Extra Insight into the iOS App

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Abstract

This document consists of examples. It also includes derivations for the monthly compound interest rate and ten-year minimum payment.

1 Examples

1.1 Increment size

If $p_{\max} = \$5{,}000$, $p_{\min} = \$3{,}000$ and $N = 20$,

$$\begin{aligned}\Delta N &= \frac{5{,}000 - 3{,}000}{20} \\ &= \frac{2{,}000}{20} \\ &= \$100/\texttt{increment}\end{aligned}$$

If $p_{\max} = \$6{,}500$, $p_{\min} = \$4{,}500$ and $N = 40$,

$$\begin{aligned}\Delta N &= \frac{6{,}500 - 4{,}500}{40} \\ &= \frac{2{,}000}{40} \\ &= \$50/\texttt{increment}\end{aligned}$$

If $p_{\max} = \$10{,}000$, $p_{\min} = \$0$ and $N = 50$,

$$\begin{aligned}\Delta N &= \frac{10{,}000 - 0}{50} \\ &= \frac{10{,}000}{50} \\ &= \$200/\texttt{increment}\end{aligned}$$

1.2 Annual interest rate

If APR $= 7\%$,

$$\begin{aligned} r &= 7 \div 100 \\ &= 0.07/\texttt{year} \end{aligned}$$

If APR $= 6.8\%$,

$$\begin{aligned} r &= 6.8 \div 100 \\ &= 0.068/\texttt{year} \end{aligned}$$

If APR $= 1.75\%$,

$$\begin{aligned} r &= 1.75 \div 100 \\ &= 0.0175/\texttt{year} \end{aligned}$$

1.3 Monthly interest rate

If APR $= 7\%$,

$$\begin{aligned} i &= \frac{0.07}{12} \\ &= 0.00583\ldots/\texttt{month} \end{aligned}$$

If APR $= 6.8\%$,

$$\begin{aligned} i &= \frac{0.068}{12} \\ &= 0.00566\ldots/\texttt{month} \end{aligned}$$

If APR $= 3.45\%$,

$$\begin{aligned} i &= \frac{0.0345}{12} \\ &= 0.00287\underline{5}/\texttt{month} \end{aligned}$$

If APR $= 7\%$ and interest is compounded,

$$
\begin{aligned}
i &\approx \left(1+\frac{0.07}{365.25}\right)^{\frac{365.25}{12}}-1 \\
&= (1.000191\ldots)^{\frac{365.25}{12}}-1 \\
&= 1.00584\ldots-1 \\
&= 0.00584\ldots/\mathtt{month}
\end{aligned}
$$

If APR $= 6.8\%$ and interest is compounded,

$$
\begin{aligned}
i &\approx \left(1+\frac{0.068}{365.25}\right)^{\frac{365.25}{12}}-1 \\
&= (1.000186\ldots)^{\frac{365.25}{12}}-1 \\
&= 1.00568\ldots-1 \\
&= 0.00568\ldots/\mathtt{month}
\end{aligned}
$$

If APR $= 3.45\%$ and interest is compounded,

$$
\begin{aligned}
i &\approx \left(1+\frac{0.0345}{365.25}\right)^{\frac{365.25}{12}}-1 \\
&= (1.0000944\ldots)^{\frac{365.25}{12}}-1 \\
&= 1.00287\ldots-1 \\
&= 0.00287\underline{9}\ldots/\mathtt{month}
\end{aligned}
$$

1.4 Absolute minimum monthly payment

If APR $= 6.8\%$, $\alpha = 0.25$ and $p = \$2{,}000$,

$$
\begin{aligned}
a_{\min_n} &= \lfloor 0.25\,(2{,}000 \cdot 0.00566\ldots) \times 100 \rceil \div 100 + 0.01 \\
&= \lfloor 0.25\,(11.333\ldots) \times 100 \rceil \div 100 + 0.01 \\
&= \lfloor 283.3\ldots \rceil \div 100 + 0.01 \\
&= 283 \div 100 + 0.01 \\
&= 2.83 + 0.01 \\
&= \$2.84
\end{aligned}
$$

If APR $= 2.61\%$, $\alpha = 0.2$ and $p = \$2{,}700$,

$$\begin{aligned}
a_{\min_n} &= \left\lfloor 0.2\left(2{,}700 \cdot 0.00217\underline{5}\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 0.2\left(5.872...\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 117.4... \right\rceil \div 100 + 0.01 \\
&= 117 \div 100 + 0.01 \\
&= 1.17 + 0.01 \\
&= \$1.18
\end{aligned}$$

If APR $= 2.61\%$ and interest is compounded, $\alpha = 0.2$ and $p = \$2{,}700$,

$$\begin{aligned}
a_{\min_n} &= \left\lfloor 0.2\left(2{,}700 \cdot 0.00217\underline{7}...\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 0.2\left(5.878...\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 117.5... \right\rceil \div 100 + 0.01 \\
&= 118 \div 100 + 0.01 \\
&= 1.18 + 0.01 \\
&= \$1.19
\end{aligned}$$

If APR $= 7\%$, $\alpha = 0$ and $p = \$35{,}221$,

$$\begin{aligned}
a_{\min_n} &= \left\lfloor 0\left(35{,}221 \cdot 0.00583...\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 0\left(15.75\right) \times 100 \right\rceil \div 100 + 0.01 \\
&= \left\lfloor 0 \right\rceil \div 100 + 0.01 \\
&= 0 \div 100 + 0.01 \\
&= 0 + 0.01 \\
&= \$0.01
\end{aligned}$$

1.5 Ten-year minimum monthly payment

If APR $= 7\%$, $\alpha = 0$ and $p = \$35{,}221$,

$$\begin{aligned} i &= \mathsf{0.00583...} > 0 \\ \alpha &= \mathsf{0}\ \textit{(given)} \\ &\quad \texttt{First case, proceeding...} \end{aligned}$$

$$\begin{aligned} a_{\min_{120}} &= \left\lceil \frac{\mathsf{35{,}221}}{120} \times 100 \right\rceil \div 100 \\ &= \lceil \mathsf{29{,}350.8...} \rceil \div 100 \\ &= \mathsf{29{,}351} \div 100 \\ &= \$\mathsf{293.51} \end{aligned}$$

If APR $= 3.45\%$ and interest is compounded, $\alpha = 0$ and $p = \$26{,}970$,

$$\begin{aligned} i &\approx \mathsf{0.002879...} > 0 \\ \alpha &= \mathsf{0}\ \textit{(given)} \\ &\quad \texttt{First case, proceeding...} \end{aligned}$$

$$\begin{aligned} a_{\min_{120}} &= \left\lceil \frac{\mathsf{26{,}970}}{120} \times 100 \right\rceil \div 100 \\ &= \lceil \mathsf{22{,}475} \rceil \div 100 \\ &= \mathsf{22{,}475} \div 100 \\ &= \$\mathsf{224.75} \end{aligned}$$

If APR $= 0\%$, $\alpha = 0.5$ and $p = \$24{,}190$,

$$\begin{aligned} i &= \mathsf{0} \\ \alpha &= \mathsf{0.5}\ \textit{(given)} \\ &\quad \texttt{Second case, proceeding...} \end{aligned}$$

$$\begin{aligned} a_{\min_{120}} &= \left\lceil \frac{\mathsf{24{,}190}}{120} \times 100 \right\rceil \div 100 \\ &= \lceil \mathsf{20{,}158.3...} \rceil \div 100 \\ &= \mathsf{20{,}159} \div 100 \\ &= \$\mathsf{201.59} \end{aligned}$$

If APR $= 6.8\%$, $\alpha = 0.5$ and $p = \$13{,}500$,

$$\begin{aligned} i &= \texttt{0.00566...} > 0 \\ \alpha &= \texttt{0.5} \textit{ (given)} \\ &\quad \texttt{Third case, proceeding...} \end{aligned}$$

$$\begin{aligned} a_{\min_{120}} &= \left\lceil \frac{\texttt{0.5}\,(\texttt{13,500}\cdot\texttt{0.00566...})\,(1+\texttt{0.5}\cdot\texttt{0.00566...})^{120}}{(1+\texttt{0.5}\cdot\texttt{0.00566...})^{120}-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{0.5}\,(\texttt{76.50})\,(\texttt{1.00283...})^{120}}{(\texttt{1.00283...})^{120}-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{0.5}\,(\texttt{76.50})\,(\texttt{1.404...})}{(\texttt{1.404...})-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{53.713...}}{(\texttt{1.404...})-1} \times 100 \right\rceil \div 100 \\ &= \lceil \texttt{13,286.4...} \rceil \div 100 \\ &= \texttt{13,287} \div 100 \\ &= \$\texttt{132.87} \end{aligned}$$

If APR $= 5.56\%$ and interest is compounded, $\alpha = 0.7$ and $p = \$20{,}000$,

$$\begin{aligned} i &\approx \texttt{0.00464...} > 0 \\ \alpha &= \texttt{0.7} \textit{ (given)} \\ &\quad \texttt{Third case, proceeding...} \end{aligned}$$

$$\begin{aligned} a_{\min_{120}} &= \left\lceil \frac{\texttt{0.7}\,(\texttt{20,000}\cdot\texttt{0.00464...})\,(1+\texttt{0.7}\cdot\texttt{0.00464...})^{120}}{(1+\texttt{0.7}\cdot\texttt{0.00464...})^{120}-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{0.7}\,(\texttt{92.874...})\,(\texttt{1.00325...})^{120}}{(\texttt{1.00325...})^{120}-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{0.7}\,(\texttt{92.874...})\,(\texttt{1.476...})}{(\texttt{1.476...})-1} \times 100 \right\rceil \div 100 \\ &= \left\lceil \frac{\texttt{95.968...}}{(\texttt{1.476...})-1} \times 100 \right\rceil \div 100 \\ &= \lceil \texttt{20,154.8...} \rceil \div 100 \\ &= \texttt{20,155} \div 100 \\ &= \$\texttt{201.55} \end{aligned}$$

If APR $= 5.56\%$, $\alpha = 0.7$ and $p = \$20{,}000$,

$$
\begin{aligned}
i &= \texttt{0.00463...} > 0 \\
\alpha &= \texttt{0.7}\ \textit{(given)} \\
&\quad \texttt{Third case, proceeding...}
\end{aligned}
$$

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{\texttt{0.7}\,(\texttt{20,000}\cdot\texttt{0.00463...})\,(1+\texttt{0.7}\cdot\texttt{0.00463...})^{120}}{(1+\texttt{0.7}\cdot\texttt{0.00463...})^{120}-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{0.7}\,(\texttt{92.666...})\,(\texttt{1.00324...})^{120}}{(\texttt{1.00324...})^{120}-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{0.7}\,(\texttt{92.666...})\,(\texttt{1.474...})}{(\texttt{1.474...})-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{95.669...}}{(\texttt{1.474...})-1} \times 100 \right\rceil \div 100 \\
&= \lceil \texttt{20,146.5...} \rceil \div 100 \\
&= \texttt{20,147} \div 100 \\
&= \$\texttt{201.47}
\end{aligned}
$$

If APR $= 4\%$, $\alpha = 0.25$ and $p = \$5{,}600$,

$$
\begin{aligned}
i &= \texttt{0.00333}\underline{\texttt{3}}\texttt{...} > 0 \\
\alpha &= \texttt{0.25}\ \textit{(given)} \\
&\quad \texttt{Third case, proceeding...}
\end{aligned}
$$

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{\texttt{0.25}\,(\texttt{5,600}\cdot\texttt{0.00333...})\,(1+\texttt{0.25}\cdot\texttt{0.00333...})^{120}}{(1+\texttt{0.25}\cdot\texttt{0.00333...})^{120}-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{0.25}\,(\texttt{18.666...})\,(\texttt{1.000833...})^{120}}{(\texttt{1.000833...})^{120}-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{0.25}\,(\texttt{18.666...})\,(\texttt{1.105...})}{(\texttt{1.105...})-1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{\texttt{5.157...}}{(\texttt{1.105...})-1} \times 100 \right\rceil \div 100 \\
&= \lceil \texttt{4,905.8...} \rceil \div 100 \\
&= \texttt{4,906} \div 100 \\
&= \$\texttt{49.06}
\end{aligned}
$$

1.6 Algorithm

Example 1.6a. *If* $APR = 7\%$*,* $\alpha = 0.33$*,* $p = \$2{,}000$ *and* $a = \$550$*,*

$$B_0 = 2{,}000\ ;$$
$$O_0 = 0\ ;$$
$$m = 1$$

$<$ `First iteration` $>$

$$\begin{aligned}
\texttt{Check:}\quad & B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 2{,}000 - \left\{550 - \lfloor 0.33\,(2{,}000 \cdot 0.00583\ldots) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 2{,}000 - \left\{550 - 3.85\right\} \overset{?}{>} 0 \\
& 2{,}000 - 546.15 \overset{?}{>} 0 \\
& 1{,}453.85 > 0 \\
& \texttt{Proceeding...}
\end{aligned}$$

$$\begin{aligned}
B_1 &= B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100\right\} \\
&= \$1{,}453.85\ ;
\end{aligned}$$

$$\begin{aligned}
O_1 &= O_0 + \lfloor (B_0 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100 \\
&= 0 + \lfloor (2{,}000 \cdot 0.00583\ldots) \times 100 \rceil \div 100 - \lfloor 0.33\,(2{,}000 \cdot 0.00583\ldots) \times 100 \rceil \div 100 \\
&= 0 + 11.67 - 3.85 \\
&= \$7.82\ ;
\end{aligned}$$

$$m = 1 + 1 = 2\ \texttt{months}$$

$<$ `Second iteration` $>$

$$\begin{aligned}
\texttt{Check:}\quad & B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 1{,}453.85 - \left\{550 - \lfloor 0.33\,(1{,}453.85 \cdot 0.00583\ldots) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 1{,}453.85 - \left\{550 - 2.80\right\} \overset{?}{>} 0 \\
& 1{,}453.85 - 547.20 \overset{?}{>} 0 \\
& 906.65 > 0 \\
& \texttt{Proceeding...}
\end{aligned}$$

$$\begin{aligned}
B_2 &= B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rceil \div 100\right\} \\
&= \$906.65\ ;
\end{aligned}$$

$$
\begin{aligned}
O_2 &= O_1 + \lfloor (B_1 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha \, (B_1 \cdot i) \times 100 \rceil \div 100 \\
&= 7.82 + \lfloor (1{,}453.85 \cdot 0.00583\ldots) \times 100 \rceil \div 100 - \lfloor 0.33 \, (1{,}453.85 \cdot 0.00583\ldots) \times 100 \rceil \div 100 \\
&= 7.82 + 8.48 - 2.80 \\
&= \$13.50 \; ;
\end{aligned}
$$

$$m = 2 + 1 = 3 \texttt{ months}$$

$$< \texttt{Third iteration} >$$

$$
\begin{aligned}
\texttt{Check :} \quad & B_2 - \left\{ a - \lfloor \alpha \, (B_2 \cdot i) \times 100 \rceil \div 100 \right\} \stackrel{?}{>} 0 \\
& 906.65 - \left\{ 550 - \lfloor 0.33 \, (906.65 \cdot 0.00583\ldots) \times 100 \rceil \div 100 \right\} \stackrel{?}{>} 0 \\
& 906.65 - \left\{ 550 - 1.75 \right\} \stackrel{?}{>} 0 \\
& 906.65 - 548.25 \stackrel{?}{>} 0 \\
& 358.40 > 0 \\
& \texttt{Proceeding...}
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= B_2 - \left\{ a - \lfloor \alpha \, (B_2 \cdot i) \times 100 \rceil \div 100 \right\} \\
&= \$358.40 \; ;
\end{aligned}
$$

$$
\begin{aligned}
O_3 &= O_2 + \lfloor (B_2 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha \, (B_2 \cdot i) \times 100 \rceil \div 100 \\
&= 13.50 + \lfloor (906.65 \cdot 0.00583\ldots) \times 100 \rceil \div 100 - \lfloor 0.33 \, (906.65 \cdot 0.00583\ldots) \times 100 \rceil \div 100 \\
&= 13.50 + 5.29 - 1.75 \\
&= \$17.04 \; ;
\end{aligned}
$$

$$m = 3 + 1 = 4 \texttt{ months}$$

$$< \texttt{Fourth iteration} >$$

$$
\begin{aligned}
\texttt{Check :} \quad & B_3 - \left\{ a - \lfloor \alpha \, (B_3 \cdot i) \times 100 \rceil \div 100 \right\} \stackrel{?}{>} 0 \\
& 358.40 - \left\{ 550 - \lfloor 0.33 \, (358.40 \cdot 0.00583\ldots) \times 100 \rceil \div 100 \right\} \stackrel{?}{>} 0 \\
& 358.40 - \left\{ 550 - 0.69 \right\} \stackrel{?}{>} 0 \\
& 358.40 - 549.31 \stackrel{?}{>} 0 \\
& -190.91 \not> 0 \\
& \texttt{Halt!}
\end{aligned}
$$

$$n = 4 \texttt{ months}$$

$$\begin{aligned} a_{\mathrm{f}} &= B_3 + \lfloor (B_3 \cdot i) \times 100 \rceil \div 100 + O_3 \\ &= 358.40 + \lfloor (358.40 \cdot 0.00583...) \times 100 \rceil \div 100 + 17.04 \\ &= 358.40 + 2.09 + 17.04 \\ &= \$377.53 \; ; \end{aligned}$$

$$\begin{aligned} B_4 &= B_3 - \left\{ a_{\mathrm{f}} - \lfloor (B_3 \cdot i) \times 100 \rceil \div 100 - O_3 \right\} \\ &= 358.40 - \left\{ 377.53 - \lfloor (358.40 \cdot 0.00583...) \times 100 \rceil \div 100 - 17.04 \right\} \\ &= 358.40 - \left\{ 377.53 - 2.09 - 17.04 \right\} \\ &= 358.40 - 358.40 \\ &= \$0 \end{aligned}$$

`Breakdown of pay :`

546.15 Prin.	+	3.85 Int.	=	550
547.20 Prin.	+	2.80 Int.	=	550
548.25 Prin.	+	1.75 Int.	=	550
358.40 Prin.	+	19.13 Int.	=	377.53

Example 1.6b. *If* $APR = 3.45\%$, $\alpha = 0.25$, $p = \$5{,}000$ *and* $a = \$200$,

$$B_0 = 5{,}000\ ;$$

$$O_0 = 0\ ;$$

$$m = 1$$

$< \texttt{First iteration} >$

$$\texttt{Check}: \quad B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0$$

$$5{,}000 - \left\{200 - \lfloor 0.25\,(5{,}000 \cdot 0.002875) \times 100 \rceil \div 100\right\} \overset{?}{>} 0$$

$$5{,}000 - \left\{200 - 3.59\right\} \overset{?}{>} 0$$

$$5{,}000 - 196.41 \overset{?}{>} 0$$

$$4{,}803.59 > 0$$

`Proceeding...`

$$B_1 = B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100\right\}$$

$$= \$4{,}803.59\ ;$$

$$O_1 = O_0 + \lfloor (B_0 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100$$

$$= 0 + \lfloor (5{,}000 \cdot 0.002875) \times 100 \rceil \div 100 - \lfloor 0.25\,(5{,}000 \cdot 0.002875) \times 100 \rceil \div 100$$

$$= 0 + 14.38 - 3.59$$

$$= \$10.79\ ;$$

$$m = 1 + 1 = 2\ \texttt{months}$$

$< \texttt{Second iteration} >$

$$\texttt{Check}: \quad B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0$$

$$4{,}803.59 - \left\{200 - \lfloor 0.25\,(4{,}803.59 \cdot 0.002875) \times 100 \rceil \div 100\right\} \overset{?}{>} 0$$

$$4{,}803.59 - \left\{200 - 3.45\right\} \overset{?}{>} 0$$

$$4{,}803.59 - 196.55 \overset{?}{>} 0$$

$$4{,}607.04 > 0$$

`Proceeding...`

$$B_2 = B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rceil \div 100\right\}$$

$$= \$4{,}607.04\ ;$$

$$
\begin{aligned}
O_2 &= O_1 + \lfloor (B_1 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rceil \div 100 \\
&= 10.79 + \lfloor (4{,}803.59 \cdot 0.002875) \times 100 \rceil \div 100 - \lfloor 0.25\,(4{,}803.59 \cdot 0.002875) \times 100 \rceil \div 100 \\
&= 10.79 + 13.81 - 3.45 \\
&= \$21.15\ ;
\end{aligned}
$$

$$m = 2 + 1 = 3 \texttt{ months}$$

$<$ `Third iteration` $>$

$$
\begin{aligned}
\texttt{Check}: \quad & B_2 - \left\{ a - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0 \\
& 4{,}607.04 - \left\{ 200 - \lfloor 0.25\,(4{,}607.04 \cdot 0.002875) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0 \\
& 4{,}607.04 - \left\{ 200 - 3.31 \right\} \overset{?}{>} 0 \\
& 4{,}607.04 - 196.69 \overset{?}{>} 0 \\
& 4{,}410.35 > 0 \\
& \texttt{Proceeding...}
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= B_2 - \left\{ a - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100 \right\} \\
&= \$4{,}410.35\ ;
\end{aligned}
$$

$$
\begin{aligned}
O_3 &= O_2 + \lfloor (B_2 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100 \\
&= 21.15 + \lfloor (4{,}607.04 \cdot 0.002875) \times 100 \rceil \div 100 - \lfloor 0.25\,(4{,}607.04 \cdot 0.002875) \times 100 \rceil \div 100 \\
&= 21.15 + 13.25 - 3.31 \\
&= \$31.09\ ;
\end{aligned}
$$

$$m = 3 + 1 = 4 \texttt{ months}$$

$\vdots$

$<$ `Twenty-sixth iteration` $>$

$$
\begin{aligned}
\texttt{Check}: \quad & B_{25} - \left\{ a - \lfloor \alpha\,(B_{25} \cdot i) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0 \\
& 47.26 - \left\{ 200 - \lfloor 0.25\,(47.26 \cdot 0.002875) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0 \\
& 47.26 - \left\{ 200 - 0.03 \right\} \overset{?}{>} 0 \\
& 47.26 - 199.97 \overset{?}{>} 0 \\
& -152.71 \not> 0 \\
& \texttt{Halt!}
\end{aligned}
$$

$$n = 26 \texttt{ months}$$

$$\begin{aligned}
a_{\mathrm{f}} &= B_{25} + \lfloor (B_{25} \cdot i) \times 100 \rceil \div 100 + O_{25} \\
&= 47.26 + \lfloor (47.26 \cdot 0.002875) \times 100 \rceil \div 100 + 141.77 \\
&= 47.26 + 0.14 + 141.77 \\
&= \$189.17\ ;
\end{aligned}$$

$$\begin{aligned}
B_{26} &= B_{25} - \Big\{ a_{\mathrm{f}} - \lfloor (B_{25} \cdot i) \times 100 \rceil \div 100 - O_{25} \Big\} \\
&= 47.26 - \Big\{ 189.17 - \lfloor (47.26 \cdot 0.002875) \times 100 \rceil \div 100 - 141.77 \Big\} \\
&= 47.26 - \Big\{ 189.17 - 0.14 - 141.77 \Big\} \\
&= 47.26 - 47.26 \\
&= \$0
\end{aligned}$$

Breakdown of pay :

196.41 Prin.	+	3.59 Int.	=	200
196.55 Prin.	+	3.45 Int.	=	200
196.69 Prin.	+	3.31 Int.	=	200
196.83 Prin.	+	3.17 Int.	=	200
⋮		⋮		⋮
47.26 Prin.	+	141.91 Int.	=	189.17

Example 1.6c. *If* $APR = 6.8\%$, $\alpha = 0.5$, $p = \$13{,}500$ *and* $a = a_{\min_{120}} = \$132.87$,

$$B_0 = \texttt{13,500}\ ;$$
$$O_0 = \texttt{0}\ ;$$
$$m = \texttt{1}$$

$$< \texttt{First iteration} >$$

$$\texttt{Check}:\quad B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100\rceil \div 100\right\} \overset{?}{>} 0$$
$$\texttt{13,500} - \left\{\texttt{132.87} - \lfloor \texttt{0.5}\,(\texttt{13,500} \cdot \texttt{0.00566...}) \times 100\rceil \div 100\right\} \overset{?}{>} 0$$
$$\texttt{13,500} - \left\{\texttt{132.87} - \texttt{38.25}\right\} \overset{?}{>} 0$$
$$\texttt{13,500} - \texttt{94.62} \overset{?}{>} 0$$
$$\texttt{13,405.38} > 0$$
$$\texttt{Proceeding...}$$

$$B_1 = B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100\rceil \div 100\right\}$$
$$= \$\texttt{13,405.38}\ ;$$

$$O_1 = O_0 + \lfloor (B_0 \cdot i) \times 100\rceil \div 100 - \lfloor \alpha\,(B_0 \cdot i) \times 100\rceil \div 100$$
$$= \texttt{0} + \lfloor (\texttt{13,500} \cdot \texttt{0.00566...}) \times 100\rceil \div 100 - \lfloor \texttt{0.5}\,(\texttt{13,500} \cdot \texttt{0.00566...}) \times 100\rceil \div 100$$
$$= \texttt{0} + \texttt{76.50} - \texttt{38.25}$$
$$= \$\texttt{38.25}\ ;$$

$$m = 1 + 1 = 2\ \texttt{months}$$

$$< \texttt{Second iteration} >$$

$$\texttt{Check}:\quad B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100\rceil \div 100\right\} \overset{?}{>} 0$$
$$\texttt{13,405.38} - \left\{\texttt{132.87} - \lfloor \texttt{0.5}\,(\texttt{13,405.38} \cdot \texttt{0.00566...}) \times 100\rceil \div 100\right\} \overset{?}{>} 0$$
$$\texttt{13,405.38} - \left\{\texttt{132.87} - \texttt{37.98}\right\} \overset{?}{>} 0$$
$$\texttt{13,405.38} - \texttt{94.89} \overset{?}{>} 0$$
$$\texttt{13,310.49} > 0$$
$$\texttt{Proceeding...}$$

$$B_2 = B_1 - \left\{a - \lfloor \alpha\,(B_1 \cdot i) \times 100\rceil \div 100\right\}$$
$$= \$\texttt{13,310.49}\ ;$$

$$O_2 = O_1 + \lfloor (B_1 \cdot i) \times 100\rceil \div 100 - \lfloor \alpha\,(B_1 \cdot i) \times 100\rceil \div 100$$
$$= \texttt{38.25} + \lfloor (\texttt{13,405.38} \cdot \texttt{0.00566...}) \times 100\rceil \div 100 - \lfloor \texttt{0.5}\,(\texttt{13,405.38} \cdot \texttt{0.00566...}) \times 100\rceil \div 100$$
$$= \texttt{38.25} + \texttt{75.96} - \texttt{37.98}$$
$$= \$\texttt{76.23}\ ;$$

$$m = 2 + 1 = 3\ \texttt{months}$$

$$< \texttt{Third iteration} >$$

$$\begin{aligned}
\texttt{Check:}\quad & B_2 - \left\{a - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 13{,}310.49 - \left\{132.87 - \lfloor 0.5\,(13{,}310.49 \cdot 0.00566...) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 13{,}310.49 - \left\{132.87 - 37.71\right\} \overset{?}{>} 0 \\
& 13{,}310.49 - 95.16 \overset{?}{>} 0 \\
& 13{,}215.33 > 0 \\
& \texttt{Proceeding...}
\end{aligned}$$

$$\begin{aligned}
B_3 &= B_2 - \left\{a - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100\right\} \\
&= \$13{,}215.33\ ;
\end{aligned}$$

$$\begin{aligned}
O_3 &= O_2 + \lfloor (B_2 \cdot i) \times 100 \rceil \div 100 - \lfloor \alpha\,(B_2 \cdot i) \times 100 \rceil \div 100 \\
&= 76.23 + \lfloor (13{,}310.49 \cdot 0.00566...) \times 100 \rceil \div 100 - \lfloor 0.5\,(13{,}310.49 \cdot 0.00566...) \times 100 \rceil \div 100 \\
&= 76.23 + 75.43 - 37.71 \\
&= \$113.95\ ;
\end{aligned}$$

$$m = 3 + 1 = 4\ \texttt{months}$$

$$\vdots$$

$$< 120^{\text{th}}\ \texttt{iteration} >$$

$$\begin{aligned}
\texttt{Check:}\quad & B_{119} - \left\{a - \lfloor \alpha\,(B_{119} \cdot i) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 131.70 - \left\{132.87 - \lfloor 0.5\,(131.70 \cdot 0.00566...) \times 100 \rceil \div 100\right\} \overset{?}{>} 0 \\
& 131.70 - \left\{132.87 - 0.37\right\} \overset{?}{>} 0 \\
& 131.70 - 132.50 \overset{?}{>} 0 \\
& -0.80 \not> 0 \\
& \texttt{Halt!}
\end{aligned}$$

$$n = 120\ \texttt{months}$$

$$\begin{aligned}
a_{\text{f}} &= B_{119} + \lfloor (B_{119} \cdot i) \times 100 \rceil \div 100 + O_{119} \\
&= 131.70 + \lfloor (131.70 \cdot 0.00566...) \times 100 \rceil \div 100 + 2{,}443.24 \\
&= 131.70 + 0.75 + 2{,}443.24 \\
&= \$2{,}575.69\ ;
\end{aligned}$$

$$\begin{aligned}
B_{120} &= B_{119} - \left\{a_{\text{f}} - \lfloor (B_{119} \cdot i) \times 100 \rceil \div 100 - O_{119}\right\} \\
&= 131.70 - \left\{2{,}575.69 - \lfloor (131.70 \cdot 0.00566...) \times 100 \rceil \div 100 - 2{,}443.24\right\} \\
&= 131.70 - \left\{2{,}575.69 - 0.75 - 2{,}443.24\right\} \\
&= 131.70 - 131.70 \\
&= \$0
\end{aligned}$$

Breakdown of pay :

94.62 Prin.	+	38.25 Int.	=	132.87
94.89 Prin.	+	37.98 Int.	=	132.87
95.16 Prin.	+	37.71 Int.	=	132.87
95.43 Prin.	+	37.44 Int.	=	132.87
⋮		⋮		⋮
131.70 Prin.	+	2,443.99 Int.	=	2,575.69

Example 1.6d. *If* $APR = 6.8\%$, $\alpha = 0.5$, $p = \$4{,}000$ *and* $a = \$5{,}000$,

$$B_0 = 4{,}000\ ;$$
$$O_0 = 0\ ;$$
$$m = 1$$

$$\texttt{Check:}\quad B_0 - \left\{a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0$$
$$4{,}000 - \left\{5{,}000 - \lfloor 0.5\,(4{,}000 \cdot 0.00566...) \times 100 \rceil \div 100 \right\} \overset{?}{>} 0$$
$$4{,}000 - \left\{5{,}000 - 11.33 \right\} \overset{?}{>} 0$$
$$4{,}000 - 4{,}988.67 \overset{?}{>} 0$$
$$-988.67 \not> 0$$
$$\texttt{Halt!}$$

$$n = 1\ \texttt{month}$$

$$\begin{aligned} a_{\mathrm{f}} &= B_0 + \lfloor (B_0 \cdot i) \times 100 \rceil \div 100 + O_0 \\ &= 4{,}000 + \lfloor (4{,}000 \cdot 0.00566...) \times 100 \rceil \div 100 + 0 \\ &= 4{,}000 + 22.67 + 0 \\ &= \$4{,}022.67\ ; \end{aligned}$$

$$\begin{aligned} \texttt{Check:}\quad & a - a_{\mathrm{f}} \\ &= 5{,}000 - 4{,}022.67 \\ &= 977.33 > 0 \end{aligned}$$

$$R = a - a_{\mathrm{f}} = 977.33\ ;$$
“Refunded \$977.33”

Breakdown of pay:

4,000 Prin. + 22.67 Int. = 4,022.67

Example 1.6e. *If* $APR = 6.8\%$, $\alpha = 0$, $p = \$4{,}000$ *and* $a = \$4{,}000$,

$$B_0 = 4{,}000\ ;$$
$$O_0 = 0\ ;$$
$$m = 1$$

$$\texttt{Check:}\quad B_0 - \Big\{a - \big\lfloor \alpha\,(B_0 \cdot i) \times 100\big\rceil \div 100\Big\} \overset{?}{>} 0$$
$$4{,}000 - \Big\{4{,}000 - \big\lfloor 0\,(4{,}000 \cdot 0.00566\ldots) \times 100\big\rceil \div 100\Big\} \overset{?}{>} 0$$
$$4{,}000 - \Big\{4{,}000 - 0\Big\} \overset{?}{>} 0$$
$$4{,}000 - 4{,}000 \overset{?}{>} 0$$
$$0 \not> 0$$
$$\texttt{Halt!}$$

$$n = 1\ \texttt{month}$$

$$\begin{aligned} a_{\mathrm{f}} &= B_0 + \big\lfloor (B_0 \cdot i) \times 100\big\rceil \div 100 + O_0 \\ &= 4{,}000 + \big\lfloor (4{,}000 \cdot 0.00566\ldots) \times 100\big\rceil \div 100 + 0 \\ &= 4{,}000 + 22.67 + 0 \\ &= \$4{,}022.67\ ; \end{aligned}$$

$$\begin{aligned} \texttt{Check:}\quad & a - a_{\mathrm{f}} \\ & = 4{,}000 - 4{,}022.67 \\ & = -22.67 < 0 \end{aligned}$$

$$E = |a - a_{\mathrm{f}}| = 22.67\ ;$$

"Pay Extra $22.67"

Breakdown of pay:

4,000 Prin.	+	22.67 Int.	=	4,022.67

1.7 Length of repayment

For $n = 4$,

$$\begin{aligned} l_{\mathrm{y}} &= \left\lfloor \frac{4}{12} \right\rfloor \\ &= \lfloor 0.333... \rfloor \\ &= 0 \texttt{ years} \end{aligned}$$

$$\begin{aligned} l_{\mathrm{m}} &= 4 - 12 \cdot 0 \\ &= 4 - 0 \\ &= 4 \texttt{ months} \end{aligned}$$

For $n = 19$,

$$\begin{aligned} l_{\mathrm{y}} &= \left\lfloor \frac{19}{12} \right\rfloor \\ &= \lfloor 1.583... \rfloor \\ &= 1 \texttt{ year} \end{aligned}$$

$$\begin{aligned} l_{\mathrm{m}} &= 19 - 12 \cdot 1 \\ &= 19 - 12 \\ &= 7 \texttt{ months} \end{aligned}$$

For $n = 27$,

$$\begin{aligned} l_{\mathrm{y}} &= \left\lfloor \frac{27}{12} \right\rfloor \\ &= \lfloor 2.25 \rfloor \\ &= 2 \texttt{ years} \end{aligned}$$

$$\begin{aligned} l_{\mathrm{m}} &= 27 - 12 \cdot 2 \\ &= 27 - 24 \\ &= 3 \texttt{ months} \end{aligned}$$

1.8 Total payments, savings and change in savings

Example 1.8a. *Assume* $APR = 6.8\%$, $\alpha = 0.5$ *and* $p = \$13{,}500$.

Total payments,

$$\texttt{Let } a = 300$$

$$\begin{aligned} \therefore n &= 49 \texttt{ months} \\ B_{48} &= \$61.91 \\ O_{48} &= \$961.95 \end{aligned}$$

$$\begin{aligned} T(a) &= (48)a + B_{48} + \lfloor (B_{48} \cdot i) \times 100 \rceil \div 100 + O_{48} \\ T(300) &= (48)300 + 61.91 + \lfloor (61.91 \cdot 0.00566...) \times 100 \rceil \div 100 + 961.95 \\ &= 14{,}400 + 61.91 + 0.35 + 961.95 \\ &= \$15{,}424.21 \end{aligned}$$

$$\texttt{Let } a = a_{\min_{120}} = 132.87$$

$$\begin{aligned} \therefore n &= 120 \texttt{ months} \\ B_{119} &= \$131.70 \\ O_{119} &= \$2{,}443.24 \end{aligned}$$

$$\begin{aligned} T_{\max} &= (119)a_{\min} + B_{119} + \lfloor (B_{119} \cdot i) \times 100 \rceil \div 100 + O_{119} \\ &= (119)132.87 + 131.70 + \lfloor (131.70 \cdot 0.00566...) \times 100 \rceil \div 100 + 2{,}443.24 \\ &= 15{,}672.30 + 131.70 + 0.75 + 2{,}443.24 \\ &= \$18{,}387.22 \end{aligned}$$

Savings,

$$\begin{aligned} s &= T_{\max} - T(300) \\ &= 18{,}387.22 - 15{,}424.21 \\ &= \$2{,}963.01 \end{aligned}$$

Total payments,

$$
\begin{aligned}
\texttt{Let } a &= 725 \\
\therefore n &= 20 \texttt{ months} \\
B_{19} &= \$113.61 \\
O_{19} &= \$388.62 \\
T(a) &= (19)a + B_{19} + \lfloor (B_{19} \cdot i) \times 100 \rceil \div 100 + O_{19} \\
T(725) &= (19)725 + 113.61 + \lfloor (113.61 \cdot 0.00566...) \times 100 \rceil \div 100 + 388.62 \\
&= 13{,}775 + 113.61 + 0.64 + 388.62 \\
&= \$14{,}277.87
\end{aligned}
$$

$$T_{\max} = \$18{,}387.22 \textit{ (page 20)}$$

Savings,

$$
\begin{aligned}
s &= T_{\max} - T(725) \\
&= 18{,}387.22 - 14{,}277.87 \\
&= \$4{,}109.35
\end{aligned}
$$

Change in savings,

$$
\begin{aligned}
\texttt{Let } s_1 &= \$2{,}963.01 \textit{ (page 20)} \\
s_2 &= \$4{,}109.35 \\
\Delta s &= |\ 4{,}109.35 - 2{,}963.01\ | \\
&= \uparrow\ \$1{,}146.34
\end{aligned}
$$

Example 1.8b. *Assume* $APR = 3.45\%$, $\alpha = 0.25$ *and* $p = \$5{,}000$.

Total payments,

$$\texttt{Let } a = 700$$

$$\begin{aligned}
\therefore n &= 8 \texttt{ months} \\
B_7 &= \$114.63 \\
O_7 &= \$43.90
\end{aligned}$$

$$\begin{aligned}
T(a) &= (7)a + B_7 + \lfloor (B_7 \cdot i) \times 100 \rceil \div 100 + O_7 \\
T(700) &= (7)700 + 114.63 + \lfloor (114.63 \cdot 0.002875) \times 100 \rceil \div 100 + 43.90 \\
&= 4{,}900 + 114.63 + 0.33 + 43.90 \\
&= \$5{,}058.86
\end{aligned}$$

$$\texttt{Let } a = a_{\min_n} = 3.60$$

$$\begin{aligned}
\therefore n &= 8{,}822 \texttt{ months} \\
B_{8{,}821} &= \$1.49 \\
O_{8{,}821} &= \$80{,}268.06
\end{aligned}$$

$$\begin{aligned}
T_{\max} &= (8{,}821)a_{\min} + B_{8{,}821} + \lfloor (B_{8{,}821} \cdot i) \times 100 \rceil \div 100 + O_{8{,}821} \\
&= (8{,}821)3.60 + 1.49 + \lfloor (1.49 \cdot 0.002875) \times 100 \rceil \div 100 + 80{,}268.06 \\
&= 31{,}755.60 + 1.49 + 0 + 80{,}268.06 \\
&= \$112{,}025.15
\end{aligned}$$

Savings,

$$\begin{aligned}
s &= T_{\max} - T(700) \\
&= 112{,}025.15 - 5{,}058.86 \\
&= \$106{,}966.29
\end{aligned}$$

Change in savings,

$$\begin{aligned}
\texttt{Let } s_1 &= \$4{,}109.35 \textit{ (page 21)} \\
s_2 &= \$106{,}966.29
\end{aligned}$$

$$\begin{aligned}
\Delta s &= |\ 106{,}966.29 - 4{,}109.35\ | \\
&= \uparrow\ \$102{,}856.94
\end{aligned}$$

Example 1.8c. *Assume APR* $= 7\%$*,* $\alpha = 0.33$ *and* $p = \$2{,}000$*.*

Total payments,

$$\text{Let } a = 550$$

$$\begin{aligned}
\therefore n &= 4 \text{ months} \\
B_3 &= \$358.40 \\
O_3 &= \$17.04
\end{aligned}$$

$$\begin{aligned}
T(a) &= (3)a + B_3 + \lfloor (B_3 \cdot i) \times 100 \rceil \div 100 + O_3 \\
T(550) &= (3)550 + 358.40 + \lfloor (358.40 \cdot 0.00583...) \times 100 \rceil \div 100 + 17.04 \\
&= 1{,}650 + 358.40 + 2.09 + 17.04 \\
&= \$2{,}027.53
\end{aligned}$$

$$\text{Let } a = a_{\min_{120}} = 18.69$$

$$\begin{aligned}
\therefore n &= 120 \text{ months} \\
B_{119} &= \$17.54 \\
O_{119} &= \$490.63
\end{aligned}$$

$$\begin{aligned}
T_{\max} &= (119)a_{\min} + B_{119} + \lfloor (B_{119} \cdot i) \times 100 \rceil \div 100 + O_{119} \\
&= (119)18.69 + 17.54 + \lfloor (17.54 \cdot 0.00583...) \times 100 \rceil \div 100 + 490.63 \\
&= 2{,}224.11 + 17.54 + 0.10 + 490.63 \\
&= \$2{,}732.38
\end{aligned}$$

Savings,

$$\begin{aligned}
s &= T_{\max} - T(550) \\
&= 2{,}732.38 - 2{,}027.53 \\
&= \$704.85
\end{aligned}$$

Change in savings,

$$\begin{aligned}
\text{Let } s_1 &= \$106{,}966.29 \textit{ (page 22)} \\
s_2 &= \$704.85
\end{aligned}$$

$$\begin{aligned}
\Delta s &= |\ 704.85 - 106{,}966.29\ | \\
&= \downarrow\ \$106{,}261.44
\end{aligned}$$

Total payments,

$$
\begin{aligned}
\texttt{Let } a &= 1{,}000 \\
\therefore n &= 3 \texttt{ months} \\
B_2 &= \$5.78 \\
O_2 &= \$11.75
\end{aligned}
$$

$$
\begin{aligned}
T(a) &= (2)a + B_2 + \lfloor (B_2 \cdot i) \times 100 \rceil \div 100 + O_2 \\
T(1{,}000) &= (2)1{,}000 + 5.78 + \lfloor (5.78 \cdot 0.00583\ldots) \times 100 \rceil \div 100 + 11.75 \\
&= 2{,}000 + 5.78 + 0.03 + 11.75 \\
&= \$2{,}017.56
\end{aligned}
$$

$$T_{\text{max}} = \$2{,}732.38 \textit{ (page 23)}$$

Savings,

$$
\begin{aligned}
s &= T_{\text{max}} - T(1{,}000) \\
&= 2{,}732.38 - 2{,}017.56 \\
&= \$714.82
\end{aligned}
$$

Change in savings,

$$
\begin{aligned}
\texttt{Let } s_1 &= \$704.85 \textit{ (page 23)} \\
s_2 &= \$714.82
\end{aligned}
$$

$$
\begin{aligned}
\Delta s &= |\ 714.82 - 704.85\ | \\
&= \uparrow\ \$9.97
\end{aligned}
$$

Total payments,

$$\texttt{Let } a = 200$$

$$\begin{aligned}\therefore n &= 11 \texttt{ months}\\ B_{10} &= \$21.42\\ O_{10} &= \$43.50\end{aligned}$$

$$\begin{aligned}T(a) &= (10)a + B_{10} + \lfloor (B_{10} \cdot i) \times 100 \rceil \div 100 + O_{10}\\ T(200) &= (10)200 + 21.42 + \lfloor (21.42 \cdot 0.00583...) \times 100 \rceil \div 100 + 43.50\\ &= 2{,}000 + 21.42 + 0.12 + 43.50\\ &= \$2{,}065.04\end{aligned}$$

$$T_{\text{max}} = \$2{,}732.38 \textit{ (page 23)}$$

Savings,

$$\begin{aligned}s &= T_{\text{max}} - T(200)\\ &= 2{,}732.38 - 2{,}065.04\\ &= \$667.34\end{aligned}$$

Change in savings,

$$\begin{aligned}\texttt{Let } s_1 &= \$714.82 \textit{ (page 24)}\\ s_2 &= \$667.34\end{aligned}$$

$$\begin{aligned}\Delta s &= |\ 667.34 - 714.82\ |\\ &= \downarrow\ \$47.48\end{aligned}$$

2 Derivation of Monthly Compound Interest Rate

Definition 2.1. *If interest is compounded, it is compounded daily. Let daily payment balance be a function of each day's previous balance and the annual interest rate:*

$$B_d = B_{d-1} + B_{d-1}\frac{r}{365.25},$$

for number of days, $d = 1, 2, 3, \ldots, 30, \frac{365.25}{12}$*, where* $\frac{365.25}{12}$ *is the average number of days per month. Treat the last day is a partial day.*

Theorem 2.2. *Monthly compound interest rate will be a function of the annual rate:*

$$i = \left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}} - 1$$

Proof. Using definition 2.1, find average monthly balance, $B_{\frac{365.25}{12}}$, and simplify.

$$\begin{aligned}
B_1 &= B_0 + B_0\frac{r}{365.25} \\
&= p + p\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right) \\
\\
B_2 &= B_1 + B_1\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right) + p\left(1 + \frac{r}{365.25}\right)\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right)\left(1 + \frac{r}{365.25}\right) \\
&= p\left(1 + \frac{r}{365.25}\right)^2 \\
\\
B_3 &= B_2 + B_2\frac{r}{365.25} \\
&= \cdots = p\left(1 + \frac{r}{365.25}\right)^2\left(1 + \frac{r}{365.25}\right) \\
&= p\left(1 + \frac{r}{365.25}\right)^3 \\
\\
&\vdots \\
\\
B_{\frac{365.25}{12}} &= B_{\frac{365.25}{12}-1} + B_{\frac{365.25}{12}-1}\frac{r}{365.25} \\
&= \cdots = p\left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}} \qquad (2.2a)
\end{aligned}$$

Write equation 2.2a as a function of i.

$$B_{\frac{365.25}{12}} = p\left(1+i\right) \tag{2.2b}$$

Combine equations 2.2a and 2.2b to solve for i.

$$p\left(1+i\right) = p\left(1+\frac{r}{365.25}\right)^{\frac{365.25}{12}}$$

$$i = \left(1+\frac{r}{365.25}\right)^{\frac{365.25}{12}} - 1 \qquad \square$$

Numerically, do not round any terms in the equation or solution itself, otherwise solutions to equations that depend on the interest rate may lose precision. The solution will be a decimal, not a percentage; this is because i is only meant to be an interim calculation.

Remark 2.3. The researcher is only aware of one instance in which interest is compounded: when loans are capitalized as students enter repayment. Nevertheless, he has mentioned interest being compounded, purely for the purpose of mathematical exploration.

3 Derivation of Ten-Year Minimum Payment

Definition 3.1. *Student loan payments are made in monthly installments. Let the monthly principal balance, be a function of each month's previous balance, the interest rate, minimum payment and proportion of interest that is paid, such that:*

$$B_m = B_{m-1} - \left[a_{\min} - \alpha\left(B_{m-1} \cdot i\right)\right], \tag{3.1}$$

for $m = 1, 2, 3, \ldots, 120$ *and* $0 \leq \alpha \leq 1$*, where* 120 *is the number of months in ten years. We want final balance,* $B_{120} = 0$*.*

Theorem 3.2. *Minimum monthly payment within ten years will depend on* i *and* α*:*

$$a_{\min_{120}} = \begin{cases} \left\lceil \dfrac{p}{120} \times 100 \right\rceil \div 100 & \text{if } i > 0 \text{ and } \alpha = 0 \\ \left\lceil \dfrac{p}{120} \times 100 \right\rceil \div 100 & \text{if } i = 0 \\ \left\lceil \dfrac{\alpha\left(p \cdot i\right)\left(1 + \alpha \cdot i\right)^{120}}{\left(1 + \alpha \cdot i\right)^{120} - 1} \times 100 \right\rceil \div 100, \text{for } \alpha \cdot i \neq 0 & \text{if } i > 0 \text{ and } 0 < \alpha \leq 1 \end{cases}$$

Proof. Simplify equation 3.1, if possible, using each case of i and α.

$$B_m = \begin{cases} B_{m-1} - a_{\min} & \text{if } i > 0 \text{ and } \alpha = 0 & (3.2a) \\ B_{m-1} - a_{\min} & \text{if } i = 0 & (3.2b) \\ B_{m-1} - \left[a_{\min} - \alpha\left(B_{m-1} \cdot i\right)\right] & \text{if } i > 0 \text{ and } 0 < \alpha \leq 1 & (3.2c) \end{cases}$$

Using cases 3.2a and 3.2b, find B_{120} and simplify.

$$\begin{aligned} B_1 &= B_0 - a_{\min} \\ &= p - a_{\min} \\[1em] B_2 &= B_1 - a_{\min} \\ &= \left(p - a_{\min}\right) - a_{\min} \\ &= p - 2a_{\min} \\[1em] B_3 &= B_2 - a_{\min} \\ &= \cdots = p - 3a_{\min} \\[1em] &\vdots \\[1em] B_{120} &= B_{119} - a_{\min} \\ &= p - 120a_{\min} \end{aligned}$$

Set $B_{120} = 0$ to solve for $a_{\min}$.

$$\begin{aligned} 0 &= p - 120a_{\min} \\ p &= 120a_{\min} \\ a_{\min} &= \frac{p}{120} \end{aligned}$$

Numerically, terms in the solution may span more than two decimal places, but cents span only two. Also, we need to ensure that one repays enough money each month. So, round terms in the solution *up* to the nearest two decimal places.

$$a_{\min} = \left\lceil \frac{p}{120} \times 100 \right\rceil \div 100$$

Let $a_{\min} = a_{\min_{120}}$ to differentiate it from the absolute minimum payment.

$$a_{\min_{120}} = \left\lceil \frac{p}{120} \times 100 \right\rceil \div 100 \qquad \square$$

Using case 3.2c, find B_{120} and simplify.

$$\begin{aligned} B_1 &= B_0 - \left[a_{\min} - \alpha\left(B_0 \cdot i\right)\right] \\ &= B_0 + \alpha\left(B_0 \cdot i\right) - a_{\min} \\ &= p + \alpha\left(p \cdot i\right) - a_{\min} \\ &= p\left(1 + \alpha \cdot i\right) - a_{\min} \end{aligned}$$

$$\begin{aligned} B_2 &= B_1 - \left[a_{\min} - \alpha\left(B_1 \cdot i\right)\right] \\ &= B_1 + \alpha\left(B_1 \cdot i\right) - a_{\min} \\ &= \left[p\left(1 + \alpha \cdot i\right) - a_{\min}\right] + \alpha\Big(\left[p\left(1 + \alpha \cdot i\right) - a_{\min}\right] \cdot i\Big) - a_{\min} \\ &= \left[p\left(1 + \alpha \cdot i\right) - a_{\min}\right]\left[1 + \alpha \cdot i\right] - a_{\min} \\ &= p\left(1 + \alpha \cdot i\right)^2 - \left(1 + \alpha \cdot i\right)a_{\min} - a_{\min} \end{aligned}$$

$$\begin{aligned} B_3 &= B_2 - \left[a_{\min} - \alpha\left(B_2 \cdot i\right)\right] \\ &= \cdots = \left[p\left(1 + \alpha \cdot i\right)^2 - \left(1 + \alpha \cdot i\right)a_{\min} - a_{\min}\right] \\ &\qquad + \alpha\Big(\left[p\left(1 + \alpha \cdot i\right)^2 - \left(1 + \alpha \cdot i\right)a_{\min} - a_{\min}\right] \cdot i\Big) - a_{\min} \\ &= \left[p\left(1 + \alpha \cdot i\right)^2 - \left(1 + \alpha \cdot i\right)a_{\min} - a_{\min}\right]\left[1 + \alpha \cdot i\right] - a_{\min} \\ &= p\left(1 + \alpha \cdot i\right)^3 - \left(1 + \alpha \cdot i\right)^2 a_{\min} - \left(1 + \alpha \cdot i\right)a_{\min} - a_{\min} \end{aligned}$$

$$\vdots$$

$$\begin{aligned}
B_{120} &= B_{119} - \left[a_{\min} - \alpha\left(B_{119} \cdot i\right)\right] \\
&= \cdots = \left[p\left(1+\alpha \cdot i\right)^{119} - \left(1+\alpha \cdot i\right)^{118} a_{\min} - \left(1+\alpha \cdot i\right)^{117} a_{\min} \right. \\
&\qquad\qquad \left. - \cdots - a_{\min}\right]\left[1+\alpha \cdot i\right] - a_{\min} \\
&= p\left(1+\alpha \cdot i\right)^{120} - \left(1+\alpha \cdot i\right)^{119} a_{\min} - \left(1+\alpha \cdot i\right)^{118} a_{\min} \\
&\quad - \cdots - \left(1+\alpha \cdot i\right) a_{\min} - a_{\min} \\
&= p\left(1+\alpha \cdot i\right)^{120} - a_{\min} \sum_{m=1}^{120} \left(1+\alpha \cdot i\right)^{m-1}
\end{aligned}$$

Set $B_{120} = 0$ to solve for $a_{\min}$.

$$0 = p\left(1+\alpha \cdot i\right)^{120} - a_{\min} \sum_{m=1}^{120} \left(1+\alpha \cdot i\right)^{m-1}$$

$$\begin{aligned}
p\left(1+\alpha \cdot i\right)^{120} &= a_{\min} \sum_{m=1}^{120} \left(1+\alpha \cdot i\right)^{m-1} \\
p\left(1+\alpha \cdot i\right)^{120} \times \left(1+\alpha \cdot i\right) &= a_{\min} \sum_{m=1}^{120} \left(1+\alpha \cdot i\right)^{m} \\
p\left(1+\alpha \cdot i\right)^{120} - p\left(1+\alpha \cdot i\right)^{120} \times \left(1+\alpha \cdot i\right) &= a_{\min} \\
&\quad - \left(1+\alpha \cdot i\right)^{120} a_{\min} \\
p\left(1+\alpha \cdot i\right)^{120}\left[1 - \left(1+\alpha \cdot i\right)\right] &= a_{\min}\left[1 - \left(1+\alpha \cdot i\right)^{120}\right] \\
p\left(1+\alpha \cdot i\right)^{120}\left[\alpha \cdot i\right] &= a_{\min}\left[\left(1+\alpha \cdot i\right)^{120} - 1\right]
\end{aligned}$$

$$a_{\min} = \frac{\alpha\left(p \cdot i\right)\left(1+\alpha \cdot i\right)^{120}}{\left(1+\alpha \cdot i\right)^{120} - 1}, \text{ for } \alpha \cdot i \neq 0$$

Numerically, do not round any terms in the equation; round only those in the solution. However, again terms in the solution may span more than two decimal places, and we need to ensure that one repays enough money each month. So, round terms in the solution *up* to the nearest two decimal places.

$$a_{\min} = \left\lceil \frac{\alpha\left(p \cdot i\right)\left(1+\alpha \cdot i\right)^{120}}{\left(1+\alpha \cdot i\right)^{120} - 1} \times 100 \right\rceil \div 100, \text{ for } \alpha \cdot i \neq 0$$

Let $a_{\min} = a_{\min_{120}}$ to differentiate it from the absolute minimum payment.

$$a_{\min_{120}} = \left\lceil \frac{\alpha\left(p \cdot i\right)\left(1+\alpha \cdot i\right)^{120}}{\left(1+\alpha \cdot i\right)^{120} - 1} \times 100 \right\rceil \div 100, \text{ for } \alpha \cdot i \neq 0 \qquad \square$$

Remark 3.3. The reason for using $\alpha\,(B_{m-1}\cdot i)$ in equation 3.1, instead of the corresponding expression $\lfloor \alpha\,(B_{m-1}\cdot i)\times 100 \rceil \div 100$ of section 2 of "Deeper Insight into the iOS App", is because, if $i>0$ and $0<\alpha\leq 1$ and one uses the latter expression to find a simplified equation for B_{120}, factoring $p+\lfloor \alpha\,(p\cdot i)\times 100 \rceil \div 100 - a$ from the `nint` function in B_2 will not be possible. Subsequent equations (i.e., B_3, B_4, B_5, $\ldots$) will continue to expand with no way to simplify them even. The caveat to using the former expression, though, is that each numerical solution of $a_{\min_{120}}$ could potentially deviate by one cent from each solution had they been computed by using the latter expression.

Remark 3.4. Simplifying the fractional part

$$\frac{(1+\alpha\cdot i)^{120}}{(1+\alpha\cdot i)^{120}-1}, \text{ for } \alpha\cdot i\neq 0$$

is not possible. α is at most equal to 1, and APR is probably at most 10% (i.e., i is at most 0.00836... [interest is compounded]).

$$\frac{(1+1\cdot 0.00836...)^{120}}{(1+1\cdot 0.00836...)^{120}-1}=\frac{(1.00836...)^{120}}{(1.00836...)^{120}-1}=\frac{2.717...}{2.717...-1}=1.582...$$

In fact, all other quotients for values of α and i will exceed 1.582... Only if all quotients were equal to 1, could one cancel the numerator and denominator from the fractional part.