Recursion

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Introduction of Recursion

Iteration vs. Recursion

- How a computer repeats:
  - Iteration
    - for-statement
    - while-statement
  - Recursion

Definition of Recursion

- Recursion is a problem-solving process that breaks a problem into identical but smaller problems.
- Solves problem using itself
  - based on the general problem solving technique of breaking down a task into subtasks
  - Mathematical induction

- The proof consists of two steps:
  - The base case: prove that the statement holds for the first natural number n. Usually, n = 0 or n = 1, rarely, n = −1
  - The inductive step: prove that, if the statement holds for some natural number n, then the statement holds for n + 1.

Example: Factorial function

- $n! = 1 \times 2 \times 3 \times \cdots \times n$
- $n! = \begin{cases} 1, & \text{if } n = 0, 1 \\ n(n-1)! & \text{if } n > 1 \end{cases}$

Base case

Inductive step

- Implementation

```
public long factorial(int n){
        if(n<0) {
                System.err.println("Number should be positive");
                System.exit(-1);
        }
        if(n<=1) return 1;
        else return n * factorial(n-1);
}
```

n	n!
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

Tracing a Recursive Method

```java
public long factorial(int n){
    if(n<0) {
        System.err.println("Number should be positive");
        System.exit(-1);
    }
    if(n<=1) return 1;
    else return n * factorial(n-1);
}
```

```java
public long factorial(int n){
    if(n<0) {
        System.err.println("Number should be positive");
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    }
    if(n<=1) return 1;
    else return n * factorial(n-1);
}
```

```java
public long factorial(int n){
    if(n<0) {
        System.err.println("Number should be positive");
        System.exit(-1);
    }
    if(n<=1) return 1;
    else return n * factorial(n-1);
}
```

```java
public static void main(String[] args) {
    Factorial f = new Factorial();
    System.out.println(f.factorial(3));
}
```

Execution of factorial(3)

factorial(1):
n: 1
return point in main

factorial(2):
n: 2
return point in main

factorial(3):
n: 3
return point in main

main

Stack of activation records

Stack overflow

RecursionStack.java

```java
package tsp;

public class RecursionStack {

    public static void main(String[] args) {
        System.out.println(factorial(3));
    }
    public static int factorial(int n) {
        return n*factorial(n-1);
    }

}

```

Console

```
<terminated> RecursionStack [Java Application] C:₩Program Files₩Java₩jdk1.8.0_14
Exception in thread "main" java.lang.StackOverflowError
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
	at tsp.RecursionStack.factorial(RecursionStack.java:9)
```

Generic rules

- Infinite Recursion
  - If every recursive call had produced another recursive call, then a call to that method would, in theory, run forever. In practice, such a method runs until the computer runs out of resources, and the program terminates abnormally.

- The general outline of a successful recursive method definition is as follows:

  - One or more cases that include one or more recursive calls to the method being defined
    - These recursive calls should solve "smaller" versions of the task performed by the method being defined

Example of Rules

- No base

```
public static int factorial(int n) {
        return n*factorial(n-1);
}
```

- No convergence

```
public static int factorial(int n) {
        if (n == 1)
                return 1;
        else return n*(n-1)*factorial(n-2);
}
```

Time complexity of Recursion

Time efficiency of factorial

- Revised code

```
public long factorial(int n){
        System.out.println(n);
        if(n<=1) return 1;
        else return n * factorial(n-1);
}
```

- Recurrence relation of time function of factorial method.

$$t(n) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + t(n-1) & \text{if } n > 1 \end{cases}$$

where t(n) is the time requirement of factorial(n)

Recurrence relation

- Recurrence relation

$$t(n) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + t(n-1) & \text{if } n > 1 \end{cases}$$

- If n=4

$$t(4) = 1 + t(3) \quad = 1 + 2 = 4$$
$$t(3) = 1 + t(2) \quad = 1 + 2 = 3$$
$$t(2) = 1 + t(1) \quad = 1 + 1 = 2$$

- Closed form

  $t(n) = 1 + t(n-1)$ for $n > 1 \longrightarrow t(n) = 1 + n - 1 = n$

  proof by induction:

  if n=1: t(1) = 1 is the same to the result of the recurrence relation

  assume that if n=k, the close form is the correct, that is, t(n)=k,

  if n = k+1, t(n)=1+t(n-1) = 1+k=n.

  This means the this closed form satisfies the recurrence relation.

- Time complexity of a factorial function is O(n)

Time efficiency of computing x^n

- Recursive definition of x^n

$$x^n = \begin{cases} (x^{\frac{n}{2}})^2 & n \ is \ even \ and \ positive \\ x(x^{\frac{n-1}{2}})^2 & n \ is \ odd \ and \ positive \\ x^0 & n = 0 \end{cases}$$

- Recurrence relation of t(n) of x^n

$$t(n) = \begin{cases} t(n) = 1 + t(\frac{n}{2}) & n \geq 2 \\ 1 & n = 1 \\ 1 & n = 0 \end{cases}$$

  - t(0)=0, t(1)=1, t(2)=2, t(4)=3, t(8)=4, t(16)=5

- Closed form

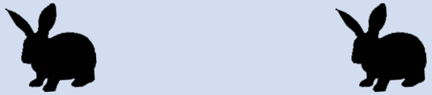
$$t(n) = 1 + \lfloor \log_2 n \rfloor$$

- Time complexity: O(log *n*)

Avoiding Recursion

Fibonacci function

- $F_n = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ F_{n-1} + F_{n-2}. & \text{otherwise} \end{cases}$

n	Number of pairs after the end of the month
0	
1	
2	
3	
4	

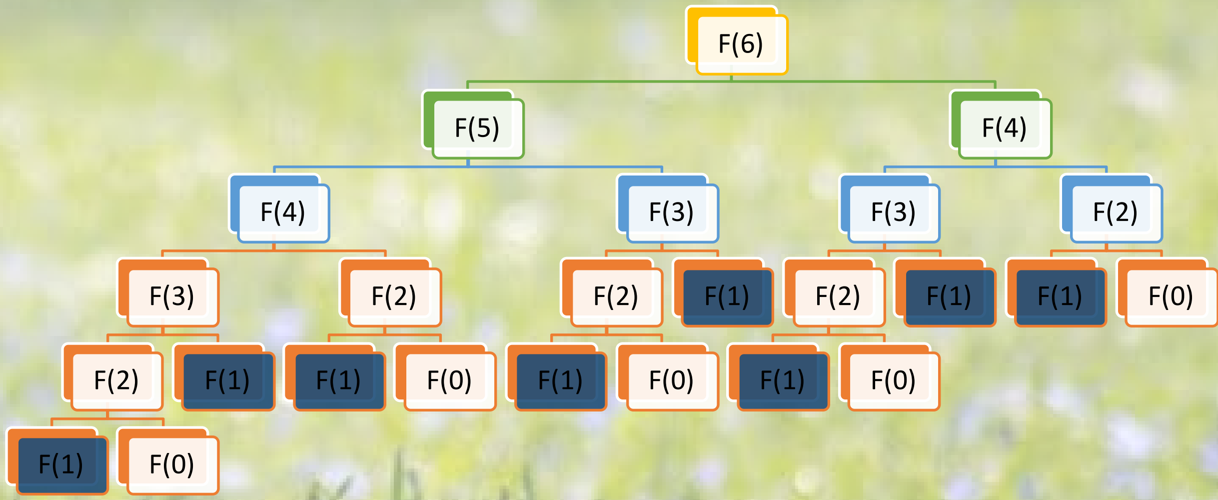
How many does it call f() for 6

```java
public class Fibonacci {
    public static void main(String[] args) {
        Fibonacci f = new Fibonacci();
        System.out.println("Recursive Fibonacci(6) is "+ f.recursive(6));
    }
    public long recursive(int n) {
        if(n<0) {
            System.err.println("Number should be positive");
            System.exit(-1);
        }
        if(n<2) return (long)1;
        return recursive(n-1)+recursive(n-2);
    }
}
```

Problems Javadoc Declaration Console

<terminated> Fibonacci [Java Application] C:₩Program Files₩Java₩jdk-14.0.2₩bin₩javaw.exe (2020. 10. 10. 오후 11:35:00 – 오후 11:35:00)

```
Recursive Fibonacci(6) is 13
```



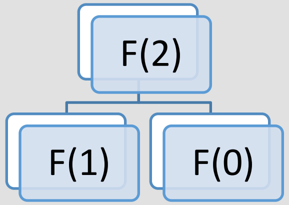
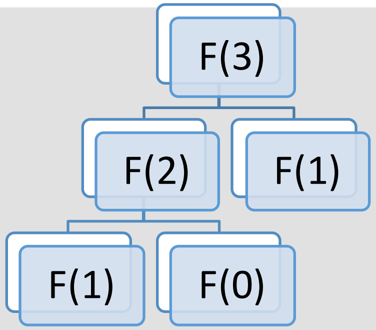
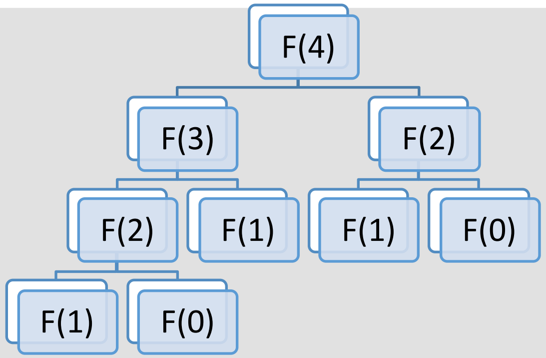
Time efficiency of Fibonacci function

- Fibonacci number

$$F_n = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ F_{n-1} + F_{n-2}. & \text{otherwise} \end{cases}$$

- Recurrence relation of time requirement for calculating Fibonacci number

$$\mathrm{t}(n) = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ 1 + t(n-1) + t(n-2). & \mathrm{n} \geq 2 \end{cases}$$

n=2	n=3	n=4
		
$T(2)=1+t(1)+t(0)=3=1+F_2>F_2$	$T(3)=1+t(2)+t(1)=5=1+F_3>F_3$	$T(4)=1+t(3)+t(2)=9=1+F_4>F_4$

Proof of the time complexity of Fibonacci function

- Growth function

$$t(n) = 1 + F_{n-1} + F_{n-2} = 1 + F_n > F_n$$
$$\therefore \Omega(F_n)$$

- Fibonacci number

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} > \frac{(\frac{1+\sqrt{5}}{2})^n - 1}{\sqrt{5}}$$

Because, $\left|\frac{1-\sqrt{5}}{2}\right| < 1$

Therefore, $\Omega(F_n) = \Omega((\frac{1+\sqrt{5}}{2})^n) = \Omega(a^n)$, exponential algorithm

Solution1: tail recursion

- Tail recursion: The last action performed by a recursive method is a recursive call.
- Tail recursion of a Fibonacci number

```java
if(n<2) return (long)1;
return recursive(n-1)+recursive(n-2);
```

```java
long prePreFibo = recursive(n-1);
long preFibo = recursive(n-2);
long currentFibo = preFibo+prePreFibo;
return currentFibo;
```

```java
currentFibo = preFibo+prePreFibo;
prePreFibo = preFibo;
preFibo = currentFibo;
return tailRecursion(n-1, preFibo, prePreFibo);
```

```java
public long tailRecursion(int n, long preFibo, long prePreFibo) {
    long currentFibo;
    if (n < 2) return n*preFibo;
    return tailRecursion(n-1, preFibo+prePreFibo, preFibo);
}
```

$$t(n) = \begin{cases} 1 & n = 1 \\ 1 + t(n-1) & n > 1 \end{cases}$$

Solution2: iteration

1. Study the function.
2. Convert all recursive calls into tail calls. (If you can't, stop. Try another method.)
3. Introduce a one-shot loop around the function body.
4. Convert tail calls into continue statements.
5. Tidy up.

Ref: http://blog.moertel.com/posts/2013-05-11-recursive-to-iterative.html

Tail recursion

```
public long tailRecursion(int n, long preFibo, long prePreFibo) {
    long currentFibo;
    if (n < 2) return n*preFibo;
    currentFibo = preFibo+prePreFibo;
    prePreFibo = preFibo;
    preFibo = currentFibo;
    return tailRecursion(n-1, preFibo, prePreFibo);
}
```

iteration

```
public long iteration(int n) {
    long currentFibo=1;
    long preFibo=1,prePreFibo=1;
    for(int i=n; i > 1 ; i--) {
        currentFibo = preFibo+prePreFibo;
        prePreFibo = preFibo;
        preFibo = currentFibo;
    }
    return currentFibo;
}
```

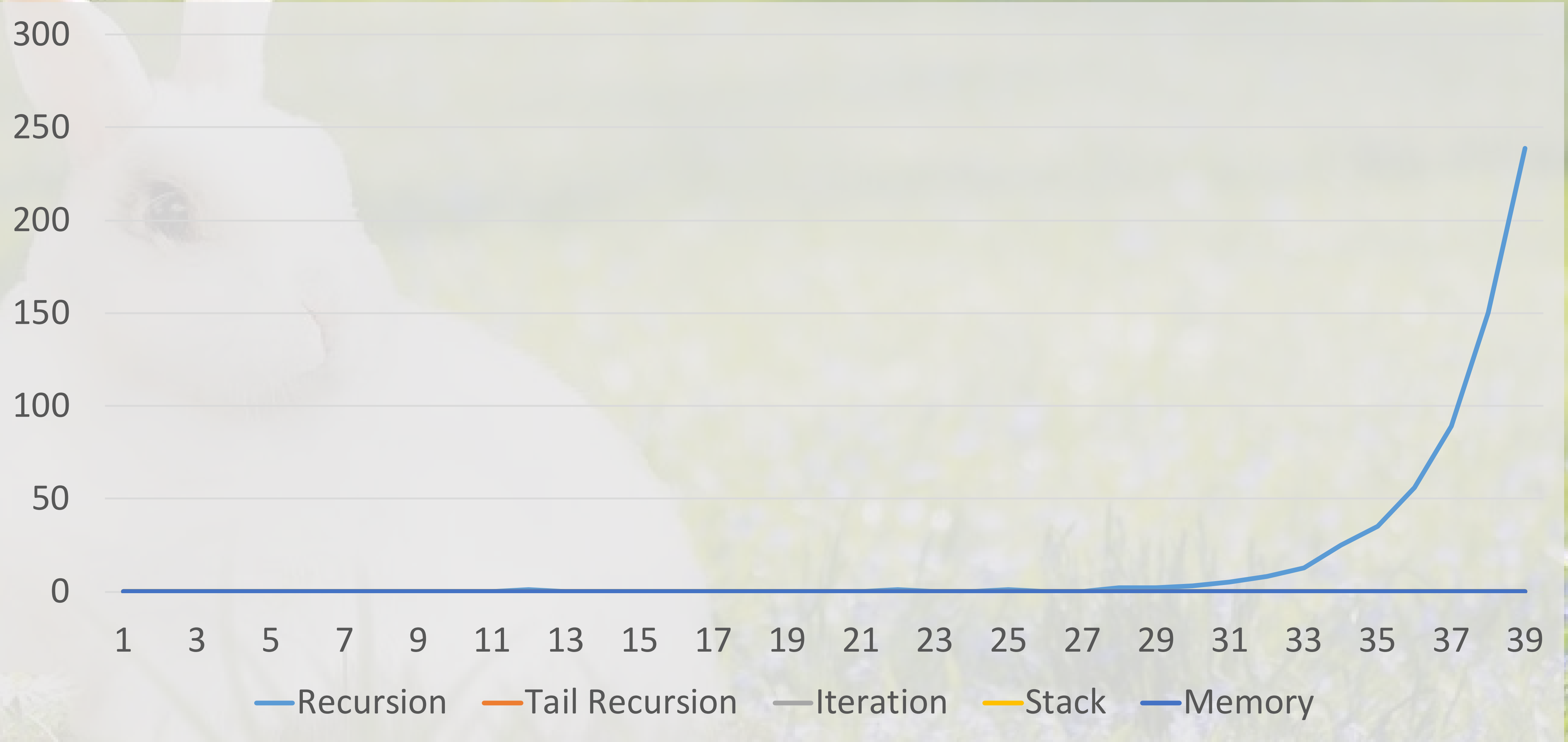
Solution 3: using a stack

```java
public long usingStack(int n) {
    ArrayDeque<Record> programStack = new ArrayDeque<>(100);
    programStack.push(new Record(n, 1, 1));
    long currentFibo = n;
    while(!programStack.isEmpty()) {
        Record topRecord = programStack.pop();
        currentFibo = topRecord.n;
        long preFibo = topRecord.pre;
        long prePreFibo = topRecord.prePre;
        if(currentFibo < 3)
            currentFibo =preFibo+prePreFibo;
        else
            programStack.push(new Record(currentFibo-1, preFibo+prePreFibo, preFibo));
    }
    return currentFibo;
}
private class Record{
    private long n;
    private long pre, prePre;
    public Record(long n, long pre, long prePre) {
        this.n = n;
        this.pre = pre;
        this.prePre = prePre;
    }
}
```

Solution 4: memorize the result

```java
private long[] fibonacci;
private int num=2;
private static final int MAX=1010;
public Fibonacci() {
    fibonacci = new long[MAX];
    fibonacci[0]=fibonacci[1]=1;
}
public long memorize(int n) {
    if(n<num) return fibonacci[n];
    else if(n==num) {
        fibonacci[n]=fibonacci[n-1]+fibonacci[n-2];
        num++;
        return fibonacci[n];
    }
    else return memorize(n-1)+memorize(n-2);
}
```

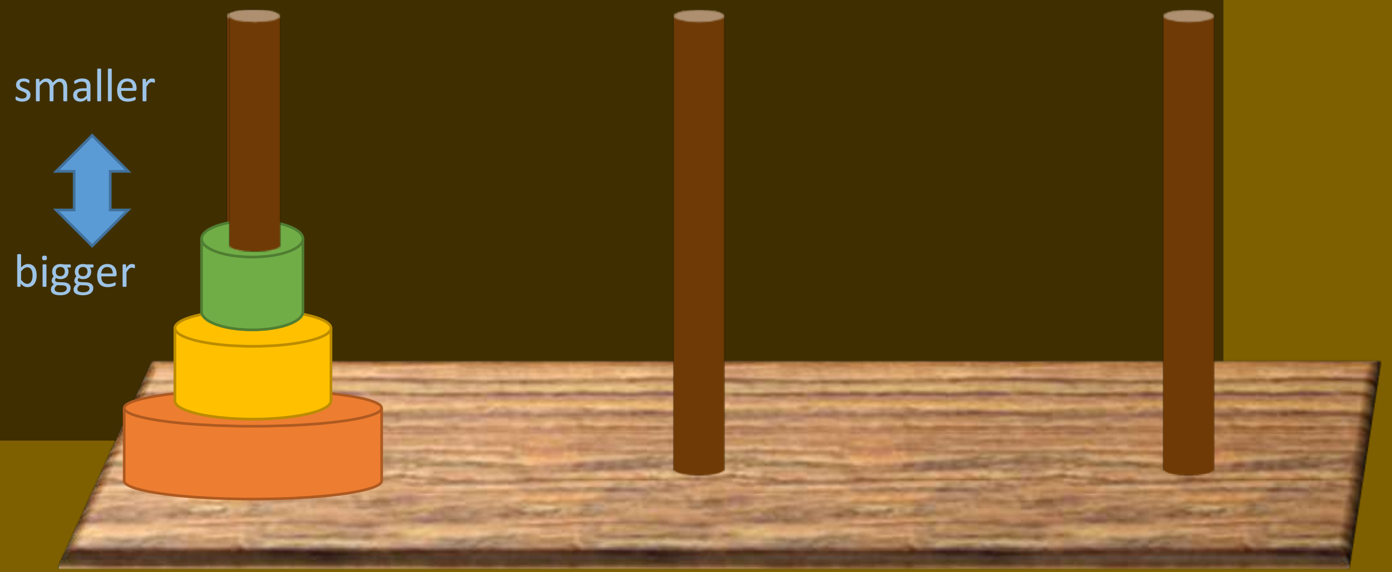
Comparison of time complexity



Other Examples

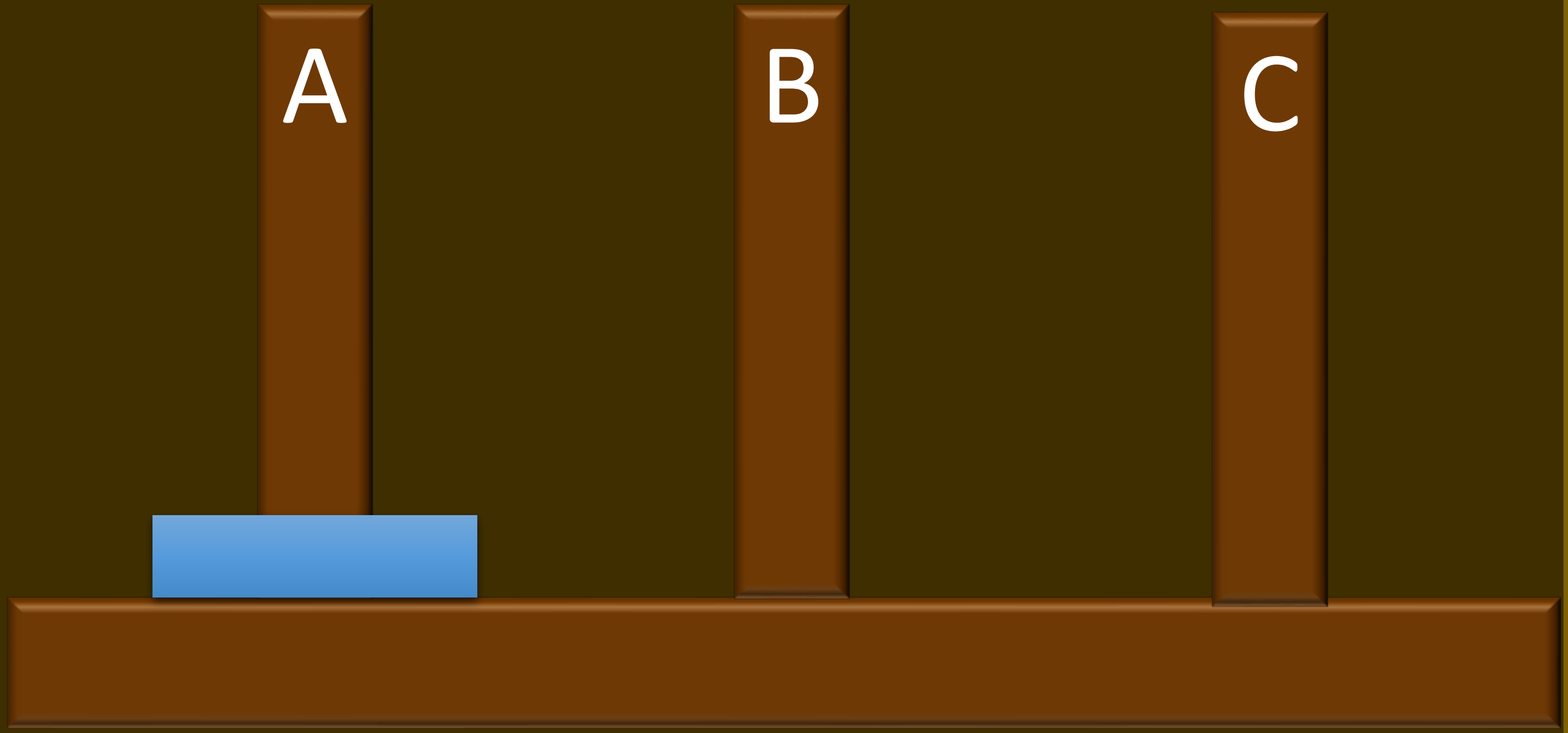
Tower of Hanoi

- Consists of
  - A board
  - Three vertical pegs
  - A progression of disks of increasing diameter
- Rule
  - Only one disk can be moved at a time.
  - No disk may be placed on top of a smaller disk.
  - Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack.
- Visualization
  - http://towersofhanoi.info/Animate.aspx



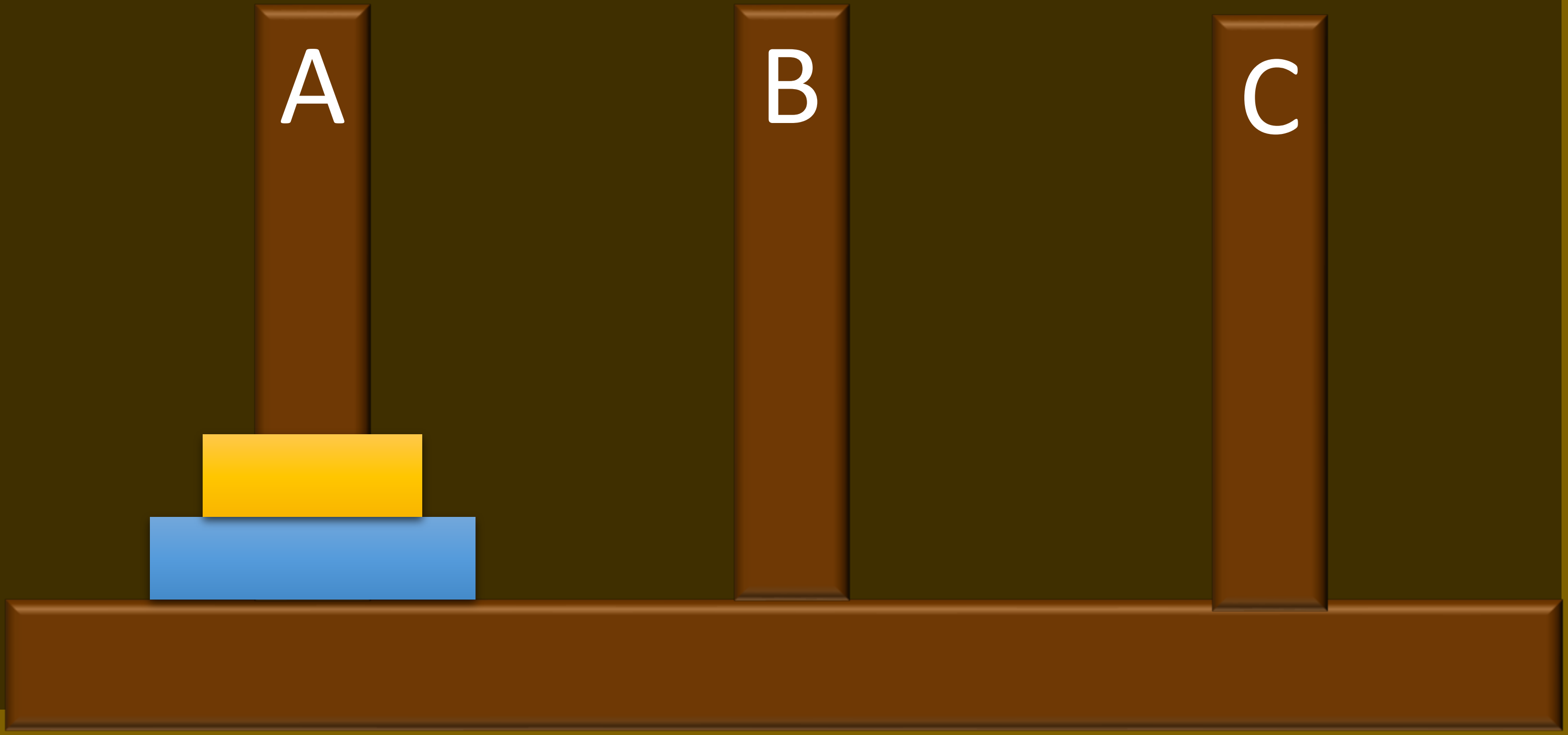
Example of tower of Hanoi

- Base case: one disk
  - Just move from the source peg to the target peg



Example of tower of Hanoi

- 2 disks



Solution of tower of Hanoi

- Base case: If the number of disks(n) is 1
  - Just move from the source to the target
- Induction step: n>1
  - Move n-1 disks from the source to another peg
  - Move the largest disk from the source to the target
  - Move n-1 disks from another peg to the target

Implementation

```java
public class Hanoi {
    public static void main(String[] args) {
        Hanoi tower = new Hanoi();
        tower.hanoi(3, "A", "C", "B");
    }
    /** solve the tower of Hanoi puzzle
     *
     * @param num the number of disks
     * @param source The source peg. You should move all disks from the source peg to target peg
     * @param target The target peg. You should move all disks from the source peg to target peg
     * @param spare The spare peg. You can use this peg for storing an unnecessary disks
     */
    public void hanoi(int num, String source, String target, String spare) {
        // base case
        if(num == 1)
            System.out.println("Move one disk from " + source + " to "+ target);
        else {
            hanoi(num-1, source, spare, target);
            System.out.println("Move one disk from " + source + " to "+ target);
            hanoi(num-1, spare, target, source);
        }
    }
}
```

Problems   Javadoc   Declaration   Console

<terminated> Hanoi [Java Application] C:₩Program Files₩Java₩jdk-14.0.2₩bin₩javaw.exe (2020. 10. 11. 오후 11:37:06 – 오후 11:37:06)

```
Move one disk from A to C
Move one disk from A to B
Move one disk from C to B
Move one disk from A to C
Move one disk from B to A
Move one disk from B to C
Move one disk from A to C
```

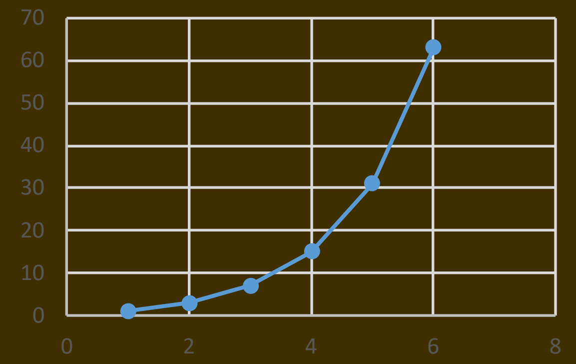
- n=1
  - 1 move
- n=2
  - 3 moves
- n=3
  - 7 moves

move(n)

=move(n-1)+1+move(n-1)

= 2move(n-1)+1

= 2^n-1



Efficiency of the algorithm

- Proof by induction that $m(n)=2^n-1$

$$\begin{aligned} m(k+1) &= 2 \times m(k) + 1 \\ &= 2 \times (2^k - 1) + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

where m(k) is the number of moves for k disks

- This cannot be solved less than the exponential time.

  Proof

  Assume that there is less moves for n disks, we denote the move as M(n).

  If n = 1, M(1)=m(1).

  Let's assume that M(n-1)=m(n-1). The largest disk is isolated on one peg and n - 1 dis ks are on another. The only way to move the largest disk is just to move the source to th e target.

  $M(n) \geq 2\ M(n-1) + 1 \geq 2\ m(n-1) + 1 = m(n)$

Example: binary search

- Binary search uses a recursive method to search an array to find a specified value. The array must be a sorted array:

  $a[0] \le a[1] \le a[2] \le \ldots \le a[\text{finalIndex}]$

  - If the value is found, its index is returned.
  - If the value is not found, -1 is returned.

https://www.cs.usfca.edu/~galles/visualization/Search.html

Solution

- Given an array A of n elements with values or records $A_0, A_1, \ldots, A_{n-1}$, sorted such that $A_0 \leq A_1 \leq \ldots \leq A_{n-1}$, and target value X,
    1. Init: L=0, R=n-1
    2. if L>R, return -1,
    3. M = (L+R)/2
    4. L=m+1 and call itself, if $A_m < X$
    5. R=m-1 and call itself, if $A_m > X$
    6. if A_m=T, Return m,

Example: X=13

A[]:	1	3	5	6	7	9	11	13	17	21
	0	1	2	3	4	5	6	7	8	9
	L				M					R

Implementation and time complexity

```java
public class MyList {
    public static void main(String[] args) {
        int[] a = {1, 3, 5, 6, 7, 9, 11, 13, 17, 21};
        System.out.println(binarySearch(13, a, 0, a.length-1));
    }
    public static int binarySearch(int x, int[] a, int l, int r) {
        if(l>r) return -1;
        int m=(l+r)/2;
        if(a[m]<x) return binarySearch(x, a, m+1, r);
        else if(a[m]>x) return binarySearch(x, a, l, m-1);
        else return m;
    }
}
```

Problems Javadoc Declaration Console

```
<terminated> MyList [Java Application] C:₩Program Files₩Java₩jdk-14.0.2₩bin₩javaw.exe  (2020. 10. 12. 오전 3:17:41 – 오전 3:17
7
```

$$t(n) < \begin{cases} 1 & n = 0 \\ 1 + \left\lfloor t\left(\frac{n}{2}\right) \right\rfloor & n > 0 \end{cases}$$

$$\therefore O(\log n)$$

proof: https://www.geeksforgeeks.org/complexity-analysis-of-binary-search/

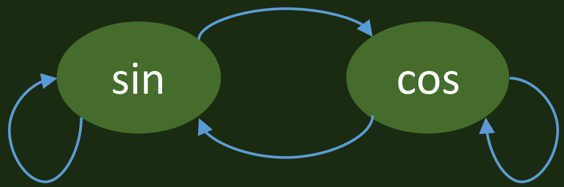
If the method cannot find the data, it is the worst case.

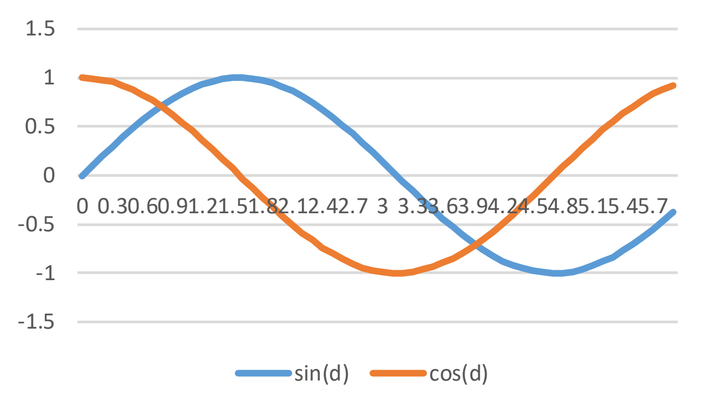
Example: Trigonometric function

- Indirect recursion

  when Method A calls a different method, this in turn calls the original calling Method A.

- Relation between trigonometric function
  - $\text{Sin}(x) = \begin{cases} x, & \text{for small } x \\ \sin(x/2)\cos(x/2) & \end{cases}$
  - $\text{Cos}(x) = \begin{cases} 1, & \text{for small } x \\ \cos(x/2)^2 \sin(x/2)^2 & \end{cases}$






```java
/**
 * Calculating the sine function
 * @param x : degree
 * @param num : the number of recursion
 * @return sin(x)
 */
public double sin(double x, int num) {
    if(num==0) return x;
    else return 2*sin(x/2, num-1)*cos(x/2, num-1);
}

/**
 * Calculating the cosine function
 * @param x : degree
 * @param num : the number of recursion
 * @return cos(x)
 */
public double cos(double x, int num) {
    if(num==0) return 1;
    else {
        double cos = cos(x/2, num-1);
        double sin = sin(x/2, num-1);
        return cos*cos-sin*sin;
    }
}
```


Thank you