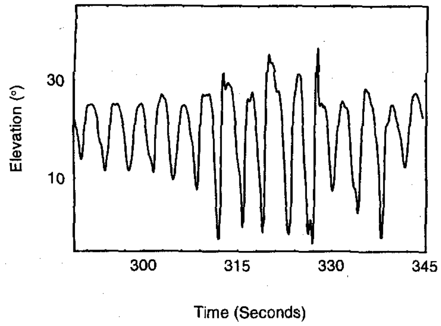

Fig. 5. Plot of conventional beamforming estimate of elevation versus time for the Spanish data set.

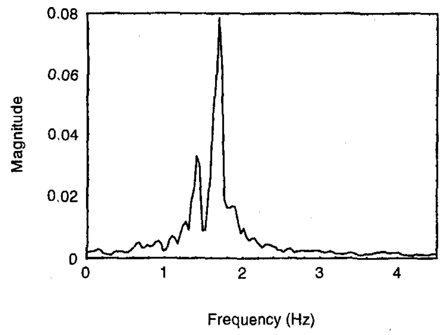

Fig. 6. Frequency spectrum obtained from one element of the Spanish data set.

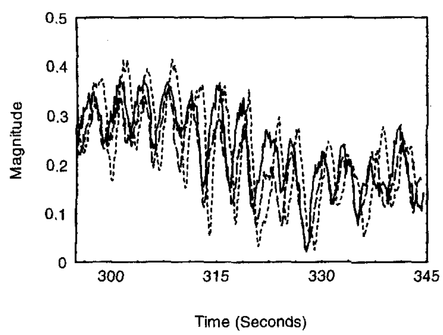

Fig. 7. Plot of signal power versus time for three of the antennas from the Spanish data set.

The periodic variation shown in Fig. 5 is not precisely sinusoidal for a number of reasons: an elevation degrees scale is used rather than a wave number scale,[1] the array is not linear, and the DOA of each of the two modes propagating are varying with time due to changes in the ionosphere.

V. Conclusion

In this communication, an effect observed on much beamformed data has been explained and quantified. The wobble, caused by the phase interaction of different signals incident on an array, was shown analytically and in simulation, to be sinusoidal for a uniform linear array. The functions derived for the magnitude of the effect demonstrate it to be most pronounced when the two signals have not been independently resolved. (Obviously increasingly overlapping beam patterns create increasing bias). Simulations showed that the bias is also dependent on the type of windowing function used on the data.

An example of real data where the wobble was observed has been presented, together with other evidence that two signals, not individually resolvable by conventional beamforming with a temporal frequency difference corresponding to the frequency of the wobble, were present.

[1] Due to the irregular shape of the array used, a wavenumber scale cannot be defined to describe the measured DOA's.

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On the Calculation of Potential Integrals for Linear Source Distributions on Triangular Domains

T. F. Eibert and V. Hansen

***Abstract*—We present analytical formulas for the singular field contributions of linear source distributions on triangular domains. The formulas can be used for near-field calculations of electric and magnetic source currents, especially for the calculation of method of moments (MOM) matrix elements due to the solution of surface integral equations. For MOM self-coupling terms involving the scalar Green function of free space, we present expressions comprising analytical solutions of both the source integrals and the testing integrals.**

I. Introduction

The calculation of electromagnetic scattering and radiation problems is often performed with the help of surface integral equation formulations. Most approaches are referring to the mixed potential integral equation (MPIE) technique with triangular subdomains introduced in [1]. Formulations dealing with inhomogeneous materials or hybrid approaches combining a surface integral equation technique with a local technique (finite elements, finite differences) require the calculation of potential integrals involving the scalar Green function and its gradient. The most severe problem of a practical implementation of such techniques is the singular behavior of the integral kernels. Usually the singular parts of the integrands are

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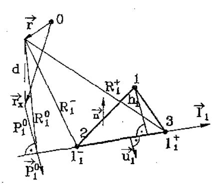

Fig. 1. Geometric quantities associated with side 1 of the source triangle.

extracted, then the source integrals with a singular contribution are evaluated analytically for given observation (testing) points [2]–[4].

In principle, all necessary integrals can be evaluated with the formulas given in [3]. We present similar formulas directly applicable to the widely used expansion functions introduced in [1] and for formulations in simplex coordinates. In the case of the self-coupling terms involving the scalar Green function of free space, we present expressions comprising an analytical solution of both the source integrals and the testing integrals. To our knowledge such expressions have not been published until now.

II. Formulation

A. Singular Potential Integrals

The singular contributions of all necessary integrals can be reduced to

$$\iint_{A'} \frac{1}{|\vec{r}-\vec{r}\,'|}da' \tag{1}$$

$$\iint_{A'} \lambda'_j \frac{1}{|\vec{r}-\vec{r}\,'|}da' \tag{2}$$

$$\iint_{A'} \lambda'_j \nabla \frac{1}{|\vec{r}-\vec{r}\,'|}da' \tag{3}$$

$$\iint_{A'} \nabla \frac{1}{|\vec{r}-\vec{r}\,'|}da'. \tag{4}$$

$\vec{r}$ is the observation point, A' is the domain of the source triangle, and the λ_j are the simplex coordinates or normalized area coordinates on the source triangle (see, e.g., [1]).

Equation (1) is given by (5) in [2] or by (19) in [3], and (4) is given by (34) in [3]. The geometric quantities used in the following equations can be seen in Fig. 1 (see also [2]). λ_j is associated to corner j which is opposite to side j. λ_j is zero along side j; it has the value one in corner j and changes linearly between these values. $\vec{u}_j$ is the outward normal of side j in the plane of the triangle. $\vec{n}$ is the unit area normal of the plane of the triangle. $\vec{l}_j$ is the unit tangent vector along side j in integration direction. The unit vectors are related by

$$\vec{n} = \vec{u}_j \times \vec{l}_j. \tag{5}$$

h_j is the distance of corner j from side j. $\vec{r}$ is the observation point. $\vec{r}_x$ is the projection of the observation point onto the plane of the source triangle. d is the height of $\vec{r}$ over this plane, and it is positive if $(\vec{r}-\vec{r}_x)$ has the same direction as $\vec{n}$. P_j^0 is the distance of $\vec{r}_x$ from side j. Different to the convention in [2] but equal to the convention in [3] (P_j^0 replaced by t_j^0) in our formulation, it is negative if $\vec{P}_j^0$ and $\vec{u}_j$ have opposite directions.

With the help of

$$\frac{h_j\lambda'_j - P_j^0}{|\vec{r}-\vec{r}\,'|} = -\vec{u}_j \cdot \nabla'_A|\vec{r}-\vec{r}\,'| \tag{6}$$

(2) becomes

$$\iint_{A'} \frac{\lambda'_j}{|\vec{r}-\vec{r}\,'|}da' = -\vec{u}_j \cdot \frac{1}{h_j}\iint_{A'} \nabla'_A|\vec{r}-\vec{r}\,'|da' + \frac{P_j^0}{h_j}\iint_{A'} \frac{1}{|\vec{r}-\vec{r}\,'|}da' \tag{7}$$

where the first integral is given by (20) in [3], and the second integral is equivalent to (1). ∇_A is the surface gradient operator. With consideration of

$$\nabla\frac{1}{|\vec{r}-\vec{r}\,'|} = -\nabla'\frac{1}{|\vec{r}-\vec{r}\,'|} = -\left(\vec{n}_A \frac{d}{|\vec{r}-\vec{r}\,'|^3} + \nabla'_A \frac{1}{|\vec{r}-\vec{r}\,'|}\right) \tag{8}$$

we can write for (3)

$$\iint_{A'} \lambda'_j \nabla \frac{1}{|\vec{r}-\vec{r}\,'|}da' = -\vec{n}_A \iint_{A'} \lambda'_j \frac{d}{|\vec{r}-\vec{r}\,'|^3}da' - \iint_{A'} \lambda'_j \nabla'_A \frac{1}{|\vec{r}-\vec{r}\,'|}da'. \tag{9}$$

For the first term with

$$\frac{h_j\lambda'_j - P_j^0}{|\vec{r}-\vec{r}\,'|^3} = \vec{u}_j \cdot \nabla'_A \frac{1}{|\vec{r}-\vec{r}\,'|} \tag{10}$$

we get

$$d\iint_{A'} \lambda'_j \frac{1}{|\vec{r}-\vec{r}\,'|^3}da' = \vec{u}_j \cdot \frac{d}{h_j}\iint_{A'} \nabla'_A \frac{1}{|\vec{r}-\vec{r}\,'|}da' + \frac{P_j^0 d}{h_j}\iint_{A'} \frac{1}{|\vec{r}-\vec{r}\,'|^3}da'. \tag{11}$$

The first integral is given by (30) in [3], and the second integral is equivalent to (26) in [3]. For the second term of (9) we can write

$$\iint_{A'} \lambda'_j \nabla'_A \frac{1}{|\vec{r}-\vec{r}\,'|}da' = \iint_{A'} \nabla'_A\left(\frac{\lambda'_j}{|\vec{r}-\vec{r}\,'|}\right)da' + \frac{\vec{u}_j}{h_j}\iint_{A'} \frac{1}{|\vec{r}-\vec{r}\,'|}da'. \tag{12}$$

The second integral is equal to (1), and the first integral can be calculated to

$$\begin{aligned}\iint_{A'} \nabla'_A&\left(\frac{\lambda'_j}{|\vec{r}-\vec{r}\,'|}\right)da' \\ &= \sum_{i=1}^{3} \vec{u}_i \int_{\partial A'} \frac{\lambda'_j}{|\vec{r}-\vec{r}\,'|}dl' \\ &= (R_{j+1}^+ - R_{j+1}^- - l_{j+1}^- f_{2j+1})\frac{\vec{u}_{j+1}}{l_{j+1}} \\ &\quad + (R_{j-1}^- - R_{j-1}^+ + l_{j-1}^+ f_{2j-1})\frac{\vec{u}_{j-1}}{l_{j-1}}\end{aligned}$$

with cyclic index calculation. f_{2j} is given by (11) in [3] and l_j is the length of side j. With the given formulas, the singular field contributions of arbitrary linear source distributions on triangular domains can be calculated.

For the often used, somewhat restricted expansion functions introduced in [1], more compact formulas can be developed. We have to solve the integrals

$$\iint\limits_{A'} \frac{\vec{r}' - \vec{r}'_j}{|\vec{r} - \vec{r}'|} da' \tag{13}$$

$$\iint\limits_{A'} (\vec{r}' - \vec{r}'_j) \times \nabla \frac{1}{|\vec{r} - \vec{r}'|} da'. \tag{14}$$

$\vec{r}_j$ are the corners of the source triangle. The solution of (13) is presented in [2] and in [3]. For (14) we can write

$$\begin{aligned} &\iint\limits_{A'} (\vec{r}' - \vec{r}'_j) \times \nabla \frac{1}{|\vec{r} - \vec{r}'|} da' \\ &= \iint\limits_{A'} (\vec{r}' - \vec{r}_x) \times \nabla \frac{1}{|\vec{r} - \vec{r}'|} da' + (\vec{r}_x - \vec{r}'_j) \\ &\times \iint\limits_{A'} \nabla \frac{1}{|\vec{r} - \vec{r}'|} da'. \end{aligned} \tag{15}$$

With the simplification of the first term

$$\begin{aligned} &\iint\limits_{A'} (\vec{r}' - \vec{r}_x) \times \nabla \frac{1}{|\vec{r} - \vec{r}'|} da' \\ &= -\iint\limits_{A'} (\vec{r}' - \vec{r}_x) \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} da' \\ &= d\vec{n}_A \times \iint\limits_{A'} \frac{\vec{r}' - \vec{r}_x}{|\vec{r} - \vec{r}'|^3} da' \\ &= -d\vec{n}_A \times \iint\limits_{A'} \nabla'_A \frac{1}{|\vec{r} - \vec{r}'|} da' \end{aligned} \tag{16}$$

the integral is reduced to contributions which are already known.

B. Singular Self-Coupling Terms

Self-coupling integrals involving the gradient of the scalar Green function usually can be circumvented. In mixed potential integral equation (MPIE) formulations of the electric field integral equation (EFIE) for ideal conducting bodies, the gradient operators are transferred to the testing and expansion functions [1]. In formulations basing on the magnetic field integral equation (MFIE) for ideal conducting bodies or on the EFIE with magnetic source currents, the self-coupling terms involving the gradient of the scalar Green function vanish due to the cross product in the integrand [see (14)]. Therefore, we only consider the integrals

$$\frac{1}{4A^2} \iint\limits_{A} \iint\limits_{A} \lambda_i \lambda'_j \frac{1}{|\vec{r} - \vec{r}'|} da' da \tag{17}$$

$$\frac{1}{4A^2} \iint\limits_{A} \iint\limits_{A} \frac{1}{|\vec{r} - \vec{r}'|} da' da \tag{18}$$

for arbitrary linear source distributions on the triangle. A is the area of the triangle. Because we consider self-coupling terms, we can write

$$\vec{r} = \lambda_1(\vec{r}_1 - \vec{r}_3) + \lambda_2(\vec{r}_2 - \vec{r}_3) + \vec{r}_3 \tag{19}$$

$$\vec{r}' = \lambda'_1(\vec{r}_1 - \vec{r}_3) + \lambda'_2(\vec{r}_2 - \vec{r}_3) + \vec{r}_3 \tag{20}$$

and therefore

$$\vec{r} - \vec{r}' = (\lambda_1 - \lambda'_1)(\vec{r}_1 - \vec{r}_3) + (\lambda_2 - \lambda'_2)(\vec{r}_2 - \vec{r}_3). \tag{21}$$

The $\vec{r}_i$ are the corners of the triangle. To simplify the following discussion we consider the more general integral

$$\begin{aligned} &\iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} f_1(\lambda'_1, \lambda'_2) f_2(\lambda_1, \lambda_2) g(\lambda_1 - \lambda'_1, \lambda_2 - \lambda'_2) \\ &d\lambda'_2 d\lambda'_1 d\lambda_2 d\lambda_1. \end{aligned} \tag{22}$$

f_1 and f_2 stand for the expansion functions and testing functions, respectively. They are equal to zero outside the triangle. g is the singular integral kernel. With the coordinate transformation

$$\lambda_1 - \lambda'_1 = u \quad \lambda_2 - \lambda'_2 = v \quad \lambda'_1 = \lambda'_1 \quad \lambda'_2 = \lambda'_2 \tag{23}$$

we get

$$\begin{aligned} &\iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} f_1(\lambda'_1, \lambda'_2) f_2(u + \lambda'_1, v + \lambda'_2) \\ &d\lambda'_1 d\lambda'_2 g(u, v) du dv. \end{aligned} \tag{24}$$

Due to the triangular integration domains of λ, λ' for (u, v), we get the integration domain illustrated in Fig. 2. It is useful to distinguish the six subdomains according to the figure. For each of the (u, v)-subdomains the inner integral of convolution type can be evaluated analytically with the integration limits

$$\begin{aligned} \text{I: } &\lambda'_1: 0 \cdots (1 - u - v) \\ &\lambda'_2: 0 \cdots (1 - u - v - \lambda_1) \\ \text{II: } &\lambda'_1: -u \cdots (1 + v) \\ &\lambda'_2: -v \cdots (1 - \lambda_1) \\ \text{III: } &\lambda'_1: 0 \cdots (1 - u) \\ &\lambda'_2: -v \cdots (1 - \lambda_1 - u - v) \\ \text{IV: } &\lambda'_1: 0 \cdots (1 + v) \\ &\lambda'_2: -v \cdots (1 - \lambda_1) \\ \text{V: } &\lambda'_1: -u \cdots 1 \\ &\lambda'_2: 0 \cdots (1 - \lambda_1) \\ \text{VI: } &\lambda'_1: -u \cdots (1 - v - u) \\ &\lambda'_2: 0 \cdots (1 - v - \lambda_1 - u). \end{aligned}$$

Finally, the complete expressions can be evaluated analytically. We get

$$\begin{aligned} &\frac{1}{4A^2} \iint\limits_{A} \iint\limits_{A} \frac{1}{|\vec{r} - \vec{r}'|} da da' \\ &= \frac{\log\left(\frac{(a-b+\sqrt{a}\sqrt{a-2b+c})(b+\sqrt{a}\sqrt{c})}{(-b+\sqrt{a}\sqrt{c})(-a+b+\sqrt{a}\sqrt{a-2b+c})}\right)}{6\sqrt{a}} \\ &+ \frac{\log\left(\frac{(b+\sqrt{a}\sqrt{c})(-b+c+\sqrt{c}\sqrt{a-2b+c})}{(b-c+\sqrt{c}\sqrt{a-2b+c})(-b+\sqrt{a}\sqrt{c})}\right)}{6\sqrt{c}} \\ &+ \frac{\log\left(\frac{(a-b+\sqrt{a}\sqrt{a-2b+c})(-b+c+\sqrt{c}\sqrt{a-2b+c})}{(b-c+\sqrt{c}\sqrt{a-2b+c})(-a+b+\sqrt{a}\sqrt{a-2b+c})}\right)}{6\sqrt{a-2b+c}} \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{1}{4A^2} \iint\limits_{A} \iint\limits_{A} \lambda'_1 \lambda_1 \frac{1}{|\vec{r} - \vec{r}'|} da\, da' \\ &= \frac{\log\left(\frac{b+\sqrt{a}\sqrt{c}}{b-c+\sqrt{c}\sqrt{a-2b+c}}\right)}{40\sqrt{c}} + \frac{\log\left(\frac{-b+c+\sqrt{c}\sqrt{a-2b+c}}{-b+\sqrt{a}\sqrt{c}}\right)}{40\sqrt{c}} \\ &+ \frac{\sqrt{a}\sqrt{a-2b+c} - \sqrt{c}\sqrt{a-2b+c}}{120(a-2b+c)^{\frac{3}{2}}} \\ &+ \frac{(2a-5b+3c)\log\left(\frac{\left(a-b+\sqrt{a}\sqrt{a-2b+c}\right)\left(c-b+\sqrt{c}\sqrt{a-2b+c}\right)}{\left(b-a+\sqrt{a}\sqrt{a-2b+c}\right)\left(b-c+\sqrt{c}\sqrt{a-2b+c}\right)}\right)}{120(a-2b+c)^{\frac{3}{2}}} \\ &+ \frac{-(\sqrt{a}\sqrt{c}) + \sqrt{a}\sqrt{a-2b+c}}{120a^{\frac{3}{2}}} \\ &+ \frac{(2a+b)\log\left(\frac{\left(b+\sqrt{a}\sqrt{c}\right)\left(a-b+\sqrt{a}\sqrt{a-2b+c}\right)}{\left(-b+\sqrt{a}\sqrt{c}\right)\left(-a+b+\sqrt{a}\sqrt{a-2b+c}\right)}\right)}{120a^{\frac{3}{2}}} \end{aligned} \tag{26}$$

$$
\begin{aligned}
&\frac{1}{4A^2}\iint\limits_A \iint\limits_A \lambda_2' \lambda_1 \frac{1}{|\vec{r}-\vec{r}\,'|} da\ da' \\
&= \frac{\log\left(\frac{b+\sqrt{a}\sqrt{c}}{b-c+\sqrt{c}\sqrt{a-2b+c}}\right)}{120\sqrt{c}} + \frac{\log\left(\frac{a-b+\sqrt{a}\sqrt{a-2b+c}}{-b+\sqrt{a}\sqrt{c}}\right)}{120\sqrt{a}} \\
&+ \frac{-(\sqrt{a}\sqrt{a-2b+c}) + \sqrt{c}\sqrt{a-2b+c}}{120(a-2b+c)^{\frac{3}{2}}} \\
&+ \frac{(2a-3b+c)\log\left(\frac{a-b+\sqrt{a}\sqrt{a-2b+c}}{b-c+\sqrt{c}\sqrt{a-2b+c}}\right)}{120(a-2b+c)^{\frac{3}{2}}} \\
&+ \frac{\sqrt{a}\sqrt{a-2b+c} - \sqrt{c}\sqrt{a-2b+c}}{120(a-2b+c)^{\frac{3}{2}}} \\
&+ \frac{(a-3b+2c)\log\left(\frac{-b+c+\sqrt{c}\sqrt{a-2b+c}}{-a+b+\sqrt{a}\sqrt{a-2b+c}}\right)}{120(a-2b+c)^{\frac{3}{2}}} \\
&+ \frac{-3\sqrt{a}\sqrt{c} + 3\sqrt{c}\sqrt{a-2b+c}}{120c^{\frac{3}{2}}} \\
&+ \frac{(3b+2c)\log\left(\frac{-b+c+\sqrt{c}\sqrt{a-2b+c}}{-b+\sqrt{a}\sqrt{c}}\right)}{120c^{\frac{3}{2}}} \\
&+ \frac{-3\sqrt{a}\sqrt{c} + 3\sqrt{a}\sqrt{a-2b+c}}{120a^{\frac{3}{2}}} \\
&+ \frac{(2a+3b)\log\left(\frac{b+\sqrt{a}\sqrt{c}}{-a+b+\sqrt{a}\sqrt{a-2b+c}}\right)}{120a^{\frac{3}{2}}}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\frac{1}{4A^2}\iint\limits_A \iint\limits_A \lambda_1' \frac{1}{|\vec{r}-\vec{r}\,'|} da\ da' \\
&= \frac{-\log\left(\frac{-b+\sqrt{a}\sqrt{c}}{a-b+\sqrt{a}\sqrt{a-2b+c}}\right)}{24\sqrt{a}} + \frac{\log\left(\frac{b+\sqrt{a}\sqrt{c}}{b-c+\sqrt{c}\sqrt{a-2b+c}}\right)}{24\sqrt{c}} \\
&+ \frac{-(\sqrt{a}\sqrt{c}) + \sqrt{a}\sqrt{a-2b+c}}{24a^{\frac{3}{2}}} \\
&+ \frac{(a+b)\log\left(\frac{b+\sqrt{a}\sqrt{c}}{-a+b+\sqrt{a}\sqrt{a-2b+c}}\right)}{24a^{\frac{3}{2}}} \\
&+ \frac{\log\left(\frac{a-b+\sqrt{a}\sqrt{a-2b+c}}{b-c+\sqrt{c}\sqrt{a-2b+c}}\right)}{24\sqrt{a-2b+c}} - \frac{\log\left(\frac{-b+\sqrt{a}\sqrt{c}}{-b+c+\sqrt{c}\sqrt{a-2b+c}}\right)}{12\sqrt{c}} \\
&+ \frac{\sqrt{a}\sqrt{a-2b+c} - \sqrt{c}\sqrt{a-2b+c}}{24(a-2b+c)^{\frac{3}{2}}} \\
&\frac{+ (a-3b+2c)\log\left(\frac{-b+c+\sqrt{c}\sqrt{a-2b+c}}{-a+b+\sqrt{a}\sqrt{a-2b+c}}\right)}{24(a-2b+c)^{\frac{3}{2}}}
\end{aligned} \tag{28}
$$

with

$$
\begin{aligned}
&a = (\vec{r}_3 - \vec{r}_1)\cdot(\vec{r}_3 - \vec{r}_1), b = (\vec{r}_3 - \vec{r}_1)\cdot(\vec{r}_3 - \vec{r}_2) \\
&c = (\vec{r}_3 - \vec{r}_2)\cdot(\vec{r}_3 - \vec{r}_2).
\end{aligned} \tag{29}
$$

The remaining integrals can easily be determined by geometric interpretations.

The computational cost for evaluating the given analytical formulas is mainly determined by calculating the log functions. By properly checking the expressions in (25)–(28), one will find that only six different log functions are necessary for the calculation of all integrals. If one likes to calculate the singular self-coupling terms basing on the formulas given in Section II-A, one has to evaluate the testing integrals by numerical quadrature. For each sampling point on the triangular subdomain, the calculation of three log functions is necessary. So the computational cost for only two sampling points is similar to the computational cost for the formulas (25)–(28). In our experience it is not recommendable to use less than seven sampling points because the results of the source integrals still show singular behavior (logarithmic singularity).

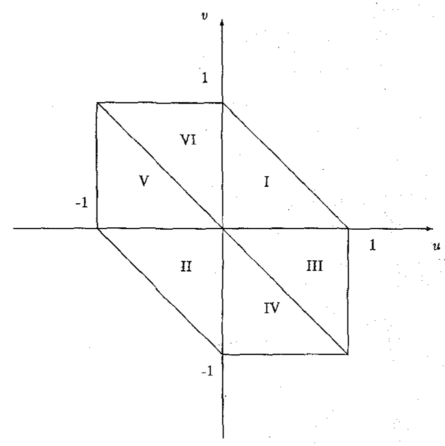

Fig. 2. Integration domain for (u, v).

III. Conclusion

We have presented analytical formulas for the calculation of the singular field contributions due to linear source distributions on triangular domains. The formulas comprise singular potential integral contributions for the scalar free-space Green function and its gradient. The formulation in simplex coordinates is well suited for numerical implementations. Closed-form expressions for the singular self-coupling terms of method of moment surface integral equation techniques with triangular subdomains are given.

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