Capacitance Computation of a Charge Conducting Plate using Method of Moments

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***Abstract*—In this paper the capacitance of a charge conducting plate has been calculated using Method of Moments with MATLAB. The result is also compared with the Least Square Approximation to analyze the charge distribution of a metallic surface.**

***Keywords*—Capacitance, Method of Moments, Least Square Approximation.**

I. Introduction

The field of Computational Electromagnetics has become popular and got rapid pace in the last two decades. There were several techniques has been proposed in the past to solve the Electromagnetics problem due to the radiation of electromagnetic objects such as transmission line , wire antennas and scatters. Some of this techniques are Methods of Moments (MOM),finite difference method (FD), Monte Carlo method (MCM), finite element method (FEM), and variation methods (VM) etc. But MOM has some certain advantages and disadvantages depending upon the formulation of the problems. Tactically MOM used to solve the differential equations. The method of moment (MOM) is a numerical procedure for solving linear operator equation by transforming it into a system of simultaneous linear algebraic equation, referred to as matrix equation. Many problems in electromagnetics can be established in the form of integral equations. An integral equation is one in which the unknown function appears in the integrand. The MOM provides a way to solve such integral equations for both the static as well as time harmonic electromagnetic fields.

As we goes from MHz to GHz of frequency the Electromagnetics Modelling(EM) has become a popular subject of interest in the last two decades. The high operating frequency of electromagnetic objects causes simultaneous electromagnetics interaction within the circuits. So it is very important to evaluate the capacitance and charge distribution of the metallic structures very accurately in the integrated circuits. The MOM helps us to transform the integral equation into a matrix equations based on the expansion of the unknowns in terms of the basis function with unknown coefficients such as charge distributions and based on this capacitance can be calculated.

This paper introduced a deep insight of the numerical solution to solve the field density distribution and capacitance of a 2D charge plate and 2D parallel plate using MOM. The convergence of the numerical solutions of the MOM has also been compared with Least Square Method.

II. Mathematical Equation

We can consider an Electromagnetics problems in terms of inhomogeneous equation

$$L(f) = g \qquad (1)$$

where L is a linear operator, g is a known function (excitation) and f is the unknown function to be determined. We expand f in a series of functions

$$f = \sum_n \alpha_n f_n \qquad (2)$$

Where α_n are constant. The set f_n is called the expansion function or basis function. For exact solution, $n \to \infty$, but in practice truncated to a finite value.

$$\sum_n \alpha_n L(f_n) = g \qquad (3)$$

We define a set of weighting functions or testing functions $w_1, w_2 .... w_N$ in the range of L and take the inner product of equation (3) with each of the w_m

$$\sum_n \alpha_n \langle w_m, \alpha f_n \rangle = \langle w_m, g \rangle \qquad (4)$$

A scalar product $\langle w, g \rangle$ is defined to be a scalar satisfying

$$\langle w, g \rangle = \langle g, w \rangle \qquad (5)$$

$$\langle b\varphi + cg, w \rangle = b\langle \varphi, w \rangle + c\langle g, w \rangle \qquad (6)$$

$$\langle g^*, g \rangle > 0 \quad if \quad g \neq 0 \qquad (7)$$

$$\langle g^*, g \rangle = 0 \quad if \quad g = 0 \qquad (8)$$

Here b and c are scalars and * indicates complex conjugation.

The inner product corresponding to our previous example is of the form

$$\langle w, g \rangle = \int_{-a}^{a}\int_{-a}^{a} w(x,y)g(x,y)dxdy \qquad (9)$$

Similarly, the testing function for our previous example is

$$w_m = \delta(x - x_m)\delta(y - y_m) \qquad (10)$$

i.e. the testing functions are Dirac delta functions. Such choice of testing function is called **point matching**.

The equation (4) can be reduced to

$$[A_{mn}]\bar{\alpha}_n = \bar{g}_m \qquad (11)$$

Where

$$[A_{mn}] = \begin{bmatrix} \langle w_{1,} Lf_1 \rangle & \langle w_{1,} Lf_2 \rangle & . & . \\ \langle w_{2,} Lf_1 \rangle & \langle w_{2,} Lf_2 \rangle & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

$$\bar{\alpha}_n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \end{bmatrix}$$

$$\bar{g}_n = \begin{bmatrix} \langle w_{1,} g \rangle \\ \langle w_{2,} g \rangle \\ . \\ . \end{bmatrix}$$

If $[A_{mn}]$ is non-singular one can write

$$\bar{\alpha}_n = [A_{mn}]^{-1}\bar{g}_m \qquad (12)$$

III. FIELD DISTRIBUTION OF CHARGE CONDUCTING SINGLE PLATE

A. *Method of Moments(MOM)*

To start with, let us consider the an example of determining the electrostatic potential due to an isolated charged conducting plate, $2a$ meters on a side and lying on the $z = 0$ plane with its center at the origin as shown in Fig.1.

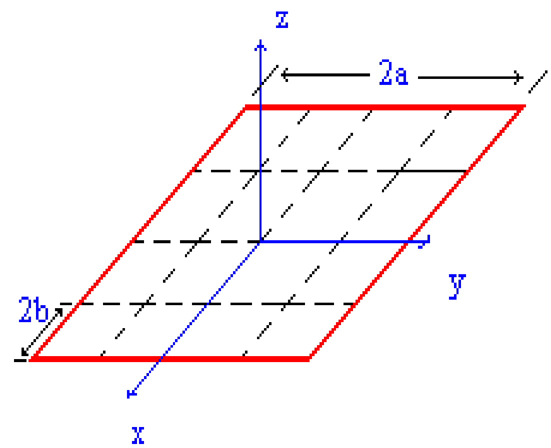

Fig. 1 Square conducting charged plate

The plate is assumed to have zero thickness. Let $\sigma(x, y)$ represent the surface charge density on the plate. Further, the boundary condition on the potential is $V = V_0$ =constant on the plate. For the charged conducting plate, the potential V at any point (x, y, z) can be written as:

$$V(x,y,z) = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{4\pi\varepsilon_0} \frac{\sigma(x',y')dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \qquad (13)$$

If $\sigma(x, y)$ is known (which is often assumed in solving simple electrostatic problems, i. e. charged density is specified), the potential function V can be computed directly. But in practical problems, often the charge distribution $\sigma(x, y)$ is not known. The equation (13) is an example of integral equation as the unknown $\sigma(x, y)$ appears under the integral. To solve the unknown charge distribution we apply the method of moments. The procedure is explained below:

We subdivide the plate into N squares of side $2b$ as shown in Fig. 1. The n^{th} square is denoted by ΔS_n and $2b = \frac{2a}{\sqrt{N}}$.

We approximate the charge distribution as:

$$\sigma(x', y') \cong \sum_{n=1}^{N} \alpha_n f_n(x', y') \qquad (14)$$

where $f_n(x', y') = \begin{cases} 1 & on\ \Delta S_n \\ 0 & on\ \Delta S_m \quad m \neq n \end{cases}$

Here, the functions $f_n(x', y')$ are called the **expansion functions** or **basis functions** and α_n are the coefficients. Basis functions can be of different types; here we have considered pulse basis functions, which have unit magnitude over some domain and are zero everywhere else. With N sufficiently large, equation (14) closely approximates the actual $\sigma(x', y')$. To solve for the charge distribution $\sigma(x', y')$ approximately, the unknown coefficients α_n are to be determined. From the given boundary condition we note that on the surface of the plate $V(x, y, 0) = V_0$ and this condition can be used to determine the unknown coefficients α_n.

Using the approximate charge distribution given by equation (8.16), let us now evaluate the equation (13) at the mid point $(x_m, y_m, 0)$ of each of the ΔS_m. The potential V_m at the midpoint of ΔS_m is given by:

$$V_m = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{4\pi\varepsilon_0} \frac{\sum_{n=1}^{N} \alpha_n f_n(x', y')\, dx'dy'}{\sqrt{(x_m - x')^2 + (y_m - y')^2}} \qquad (15)$$

The above equation can be written as

$$V_m = \sum_{n=1}^{N} \alpha_n V_{mn} \qquad (16)$$

where V_{mn} is the potential at the center of ΔS_m due to unit charge placed on ΔS_n.

For $m = n$,

$$V_{mn} = \int_{-b}^{b}\int_{-b}^{b} \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{x'^2 + y'^2}}\, dx'dy' = \frac{2b}{\pi\varepsilon_0} \ln(1 + \sqrt{2}) \qquad (17)$$

For $m \neq n$, treating the unit charge on ΔS_n as a point charge located at the mid point (x_n, y_n) of ΔS_n,

$$V_{mn} = \frac{b^2}{\pi\varepsilon_0 \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}} \qquad (18)$$

As $V_m = \sum_{n=1}^{N} \alpha_n V_{mn} = V_0$, considering the potentials at all the N sub sections we can write

$$\begin{bmatrix} V_{11} & V_{12} & . & . & V_{1N} \\ V_{21} & V_{22} & . & . & V_{2N} \\ . & . & . & . & . \\ . & . & . & . & . \\ V_{N1} & V_{N2} & . & . & V_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_N \end{bmatrix} = \begin{bmatrix} V_0 \\ V_0 \\ . \\ . \\ V_0 \end{bmatrix}$$

Or,

$$[V]\overline{\alpha} = \overline{V}_0 \qquad (19)$$

The unknown coefficients α_n s can be computed as

$$\overline{\alpha} = [V]^{-1}\overline{V}_0 \qquad (20)$$

Fig. 2 and Fig. 3 shows the charge distribution obtained using the MOM technique for $V_0 = 1V$, $2a = 1$ m.

Approximate charge density along the side of the plate

Another parameter of interest is the capacitance of the plate.

$$C = \frac{Q}{V_0} = \frac{1}{V_0} \int_{-a}^{a}\int_{-a}^{a} \sigma(x, y)\, dxdy \qquad (21)$$

With α_n known, the capacitance of the plate can be approximated as:

$$C \approx \frac{1}{V_0} \sum_{n=1}^{N} \alpha_n \Delta s_n \qquad (22)$$

In this example we have used pulse type basis function and point matching, that is, Dirac delta function as testing or weighting function.

B. Least Square Approximatioin

The charge density and the value of the true capacitance of the above problem can be found using Least Square Approximation. In the method of least squares, the solution of $\alpha_n = V_0$ is attempted by minimizing the functional $F^1(\alpha_n)$ which can be given by

$$F^1(\alpha_n) = ||V \alpha_n - V_0||^2 \qquad (23)$$

The optimal solution of α_n is given by

$$\alpha_n = (V^T V)^{-1}(V^T V_0) \qquad (24)$$

IV. EXPERIENTAL RESULT

With the reference of the Figure 3 the charge density for different number of section has been approximated. For computing the more accurate result a maximum number of 65×65 subsections has been considered. Using the (20),

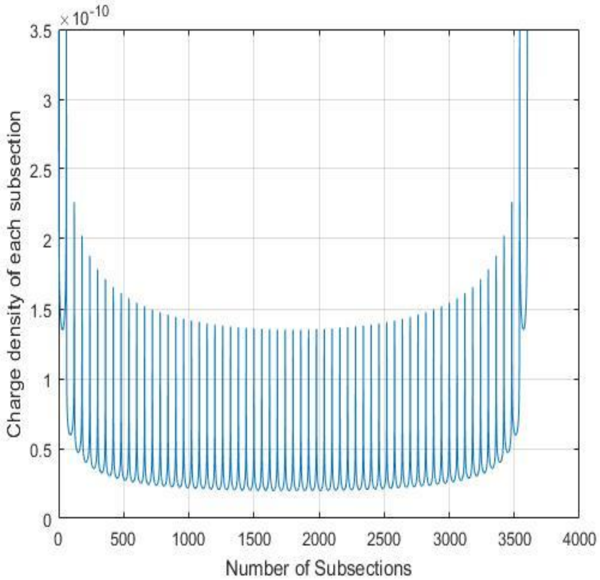

Fig 2 . Approximate charge density on each subsection adjacent to the centerline of a unit square conducting plate

adding the known boundary condition(1V), the unknown charge distribution has been calculated with a MATLAB program which helps to find the capacitance of the charge conducting plate. From thefigure 2 the charge density of each number of subsection has been determined.

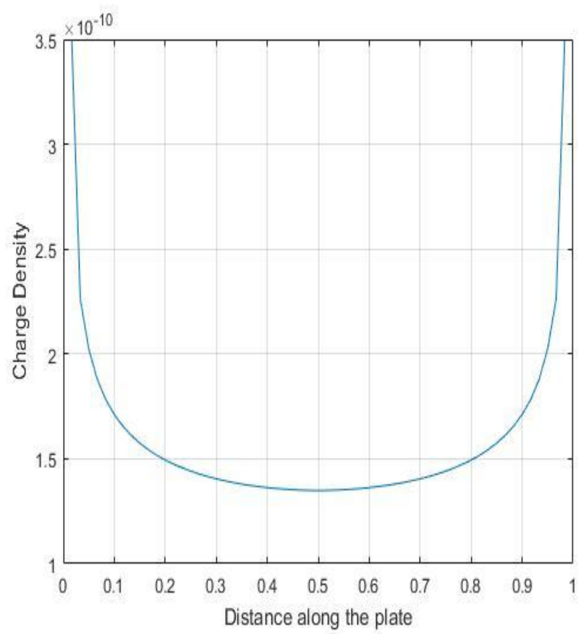

Fig 2 . Approximate charge density on subsection adjacent to the centerline of a unit square conducting plate

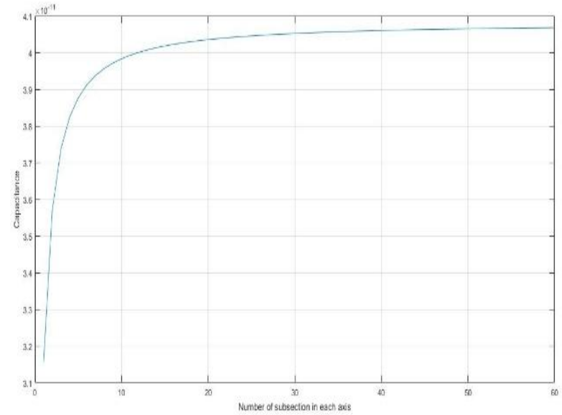

Fig 3. Capacitance with different number of subsection

Figure 3 shows the approximate charge density on subsection adjacent to the centerline of a unit square conducting plate which is asymptotic in nature. Figure 4 shows that after increasing the certain no. of subsection (in this case 10×10) the capacitance of the unit square conducting converges. The Table-1 shows the value of the capacitance calculated by (22) and (24).

Table-1 shows the true value of capacitance has been found in Method of Moment and Least Square Approximation for different number of subsection. The value of the capacitance found in both the cases are exactly same. But least square approximation takes more computational load to obtain the strong convergence of the residuals. It is also evident that even through V is unbounded $\alpha_n = V_0$converge in the mean to zero monotonically.

Number of Subsections (N^2)	**Capacitance(pF) Using Method of Moments**	**Capacitance(pF) Using Least Square Approximation**
1	31.55	31.55
4	35.70	35.70
9	37.37	37.37
16	38.23	38.23
36	39.12	39.12
100	39.83	39.83
225	40.18	40.18
400	40.34	40.35
625	40.45	40.45
1225	40.56	40.56
2025	40.62	40.62
2500	40.64	40.64
3025	40.66	40.66
3600	40.67	40.67
4225	40.69	40.69

Table-1 Comparison of the experimental result of Method of Moments and Least Square Approximation

It is important to note that the sequence of solution of α_n inboth the method converge asymptotically. In case of asymptotic convergence the solution tend to converge more and more as much as basis function are chosen. As the method of least square guarantees monotonic convergence, it is better to use least square approximation when the exact solutions is not known. In this scenario it is easier to established relative errors estimates the solution.

V. CONCLUSION

In this paper two different method has been proposed to evaluate the charge distribution and capacitance of a charge conducting single plate to analyze the computational load of the each method. The experimental result of the capacitance for different number of subsection demonstrates that the value of the capacitance converges with increasing the number of subsection of the unit square plate. As both of the methods-Method of moments and Least Square approximation convey exactly same result, more computational work is needed for the Least Square Approximation to obtain the strong convergence of the residuals by minimizing the relative error. The steeper slope of the charge distribution of the conducting surface exhibit that the free charge distribution fetch up to the edge of the metallic surface and comparatively flat at the middle section of the conducting plate. The Converge value of the capacitance is 40.69 pF in both cases.

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