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On the capacitance of a cube

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Abstract

The capacitance of a metallic cube is analyzed using the method of subareas and modern computing tools. This calculation extends the work of Reitan and Higgins in order to obtain a more accurate value for the capacitance.

Keywords: Capacitance; Method of moments; Method of subareas; MATLAB

1. Introduction

In an early paper, Reitan and Higgins analyzed the capacitance of a cube and introduced and advanced the technique of using the *method of subareas* in order to calculate the capacitance of a cube [1]. This method was later extended by these authors and others to examine the capacitance of other irregular shapes. At the time of these calculations, the analysis was carried out using desk calculators and it was very tedious to perform. The purpose of the present note is to update this calculation using techniques and tools that are available at the present time.

At the current time, students typically encounter the *method of moments* or the *measured equation of invariance method* early in their electromagnetic theory courses [2–7]. Both methods grew out of the *method of subareas*. Hence, we believe that the calculation of the capacitance of a cube can now be introduced at the undergraduate level. Using the modern powerful tools that are available to students at the present time, this can also be a motivational mechanism for the student.

In Section 2, the mathematical formalism required to analyze the capacitance of a cube is presented. The results of the calculation and the concluding comments are presented in Section 3.

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2. Mathematical formalism

In order to use the *method of subareas*, we divide the total surface area a^2 of one side of a cube into a number of subareas as shown in Fig. 1. The area of a subarea is $\Delta A = (a/N)^2$. In addition, the origin of the coordinate system has been chosen in order that no corners of the cube are centered at the origin. Each of the six sides of the cube have been divided into $N \times N$ subareas. We initially assume that the total charge Q is uniformly distributed on the surface of the cube. However, in order to calculate the potential we then assume that the total charge in a particular subarea $\Delta Q(i,j,k)$ is concentrated at its center and has a value that is equal to

$$\Delta Q(i,j,k) = \frac{Q}{6a^2}\Delta A = \rho_{\mathrm{s}}(i,j,k)\left(\frac{a}{N}\right)^2 \tag{1}$$

We will now make an assumption at this stage concerning the symmetry of a cube. If we can calculate the charge on one of these sides, we expect that all six sides will be identical. This is the only assumption of symmetry that we are making.

We calculate the electrical potential on one of the surfaces, say $i = 1$ due to all of the other $6 \times N \times N$ distributed charges. The electrical potential at the center of a subarea caused by the electrical charge in another subarea is given by

$$V(i=1,j,k) = \sum_{i=1}^{6}\sum_{j=1}^{\hat{N}}\sum_{k=1}^{N}\frac{1}{4\pi\varepsilon_0}\int\int_{\Delta A}\frac{\rho_{\mathrm{s}}(i,j,k)}{[r_{\mathrm{a}} - r_{\mathrm{b}}]}\,\mathrm{d}\zeta\,\mathrm{d}\xi \tag{2}$$

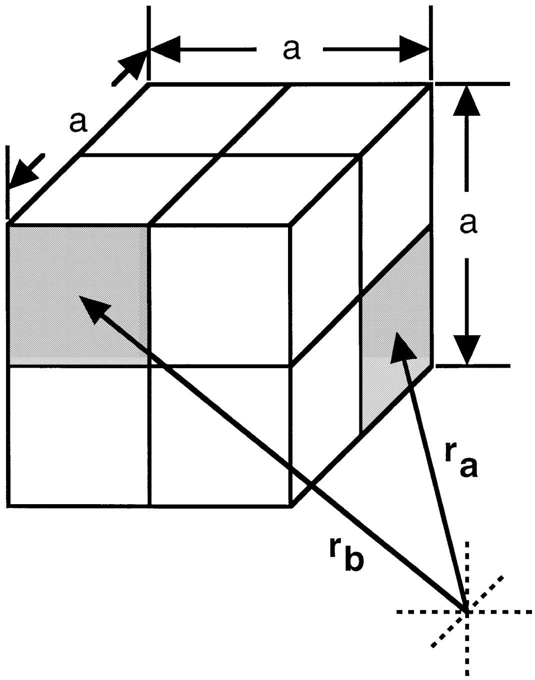

Fig. 1. Configuration of the subareas of a cube with $N = 2$.

Using this notation for the voltage and for the charge, the letter "i" refers to the side that is being evaluated. The remaining two letters locate the particular subarea.

We evaluate the integral by replacing it with a summation over all of the six subareas. In Fig. 1, there will be $6 \times N \times N = 24$ terms that will contribute to the potential $V(i=1,j,k)$ at one subarea.

$$V(i=1,j,k) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{6}\sum_{j=1}^{\hat{N}}\sum_{k=1}^{N}\frac{\rho_{\mathrm{s}}(i,j,k)}{[r_{\mathrm{a}} - r_{\mathrm{b}}]}\left(\frac{a}{N}\right)^2 \tag{3}$$

Twenty-three of the terms in Fig. 1 can be easily evaluated and will create no problem in the analysis. However, the 24th term will have a singularity. This singularity arises when we try to evaluate the potential at the same subarea where the charge resides, $V(i=1,j,j)$. There are standard methods that will be employed in order to remove the singularity [4–7]. The square subarea is replaced with a circular subarea that has the same area ΔA as the square subarea. Rather than evaluating the potential at the center of this circle where the charge resides, we evaluate the potential at the edge of the circle. We evaluate this potential in some detail:

$$\begin{aligned} V(i=1,j,j) &= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{b}\frac{\rho_{\mathrm{s}}(i=1,j,j)r\,\mathrm{d}r\,\mathrm{d}\varphi}{r} = \frac{\rho_{\mathrm{s}}(i=1,j,j)}{4\pi\varepsilon_0}(2\pi b) \\ &= \frac{\rho_{\mathrm{s}}(i=1,j,j)}{4\pi\varepsilon_0}\left(2\pi\sqrt{\frac{\Delta A}{\pi}}\right) = \frac{\rho_{\mathrm{s}}(i=1,j,j)}{4\pi\varepsilon_0}(2\sqrt{\pi})\left(\frac{1}{\sqrt{\Delta A}}\right)\Delta A \\ &= \frac{\rho_{\mathrm{s}}(i=1,j,j)}{4\pi\varepsilon_0}(2\sqrt{\pi})\left(\frac{1}{\sqrt{\left[\frac{a}{N}\times\frac{a}{N}\right]}}\right)\Delta A = \frac{\rho_{\mathrm{s}}(i=1,j,j)}{4\pi\varepsilon_0}(2\sqrt{\pi})\left(\frac{N}{a}\right)\Delta A \end{aligned} \tag{4}$$

Therefore, all of the potentials $V(i=1,j,k)$ can now be evaluated in terms of the charges that reside on each individual subarea of the six surfaces. The incremental capacitance that exists between this subarea and the rest of the cube can also be evaluated. The voltage on each subarea will be set equal to $V=1$. The charge densities can be calculated by inverting the resulting matrix.

3. Results

In order to evaluate the capacitance of a cube, we use MATLAB. The dot-m program that effects this calculation, is available from the authors. Our choice of this software program is based on its wide availability in universities and industries and its ease of use. In the original calculation, Reitan and Higgins were forced to examine regions of symmetry that would exist at the eight corners of the cube. This symmetry allowed them to analyze a larger number of total subareas ($6 \times N \times N = 216$). It also, unfortunately, required that the cube be subdivided only into an even number of subareas (6, 24, 216). The MATLAB program does not make this restriction and the maximum number of subareas is dictated only by the speed and the availability of the computer. The calculations presented below have included both odd and even divisions and the number of subareas range from 6 to 2400.

In Fig. 2, the calculated charge distribution on one of the plates is calculated. Initially, the voltage on all of the some areas was specified to have a numerical value of 1. The calculation was repeated for two values of the parameter which indicates the number of subareas $N \times N$ in which the area of one side has been subdivided. The results of this calculation indicate that the distribution of charge on that plate is very nonuniform. In addition, since the voltage on each area was chosen to have the numerical value of 1, these graphs also indicate the spatial distribution of the incremental capacitances.

The total capacitance of the cube as a function of the number of subareas is shown in Fig. 3. As originally noted by Reitan and Higgins, the capacitance approaches a constant value of $C = 0.7283a$ μμf as the value of the number of subareas increases, where a is the size of the cube. This value should be updated to be $C = 0.7345a$ μμf.

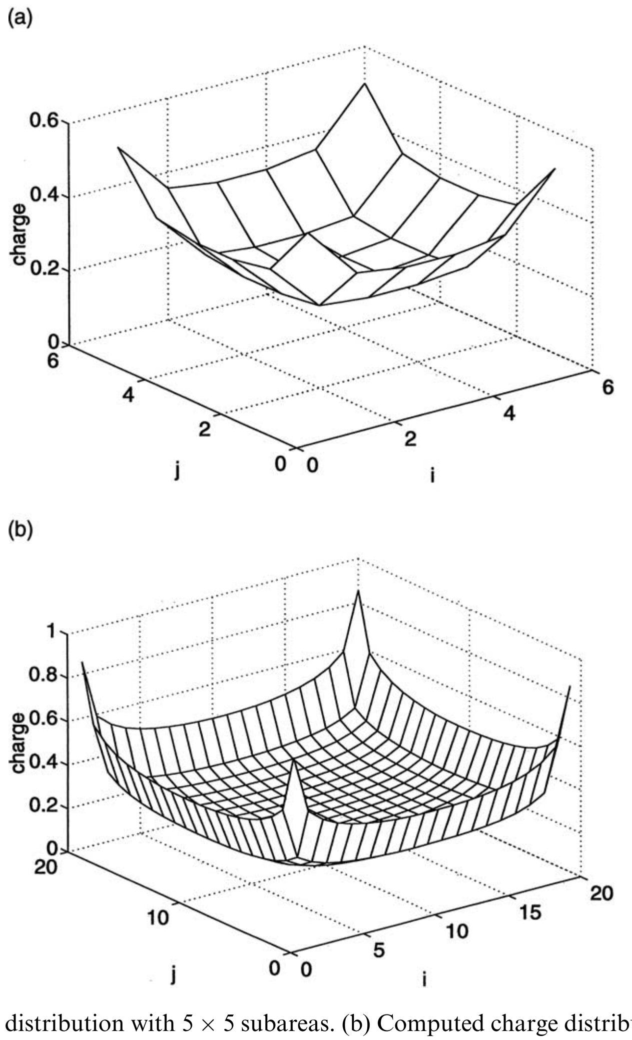

Fig. 2. (a) Computed charge distribution with 5×5 subareas. (b) Computed charge distribution with 19×19 subareas.

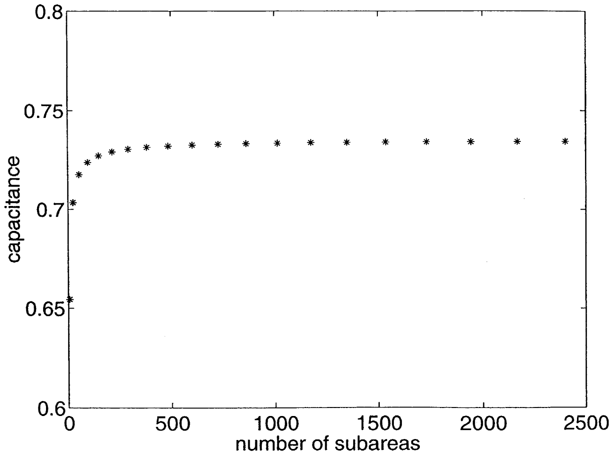

Fig. 3. Computed capacitance of a unit cube ($a = 1$) as a function of the total numbers $6 \times N \times N$ subareas on the unit cube.

We remark that the technique introduced by Reitan and Higgins in Ref. [1] later formed the foundation of the *method of moments* that is now widely used in electromagnetic theory calculations [2–7].

This calculation using present-day computers is a calculation that many students in electromagnetic theory can easily perform. The students who perform those calculation will be introduced to many modern techniques that they may later encounter in practice.

This paper is dedicated to the late Professor Thomas J. Higgins who passed away in the summer of 1999 after a lengthy and very productive academic career at the University of Wisconsin.

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