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Capacitors and the method of moments

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Abstract

The inhomogeneous charge distribution in several plate capacitors is calculated using the Method of Moments. Effects of global and local "inhomogeneities" in the capacitor are simulated.

1. Introduction

Problems that an undergraduate encounters in the first course in electromagnetic theory are typically highly idealized and soon become mathematically intractable. The required computer sophistication needed to obtain a pictorial solution to a simple practical problem has in the past been "beyond the scope of this course." The purpose of the present note is to take a somewhat more difficult but realistic electrostatics problem in order to show that it can be easily treated using the widely available software, for example MATLAB together with all of its graphical capabilities. This software is widely available and most students have previous experience with it. In particular, we pose and then solve the problem: "Accurately sketch the charge density distribution and the evolution of the voltage within the capacitor." The solution of this problem would introduce and will fortify the student's understanding of electromagnetic theory and the numerical solution of integral equations. It will also provide an introduction to the subject of the "method of moments." This topic is of the current interest in the electromagnetic theory community at the graduate level and beyond [1]. However, it is now starting to appear in educational articles [2] and in undergraduate textbooks [3].

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Though we focus our study to a capacitor that has a relatively simple shape, the procedure can also be applied to a capacitor with an arbitray shape. In Section 2, we summarize the procedure that allows one to calculate the capacitance of a capacitor. This includes a discussion of the singular term that may be encountered using this procedure. The procedure is applied to several examples in Section 3. Section 4 is the conclusion.

2. Procedure

With reference to Fig. 1, we subdivide the square area $a \times a$ of the top plate and of the bottom plate of a parallel-plate capacitor into $2 \times N \times N$ sub-areas. The charge in a particular sub-area ΔQ is assumed to be *concentrated* at its center and have a value

$$\Delta Q = \rho\left(\frac{a}{N} \times \frac{a}{N}\right) \tag{1}$$

where ρ is the charge density in that particular sub-area. The potential at the center of a sub-area due to the charge ΔQ in another sub-area can be computed from

$$V(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon} \int \int_{\Delta A} \frac{\rho(\mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|}\, \mathrm{d}s = \frac{1}{4\pi\varepsilon} \frac{\Delta Q(\mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{2}$$

In Fig. 1, there will be $2 \times N \times N = 8$ sub-areas (i.e., point charges) that will contribute to this potential $V(i,k)$. Seven of the terms will have $\mathbf{r_j}$ not equal to $\mathbf{r_k}$ and will create no difficulty in the calculation. However, there will be a singularity in the remaining term when $\mathbf{r_j} = \mathbf{r_j}$. In order to remove this singularity, we use the standard procedure [3] of evaluating the potential at the edge of a circle of radius b if this point charge is located at the center of the circle. The value of this point charge is assumed to be equal to the total charge that is located in that particular sub-area. This implies that the area of the circle equals the area of the sub-area. The potential at the center of the circle is assumed to be equal to the value that it has at the edge of the circle. This term will be evaluated in full detail.

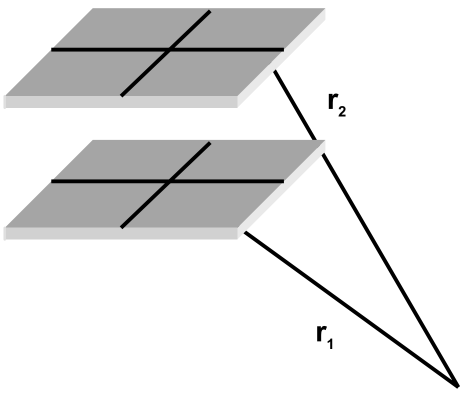

Fig. 1. A parallel plate capacitor that is divided into $2 \times N \times N = 8$ sub-areas.

$$V(i,i) = \frac{1}{4\pi\varepsilon}\int_0^{2\pi}\int_0^b \frac{\rho r\,\mathrm{d}r\,\mathrm{d}\varphi}{r} = \frac{\rho}{4\pi\varepsilon}(2\pi b) = \frac{\rho}{4\pi\varepsilon}\left(2\pi\sqrt{\frac{\Delta A}{\pi}}\right) = \frac{\rho}{4\pi\varepsilon}(2\sqrt{\pi})\sqrt{\Delta A} \tag{3}$$

Therefore, the potential at the centers of all $2 \times N \times N = 8$ sub-areas in Fig. 1 can be computed.

The calculation outlined above is straightforward, but it has made the tacit assumption that the charge density ρ is a priori, a quantity that is already known. Since the plates are made of a conducting material, a student might feel more comfortable being told that the potential of the entire plate had a specified value, say V_1 or V_2. The unknown quantity in this case would be the actual charge distribution that exists on that plate. With the specified values for the voltage difference between the two plates being a constant, the student will, in essence, calculate the capacitance between the two plates of the capacitor.

In Fig. 1, the two plates were subdivided into a total of 8 subareas. In practice, one would subdivide each of the plates into a large number of sub-areas. In the MATLAB program that is available from the authors, there are more than 400 sub-areas in each surface. The sub-areas that are located at the central regions of the two plates have a certain element of symmetry. The sub-areas at the edges of a plate lose this symmetry. In a calculation, the voltage is specified to have a constant value on the two plates. However, the required charges in the sub-areas at the edge of a plate must be larger than that at the center in order to ensure this constant voltage on the plates. Therefore, one must be cognizant of this lack of symmetry at the edges in the interpretation of the numerical calculation.

The *modus operandi* of any computer program that will effect the above calculation will have the following basic steps:

(1) Identify the sub-areas of the two surfaces. As will be shown below, metallic perturbations or inhomogeneous nonparallel capacitor plates can easily be introduced at this stage.
(2) Calculate the matrix elements that relate the potential in a particular sub-area to the charge density in a different sub-area or in the same sub-area. In the latter case, the singular term must be removed by the method discussed previously.
(3) Set the voltage on each plate.
(4) Calculate and plot the charge distribution on the top and on the bottom plates. This step will involve the inversion of a very large matrix. MATLAB, which has been used by us, is well-equipped to handle this calculation and to present the results in pictorial form.

3. Examples

The capacitance of a single conducting flat plate has been computed elsewhere [4] and shall be used as a benchmark in order to demonstrate the efficacy of the program. In particular, it was stated that the capacitance of a square plate that had an area of 1 m^2 had a capacitance of 40.82 ± 0.001 pF. The value that we computed using MATLAB for a square plate that was subdivided into 32 sub-areas was 43.855 pF.

We consider four capacitors with different configurations. They are: a normal parallel capacitor, a capacitor with nonparallel plates, a parallel plate capacitor with a concave perturbation at the center and a parallel plate capacitor with a convex perturbation at the center respectively. The charge distribution for each of the four capacitors is calculated and displayed in juxtaposition with the capacitor. A factor of $4\pi\varepsilon$ is absorbed into the value of the charge distribution. In the calculation, the voltages are equal to $\pm\frac{1}{2}$ V at the top and bottom plates. Therefore, except for the factor of $4\pi\varepsilon$ the distribution of the capacitance is displayed. Since the charge distribution is now stored in the computer, it is also possible to display the voltage distribution at several locations between the metallic surfaces. The results of this calculation will also be displayed in juxtaposition with the capacitor. Peculiarities that will be encountered for each capacitor will be discussed below. We have arbitrarily subdivided the top and bottom plates into $2 \times N \times N = 800$ sub-areas. The calculations were performed on a PC using MATLAB.

A capacitor with parallel plates that have a uniform separation is shown in Fig. 2a. This is, of course, is the normal capacitor that the student has frequently encountered in other classes. The plate has been divided into $N \times N$ sub-areas.

In the work to follow, we minimize the contributions of the edges and the corners of the plate by first calculating the charge distribution at all $N \times N$ points. At the edges and at the corner, there is a significant loss of symmetry. In order to compensate for this loss of symmetry, we eliminated an outer ring containing the outer two points when the charge distribution is actually shown. In addition, these points are eliminated in the later calculation of the voltage distribution between the plates. We note that the calculated charge distribution shown in Fig. 2b is nonuniform on both the top and the bottom plates.

Since the numerical values for the local charge distribution have been calculated in the program, it is possible to add them up in order to calculate the total charge Q_T that is stored on one of the plates. The charge distribution ρ can be estimated from the ratio of this total charge divided by the number of selected sub-areas.

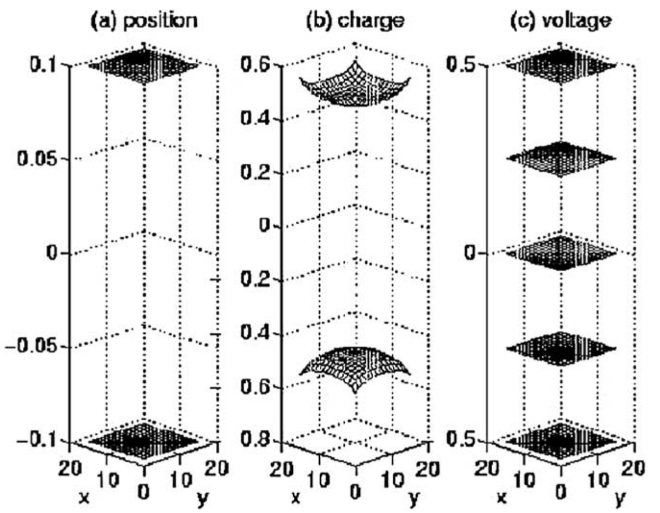

Fig. 2. A parallel plate capacitor. (a) Numerical depiction of a parallel plate. capacitor. (b) Calculated charge distribution for a parallel plate capacitor. A factor of $4\pi\varepsilon$ has been absorbed in the charge distribution. (c) Voltage distribution at several locations within a parallel plate capacitor.

$$\rho = \frac{Q_T}{(N-4)\times(N-4)} = \frac{126.5624}{(20-4)\times(20-4)} \approx 0.4944 \tag{4}$$

This estimate is in good agreement with the numerical values shown in Fig. 2b. At this point, the instructor could extend this calculation to include a discussion of any numerical errors that may be encountered. This topic is beyond the scope of this paper.

The potential distribution at three parallel planes uniformly separated between the top and the bottom plates are also calculated using the just obtained charge distribution on the two plates. The results of ths calculation are summarized in Fig. 2c. As expected, the potential increases linearly from the bottom plate where the applied voltage is $V = -\frac{1}{2}$ volt to the top plate where the applied voltage is $V = +\frac{1}{2}$ volt. The potential at the mid plane between the two metallic plates is equal to 0. In addition, the potential distribution is almost uniform in each of the planes that is parallel to these plates.

A capacitor with a nonuniform separation is shown in Fig. 3a. As shown in Fig. 3b, the magnitude of the computed local charge density increases as the separation of the plates decreases. This is true for the charge that resides on either plate. The variation of the potential distribution between the two plates is determined by the charge distribution that has just been computed as shown in Fig. 3c. Note that it is equal to 0 at the midpoint between the plates. This particular example was analyzed in an undergraduate course in electromagnetics.

In Fig. 4a, a capacitor with a small localized spatially concave perturbation in both plates is shown. The shape of the perturbation has been chosen to have the same square shape as the sub-areas. If a different shape had been chosen, there would be an anomalous alteration in the computed charge distribution at the edge of the perturbation that is shown in Fig. 4b. This effect would be minimized by increasing the number of sub-areas. We note that the charge distribution

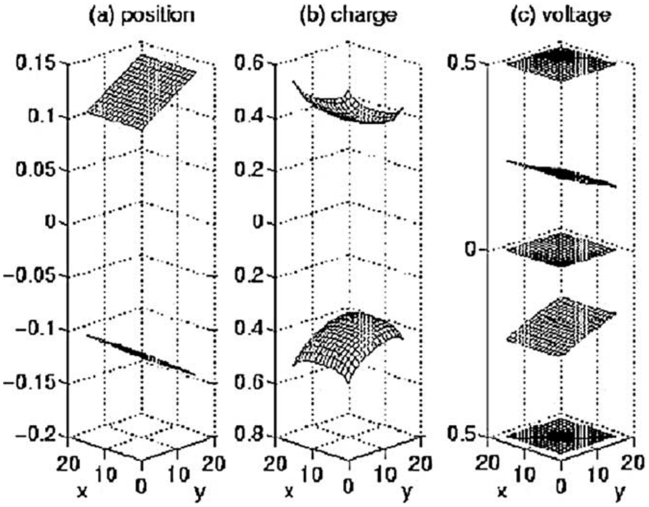

Fig. 3. A spatially inhomogeneous capacitor. The separation of the plates increases as y increases. (a) Numerical depiction of a spatially inhomogeneous capacitor. (b) Calculated charge distribution for a spatially inhomogeneous capacitor. A factor of $4\pi\varepsilon$ has been absorbed in the charge distribution. (c) Voltage distribution at several locations within a spatially inhomogeneous capacitor.

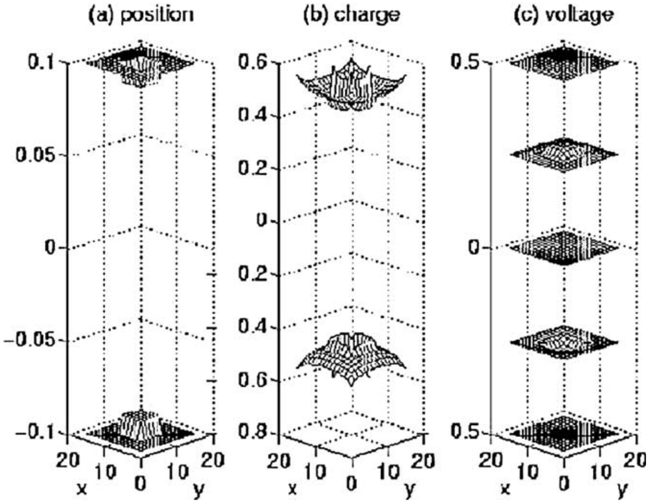

Fig. 4. A parallel plate capacitor that contains a localized concave perturbation. (a) Numerical depiction of a parallel plate capacitor that contains a localized concave perturbation. (b) Calculated charge distribution for a parallel plate capacitor that contains a localized concave perturbation. A factor of $4\pi\varepsilon$ has been absorbed in the charge distribution. (c) Voltage distribution at several locations within a parallel plate capacitor that contains a localized concave perturbation.

in both the top and bottom plates are affected by the perturbation. The computed voltage distribution between the two plates is shown in Fig. 4c.

In Fig. 5a, a capacitor with a small localized spatially convex perturbation in both plates is shown. The shape of the perturbation has also been chosen to have the same square shape as the

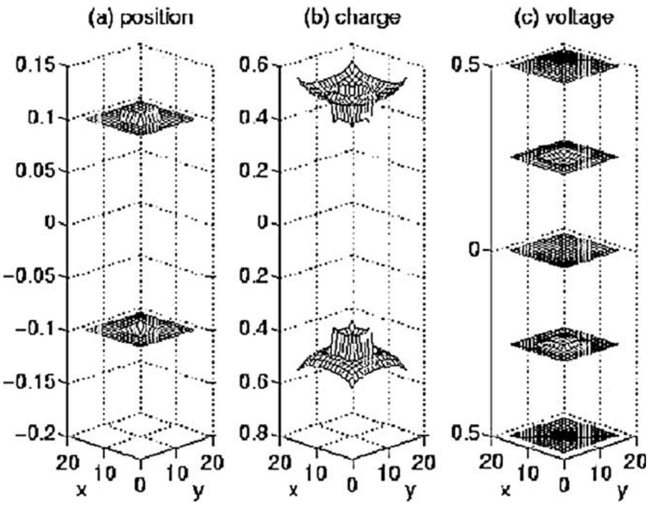

Fig. 5. A parallel plate capacitor that contains a localized convex perturbation. (a) Numerical depiction of a parallel plate capacitor that contains a localized convex perturbation. (b) Calculated charge distribution for a parallel plate capacitor that contains a localized convex perturbation. A factor of $4\pi\varepsilon$ has been absorbed in the charge distribution. (c) Voltage distribution at several locations within a parallel plate capacitor that contains a localized convex perturbation.

sub-areas. This effect would be minimized by increasing the number of sub-areas. We note that the charge distribution in both the top and bottom plates are affected by the perturbation. The computed voltage distribution between the two plates is shown in Fig. 5c.

4. Conclusion

We believe that the powerful tool of numerical calculation will help students in understanding a fairly difficult topic since most students are computer literate. There are several extensions to the series of calculations that can be easily incorporated into the program. For example, the distribution of the electric field between the plates could be assigned as an example. The detailed MATLAB programs that have been employed here are available from the authors.

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