ps8

November 11, 2025

1 Econ 3110: Problem Set 8

1.1 Question 1 (10 points)

Suppose there are n firms in the market for lawn chairs. These firms engage in Stackelberg competition, i.e., they choose their output sequentially. All firms face the same constant marginal cost $c > 0$ and have zero fixed costs of production. The market's inverse demand is $P = a - BQ$, where $Q = \sum_{i=1}^{n} q_i$ and q_i represents the quantity supplied by firm i.

1.1.1 (a) (3 points) Let n = 2. Find the equilibrium quantities and price. These will be functions of (a, b, c).

We solve this Stackelberg duopoly by backward induction, starting with the follower, Firm 2.

1. Firm 2's Problem (Follower): Firm 2 chooses its quantity q_2 to maximize its profit, taking Firm 1's quantity q_1 as given.

$$\max_{q_2} \pi_2 = P(Q)q_2 - cq_2 = (a - B(q_1 + q_2))q_2 - cq_2$$

To find the maximum, we take the first-order condition (FOC) with respect to q_2:

$$\frac{\partial \pi_2}{\partial q_2} = a - Bq_1 - 2Bq_2 - c = 0$$

Solving for q_2, we get Firm 2's reaction function:

$$q_2(q_1) = \frac{a - c - Bq_1}{2B}$$

2. Firm 1's Problem (Leader): Firm 1 anticipates Firm 2's reaction. It substitutes Firm 2's reaction function into its own profit function and chooses q_1 to maximize its profit.

$$\max_{q_1} \pi_1 = (a - B(q_1 + q_2(q_1)))q_1 - cq_1$$

Substitute $q_2(q_1)$:

$$\pi_1 = \left(a - B\left(q_1 + \frac{a - c - Bq_1}{2B}\right)\right)q_1 - cq_1$$

$$\pi_1 = \left(a - B\left(\frac{2Bq_1 + a - c - Bq_1}{2B}\right)\right)q_1 - cq_1$$

$$\pi_1 = \left(a - \frac{a - c + Bq_1}{2}\right)q_1 - cq_1 = \left(\frac{2a - a + c - Bq_1}{2}\right)q_1 - cq_1 = \left(\frac{a - c - Bq_1}{2}\right)q_1$$

Taking the FOC with respect to q_1:

$$\frac{\partial \pi_1}{\partial q_1} = \frac{a-c}{2} - Bq_1 = 0$$

Solving for q_1 gives the leader's equilibrium quantity:

$$q_1^* = \frac{a-c}{2B}$$

3. Equilibrium Quantities and Price: Now we find the follower's equilibrium quantity by plugging q_1^* into its reaction function:

$$q_2^* = \frac{a-c-B(\frac{a-c}{2B})}{2B} = \frac{a-c-\frac{a-c}{2}}{2B} = \frac{\frac{a-c}{2}}{2B} = \frac{a-c}{4B}$$

Total equilibrium quantity is:

$$Q^* = q_1^* + q_2^* = \frac{a-c}{2B} + \frac{a-c}{4B} = \frac{3(a-c)}{4B}$$

The equilibrium price is:

$$P^* = a - BQ^* = a - B\left(\frac{3(a-c)}{4B}\right) = a - \frac{3}{4}(a-c) = \frac{4a-3a+3c}{4} = \frac{a+3c}{4}$$

1.1.2 (b) (4 points) Extend your solution in part (a) to the case with n > 2 firms.

We continue to solve by backward induction, starting with the last mover, firm n.

Firm n's Problem: Firm n takes the total output of the first $n-1$ firms, $Q_{n-1} = \sum_{i=1}^{n-1} q_i$, as given.

$$\max_{q_n} \pi_n = (a - B(Q_{n-1} + q_n))q_n - cq_n$$

The FOC gives its reaction function:

$$q_n(Q_{n-1}) = \frac{a-c-BQ_{n-1}}{2B}$$

Firm n-1's Problem: Firm $n-1$ anticipates firm n's reaction. It takes Q_{n-2} as given. The total quantity from firm $n-1$'s perspective is $Q = Q_{n-2} + q_{n-1} + q_n(Q_{n-2} + q_{n-1})$. Following the same logic as in part (a), firm $n-1$ sees its residual inverse demand as $P = \frac{a-c-BQ_{n-2}}{2} - \frac{B}{2}q_{n-1}$. Its profit maximization problem is equivalent to that of a leader in a 2-firm Stackelberg game, facing this residual demand. Thus, its optimal quantity will be:

$$q_{n-1} = \frac{(\frac{a-c-BQ_{n-2}}{2}) - 0}{2(\frac{B}{2})} = \frac{a-c-BQ_{n-2}}{2B}$$

Generalizing for Firm i: By induction, we can see a pattern. Each firm i acts as a Stackelberg leader to all subsequent firms ($j > i$). The problem for any firm i is to choose q_i taking $Q_{i-1} = \sum_{j=1}^{i-1} q_j$ as given. The total quantity produced by firms $j > i$ will be a function of $Q_i = Q_{i-1} + q_i$. It can be shown that this leads to the following recursive relationship:

$$q_i = \frac{a-c-BQ_{i-1}}{2B}$$

Solving Recursively: 1. **Firm 1 ($i = 1$):** $Q_0 = 0$. $q_1^* = \frac{a-c}{2B}$ 2. **Firm 2 ($i = 2$):** $Q_1 = q_1^* = \frac{a-c}{2B}$. $q_2^* = \frac{a-c-B(\frac{a-c}{2B})}{2B} = \frac{(a-c)/2}{2B} = \frac{a-c}{4B}$ 3. **Firm 3 ($i = 3$):** $Q_2 = q_1^* + q_2^* = \frac{3(a-c)}{4B}$. $q_3^* = \frac{a-c-B(\frac{3(a-c)}{4B})}{2B} = \frac{(a-c)/4}{2B} = \frac{a-c}{8B}$

The general formula for the quantity of firm i is:

$$q_i^* = \frac{a-c}{2^i B}$$

Total Equilibrium Quantity and Price: Total quantity is the sum of a geometric series:

$$Q_n^* = \sum_{i=1}^{n} q_i^* = \sum_{i=1}^{n} \frac{a-c}{2^i B} = \frac{a-c}{B} \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i$$

Using the formula for a finite geometric series, $\sum_{i=1}^{n} r^i = \frac{r(1-r^n)}{1-r}$:

$$Q_n^* = \frac{a-c}{B} \left(\frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} \right) = \frac{a-c}{B} \left(1 - \frac{1}{2^n}\right) = \frac{a-c}{B} \left(\frac{2^n - 1}{2^n}\right)$$

The equilibrium price is:

$$P_n^* = a - BQ_n^* = a - B\left(\frac{a-c}{B}\frac{2^n-1}{2^n}\right) = a - (a-c)\left(1 - \frac{1}{2^n}\right) = a - (a-c) + \frac{a-c}{2^n} = c + \frac{a-c}{2^n}$$

1.1.3 (c) (3 points) How does the equilibrium quantity supplied by firm i compare to firm i-1? Provide economic intuition behind this result.

Comparison: From part (b), we have the equilibrium quantities:

$$q_i^* = \frac{a-c}{2^i B} \quad \text{and} \quad q_{i-1}^* = \frac{a-c}{2^{i-1} B}$$

Comparing them, we see that:

$$q_i^* = \frac{1}{2}\left(\frac{a-c}{2^{i-1} B}\right) = \frac{1}{2} q_{i-1}^*$$

The quantity supplied by firm i is exactly half the quantity supplied by firm $i-1$. Thus, $q_i^* < q_{i-1}^*$ for all $i > 1$.

Economic Intuition: This result highlights the **first-mover advantage** inherent in Stackelberg competition. The first firm (the leader) gets to choose its output knowing how all subsequent firms will react. It maximizes its profit by choosing a large output, which reduces the residual demand remaining for the next firm. Firm 2, facing this smaller market, chooses a smaller optimal output. This logic continues down the line: each firm i faces a smaller residual market than firm $i-1$ did, because the cumulative output of firms 1 through $i-1$ is larger than the cumulative output of firms 1 through $i-2$. This strategic commitment by early movers to a high output level effectively "crowds out" later movers, forcing them to produce less.

1.2 Question 2 (8 points)

Jon's indirect utility function is $v(p, M) = \frac{M}{\sqrt{p_1 p_2}}$, where $p_i > 0$ is the price of good i and $M > 0$ denotes his wealth.

1.2.1 (a) (4 points) Derive the expenditure function e(p, U) and the Hicksian demand functions $h_1(p, U)$ and $h_2(p, U)$.

1. Expenditure Function $e(p, U)$: The expenditure function is the minimum amount of money required to achieve a certain utility level U at given prices p. We find it by inverting the indirect utility function, $U = v(p, M)$, to solve for wealth (M).

$$U = \frac{M}{\sqrt{p_1 p_2}}$$

Solving for M gives the expenditure, $e(p, U)$:

$$M = U\sqrt{p_1 p_2}$$

So, the expenditure function is:

$$e(p, U) = U p_1^{1/2} p_2^{1/2}$$

2. Hicksian Demand Functions $h_i(p, U)$: We can derive the Hicksian (compensated) demand functions using Shephard's Lemma, which states that $h_i(p, U) = \frac{\partial e(p,U)}{\partial p_i}$.

For good 1:

$$h_1(p, U) = \frac{\partial}{\partial p_1}(U p_1^{1/2} p_2^{1/2}) = U \cdot \frac{1}{2} p_1^{-1/2} p_2^{1/2} = \frac{U}{2}\sqrt{\frac{p_2}{p_1}}$$

For good 2:

$$h_2(p, U) = \frac{\partial}{\partial p_2}(U p_1^{1/2} p_2^{1/2}) = U p_1^{1/2} \cdot \frac{1}{2} p_2^{-1/2} = \frac{U}{2}\sqrt{\frac{p_1}{p_2}}$$

1.2.2 (b) (4 points) Show that if $U = v(p, M)$, then $h_i(p, U) = x_i(p, M)\ \forall\ i \in \{1, 2\}$, i.e., the Hicksian and Marshallian demands coincide.

First, we must find the Marshallian (uncompensated) demand functions, $x_i(p, M)$, using Roy's Identity: $x_i(p, M) = -\frac{\partial v/\partial p_i}{\partial v/\partial M}$.

The partial derivatives of $v(p, M) = M(p_1 p_2)^{-1/2}$ are:

$$\frac{\partial v}{\partial M} = (p_1 p_2)^{-1/2} = \frac{1}{\sqrt{p_1 p_2}}$$

$$\frac{\partial v}{\partial p_1} = M\left(-\frac{1}{2}\right)(p_1 p_2)^{-3/2} p_2 = -\frac{M p_2}{2(p_1 p_2)^{3/2}} = -\frac{M}{2 p_1 \sqrt{p_1 p_2}}$$

$$\frac{\partial v}{\partial p_2} = M\left(-\frac{1}{2}\right)(p_1 p_2)^{-3/2} p_1 = -\frac{M p_1}{2(p_1 p_2)^{3/2}} = -\frac{M}{2 p_2 \sqrt{p_1 p_2}}$$

Now we apply Roy's Identity: For good 1:

$$x_1(p, M) = -\frac{-\frac{M}{2p_1\sqrt{p_1 p_2}}}{\frac{1}{\sqrt{p_1 p_2}}} = \frac{M}{2p_1}$$

For good 2:

$$x_2(p, M) = -\frac{-\frac{M}{2p_2\sqrt{p_1 p_2}}}{\frac{1}{\sqrt{p_1 p_2}}} = \frac{M}{2p_2}$$

Finally, we substitute $U = v(p, M) = \frac{M}{\sqrt{p_1 p_2}}$ into the Hicksian demand functions from part (a) to show they equal the Marshallian demands.

For good 1:

$$h_1(p, U) = \frac{U}{2}\sqrt{\frac{p_2}{p_1}} = \frac{1}{2}\left(\frac{M}{\sqrt{p_1 p_2}}\right)\sqrt{\frac{p_2}{p_1}} = \frac{M}{2\sqrt{p_1 p_2}}\frac{\sqrt{p_2}}{\sqrt{p_1}} = \frac{M}{2p_1} = x_1(p, M)$$

For good 2:

$$h_2(p, U) = \frac{U}{2}\sqrt{\frac{p_1}{p_2}} = \frac{1}{2}\left(\frac{M}{\sqrt{p_1 p_2}}\right)\sqrt{\frac{p_1}{p_2}} = \frac{M}{2\sqrt{p_1 p_2}}\frac{\sqrt{p_1}}{\sqrt{p_2}} = \frac{M}{2p_2} = x_2(p, M)$$

Thus, we have shown that $h_i(p, U) = x_i(p, M)$ for $i = 1, 2$. This occurs because the underlying preferences (Cobb-Douglas) are homothetic, which implies that the income effect of a compensated price change is zero.

1.3 Question 3 (12 points)

For each of the following claims, state whether they are true or false (choose false if it is not always true).

1.3.1 (a) (3 points) A zero-sum game cannot have a pure-strategy Nash Equilibrium (PSNE).

False.

A zero-sum game can have a PSNE if there is a saddle point in the payoff matrix. Consider the following zero-sum game where payoffs are (Player 1, Player 2): | | Left | Right | | :— | :—: | :—: | | **Up** | (1, -1) | (2, -2) | | **Down** | (0, 0) | (3, -3) |

To find a PSNE, we check for best responses: - If Player 1 chooses Up, Player 2's best response is Left (payoff of -1 is better than -2). - If Player 1 chooses Down, Player 2's best response is Left (payoff of 0 is better than -3). Therefore, Left is a strictly dominant strategy for Player 2. - Given that Player 2 will play Left, Player 1's best response is Up (payoff of 1 is better than 0). The strategy profile (Up, Left) is a Pure-Strategy Nash Equilibrium.

1.3.2 (b) (3 points) If a game has a PSNE, then it must have a mixed-strategy Nash Equilibrium (MSNE).

True.

A pure strategy is a special, or degenerate, case of a mixed strategy. A mixed strategy assigns a probability to each pure strategy. A pure strategy, say s_i, can be represented as a mixed strategy that assigns probability 1 to s_i and probability 0 to all other pure strategies. Therefore, any pure-strategy Nash Equilibrium is, by definition, also a mixed-strategy Nash Equilibrium.

1.3.3 (c) (3 points) Consider a game with two players, call them i and j. Each player has a strictly dominant pure strategy. Claim: this game has a unique PSNE.

True.

A strictly dominant strategy is one that provides a strictly higher payoff than any other strategy, regardless of what the other player does. A rational player will always play their strictly dominant strategy. Let s_i^* be the strictly dominant strategy for player i, and s_j^* be the strictly dominant strategy for player j. The strategy profile (s_i^*, s_j^*) must be a Nash Equilibrium, because neither player has an incentive to deviate; any deviation would lead to a strictly lower payoff. This equilibrium is unique because no other strategy profile can be an equilibrium. For any other profile (s_i, s_j) where $s_i \neq s_i^*$ (or $s_j \neq s_j^*$), player i (or player j) could improve their payoff by switching to their dominant strategy.

1.3.4 (d) (3 points) Firms i and j are deciding when to release a new product: early (E) or late (L)... Claim: This game has a unique PSNE.

True.

We can represent this game with the following payoff matrix, with payoffs listed as (Firm i, Firm j):

	Firm j: Early	Firm j: Late
Firm i: Early	(3, 3)	(7, 1)
Firm i: Late	(1, 7)	(5, 5)

We can find the equilibrium by checking for dominant strategies.

For Firm i: - If Firm j chooses Early, Firm i prefers Early (payoff $3 > 1$). - If Firm j chooses Late, Firm i prefers Early (payoff $7 > 5$). Thus, choosing Early is a strictly dominant strategy for Firm i.

For Firm j: By symmetry, the same logic applies: - If Firm i chooses Early, Firm j prefers Early (payoff $3 > 1$). - If Firm i chooses Late, Firm j prefers Early (payoff $7 > 5$). Thus, choosing Early is also a strictly dominant strategy for Firm j.

Since both firms have a strictly dominant strategy (Early), the unique Pure-Strategy Nash Equilibrium is (Early, Early). This is a classic Prisoner's Dilemma scenario.