ps5

October 8, 2025

1 Econ 3110: Problem Set 5 Solutions

1.1 Question 1 (5 points)

1.1.1 (a) Net lifetime utility

The gross lifetime utility from buying the Switch 2 at time $\bar{t}$ is the discounted sum of flow utilities:

$$U(\bar{t}) = \sum_{t=\bar{t}}^{\bar{t}+9} \beta^t u^{**} + \sum_{t=\bar{t}+10}^{\infty} \beta^t u^*$$

We can simplify this using the formulas for geometric series:

$$U(\bar{t}) = u^{**} \left(\frac{\beta^{\bar{t}} - \beta^{\bar{t}+10}}{1-\beta} \right) + u^* \left(\frac{\beta^{\bar{t}+10}}{1-\beta} \right)$$

The utility cost of purchasing at $\bar{t}$ is $c_{\bar{t}} = \frac{5}{1-\beta} + \frac{4}{\bar{t}}$. The present discounted value of this cost at $t = 0$ is $\beta^{\bar{t}} c_{\bar{t}}$.

The net lifetime utility, $V(\bar{t})$, from the perspective of today ($t = 0$) is:

$$V(\bar{t}) = U(\bar{t}) - \beta^{\bar{t}} c_{\bar{t}} = u^{**} \left(\frac{\beta^{\bar{t}} - \beta^{\bar{t}+10}}{1-\beta} \right) + u^* \left(\frac{\beta^{\bar{t}+10}}{1-\beta} \right) - \beta^{\bar{t}} \left(\frac{5}{1-\beta} + \frac{4}{\bar{t}} \right)$$

$$V(\bar{t}) = \frac{\beta^{\bar{t}}}{1-\beta} \left[u^{**}(1-\beta^{10}) + u^* \beta^{10} - 5 - \frac{4(1-\beta)}{\bar{t}} \right]$$

1.1.2 (b) Optimal choice of $\bar{t}$

Suppose $u^* = u^{**} = 6$. The net lifetime utility expression from part (a) simplifies significantly. The two utility terms in the summation become a single infinite geometric series:

$$U(\bar{t}) = \sum_{t=\bar{t}}^{\infty} \beta^t (6) = \frac{6\beta^{\bar{t}}}{1-\beta}$$

The net utility is:

$$V(\bar{t}) = \frac{6\beta^{\bar{t}}}{1-\beta} - \beta^{\bar{t}} \left(\frac{5}{1-\beta} + \frac{4}{\bar{t}} \right) = \beta^{\bar{t}} \left(\frac{6-5}{1-\beta} - \frac{4}{\bar{t}} \right) = \beta^{\bar{t}} \left(\frac{1}{1-\beta} - \frac{4}{\bar{t}} \right)$$

To find the $\bar{t}$ that maximizes this, we can treat $\bar{t}$ as a continuous variable and take the derivative with respect to $\bar{t}$ and set it to zero:

$$\frac{dV}{d\bar{t}} = (\ln\beta)\beta^{\bar{t}}\left(\frac{1}{1-\beta} - \frac{4}{\bar{t}}\right) + \beta^{\bar{t}}\left(\frac{4}{\bar{t}^2}\right) = 0$$

Dividing by $\beta^{\bar{t}}$ (which is non-zero):

$$(\ln\beta)\left(\frac{1}{1-\beta} - \frac{4}{\bar{t}}\right) + \frac{4}{\bar{t}^2} = 0$$

$$\frac{\ln\beta}{1-\beta} - \frac{4\ln\beta}{\bar{t}} + \frac{4}{\bar{t}^2} = 0$$

Multiplying by $\bar{t}^2$ gives a quadratic equation in $\bar{t}$:

$$\left(\frac{\ln\beta}{1-\beta}\right)\bar{t}^2 - (4\ln\beta)\bar{t} + 4 = 0$$

Using the quadratic formula, $\bar{t} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$:

$$\bar{t} = \frac{4\ln\beta \pm \sqrt{(-4\ln\beta)^2 - 4(\frac{\ln\beta}{1-\beta})(4)}}{2(\frac{\ln\beta}{1-\beta})} = \frac{4\ln\beta \pm \sqrt{16(\ln\beta)^2 - \frac{16\ln\beta}{1-\beta}}}{2(\frac{\ln\beta}{1-\beta})}$$

$$\bar{t} = (2\ln\beta \pm 2\sqrt{(\ln\beta)^2 - \frac{\ln\beta}{1-\beta}})\frac{1-\beta}{\ln\beta} = 2(1-\beta)\left(1 \pm \sqrt{1 - \frac{1}{(1-\beta)\ln\beta}}\right)$$

1.1.3 (c) Economic intuition

A higher β means an individual is more patient and values future utility streams more. This increased valuation of the future makes the prospect of starting the flow of utility from the Switch 2 more attractive sooner. While waiting reduces a small part of the cost (the $\frac{4}{t}$ term), a more patient individual finds the opportunity cost of delaying the entire stream of future utility to be higher. Therefore, a higher β encourages an earlier purchase to begin enjoying the benefits sooner, causing the optimal $\bar{t}$ to decrease.

1.1.4 (d) Implied discount rate

The discount factor for one period is β. The discount factor for n periods is β^n. The discount rate r is related by $\beta^n = \frac{1}{1+r_{total}}$. So, $r_{total} = \frac{1}{\beta^n} - 1$.

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If a period is one day:
The implied yearly discount factor is: 1.9885e-17
The implied annual discount rate is: 50290280116685232.0000 or
5029028011668523008.00%

If a period is one month:
The implied yearly discount factor is: 0.2824
The implied annual discount rate is: 2.5407 or 254.07%
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When each period represents one month instead of one day, the implied discount rate for a year is significantly lower. This is because the high discount rate is compounded 12 times instead of 365 times, leading to less severe discounting over the same one-year horizon.

1.2 Question 2 (4 points)

1.2.1 (a) Demand functions and quantities

The utility function is $u(S, G) = \sqrt{SG} = S^{0.5}G^{0.5}$, which is a Cobb-Douglas utility function. For a general Cobb-Douglas function $u(S, G) = S^a G^b$, the demand functions are:

$$S^*(p_S, p_G, M) = \frac{a}{a+b}\frac{M}{p_S} \quad \text{and} \quad G^*(p_S, p_G, M) = \frac{b}{a+b}\frac{M}{p_G}$$

Here, $a = 0.5$ and $b = 0.5$, so $a + b = 1$. The demand functions are:

$$S^*(p_S, p_G, M) = \frac{0.5}{1}\frac{M}{p_S} = \frac{M}{2p_S}$$

$$G^*(p_S, p_G, M) = \frac{0.5}{1}\frac{M}{p_G} = \frac{M}{2p_G}$$

Given $p_S = 300$, $p_G = 300$, and $M = 6000$, the quantities demanded are:

$$S^* = \frac{6000}{2(300)} = \frac{6000}{600} = 10$$

$$G^* = \frac{6000}{2(300)} = \frac{6000}{600} = 10$$

1.2.2 (b) Indirect utility function

Substitute the demand functions into the utility function:

$$v(p_S, p_G, M) = u(S^*, G^*) = \sqrt{\left(\frac{M}{2p_S}\right)\left(\frac{M}{2p_G}\right)} = \sqrt{\frac{M^2}{4p_S p_G}} = \frac{M}{2\sqrt{p_S p_G}}$$

1.2.3 (c) Compensating Variation (CV)

First, calculate the cost of the original bundle $(S = 10, G = 10)$ at the new prices $(p'_S = 400, p'_G = 150)$:

$$M' = p'_S S + p'_G G = 400(10) + 150(10) = 4000 + 1500 = 5500$$

The wealth needed to afford the original bundle is \$5500.

Compensating variation (CV) is the change in income that would make the consumer indifferent between the old prices and the new prices. First find the original utility level:

$$u_0 = u(10, 10) = \sqrt{10 \times 10} = 10$$

Now find the income M_{CV} needed to achieve $u_0 = 10$ at the new prices using the indirect utility function:

$$10 = v(400, 150, M_{CV}) = \frac{M_{CV}}{2\sqrt{400 \times 150}} = \frac{M_{CV}}{2\sqrt{60000}} = \frac{M_{CV}}{200\sqrt{6}}$$

$$M_{CV} = 2000\sqrt{6}$$

The CV is the difference between this hypothetical income and the original income:

$$CV = M_{CV} - M = 2000\sqrt{6} - 6000$$

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The income needed to achieve the original utility is M_CV = $4898.98
The Compensating Variation (CV) is $-1101.02
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1.2.4 (d) Equivalent Variation (EV)

First, find Ash's optimal bundle at the new prices ($p'_S = 400, p'_G = 150$) and original income ($M = 6000$):

$$S'^* = \frac{6000}{2(400)} = 7.5$$

$$G'^* = \frac{6000}{2(150)} = 20$$

Next, calculate the utility level at this new bundle:

$$u_1 = u(7.5, 20) = \sqrt{7.5 \times 20} = \sqrt{150} = 5\sqrt{6}$$

Equivalent variation (EV) is the change in income at the *original* prices that would yield the same utility as the new price regime. We find the income M_{EV} that gives utility u_1 at the old prices ($p_S = 300, p_G = 300$):

$$5\sqrt{6} = v(300, 300, M_{EV}) = \frac{M_{EV}}{2\sqrt{300 \times 300}} = \frac{M_{EV}}{600}$$

$$M_{EV} = 3000\sqrt{6}$$

The EV is the difference between the original income and this hypothetical income:

$$EV = M - M_{EV} = 6000 - 3000\sqrt{6}$$

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The income at original prices yielding the new utility is M_EV = $7348.47
The Equivalent Variation (EV) is $-1348.47
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1.3 Question 3 (5 points)

1.3.1 (a) Marginal Rate of Substitution (MRS)

The utility function is $u(x, y) = (x^\rho + y^\rho)^{1/\rho}$. The marginal utilities are:

$$MU_x = \frac{\partial u}{\partial x} = \frac{1}{\rho}(x^\rho + y^\rho)^{1/\rho - 1}(\rho x^{\rho-1}) = (x^\rho + y^\rho)^{1/\rho - 1} x^{\rho - 1}$$

$$MU_y = \frac{\partial u}{\partial y} = \frac{1}{\rho}(x^\rho + y^\rho)^{1/\rho - 1}(\rho y^{\rho-1}) = (x^\rho + y^\rho)^{1/\rho - 1} y^{\rho - 1}$$

The MRS is the ratio of the marginal utilities:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{(x^\rho + y^\rho)^{1/\rho - 1} x^{\rho-1}}{(x^\rho + y^\rho)^{1/\rho - 1} y^{\rho-1}} = \frac{x^{\rho-1}}{y^{\rho-1}} = \left(\frac{x}{y}\right)^{\rho - 1}$$

1.3.2 (b) Elasticity of substitution

The elasticity of substitution ϵ is defined as $\epsilon = \frac{d\ln(y/x)}{d\ln(MRS_{x,y})}$. From part (a), we have $MRS_{x,y} = (x/y)^{\rho-1} = (y/x)^{1-\rho}$. Take the natural logarithm of both sides:

$$\ln(MRS_{x,y}) = (1 - \rho)\ln(y/x)$$

Rearrange the expression to get $\ln(y/x)$ in terms of $\ln(MRS_{x,y})$:

$$\ln(y/x) = \frac{1}{1-\rho}\ln(MRS_{x,y})$$

Now, we can find the elasticity by differentiating:

$$\epsilon = \frac{d\ln(y/x)}{d\ln(MRS_{x,y})} = \frac{1}{1-\rho}$$

1.3.3 (c) Case $\rho \to 1$

As $\rho \to 1$, the denominator of the elasticity expression approaches zero: $1-\rho \to 0$. Therefore, $\epsilon = \frac{1}{1-\rho} \to \infty$. An infinite elasticity of substitution means the goods are **perfect substitutes**. When $\rho = 1$, the utility function becomes $u(x,y) = x+y$, which represents perfect substitutes.

1.3.4 (d) Case $\rho \to -\infty$

As $\rho \to -\infty$, the denominator of the elasticity expression approaches infinity: $1-\rho \to \infty$. Therefore, $\epsilon = \frac{1}{1-\rho} \to 0$. An elasticity of substitution of zero means the goods are **perfect complements**. This utility function approaches the Leontief form, $u(x,y) = \min\{x,y\}$, as $\rho \to -\infty$.

1.4 Question 4 (5 points)

1.4.1 (a) Aggregate demand from inverse demand

The inverse demand for consumer j is $p_x = M_j - \frac{x_j}{\beta}$. We can find the direct demand function for consumer j by solving for x_j:

$$\frac{x_j}{\beta} = M_j - p_x \implies x_j(p_x, M_j) = \beta(M_j - p_x)$$

Demand is only positive if $M_j > p_x$. The aggregate demand X is the sum of individual demands for all consumers who are in the market at a given price p_x.

$$X(p_x) = \sum_{j \text{ s.t. } M_j > p_x} x_j = \sum_{j \text{ s.t. } M_j > p_x} \beta(M_j - p_x)$$

1.4.2 (b) Aggregate demand with quasilinear utility

The utility function is $u_j(x,y) = j\ln(x) + y$. The prices are p_x and $p_y = 1$. The MRS is $MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{j/x}{1} = \frac{j}{x}$. Setting $MRS = \frac{p_x}{p_y}$:

$$\frac{j}{x_j} = \frac{p_x}{1} \implies x_j(p_x) = \frac{j}{p_x}$$

(The condition $M_j > j$ ensures an interior solution). The aggregate demand is the sum of individual demands:

$$X(p_x) = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{j}{p_x} = \frac{1}{p_x}\sum_{j=1}^{n} j$$

Using the formula for the sum of the first n integers, $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$:

$$X(p_x) = \frac{n(n+1)}{2p_x}$$

1.4.3 (c) Aggregate demand with Cobb-Douglas utility

The utility function $u_j(x,y) = \sqrt{xy} = x^{0.5}y^{0.5}$ is the same for all consumers. The demand for good x for a consumer with income M_j is:

$$x_j(p_x, M_j) = \frac{0.5}{0.5+0.5}\frac{M_j}{p_x} = \frac{M_j}{2p_x}$$

The aggregate demand is the sum of these individual demands:

$$X(p_x) = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{M_j}{2p_x} = \frac{1}{2p_x}\sum_{j=1}^{n} M_j$$

1.4.4 (d) Condition for same aggregate demand

For the aggregate demands in parts (b) and (c) to be the same, we set the expressions equal:

$$\frac{n(n+1)}{2p_x} = \frac{1}{2p_x}\sum_{j=1}^{n} M_j$$

Multiplying both sides by $2p_x$ gives the sufficient condition:

$$n(n+1) = \sum_{j=1}^{n} M_j$$

The aggregate demands are the same if the sum of all consumers' incomes equals the n^{th} triangular number.

1.5 Question 5 (5 points)

1.5.1 (a) Equilibrium price and quantity

Demand: $Q_d = a - b\sqrt{P}$ Supply: $Q_s = c + d\sqrt{P}$ In equilibrium, $Q_d = Q_s$:

$$a - b\sqrt{P} = c + d\sqrt{P}$$

$$a - c = (b+d)\sqrt{P}$$

Solving for the equilibrium price P^*:

$$\sqrt{P^*} = \frac{a-c}{b+d} \implies P^* = \left(\frac{a-c}{b+d}\right)^2$$

Substitute $\sqrt{P^*}$ back into the supply equation to find the equilibrium quantity Q^*:

$$Q^* = c + d\left(\frac{a-c}{b+d}\right) = \frac{c(b+d)+d(a-c)}{b+d} = \frac{cb+cd+ad-cd}{b+d} = \frac{ad+bc}{b+d}$$

1.5.2 (b) Equilibrium with a tax

With a tax rate τ paid by buyers, the price they pay is $P_b = (1+\tau)P_s$, where P_s is the price sellers receive. The demand equation becomes a function of P_s:

$$Q_d = a - b\sqrt{(1+\tau)P_s}$$

Setting demand equal to supply, $Q_s = c + d\sqrt{P_s}$:

$$a - b\sqrt{1+\tau}\sqrt{P_s} = c + d\sqrt{P_s}$$
$$a - c = (b\sqrt{1+\tau} + d)\sqrt{P_s}$$

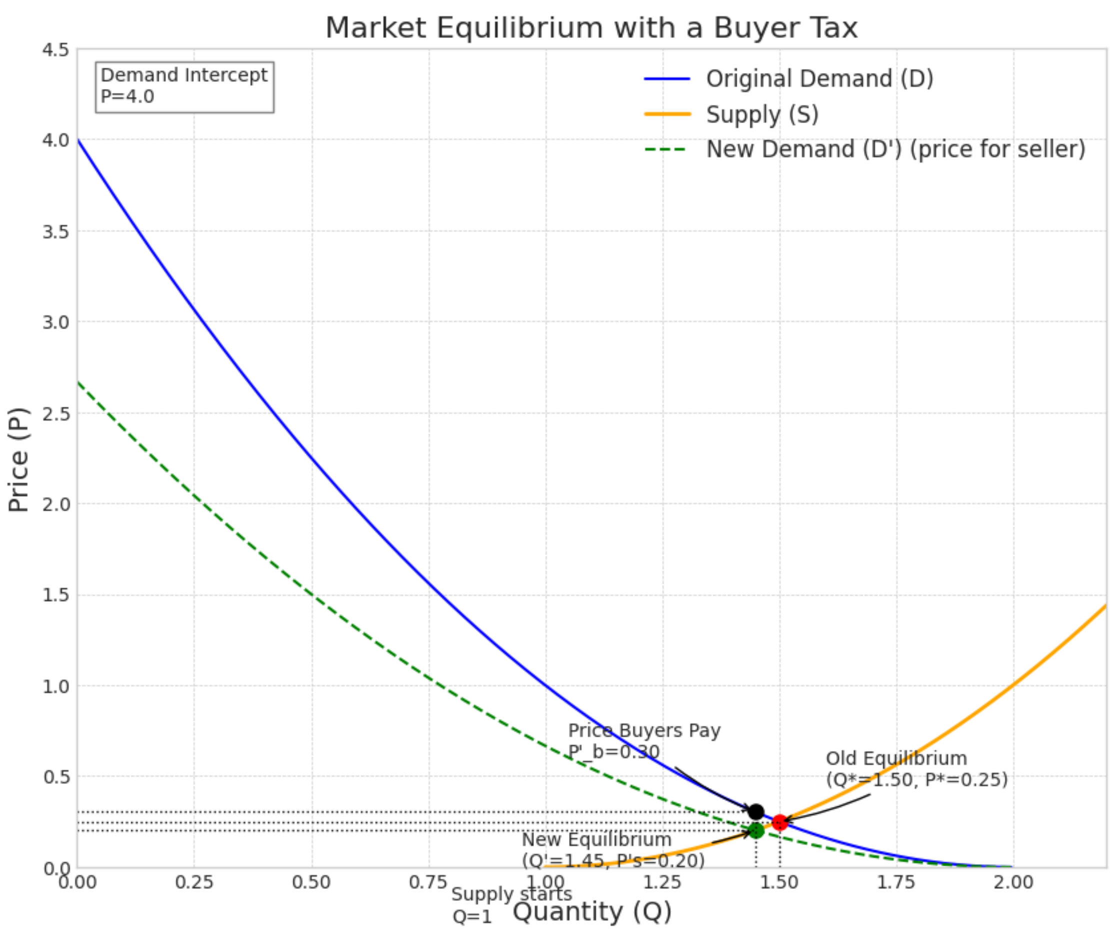
The new equilibrium price for sellers, P_s', is:

$$P_s' = \left(\frac{a-c}{b\sqrt{1+\tau}+d}\right)^2$$

The new equilibrium quantity, Q', is:

$$Q' = c + d\sqrt{P_s'} = c + d\left(\frac{a-c}{b\sqrt{1+\tau}+d}\right) = \frac{ad + bc\sqrt{1+\tau}}{d + b\sqrt{1+\tau}}$$

Illustration:



1.5.3 (c) Consumer surplus calculation

Given $a = 2, b = 1, c = 1, d = 1, \tau = 1/2$.

Before Tax:

$$P^* = \left(\frac{2-1}{1+1}\right)^2 = \left(\frac{1}{2}\right)^2 = 0.25$$

To calculate consumer surplus, we integrate the area between the demand curve and the equilibrium price. The inverse demand curve is $P_d(Q) = (a - Q/b)^2 = (2 - Q)^2$. The maximum price (choke price) is when $Q = 0$, so $P_{max} = 4$.

$$CS_{old} = \int_{P^*}^{P_{max}} Q_d(P)dP = \int_{0.25}^{4} (2 - \sqrt{P})dP$$

$$CS_{old} = \left[2P - \frac{2}{3}P^{3/2}\right]_{0.25}^{4} = \left(2(4) - \frac{2}{3}(4)^{3/2}\right) - \left(2(0.25) - \frac{2}{3}(0.25)^{3/2}\right)$$

$$CS_{old} = \left(8 - \frac{16}{3}\right) - \left(0.5 - \frac{2}{3}(0.125)\right) = \frac{8}{3} - \left(\frac{1}{2} - \frac{1}{12}\right) = \frac{8}{3} - \frac{5}{12} = \frac{32-5}{12} = \frac{27}{12} = 2.25$$

After Tax: The tax is $\tau = 0.5$, so $1 + \tau = 1.5$. The new price for sellers is:

$$P'_s = \left(\frac{2-1}{1\sqrt{1.5}+1}\right)^2 = \frac{1}{(1+\sqrt{1.5})^2}$$

The new price for buyers is $P'_b = (1.5)P'_s = \frac{1.5}{(1+\sqrt{1.5})^2}$.

$$CS_{new} = \int_{P'_b}^{4} (2 - \sqrt{P})dP = \left[2P - \frac{2}{3}P^{3/2}\right]_{P'_b}^{4} = \frac{8}{3} - \left(2P'_b - \frac{2}{3}(P'_b)^{3/2}\right)$$

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Consumer surplus before tax (CS_old): 2.2500
The new price for buyers is P'_b: 0.3031
Consumer surplus after tax (CS_new): 2.1718
The percentage change in consumer surplus is: -3.48%
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1.6 Question 6 (6 points)

1.6.1 (a) Returns to scale

Let's scale all inputs by a factor $t > 1$. The new output is:

$$Y(tK, tL) = (tK)^\alpha (tL)^\beta = t^\alpha K^\alpha t^\beta L^\beta = t^{\alpha+\beta} K^\alpha L^\beta = t^{\alpha+\beta} Y(K, L)$$

The returns to scale depend on the value of $\alpha + \beta$: * **Increasing returns-to-scale (IRS):** if $\alpha + \beta > 1$ * **Decreasing returns-to-scale (DRS):** if $\alpha + \beta < 1$ * **Constant returns-to-scale (CRS):** if $\alpha + \beta = 1$

1.6.2 (b) MRTS

$$MP_L = \frac{\partial Y}{\partial L} = \beta K^\alpha L^{\beta-1}$$

$$MP_K = \frac{\partial Y}{\partial K} = \alpha K^{\alpha-1} L^\beta$$

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\beta K^\alpha L^{\beta-1}}{\alpha K^{\alpha-1} L^\beta} = \frac{\beta}{\alpha}\frac{K}{L}$$

1.6.3 (c) Optimal choice of labor and capital

With $\alpha \in (0,1)$ and $\beta = 1-\alpha$, we have $\alpha+\beta = 1$, so the production technology exhibits constant returns to scale (CRS). A profit-maximizing firm with a CRS production function faces a dilemma: if the maximum profit is positive, it would want to produce an infinite amount; if it's negative, it would produce zero. A non-trivial solution exists only if the maximum profit is exactly zero. This occurs when the price equals the marginal cost. The conditions for profit maximization are $p \cdot MP_L = w$ and $p \cdot MP_K = r$. Dividing these two conditions gives the tangency condition: $\frac{MP_L}{MP_K} = \frac{w}{r}$, so $\frac{\beta}{\alpha}\frac{K}{L} = \frac{w}{r}$. This defines the optimal capital-labor ratio: $\frac{K}{L} = \frac{\alpha w}{\beta r}$. However, because of CRS, the absolute levels of L^* and K^* are indeterminate from profit maximization alone. Any combination of K and L that satisfies the optimal ratio will result in zero economic profit. The scale of the firm is not determined.

1.6.4 (d) Labor demand with fixed capital

With $K = 4, p = 6, w = 2, r = 2$, and $\alpha = 1/2$ (so $\beta = 1/2$), the production function is $Y = 4^{1/2}L^{1/2} = 2\sqrt{L}$. The firm's profit function is:

$$\pi = pY - wL - rK = 6(2\sqrt{L}) - 2L - 2(4) = 12\sqrt{L} - 2L - 8$$

To maximize profit, we take the derivative with respect to L and set it to zero:

$$\frac{d\pi}{dL} = 12\left(\frac{1}{2\sqrt{L}}\right) - 2 = 0$$

$$\frac{6}{\sqrt{L}} = 2 \implies \sqrt{L} = 3 \implies L = 9$$

The optimal level of labor is 9 units.

1.6.5 (e) Change with a fixed cost

A fixed cost $\mathcal{F}$ does not depend on the level of inputs, so it does not affect the profit-maximizing choice of labor. The derivative of the profit function with respect to L is unchanged, so the level of labor that maximizes operating profit is still $L = 9$. However, the firm will only choose to operate if its total profit is non-negative. First, let's calculate the maximized operating profit (profit before the fixed cost $\mathcal{F}$):

$$\pi_{op} = 12\sqrt{9} - 2(9) - 8 = 12(3) - 18 - 8 = 36 - 26 = 10$$

Now consider the total profit, $\pi_{total} = \pi_{op} - \mathcal{F}$. * **If** $\mathcal{F} = 5$: $\pi_{total} = 10 - 5 = 5$. Since profit is positive, the firm will operate and hire $L = 9$. The answer does not change. * **If** $\mathcal{F} = 15$: $\pi_{total} = 10 - 15 = -5$. Since profit is negative, the firm is better off shutting down (producing zero) to avoid the variable costs and earn a profit of $-\mathcal{F}$ if the cost is sunk, or 0 if it is avoidable. Assuming the firm can avoid all costs by not producing, it will shut down. In this case, the labor hired is $L = 0$. The answer changes from 9 to 0.