

Precise Calculation of Capacitance of parallel Plate Capacitors by means of Quadruple Integrals in Method of Moments technique

SAEED SARKARATI*

Shahid Beheshti University
saeedsarkarati@gmail.com

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Abstract

In this paper capacitance of parallel plate air gapped rectangular capacitor has been calculated. Because of its generality and simplicity MOM Technique is utilized. Due to precision increment, applying quadruple integrals instead of double integrals has been suggested. A neat form for analytical solution of integrals needed for MOM technique is presented. The results show very small error for capacitance calculation even with rough boundary tessellation. Described formulae and codes can easily be used for similar purposes.

I. INTRODUCTION

FROM early water-filled Leyden jar upto modern supercapacitors, precise calculation of capacitance is a challenging problem for scientists and researchers.

Among various geometries of capacitors, parallel plate air-gapped capacitor is a typical capacitor which its capacitance has been calculated by a variety of methods such as finite difference and finite element methods, monte carlo technique and method of moments. Except stochastic methods, the other ones make an equation matrix which must be solved to get the capacitance of desired geometry. Size of this matrix is a critical parameter resulting time needed for capacitance calculation.

In method of moments (MOM) the elements are placed only on boundary surfaces. And in special case of parallel plate air-gapped capacitor only two conducting plate surface are involved in generating matrix elements. So the

matrix size is very small compared with matrix size in finite difference and finite element methods. Although these two techniques conclude sparse matrices, huge amount of their matrix sizes can not be compensated by using improved fast sparse matrix manipulation techniques.

To do calculations in MOM it is needed firstly to make coupling matrix which consists of mutual and self coupling coefficients between boundary segments or tiles. And then matrix equation must be solved. Various geometries for boundary segments have been suggested by authors. triangular tiles, square tiles, rectangular tiles and arbitrary shaped tiles have been considered as boundary segments in MOM by researchers and scientists.

To obtain coupling matrix coefficients, it is needed to integrate over segment plates. These integral calculations can be done numerically or analytically. Numerical integration can generally be used for every segments. It can be done by stochastic or direct methods;

*A thank you or further information

and there are many optimised packages and libraries for doing it; But speed of calculation is obviously large against using exact formulae obtained from analytical calculations. In the other words we usually expect to spend cpu time for solving matrix equation, but numerical integration makes generating matrix time more larger. Although some authors have suggested mixed methods to obtain simplicity and speed together.

In this paper we have used an integral transformation to solve quadruple integrals which give us exact formulae for coupling matrix coefficients with no approximation. These analytical Terms have been used to generate coupling matrix. Matrix equation have been solved and electric capacitance has been calculated for parallel plate air gapped capacitor and the results have been compared with approximated integral calculation methods. Charge distribution over capacitor plates also has been investigated.

An understandable path to solving quadruple integrals and very neat form for analytical calculation are presented here to help researchers to calculate their own desired geometries.

A python code is written for doing the calculation and it is now available in our website and can be freely used by others. NUMPY and SCIPY libraries have been used to do numerical calculation easily and relatively fast.

II. METHOD OF MOMENTS FOR CAPACITANCE CALCULATION

Basic ideas for MOM technique firstly suggested by great physicist James Clerk Maxwell, who wanted to calculate electric capacity of a metal square. It is worth to have a look at his work which can be found here <https://archive.org/details/electricalresear00caveuoft/page/426>.

He divided the square into 36 equal squares and assumed uniform charge density for each one, Then he assumed electric potential at the middle of each square segment equal to 1. To keep potential of all segments to one, electric

A	B	C	C	B	A
B	D	E	E	D	B
C	E	F	F	E	C
C	E	F	F	E	C
B	D	E	E	D	B
A	B	C	C	B	A

Figure 1: ss

charge of each segment must be different to others. There are 36 Values for 36 segments. but geometric symmetry shows us these 36 values can be grouped into 6 distinct values. He then calculated electric charges of each segment and consequently electric capacitance.

Generally to calculate electric capacitance of a parallel plate capacitor, at first two constant voltage are proposed for two plate (usually 1V and -1V). Now each plate must be divided to segments. Electric potential of each segment must be equal to its plate. On the other hand potential of each segment can be calculated by means of electric charge of all segments and coupling coefficients.

$$V_i = \sum_j P_{ij} Q_j \quad (1)$$

This formula can be written in matrix form.

$$\mathbf{V} = \mathbf{PQ} \quad (2)$$

Now it is needed to solve this matrix equation to calculate charge of each segment, And obviously capacitance can be obtained when electric potential and electric charge are known.

Coupling coefficient between two segment depends on shape and location of two segments. It can be approximated roughly by potential formula of a unit point charge.

$$P_{ij} = \frac{1}{4\pi\epsilon_0 d_{ij}} \quad (3)$$

where d_{ij} is the distance between centers of two segments.

In literature this approximation is used in a method called "surface charge simulation method", although this method can be categorized in varieties of MOM. Really for self

coupling coefficient this formula is not useful. Self coupling is calculated by integrating over the area to find mean distance of all the points of area to its center.

$$P_{ii} = \frac{1}{S_i} \iint \frac{dx dy}{4\pi\epsilon_0 \sqrt{(x - x_{ci})^2 + (y - y_{ci})^2}} \quad (4)$$

Where (x_{ci}, y_{ci}) is the center point of domain i and S_i is the segment area. To find out less approximated results in capacitance calculation it is better to apply double integral formula, not only for self coupling, but also in mutual coupling. It leads us to double integral formula.

$$P_{ij} = \frac{1}{S_i} \iint \frac{dx dy}{4\pi\epsilon_0 \sqrt{(x - x_{cj})^2 + (y - y_{cj})^2 + z^2}} \quad (5)$$

Two domains are assumed to be parallel. z is distance of two domains and is constant over integration. The integral must be calculated over domain i , and distance of each point to center of other domain is considered in this formula. These integrals can be calculated analytically and have been applied to find capacitance of parallel plate capacitors by nishiyama and nakamura.

Obviously this formula is not suitable where two domains are relatively close according to their dimensions. In this case we can not propose center of one domain as representative of all points. Really it is better to find all mutual distances between points of two segments. It can be done by using quadruple integral instead of double integral.

$$P_{ij} = \frac{1}{S_i S_j} \iiint \frac{dx_i dy_i dx_j dy_j}{4\pi\epsilon_0 d_{ij}} \quad (6)$$

where d_{ij} is

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + z^2} \quad (7)$$

III. QUADRUPLE INTEGRATION FOR RECTANGULAR SEGMENTS

To obtain coupling coefficient described in Eq(6) one can use numerical integration

method or solve it analytically. Although there are many improvements in numerical techniques, but analytical solutions still take less process time. Firstly it is done by Eibert and Hansen for triangular domains. Analytical solution for rectangular domains presented by Lopez-Pena and Mosig, with a small mistake in driven formula. Recently this integral has been performed by Maccarrone and Paffuti, and its result has been used to find capacitance and forces for two square electrodes.

To perform integral in Eq(6) we can use this transformation.

$$\frac{1}{d_{ij}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 d_{ij}^2} du \quad (8)$$

And Eq(6) can be rewritten in this form.

$$P_{ij} = \frac{2}{\sqrt{\pi} S_i S_j} \int_0^\infty \iiint \frac{e^{-u^2 d_{ij}^2} dx_i dy_i dx_j dy_j}{4\pi\epsilon_0} du \quad (9)$$

For simplicity we omit constants from formula.

$$I = \int_0^\infty \iiint \int e^{-u^2 d_{ij}^2} dx_i dy_i dx_j dy_j du \quad (10)$$

There is no way to find primitive function of $e^{-u^2 d_{ij}^2}$ over these five integrals, but we can find primitive function over four inner integrals. If we assume J as the primitive function of quadruple integral.

$$J = \iiint \int e^{-u^2 d_{ij}^2} dx_i dy_i dx_j dy_j \quad (11)$$

Then J can be calculated analytically.

$$J = \frac{e^{-u^2(x^2+y^2+z^2)}}{4u^4} + \frac{\sqrt{\pi}e^{-u^2(y^2+z^2)}x \operatorname{erf}(ux)}{4u^3} + \frac{\sqrt{\pi}e^{-u^2(x^2+z^2)}y \operatorname{erf}(uy)}{4u^3} + \frac{\pi e^{-u^2z^2}x y \operatorname{erf}(ux) \operatorname{erf}(uy)}{4u^2} \quad (12)$$

Where $x = x_i - x_j$ and $y = y_i - y_j$. Now these

integrals must be calculated separately.

$$\begin{aligned}
 I_1 &= \left[\int_0^\infty \frac{e^{-u^2(x^2+y^2+z^2)}}{4u^4} \right]_{D_0}^{D_1} \\
 I_2 &= \left[\int_0^\infty \frac{\sqrt{\pi} e^{-u^2(y^2+z^2)} x \operatorname{erf}(ux)}{4u^3} \right]_{D_0}^{D_1} \\
 I_3 &= \left[\int_0^\infty \frac{\sqrt{\pi} e^{-u^2(x^2+z^2)} y \operatorname{erf}(uy)}{4u^3} \right]_{D_0}^{D_1} \\
 I_4 &= \left[\int_0^\infty \frac{\pi e^{-u^2 z^2} x y \operatorname{erf}(ux) \operatorname{erf}(uy)}{4u^2} \right]_{D_0}^{D_1} \quad (13)
 \end{aligned}$$

We want to do calculation over rectangular segments. So limits of integration are over two rectangles.

$$\begin{aligned}
 D_0 : x_i = a_0, y_i = b_0, x_j = c_0, y_j = d_0 \\
 D_1 : x_i = a_1, y_i = b_1, x_j = c_1, y_j = d_1 \quad (14)
 \end{aligned}$$

Finally the answer can be calculated.

$$\begin{aligned}
 I_1 &= \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \frac{\sqrt{\pi}}{6} (x^2 + y^2 + z^2)^{\frac{3}{2}} \\
 I_2 &= \sum_{i,j,k,l=0}^1 -A_{i,j,k,l} \frac{\sqrt{\pi}}{4} x \times \\
 &\quad \left((y^2 + z^2) \sinh^{-1}\left(\frac{x}{\sqrt{y^2 + z^2}}\right) + x \sqrt{x^2 + y^2 + z^2} \right) \\
 I_3 &= \sum_{i,j,k,l=0}^1 -A_{i,j,k,l} \frac{\sqrt{\pi}}{4} y \times \\
 &\quad \left((x^2 + z^2) \sinh^{-1}\left(\frac{y}{\sqrt{x^2 + z^2}}\right) + y \sqrt{x^2 + y^2 + z^2} \right) \\
 I_4 &= \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \frac{\sqrt{\pi}}{2} x y \left(x \sinh^{-1}\left(\frac{y}{\sqrt{x^2 + z^2}}\right) + \right. \\
 &\quad \left. y \sinh^{-1}\left(\frac{x}{\sqrt{y^2 + z^2}}\right) - z \tan^{-1}\left(\frac{x y}{z \sqrt{x^2 + y^2 + z^2}}\right) \right) \quad (15)
 \end{aligned}$$

In above expressions $x = a_i - c_j$ and $y = b_k - d_l$. The amount of $A_{i,j,k,l}$ depends on sum of i to l . If this summation is an odd number $A_{i,j,k,l}$ becomes -1, otherwise its value equals to 1.

$$A_{i,j,k,l} = \begin{cases} 1 & \text{if } i+j+k+l \text{ is even} \\ -1 & \text{if } i+j+k+l \text{ is odd} \end{cases} \quad (16)$$

Now it is needed to find sum of I_1 to I_4 .

$$\begin{aligned}
 I &= I_1 + I_2 + I_3 + I_4 = \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \\
 &\quad \left[\frac{\sqrt{\pi}}{12} \left((-x^2 - y^2 + 2z^2) \sqrt{x^2 + y^2 + z^2} \right) \right. \\
 &\quad + \frac{\sqrt{\pi}}{4} \left(y(x^2 - z^2) \sinh^{-1}\left(\frac{y}{\sqrt{x^2 + z^2}}\right) \right) \\
 &\quad + \frac{\sqrt{\pi}}{4} \left(x(y^2 - z^2) \sinh^{-1}\left(\frac{x}{\sqrt{y^2 + z^2}}\right) \right) \\
 &\quad \left. - \frac{\sqrt{\pi}}{2} x y z \tan^{-1}\left(\frac{x y}{z \sqrt{x^2 + y^2 + z^2}}\right) \right] \quad (17)
 \end{aligned}$$

As we know " $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$ ", the above formula can be rewritten in this form.

$$\begin{aligned}
 I &= \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \left[\frac{\sqrt{\pi}}{12} \left((-x^2 - y^2 + 2z^2) \sqrt{x^2 + y^2 + z^2} \right) \right. \\
 &\quad + \frac{\sqrt{\pi}}{4} \left(y(x^2 - z^2) \ln\left(\frac{y + \sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + z^2}}\right) \right. \\
 &\quad \left. + x(y^2 - z^2) \ln\left(\frac{x + \sqrt{x^2 + y^2 + z^2}}{\sqrt{y^2 + z^2}}\right) \right) \\
 &\quad \left. - \frac{\sqrt{\pi}}{2} x y z \tan^{-1}\left(\frac{x y}{z \sqrt{x^2 + y^2 + z^2}}\right) \right] \quad (18)
 \end{aligned}$$

In this paper we prefer to use hyperbolic form of formula, which introduced in eq(17). In the case that two segments are coplanar, mutual coupling obtains by taking limit of eq(17) when z goes to zero.

$$\begin{aligned}
 I_{\text{coplanar}} &= \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \left[\frac{\sqrt{\pi}}{12} \left((-x^2 - y^2) \sqrt{x^2 + y^2} \right) \right. \\
 &\quad + \frac{\sqrt{\pi}}{4} \left(y(x^2) \sinh^{-1}\left(\frac{y}{x}\right) + x(y^2) \sinh^{-1}\left(\frac{x}{y}\right) \right) \left. \right] \quad (19)
 \end{aligned}$$

Last formula can not be used if horizontal or vertical distance of two rectangles vanishes to zero. In this case mutual coupling can be calculated with this formula.

$$I_{\text{commonrow}} = \sum_{i,j,k,l=0}^1 A_{i,j,k,l} \left[\frac{\sqrt{\pi}}{12} (-y^3) \right] \quad (20)$$

And finally self coupling of a rectangular segment takes this form.

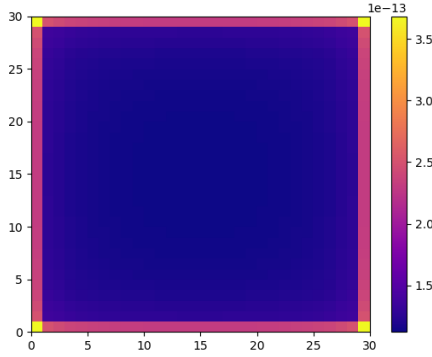


Figure 2: Charge distribution on one plate of parallel plate air gapped capacitor

$$I_{SC} = \frac{\sqrt{\pi}}{3}(x^3 + y^3) + \frac{\sqrt{\pi}}{3} \left((-x^2 - y^2) \sqrt{x^2 + y^2} \right) + \sqrt{\pi} \left(y(x^2) \sinh^{-1}\left(\frac{y}{x}\right) + x(y^2) \sinh^{-1}\left(\frac{x}{y}\right) \right) \quad (21)$$

Now x and y are length and width of rectangular region.

IV. RESULTS

As mentioned above in MOM a set of equation is solved to obtain charge amount of every tile. So charge density distribution is obtained directly in this method. As an example this method applied to an air gapped parallel plate capacitor with $1m \times 1m$ plate dimensions and a plate separation of 10 cm. In fig. 2 charge distribution has been showed. Calculation of chare density in some research areas such as high voltage engineering is a bottleneck of designing high voltage apparatus.

Capacitance of this capacitor has been calculated in three methods. In the first one, point charge approximation is used for mutual coupling and double integration for self coupling. In the second mehod both of self and mutual couplings are calculated by means of double integration. The third method uses quadraple integration for calculating coupling coefficients.

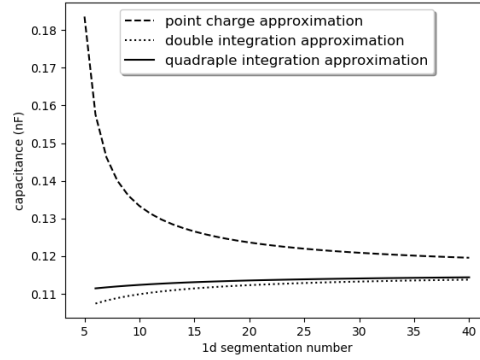


Figure 3: Capacitane of quadraple integration vs two other methods

In fig.3 results of capacitance extraction are shown for these three methods versus number of segmentation of square plate in each dimension (n). Total number of tiles is $2 * n * n$. Because of far answers of point charge method in small n 's, the first five small n 's are omitted. It can be seen that point charge approximation is out of accuracy and two others are more compatible.

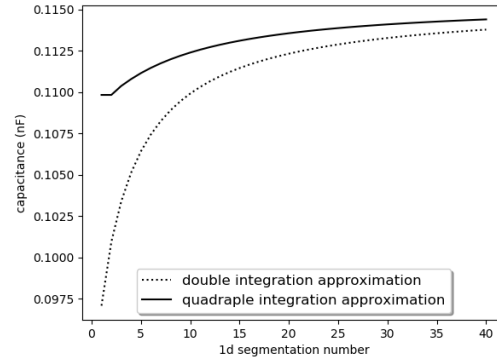


Figure 4: comparision between results extracted from double and quadraple integrations

In fig.4 double and quadraple integration methods are compared. Clearly quadraple integration results more accurate advantages. Even in rough segmentations a meaningful answer has been achieved from quadraple integration.