Firm Entry and Exit in Continuous Time

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Abstract

In this paper, I develop and analyze a model of firms' entry and exit in a continuous-time setting. I build my analysis based on Hopenhayn (1992) firm dynamics framework and use the continuous-time structure to solve the model. Solving the model in continuous time brings in many advantages, such as lower computational cost and the model's tractability. However, there are some challenges too. One of the major challenges is to have entry cost in the model, i.e., to obtain a Hamilton-Jacobi-Bellman equation that incorporates the entry cost. I use a form of exit cost as the future value of the entry cost to avoid this problem. To do so, I have to keep track of the firms' age distribution in addition to the distribution of the shocks, which makes my model richer than Hopenhayn's (1992). To solve for the joint stationary distribution of the firms, I introduce a simple process for aging and obtain the Kolmogorov forward equation using the age and shock processes. Another important contribution of this paper is to introduce a way to deal with the Kolmogorov equation in two states with discontinuity and combine them into one equation that governs the state of the economy. The results obtained in this paper are in line with those reported in Hopenhayn (1992). However, the methods, tools, and the way of approaching the model differs depending on whether I solve the model in discrete or continuous time. The tools and procedures developed in this paper can easily be extended to other optimal stopping time problems.

1 Introduction

Understanding the firm dynamics is one of the most important questions that macroeconomists try to address. In particular, analyzing the distribution of variables such as productivity, size, and age of firms are of special interest in macroeconomics. One of the leading works on firm

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dynamics in the literature is the paper by Hopenhayn (1992) which introduces endogenous entry and exit for firms in a partial equilibrium setting. The only source of firm heterogeneity is the individual productivity shocks, which lead to the allocation of resources across the firms. This paper develops methods and tools to solve Hopenhayn (1992) model in a continuous-time framework.

Solving macroeconomic models in continuous-time is attractive for mainly two reasons: 1. It is more efficient quantitatively, i.e., the run-time for solving the models in continuous time, in general, is much shorter than its conventional counterpart based on iteration of Bellman equation, and consequently the models can be made richer by adding other shocks, states or frictions at almost no extra cost. 2. It often gives us extra analytical power to handle the model differently and see through it from different angles, e.g., possibility of ignoring some of the constraints because they never bind in the interior of the state space in a continuous time setting (See Achdou et. al. 2015 for a good example on this).

In continuous time, the economy breaks down to a system of partial differential equations (PDEs). To be more specific, in order to solve the model in continuous time framework, I need to derive a Hmilton-Jacobi-Bellman (HJB) equation which gives the evolution of the value function as provided with the Bellman equation in its discrete time counterpart, and a Kolmogorov Forward equation (KFE) which gives the evolution of the distribution of the variables of interest. These PDEs are well known in optimal control theory (OCT), and their solution concepts are developed in this context. Especially, the problems that we are dealing with in the continuous time version of the heterogeneous agent models are known as mean field games (Lasry and Lions (2007)). Achdou et. al. (2015) develop methods to solve some main heterogeneous agent models in continuous time. They provide analytical tools to solve and analyze three leading works on heterogeneous agent models in the literature, Aiyagari (1994), Bewley (1986), and Huggett (1993).

The presence of entry and exit in the model makes the problem different than the standard optimal control problem. These kinds of problems are closely related to stopping time problems in the optimal control theory, which is explained in details in Kushner and Dupuis (2013). In this case we will have a Hamilton-Jacobi- Bellman variational inequality (HJBVI) instead of the standard HJB equation. Replacement of HJB with HJBVI in these models is due to the firms optimal decision on exit time based on when their expected future value would fall below a threshold, which is the firm's entry cost in this case. See Achdou et. al. (2014) for an example of a model that breaks down to stopping time problem. They, however, do not discuss the solution methods for these problems. The HJBVI can be efficiently solved as a Linear

Complimentarity Problem (LCP). Huang and Pang (1998) first developed techniques to solve HJB equation using LCP.

In this paper, I will develop tools to solve the Hopenhayn (1992) economy in continuous time. First, I ignore the firms entry costs and solve the stationary equilibrium of the simplified model with no entry cost. The reason is that, having entry cost in the model makes it challenging to translate the model into HJBVI and KFE in a straightforward way. In order to deal with this problem, I changed the presentation of the problem to fit into the full model with entry cost. The idea used here is to convert the entry cost to exit cost in its future value form. So, in this setting, instead of paying an entry cost in advance, firms will have to pay a cost equivalent to the future value of the entry cost at the time of exit. This treatment has its own complexities. The main difficulty arises with the need to know the firms' age when they exit the industry. That is, we need to keep track of the age distribution too, in order to include the entry cost in my model economy. For this reason, we need to introduce the aging process of the firms. More specifically, we need to differentiate between the age process when the firms choose to continue production and when they choose to shut down. We will have two simple processes for age, both of which only include the drift part and no diffusion part. To solve for the distribution of firms, it is easier to combine two KFEs into a single equation. I have dealt with this issue in a straightforward way, which will be explained in section 4.

The paper is organized as follows. Section 2 explains the basic Hopenhayn (1992) model and its characteristics that are developed in a discrete time setting. Section 3 describes the continuous time formulation of the Hopenhayn economy without the entry cost, and explains their solution methods, followed by a numerical example. Section 4 gives the analysis the model with entry cost, and obtains the stationary equilibrium of the model and reports some quantitative results. Finally, section 5 concludes.

2 A Glance at the Hopenhayn Model

In this section, I will give a short summary of Hopenhayn (1992) and will try to touch the basic parts of the model based on which we will develop the continuous time version of it. This is a mere introduction of the environment of the model, and the reader can refer to the original paper for all the detailed analysis.

There is a continuum of competitive firms in the economy, i.e., price takers in both input and output markets. The firms produce a homogeneous good for which there is an aggregate demand given by an inverse demand function $P(Q)$, where Q is the total industry output.

Firms use a single input, say labor, and its price is given by function $W(N)$, where N is the total labor demand. The inverse demand function, $P(\cdot)$, is assumed to be strictly decreasing and satisfy $\lim_{x\to 0} P(x) = 0$. The input price function, $W(\cdot)$, is assumed to be increasing and strictly bounded above zero. Both $P(\cdot)$ and $W(\cdot)$ are continuous.

The production function is given by $q(z, n)$, where n is the labor input of the firm and $z \in S = [0, 1]$ is the productivity shock. The productivity shock follows a Markov process independent across firms which is standard in the discrete time models literature. Given the output and input prices, p and w, an individual firm chooses labor input and output to maximize lifetime profits. Therefore, the labor demand depends on prices and the period's productivity shock, $n(z, p, w)$. As a result, an individual firm's output supply is given by $q(z, p, w)$, and the profit function is given by $\pi(z, p, w)$. Every period that a firm decides to produce, it must pay a fixed cost, c_f. The fixed cost is a crucial component of the model that can be interpreted as an outside option for the firms, without which there would be no exit decision made by the firms. We have $\pi(z, 0, w) = -c_f$. Firms discount future profits with $0 < \beta < 1$. There are some other standard assumptions made on the functions $n(\cdot)$, $q(\cdot)$, $\pi(\cdot)$, and the shock process, to ensure that the problem is well-behaved, and to guarantee that there exists a reservation productivity level below which firms will decide to shut down. I skip restating these assumptions here. See assumptions A.2 - A.5 in Hopenhayn (1992).

Entrant firms should pay an entry cost, c_e in order to observe their productivity shock and start production. The exit decision is also made before realization of the period's productivity shock. After the shock is realized, firms decide on their labor demand, n and output supply, q. Then, the input and output prices are determined competitively to clear the markets. I denote the distribution of the firms over shocks at period t by $g_t(A)$, $A \subset S$, where g is the state of the economy. The aggregate output supply and labor demand are $Q^s(g, p, w) = \int q(z, p, w)g(dz)$, and $N^d(g, p, w) = \int n(z, p, w)g(dz)$, respectively.

Firms will maximize the expected discounted profits,

$$E \sum_{t=0}^{\infty} \beta^t \pi(n_t, q_t)$$

Since there is no aggregate uncertainty in the model, given the initial distribution g_0, the price sequence, $\{p_t, w_t\}$, follows a deterministic path. Therefore, given the price sequence $\{p_t, w_t\}$, the recursive formulation of the incumbent firm's value is given by the following:

$$v_t(z, p_t, w_t) = \max \{\pi(z, p_t, w_t)\} + \beta \max \{0, \mathbb{E}_t [v_{t+1}(z', p_{t+1}, w_{t+1})|z]\}, \tag{2.1}$$

where the conditional expectation is based on the Markov transition probabilities. For the new entrants the value function is:

$$v_t^e(p_t, w_t) = \mathbb{E}\left[v_t(z, p_t, w_t)\right], \tag{2.2}$$

where the unconditional expectation is based on the initial distribution of the shocks. Free entry condition requires $v_t^e(p_t, w_t) \leq c_e$, with equality if there is positive entry.

Firms will base their exit decision on a reservation productivity shock below which they will choose to exit the industry. The reservation rule is given by:

$$\underline{z} = \min\left\{\inf\left\{z \in [0,1] : \mathbb{E}_t\left[v_{t+1}(z', p_{t+1}, w_{t+1})|z\right] \geq 0\right\} \bigcup \{1\}\right\}. \tag{2.3}$$

I skip the definition of the competitive equilibrium and its existence argument provided in Theorem 1 in Hopenhayn (1992) and jump into the stationary equilibrium which is of direct relevance in this paper. Since the prices can be determined using the distribution of the firms, g, we can write the Bellman equation (2.1) as follows:

$$v(z, g) = \max\left\{\pi(z, g)\right\} + \beta \max\left\{0, \mathbb{E}\left[v(z', g)|z\right]\right\}. \tag{2.4}$$

The reservation productivity level is such that $\mathbb{E}[v(z', g)|\underline{z}] = 0$. The stationary equilibrium, i.e. invariant distribution of firms with positive entry can exist if $\mathbb{E}[v(z, g)] = c_e$, where again the unconditional expectation is over the initial distribution of the shocks. Theorem 2 in Hopenhayn provides the existence of a stationary competitive equilibrium, and Theorem 3 states that there exists a real number $c^* > 0$, such that $c_e < c^*$ is both necessary and sufficient for an stationary equilibrium with positive entry and exist. The latter is important for my results since we will be using two specifications of the model, one with and the other without the entry cost. The second specification justifies this statement.

Sections 3 and 4 provide the continuous time version of the model discussed in this section. Since the continuous time formulation of the model is different for the case with entry cost and without it, we will explain these two structures in two separate sections. First we introduce the model without entry cost, which is a standard stopping time problem. Next, I will discuss the model with entry cost, which is different than the standard models and needs to be treated differently. For this case I will keep track of the joint distribution of the shocks and age in order to have the entry cost in my continuous time formulation.

3 Continuous-Time Model Without Entry Cost

Here, a version of the firm entry/exit model with no entry cost, i.e. $c_e = 0$, is explained in a continuous time framework. The state of the economy is the distribution of the shocks. Firms expected present discounted profits are given by

$$\max_{\tau, n_t} \mathbb{E}_0 \int_0^\tau e^{-\rho t} \pi(z_t, n_t) dt \tag{3.1}$$

where the profit function $\pi(\cdot)$ is continuous and strictly increasing in z. I let the shocks process to follow a diffusion process as follows

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t \tag{3.2}$$

where μ and σ are the drift and diffusion of the process respectively, and dW_t is the Brownian motion component of the diffusion process. This is the continuous time analogue of the Markov process with no jumps, i.e. it has continuous paths. Later on, when we provide numerical examples in section 4, I will use Ornstein-Uhlenbeck process (O-U process) which is a stationary process, and is similar to AR(1) process in its continuous time form.

Given the evolution of the prices, p_t and w_t, for $t \geq 0$, firms will maximize (3.1) subject to (3.2). Let $g(z,t)$ denote the distribution of the shocks at time t. For all $t \geq 0$, the distribution satisfies

$$\int_S g(z,t)dz = 1. \tag{3.3}$$

The prices are determined by the functions $P(Q_t)$ and $W(N_t)$, where Q_t and N_t are aggregate output supply and input demand respectively. We have

$$\begin{aligned} N_t &= \int_S n_t(z, p_t, w_t) g(dz, t), \\ Q_t &= \int_S q_t(z, n_t(z, p_t, w_t)) g(dz, t). \end{aligned} \tag{3.4}$$

This implies that the evolution of the aggregates and consequently the prices are governed by the evolution of the shocks distribution.

3.1 Stationary Equilibrium

Now, I will derive the equations/inequalities that characterize the stationary equilibrium of the model. These are the so-called HJBVI and KFE. We start with the discrete time model and take the period's length to be equal to Δ. Given this, the firm's discounting factor can be interpreted as a function of Δ, and can be approximated by $\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta$ when Δ gets arbitrarily close to zero. Ignoring the firm's exit decision, from equation (3.1) we can get the Bellman equation

$$v(z_t) = \max_{n_t} \pi(z_t, n_t)\Delta + (1 - \rho\Delta)\mathbb{E}[v(z_{t+\Delta})|z_t], \tag{3.5}$$

which yields

$$\rho\Delta v(z_t) = \max_{n_t} \pi(z_t, n_t)\Delta + (1 - \rho\Delta)\left\{\mathbb{E}[v(z_{t+\Delta})|z_t] - v(z_t)\right\}.$$

After dividing by Δ and taking the limit as $\Delta \to 0$, we have

$$\rho v(z_t) = \max_{n_t} \pi(z_t, n_t) + \frac{\mathbb{E}[dv(z_{t+\Delta})|z_t]}{dt}. \tag{3.6}$$

By Ito's Lemma:

$$dv(z_t) = \left(\dot{v}(z_t) + v'(z_t)\mu(z_t) + \frac{1}{2}v''(z_t)\sigma^2(z_t)\right) dt + \sigma(z_t)v'(z_t)dW_t.$$

Taking expectation conditional on z_t gives

$$\mathbb{E}\left[dv(z_t)|z_t\right] = \left(v'(z_t)\mu(z_t) + \frac{1}{2}v''(z_t)\sigma^2(z_t)\right) dt. \tag{3.7}$$

Finally, plugging back equation (3.7) into equation (3.6) and suppressing the time subscript, we get a form of HJB equation

$$\rho v(z) = \max_{n} \pi(z, n) + v'(z)\mu(z) + \frac{1}{2}v''(z)\sigma^2(z). \tag{3.8}$$

Let's assume that the exiting firms secure a fixed reservation value of $\underline{v} \in \mathbb{R}$. This assumption can be made without loss of generality, since we can always ignore the reservation value and set $\underline{v} = 0$. Firms decide to exit the industry if their value falls below the reservation value $v(z_t) < \underline{v}$. Note that a negative value for $\underline{v}$ means that the exit decision is costly for firms, and therefore rises their tolerance of bad shocks, i.e. more negative $\underline{v}$ translates into lower exit cutoff shock, and vice versa. Equation (3.8) combined with the exit decision, gives the HJBVI,

$$\min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z, n), v(z) - \underline{v}\right\} = 0. \tag{3.9}$$

In order to be able to solve the HJBVI, we need to clarify the boundary conditions for $v(z)$. We know that the shock process gets reflected from some upper and lower bounds since the state space is bounded, i.e. $S = [\underline{s}, \overline{s}] = [0, 1]$. As a result, the boundary conditions are

$$v'(\underline{s}) = v'(\overline{s}) = 0. \tag{3.10}$$

We now turn to Kolmogorov Forward equation which describes the evolution of the shocks distribution. KFE depends only on the diffusion process for shocks and the entry/exit decisions have nothing to do with the distribution of the shocks. Derivation of the KFE using any diffusion process also follows from Ito's Lemma. It is out of the scope of this paper to derive the KFE, however, the procedure for its derivation can be found in several sources including Rackauckas (2014). For my model, the KFE becomes

$$\frac{1}{2}\frac{d^2}{dz^2}\left(\sigma^2(z)g(z)\right) - \frac{d}{dz}(\mu(z)g(z)) = 0. \tag{3.11}$$

The distribution satisfies the following condition

$$\int_S g(z)dz = 1, \quad g \geq 0. \tag{3.12}$$

Also, the prices are given by

$$\begin{aligned} w &= W(N) = W\left(\textstyle\int_S ng(dz)\right), \\ p &= P(Q) = P\left(\textstyle\int_S q(z,n)g(dz)\right). \end{aligned} \tag{3.13}$$

The stationary competitive equilibrium of the economy is fully characterized by the equations (A.1)-(A.4). The following proposition deals with the existence of the stationary equilibrium with positive entry. This proposition is equivalent to the combination of the theorems 2 and 3 in Hopenhayn (1992).The proof for proposition 1 can be obtained from the proofs provided for theorems 2 and 3 in Hopenhayn (1992).

Proposition 1. *There exists a range of fixed costs, $[\underline{c_f}, \overline{c_f}]$, for which there exists a stationary equilibrium with positive entry/exit for the economy.*

The HJBVI, (A.1) can be written as a Linear Complimentarity Problem (LCP) for which there are standard procedures to solve. See `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` for derivation of the LCP form. The algorithm for solving for stationary equilibrium is described below. We start with an initial guess on price, p_0 and wage, w_0, and then for $i = 0, 1, 2, ...$ we do as follows:

1. Given prices, solve for the optimal labor all over the shock grid, i.e. $n_i = \text{argmax}\ \pi(z, n)$. Then solve for q, and π given the labor choice.

Table 1: Functions used for the quantitative exercise

Production Technology:	$q(z,n) = zn^{\alpha}$
Inverse Demand Function:	$P(Q) = Q^{\frac{1}{1-\gamma}}$
Wage Function:	$W(N) = 1$
Profit Function:	$\pi(z,n) = pq - wn - c_f$
Shock Process (O-U):	$dz = \theta(\overline{z} - z)dt + \sigma dW$

2. Solve for the value function, v_i, from (A.1) using its LCP form considering the boundary conditions on (A.2).

3. Solve for the distribution, g_i, from (A.3) and (3.12).

4. Solve for the aggregate quantity supply, Q_i, and labor demand, N_i, using distribution on the shocks. Update the price and wage given the aggregates.

5. Given the newly obtained values for price and wage, return to step 1. Stop if $|p_{i+1} - p_i| < \epsilon$ and $|w_{i+1} - w_i| < \epsilon$ for some desirably small ϵ.

This gives us the stationary equilibrium of the economy, $(p_i, w_i, v_i, g_i, z_i^c, Q_i, N_i)$. Note that, in this case we do not need to know the cutoff shocks to solve the KFE. That is because of the fact that the shock distribution is independent of firms' entry-exit decision. In the next section we will see the case where the joint distribution of shocks and age depend on the firms' exit decision, which makes things rather complicated. As stated earlier, in this section I have tried to explain the easiest and basic case in order to have a grasp on the model in continuous time. I will turn to keep control of age, and have a non zero entry cost in the model in section 3. Note also that, it is not hard nor would it be costly to enrich the model by adding, for example, physical capital or assets, or other types of state or non-state variables. To solve for the model in a general equilibrium setting would be another easy extension of this model.

3.2 A Quantitative Work

Now I will run the simplified model using the following functions that satisfy the model assumptions mentioned earlier in the paper. The functions are reported in Table 1. The parameter values used in this exercise are reported in Table 2.

The results of the model for given functions and parameter values are shown in Figure 3.1. For this case, I have chosen the fixed cost $c_f = 0.35$ to match the exit rate of about 10% as

Table 2: Parameter values

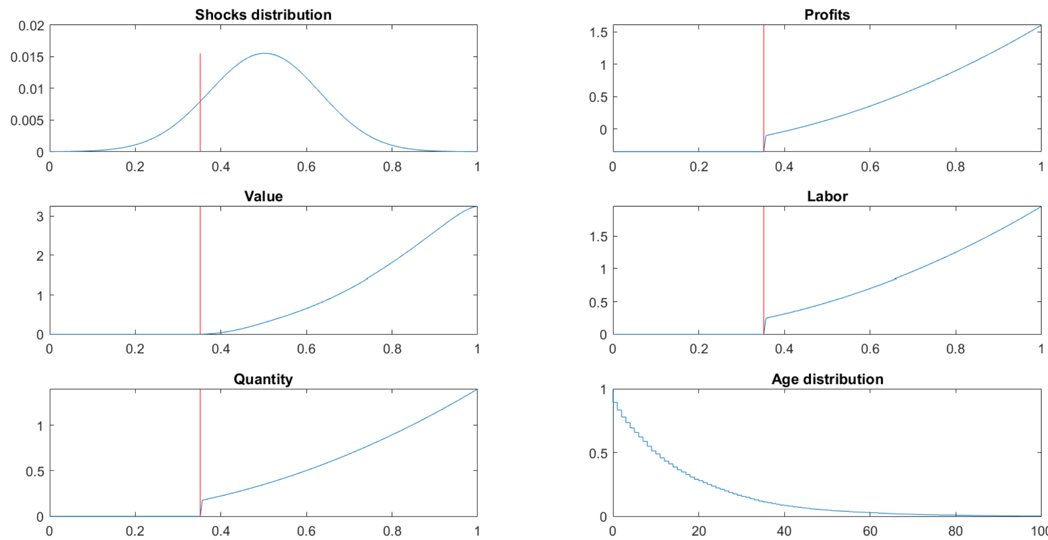
Model parameters	Values
α	0.5
γ	2
θ	0.3
$\overline{z}$	0.5
σ	0.1
ρ	0.05

we observe in the data. In Figure 3.1, I have reported the distribution of the shocks which is obtained through KFE, and the value of the firms for different shocks which is obtained from HJBVI. I have also reported the profits, labor demand and quantity supply, for different shocks. Note that all values for profits, labor demand and quantity supply are increasing in the shocks. This means, the firms with higher shocks will hire more in order to take advantage of the high productivity, and as a result the output and profits will also increase. The bottom-right panel in the Figure 3.1 displays the age distribution of the firms in the economy which is the result of simulating 10,000 samples.

We compared the age distribution of the firms for my model with the age distribution in the U.S data. As we can see in Figure 3.2, the results of the model is very close to the data except for slight differences for the bins 0 and $16-20$. These results will be very helpful when we include the dynamics of aging in the model in the next section. Figure 3.2 or the bottom-right panel of Figure 3.1 can be the point of comparison to see how the model with entry cost performs.

Since the cutoff for exit varies with the fixed cost, it would be interesting to see the correspondence of fixed cost and the exit cutoff shock. I have run the model with different values for the fixed cost to see how the exit rule responds to changes in the fixed cost. As we expected, the exit cutoff shock increases with the fixed cost. That is, with firms paying more fixed cost each period, more firms will have to shut down and only more productive firms can survive. As displayed in Figure 3.3, for lower values of the fixed cost (close to zero), no exit will happen because no shock is low enough to make the present value of the firms' expected profits become negative. Therefore, all the firms in the market will make positive profits independent of their productivity shock, so no firm will leave the market. On the other side, for the fixed cost high enough (in this case, more than 1.6), the exit rate will be 100%. This means that the fixed

Figure 3.1: Results of the model



cost is so high that even the firms with highest productivity cannot make positive profits. As a result, no firm can stay in the market when fixed cost is too high.

Another important thing here is the rate at which the slope of this function changes. For the first part it is convex, meaning that the slope is increasing, i.e. the exit rule increases with a positive pace. The slope is about zero for the low values and it increases to about 2 around the point $c_f = 0.55$. This means, adding to the fixed cost, makes firms exit at even higher rates. After the point $c_f = 0.55$, the function becomes concave and the slope decreases. The slope goes down from 2 and becomes zero after the point $c_f = 1.6$. That is, higher fixed cost increases the exit rate by less as we move forward. This is because we already have high exit rate and the firms in the economy are very productive, so adding to the fixed cost won't decrease the number of the firms in the economy at a higher rate as we move ahead. What Figure 3.3 shows is consistent with the Proposition 1 according to which there exists a range for the fixed cost that we get an stationary equilibrium with positive entry/exit. It could be very valuable to analytically derive and explain the shape of this function (a step that I have skipped in this paper).

Figure 3.2: Age distributions

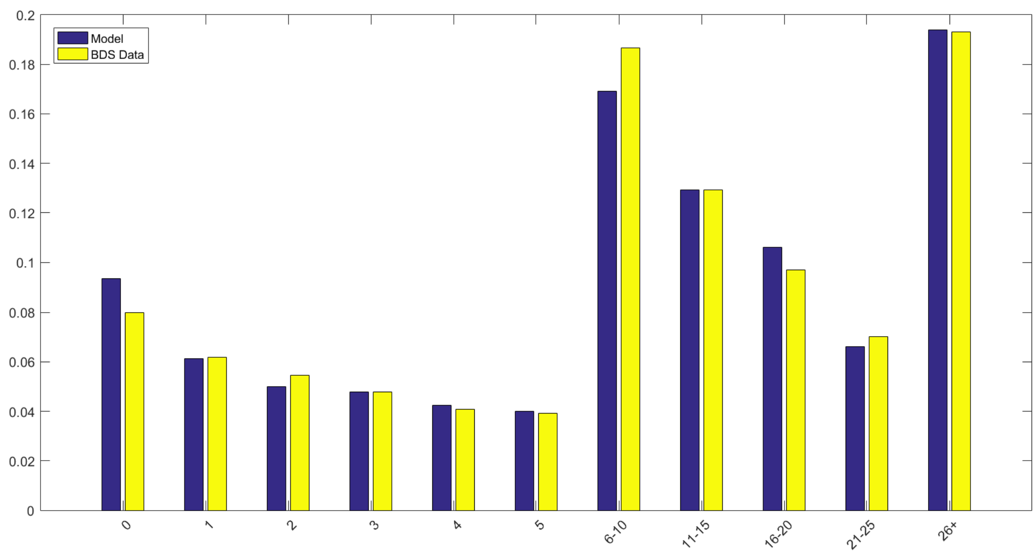
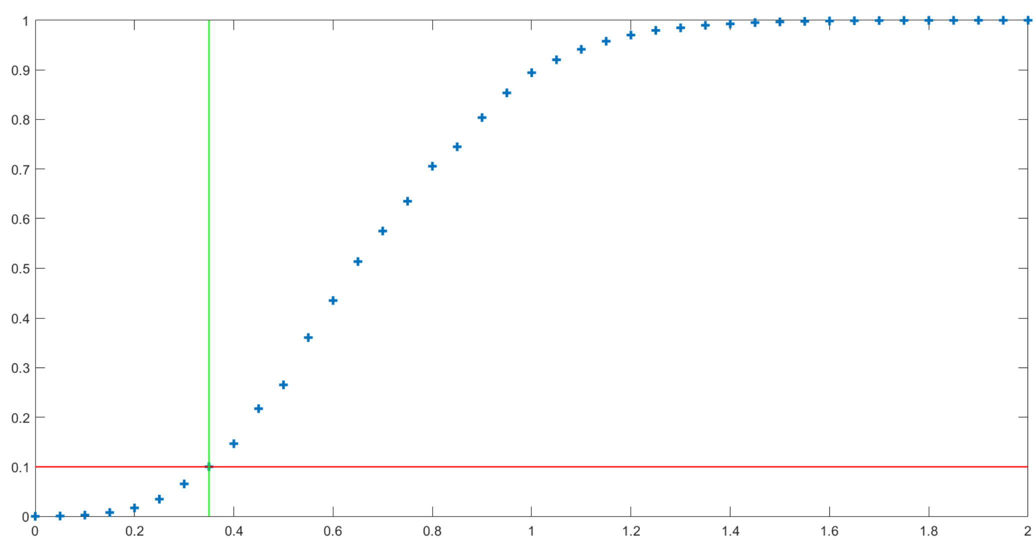


Figure 3.3: Exit rule for different values of fixed cost



4 Continuous-Time Model With Entry Cost

Now I turn to the analysis of the model with entry cost. The model discussed in this section keeps track of the evolution of the age as well as the shocks. This is a way to include the age in the model as a state variable that will show up in the value function. The dependence of the value function to the age will be through the entry cost. First we will explain the stationary equilibrium in this setting, and then we will show some quantitative results. As we will see, the model with entry cost completely encompasses the model without entry cost. i.e. the latter is a special case of the former.

4.1 Stationary Equilibrium of the Model

The analysis in this part has lots of similarities to the model without entry cost, hence I will not repeat those parts in this section. In this case firms will have to pay a fixed cost, $c_e > 0$ only once when they enter the market. The rest of the firms' activities are similar to the case with no entry cost discussed in the previous section. In order to have the entry cost in the model in the continuous time setting, I assume an equivalent characterization of the problem. We let the firms to pay the future value of the entry cost when they exit the market. To do so, we need to keep track of the firms' age, and then calculate the future value of the entry cost using the discount factor at the age that firms decide to exit the market. Here, the state of the economy is the joint distribution of the age and shocks. Firms expected present discounted profits are

$$\max_{\tau, n_t} \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} \pi(z_t, n_t) dt - c_e \tag{4.1}$$

where the profit function $\pi(\cdot)$ is continuous and strictly increasing in z as we had before. The shocks process is given by equation 3.2. In order to include the evolution of the firm's age as a state of the economy, or in order to have entry cost in the model, I re-write the firm's objective function as

$$\max_{\tau, n_t} \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} \pi(z_t, n_t) dt - e^{-\rho \tau} \left(c_e^*(\tau) \right), \tag{4.2}$$

where $c_e^*(\tau) = e^{\rho \tau} c_e$ is the age-dependent equivalent of the entry cost, i.e. it's future value evaluated at the exit time. This will lead us to a new HJBVI that depends on both age and shocks.

An important task here is to create a process for the evolution of age. Then, based on the process for age, the most challenging task is to obtain the HJBVI and KFE. The process for age has two components depending on whether the firms decide to continue or shut down:

1. If the firms survive, they will continue, and their age will simply evolve according to the following:
$$da_t = (1)dt \tag{4.3}$$

2. If the firms exit the market, their age will be set to zero, which is equivalent of replacing the dead firms with newborn ones that enter the market in the stationary equilibrium. That is, for any firm that exits at any age, there is a new firm entering the market with age zero. This can be written as
$$da_t = (\frac{-a_t}{dt})dt \tag{4.4}$$

The drift part of these processes, $\mu(a)$, are shown in the parentheses, and there's no diffusion part here. Solving for the HJBVI is relatively straightforward compared to the KFE, because in HJBVI we only need to use the continuation process, equation 4.3, and the other part, i.e exit decisions, will be taken care of by the HJBVI itself. HJBVI can be written as follows:

$$\begin{aligned} \min \Bigg\{ & \rho v(z,a) - \frac{\partial}{\partial a}v(z,a) - \frac{\partial}{\partial z}v(z,a)\mu(z) \\ & -\frac{1}{2}\frac{\partial^2}{\partial z^2}v(a,z)\sigma^2(z) - \pi(z,n), v(a,z) + e^{\rho a}.c_e \Bigg\} = 0 \end{aligned} \tag{4.5}$$

The way of obtaining this HJBVI, is exactly the same as the one discussed above. Again, here the first part in the left is the continuation value, and the second part is the exit value. Note that for the first part I use the continuation process, equation B.1, with $\mu(a) = 1$. The solution for this HJBVI will provide us with the exit cutoff shock. Because the shocks are independent of age, there is only one exit cutoff for the firms across the age distribution. From this I obtain the mass or probability of entry/exit.

We will have two Kolmogorov forward equations, one for each age process, that need to be combined carefully. Note that the shocks process remains the same for both cases. Given the exit rule obtained from HJBVI, we will use the corresponding KFE. That is, for any grid point in the shock-age distribution, I will use the KFE^X if the point lies in the exit region, and KFE^N if the point is in the no exit region. The general form for the KFEs is as follows

$$\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) - \frac{\partial}{\partial a}(\mu(a)g(a,z)) = 0. \tag{4.6}$$

This gives us the following two KFEs:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) + \frac{a}{da}\left(\frac{\partial}{\partial a}g(a,z)\right) + g(a,z) = 0 \\
\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) - \frac{\partial}{\partial a}g(a,z) = 0
\end{aligned}
\tag{4.7}
$$

where the first equation is KFE^X and the second one is KFE^N. The distribution satisfies the following condition

$$
\int_0^\infty \int_S g(a,z)dadz = 1, \quad g \geq 0, \tag{4.8}
$$

and the prices are given by

$$
\begin{aligned}
w = W(N) = &\quad W\left(\int n.g(da,dz)\right), \\
p = P(Q) = &\quad P\left(\int q.g(da,dz)\right).
\end{aligned}
\tag{4.9}
$$

The stationary competitive equilibrium is fully characterized by the equations (B.1), (B.4)-(B.5). Similar to the case with no entry cost, the following proposition is about the existence of the stationary equilibrium with positive exit/entry rate. This is also equivalent to the theorems 2 and 3 in Hopenhayn (1992).

Proposition 2. *For any positive value of fixed cost, there is a positive real number $\tilde{c_e}$, such that for any entry cost $c_e < \tilde{c_e}$, there exists a stationary equilibrium with positive exit rate for the economy.*

In the next part I will report a wide range of fixed costs where for each one of them there is a range of entry costs with positive exit rate. This result is in full agreement with the Proposition 2. In order to solve the stationary equilibrium of the model, I will follow the exact same steps of the algorithm explained in the previous section with the exception of the 3^{rd} step which will become as follows:

3-1 Solve for the exit cutoff, z_i^c, using the value function, or get the exit region where the value is equal to the entry cost at those shock-age grid points.

3-2 Solve for the distribution, g_i, by solving KFE^{N}and KFE^X, (B.4) jointly at their corresponding regions.

In general it is very hard to deal with this kind of problem even with few states in the model. The HJBVI part can be solved easily as a LCP, the same way as the case without age and entry cost. The challenge here is to solve KFEs jointly in their regions. I will explain a couple tricks that can be used to make it easier to solve the KFEs.

First, we can use the fact that the shock distribution is independent of the aging process and the exit decisions. Therefore, we can solve for two separate distributions, one for the evolution of the shocks, and the other one for the evolution of the age. It is much easier to solve for these two (2-dimensional) distributions separately, and then combine them into one (3-dimensional) joint distribution. One main advantage of using this method, is that it avoids all the computational difficulties that arises from handling large matrices. In other words, if we use N_z as the number of shock grid points, and N_a as the number of age grid points in my computation, using this method reduces the computational difficulty from an order of $N_z \times N_a$ to an order of $N_z + N_a$. It also avoids the computational errors that may accumulate because of having many more linear equations to solve.[1]

The second trick uses an altered structure for the KFEs and combines them into one. The idea behind the KFE and HJB equation is that they capture the interactions between single objects (decision makers) whose process is given, and translate all these interactions into aggregate outcomes and distribution. Therefore, the KFE is a measure in the aggregate level. We will use this fact to simplify things even more. After solving the HJBVI, we get a cutoff shock for exit, z^c. Using this cutoff value and the distribution of shocks we can determine the exit rate, i.e. the probability that a firm will exit independent of its age. Let's denote this probability by λ. Knowing the fact that the KFEs give the distribution for the whole economy, we can convert these two equations into one. The simplified KFE will be:[2]

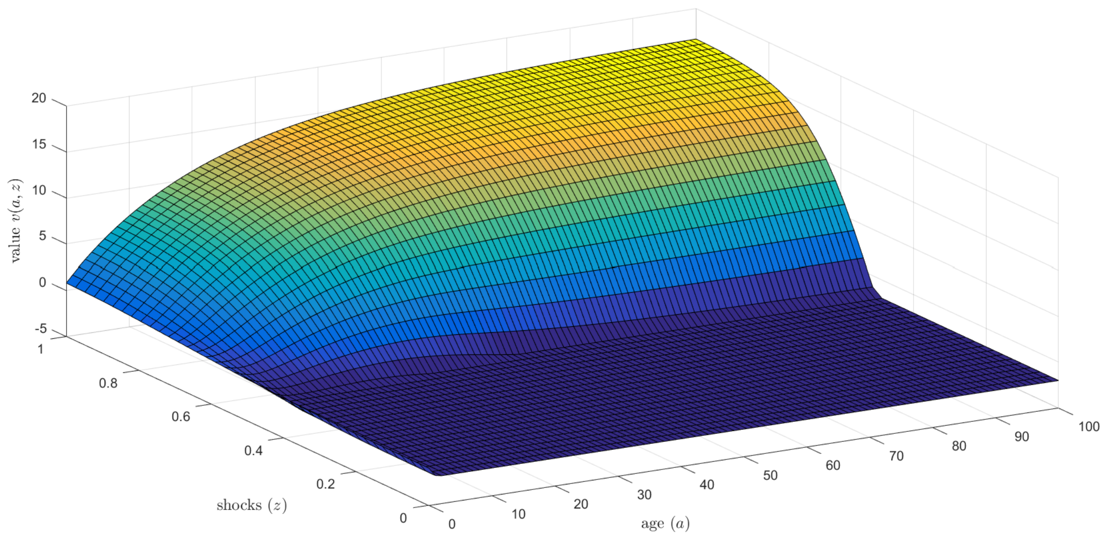
$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) - \lambda\left[\frac{\partial}{\partial a}g(a,z)\right] \\ +(1-\lambda)\left[\frac{a}{da}\left(\frac{\partial}{\partial a}g(a,z)\right) + g(a,z)\right] = 0 \end{aligned} \tag{4.10}$$

Now we have one KFE which will give us the joint shock-age distribution and together with HJBVI will give us the stationary equilibrium of the economy. I have solved all these differential

[1]The separating method introduced here produces more accurate results compared to the joint method mainly because in the separate method we shut down all the interactions between the multipliers of the shocks and age processes.

[2]This is to be analytically proven in the upcoming versions of the paper. For now I proceed with the argument provided above.

Figure 4.1: Value function with $c_e = 2$ and $c_f = 1$



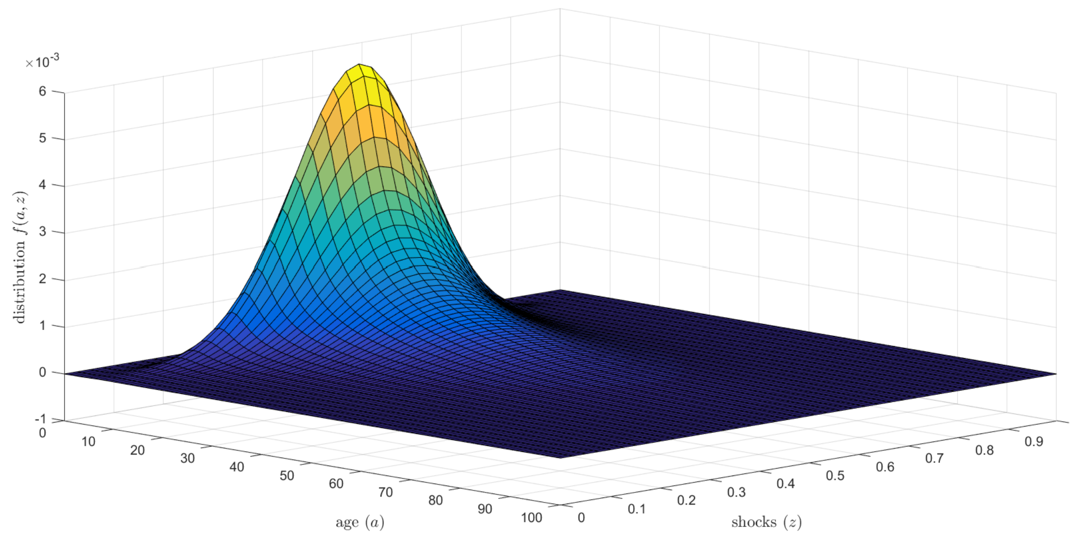
equations using the Finite Difference Method (FDM). To solve the KFE, I will use a kind of FDM algorithm developed in Achdou et. al. (2015), and to solve the HJBVI I will solve it as a Linear Complementarity Problem (LCP), which is based on the FDM. It can be proven that under the conditions of the model, the FDM solution converges to the viscosity solution of the HJB. A proof is provided in Barles and Souganidis (1991) for the regular HJB equation (not for the variational inequality), but the case for the HJBVI immediately follows, because we just need converging results for the HJB differential equation part of the variational inequality, and then taking the minimum of its solution and the entry cost won't affect the result. In the next subsection I will provide some quantitative results that will clarify the function of the model.

4.2 Numerical Results

In this part I will report some numerical results of the model with entry cost. The structure of the model here is similar to that of the model without entry cost except for having the age in the value function. To be consistent with the results obtained in section 3, the functions used here are the ones shown in Table 1, and all the parameter values are those given in Table 2. Figure 4.1 gives the value function across shocks and age for entry cost, $c_e = 2$, and fixed cost, $c_f = 1$. As expected, the value function is equal to the fixed entry cost for the shocks below some threshold.

Figure 4.2 shows the shock-age distribution using the same values for entry and fixed costs.

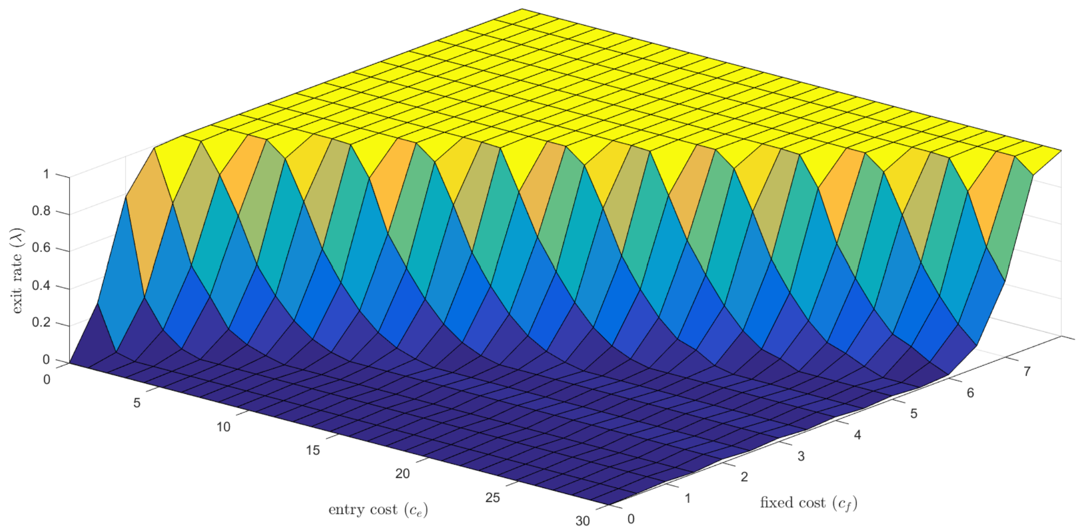
Figure 4.2: Distribution of firms across shocks and age for $c_e = 2$ and $c_f = 1$



Again, for the exit/entry rate of around 11% this is what we expect for the distribution. That is, more young firms and less and less older firms. With lower probability of exit, the distribution's peak will shift forward in the age dimension. For example for the exit rates below 1% we will see more and more older firms compared to the younger ones. That is, more firms will survive in the market and most of them will get to live longer, and as a result, the economy will be populated with older firms.

We have run the model for several values of fixed cost and entry cost. We have used a wide range of entry costs from 0 to 30, and correspondingly we chose the values for the fixed costs to vary from 0 to 8. According to Proposition 2, for any given fixed cost, there exits an interval $(0, \tilde{c_e}]$ for entry costs, where there is a positive exit rate in the model. Note that the entry cost interval depends on the value picked for the fixed cost, and we expect these intervals to move together, i.e. lower values of the fixed cost will require lower values of the entry cost, and as a result lower $\tilde{c_e}$, in order to have a positive exit rate in the economy, and vice versa. Figure 4.3 shows the exit rate for different values of entry cost and fixed cost. The results shown in Figure 4.3 are fully consistent with Proposition 2. Also, it shows that, given any value for entry cost, the exit rate increases with the fixed cost. This is the same result reported in section 3, using the model without entry cost. Now we can also see the response of the exit rate to the changes in entry cost. Given any fixed cost, it is obvious from Figure 4.3 that the exit rate is decreasing in the entry cost. This is also consistent with the analysis in Hopenhayn (1992). Having paid

Figure 4.3: Response of exit/entry rate to changes in the fixed cost and entry cost



higher entry cost, firms will exit in a lower rate because the shock threshold needs to be even lower to drop the expected discounted payoffs below the value of the entry cost and force the firms to shut down.

5 Conclusion

In this paper I developed tools to numerically solve and analyze a model of firms entry and exit in a continuous time framework. The model I used here is Hopenhayn's (1992). Because of the structural differences between the models in discrete and continuous time, I differentiated between the model without entry cost and with entry cost. As my benchmark model I derived the stationary equilibrium of the model with no entry cost in continuous time, and then I turned to the full model with entry cost while keeping track of age distribution in the model. Introducing the aging process in the model brings in lot's of analytical advantages, which can be easily adopted in other models. I also developed a straightforward way of dealing with Kolmogorov Forward equation when we have two processes for the evolution of the variable in charge (age in this case). The process for combining the Kolmogorov forward equations introduced in this paper can be extended to cases with multiple states instead of two. As a result, we can easily handle models with more states to happen, instead of just having exit and continuation decisions as we had in this paper.

One of the main advantages of the full model is that it gives us a clear and bigger picture of what we expect from the model. We can easily see the relationship between the variables and the effect of parameters on the model. As I showed earlier, the exit rate is increasing in the fixed cost and is decreasing in the entry cost. We saw in both models that the slope of the exit rate for changes in the fixed cost increases up to some point and then decreases, which produces a s-shaped curve. In a similar way, but going to an opposite direction in the entry cost dimension, we have the the same response of the exit rate to the changes in the entry cost, and this produces an inverse s-shaped curves. Another result that can be drawn is that, when there is no fixed cost, $c_f = 0$, there can be no entry cost that can guarantee a positive exit rate. That is, with no fixed cost, there is no reason to exit the market. Positive exit rate in this case is only possible through some negative values for the entry cost.

There could be some extensions done on my work. It is easy to run the model with different parameter values and see the effects of changes in some basic parameters (e.g. discounting factor, parameters of the production function, etc.) on the variables of interest such as exit rate, labor demanded, quantity supplied, etc. It is also very interesting to solve this model in a general equilibrium setting instead of partial equilibrium. To do so, we would need to introduce the household side in the economy and calibrate the model accordingly. Another interesting extension of the model would be addition of new state and non-state variables. The first example that comes to mind is the addition of physical capital and/or bonds to the model. Adding any of these features to the model would make it richer and can produce lots of interesting results.

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Appendix

A Model Without Entry Cost

Stationary equilibrium in this case is characterized by the following set of equations:
1) HJBVI:

$$\min\left\{\rho v(z)-v'(z)\mu(z)-\frac{1}{2}v''(z)\sigma^2(z)-\pi(z,n),v(z)-\underline{v}\right\}=0. \tag{A.1}$$

2) Boundary conditions:

$$v'(0)=v'(1)=0. \tag{A.2}$$

3) KFE:

$$\frac{1}{2}\frac{d^2}{dz^2}\left(\sigma^2(z)g(z)\right)-\frac{d}{dz}(\mu(z)g(z))=0. \tag{A.3}$$

4) Prices:

$$\begin{aligned} w&=W(N)=W\left(\textstyle\int_S ng(dz)\right),\\ p&=P(Q)=P\left(\textstyle\int_S q(z,n)g(dz)\right). \end{aligned} \tag{A.4}$$

To solve this, we start with an initial guess on price, p_0 and wage, w_0, and then for $i=0,1,2,...$ we do as follows:

1. Given prices, solve for the optimal labor all over the shock grid, i.e. $n_i=\text{argmax}\ \pi(z,n)$. Then solve for q, and π given the labor choice.

* *This step is obvious*

2. Solve for the value function, v_i, from (A.1) using its LCP form considering the boundary conditions on (A.2).

* *This step is identical to those explained in (`http://www.princeton.edu/~moll/HACTproject/option_simple.pdf`)*

3. Solve for the distribution, g_i, from (A.3).

* *This step can be solved either by a direct application of finite diference method on the Kolmogorov Forward equation, or by using the transpose of the intensity matrix* $[\mathbf{A}]$ *from the HJB equation. See section 2 in (`http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf`). Note that, we don't need an upwind scheme here.*

4. Solve for the aggregate quantity supply, Q_i, and labor demand, N_i, using distribution on the shocks. Update the price and wage given the aggregates.

* *This step is clear*

5. Given the newly obtained values for price and wage, return to step 1. Stop if $|p_{i+1} - p_i| < \epsilon$ and $|w_{i+1} - w_i| < \epsilon$ for some desirably small ϵ.

B Model With Entry Cost

Stationary equilibrium in this case is characterized by the following set of equations:
1) HJBVI:

$$\begin{aligned} \min \Big\{ & \rho v(z,a) - \frac{\partial}{\partial a} v(z,a) - \frac{\partial}{\partial z} v(z,a)\mu(z) \\ & -\frac{1}{2}\frac{\partial^2}{\partial z^2} v(a,z)\sigma^2(z) - \pi(z,n), v(a,z) + e^{\rho a}.c_e \Big\} = 0 \end{aligned} \tag{B.1}$$

2) Boundary conditions:

$$v'(0,a) = v'(1,a) = 0, \qquad \forall a. \tag{B.2}$$

3) KFE - Exit (KFE^X):

$$\begin{aligned} & \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) \\ & \qquad + \frac{a}{da}\left(\frac{\partial}{\partial a} g(a,z)\right) + g(a,z) = 0 \end{aligned} \tag{B.3}$$

4) KFE - No Exit (KFE^N):

$$\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) - \frac{\partial}{\partial a} g(a,z) = 0 \tag{B.4}$$

5) Prices:

$$\begin{aligned} w = W(N) = & \quad W\left(\int n.g(da,dz)\right), \\ p = P(Q) = & \quad P\left(\int q.g(da,dz)\right). \end{aligned} \tag{B.5}$$

All steps are the same as the case without entry cost, except the third step which changes as follows:

3-1 Solve for the exit cutoff, z_i^c, using the value function, or get the exit region where the value is less than or equal to the entry cost at those shock-age grid points.

3-2 Solve for the distribution, g_i, by solving KFE^{N}and KFE^X, (B.4) jointly at their corresponding regions.

Step 3-1 is clear and can be solved for using the value function obtained in step 2. Step 3-2 is somewhat challenging and it is different that what we have seen so far. I have used two slightly different methods to deal with it in an easily accessible way.

B.1 First Method

Here I will exploit what I have discussed in the paper and combine the two KFEs into one using the exit rate obtained in step 3-1. That is, I replace B.3 and B.4 by the following single KFE:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g(a,z)\right) - \frac{\partial}{\partial z}(\mu(z)g(a,z)) - \lambda\left[\frac{\partial}{\partial a}g(a,z)\right]& \\ +(1-\lambda)\left[\frac{a}{da}\left(\frac{\partial}{\partial a}g(a,z)\right) + g(a,z)\right] &= 0\end{aligned} \tag{B.6}$$

Now, we have got one HJB(VI) and one KFE, both of which need to be solved in two states (a,z). For this, refer to section 5 in (`http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf`) which deals with solving HJB and KFE in a general diffusion setting in two states.

B.2 Second Method

The first method is generally how I deal with similar problems. However, another method can be used to solve this specific problem. Because of the nature of the state space in this problem, we can separate z from a. That is, instead of using B.6 we could solve for the distribution over shocks and age separately. The separate KFEs would be the following:
1) Shock KFE:

$$\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(z)g_z(z)\right) - \frac{\partial}{\partial z}(\mu(z)g_z(z)) = 0 \tag{B.7}$$

2) Age KFE:

$$-\lambda\left[\frac{\partial}{\partial a}g_a(a)\right]+(1-\lambda)\left[\frac{a}{da}\left(\frac{\partial}{\partial a}g_a(a)\right)+g_a(a)\right]=0 \tag{B.8}$$

We can solve this by solving two single-state equations. See section 2 in (`http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf`). These equations give us two distributions that can be combined accordingly.