad i = 1, ..., N. \tag{5.1.6}$$

This constrains the number of independent non-zero components of J_{ijkl} to $\frac{N!}{4!(N-4)!}$, and allows to define the disorder average of two arbitrary couplings [2]:

$$\overline{J_{i_1 i_2 i_3 i_4} J_{j_1 j_2 j_3 j_4}} = \frac{3! J^2}{N^3} \sum_{\sigma} \text{sgn}\sigma \delta_{i_1 \sigma(j_1)} \delta_{i_2 \sigma(j_2)} \delta_{i_3 \sigma(j_3)} \delta_{i_4 \sigma(j_4)}, \tag{5.1.7}$$

where the sum is performed over all possible permutations. This sum essentially checks whether the indices coincide. In the case of three coincident indices, we have

$$\sum_{k,l,m=1}^{N} \overline{J_{iklm} J_{jklm}} = \frac{3! J^2}{N^3} \sum_{k,l,m=1}^{N} \delta_{ij}\delta_{kk}\delta_{ll}\delta_{mm} + \cdots = \frac{3! J^2}{N^3} \left(N^3 \delta_{ij} + \mathcal{O}(N^2) \right) = 3! J^2 \delta_{ij} + \mathcal{O}\left(\frac{1}{N} \right). \tag{5.1.8}$$

This particular result will be very useful in the coming applications. In the following, we will consider two possibilities for the interval on which the euclidean time runs: the real line $\tau_{line} \in (-\infty, \infty)$ and the euclidean circle $\tau_{circle} \in \left[-\frac{\beta}{2}, \frac{\beta}{2} \right), \tau + \beta \sim \tau$. These correspond respectively to zero-temperature quantum mechanics, and the thermal state with inverse temperature $\beta = \frac{1}{T}$ (notice that $\beta \to \infty$ as $T \to 0$). To go from one time to the other, we use the map

$$\tau_{line} = \tan\left(\frac{\pi \tau_{circle}}{\beta} \right). \tag{5.1.9}$$

This mapping is real and monotonic, meaning that it preserves the order of times $\frac{d\tau_{line}}{d\tau_{circle}} > 0$. Coming back to the action (5.1.1), notice that if the interaction term (the second member of the right-hand side) is turned off, the hamiltonian of the free theory is zero, $H_0(\tau) = 0$. We thus have constant operators in the free theory: $\chi_i(\tau) = e^{\tau H_0} \chi_i(0) e^{-\tau H_0}$, which allows us to compute 2-point functions using the anticommutation relations in eq. (5.1.2). For the zero-temperature free theory, they read:

$$\langle 0 | \text{T} \chi_i(\tau) \chi_j(0) | 0 \rangle = \theta(\tau) \langle 0 | \chi_i \chi_j | 0 \rangle - \theta(-\tau) \langle 0 | \chi_j \chi_i | 0 \rangle = \frac{1}{2} \text{sgn}(\tau) \delta_{ij}. \tag{5.1.10}$$

Here, $|0\rangle$ is the vacuum state in the free theory. For the finite temperature free theory we have

$$\langle \text{T} \chi_i(\tau) \chi_j(0) \rangle_\beta = \frac{1}{2} \text{sgn}\left(\sin\left(\frac{\pi \tau}{\beta} \right) \right) \delta_{ij}, \tag{5.1.11}$$

where $\langle ... \rangle_\beta$ is the averaging over the thermal distribution along with the quantum averaging:

$$\langle ... \rangle_\beta := \frac{\text{tr}[e^{-\beta H} ...]}{\text{tr}[e^{-\beta H}]}. \tag{5.1.12}$$

The trace is taken over the Hilbert space generated by the fermionic creation operator $c_i^\dagger$ acting on the vacuum state $|0\rangle$ annihilated by all c_i's. These operators are defined as:

$$c_i = \frac{1}{\sqrt{2}} (\chi_{2i} - i \chi_{2i+1}), \quad c_i^\dagger = \frac{1}{\sqrt{2}} (\chi_{2i} + i \chi_{2i+1}), \quad i = 1, ..., \frac{N}{2} \text{ for } N \text{ even}. \tag{5.1.13}$$

More details in annex A of [2]. Now, recall that a 2-point function is also called a propagator. In our case, the thermal fermion propagator is antiperiodic. Assuming $\tau > 0$, we have

$$\begin{aligned} \mathrm{tr}\left[e^{-\beta H}\chi(\tau+\beta)\chi(0)\right] &= \mathrm{tr}\left[\chi(\tau)e^{-\beta H}\chi(0)\right] \\ &= \mathrm{tr}\left[e^{-\beta H}\chi(0)\chi(\tau)\right] \\ &= -\mathrm{tr}\left[e^{-\beta H}\chi(\tau)\chi(0)\right]. \end{aligned} \tag{5.1.14}$$

To go from the first line to the second, we used a property of the trace, namely $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$. The last line comes from the anticommutation rule (5.1.2).
The last specificity of the model is the use of averaged correlation functions:

$$G_0(\tau) := \frac{1}{N}\sum_{i=1}^{N}\langle \mathrm{T}\chi_i(\tau)\chi_i(0)\rangle = \frac{1}{2}\mathrm{sgn}\tau \tag{5.1.15}$$

$$G_0^\beta(\tau) := \frac{1}{N}\sum_{i=1}^{N}\langle \mathrm{T}\chi_i(\tau)\chi_i(0)\rangle_\beta = \frac{1}{2}\mathrm{sgn}\left(\sin\frac{\pi\tau}{\beta}\right). \tag{5.1.16}$$

It is interesting to note that both propagators coincide if we use τ_{circle} for the finite temperature propagator (5.1.16). Both these functions are antisymmetric, as are all fermion Green functions: $G(\tau) = -G(-\tau)$.
The appearance of averaged n-point functions is interesting when compared to the standard QFTs we discussed earlier. In the SYK model, we have a multitude of theories with random couplings that we average over.

5.2 Two-point Function and Diagrammatics

Now, let us move on to the interacting theory. The interaction term is

$$H(\tau) = \frac{1}{4!}\sum_{i,j,k,l} J_{ijkl}\chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau). \tag{5.2.1}$$

We'll calculate averaged over disorder loop corrections to the free propagators in Lorentzian time t for now. To do so, we'll expand the evolution operator and calculate a few orders in J. The time-evolution operator is given by:

$$\begin{aligned} U(t_1,t_2) &:= \mathrm{T}\exp\left[-i\int_{t_2}^{t_1} dt H(t)\right] \\ &= 1 - i\int_{t_2}^{t_1} dt H(t) - \frac{1}{2}\int_{t_2}^{t_1} dt \int_{t_2}^{t} dt' H(t)H(t') + ... \end{aligned} \tag{5.2.2}$$

It is then possible to write the exact propagator $G(t)$ as:

$$\begin{aligned} G(t)\delta_{ij} &= \left\langle \mathrm{T}U^\dagger(t,+\infty)\chi_i(t)U(t,0)\chi_j(0)U(0,-\infty)\right\rangle \\ &= \frac{\langle \mathrm{T}\chi_i(t)\chi_j(0)U(+\infty,-\infty)\rangle}{\langle U(+\infty,-\infty)\rangle}. \end{aligned} \tag{5.2.3}$$

This expression comes from the fact that the vacuum state in the free theory isn't the same as in the one in the interacting one, and the same goes for the fields χ_i. To end up with the final result, we go through computations in the interacting picture. For more details, see [7, 10] (respectively ex 9.5 and pages 82-87). Note that in our case, the hamiltonian of the free

theory is trivial $H_0 = 0$, and therefore it is not really necessary to use this picture. Expanding this expression and averaging it over disorder gives:

$$G(t)\delta_{ab} = \Big\langle \mathcal{T}\Big[\chi_a(t)\chi_b(0) - \frac{i}{4!}\sum_{i,j,k,l}\overline{J_{ijkl}}\int_{-\infty}^{+\infty} dt'\chi_a(t)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l' - \\ - \frac{1}{2}\frac{1}{(4!)^2}\sum_{i,j,k,l,p,q,r,s}\overline{J_{ijkl}J_{pqrs}}\int_{-\infty}^{+\infty}dt'\int_{-\infty}^{+\infty}dt''\chi_a(t)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'\chi_p''\chi_q''\chi_r''\chi_s'' + \mathcal{O}(J^3)\Big]\Big\rangle. \tag{5.2.4}$$

Note that the indices change from eq. (5.2.3) because of the introduction of the couplings. In the above, we slightly altered the notation to lighten the expression by denoting $\chi_i(t') \equiv \chi_i'$ and $\chi_i(t'') \equiv \chi_i''$. Also note that we were able to compute the disorder averages on the numerator and denominator separately thanks to the large N limit.
Using the rules we derived in the previous subsection, namely eqs. (5.1.4), (5.1.5) and (5.1.7), we see that odd orders in J_{ijkl} vanish after the averaging. In addition, the disconnected part (diagramatically, cf. our discussion of Feynman diagrams) is cancelled by the denominator of eq. (5.2.3). Substracting the free propagator $G_0(t)$, we end up with

$$\begin{aligned} G(t) - G_0(t) &= \frac{2\cdot 4\cdot 4!}{2(4!)^2}\frac{1}{N}\sum_{i,j,k,m,n}\overline{J_{ikmn}J_{jkmn}}\delta_{ij}\int dt'dt''G_0(t-t')G_0(t'-t'')^3G_0(t'') + \mathcal{O}(J^4) \\ &= J^2\int dt'dt''G_0(t-t')G(t'-t'')^3G(t'') + \mathcal{O}\left(\frac{J^2}{N}\right) + \mathcal{O}(J^4). \end{aligned} \tag{5.2.5}$$

Let us break down the steps we took to get from $G(t) - G_0(t)$ to the final expression. First, we reintroduced the averaged correlator, which made the first term of (5.2.4) become $\sum_{a=1}^{N}\chi_a(t)\chi_a(0)$. This is equal to $G_0(t)$ as defined in eq. (5.1.16), and therefore this term is cancelled by the substraction. We then have to work with the bottom line of the equation, to which we apply Wick's theorem. Thinking about the possible contractions for some time, one notices that contracting any two χ's with the same argument makes the corresponding term vanish, because of the antisymmetry of the couplings. In addition, contracting such that the resulting propagators are $G_0(t)G_0(t'-t'')^4$ also results in a vanishing term, and the two χ_a's cannot be contracted with two other chi's that have the same argument. Finally, since each χ_a has four possible contractions with one of the χ' and four others with the χ'', and that the remaining χ' and χ'' have to match up with each other, we end up with $4\cdot 4\cdot 3\cdot 2$ possibilities that yield the same result. Multiply that by 2 due to the symmetry under exchange $t' \leftrightarrow t''$, and remember that a fermion Green's function is odd so the minus sign in front of the expression cancels out to get to the rhs on the first line. Obtaining the second line is just a matter of using eq. (5.1.8).
As was done with correlation functions in QFT, it is possible to represent (5.1.5) schematically, in the form of a melonic diagram (Fig. 4). We also have a diagrammatic explanation as to why we can't have propagators at equal times in Fig. 5.

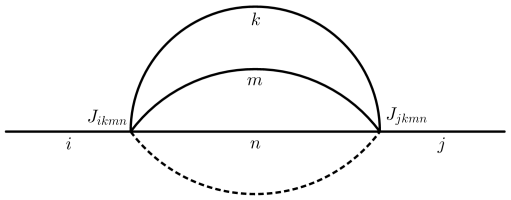


Figure 4: Melonic diagram. [2]

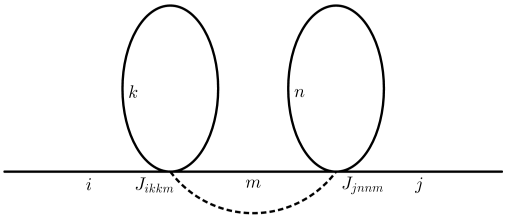


Figure 5: Double tadpole diagram that is identically zero. [2]

The presence of the dashed line, which does not correspond to any propagator, is a little unusual. In fact, it tells us which couplings are contracted together in the disorder average. This is best seen once we think about the large N limit and the higher-order corrections to the propagator that survive in it (see Fig. 6): in the vanishing diagrams (all except (a), (b) and (e)), contractions are made between couplings that do not have three coincident indices, leading to smaller powers of N obtained through the averaging, which do not cancel the N^{-3n} that appears in (5.1.7).

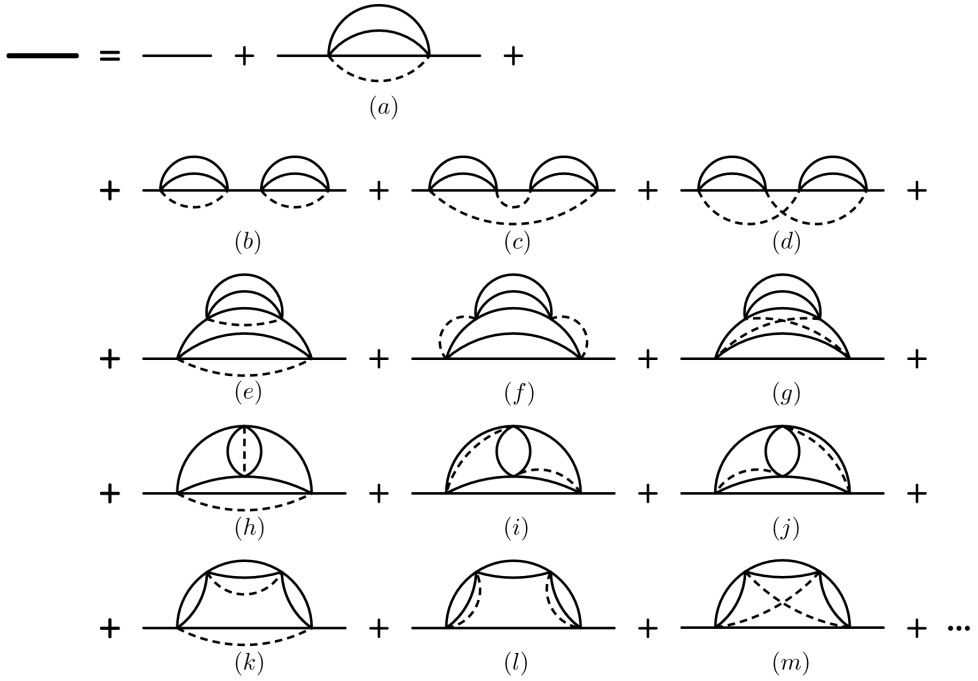


Figure 6: Second-order (a) and fourth-order (b–m) corrections to the propagator. The only diagrams that survive in the limit $N \to \infty$ are (a), (b) and (e). [2]

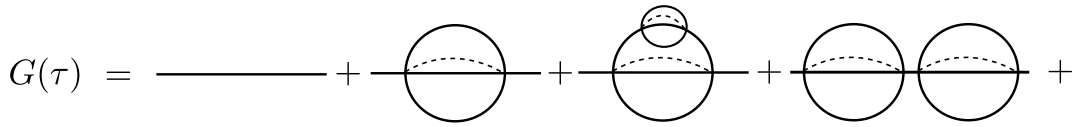
The diagrams are proportional to powers of J and N. As mentioned earlier, the power of N depends on the way the different couplings are contracted together, and one needs to inspect them on a case-by-case basis, computing the averages using eq. (5.1.7). Note that we only ever need to compute the disorder average of two couplings, because of Wick's theorem. The powers of J, on the other hand, are given by the number of vertices of the diagram, each contributing J to the total power.

5.3 Schwinger–Dyson Equation and IR Limit

We turn back to euclidean time for now. Recall the Schwinger-Dyson (SD) equation (3.3.16). We can now write it in the limit $N \longrightarrow +\infty$ using the results derived earlier:

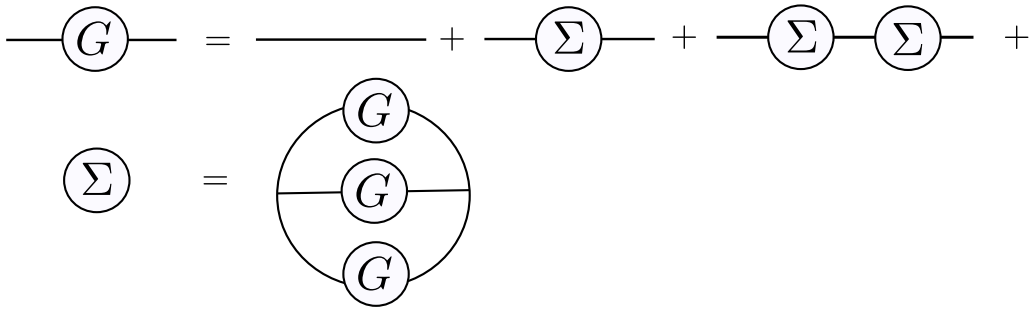
$$\begin{aligned} G(\tau_1, \tau_2) &= G_0(\tau_1, \tau_2) + \int d\tau_3 d\tau_4 G_0(\tau_1, \tau_3)\Sigma(\tau_3, \tau_4)G(\tau_4, \tau_2), \\ \Sigma(\tau_1, \tau_2) &\equiv J^2 G(\tau_1, \tau_2)^3, \end{aligned} \tag{5.3.1}$$

which holds (with the appropriate integration boundaries for τ) for both zero- and finite-temperature propagators. Another way to obtain this result is by looking at Fig. 6 in the large N limit, which looks like [3]:



,

and to summarize it in the closed set of consistency equations [3]:



,

where Σ, as in eq. (5.3.1), is the self energy and is defined to contain all the iterated melon diagrams. Although we give it a fancy name, we'll see later than it is no more than a Lagrange multiplier enforcing a delta function condition.
We introduce a matrix multiplication notation for the bilinear kernels (things we integrate against, basically):

$$(AB)(\tau_1, \tau_2) = \int d\tau_3 A(\tau, \tau_3)B(\tau_3, \tau_2), \tag{5.3.2}$$

From this, we can write

$$\begin{aligned} G &= G_0 + G_0\Sigma G_0 + G_0\Sigma G_0\Sigma G_0 + \cdots \\ &= G_0\big[1 + \Sigma G_0 + \Sigma G_0\Sigma G_0 + \cdots\big] \\ &= G_0\big[1 - \Sigma G_0\big]^{-1} \\ &= \big[(G_0)^{-1} - \Sigma\big]^{-1}, \end{aligned} \tag{5.3.3}$$

where we have resummed the geometric series in the third line. The reason for the appearance of exact propagators G in the equations and schematic diagrams above is that corrections to each propagator of the melon endlessly grow upwards and to the right. In other words, each propagator itself is corrected, making them all (except the first one, see Fig. 7) exact propagators

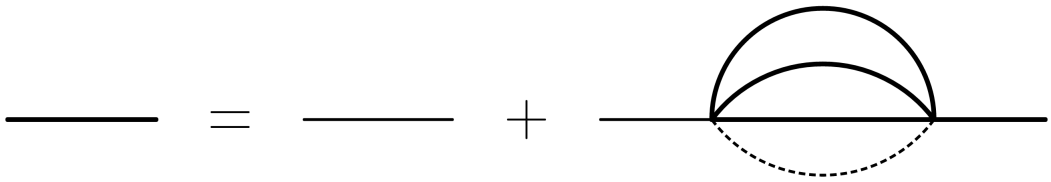


Figure 7: Schwinger–Dyson equation which sums up melonic diagrams. Thin lines correspond to free theory (or tree-level) propagators, while thick lines correspond to exact ones. [2]

There is translational invariance in time, and therefore the kernels only depend on the time difference: $G(\tau_1, \tau_2) = G(\tau_1 - \tau_2)$, $\Sigma(\tau_1, \tau_2) = \Sigma(\tau_1 - \tau_2)$. Coming back to eq. (5.3.1), translational invariance motivates the use of a Fourier transform which gives:

$$G^{-1}(\omega) = -i\omega - \Sigma(\omega). \tag{5.3.4}$$

To get the first term, we use the explicit form of the free propagator

$$G_0(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \frac{1}{2} \mathrm{sgn}\tau = \frac{i}{w} \iff \frac{1}{G_0(\omega)} = -i\omega. \tag{5.3.5}$$

In general, the SD equation is hard to solve analytically. In some cases however, such as the low frequency limit $\omega \ll J$ (i.e $J\tau \gg 1$) and strong coupling $\beta J \gg 1$, it is possible to find an analytical solution. Consider the zero-temperature case $\beta = +\infty$. A dimensional analysis tells us that in the $N \to +\infty$ limit, the exact propagator should decay as $G(\tau) \sim \tau^{-\frac{1}{2}}$. This leads to the lhs vanishing in eq. (5.3.1) and we end up with:

$$0 = G_0(\tau_1, \tau_2) + \int_{-\infty}^{\infty} d\tau_3 \int_{-\infty}^{\infty} d\tau_4 G_0(\tau_1, \tau_3)\Sigma(\tau_3, \tau_4)G(\tau_4, \tau_2). \tag{5.3.6}$$

Differentiating eq. (5.3.6) over τ_1, using $\partial_{\tau_1} G_0(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2)$ and integrating over τ_3, we obtain the identity

$$\int d\tau \Sigma(\tau_1, \tau)G(\tau, \tau_2) = -\delta(\tau_1 - \tau_2). \tag{5.3.7}$$

We get the same result by dropping the free propagator term in eq. (5.3.4):

$$\frac{1}{G(\omega)} \approx -\Sigma(\omega), \quad \text{i.e.} \quad \Sigma(\omega)G(\omega) \approx -1. \tag{5.3.8}$$

Notice that this is simply the Fourier transform of eq. (5.3.7). The limit (infrared or IR, i.e. $J\tau \gg 1$) also leads to invariance under reparametrizations of time of the SD equation (5.3.6). Indeed, for $\tau \to f(\tau)$, $f'(\tau) > 0$, we have

$$\begin{aligned} G(\tau_1, \tau_2) &\to G\left[f(\tau_1), f(\tau_2)\right] f'(\tau_1)^{\Delta} f'(\tau_2)^{\Delta}, \\ \Sigma(\tau_1, \tau_2) &\to \Sigma\left[f(\tau_1), f(\tau_2)\right] f'(\tau_1)^{3\Delta} f'(\tau_2)^{3\Delta}, \end{aligned} \tag{5.3.9}$$

with $\Delta = \frac{1}{4}$ the anomalous conformal dimension. In the general model with a q-legs vertex, this dimension is $\Delta = \frac{1}{q}$. This value is determined so that the previous equation still hold:

$$\begin{aligned} \int df(\tau)\Sigma\left[f(\tau'), f(\tau)\right] G\left[f(\tau), f(\tau'')\right] &= \frac{\int d\tau \Sigma(\tau', \tau)G(\tau, \tau'')}{f'(\tau')^{\Delta} f'(\tau'')^{3\Delta}} \\ &\overset{!}{=} \frac{-\delta(\tau' - \tau'')}{f'(\tau')} \\ &= -\delta\left[f(\tau') - f(\tau'')\right]. \end{aligned} \tag{5.3.10}$$

Note that the reparametrizations should respect the orientation of the euclidean circle. Otherwise, we can't go from the second to the third line. In addition, note that reintroducing the tree-level propagator G_0 (or its inverse, depending on the version of the equation under consideration) breaks the reparametrization invariance.
The appearance of the anomalous conformal dimension in the infrared limit hints at a conformal ansatz to solve the SD equation:

$$G_c(\tau_1, \tau_2) = B\frac{\text{sgn}(\tau_{12})}{|J\tau_{12}|^{2\Delta}}, \tag{5.3.11}$$

with $\tau_{12} = \tau_1 - \tau_2$ and B a constant we need to fix. The following integral will prove to be useful in our computation:

$$\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\frac{\text{sgn}(\tau)}{|\tau|^{2D}} = 2i\Gamma(1-2D)\cos(\pi D)|\omega|^{2D-1}\text{sgn}(\omega). \tag{5.3.12}$$

With it, we find out that the ansatz solves the equation, and that $B = \frac{1}{(4\pi)^{\frac{1}{4}}}$, (more details in [3]) i.e.

$$G_c(\tau) = \frac{1}{(4\pi)^{\frac{1}{4}}}\frac{\text{sgn}(\tau)}{|J\tau|^{2\Delta}}. \tag{5.3.13}$$

We see that this solution decays as $J\tau_{12} \to +\infty$, which is consistent with the approximation in which we obtained the "reduced" SD equation (5.1.6). Since we have reparametrization invariance, and that the map between τ_{line} and τ_{circle} is monotonic and real, we can simply use eq. (5.3.9) and obtain

$$G_c^{\beta}(\tau) = \frac{\pi^{\frac{1}{4}}}{\sqrt{2\beta J}}\frac{\text{sgn}\left(\sin\left(\frac{\pi\tau}{\beta}\right)\right)}{|\sin\left(\frac{\pi\tau}{\beta}\right)|^{2\Delta}}, \quad \tau \in \left[-\frac{\beta}{2}, \frac{\beta}{2}\right). \tag{5.3.14}$$

The sinus inside the sgn function is obtained through the identity $\text{sgn}\left(\tan\left(\frac{\pi\tau}{\beta}\right)\right) = \text{sgn}\left(\sin\left(\frac{\pi\tau}{\beta}\right)\right) = \text{sgn}(\tau)$ for $\tau \in \left[-\frac{\beta}{2}, \frac{\beta}{2}\right)$. Note that both zero- and finite-temperature propagators coincide in the $\tau \ll \beta$ limit. Furthermore, $G_c(\tau)$ and $G_c^{\beta}(\tau)$ are approximately equal to the exact propagators for large times $\tau \gg \frac{1}{J}$. In the ultraviolet limit ($\tau \ll \frac{1}{J}$) exact propagators are approximately equal to the free ones. In between those regions, the exact propagators interpolate between the two options.
We wrap up this subsection by performing the Wick rotation (or analytic continuation) $\tau = it$ and obtaining the two-point function of the finite-temperature theory:

$$G_c^{\beta}(t) = \frac{(\pi)^{\frac{1}{4}}}{\sqrt{2\beta J}}\frac{1}{|\sinh\left(\frac{\pi t}{\beta}\right)|^{2\Delta}} \propto e^{-\frac{2\pi\Delta}{\beta}t}, \quad \text{as} \quad t \gg \frac{1}{J}. \tag{5.3.15}$$

We see that this function "dissipates" after the time $t_d = \frac{\beta}{2\pi\Delta} \sim \beta$, which is unusual for a 1D system.

5.4 Effective Action

We want to derive the effective action and Schwinger-Dyson equations from the path integral, instead of "guessing" them. An effective action is essentially a "reduced" version of the full action, tailored to the problem we have at hand. In other words, we get an action with less degrees of freedom, which is hopefully easier to use in computations.

We start by using the Gaussian distribution assumption for the coupling constants J_{ijkl} to give the averaging rule (for a derivation of $\mathcal{D}J_{ijkl}$, see [3]):

$$\overline{f(J_{ijkl})} \equiv \int \mathcal{D}J_{ijkl} f\left(J_{ijkl}\right), \quad \text{where} \quad \mathcal{D}J_{ijkl} \equiv \exp\left[-\frac{N^3}{12J^2}\sum_{i<j<k<l} J_{ijkl}^2\right] \prod_{i<j<k<l} \sqrt{\frac{N^3}{3!J^2}}\frac{dJ_{ijkl}}{\sqrt{2\pi}}. \tag{5.4.1}$$

We have two ways of realizing the disorder average:

1. Averaging the partition function by directly computing $\overline{Z}$, treating J as a microscopic degree of freedom. This is called annealed disorder.
2. Averaging the free energy $\overline{\log Z}$, which is usually more complicated to do but has more physical relevance in e.g. condensed matter theory, or for the computation of thermodynamic quantities. The computation is done through the use of the "replica trick": $\beta\overline{F} = -\overline{\log Z} = -\lim_{M\to 0} \partial_M \overline{Z^M}$.

Luckily for us, both give the same effective action in the case of the SYK model [2]. For simplicity, we will therefore calculate the disorder average of the partition function

$$\begin{aligned}
\overline{Z} &= \int \mathcal{D}J_{ijkl}\mathcal{D}\chi_i \exp\left[\int d\tau \left(\frac{1}{2}\sum_{i=1}^{N}\chi_i\partial_\tau\chi_i - \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l\right)\right] \\
&= \int \mathcal{D}\chi_i \exp\left[\frac{1}{2}\sum_i \int d\tau \chi_i\partial_\tau\chi_i + \frac{3!J^2}{2N^3}\frac{1}{4!}\sum_{i,j,k,l}\left(\int d\tau \chi_i\chi_j\chi_k\chi_l\right)^2\right] \\
&= \int \mathcal{D}\chi_i \exp\left[\frac{1}{2}\sum_i \int d\tau \chi_i\partial_\tau\chi_i + \frac{NJ^2}{8}\int d\tau d\tau' \left(\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')\right)^4\right] \\
&= \int \mathcal{D}\chi_i \exp\left[\int d\tau d\tau' \left(\frac{N}{2}G_0^{-1}(\tau,\tau')\Xi(\tau,\tau') + \frac{NJ^2}{8}\Xi(\tau,\tau')^4\right)\right]
\end{aligned} \tag{5.4.2}$$

To go from the first to the second line, we performed the Gaussian integrals using

$$\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}. \tag{5.4.3}$$

To perform the next step, we turn the square into a double integral and remember that Majorana operators at different times commute. The factor of N^{-3} is absorbed into the expression, leading to the N factor appearing. Finally, we introduce the inverse free propagator $G_0^{-1}(\tau,\tau')$ and mean field variable $\Xi(\tau,\tau')$:

$$\begin{aligned}
G_0^{-1}(\tau,\tau') &= \delta(\tau-\tau')\partial_\tau, \\
\Xi(\tau,\tau') &= \frac{1}{N}\sum_{i=1}^{N}\chi_i(\tau)\chi_i(\tau').
\end{aligned} \tag{5.4.4}$$

Note that we swept under the rug the fact that Majorana fermions are not represented by either normal or Grassmann numbers, and treat them as the latter (because we can rewrite Majoranas as sums of Dirac fermions, which are represented by Grassmann variables). Now, a

few more not so enlightening steps [2] lead us to:

$$\begin{aligned}\overline{Z} &= \int \mathcal{D}G\mathcal{D}\Sigma \int \mathcal{D}\chi_i \exp\left\{\frac{N}{2}\int d\tau d\tau' \left[\left(G_0^{-1}(\tau,\tau') + \Sigma(\tau,\tau')\right)\Xi(\tau,\tau') + \frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')\right]\right\} \\ &= \int \mathcal{D}G\mathcal{D}\Sigma \int \mathcal{D}\chi_i \exp\left[\frac{1}{2}\sum_i \int d\tau d\tau' \chi_i(\tau)\Big(\delta(\tau-\tau')\partial_\tau + \Sigma(\tau,\tau')\Big)\chi_i(\tau') + \right. \\ &\qquad \left. + \frac{N}{2}\int d\tau d\tau' \left(\frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')\right)\right],\end{aligned} \tag{5.4.5}$$

which we integrate over $\chi_i(\tau)$ to obtain the effective action:

$$\overline{Z} = \int \mathcal{D}G\mathcal{D}\Sigma e^{-NI_{eff}[G,\Sigma]} \tag{5.4.6}$$

$$I_{eff} = -\frac{1}{2}\log\det\Big(-\delta(\tau-\tau')\partial_\tau - \Sigma(\tau,\tau')\Big) + \frac{1}{2}\int d\tau d\tau' \left(\Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{4}G(\tau,\tau')^4\right) \tag{5.4.7}$$

The factor of N in front of the action is important, because it tells us that N acts as $\hbar^{-1}$, and therefore that the $N \to +\infty$ limit is the classical limit. The effective action reproduces the Schwinger-Dyson equation (5.3.1) after variations over G and Σ. In fact, variation w.r.t G yields the self-energy and variation w.r.t Σ yields the equation itself

$$\begin{aligned}\delta_\Sigma I_{eff} &= -\frac{1}{2}\mathrm{tr}\log\left(1 - (-\partial_\tau - \Sigma)^{-1}\delta\Sigma\right) + \frac{1}{2}\int d\tau d\tau' G(\tau,\tau')\delta\Sigma(\tau,\tau') = \\ &= \frac{1}{2}\int d\tau d\tau' \left[G(\tau,\tau') - \left[G_0^{-1}(\tau,\tau') - \Sigma(\tau,\tau')\right]^{-1}\right]\delta\Sigma(\tau,\tau') \quad \Longrightarrow \quad G^{-1} = G_0^{-1} - \Sigma.\end{aligned} \tag{5.4.8}$$

In practice, this result enables us to wave away rigor concerns related to our calculations, because the only thing we want from the effective action is to recover the SD equation. Note that the solution of eq. (5.3.1) is a saddle point of the effective action (5.4.7): it is maximum in G and minimum in Σ. Saddle points are usually to be treated with caution, but in our case the solution of the SD equation does converge to this point [2, 3].
The entropy and free energy of the system can be computed from the effective action, determining the system's thermodynamic properties:

$$\beta F = \beta E_0 + N\left[-S_0 - \frac{2\pi^2 C}{\beta J} + \mathcal{O}\left(\frac{1}{(\beta J)^2}\right)\right] + \frac{3}{2}\log(\beta J) + \mathrm{cst} + \mathcal{O}\left(\frac{1}{N}\right), \tag{5.4.9}$$

with E_0 the ground state energy, $S_0 \approx 0.232$ the low (zero) temperature entropy per site and C a numerical coefficient. The entropy of the system, given by the relationship $\beta F = \beta E - S$, is large ($S \sim N$) even at low temperatures. The actual formula for S_0 is

$$S_0 = \pi\int_\Delta^{\frac{1}{2}} dx \left(\frac{1}{2} - x\right)\tan(\pi x), \tag{5.4.10}$$

with Δ the conformal dimension.

5.5 Schwarzian Theory

We mentioned previously that the reparametrization invariance of the SD equation was broken in the presence of the (inverse) free theory propagator. We will study this phenomenon in more

detail. To do so, let us first change variables in the effective action (5.4.7), $\Sigma \to \Sigma - G_0^{-1}$, and separate it into two parts, one that is conformally invariant and the other which is not: $I_{eff} = I_{CFT} + I_S$

$$I_{CFT} = -\frac{1}{2}\log\det\Big(-\Sigma(\tau,\tau')\Big) + \frac{1}{2}\int d\tau d\tau' \left(\Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{4}G(\tau,\tau')^4\right), \tag{5.5.1}$$

$$I_S = -\frac{1}{2}\int d\tau d\tau' G_0^{-1}(\tau,\tau')G(\tau,\tau'). \tag{5.5.2}$$

We see that the conformal part I_{CFT} is the part that reproduces the IR limit SD equations (5.3.6) and (5.3.8) which are invariant under time reparametrization. The delta function in the inverse tree-level propagator picks up small time differences $|\tau - \tau'| \ll J^{-1}$, which we neglect in the limit. Conformal invariance is therefore an emergent property in the deep IR limit, and it disappears as soon as one moves away from it.
However, the non-invariant part of the effective action can't be neglected, as it contains essential information about the theory. Let us examine this claim in detail, by considering fluctuations of eq. (5.4.7) near the saddle point $(\tilde{G}, \tilde{\Sigma})$. It is important to remember that G_c is not $\tilde{G}$, but its IR limit. The functional integration measure does not change if we parametrize the fluctuations in the form

$$\begin{aligned} G &= \tilde{G} + \frac{\delta G}{|\tilde{G}|}, \\ \Sigma &= \tilde{\Sigma} + |\tilde{G}|\delta\Sigma. \end{aligned} \tag{5.5.3}$$

In that case, we get:

$$\begin{aligned} I_{eff} &\approx \frac{1}{4}\int d\tau_1 d\tau_2 d\tau_3 d\tau_4\, \delta\Sigma(\tau_1,\tau_2)\left(\big|\tilde{G}(\tau_1,\tau_2)\big|\tilde{G}(\tau_1,\tau_3)\tilde{G}(\tau_2,\tau_4)\big|\tilde{G}(\tau_3,\tau_4)\big|\right)\delta\Sigma(\tau_3,\tau_4) \\ &+ \frac{1}{2}\int d\tau_1 d\tau_2 \left(\delta G(\tau_1,\tau_2)\delta\Sigma(\tau_1,\tau_2) - \frac{3J^2}{2}\delta G(\tau_1,\tau_2)^2\right) \\ &\equiv -\frac{1}{12J^2}\langle\delta\Sigma|K|\delta\Sigma\rangle + \frac{1}{2}\langle\delta G|\delta\Sigma\rangle - \frac{3J^2}{4}\langle\delta G|\delta G\rangle. \end{aligned} \tag{5.5.4}$$

Let us clarify the introduced quantities: K is an operator that acts on the space of antisymmetric 2-point functions. Its integral kernel is

$$\begin{aligned} K(\tau_1,\tau_2,\tau_3,\tau_4) &\equiv -3J^2\big|\tilde{G}(\tau_1,\tau_2)\big|\tilde{G}(\tau_1,\tau_3)\tilde{G}(\tau_2,\tau_4)\big|\tilde{G}(\tau_3,\tau_4)\big|, \\ K|A\rangle &= \int d\tau_3 d\tau_4\, K(\tau_1,\tau_2,\tau_3,\tau_4)A(\tau_3,\tau_4). \end{aligned} \tag{5.5.5}$$

This kernel is antisymmetric under exchange of $\tau_1 \leftrightarrow \tau_2$ and $\tau_3 \leftrightarrow \tau_4$, but symmetric under $(\tau_1,\tau_2) \leftrightarrow (\tau_3,\tau_4)$. We also introduce the identity operator

$$\begin{aligned} I(\tau_1,\tau_2,\tau_3,\tau_4) &\equiv \frac{1}{2}\left[\delta(\tau_1-\tau_3)\delta(\tau_2-\tau_4) - \delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3)\right], \\ I|A\rangle &= |A\rangle, \end{aligned} \tag{5.5.6}$$

and we've changed our notation from eq. (5.3.2) to

$$\langle A|B\rangle \equiv \int d\tau_1 d\tau_2 A^*(\tau_1,\tau_2)B(\tau_1,\tau_2). \tag{5.5.7}$$

Since Σ is in essence just a Lagrange multiplier (it does not appear in physical quantities), we can integrate out its fluctuations from the functional integral to obtain the semiclassical approximation

$$I_{eff}[\delta G] = -\frac{1}{N}\log\int \mathcal{D}\delta\Sigma e^{-NI_{eff}[\delta G,\delta\Sigma]} \simeq \frac{3J^2}{4}\langle\delta G|(K^{-1}-I)|\delta G\rangle. \tag{5.5.8}$$

To obtain this expression, recall the Gaussian integral

$$\int d^n x \exp\left[-\frac{1}{2}x^T A x + b^T x\right] = \left(\frac{(2\pi)^n}{\det(A)}\right)^{\frac{1}{2}} \exp\left[b^T A^{-1} b\right], \tag{5.5.9}$$

with $x = \delta\Sigma$, $A = \frac{N}{6J^2}K$ and $b = \frac{N}{2}\delta G$. The semiclassical limit is the $N \to +\infty$ limit, which explains why the only term we are left with is the right-hand side of eq. (5.5.8). Now, let us go to the conformal (IR) limit. The naive thought is that the non-invariant part of the action vanishes so that the action we're left with is I_{CFT} (5.5.1), which means that $\tilde{G} \approx G_c$. In this limit, the fluctuations are then

$$I_{eff}[\delta G] \approx I_{CFT}[\delta G] \approx \frac{3J^2}{4}\langle \delta G | K_c^{-1} - I | \delta G \rangle, \tag{5.5.10}$$

with K_c the kernel (5.5.5) with the saddle propagators $\tilde{G}$ replaced with G_c. As hinted by the word naive, this method does not appropriately treat all fluctuations around the saddle point. To see this, consider fluctuations δG such that conformal symmetry (time reparametrization invariance (5.3.9)) is conserved. We then have $G = G_c + \frac{\delta G}{|G_c|}$ and $\Sigma = J^2 G_c^3 + 3J^2 |G_c| \delta G$, which solve the conformal Schwinger-Dyson equation (5.3.7):

$$\begin{aligned}
&\int d\tau_4 \Big(\Sigma_c(\tau_3,\tau_4) + 3J^2 |G_c(\tau_3,\tau_4)| \delta G(\tau_3,\tau_4)\Big)\left(G_c(\tau_4,\tau_2) + \frac{\delta G(\tau_4,\tau_2)}{|G_c(\tau_4,\tau_2)|}\right) = -\delta(\tau_3-\tau_2) \\
&\int d\tau_4 \left(\frac{\delta G(\tau_4,\tau_2)}{|G_c(\tau_4,\tau_2)|}\Sigma_c(\tau_3,\tau_4) + 3J^2 G_c(\tau_2,\tau_4)|G_c(\tau_3,\tau_4)|\right) = -\delta(\tau_3-\tau_2) - \Sigma_c(\tau_3,\tau_4)G_c(\tau_2,\tau_4) \\
&\int d\tau_3 d\tau_4 \left(\frac{\delta G(\tau_4,\tau_2)}{|G_c(\tau_4,\tau_2)|}\Sigma_c(\tau_3,\tau_4)G_c(\tau_3,\tau_1) + 3J^2 G_c(\tau_1,\tau_3)G_c(\tau_2,\tau_4)|G_c(\tau_3,\tau_4)|\delta G(\tau_3,\tau_4)\right) = 0,
\end{aligned} \tag{5.5.11}$$

To go to the last line, we multiplied by $G_c(\tau_3,\tau_1)$ and integrated over τ_3. This reduces to

$$(I - K_c)\delta G = 0 \tag{5.5.12}$$

This means that this choice of fluctuations leads the conformally-invariant action to vanish, and that we can't neglect the non-invariant part (5.5.2). To take into account, we have to move away from the IR limit and study its behavior under the conformal transformations (5.3.9). Consider the zero-temperature case, and let us start by expanding the conformal propagator:

$$G_c(\tau_1,\tau_2) \to G_c\left[f(\tau_1), f(\tau_2)\right] \approx \frac{\mathrm{sgn}(\tau_1-\tau_2)}{(4\pi)^{\frac{1}{4}} J^{2\Delta}} \frac{f'(\tau_1)^\Delta f'(\tau_2)^\Delta}{|f(\tau_1)-f(\tau_2)|^{2\Delta}}, \tag{5.5.13}$$

around $\tau = \frac{\tau_1+\tau_2}{2}$ in powers of τ_{12}:

$$G(\tau_1,\tau_2) = G_c(\tau_1,\tau_2)\left(1 + \frac{\Delta}{6}\tau_{12}^2 \mathrm{Sch}\left[f(\tau),\tau\right] + \mathcal{O}(\tau_{12}^3)\right), \quad \text{with} \quad \mathrm{Sch}\left(f(z),z\right) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{5.5.14}$$

The reason for this choice of power expansion is because the delta function in the inverse free propagator $G_0^{-1}(\tau_1,\tau_2)$ in eq. (5.5.2) will pick up terms with $\tau_{12} \approx 0$. Substracting the untransformed conformal propagator from eq. (5.5.14) and substituting the result in (5.5.2) yields

$$\begin{aligned}
I_S = -\frac{1}{2}\langle G_0^{-1}|\delta G\rangle &= -\frac{1}{2}\int d\tau d\tau_{12}\, G_0^{-1}(\tau_{12})\tilde{G}(\tau_{12})\left[\frac{\Delta}{6}\tau_{12}^2 \mathrm{Sch}\left[f(\tau),\tau\right] + \mathcal{O}(\tau_{12}^3)\right] \approx \\
&\approx -\frac{\Delta}{12}\int d\tau_{12}\delta(\tau_{12})\partial_{\tau_{12}}\left(\tau_{12}^2\tilde{G}(\tau_{12})\right)\int d\tau\, \mathrm{Sch}\left[f(\tau),\tau\right)] = \\
&= -\frac{1}{J}\underbrace{\frac{\Delta}{12}\int d\eta\delta(\eta)\partial_\eta\left(\eta^2\tilde{G}(\eta)\right)}_{C}\int d\tau\, \mathrm{Sch}\left[f(\tau),\tau\right)],
\end{aligned} \tag{5.5.15}$$

with $\eta = J\tau_{12}$. The integral over $d\eta$ is not defined, because

$$C = \frac{\Delta}{12}\int d\eta\delta(\eta)\left[(\eta^2 g(\eta))'\text{sgn}\eta + \eta^2 g(\eta)\delta(\eta)\right] = \frac{\Delta}{12}\int d\eta g(\eta)\eta^2\delta(\eta)^2 = \frac{\Delta}{12}\delta(0)\cdot g(0)\cdot 0^2\text{“} = \text{”}0\cdot\infty, \tag{5.5.16}$$

where we distinguished the relevant part of the saddle point value, $\tilde{G}(\eta) = g(\eta)\text{sgn}(\eta)$.
There is no way to find an analytical expression for $g(\eta)$ at all times, which makes this undefinition hard to resolve. There are two known ways to patch up the system in order to get a working expression, which both lead to I_S having the form of eq. (5.5.15) with $C \approx 0.48 \times \frac{\Delta}{12}$ [2]. For zero-temperature, it is:

$$I_S \approx -\frac{C}{J}\int_{-\infty}^{\infty}\text{Sch}\left[f(\tau),\tau\right]d\tau, \tag{5.5.17}$$

and for finite-temperature:

$$I_S \approx -\frac{C}{J}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\text{Sch}\left[\tan\left(\frac{\pi\varphi(\tau)}{\beta}\right),\tau\right]d\tau, \tag{5.5.18}$$

with $\varphi(\tau)$ a function that maps the circle onto itself. The numerical coefficient C is the same that appeared in eq. (5.4.9). The reason for that is because the low energy dynamics of the SYK model are determined by the Schwarzian action.

5.6 Four-point Functions

In contrast to the previous subsections which were rather technical, we will now focus on the conceptual role of the four-point functions and on physical understanding. For a proper (and rather tedious) mathematical treatment, see e.g. [2, 16]. Note that this implies that we mostly step away from the path integral formalism used up to this point. Recall the form of the 2-point function in the conformal limit, eq. (5.3.13), and notice that it looks oddly similar to eq. (4.2.9). This hints us towards an interpretation of $\chi_i(\tau)$ as a conformal primary operator of weight (or dimension) $\frac{1}{4}$. We can expect the theory to have other fields that are primaries with nontrivial anomalous scaling dimensions, because the model is interacting. Now, we know from our discussion on conformal field theory that in $d \geq 3$ dimensions, a CFT is determined by the "CFT data", i.e. the set of primary conformal weights and the 3-point function (OPE) coefficients C_{ijk} in eq. (4.3.5). The SYK model being unidimensional leads to a few caveats when applying the results we derived earlier, but the CFT data still allows us to almost solve the theory. Now, as we have seen, the 4-point function can be decomposed in conformal blocks

$$\langle\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle = \langle\chi_i(\tau_1)\chi_i(\tau_2)\rangle\langle\chi_j(\tau_3)\chi_j(\tau_4)\rangle\sum_h (C^h_{\chi_i\chi_j})^2 z^h\,{}_2F_1(h,h,2h,z), \tag{5.6.1}$$

where we replace $f_{\phi\phi\Phi}$ by $C^h_{\chi_i\chi_j}$ and Φ with h in (4.5.3) so that h runs over conformal primaries, z is the anharmonic ratio

$$z = \frac{\tau_{12}\tau_{24}}{\tau_{13}\tau_{24}}, \tag{5.6.2}$$

and $z^h\,{}_2F_1(h,h,2h,z)$ is the conformal block summing the contribution of the $SL(2,\mathbb{R})$ descendants of the primary h. Although we haven't mentioned it before, the appearance of $SL(2,\mathbb{R})$ is normal, as it is the group associated with conformal symmetry in one dimension. Writing the four-point function in this form would allow us to simply read off the scaling dimensions h and the OPE coefficients $C^h_{\chi_i\chi_j}$.
As in most of the previous computations, we're interested in the large N limit, where the model

has the effective action (5.4.7). Now, notice that the four-point function of fermions is a two-point function of the bilocal G's, which in this limit is the product of the saddle-point values $\tilde{G}$

$$\begin{aligned}\frac{1}{N^2}\sum_{i,j}\langle \mathrm{T}\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle &= \langle G(\tau_{12})G(\tau_{34})\rangle \\ &\sim \tilde{G}(\tau_{12})\tilde{G}(\tau_{34}).\end{aligned} \tag{5.6.3}$$

In the IR limit and the language of conformal block decomposition (as in eq. (5.6.1)), this is simply the contribution of the identity operator $h = 0$. This means that the OPE coefficients $C^h_{\chi_i\chi_j}$ need to be suppressed by $\frac{1}{\sqrt{N}}$ for all the other operators. To get the leading quantum correction, we study, as was done earlier for the Schwarzian action, the small fluctuations around the saddle in eq. (5.4.7). We parametrize the corrections as

$$\begin{aligned}\frac{1}{N^2}\sum_{i,j}\langle \mathrm{T}\chi_i(\tau_1)\chi_j(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle &= \tilde{G}(\tau_{12})\tilde{G}(\tau_{34}) \\ &\times\left[1+\frac{1}{N}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4)+...\right].\end{aligned} \tag{5.6.4}$$

What we are looking for is $\mathcal{F}$. The fluctuations have the same form as the ones from the Schwarzian action derivation

$$\begin{aligned}G &= \tilde{G}+\frac{\delta G}{|\tilde{G}|}, \\ \Sigma &= \tilde{\Sigma}+|\tilde{G}|\delta\Sigma,\end{aligned} \tag{5.6.5}$$

such that

$$\frac{1}{N^2}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) = \tilde{G}(\tau_{12})^{-\frac{1}{2}}\tilde{G}(\tau_{34})^{-\frac{1}{2}}\langle\delta G(\tau_1,\tau_2),\delta G(\tau_3,\tau_4)\rangle. \tag{5.6.6}$$

Taking the expanded effective action (5.5.4), we remember that we can compute the two-point function in eq. (5.6.6) through the action (5.5.8). Thus we have

$$\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) = \tilde{G}(\tau_{12})^{-\frac{1}{2}}(K_c^{-1}-I)^{-1}\tilde{G}(\tau_{12})^{-\frac{3}{4}}. \tag{5.6.7}$$

We now see that the object we need to understand is the inverse of $K_c^{-1}-I$. Notice that we can write

$$(K_c^{-1}-I)^{-1} = (I-K_c)^{-1}K_c = \sum_n K_c^n K_c, \tag{5.6.8}$$

which, following the diagrammatics introduced previously, can be represented by ladder diagrams with n rungs such as the ones in Fig. 8. In the same way as the melonic diagrams for two-point functions, the ladders give the leading $\frac{1}{N}$ contribution to the four-point function.

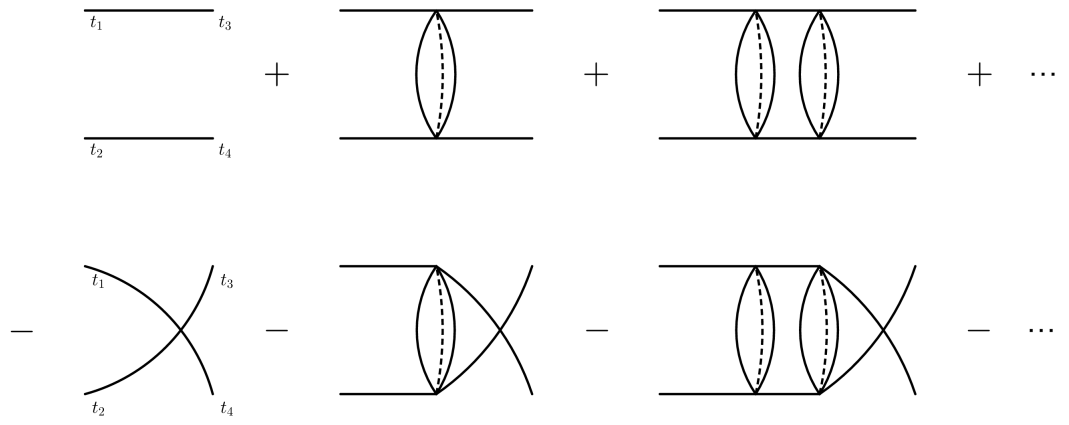


Figure 8: Sum of the ladder diagrams which contribute to $\mathcal{F}$ [2]

To understand the inverse of $K_c - I$, we want to diagonalize it and be able to write it as a spectral decomposition by finding a complete set of eigenfunctions to the kernel K_c. If we find such functions $\Psi_\lambda(\tau_1, \tau_2)$ which satisfy

$$K_c \Psi_\lambda = k_c(\lambda)\Psi_\lambda, \tag{5.6.9}$$

and a completeness relation

$$\mathbb{1} = \sum_\lambda \frac{1}{\langle \Psi_\lambda | \Psi_\lambda \rangle} |\Psi_\lambda\rangle\langle\Psi_\lambda|, \tag{5.6.10}$$

with $\mathbb{1}$ the identity on the space of antisymmetric functions of two variables, we can write the expansion

$$(K_c - I)^{-1} = \sum_\lambda \frac{k_c(\lambda)}{1 - k_c(\lambda)} \frac{1}{\langle \Psi_\lambda | \Psi_\lambda \rangle} |\Psi_\lambda\rangle\langle\Psi_\lambda|. \tag{5.6.11}$$

If this all feels a little sketchy for now, it is normal. We need to specify what values λ runs over and what the inner product $\langle .|. \rangle$ is.
We start by using conformal invariance to find the eingenfunctions of K_c. In particular, the $SL(2, \mathbb{R})$ generators

$$L'_{-1}\tau = \partial_\tau, \quad L_0^\tau = -\tau\partial_\tau - \Delta, \quad L_1^\tau = \tau^2\partial_\tau + 2\Delta\tau, \tag{5.6.12}$$

which have the commutation relations

$$[L_m^\tau, L_n^\tau] = (m - n)L_{m+n}^\tau \quad \text{for} \quad m, n = -1, 0, 1, \tag{5.6.13}$$

satisfy $(L_m^{\tau_1} + L_m^{\tau_2})K_c = K_c(L_m^{\tau_3} + L_m^{\tau_4})$. Thus K_c commutes with the Casimir operator

$$C^{\tau_1,\tau_2} = (L_0^{\tau_1} + L_0^{\tau_2})^2 - \frac{1}{2}(L_{-1}^{\tau_1} + L_{-1}^{\tau_2})(L_1^{\tau_1} + L_1^{\tau_2}) - \frac{1}{2}(L_1^{\tau_1} + L_1^{\tau_2})(L_{-1}^{\tau_1} + L_{-1}^{\tau_2}), \tag{5.6.14}$$

and eigenfunctions of C^{τ_1,τ_2} are also eigenfunctions of K_c. Note that, altough we made it disappear, the conformal dimension Δ of K_c is $\frac{1}{2}$. Now, it is much easier to diagonalize the Casimir. In particular,

$$C^{\tau_1,\tau_2}\Psi_\lambda = \lambda(\lambda - 1)\Psi_\lambda \tag{5.6.15}$$

and eq. (5.6.9) both yield the same solutions. Since there are multiple solutions, the general one is a linear combination of conformally invariant three-point functions

$$\Psi_\lambda(\tau_1, \tau_2) = \int d\tau_0 g_\lambda(\tau_0) f_\lambda^{\tau_0}(\tau_1, \tau_2), \quad f_\lambda^{\tau_0}(\tau_1, \tau_2) = \frac{\mathrm{sgn}\tau_{12}}{|\tau_{01}|^\lambda |\tau_{02}|^\lambda |\tau_{12}|^{1-\lambda}}. \tag{5.6.16}$$

Any of the solutions allows us to determine $k_c(\lambda)$ by evaluating eq. (5.6.9), which gives us

$$k_c(\lambda) = -\frac{3}{2}\frac{\tan(\frac{\pi}{2}(\lambda - \frac{1}{2}))}{\lambda - \frac{1}{2}}. \tag{5.6.17}$$

We now want to find a subset of these solutions which form a complete basis of antisymmetric eigenfunctions for our choice of inner product. We'll skip the details, but it is equivalent to requiring that $\lambda(\lambda - 1) \in \mathbb{R}$, and gives the constraints:

$$\lambda = \frac{1}{2} + is, \quad s \in \mathbb{R} \quad \text{and} \quad \lambda = 2n, \quad n \in \mathbb{Z}^+. \tag{5.6.18}$$

This leads to the following expression for the inverse:

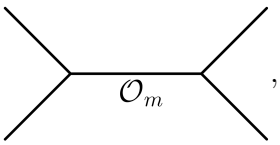
$$(K^{-1} - I)^{-1}(\tau_1, ..., \tau_4) = \int_{-\infty}^{\infty} ds \alpha_{\lambda_s}(\tau_1, ..., \tau_4) \frac{k_c(\lambda_s)}{1 - k_c(\lambda_s)} + \sum_{n=1}^{\infty} \beta_{\lambda_n}(\tau_1, ..., \tau_4) \frac{k_c(\lambda_n)}{1 - k_c(\lambda_n)}, \tag{5.6.19}$$

where the s and n subscripts indicate which subset of the λ's is summed (or integrated) over. The functions α and β account for the dependence on the eigenfunctions and the measure factors coming from the inner product. It should also be noted that the $n = 1$ term in the discrete part diverges, because $k_c(2) = 1$. This could have been expected, as this corresponds to the reparametrizations modes which are zero modes of the action. These values actually have to be shifted by $(\beta J)^{-1}$ corrections that come from the Schwarzian part of the effective action. In fact, the full $\mathcal{F}$ is actually of the form

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{CFT} + \mathcal{F}_S \tag{5.6.20}$$

We will come back to this later, when we discuss (O)TOC's. Note that we will however not go in the full details of the derivation of $\mathcal{F}_S$, the so-called soft mode contribution.
Now, coming back to our inverse kernel and our attemps to make it look like eq. (5.6.1), we still need to complete a few steps. These are outlined in [2], but basically boil down to rewriting eq. (5.6.19), interpreting it as a contour integral and using relatively standard tricks to end up with

$$\mathcal{F}_{CFT} = \sum_{m=1}^{\infty} c_{h_m}^2 z^{h_m} {}_2F_1(h_m, h_m, 2h_m, z) = \sum_{m=0}^{\infty} c_m^2$$



, (5.6.21)

where the c_{h_m} are known analytic functions of the conformal weights h_m, for which there is no explicit expression. We have the relationship $C^{h_m}_{\chi_i\chi_j} = \frac{c_{h_m}}{\sqrt{N}}$. Note that for $m \gg 1$, we have

$$h_m \approx 2\Delta + 1 + 2m + \frac{3}{2\pi m}, \tag{5.6.22}$$

which shows that the operators of the OPE are built from two fermion fields, $2m+1$ derivatives and an anomalous part that corresponds to the interactions:

$$\mathcal{O}_m = \sum_{i=1}^{N} \sum_{k=0}^{2m+1} d_{mk} \partial_\tau^k \chi_i \partial_\tau^{2m+1-k} \chi_i, \tag{5.6.23}$$

with d_{mk} numerical coefficients. A more complete treatment of these operators can be found in [17].

5.7 TOC and OTOC

We come back to the comments made around eq. (5.6.20). The TOC and OTOC are respectively defined as

$$\text{TOC(t)} := \frac{1}{N^2} \sum_{i,j}^{N} \text{tr}\Big[\chi_i(t)\rho^{\frac{1}{2}}\chi_i(t)\chi_j(0)\rho^{\frac{1}{2}}\chi_j(0)\Big], \tag{5.7.1}$$

and

$$\text{OTOC(t)} := \frac{1}{N^2} \sum_{i,j}^{N} \text{tr}\Big[\rho^{\frac{1}{4}}\chi_i(t)\rho^{\frac{1}{4}}\chi_j(0)\rho^{\frac{1}{4}}\chi_i(t)\rho^{\frac{1}{4}}\chi_j(0)\Big]. \tag{5.7.2}$$

In both the equations above, the density matrix is $\rho \equiv \frac{1}{Z}e^{-\beta H}$, with Z the partition function (5.4.6). It can be shown that the leading order corrections $(1/N)$ are given by $\mathcal{F}_S$ and are [2]:

$$\begin{aligned} \text{TOC}(t) &= \tilde{G}\left(\frac{\beta}{2}\right)\tilde{G}\left(\frac{\beta}{2}\right) + \mathcal{F}\left(\frac{\beta}{2} + it, it, 0, -\frac{\beta}{2}\right) \\ &\approx \tilde{G}\left(\frac{\beta}{2}\right)\tilde{G}\left(\frac{\beta}{2}\right) + \mathcal{F}_S\left(\frac{\beta}{2} + it, it, 0, -\frac{\beta}{2}\right) \approx \frac{\sqrt{\pi}}{2\beta J} + \frac{\text{const}}{N}. \end{aligned} \tag{5.7.3}$$

and

$$\begin{aligned}\text{OTOC}(t) &= \tilde{G}\left(\frac{\beta}{2}\right)\tilde{G}\left(\frac{\beta}{2}\right) + \mathcal{F}\left(\frac{\beta}{4}+it, -\frac{\beta}{4}+it, 0, -\frac{\beta}{2}\right) \\ &= \tilde{G}\tilde{G} + \mathcal{F}_S + \mathcal{F}_{CFT} + \mathcal{F}_0 + \mathcal{O}\left(\frac{1}{N^2}\right) \\ &\approx \frac{\sqrt{\pi}}{2\beta J}\left[1 - \frac{\Delta^2}{2C}\frac{\beta J}{N}e^{\frac{2\pi}{\beta}t}\right], \quad \text{for} \quad \beta \ll t \ll \beta \log\frac{N}{\beta J}\end{aligned} \tag{5.7.4}$$

As mentioned in the previous subsection, the $\mathcal{F}_{CFT}$ contributions are multiplied by a factor $(\beta J)^{-1}$, making them subleading. In the TOC case, they do not contribute meaningfully, and eq. (5.7.3) is the correct approximation. Still, one should account for them for the OTOC, and we get

$$\text{OTOC}(t) \approx \frac{2\pi}{\beta J}\left[1 - \text{const}\frac{\beta J}{N}e^{\kappa t}\right], \quad \text{for} \quad \beta \ll t \ll \beta \log\frac{N}{\beta J}, \tag{5.7.5}$$

where "const" > 0 and κ is the corrected Lyapunov exponent

$$\kappa \approx \frac{2\pi}{\beta}\left(1 - \frac{6.05}{\beta J} + ...\right). \tag{5.7.6}$$

Note the constrained time interval in which all these expressions hold. At greater times, other types of diagrams (multiple parallel ladders, for one) generate significant corrections to the OTOC.

6 AdS/CFT and the gravity dual of the SYK model

The motivation for the study of the SYK model and in particular its time-ordered and out-of-time-ordered correlation functions in this project is the link that can be made with 2d dilaton gravity. This connection is part of a much wider and deeper topic which originates from string theory, namely the anti-de Sitter/conformal field theory (AdS/CFT) correspondence.
String theory is not the subject of this project, and neither are general relativity and the specific model of two-dimensional dilaton gravity (Jackiw-Teitelboim, or JT). This theory, which appears in the near-horizon limit of an extremal black hole, has the remarkable property of exhibiting chaotic behavior when coupled to matter. In fact, the correlation functions of the boundary operators corresponding to bulk matter fields in this gravity model are similar to the correlators of the χ_i's in the SYK model. It is important to note that the two models only coincide in the low energy limit.
In the following, we'll use the results outlined in [2] to compare the (out-of-)time-ordered correlation functions of both models, and then expand on a possible bulk dual of the SYK model by performing a brief review of the results of [17, 18, 19].

6.1 Comparison with 2dDilaton Gravity

Let us start by recalling the equations derived in the previous section. In the low (but non-zero) temperature and large coupling limit $1 \ll \beta J \ll N$, the exact propagator (5.3.14) (which is conformal, as this is the IR limit of the finite-temperature theory) exponentially decays in Lorentzian time:

$$G_c^\beta(t) \approx \frac{\pi^{\frac{1}{4}}}{\sqrt{2\beta J}} \frac{\mathrm{sgn}(t)}{|\sinh\left(\frac{\pi t}{\beta}\right)|^{\frac{1}{2}}} \sim e^{-\frac{t}{t_d}}, \quad \text{for} \quad t \gg t_d = \frac{2\beta}{\pi}. \tag{6.1.1}$$

The time-ordered correlator (5.7.3) is then approximately a product of 2-point functions:

$$\mathrm{TOC}(t) \approx G_c^\beta\left(-\frac{i\beta}{2}\right) G_c^\beta\left(-\frac{i\beta}{2}\right) \approx \frac{\sqrt{\pi}}{2\beta J}, \quad \text{for} \quad t \gg t_d, \tag{6.1.2}$$

and the OTOC (5.7.4) rapidly saturates, with the approximate expression:

$$\mathrm{OTOC}(t) \approx \frac{\sqrt{\pi}}{2\beta J}\left[1 - \frac{\Delta^2}{2C}\frac{\beta J}{N} e^{\kappa t}\right], \quad \text{for} t_d \ll t \ll t_* = \beta \log\left(\frac{N}{\beta J}\right), \tag{6.1.3}$$

with C is a positive numerical constant, $\Delta = \frac{1}{4}$ is the anomalous conformal dimension we found in the IR and κ is the Lyapunov exponent given by eq. (5.7.6).
Now, as mentioned earlier, one can find a relationship between the SYK model and JT gravity. In particular, the effective action of JT gravity is the same as the one from the SYK model in the IR limit. One then finds that in the semiclassical limit, the correlation functions (most notably the four-point ones) of JT gravity behave similarly to the ones of the SYK model [2, 3]:

$$G(t) \approx \left(\frac{\pi}{\beta \sinh\frac{\pi t}{\beta}}\right)^{2\Delta} \sim e^{-t/t_d}, \qquad \text{for} \quad t \gg t_d = \frac{\beta}{2\pi\Delta}, \tag{6.1.4}$$

$$\mathrm{TOC}(t) \approx \left(\frac{\pi}{\beta}\right)^{4\Delta}, \qquad \text{for} \quad t \gg t_d, \tag{6.1.5}$$

$$\mathrm{OTOC}(t) \approx \left(\frac{\pi}{\beta}\right)^{4\Delta}\left[1 - 2\Delta^2 \frac{\beta G_N}{\bar{\phi}_r} e^{\kappa t}\right], \qquad \text{for} \quad t_d \ll t \ll t_* = \beta \log\frac{\bar{\phi}_r}{\beta G_N}, \tag{6.1.6}$$

with Δ the conformal dimension of operators dual to free matter fields in the bulk of the theory, G_N the 2d Newton constant, $\bar{\phi}_r$ the boundary value of the dilaton and $\kappa \approx \frac{2\pi}{\beta}$ the Lyapunov exponent.
While these comparisons seem encouraging, it is important to notice that the SYK model only exhibits this kind of behavior in the low-energy limit (where it is described by the Schwarzian action) and that therefore JT gravity is not a complete gravity dual.

6.2 Complete Dual?

As JT gravity does not incorporate the behavior at higher energies, it begs the question "Does such a dual exist?" and, if yes, "What does it look like?". Unfortunately for us, both questions are still open. The answer to the first one is believed to be that yes, every CFT has an AdS dual, although not necessarily one that makes calculations easier (i.e. whether it is weakly-coupled and has low curvature). Now, as far as we could find, developments regarding the second question seem to have stalled a little since 2018 in the case of the regular SYK model we treated. Back in 2017, Gross and Rosenhaus [17] have detailed a method to compute all-point correlation functions in the SYK model from the fermion two-,four- and six-point functions, which in principle would allow to determine the complete tree-level AdS dual lagrangian. As they however note, there should be a stringy interpretation of the bulk, which has since then not been formulated. Other works [18, 19] have tried to identify the bulk spacetime of the model, and found out that redefining spacetime through non-local transformations allowed to resolve the issue of the canonical spectrum of the SYK being lorentzian even though the CFT is euclidean by placing the theory in an euclidean AdS dual setting. In addition, they have provided an alternate three-dimensional view of the dual spacetime at the linearized level, opening up a potential new path to obtaining a complete dual.
Overall, there is no definite answer to what the dual actually is past the leading corrections. There are multiple culprits that can be blamed for this, the randomness of the couplings being a major one, as their bulk interpretation is at best unclear and at worst not understood at all. Sarosi also reported that a string theory bulk interpretation might not even be possible due to the small number of states present in the SYK model [3].

7 Conclusion

Through reminding some classical field theory results such as Noether's theorem and the introduction of Lie theory and quantum field theory, we explored conformal field theory in $d \geq 3$ dimensions and some of its properties, which then allowed us to analyze the SYK model and in particular its near-conformal behavior in the IR. Most notably, the form of two-point functions, the effective and Schwarzian actions of the model were determined. Subsequently, a more qualitative approach was chosen to describe the conduct of four-point functions when taking the leading-order corrections into account, and results regarding (out-of-)time-ordered correlation functions were cited in order to give context to the SYK/JT connection. It was stressed that it is not an AdS/CFT correspondence in the usual sense, and a follow-up discussion about the advances of research regarding a complete bulk dual interpretation of the SYK model was done. Although these seem to have stalled, it is important to note that the SYK model and its generalizations are still active topics of research and are being used in many interesting applications, for example in the study of traversable wormholes [20, 21, 22, 23], strange metals [24, 25, 26], the Eigenstate Thermalization Hypothesis [27], and so on.

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