Report on: “On the independence Jeffreys’ prior for truncated-exponential skew-symmetric models”

The paper studies the independence Jeffreys prior for TESS distributions. The authors obtain an expression for the independence Jeffreys prior and show that this prior is improper. However, the propriety of the posterior distribution is not formally established. Without this proof, the use of this prior is not justified. The simulation study is also limited to fitting models without location and scale parameters, which makes difficult to compare the fit of skew symmetric models as the parameters have different roles. I believe that the contribution of the paper could be strengthened by proving the propriety of the posterior, and presenting a more balanced and thorough simulation study. Please, find some specific comments below.

Comments

1. Page 3. What is a “strong stochastic property”?

2. Dette et al. (2018) have also studied priors based on distances for skew-symmetric models, exhibiting other types of tail behaviour. The introduction of this paper presents a literature review on priors for skew-symmetric models. Another recent review can be found in Ghaderinezhad et al. (2020).

3. The results and properties presented in pages 4-5 can be summarized in two Propositions, with proofs and details presented in an Appendix, one for the properties of the Fisher information and the prior, and another one for the propriety of the posterior.

4. Page 6. The Fisher information matrix can be presented in another Proposition, with calculations presented in the Appendix.

5. Page 6. I find very intriguing that the independence Jeffreys prior of the scale parameter is not proportional to $\dfrac{1}{\sigma}$, as this is the case for location-scale models. See Lehmann and Casella (2006), for example. It seems like there is a calculation error (probably due to ignoring the Jacobian):

$$
\begin{aligned}
I(\sigma) &= E\left(\frac{\partial}{\partial\sigma}\log(f_Y(Y;\mu,\sigma,\lambda))\right)^2 \\
&= \frac{1}{\sigma^2}E\left(1+T\frac{f_X'(T)}{f_X(T)}+\lambda T f_X(T)\right)^2 \\
&= \frac{\lambda}{\sigma^3(1-\exp(-\lambda))}\int_{-\infty}^{\infty}\left(1+t\frac{f_X'(t)}{f_X(t)}+\lambda t f_X(t)\right)^2 f_X(t)\exp\left(-\lambda F_X(t)\right)dx,
\end{aligned}
$$

In view of this, it seems appropriate to double-check other calculations.

6. The tail characterization you have obtained is that the tails of the Jeffreys prior are of order $o(\lambda^{-1})$ (little-o) rather than $O(\lambda^{-1})$ (Big-o), as you have only shown $\leq c_1\lambda^{-1}$, but not $\geq c_2\lambda^{-1}$. This is not too difficult to prove, though, as you can simply take the limit of the Jeffreys prior and $|\lambda|$ and show that this is a finite number.

7. Plots should be presented in a neater format. For instance,

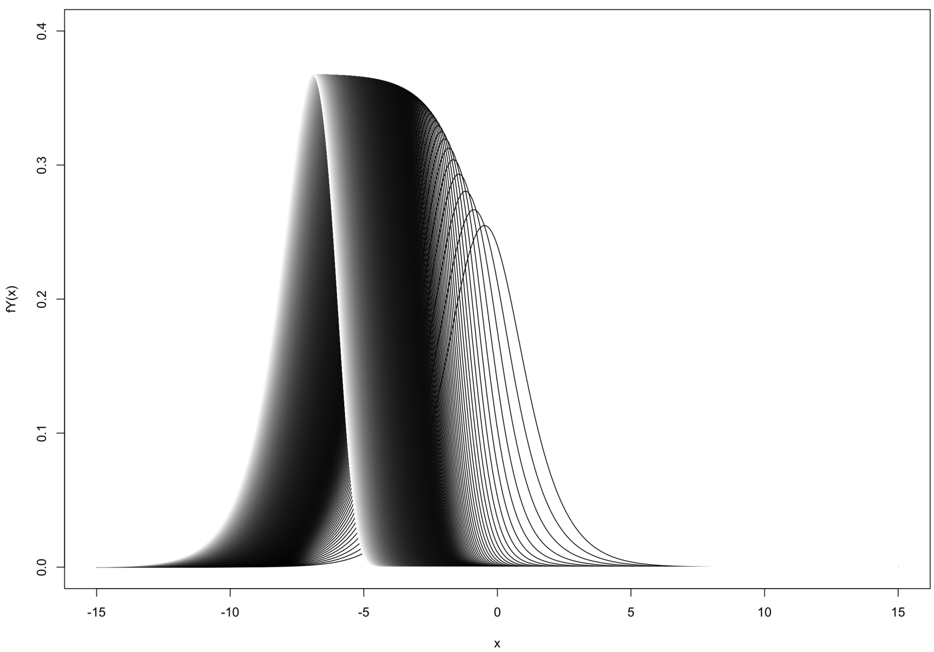
```
f.info <- Vectorize(function(lambda){
 if(lambda == 0) val <- 1/12
 if(lambda!=0)  val <- 1/lambda^2 - exp(-lambda)/(1-exp(-lambda))^2
return(val)
})
curve(f.info,-5,5,n=1000)


j.prior <- Vectorize(function(lambda){
  if(lambda == 0) val <- 1/sqrt(12)
  if(lambda!=0)  val <- sqrt(1/lambda^2 - exp(-lambda)/(1-exp(-lambda))^2)
```

```
    return(val)
})
curve(j.prior,-50,50,n=1000)
```

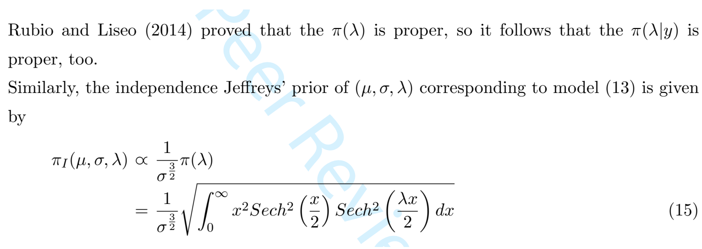
8. The expressions for the posterior distributions are probably unnecessary as it is well known that the posterior is proportional to the likelihood $\times$ the prior.

9. Page 7. Properties (i)–(iii) only concern the prior on λ, and these properties are virtually the same as those presented in Page 5.

10. How flexible is the TESL distribution? I have plotted the density for $\lambda = 0, 1, \ldots, 1000$ and it seems like the amount of skewness this distribution can capture is limited



11. The Jeffreys prior:


Rubio and Liseo (2014) proved that the $\pi(\lambda)$ is proper, so it follows that the $\pi(\lambda|y)$ is proper, too.
Similarly, the independence Jeffreys' prior of (μ, σ, λ) corresponding to model (13) is given by

$$\begin{aligned} \pi_I(\mu,\sigma,\lambda) &\propto \frac{1}{\sigma^{\frac{3}{2}}}\pi(\lambda) \\ &= \frac{1}{\sigma^{\frac{3}{2}}}\sqrt{\int_0^\infty x^2 Sech^2\left(\frac{x}{2}\right) Sech^2\left(\frac{\lambda x}{2}\right)dx} \end{aligned} \quad (15)$$

does not coincide with the expression presented in Rubio and Liseo (2014) in terms of the power of σ, which also relates to my comment 5.

12. Page 8 "Figure 2 shows that the posterior distribution derived in (11) is proper". Plots cannot be used to prove the propriety of a posterior distribution. Propriety of the posterior distribution needs to be formally established by showing that the integral of the product of the likelihood and the prior (this is, the marginal likelihood) is finite:

$$m(\mathbf{y}) = \int \pi_I(\mu, \sigma, \lambda \mid \mathbf{y}) d\mu d\sigma d\lambda < \infty.$$

However, this may not be an easy task as the prior on λ is improper. Unfortunately, without this proof, the use of the independence Jeffreys prior is not theoretically justified.

13. Simulations should be done using location and scale parameters as the skewness parameter in the TESL distribution seems to be acting as a location parameter as well. This is, regardless of the true values, the fitted models should contain location and scale parameters, which is more realistic.

14. More details about the Metropolis-Hastings algorithm used to sample from the posteriors would be welcomed (instrumental distribution, acceptance rate).

15. I could not understand what exactly Figures 3–4 are showing.

16. Software can be shared in a repository like GitHub. This allows for reproducibility of the results.

17. The conclusions section contains two arguable points. First, the propriety of the posterior was not formally established. Second, the comparison in the simulation is not entirely fair, as the models are compared without location and scale parameters.

18. A careful proofread is recommended.

References

H. Dette, C. Ley, and F.J. Rubio. Natural (non-) informative priors for skew-symmetric distributions. *Scandinavian Journal of Statistics*, 45(2):405–420, 2018.

F. Ghaderinezhad, C. Ley, and N. Loperfido. Bayesian inference for skew-symmetric distributions. *Symmetry*, 12 (4):491, 2020.

E.L. Lehmann and G. Casella. *Theory of point estimation*. Springer Science & Business Media, 2006.

F.J. Rubio and B. Liseo. On the independence Jeffreys prior for skew-symmetric models. *Statistics & Probability Letters*, 85:91–97, 2014.