ISE 5113: Advanced Analytics and Metaheuristics

Homework #1
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1 Smullyan's Island Revisited (6 points)

There are $2^3 = 8$ possibilities listed in the left part of the Table 1. Based on the statements from Gregory, Tywin and Catelyn, the True-or-False determinations are made which is showed on the middle part of the table. The last column is established according to the possibilities and statement evaluation. It is found that the only paired matched in the case that Gregor is truth-teller while Tywin and Catelyn are both liars. It is highlighted in the table below.

Table 1. Truth table of Question 1

Possibilities			***Evaluation***			***Matched***
Gregor	***Tywin***	***Catelyn***	***Gregor***	***Tywin***	***Catelyn***	
T	T	T	F	F	F	×
T	T	F	F	F	F	×
T	F	T	F	F	F	×
T	F	F	T	F	F	✓
F	T	F	T	F	F	×
F	T	T	F	F	F	×
F	F	T	T	F	T	×
F	F	F	F	T	T	×

2 Wilding and Humans (10 points)

There are originally $4^2 = 16$ possibilities for the identities of Arya and Sansa. While Jon knew that one of two is a wilding and another one is a human, the possibilities are dropped to half of total cases which is 8. The next step is to determine from the truth table. The (T) in the left part of the Table 2 stands for telling the truth while the (F) on the left part of the Table 2 stand for lying. It was given that what combination of Human/Wilding and Sane/Insane tell truth or lie. Arya and Sansa's statements were evaluated as truth or lie, then compared with the assumptions. There are two paired possibilities with the statements highlighted in the table 2 in green. The two possibilities both give Arya as wilding and Sansa as human. Although we cannot evaluate their sanity, but the problem is solved, because we were asked only about the wilding-human determination apart from the sanity.

Table 2. The truth table for Question 2

Arya	***Sansa***	***Arya***	***Sansa***	***Matched***
Insane-Wilding (T)	Insane-Human (F)	T	F	✓
Sane-Wilding (F)	Insane-Human (F)	F	T	×
Sane-Wilding (F)	Sane-Human (T)	F	T	✓
Insane-Wilding (T)	Sane-Human (T)	F	T	×
Insane-Human (F)	Insane-Wilding (T)	T	F	×
Sane-Human (T)	Insane-Wilding (T)	F	T	×
Sane-Human (T)	Sane-Wilding (F)	F	T	×
Insane-Human (F)	Sane-Wilding (F)	F	T	×

3 Working Capital Management (20 points)

3.1 Assumptions

Dividends are paid at the end of the month with principal.
No money should be out of vault at the end of month 6. All the invested money should be paid back.

3.2 Decision Variables

The amount of money in thousand(s) dollars that will be invested in three different plans over six months are decision variables. The investment variable has distinguished by two indexes: month numbering as first index denoted in "month" which are numbered in 1 through 6 for six months, and plan notation as the second index denoted in "plan" which are indicated as 1 for one-month plan, 3 for three-month plan and 6 for six-month plan. For example, *X[4,3]* stands for the investments in a three-month plan on 4th month. Based on the assumption, all the money should be paid back at the end of the study period. It means that we are not able to invest any money on 6-monthe plan at 2nd month. Because it is not possible for us to collect the invested money and its dividend at the end of period. Same argument is valid for 3rd month and so on.

$$X_{i,j} \geq 0, \qquad i \text{ is month of investment, and } j \text{ is plan indicator}$$
$$X_{i,6} = 0 \qquad for\ i \geq 2$$
$$X_{i,3} = 0 \qquad for\ i \geq 5$$

3.2.1 Intermittent Variables

There is another variable named *"Money in Pocket"* which is defined as paid back invested money. After adding or subtracting the revenues or expenditures which is given as net cash flow, it is possible to reinvestment of the available cash to increase the profit.

$$(Money\ in\ Pocket)_i = \sum_i \sum_j \left(1 + (interest\ rate)_j\right). X_{i-j,j}$$
$$Investment_i = (Money\ in\ Pocket)_i + (Net\ Cash\ Flow)_i$$

Other formulated variables are:

$$(Interest\ Gained)_j = \sum_i \left(1 + (interest\ rate)_j\right). X_{i,j}$$

$$Final\ Cash = \sum_j \left(1 + (interest\ rate)_j\right). X_{7-j,j}$$
$$Total\ interest\ income = Final\ Cash - 300 - \sum_i (Cash\ Flow)_i$$

They are defined to aid the displaying purpose. They are acceptable to be located in both constrains area and variable area.

3.3 Objective and objective function

Maximize the total amount of money that will be obtained at the end of sixth month investment period. It is similar to maximizing the total interest income over six months. Objective function is showing as:

$$Objective: \quad Maximize \left\{ Final\ Cash = \sum_j \big(1 + (interest\ rate)_j\big). X_{7-j,j} \right\}$$

3.4 Constraints

The main idea of the constrains is that the investments amount is less or equal the amount of available money. The cash flows are considered in here as well. The constrains are set as follows:

```
subject to Cash_Back1:
        Money_in_Pocket [1] = 300;
subject to Cash_Back2:
        Money_in_Pocket [2] = (1+int[1]) * X[1,1];
subject to Cash_Back3:
        Money_in_Pocket [3] = (1+int[1]) * X[2,1];
subject to Cash_Back4:
        Money_in_Pocket [4] = (1+int[1]) * X[3,1] + (1+int[3]) * X[1,3];
subject to Cash_Back5:
        Money_in_Pocket [5] = (1+int[1]) * X[4,1] + (1+int[3]) * X[2,3];
subject to Cash_Back6:

        Money_in_Pocket [6] = (1+int[1]) * X[5,1] + (1+int[3]) * X[3,3];
```

In addition, the non-negativity for all investments and interests are included. They are specified in the decision variables section in the code.

3.5 Results

The results are solved in AMPL and are shown in the table 3 below:

Table 3. The investment distributions and interest incomes summary

Investment Distributions for Three Plans				
Name of Plan		Investment amount (unit: thousands dollars)	Interest Income (unit: thousands dollars)	Total Interest Incomes (unit: thousands dollars)
1-month investment plan	1st month investments	50.788	0.433	10.151
	2nd month investments	22.886		
	3rd month investments	0.000		
	4th month investments	0.000		
	5th month investments	0.000		
	6th month investments	13.000		
3-month investment plan	1st month investments	199.212	9.718	
	2nd month investments	40.157		
	3rd month investments	0.000		
	4th month investments	223.395		
6-month investment plan	1st month investments	0.000	0.000	

4 Seasonal Demand (14 points)

4.1 Assumption

The operations are going to be the long term runs.
The operation is circular which means the first season come after the next season forever.

4.2 Decision Variables

Storages and productions of each season are the decision variables. They determine the costs of the inventory. The numbering of the decision variables from 1 to 4 are assigned over 4 seasons.

$$Storage_s \geq 0, \qquad s \text{ is season index}$$

$$0 \leq Production_s \leq 1200, \qquad s \text{ is season index}$$

4.3 Objective and objective function

The objective in this question is to minimizing the total inventory costs over a year. Objective function is showing as

$$Minimize \left\{ Total\ Cost = 0.15 * \sum_{s=1}^{4} Storage_s \right\}$$

4.4 Constraints

The constrains include the production capacity limit which was mentioned in decision variable section above. As the business operation is a circular process, the inventory of season is equal to the difference between the production demand which is the sold commodity and summation of the produced production and available commodities in the warehouse at current season. Other constrain is sufficiency of products which at least is the total amount of products produced in the current season plus current inventory. In terms of AMPL syntax they are as follow:

```
subject to production {i in season}:  Production [i] <= p [i];

subject to meeting_demands_season {i in season}:

                Storage [i] + Production [i] >= d [i];

subject to Storage_Define_1: Storage [1] = Storage [4] + Production [4] - d [4];

subject to Storage_Define_2: Storage [2] = Storage [1] + Production [1] - d [1];

subject to Storage_Define_3: Storage [3] = Storage [2] + Production [2] - d [2];

subject to Storage_Define_4: Storage [4] = Storage [3] + Production [3] - d [3];
```

In addition, the non-negativity for all the variables are defined in the decision variable section.

4.5 Results

The best production plans in each season and the corresponding inventories are summarized in the Table 4 below for sake of the lowest inventory cost over years. The total costs in a year is estimated as 450 dollars per year.

Table 4. Production Plan and Storages Distributions

The Production Plan and Storages Distributions of Sandals (Unit: pairs)			
	Demands	Production	Storage (Inventory)
Season 1	2800	1200	1600
Season 2	500	650	0
Season 3	100	1200	150
Season 4	850	1200	1250
Total Inventory Cost: $450			

5 Golden Canning Co (50 points)

5.1 Part (a)

Jaggers state that the whole tomato production is limited to 800,000 ponds is because of the quality and the availability of the tomato. The minimum average required quality for canned Whole Tomato is 8 points which could be a mixer of the grade A tomato with the average point of 9 point and the grade B tomatoes with average point of 5. So, the maximum canned whole tomato productions can be achieved to mix all the grade A tomatoes with corresponding amount of grade B tomatoes and some grade B tomatoes would be excess. At the same time, the produced canned whole tomato products have to meet minimum 8 points quality. The calculations are showing below. If m_A and m_B is mass of the consumed grade A and grade B tomatoes, respectively:

$$9 \times m_A + 5 \times m_B = 8 \times m_{canned\ whole\ tomato} \qquad \text{Equation 1}$$

Here,

$$m_{canned\ whole\ tomato} = m_A + m_B \qquad \text{Equation 2}$$

Substituting above Equation 2 into Equation 1, we can get:

$$9 \times m_A + 5 \times m_B = 8 \times (m_A + m_B) \qquad \text{Equation 3}$$

Simplifying Equation 3:

$$m_A = 3m_B \qquad \text{Equation 4}$$

The availability or maximum amount of the grade A tomatoes is:

$$m_A = 20\% \times 3{,}000{,}000\ lb = 600{,}000\ lb \qquad \text{Equation 5}$$

We can plug Equation 5 in Equation 4 to solve m_B. The m_B is calculated as:

$$m_B = 200{,}000\ lb \qquad \text{Equation 6}$$

The $m_{canned\ whole\ tomato}$ can be calculated by plugging Equation 5 and 6 in Equation 2:

$$m_{canned\ whole\ tomato} = 600{,}000\ lb + 200{,}000\ lb = 800{,}000\ lb \qquad \text{Equation 7}$$

5.2 Part (B)

Bollman assumed the linear relationship between the costs of tomatoes and points-scaled qualities of tomatoes in each grade to obtain the tomato costs in each grade.

$$Tomato\ Pice\ (¢/lb) = 1.036 * Tomato\ Quality\ (ponits)$$

After obtaining the tomato cost in each grade, she calculated the cost of each product in unit of case. Tomato price of tomato Juice as an example:

$$9m_A + 5m_B = 6(m_A + m_B) \qquad => \qquad m_A/m_B = 1/3$$

That means the tomato juice is mixed with the $m_A{:}\,m_B = 1{:}\,3$. So, we can obtain the price of the tomato juice per pond cost by:

$$Cost\ of\ tomato\ juice\ per\ pond = \frac{1\ lb \times Price\ of\ Grade\ A\ tomato + 3\ lbs \times Price\ of\ Grade\ B\ tomato}{1\ lb + 3\ lbs}$$

$$= \frac{1lb\ \times\ 9.32\ cents + 3lb\ \times\ 5.18\ cents}{1\ lb + 3\ lbs} = 6.215\ cents/lb$$

From Table 2 in the problem statement, we can extract that each tomato juice case is produced from 20 lbs tomatoes. So, the tomato price of the tomato juice for each case in table 3 of the problem statement can be calculated as:

$$\begin{aligned} Cost\ of\ tomato\ juice\ per\ pond &= Cost\ of\ tomato\ juice\ per\ pond\ \times\ weights\ in\ lbs\ needed\ for\ each\ case \\ &= 6.215\ cents/lbs \times\ 20\ lbs\ per\ case = 124.3\ cents = 1.24\ dollars \end{aligned}$$

The generalized equation to calculate the tomato cost for each product in table 3 of the problem statement is expressed as:

$$Cost\ of\ tomato\ juice\ per\ pond =$$

$$\frac{portion\ of\ Grade\ A \times Price\ of\ Grade\ A + portion\ of\ Grade\ B \times Price\ of\ Grade\ B}{portion\ of\ Grade\ A + portion\ of\ Grade\ A} \times ponds\ needed\ for\ each\ case$$

The reason why she concludes that the company should use 2,000,000 lbs of "B" tomatoes to produce paste and use remaining 400,000 lbs of B and all of "A" in juice is due to the marginal profit consideration and the qualities of each products:

(1) Based on her calculation in table 3 of the problem statement, the canned whole tomatoes has negative marginal profits so that it should be produced at all.

(2) The largest marginal profit product is tomato paste. Then we would have to meet the demand of 80,000 cases tomato paste to make maximized profit for tomato paste.

$$The\ amount\ of\ tomato\ needed\ to\ produce\ paste = demands\ of\ paste\ in\ cases\ \times$$
$$weights\ of\ in\ each\ paste\ case = 80{,}000\ cases\ \times 25lb/case = 2{,}000{,}000\ lbs$$

(3) The tomato paste can be produced from grade B since the required points-scaled qualification is low.

(4) So the company should use 2,000,000 lbs of grade B tomato to produce paste.

(5) Juice should meet minimum 6 point-scaled quality which is bigger than the grade B averaged points (5 points). So, the juice should be produced from mix of Grade A tomato and Grade B tomato. So the rest of Grade B tomato (remaining grade B tomato: 3,000,000 lbs × 80% - 2,000,000 lbs = 400,000 lbs) will be mixed with all the grade A tomato.

The wrong reasoning from Bollman is:

First, the linear relationship between the points-scaled quality and price are assumed.

Second, she did not count the fixed cost or overhead in the marginal profit calculations.

Third, the miscalculation of the tomato cost of juice. It results to the miscalculation of the juice marginal calculation. It is explained as following reasoning: if remaining 400,000 lbs of grade B tomato will be mixed with all the grade A tomato to produce juice. The Bollman intended scaled points quality of juice will be calculated as:

$$Bollman's\ intended\ juice\ points$$
$$= \frac{400{,}000\ lbs\ of\ \text{grade B tomato} \times 5\ points\ of\ grade\ B\ tomato + 600{,}000\ lbs\ of\ \text{grade A tomato} \times 9\ points\ of\ grade\ A\ tomato}{400{,}000\ lbs\ of\ \text{grade B tomato} + 600{,}000\ lbs\ of\ \text{grade A tomato}}$$
$$= 7.4\ points$$

Then the tomato cost of Bollman's intended juice price per case would be:

$$Bollman\ intended\ juice\ price\ per\ case = \frac{9.32\ cents/lb}{9\ points} \times 7.4\ points \times 20\ (lbs/case)$$
$$= 153.3\ cents = \$1.53$$

If tomato cost of juice is 1.53 dollars, the marginal profit of Bollman's intended juice price per case would be $\$1.32 - 1.53 = \$ - 0.21$. Then the tomato juice is not profitable anymore.

5.3 Part (C)

The solutions for *part c(i)* to *part c(vi.)* are stated in the result section. Here, we will start from the coding approaches for solutions.

5.3.1 Assumption

1. The products qualities in each types of products (whole, juice, paste) are homogeneous. In other words, all of produced canned whole tomatoes are same. So are juices' and pastes'.
2. There are no defective products in the whole processes.
3. There is no production capacity limitation.

5.3.2 Decision Variables

The decision variables are

- X[*i, j*]: amount of grade A or B tomatoes (*j* is index of tomato grades in Tomatos: A, B) in ponds are used to produce specific *i* product (*i* is index of Products: whole, juice, paste).
- Wasted_Fruits: the leftover tomatoes in ponds. The wasted fruits calculation in part (c) in here exclude the chance of purchasing additional A-grade tomatoes.
- Net_Profit: the total profits that Golden Can Co. would make. Here, we considered the expenses of fix/overhead costs, variable costs, fruits costs and wasted fruits costs (The tomato leftovers are purchased while they won't make profits. Then they became costs).

They are listed in here:

```
var X {i in Products, j in Tomatos} >= 0;
var Wasted_Fruits >= 0
    Wasted_Fruits = sum {j in Tomatos} Availability [j]
                            - sum {i in Products} sum{j in Tomatos} X[i,j]
    Net_Profit = sum {i in Products} ((Profit [i]) *
                      (sum {j in Tomatos} (X[i,j] / lb_per_case[i])))
                            - Wasted_Fruits * Average_Price
```

5.3.3 Objective and objective function

Objective: the objective is to maximize the overall profits for Golden Can Co.

Objective function is showing as: `Net_Profit =`

```
sum {i in Products} ((Profit [i]) * (sum {j in Tomatos} (X[i,j] / lb_per_case[i])))
                                        - Wasted_Fruits * Average_Price
```

5.3.4 Constraints

The constrains are considered from the following aspect: market demands, availabilities of tomatoes each grade, Quality controls of each product. They are:

```
subject to Market {i in Products}:

        sum {j in Tomatos} X[i,j]  <= Demand [i] * lb_per_case [i];
```

```
subject to Production {j in Tomatos}:

        sum {i in Products} X[i,j] <= Availability [j];

subject to Quality_Control {i in Products}:

        sum {j in Tomatos} (X[i,j] * Quality [j]) >=

              req_Qulity [i] * sum {j in Tomatos} X[i,j];
```

In addition, the wasted tomato is set in the constraint section. It is reasonable to be set in either variable section or constraint section. It is expressed as:

```
Wasted_Fruits = sum {j in Tomatos} Availability [j] -

       sum {i in Products} sum {j in Tomatos} X[i,j];
```

In addition, the non-negativity for all the variables are defined in the decision variable section.

5.3.5 *Results*

i. How whole, juice, and paste should be made?
The amounts of listed products in unit of case is summarized in the table below:

Table 5. The Tomato Usages and Expected Productions for Each Product

Product	Grade A (lbs)	Grade B (lbs)	Grade A+B (lbs)	Productions in cases after rounding
Whole tomato	525,000	175,000	700,000	38,888
Tomato Juice	75,000	225,000	300,000	15,000
Tomato Paste	0	2,000,000	20,00,000	80,000

ii. Contributions to the profits from each product are summarized in the table 6 below:

Table 6 product profit distributions and contributions

Product	Profit Contribution ($)	Profit Contributions ($/case)	Total Profit Contributions (%)	Total Profits ($)
Whole tomato	15,556	0.40	34.3%	45,356
Tomato Juice	1,800	0.12	4.0%	
Tomato Paste	28,000	0.35	61.7%	

iii. Are there any tomatoes left over? If so, of what grade?

From the AMPL, Wasted_Fruits = 0, meaning there is no tomatoes left over.

iv. The average quality point count of whole, juice, and paste?

The average quality point of each product can be calculated from the equation below:

$$\text{Average Quality Point [i]} = \frac{X[i,A] \times Quality\ [A] + X[i,B] \times Quality\ [B]}{X[i,A] + X[i,B]}$$

Where X[i, A] and X[i, B] stand for the amount of tomatoes in grade A and B in pounds that are used to produce a specific product i.

From there, the average quality point of each product is calculated as follow:

Table 7. The Average Quality Point for Each Product

Product	Average Quality Point
Whole tomato	8
Tomato Juice	6
Tomato Paste	5

v. What would be the worth of one additional pound of A-grade tomatoes?

The shadow price of constraint Production or the availability of the A-grade tomato is 3.24 cents. That means if we add additional pound of A-grade tomatoes.

vi. Should Golden Canning Co buy the extra 80,000 pounds of A-grade tomatoes at the offered price?

Production or the availability sensitivity shows that the A-grade tomatoes has upper bound of 750,000 lbs. It means that we can ask for 750,000 lbs - 600,000 lbs = 150,000 lbs more A-grade grade A without changing the optimum solution. Therefore; we can take the offer of 80,000 pounds of A-grade tomatoes.

5.4 Part (D)

i. If Thomas's contribution figures were used, the profit of unit products in each types would be changes. From part C to Thoma's contribution figures, the changes are summarized from the table below:

Table 8 Thomas's contribution method changes from part c.

Profit Methods	Profit of Whole Tomatoes ($/case)	Profit of Juice ($/case)	Profit of Paste ($/case)
Thomas's contribution	0.12	-0.09	0.12
Part (c)	0.40	0.12	0.35

The differences from part c to Thomas's contributions is the overhead. Then changes in AMPL code is set the overhead of Thomas's contribution methods from 0 to corresponding OHD in the table 2 of the problem statement for all products. The corresponding results are showed in the following table:

Table 9. The Outcome Summary of Using Thomas's Contribution Figures.

Product	Grade A (lbs)	Grade B (lbs)	Grade A+B (lbs)	Production in cases (after rounding)	Profit Contribution ($)	Profit Contributions (%)	Total Profits ($)
Whole tomato	525,000	175,000	700,000	38,888	4,667	36.10%	12,917
Tomato Juice	75,000	225,000	300,000	15,000	-1,350	-10.50%	
Tomato Paste	0	2,000,000	2,000,000	80,000	9,600	74.30%	

Comparing the Thomas' contribution productions to the productions in part C, the amounts of each type of products in unit of cases are exactly the same. Even there is no difference on quantities, the calculated profits are different due to the unit price contributions of each product, coming from the overhead.

ii. If Bollman's product mix used, the difference from part (c) to here would be the unit profit of each product per case. They are listed in the table below:

Table 10.Unit profit Differences between the part c method and Bollman's profit figure.

Profit Methods	Profit of Whole Tomatoes ($/case)	Profit of Juice ($/case)	Profit of Paste ($/case)
Part (c)	0.40	0.12	0.35
Bollman's profit figure	-0.01	0.08	0.55

This difference in unit profit of each product per case from part c to here is caused by few parameters: tomatoes cost. the tomato costs will be modified with changes of original tomato prices for both grade A and grade B tomatoes while they are not used in the calculation at all. However, Fruit_Cost parameters for each product in the code will be modified. The outcomes from Bollman's profit contributions method are summarized below:

Table 11. The Outcome Summary of Using Bollman's Contribution Figures.

Product	Grade A (lbs)	Grade B (lbs)	Grade A+B (lbs)	Productions in cases after rounding	Profit Distribution ($)	Profit Contributions	Total Profits ($)
Whole tomato	0	0	0	0	0	0.0%	48,000
Tomato Juice	600,000	400,000	1,000,000	50,000	4,000	8.3%	
Tomato Paste	0	2,000,000	2,000,000	80,000	44,000	91.7%	

From the table, the total product mix is 0 case+ 50,000 cases +80,000 cases=130,000 cases while the total product mix from part c is 38,888 cases + 15,000 cases + 80,000 cases =133,888 case. So the production mix in here is less than that of in part c.

However, as we analyzed in the part (b) of Question 5, Bollman ignored the corresponding changes of average point quality of the tomato juice. So, this the summarized outcomes from Bollman's contributions figures are off the reality.

iii. Unlimited Grade- A tomato scenario:

Suppose an unlimited supply of A-grade tomatoes were available at $0.085 per pound. From the part (C) v , we concluded that the production or the availability sensitivity shows that the A-grade tomatoes has upper bound of 750,000 lbs. It means that we can ask for <u>up to</u> 750,000 lbs - 600,000 lbs = 150,000 lbs A-grade tomatoes with 8.5 cents/lb price with the increasing profit trend. So the Golden Canning should purchase additional 150,000 lbs of A-grade tomatoes. Since the new Grade-A tomatoes with higher price are added, the fruit prices of each product are changed too. Here is the summary of this part:

Table 12. Outcomes of Unlimited Grade-A tomato Scenario.

Product	Grade A (lbs)	Grade B (lbs)	Grade A+B (lbs)	Productions in cases after rounding	Profit Distribution ($)	Profit Contributions	Total Profits ($)
Whole tomato	693,750	231,250	925,000	51,388	19,528	42.2%	46,253
Tomato Juice	56,250	168,750	225,000	11,250	1,125	2.4%	
Tomato Paste	0	2,000,000	2,000,000	80,000	25,600	55.3%	

The product mix is 51,388 cases + 11,250 cases + 80000 cases = 142,638 cases in total.