CS201: Data Structures and Discrete Mathematics I

Basic Set Theory

Sets

- A set is a collection of distinct objects.
- For example (let A denote a set):

  A = {apple, pear, grape}

  A = {1, 2, 3, 4, 5}

  A = {1, b, c, d, e, f}

  A = {(1, 2), (3, 4), (9, 10)}

  A = {<1, 2, 3>, <3, 4, 5>, <6, 7, 8>}

  A = a collection of anything that is meaningful.

Members and Equality of Sets

- The objects that make up a set are called members or elements of the set.
- Two sets are equal iff they have the same members.
  - That is, a set is completely determined by its members.
  - Order and repetition do not matter in a set.

Set notations

- The notation of {...} describes a set. Each member or element is separated by a comma.
  - E.g., S = {apple, pear, grape}
  - S is a set
  - The members of S are: apple, pear, grape
- Order and repetition do not matter in a set.
- The following expressions are equivalent:
  - {1, 3, 9}
  - {9, 1, 3}
  - {1, 1, 3, 3, 9}

The membership symbol $\in$ and the empty set $\emptyset$

- The fact that x is a member of a set S can be expressed as
  - $x \in S$
  - Reads:
    - x is in S, or
    - x is a member of S, or
    - X belongs to S
- An example, $S = \{1, 2, 3\}$, $1 \in S$, $2 \in S$, $3 \in S$
- The negation of $\in$ is written as $\notin$ (is not in).
- The empty set is a set without any element
  - Denoted by $\{\}$ or $\emptyset$
  - For any object x, $x \notin \emptyset$

Defining a Set by membership properties

- Notation
  - $S = \{x \in T \mid P(x)\}$ (or $S = \{x \mid x \in T \text{ and } P(x)\}$)
  - The members of S are members of an already known set T that satisfy property P.
- An example:
  - Let Z be the set of integers
  - Let Z_+ be the set of positive integers.
  - $Z_+ = \{x \in Z \mid x > 0\}$

Sets of numbers

- Z = The set of all integers

  $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

- Z_+ = The set of positive integers

  $Z_+ = \{1, 2, 3\ldots\} = \{x \mid x \in Z \text{ and } x > 0\} = \{x \in Z \mid x > 0\}$

- Z_- = The set of negative integers

  $Z_- = \{\ldots, -3, -2, -1\} = \{-1, -2, -3\ldots\} = \{x \in Z \mid x < 0\}$

- R = The set of all real numbers
- Q = the set of all rational numbers

  $Q = \{x \in R \mid x = p/q \text{ and } p, q \in Z \text{ and } q \neq 0\}$

- We can use “;” to replace “and”

Subsets

- A is a subset ($\subseteq$) of B, or B is a superset of A iff every member of A is a member of B.
  - $A \subseteq B$ iff forall x if $x \in A$, then $x \in B$
- An example:
  - (-2, 0, 6} $\subseteq$ {-3, -2, -1, 0, 1, 3, 6}
- Negation: A is not a subset of B or B is not a superset of A iff there is a member of A that is not a member of B
  - $A \nsubseteq B$ iff there exist x, $x \in A$, $x \notin B$

Obvious subsets

- $S \subseteq S$

  $\emptyset \subseteq S$
  - By contradiction:
  - if $\emptyset \not\subseteq S$ then there exist x, $x \in \emptyset$ and $x \notin S$.

Proper subsets

- A is a proper subset ($\subset$) of B, or B is a proper superset of A iff A is a subset of B and A is not equal to B.
  - $A \subset B$ iff $A \subseteq B$ and $A \neq B$
- Examples:
  - $\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}$
  - $Z_+ \subset Z \subset Q \subset R$
  - If $S \neq \varnothing$ then $\varnothing \subset S$

Power sets

- The set of all subsets of a set is called the power set of the set
- The power set of S is denoted by P(S).
- Example:
  - $P(\varnothing) = \{\varnothing\}$
  - $P(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$
  - $P(S) = \{\varnothing, \ldots, S\}$
  - What is $P(\{1, 2, 3\})$?
- How many elements does the power set of S have? Assume S has n elements.

$\in$ and $\subseteq$ are different

- Examples:

  $1 \in \{1\}$ is true

  $1 \subseteq \{1\}$ is false

  $\{1\} \subseteq \{1\}$ is true

- Which of the following statement is true?

  $S \subseteq P(S)$

  $S \in P(S)$

Mutual inclusion and set equality

- Sets A and B have the same members iff they mutually include
  - $A \subseteq B$ and $B \subseteq A$
- That is, $A = B$ iff $A \subseteq B$ and $B \subseteq A$
- Mutual inclusion is very useful for proving the equality of sets
- To prove an equality, we prove two subset relationships.

An example: equality of sets

- Recall that Z = the set of (all) integers
- Let $A = \{x \in Z \mid x = 2m \text{ for } m \in Z\}$
- Let $B = \{x \in Z \mid x = 2n-2 \text{ for } n \in Z\}$
- To show $A \subseteq B$, note that

  $2m = 2(m+1) - 2 = 2n-2$
- To show that $B \subseteq A$, note that

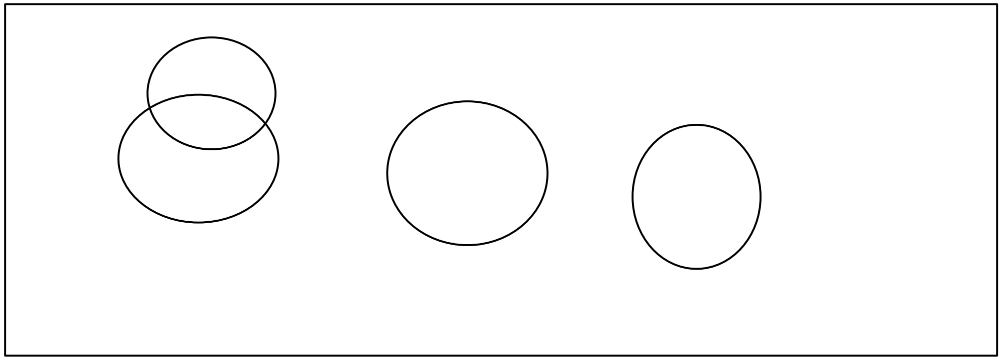
  $2n-2=2(n-1) = 2m$
- That is, A = B. (A, B both denote the set of all even integers.

Universal sets

- Depending on the context of discussion
  - Define a set of U such that all sets of interest are subsets of U.
  - The set U is known as a universal set
- Examples:
  - When dealing with integers, U may be Z.
  - When dealing with plane geometry, U may be the set of points in the plane

Venn diagram

- Venn diagram is used to visualize relationships of some sets.
- Each subset (of U, the rectangle) is represented by a circle inside the rectangle.



Set operations

- Let A, B be subsets of some universal set U.
- The following set operations create new sets from A and B.
- Union:

  $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
- Intersection:

  $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
- Difference:

  $A - B = A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\}$
- Complement

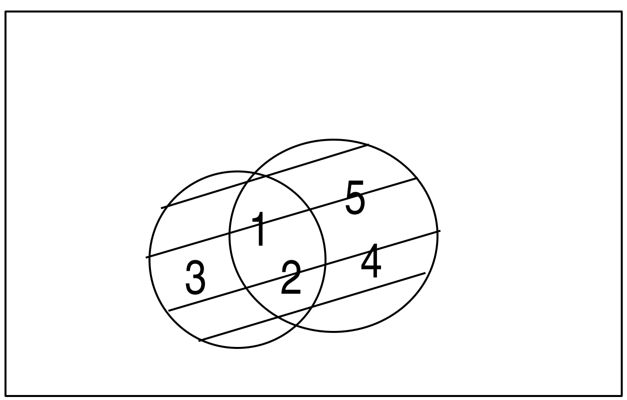
  $A' = U - A = \{x \in U \mid x \notin A\}$

Set union

- An example

  $\{1, 2, 3\} \cup \{1, 2, 4, 5\} = \{1, 2, 3, 4, 5\}$

  The venn diagram

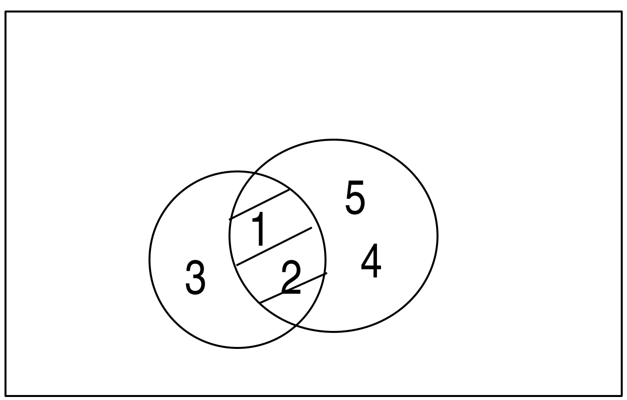


Set intersection

- An example

  $\{1, 2, 3\} \cap \{1, 2, 4, 5\} = \{1, 2\}$

  The venn diagram

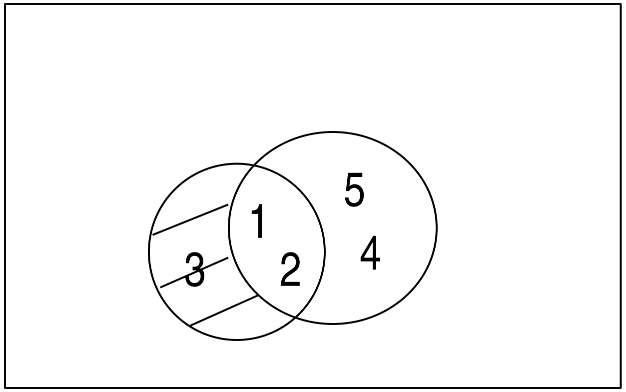


Set difference

- An example

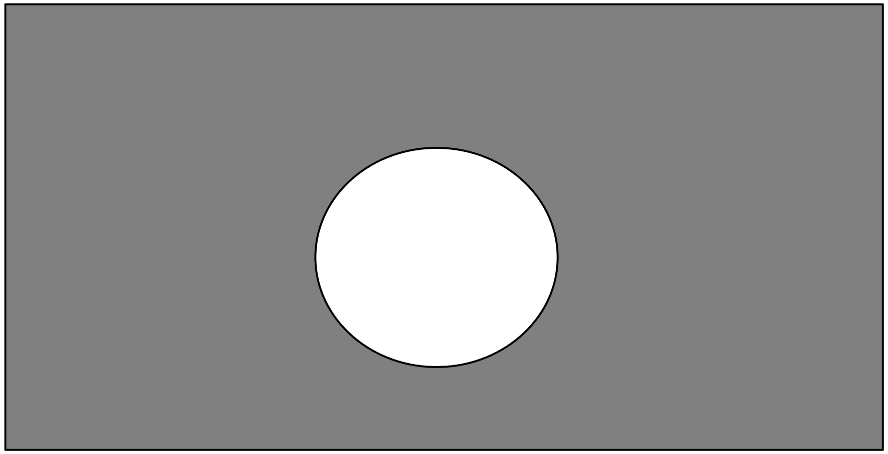
  {1, 2, 3} - {1, 2, 4, 5} = {3}

  The venn diagram



Set complement

- The venn diagram



Some questions

- Let $A \subseteq B$.
  - What is $A - B$?
  - What is $B - A$?
  - If $A, B \subseteq C$, what can you say about $A \cup B$ and C?
  - If $C \subseteq A, B$, what can you say about C and $A \cap B$?

Basic set identities (equalities)

- Commutative laws

  $A \cup B = B \cup A$

  $A \cap B = B \cap A$

- Associative laws

  $(A \cup B) \cup C = A \cup (B \cup C)$

  $(A \cap B) \cap C = A \cap (B \cap C)$

- Distributive laws

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Basic set identities (cont …)

- Identity laws

  $\emptyset \cup A = A \cup \emptyset = A$

  $A \cap U = U \cap A = A$

- Double complement law

  $(A')' = A$

- Idempotent laws

  $A \cup A = A$

  $A \cap A = A$

- De Morgan's laws

  $(A \cup B)' = A' \cap B'$

  $(A \cap B)' = A' \cup B'$

Proof methods

- There are many ways to prove set identities
- The methods include
  - Applying existing identities
  - Using mutual inclusion
- Prove $(A \cap B) \cap C = A \cap (B \cap C)$ using mutual inclusion
  - First show: $(A \cap B) \cap C \subseteq A \cap (B \cap C)$
  - Let $x \in (A \cap B)$ and $x \in C$
    - $(x \in A$ and $x \in B)$ and $x \in C$
    - $x \in A$ and $x \in (B \cap C)$
    - $x \in A \cap (B \cap C)$
  - Then show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

More proof examples

- Let B = {x | x is a multiple of 4}

  A = {x | x is a multiple of 8}

  Then we have $A \subseteq B$
- Proof: let $x \in$ A. We must show that x is a multiple of 4. We can write x = 8m for some integer m. We have
  - x = 8m = 2*4m = 4 k, where k = 2m,
  - so k is a integer.
  - Thus, x is a multiple of 4, and
    - therefore $x \in B$

More proof examples

- Prove $\{x \mid x \in Z \text{ and } x \geq 0 \text{ and } x^2 < 15\}$
  $= \{x \mid x \in Z \text{ and } x \geq 0 \text{ and } 2x < 7\}$

Proof:

Let $A = \{x \mid x \in Z \text{ and } x \geq 0 \text{ and } x^2 < 15\}$

$B = \{x \mid x \in Z \text{ and } x \geq 0 \text{ and } 2x < 7\}$

Let $x \in A$. x can only be 0, 1, 2, 3

2x for 0, 1, 2, 3 are all less then 7.

Thus, $A \subseteq B$.

Likewise, we can also show that $B \subseteq A$

More proof examples

- Prove:
  $[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') = \varnothing$

Proof:

$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)')$

$= ([A \cup (B \cap C)] \cap [A' \cup (B \cap C)]) \cap (B \cap C)'$

$= ([(B \cap C) \cup A] \cap [(B \cap C) \cup A']) \cap (B \cap C)'$

$= [(B \cap C) \cup (A \cap A')] \cap (B \cap C)'$

$= [(B \cap C) \cup \varnothing] \cap (B \cap C)'$

$= (B \cap C) \cap (B \cap C)'$

$= \varnothing$

Pigeonhole principle

- If more than k pigeons fly into k pigeonholes, then at least one hole will have more than one pigeon.
- **Pigeonhole principle**: if more than k items are placed into k bins, then at least one bin contains more than one item.
- Simple, and obvious!!
- To apply it, may not be easy sometimes. Need to be clever in identifying pigeons and pigeonholes.

Example

- How many people must be in a room to guarantee that two people have last names that begin with the same initial?

  27 since we have 26 letters

- How many times must a single die be rolled in order to guarantee getting the same value twice?

  7

Another example

- Prove that if four numbers are chosen from the set {1, 2, 3, 4, 5, 6}, at least one pair must add up to 7.

Proof: There are 3 pairs of numbers from the set that add up to 7, i.e.,

(1, 6), (2, 5), (3, 4)

Apply pigeonhole principle: bins are the pairs, and the numbers are the items.

Infinite sets

- In a finite set, we can always designate one element as the first member, s_1, another element as the second member, s_2 and so on. If there are k elements in the set we can list them as
  - $s_1, s_2, \ldots, s_k$
- A set that is not finite is called a infinite set.
- If a set is infinite, we may still be able to select a first element s1, a second element s2 and so on:
  - $s_1, s_2, \ldots$
  - Such an infinite set is said to be **denumerable**.
- Both finite and denumerable sets are **countable**.
- Countable does not mean we can give a total number, but means that we can say, "here is the first one" and "here is the second one" and so on.

Countable sets: examples

- The set of positive integer numbers are countable.
- To prove it, we need to give a counting scheme, in this case,

  1, 2, 3, 4, ….

- The set of positive rational numbers are countable

1/1	1/2	1/3	1/4 …
2/1	2/2	2/3	2/4…
3/1	3/2	3/3	3/4 …

Uncountable sets

- There are also some sets that are uncountable.
  - The set is so large, and there is no way to count out the elements.
- One example: The set of real numbers between 0 and 1 is uncountable.

- A computer can only manage finite sets.

Summary

- Sets are extremely important for Computer Science.
- A set is simply an unordered list of objects.
- Set operations: union, intersection, difference.
- To prove set equalities
  - Applying existing identities
  - Using mutual inclusion
- Pigeonhole principle