Journal of Electrostatics 83 (2016) 22–27

Contents lists available at ScienceDirect

Journal of Electrostatics

journal homepage: www.elsevier.com/locate/elstat

Capacitance and potential coefficients at large distances

Francesco Maccarrone [a, *], Giampiero Paffuti [a, b]

[a] University of Pisa, Physics Department, Largo Pontecorvo 7, Pisa, Italy
[b] INFN sezione di Pisa, L.go Pontecorvo 3 Ed. C, I-56127, Pisa, Italy

ARTICLE INFO

Article history:
Received 9 May 2016
Received in revised form
9 July 2016
Accepted 9 July 2016

Keywords:
Green's function method
Coefficients of capacitance
Coefficients of potential
Asymptotic formulas for capacitance

ABSTRACT

The design of capacitive sensors and devices for new and emerging applications would benefit from simple and reliable methods to estimate the capacitance between conductors in terms of the capacitance of the isolated bodies and of the distance between them. The coefficients of potential and capacitance of a pair of conductor are approximated with the first terms of an expansion formula in the inverse of their distance. The form given applies to conductors of generic shapes and position in space. A comparison with the exact value for two spheres shows agreement even for rather small distances.

1. Introduction

It is well-known that a formal calculation of capacitance coefficients of a system of conductors can be achieved through the Green's function method [1], but closed and exact analytic expressions are possible only for certain symmetries. Moreover, numerical algorithms [2] are generally cumbersome ad time-consuming, being bound, for each single calculation, to an individual choice of parameters. On the other hand, the wide spread of capacitive sensors and devices makes it interesting to have simple and roughly reliable methods to estimate the capacitance of a couple of conductors in terms of the capacitance of the isolated bodies and of the distance among the conductors. Particularly, it could be useful to dispose of analytical forms describing accurate asymptotic behaviours when a quite wide separation among electrodes with respect to their dimensions occurs. The application of the Green's function method in conjunction with a perturbative approach gives the first terms of an expansion of the coefficient of capacitance for a generic pair of conductors in power of the inverse of the distance of their centres of charge, having as parameters the electrical polarizability, the electric quadrupole coefficient and the intrinsic capacitance of each of the two bodies.

Asymptotic formulas for coefficients of capacitance and potential in particular cases were given since the unprecedented work of J.C.Maxwell [3] both in the near limit and in the far limit regimes, i.e. when the two conducting bodies are very near or very far each other. Specifically, in Maxwell's Treatise the case of two spherical conductors is considered and the exact solution has been written expanding it in power of the distance between the center of the spheres providing manageable formulas which differ in the regions of validity less than one per cent from the exact solution. Even recently, some works were devoted to a critical analysis of approximate formulas in the two regimes [4] [5], for the case of spheres. Valuable would be an approximate formula for the coefficients of capacitance and potentials in the far limit with no restriction on the shape and the orientation of the conductors. The results given has general validity and its comparison with cases in which the coefficients are known analytically or numerically are satisfactory. The calculus appears also useful for applications as the precise knowledge of the asymptotic form in the far limit together with the discrepancy of the measured value should indicates the presence and the magnitude of external disturbances in the system.

The paper is organized as follows. Section 2.1 provides the derivation of the result based on a physical description of the system. The result is confirmed in a more mathematical way in Section 2.2. Section 3 contains a comparison with some analytical results and a brief discussion on the accuracy of the result with respect to possible experimental measures.

* Corresponding author.
E-mail address: francesco.maccarrone@unipi.it (F. Maccarrone).

http://dx.doi.org/10.1016/j.elstat.2016.07.003
0304-3886/

2. Method

2.1. Physical derivation of the result

The charges Q_i on the surfaces of a system of conductors ($i = 1,2,\ldots,N$) at equilibrium are a linear superposition of the uniform potentials V_i of the bodies:

$$Q_i = C_{ij}V_j; \qquad V_i = M_{ij}Q_j. \tag{1}$$

Repeated indices are summed. The coefficients C_{ij} form the capacitance matrix C and depend on the geometry of the problem (shapes, distances etc.). The matrix $M = C^{-1}$ is the potential matrix. C and M are symmetric matrices.

Our goal is to give a simple expression for these matrices in the case of a system of two conductors, a,b, at large distance R, specified below, compared to their sizes, as a function of few *intrinsic parameters* of the separated bodies. The leading order in the expansion is given in textbooks [6]:

$$M = \begin{pmatrix} \frac{1}{C_a} & \frac{1}{R} \\ \frac{1}{R} & \frac{1}{C_b} \end{pmatrix}; \qquad C = \begin{pmatrix} C_a\left(1+\frac{C_aC_b}{R^2}\right) & -\frac{C_aC_b}{R} \\ -\frac{C_aC_b}{R} & C_b\left(1+\frac{C_aC_b}{R^2}\right) \end{pmatrix}. \tag{2}$$

The computation will be extended to the next order, $1/R^3$ for C_{ab} and $1/R^4$ for diagonal elements.

The simplest starting point is to consider the energy of the system expressed in terms of the matrix M

$$U = \frac{1}{2}M_{aa}Q_a^2 + M_{ab}Q_aQ_b + \frac{1}{2}M_{bb}Q_b^2. \tag{3}$$

Up to order $1/R^4$ this energy can be written as the sum of three contributions: (a) a self-energy term, (b) an interaction term, and (c) an induced dipole term. In order to define uniquely the coefficients of the expansion it is necessary to specify more precisely the distance R. Our choice is to define R as the distance between the centres of charge of the *isolated* bodies. This choice is particularly convenient as the intrinsic dipole vanishes and the form of the coefficients results simpler (see also Section 2.2). For those reasons, it is maintained at each step of the computation.

a) A self energy term:

$$U_{self} = \frac{1}{2}\frac{Q_a^2}{C_a} + \frac{1}{2}\frac{Q_b^2}{C_b} \tag{4}$$

C_a,C_b are the self-capacitance of the conductors, and are supposed known.

b) The interaction energy for fixed charge distribution, i.e. for the distribution of charges in equilibrium for *isolated* conductors. With the convention for R specified above, the interaction energy is

$$U_{int} = Q_b\Phi_a(b) + \frac{1}{6}Q_bD_{ij}^{(b)}\partial_i\partial_j\Phi_a(b) + Q_a\Phi_b(a) + \frac{1}{6}Q_aD_{ij}^{(a)}\partial_i\partial_j\Phi_b(a). \tag{5}$$

$D_{ij}^{(a,b)}$ denotes the quadrupole for unit charge of the conductors and $\Phi_a(b)$ is the value of the potential generated by the conductor a at the center of the conductor b. The same convention is used for derivatives and similarly for $\Phi_b(a)$. For our needs is sufficient to take only the first few terms of the multipole expansion

$$\Phi_a(b) = \frac{Q_a}{R} + \frac{1}{2}\frac{1}{R^3}Q_aD_{ij}^{(a)}n_in_j \tag{6}$$

and the similar expression for $\Phi_b(a)$. $\mathbf{n} = \mathbf{R}/R$ is the unit vector between the two centres. Let us note that our choice for R has excluded the dipole interaction term in (5) and consequently, the term proportional to $1/R^2$ in (6).

Substituting (6) in (5) and taking the derivatives an easy computation gives

$$U_{int} = Q_aQ_b\left(\frac{1}{R} + \frac{1}{R^3}\,\frac{1}{2}\left(D_{ij}^{(a)} + D_{ij}^{(b)}\right)n_in_j\right). \tag{7}$$

c) A third term is due to induced dipoles in the conductors:

$$U_{dip} = -\frac{1}{2}\alpha_{ij}^{(a)}E_i^{(b)}(a)E_j^{(b)}(a) + (a \leftrightarrow b) \tag{8}$$

E is the electric field and the notation is the same as for the potentials. It is worth to note that even if the dipole of the isolated body vanishes, an external electric field at the chosen coordinate origin gives rise to an induced dipole. In (8) $E_j^{(a)}(b)$ is the j-th cartesian component of the electric field generated by conductor a at points of the conductor b. In our approximation, it is sufficient to take only the leading coulombic term $\mathbf{E} = Q\mathbf{n}/R^2$ and substituting in (8) we have

$$U_{dip} = -\frac{1}{2}\frac{1}{R^4}n_in_j\left(Q_a^2\alpha_{ij}^{(a)} + Q_b^2\alpha_{ij}^{(b)}\right). \tag{9}$$

The sum of the three contributions (4, 7, 9) has the form (3) with the matrix M given by

$$M = \begin{pmatrix} \frac{1}{C_a} - \frac{\alpha^{(b)}}{R^4} & \frac{1}{R} + \frac{1}{2}\frac{D}{R^3} \\ \frac{1}{R} + \frac{1}{2}\frac{D}{R^3} & \frac{1}{C_b} - \frac{\alpha^{(a)}}{R^4} \end{pmatrix}. \tag{10}$$

with the notation

$$\alpha^{(I)} = \alpha_{ij}^{(I)}n_in_j \qquad D = n_in_j\left(D_{ij}^{(a)} + D_{ij}^{(b)}\right). \tag{11}$$

Inversion of (10) gives to order $1/R^4$

$$\begin{aligned} C_{aa} &= C_a\left(1 + \frac{C_aC_b}{R^2} + \frac{1}{R^4}\left(C_a^2C_b^2 + C_aC_bD + C_a\alpha^{(b)}\right)\right) \\ C_{bb} &= C_b\left(1 + \frac{C_aC_b}{R^2} + \frac{1}{R^4}\left(C_a^2C_b^2 + C_aC_bD + C_b\alpha^{(a)}\right)\right) \\ C_{ab} &= C_{ba} = -\frac{C_aC_b}{R}\left(1 + \frac{D}{2R^2} + \frac{C_aC_b}{R^2}\right). \end{aligned} \tag{12}$$

The first order in this expansion in given in any textbook, see e.g.

Ref. [6]. Let us note *en passant*, that for a couple of conductors with opposite charges $\pm Q$ the relative capacitance, defined as $\Delta V = Q/C$ is given in general by

$$\frac{1}{C} = (M_{11} + M_{22} - 2M_{12}) \tag{13}$$

and at large distances

$$\frac{1}{C} \to \frac{1}{C_a} + \frac{1}{C_b}$$

i.e. the system of two conductors has a capacitance which is the series configuration of the single capacitors.

2.2. Mathematical proof of the result

While the presentation given in Section.2.1 is satisfactory from a physical point of view, the complexity of the Dirichlet problem motivates a more formal mathematical proof. Moreover, this proof will clarify that the expansion of the interaction energy of two charged bodies in 1/R given above applies not only for conductors, but to every system.

This proof clarifies where we use the fact that we are dealing with conductors: the expansion given above of the interaction energy of two charged bodies in 1/R applies not only to conductors, but to every system.

2.3. Statement of the problem and notations

We will need two well known (see e.g. Ref. [7]) facts about the Dirichlet problem for the Laplace equation:

a) The existence of a Green's function $G(\mathbf{x},\mathbf{y})$ satisfying the equation

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = -4\pi\,\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{14}$$

with the boundary condition $G(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{x},\mathbf{y}$ on the boundary. In terms of this function the solution of the Laplace equation has the form

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y \tag{15}$$

For the particular case of a conductor $\Phi(\mathbf{y})$ has a constant value V at points $\mathbf{y}\in S$ on the surface of the conductor. Here and in the following ∂_{ny} is the normal derivative and dA_y is the area element at the point $\mathbf{y}$.

b) The Green's function is symmetric for exchange of variables

The physics of a single isolated conductor is taken as known, i.e. we assume to know the functions G_a,G_b for the two relative Dirichlet problems. For a given conductor at potential V the charge density is computed from the discontinuity of the electric field

$$\sigma(\mathbf{x}) = \frac{V}{(4\pi)^2}\oint_S \frac{\partial}{\partial n_x}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y\,. \tag{16}$$

Integrating, one finds the total charge on the conductor:

$$Q = \oint \sigma(\mathbf{x})\mathrm{d}A_x = \frac{V}{(4\pi)^2}\oint\oint \frac{\partial}{\partial n_x}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y\mathrm{d}A_x \tag{17}$$

and, eventually, the formal expression of the capacitance Q/V:

$$C = \frac{1}{(4\pi)^2}\oint\oint \frac{\partial}{\partial n_x}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y\mathrm{d}A_x \tag{18}$$

The coordinates $\mathbf{X}$ for the center of charges are

$$Q\,\mathbf{X} = \oint \mathbf{x}\sigma(\mathbf{x})\mathrm{d}A_x = \frac{V}{(4\pi)^2}\oint\oint \mathbf{x}\frac{\partial}{\partial n_x}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y\mathrm{d}A_x \tag{19}$$

and it is always possible with a coordinates shift to set $\mathbf{X}$=0. Moreover, as Green's functions are symmetric for an exchange of their arguments one can rewrite with the same origin:

$$\frac{1}{(4\pi)^2}\oint\oint \mathbf{y}\frac{\partial}{\partial n_x}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_y\mathrm{d}A_x = 0 \tag{20}$$

2.4. Computation of M_{ba}

It is easier in this analysis to consider the problem of a conductor with charge Q_a and a second conductor with zero charge $Q_b = 0$. We will compute V_a,V_b and then derive M_{aa} and M_{ba}. The rest of the matrix follows by symmetry and interchange of the role of the conductors. To compute V_b, and hence M_{ba}, we need a solution of the Laplace equation constant on the conductor b. In the region of the conductor b it is present the potential generated by the charged conductor a. The idea is to write by inspection a solution with constant potential by cancelling this contribution in the region near the second conductor. This first step in general gives to b a charge $\tilde{Q}$ which can be easily removed by adding a solution of type (15) with potential $V_b = -\tilde{Q}/C_b$.

With respect to a reference system with origin in the center of charges of b the potential generated by conductor a is the function $\Phi_a(\mathbf{R}+\mathbf{x})$ and the proposed formula for the solution of the Laplace equation near conductor b is

$$\Phi(\mathbf{x}) = \Phi_a(\mathbf{R}+\mathbf{x}) - \frac{1}{4\pi}\oint_{S_b}[-\Phi_a(\mathbf{R}+\mathbf{y})]\,\partial_{n_y}G_b(\mathbf{x},\mathbf{y})\mathrm{d}A_y - \frac{\tilde{Q}}{C_b}\left[-\frac{1}{4\pi}\oint_{S_b}\partial_{n_y}G_b\mathrm{d}A_y\right] \tag{21}$$

G_b being the Green's function of conductor b.

The sum of the first two terms of Eq. (21) gives zero potential on the surface S_b, by definition. It is worth to note that in order to guarantee the continuity of the solution we extend to zero the value of the potential inside conductor b. This introduce a discontinuity in the first derivative i.e a non zero charge distribution on S_b and, consequently, a charge $\tilde{Q}$ on conductor b. In order to respect the condition of null charge we add the third term describing, in accordance with (15), a constant potential $-\tilde{Q}/C_b$ and cancelling exactly total charge Q_b. In conclusion each term of the guess solution satisfies Laplace's equation outside S_b, with the correct boundary condition of constant potential and null charge on the conductor b and, then is the solution of the electrostatic problem around body b at the actual degree of approximation.

The value of $\tilde{Q}$ induced by the first two term in (21) can be computed by integrating the surface charge density $\sigma_b(\mathbf{x})$ obtained by the discontinuity of the normal component of the electric field

$$\sigma_b(\mathbf{x}) = -\frac{1}{4\pi}\partial_{n_x}\Phi_a(\mathbf{R}+\mathbf{x}) - \frac{1}{(4\pi)^2}\oint_{S_b}\Phi_a(\mathbf{R}+\mathbf{y})\,\partial_{n_x}\partial_{n_y}G_b\,\mathrm{d}A_y. \tag{22}$$

As Φ_a is the potential generated by conductor a, the integral of the first term is null (Gauss's law) and the second fixes $\tilde{Q}$:

$$\tilde{Q} = -\frac{1}{(4\pi)^2}\oint_{S_b}\Phi_a(\mathbf{R}+\mathbf{y})\,\partial_{n_x}\partial_{n_y}G_b\,\mathrm{d}A_y\mathrm{d}A_x \tag{23}$$

and the potential

$$V_b = -\frac{\tilde{Q}}{C_b}\,. \tag{24}$$

It is worth to note that the total charge Q_b is zero by construction and this implies that the term of monopole is absent and, thus, that the perturbation to the external potential Φ_a is asymptotically negligible.

The potential Φ_a has to be evaluated on the surface S_b of conductor b and as R is assumed large with respect to the dimension of conductors we can use the Taylor expansion

$$\begin{aligned}\Phi_a(\mathbf{R}+\mathbf{y}) &= \Phi_a(\mathbf{R}) + \partial_i\Phi_a y_i + \frac{1}{2}\partial_i\partial_j\Phi_a y_i y_j + \dots \\ &= \Phi_a(\mathbf{R}) - E_i y_i - \frac{1}{6}\partial_i E_j\left(3y_i y_j - y^2\,\delta_{ij}\right) + \dots\end{aligned} \tag{25}$$

The term proportional to $1/R^2$ is absent for our choice of the center of charges in conductor a.

The first term in (25) inserted in (23,24) gives the contribution

$$V_b^{(0)} = \frac{\Phi_a(\mathbf{R})}{C_b}\frac{1}{(4\pi)^2}\oint_{S_b}\partial_{n_x}\partial_{n_y}G_b\mathrm{d}A_y\mathrm{d}A_x$$

and using the multipole expansion (6) and Eq. (18) for the capacity:

$$V_b^{(0)} = \Phi_a(\mathbf{R}) = \frac{Q_a}{R} + \frac{Q_a}{2}\frac{n_i n_j}{R^3}D_{ij}^{(a)}. \tag{26}$$

As already noticed $D_{ij}^{(a)}$ denotes the electric quadrupole for unit charge, i.e.

$$D_{ij}^{(a)} = \oint_{S_a}\left(3x_i x_j - x^2\,\delta_{ij}\right)\sigma(\mathbf{x})\mathrm{d}A_x \tag{27}$$

which can be calculated for the isolated conductor. Let us explicitly note that for unit charge the potential of the isolated conductor is $1/C_a$, then, expressing $\sigma(\mathbf{x})$ through the Green's function G_a, see (16), we can write

$$D_{ij}^{(a)} = \frac{1}{C_a}\oint_{S_a}\left(3x_i x_j - x^2\,\delta_{ij}\right)\frac{\partial}{\partial n_x}\frac{\partial G_a(\mathbf{x},\mathbf{y})}{\partial n_y}\mathrm{d}A_x\mathrm{d}A_y. \tag{28}$$

This is the first point where the conducting nature of the components enters in the evaluation of the parameter. In the general case the quadrupole moment of a system *is not* proportional to the charge. This is true for conductors, as the charge distribution is fixed by the solution of the Dirichlet problem and the property is evident in the form (28), where all references to charges have disappeared.

The second term in (25), gives a null contribution

$$V_b^{(1)} = -\frac{1}{C_b}\frac{1}{(4\pi)^2}\oint_{S_b}E_j^{(a)}y_j\,\partial_{n_x}\partial_{n_y}G_b\mathrm{d}A_y\mathrm{d}A_x = 0$$

for our choice of origin in conductor b and the symmetry of the Green's function.

For the contribution of the third term in (25) at our approximation we need only the coulombic part of this field, $\mathbf{E} = Q_a\mathbf{n}/R^2$. With this approximation the insertion of the third term in (23,24) gives the contribution

$$\begin{aligned}V_b^{(2)} &= -\frac{1}{6C_b}\frac{\partial_i E_j}{(4\pi)^2}\oint_{S_b}\left(3y_i y_j - \delta_{ij}y^2\right)\partial_{n_x}\partial_{ny}G_b\mathrm{d}A_y\mathrm{d}A_x \\ &= \frac{Q_a}{6C_b}\frac{3R_iR_j - \delta_{ij}R^2}{(4\pi)^2R^5}\oint_{S_b}\left(3y_i y_j - \delta_{ij}y^2\right)\partial_{n_x}\partial_{n_y}G_b\mathrm{d}A_y\mathrm{d}A_x \\ &= \frac{Q_a}{2C_b}\frac{n_i n_j}{(4\pi)^2R^3}\oint_{S_b}\left(3y_i y_j - \delta_{ij}y^2\right)\partial_{n_x}\partial_{n_y}G_b\mathrm{d}A_y\mathrm{d}A_x\,.\end{aligned}$$

Due to the symmetry of Green's function and of its mixed derivatives with respect to the exchange of the arguments x and y, the above expression can be simplified finding the quadrupole moment of conductor b:

$$V_b^{(2)} = \frac{Q_a}{2}\frac{n_i n_j}{d^3}\,D_{ij}^{(b)} \tag{29}$$

with

$$D_{ij}^{(b)} = \frac{1}{C_b}\frac{1}{(4\pi)^2}\oint_{S_2}\left(3x_i x_j - \delta_{ij}x^2\right)\partial_{n_x}\partial_{n_y}G_b\mathrm{d}A_x\mathrm{d}A_y$$

symmetric with the term originating from the quadrupole moment of body a in the expansion of $\Phi_1(d)$. $D^{(b)}$ is the quadrupole for unit charge of conductor b, as explained for the analogous case of conductor a in passing from (27) to (28). As already noticed this definition has a meaning only for conductors.

Finally, the potential on conductor b, summing the two contributes $V_b^{(0)}$ and $V_b^{(2)}$ is:

$$V_b = Q_a\left(\frac{1}{R} + \frac{1}{2}\frac{n_i n_j}{R^3}\left(D_{ij}^{(a)} + D_{ij}^{(b)}\right)\right) \equiv M_{ba}Q_a \tag{30}$$

and the factor multiplying Q_a express, at the order $1/R^3$ the off-diagonal potential coefficient M_{ba}.

2.5. Calculation of M_{aa}

The computation of M_{aa} requires the estimation of the effect of the potential generated by conductor b on conductor a. As in the present configuration $Q_b = 0$, the first effect is due to a dipole potential. The choice of the center of charges on conductor b assures that the intrinsic dipole vanishes but allows an induced dipole, and we need the charge density to compute it. Following the previous results the potential (21) induces a charge distribution on potential b. The third term in this potential has the form of the usual equilibrium potential for isolated conductor b and consequently the dipole moment induced by this term vanishes for our choice of coordinates. The induced dipole moment is produced only by the first two term in (21) and these terms produce a density charge $\sigma(\mathbf{x})$ written in (22). To the order needed it is sufficient to consider only the first order terms in the Taylor expansion of the potential Φ_a (25) and we have

$$\sigma_b(\mathbf{x}) \simeq \frac{1}{4\pi} E_j \partial_{n_x} x_j - \frac{1}{(4\pi)^2} \oint_{S_b} \Phi_a(R) \partial_{n_x} \partial_{n_y} G_b \, \mathrm{d}A_x \, \mathrm{d}A_y + \frac{1}{(4\pi)^2} \oint_{S_b} E_j y_j \partial_{n_x} \partial_{n_y} G_b \, \mathrm{d}A_x \, \mathrm{d}A_y \, .$$

Again the second term in this expression refers to the equilibrium of the isolated conductor and gives a null contribution to the dipole moment. We obtain finally for the components of the induced dipole

$$\mu_i^{(b)} = \oint_{S_b} x_i \sigma_b(\mathbf{x}) \mathrm{d}A_x = E_j \left\{ \frac{1}{4\pi} \oint_{S_2} x_i \partial_{n_x} x_j dA_x + \frac{1}{(4\pi)^2} \oint_{S_b} \times \oint_{S_b} x_i y_j \partial_{n_x} \partial_{n_y} G_b \, \mathrm{d}A_y \mathrm{d}A_x \right\} \equiv \alpha_{ij}^{(b)} E_j \, . \tag{31}$$

An elementary application of Gauss theorem allows the computation of the first integral giving the final expression for the polarizability

$$\alpha_{ij}^{(b)} = \frac{\Omega_b}{4\pi} \delta_{ij} + \frac{1}{(4\pi)^2} \oint \oint_{S_b} x_i y_j \partial_{n_x} \partial_{n_y} G_b \mathrm{d}A_y \mathrm{d}A_x \Big\} \tag{32}$$

where Ω_b is the volume of the conductor b.

From the dipole moment at this order of approximation it is a trivial matter to deduce the change on the conductor a: is the dipole potential generated at position $-\mathbf{R}$ by conductor b, i.e.

$$\delta V_a = -\frac{\boldsymbol{\mu}\mathbf{R}}{R^3} = -Q_a \alpha_{ij}^{(b)} \frac{n_i n_j}{R^4} \, . \tag{33}$$

From this result, adding the leading term, we have for M_{aa}

$$M_{aa} = \frac{1}{C_a} - \alpha_{ij}^{(b)} \frac{n_i n_j}{R^4} \, . \tag{34}$$

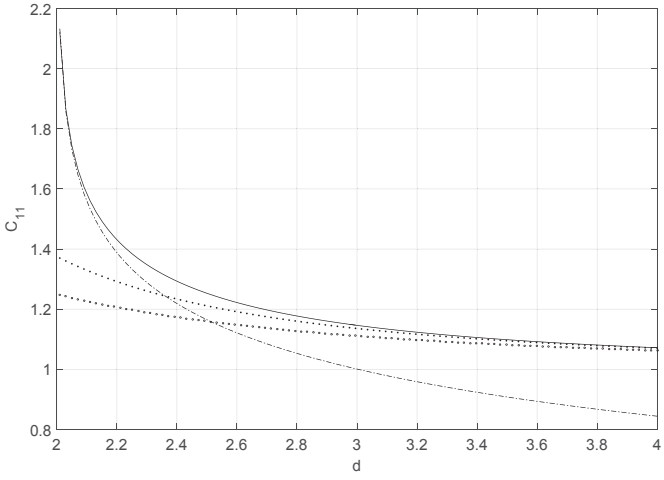

Fig. 1. C_{aa} for two spheres of equal radius $a = 1$ as a function of the distance between centres. The continuous line is the exact result [3]. The points represent our approximate result at the order $1/d^4$ and the small circles the same at the leading order $1/d^2$. The dash-dot line is the approximate result given by Eq. (17) of Ref. [8].

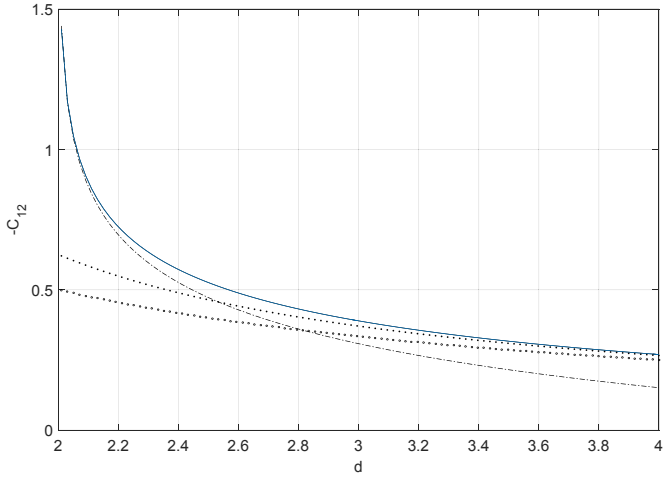

Fig. 2. $-C_{ab}$ for two spheres of equal radius $a = 1$. The continuous line is the exact result. The points represent our approximate result at the order $1/d^3$ and the small circles the same at the leading order $1/d$. The dash-dot line is the approximate result given by Eq. (16) of Ref. [8].

The result for M_{bb} is clearly obtained interchanging the role of a,b in the computation.

Here again we note a peculiarity of the energy problem for conductors: the polarizability is fixed by the Green's function of the *isolated* conductor. The functions G_a,G_b contain all the information needed to describe the composed system up to order $1/R^4$, they determine the capacity, the quadrupole moments and the polarizability.

Collecting the results (30) and (34) we recover the anticipated result (10).

3. Results and discussion

In summary, we have found (Eq. (10) and Eq. (12)) an expression for the potential and capacitance coefficients for two generic conducting bodies at the order $1/R^4$.

In order to evaluate the usefulness of the given formulas we make a check for a system in which the exact solution is disposable. In fact, the matrix of the coefficient of capacitance can be calculated analytically only for few geometries. Let us consider the classical case of two spheres of equal radius a and denote with $d > 2a$ the distance between the centres. The intrinsic capacitance of each sphere is $C_a = C_b = a$ and the expressions for the coefficients C_{aa} and C_{ab} can be given as a series involving hyperbolic functions [8]. A graphical comparison between the analytic solution and our approximation is shown in Figs. 1 and 2 where it is evident that the approximated formula fails for the closest approaches but its discrepancies from the exact formula even for distances of the order of few tens per cent of the sum of radii is less than few per cent. At a distance $d = 3a$, that is at a gap equal to each radius, the deviation of C_{aa} of (12) from the exact value is only one per cent It is also interesting to note, as it is shown in figure, that the formulas given by Ref. [8] very accurate for the closest gap begins to fail just in the region where our approximation begins to operate well.

This system offers also the opportunity to check the general formula for the polarizability tensor. For a sphere, rotational symmetry imposes $D_{ij} = 0$ and $\alpha_{ij} = \alpha\delta_{ij}$. The standard result [6] for α is $\alpha = a^3$. Using the known Green's function for a sphere

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{a}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|} = \frac{1}{\left(x^2+y^2-2xy\cos\gamma\right)^{1/2}} - \frac{1}{\left(\frac{1}{a^2}x^2y^2+a^2-2xy\cos\gamma\right)^{1/2}} \tag{35}$$

with $\mathbf{x}\cdot\mathbf{y} = xy\cos\gamma$, the insertion in (31) and a straightforward integration confirms the result. Substituting the values of $\alpha = a^3$ and $D_{ij} = 0$ valid for spheres in Eq. (12) we find the same result of Eq. (2.1) of [4] up to order $1/R^4$. Also the expression of energy given by Ref. [4] is reproduced to the same order by our (3), using the matrix of potential coefficients (10). The comparison of the asymptotic expansion with more complicated systems will be considered elsewhere.

The reader has surely noted that up to order $1/R^4$ the diagonal terms of the C matrix have only even powers of R while the off diagonal ones have only odd powers, and similarly for M. Up to this order this is due to our judicious choice of the definition of R. For particular systems like two spheres this property is valid to all orders in $1/R$. It is possible to show that the same property holds for another interesting system, the condenser made by two equal circular parallel plates. In the general case this property no longer seems to hold at higher order in $1/R$: as an example an octupole moment of conductor a induces a term proportional to $1/R^4$ in M_{ab}. The regularity in simple systems is due to the high symmetry of the conductors.

Let us note that the only effect of an octupole contribution is a change in M_{ab} of the form $\delta M_{ab} = S/R^4$ and a corresponding term

$$\delta C_{ab} = -C_a C_b \frac{1}{R^4} S$$

in the off diagonal terms in the matrix C, S being proportional to the sum of the octupole moments of the conductors. Due to this contribution we note that if the expansion is used as a tool to measure polarizability, the convenient quantity to measure are diagonal coefficients as C_{aa}, which are not affected by this correction. The value of the usual relative capacity (13) depends on this additive term and is not suited to detected polarizability effects. On the other side the measure of off diagonal term C_{ab} is better suited for a measure of quadrupole contribution.

4. Conclusion

In this paper we presented a perturbative calculation giving an expansion of the coefficients of potential and of capacitance between a couple of conducting bodies to the successive order in the distance with respect to that commonly found in the literature [6]. The form in which the result is presented is, to our knowledge, a new one. We were not able to find a similar result elsewhere. The resulting formulas are useful as they allow almost accurate values of the coefficient even when the distance between the conductors is only slightly larger with respect to their dimensions. Moreover, the procedure is not limited to simple shapes or to particular orientation of the two bodies. The only request is the knowledge of the tensors of polarizability and of quadrupole distribution of charge and the intrinsic capacitance of the conductor. For a particular body, the three constant must be numerically estimated only once and this is possible with a great degree of precision. After that, the application of the formulas (12) and the similar expressions for the M-matrix is straightforward.

References

[1] M. Uehara, Green's functions and coefficients of capacitance, Am. J. Phys. 54 (2) (1986) 184–185, http://dx.doi.org/10.1119/1.14690.
[2] W.C. Chew, L. Jiang, A complete variational method for capacitance extractions, Prog. Electromagn. Res. 56 (2006) 19–32.
[3] J.C. Maxwell, A Treatise on Electricity and Magnetism, Vol. 1 §146, Clarendon Press, 1892.
[4] J. Lekner, Electrostatics of two charged conducting spheres, Proc. R. Soc. A, R. Soc. (2012) p. rspa20120133.
[5] S. Banerjee, M. Levy, Exact closed-form solution for the electrostatic interaction of two equal-sized charged conducting spheres, J. Phys. Conf. Ser. 646 (2015) 012016. IOP Publishing.
[6] L. Landau, E. Lifshitz, Electrodynamics of Continuous Media; §2 prob.3, Pergamon, 1984.
[7] J.D. Jackson, Classical Electrodynamics; §1.9, Wiley, 1999.
[8] J. Lekner, Capacitance coefficients of two spheres, J. Electrost. 69 (1) (2011) 11–14.