Resolution Control in Half-space Time-reversal Wave Focusing

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Abstract

In wave focusing subsurface geophysical applications, the recordings at the mirror, situated on the surface of a half-space, may have to be time-reversed while flipping the character of the boundary conditions due to equipment/sensor limitations. For example, the recording sensors at the time-reversal mirror may record Dirichlet data, but the transmitting equipment may be able to accommodate Neumann data only. Under certain conditions, such flipping may worsen the focusing resolution. We study the relation between the wavefields generated by the recording-transmitting pairs D_r-to-D_t and D_r-to-N_t, and propose a filter to improve the resolution imposed by the aforementioned equipment constraints in the D_r-to-N_t case.

Keywords: Time-reversal, wave focusing, half-space, subsurface resolution

1 Introduction

A wave source is located at point $(x_0, 0, 0)$ in a half-space (Fig. 1), and the time-reversal mirror is on the surface of the half-space. The mirror is capable of recording either Dirichlet or Neumann data generated by the wave source at x_0. We are interested in the time-reversed field ψ generated by time-reversing either the Dirichlet or the Neumann data; the time-reversed field is governed by the Helmholtz equation:

$$\begin{aligned} \operatorname{div}(\mu \,\mathrm{grad}\psi) + \rho\,\omega^2\psi &= 0, \mathbf{x} \in \Omega \text{ and} \\ &= P \text{ or } \mathcal{N}\psi = Q, \mathbf{x} \in \Gamma, \end{aligned} \tag{1}$$

where $\mathcal{N}[\,] = \mu\,\mathrm{grad}[\,]\cdot\mathbf{n}$; $\mathbf{x} = (x^1, x^2, x^3)$; μ is shear modulus; ρ is density; and P and Q are the Dirichlet and Neumann prescribed data, respectively (Fig. 1). We are interested in assessing the resolution at the source location x_0 when the recorded Dirichlet data are time-reversed as Dirichlet data (D_r-to-D_t) and when they are time-reversed as Neumann data (D_r-to-N_t). None of the above time-reversals will result in perfect focusing: even the D_r-to-D_t case will result in loss of resolution owing to the incomplete time-reversal mirror (or the unboundedness of the physical domain), and the non-reversal of the source (no sink). Of particular interest to geophysical applications is the D_r-to-N_t case, whereby geophones record Dirichlet data and actuators apply Neumann data during the time-reversal phase [1].

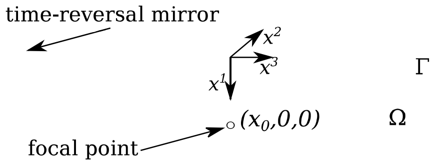

Figure 1: Half-space time-reversal wave focusing problem

2 Resolution in time-reversal focusing

We consider two cases for the time-reversal phase (D_r-to-D_t and D_r-to-N_t), as also proposed in [2] and [3]. Their resolutions, along the depth direction, are of interest. The resolution in the D_r-to-D_t case, defined as the wavefield support above half-maximum strength, is, approximately, 1.1λ (Fig. 2(a)). By comparison, the diffraction limit of a closed cavity problem is $\lambda/2$. As shown in Fig. 2(b), the resolution of the D_r-to-N_t case is fairly poor, amounting to multiple wavelengths. Due to practical applications, the interest is in improving the resolution of the D_r-to-N_t case.

3 Resolution control - filter design

Let g_D and g_N denote the Green's functions for a half-space, corresponding to homogeneous Dirichlet or Neumann surface data, respectively. Furthermore, let p denote the Dirichlet data recorded at the mirror, and let u denote the wavefield generated by time-reversing the p data, i.e., by setting $P = p^*$ in (1), where p^* is the conjugated p. Then, it can be shown that:

$$u = (\mathcal{N}g_D, p^*)_\Gamma \text{ or } \hat{u} = 2\mu \frac{\partial \hat{g}}{\partial x^1}\hat{p}^*\bigg|_{\xi^1=0}, \tag{2}$$

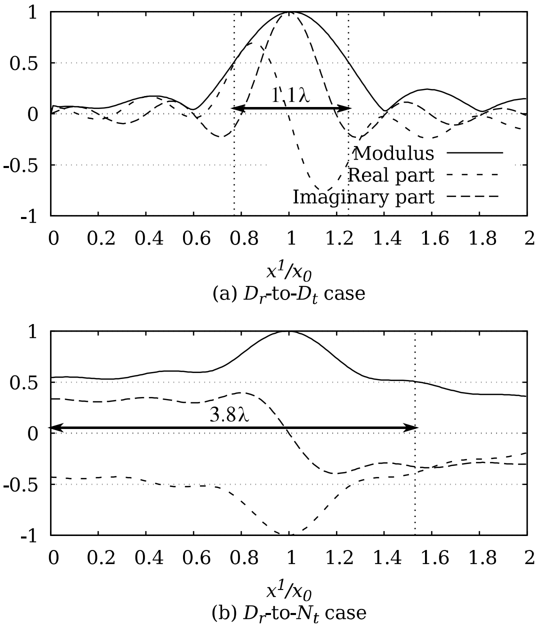

Figure 2: Resolution along the depth direction for cases D_r-to-D_t and D_r-to-N_t; $\lambda = 0.4$, $x^2 = x^3 = 0$

where a caret is used for the doubly Fourier-transformed function, i.e., $\widehat{(\cdot)} = \mathcal{F}_{x^2}\mathcal{F}_{x^3}(\cdot)$; $\hat{g}(\xi^1; x^1) = -\frac{\mathrm{i}e^{\mathrm{i}\alpha|\xi^1 - x^1|}}{2\alpha}$, is the full-space Green's function, with $\alpha = \sqrt{(\omega/c)^2 - k^2}$ for $\omega > k$ and $\alpha = \mathrm{i}\sqrt{k^2 - (\omega/c)^2}$ for $\omega < k$; $k = \sqrt{k_2^2 + k_3^2}$, and, k_2 and k_3 are the spatial wavenumbers; $c = \sqrt{\frac{\mu}{\rho}}$ is the wave velocity; and $(a, b)_\Gamma = \int_\Gamma ab\, d\Gamma$.

Similarly, let v denote the wavefield generated by time-reversing the p data, when applied as Neumann data, i.e., $Q = \frac{\mu}{x_0}p^*$. Then:

$$v = -\left(g_N, \frac{\mu}{x_0}p^*\right)_\Gamma \quad \text{or } \hat{v} = -2\frac{\mu}{x_0}\hat{g}\hat{p}^*\Big|_{\xi^1=0}, \quad (3)$$

In deriving (2) and (3), relations between g_D, g_N and g have been taken into account. Then, with the aid of the translational symmetry of the full-space Green's function g:

$$\mathcal{N}_x\left[g_N(\mathbf{x}; \boldsymbol{\xi})\right] = -\mathcal{N}_\xi\left[g_D(\mathbf{x}; \boldsymbol{\xi})\right], \forall \mathbf{x} \in \Gamma, \quad (4)$$

it follows that:

$$\begin{aligned} -\,x_0\frac{\partial}{\partial x^1}v(\mathbf{x}) &= x_0\frac{\partial}{\partial x^1}\left(g_N, \frac{\mu}{x_0}p^*\right)_\Gamma \\ &= -\left(\mathcal{N}_x g_N, p^*\right)_\Gamma = \left(\mathcal{N}_\xi g_D, p^*\right)_\Gamma = u(\mathbf{x}). \quad (5) \end{aligned}$$

We would like for the v field (D_r-to-N_t) to have the resolution of the u field (D_r-to-D_t), i.e., a λ-level resolution quality. Therefore, we require that:

$$-x_0\left(g_N, \frac{\mu}{x_0}f\left(p^*\right)\right)_\Gamma = \left(\mathcal{N}g_D, p^*\right)_\Gamma = u, \quad (6)$$

where $f(\cdot)$ is the sought filter to be applied on the Dirichlet data prior to being time-reversed as Neumann data. Using $\left(\frac{\partial}{\partial x^1} - i\alpha\right)\hat{g} = 0$ and (5), it can be shown that:

$$\begin{aligned} \hat{u} = &-x_0\frac{\partial}{\partial x^1}\left(-2\frac{\mu}{x_0}\hat{g}\hat{p}^*\right)_{\xi^1=0} = 2\mu\hat{g}(\mathrm{i}\alpha)\hat{p}^*|_{\xi^1=0} \\ &\Rightarrow f\left(p^*\right) = \mathcal{F}_{x^2}^{-1}\mathcal{F}_{x^3}^{-1}\left[-\mathrm{i}\alpha\mathcal{F}_{x^2}\mathcal{F}_{x^3}p^*\right]. \quad (7) \end{aligned}$$

The filter's (7) effect, when applied to the D_r-to-N_t case, is shown in Fig. 3.

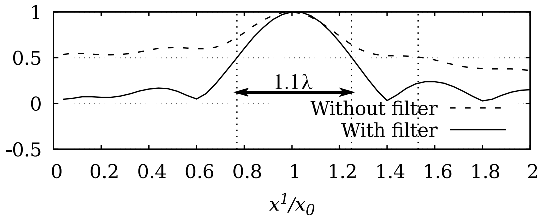

Figure 3: Resolution along the depth direction for case D_r-to-N_t with and without filter; $\lambda = 0.4$, $x^2 = x^3 = 0$

4 Conclusions

A resolution-improving filter was derived that can be applied in certain time-reversal wave focusing applications when equipment limitations enforce a flipping in the character of the boundary conditions between the recording and the time-reversal phases. The filter is capable of rendering resolution of order λ.

References

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