CS201: Data Structures and Discrete Mathematics I

Relations and Functions

Relations

Ordered n-tuples

- An ordered n-tuple is an ordered sequence of n objects
- $(x_1, x_2, \ldots, x_n)$
  - First coordinate (or component) is x_1
  - …
  - n-th coordinate (or component) is x_n
- An ordered pair: An ordered 2-tuple
  - (x, y)
- An ordered triple: an ordered 3-tuple
  - (x, y, z)

Equality of tuples vs sets

- Two tuples are equal iff they are equal coodinate-wise
  - $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ iff

    $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$
- $(2, 1) \neq (1, 2)$, but $\{2, 1\} = \{1, 2\}$
- $(1, 2, 1) \neq (2, 1)$, but $\{1, 2, 1\} = \{2, 1\}$
- $(1, 2\text{-}2, a) = (1, 0, a)$
- $(1, 2, 3) \neq (1, 2, 4)$ and $\{1, 2, 3\} \neq \{1, 2, 4\}$

Cartesian products

- Let A_1, A_2, …A_n be sets
- The cartesian products of A_1, A_2, …A_n is
  - $A_1 \times A_2 \times \ldots \times A_n$
    $= \{ (x_1, x_2, \ldots, x_n) \mid x_1 \in A_1$ and $x_2 \in A_2$ and …
    and $x_n \in A_n)$
- Examples: A = {x, y}, B = {1, 2, 3}, C = {a, b}
- AxB={(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}
- AxBxC = {(x, 1, a), (x, 1, b), …, (y, 3, a), (y, 3, b)}
- Ax(BxC) = {(x, (1, a)), (x, (1, b)), …, (y, (3, a)), (y, (3, b))}

Relations

- A relation is a set of ordered pairs
  - Let x R y mean x is R-related to y
  - Let A be a set containing all possible x
  - Let B be a set containing all possible y

  Relation R can be treated as a set of ordered pairs

  $R = \{(x, y) \in AxB \mid x \, R \, y\}$
- Example: We have the relation “is-capital-of” between cities and countries:

  Is-capital-of = {(London, UK), (WashingtonDC, US), …}

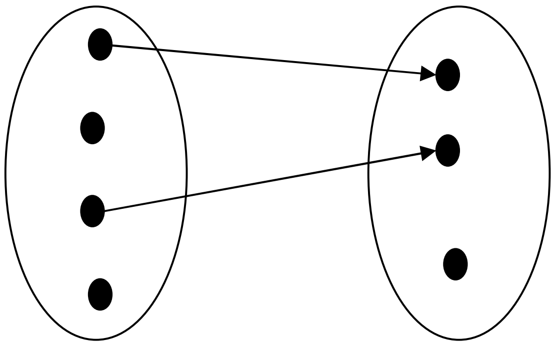
Relations are sets

- $R \subseteq AxB$ as a relation from A to B
- R is a relation from A to B iff $R \subseteq AxB$
  - Furthermore, x R y iff $(x, y) \in R$.
- If the relation R only involves two sets, we say it is a **binary relation**.
- We can also have an n-ary relation, which involves n sets.

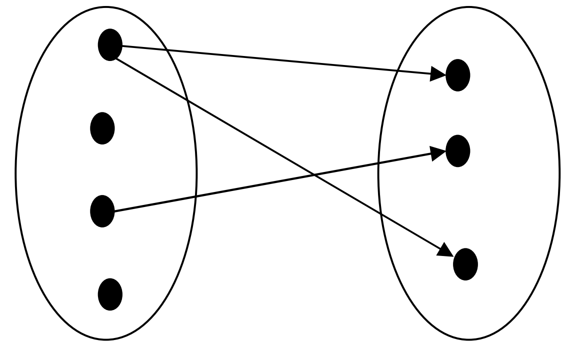
Various kinds of binary relations

- One-to-one relation: each first component and each second component appear only once in the relation.
- One-to-many relation: if some first component s_1 appear more than once.
- Many-to-one relation: if some second component s_2 is paired with more than one first component.
- Many-to-many relation: if at least one s_1 is paired with more than one second component and at least one s_2 is paired with more than one first component.

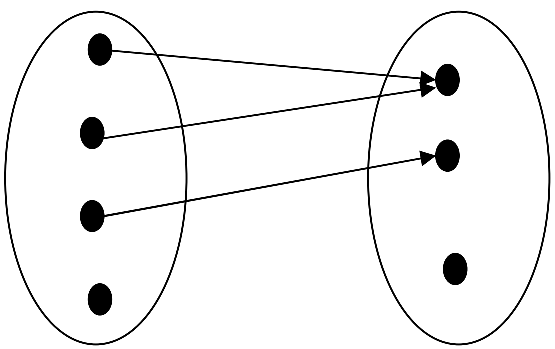
Visualizing the relations



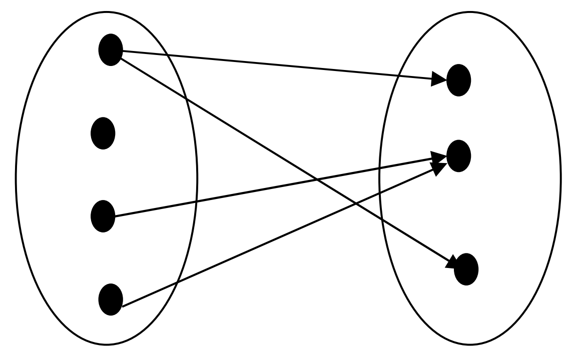
One-to-one



One-to-many



Many-to-one



Many-to-many

Binary relation on a set

- Given a set A, a binary relation R on A is a subset of AxA ($R \subseteq AxA$).
- An example:
  - A = {1, 2}. Then AxA={(1,1), (1,2), (2,1), (2,2)}. Let R on A be given by $x\ R\ y \leftrightarrow x+y$ is odd.
  - then, $(1, 2) \in R$, and $(2, 1) \in R$

Properties of Relations: Reflexive

- Let R be a binary relation on a set A.
  - R is reflexive: iff for all $x \in A$, $(x, x) \in R$.
- Reflexive means that every member is related to itself.
- Example: Let A = {2, 4, a, b}
  - R = {(2, 2), (4, 4), (a, a), (b, b)}
  - S = {(2, b), (2, 2), (4, 4), (a, a), (2, a), (b, b)}
- R, S are reflexive relations on A.
- Another example: the relation $\leq$ is reflexive on the set Z_+.

Symmetric relations

- A relation R on a set A is symmetric iff for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- R, T are symmetric relations on A.
- S is not a symmetric relation on A.
- The relation $\leq$ is reflexive on the set Z_{+}, but not symmetric. E.g., $3 \leq 4$ is in, but not $4 \leq 3$

Anti-symmetric relations

- A relation R on a set A is anti-symmetric iff for all $x, y \in A$. if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- R, S are anti-symmetric relations on A.
- T is not an anti-symmetric relation on A.
- The relation $\leq$ is reflexive on the set $Z_{+,}$ but not symmetric. It is anti-symmetric.

Transitive relations

- A relation R on a set A is transitive iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- R, S are transitive relations on A.
- T is not a transitive relation on A.
- The relation $\leq$ is reflexive on the set $Z_{+,}$ but not symmetric. It is also anti-symmetric, and transitive (why?).

Transitive closure

- Let R be a relation on A
- The smallest transitive relation on A that includes R is called the transitive closure of R.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- The transitive closures of R and S are themselves
- The transitive closure of T is $T \cup \{(2, 2), (b, b)\}$

Equivalence relations

- A relation on a set A is an equivalence relation if it is
  - Reflexive.
  - Symmetric
  - Transitive.
- Examples of equivalence relations
  - On any set S, $x \; R \; y \leftrightarrow x = y$
  - On integers ≥ 0, $x \; R \; y \leftrightarrow x+y$ is even
  - On the set of lines in the plane, $x \; R \; y \leftrightarrow x$ is parallel to y.
  - On $\{0, 1\}$, $x \; R \; y \leftrightarrow x = y^2$
  - On $\{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.

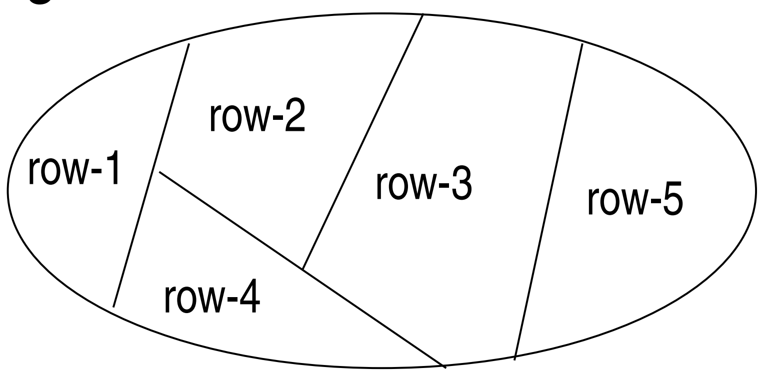
Congruence relations are equivalence relations

- We say x is congruent modulo m to y
  - That is, x C y iff m divides x-y, or x-y is an integral multiple of m.
  - We also write $x \equiv y \pmod{m}$ iff x is congruent to y modulo m.
- Congruence modulo m is an equivalent relation on the set Z.
  - Reflexive: m divides x-x = 0
  - Symmetry: if m divides x-y, then m divides y-x
  - Transitive: if m divides x-y and y-z,

    then m divides (x-y)+(y-z) = x-z

An important feature

- Let us look at the equivalence relation:
  - S = {x | x is a student in our class}
  - x R y ↔ "x sits in the same row as y"
- We group all students that are related to one another. We can see this figure:



- We have partitioned S into subsets in such a way that everyone in the class belongs to one and only one subset.

Partition of a set

- A partition of a set S is a collection of nonempty disjoint subsets (S_1, S_2, .., S_n) of S whose union equals S.
  - $S_1 \cup S_2 \cup \ldots \cup S_n = S$
  - If $i \neq j$ then $S_i \cap S_j = \emptyset$ ($S_i \cap S_j$ are disjoint)
- Examples, let A = {1, 2, 3, 4}
  - {{1}, {2}, {3}, {4}}   a partition of A
  - {{1, 2}, {3, 4}}   a partition of A
  - {{1, 2, 3}, {4}}   a partition of A
  - {{}, {1, 2, 3}, {4}}   not a partition of A
  - {{1, 2}, {3, 4}, {1, 4}}   not a partition of A

Equivalent classes

- Let R be an equivalence relation on a set A.
  - Let $x \in A$
- The equivalent class of x with respect to R is:
  - $R[x] = \{y \in A \mid (x, y) \in R\}$
  - If R is understood, we write [x] instead of R[x].
- Intuitively, [x] is the set of all elements of A to which x is related.

Theorems on equivalent relations and partitions

Theorem 1: An equivalence relation R on a set A determines a partition of A.

- i.e., the distinctive equivalence classes of R form a partition of A.

Theorem 2: a partition of a set A determines an equivalence relation on A.

- i.e., there is an equivalence relation R on A such that the set of equivalence classes with respect to R is the partition.

An equivalent relations induces a partition

- Let A = {0, 1, 2, 3, 4, 5}
- Let R be the congruence modulo 3 relation on A
- The set of equivalence classes is:
  - {[0], [1], [2], [3], [4], [5]} =
    {{0, 3}, {1, 4}, {2, 5}, {3, 0}, {4, 1}, {5, 2}} =
    {{0, 3}, {1, 4}, {2, 5}}
- Clearly, {{0, 3}, {1, 4}, {2, 5}} is a partition of A.

An partition induces an equivalent relation

- Let A = {0, 1, 2, 3, 4, 5}
- Let a partition P = {{0, 5}, {1, 2, 3}, {4}}
- Let R =

  $\{\{0, 5\} \times \{0, 5\} \cup \{1, 2, 3\} \times \{1, 2, 3\} \cup \{4\} \times \{4\}\}$

  $= \{(0, 0), (0, 5), (5, 0), (5, 5), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$
- It is easy to verify that R is an equivalent relation.

Partial order

- A binary relation R on a set S is a partial order on S iff R is
  - Reflexive
  - Anti-symmetric
  - Transitive
- We usually use $\leq$ to indicate a partial order.
- If R is a partial order on S, then the ordered pair (S, R) is called a **partially ordered set** (also known as **poset**).
- We denote an arbitrary partially ordered set by (S, $\leq$).

Examples

- On a set of integers, x R y $\leftrightarrow$ x $\leq$ y is a partial order ($\leq$ is a partial order).
- for integers, a, b, c.
  - a $\leq$ a (reflexive)
  - a $\leq$ b, and b $\leq$ a implies a = b (anti-symmetric)
  - a $\leq$ b and b $\leq$ c implies a $\leq$ c (transitive)
- Other partial order examples:
  - On the power set P of a set, A R B $\leftrightarrow$ A $\subseteq$ B
  - On Z_+, x R y $\leftrightarrow$ x divides y.
  - On {0, 1}, x R y $\leftrightarrow$ x = y^2

Some terminology of partially ordered sets

- Let $(S, \leq)$ be a partially ordered set
- If $x \leq y$, then either $x = y$ or $x \neq y$.
- If $x \leq y$, but $x \neq y$, we write $x < y$ and say that x is a **predecessor** of y, or y is a **successor** of x.
- A given y may have many predecessors, but if $x < y$ and there is no z with $x < z < y$, then x is an immediate predecessor of y.

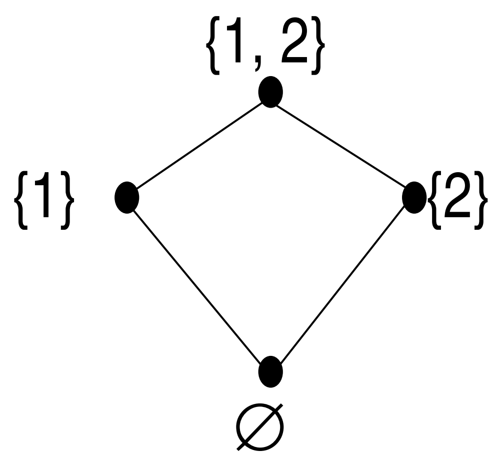
Visualizing partial order: Hasse diagram

- Let S be a finite set.
- Each of the element of S is represented as a dot (called a **node**, or **vertex**).
- If x is an immediate predecessor of y, then the node for y is placed above node x, and the two nodes are connected by a straight-line segment.
- The Hasse diagram of a partially ordered set conveys all the information about the partial order.
- We can reconstruct the partial order just by looking at the diagram

An example Hasse diagram

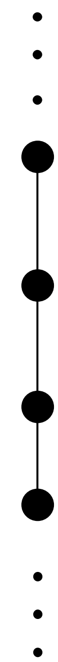
- $\forall \subseteq$ on the power set P({1, 2}):
  - Poset: (P({1, 2}), $\subseteq$)
- P({1, 2}) = {$\emptyset$, {1}, {2}, {1, 2}}
- $\forall \subseteq$ consists of the following ordered pairs

  ($\emptyset$, $\emptyset$), ({1}, {1}), ({2}, {2}), ({1, 2}, {1, 2}),
  ($\emptyset$, {1}), ($\emptyset$, {2}), ($\emptyset$, {1, 2}), ({1}, {1, 2}),
  ({2}, {1, 2})



Total orders

- A partial order on a set is a **total order** (also called **linear order**) iff any two members of the set are related.
- The relation $\leq$ on the set of integers is a total order.
- The Hasse diagram for a total order is on the right

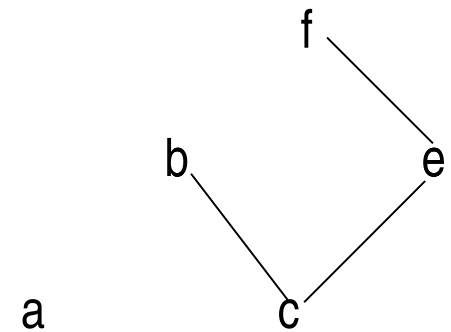


Least element and minimal element

- Let $(S, \leq)$ be a poset. If there is a $y \in S$ with $y \leq x$ for all $x \in S$, then y is a **least element** of the poset. If it exists, is unique.
- An element $y \in S$ is **minimal** if there is no $x \in S$ with $x < y$.
- In the Hasse diagram, a least element is below all orders.
- A minimal element has no element below it.
- Likewise we can define **greatest element** and **maximal element**

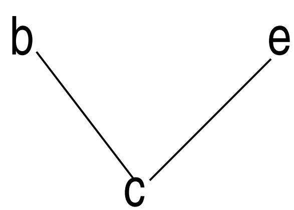
Examples: Hasse diagram

- Consider the poset:

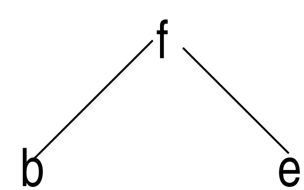


- The maximal elements are a, b, f
- The minimal elements are a, c.

A least element but no greatest element



A greatest element but no least element



Summary

- A binary relation on a set S is a subset of SxS.
- Binary relations can have properties of reflexivity, symmetry, anti-symmetry, and transitivity.
- Equivalence relations. A equivalence relation on a set S defines a partition of S.
- Partial orders. A partial ordered set can be represented graphically.

Functions

High school functions

- Functions are usually given by formulas
  - $f(x) = \sin(x)$
  - $f(x) = e^x$
  - $f(x) = x^3$
  - $f(x) = \log x$
- A function is a computation rule that changes one value to another value
- Effectively, a function associates, or relates, one value to another value.

“general” functions

- We can think of a function as relating one object to another (need not be numbers).
- A relation f from A to B is a **function** from A to B iff
  - for every $x \in A$, there exists a unique $y \in B$ such that x f y, or equivalently $(x, y) \in f$
- Functions are also known as transformations, maps, and mappings.

Notational convention

- Sometimes functions are given by stating the rule of transformation, for example,
  - $f(x) = x + 1$
- This should be taken to mean

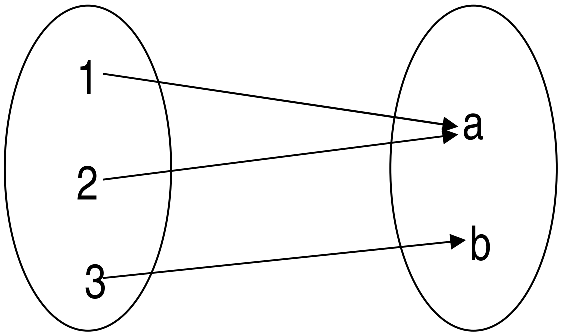
  $f = \{(x, f(x)) \in AxB \mid x \in A\}$

  where A and B are some understood sets.

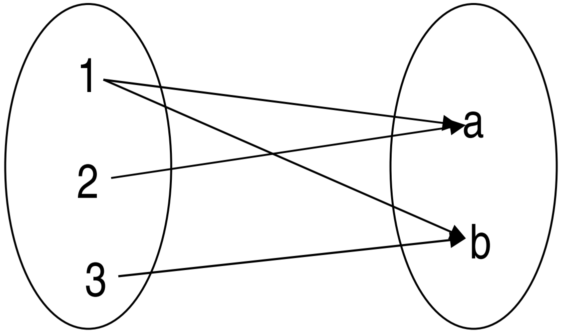
Examples

- Let A = {1, 2, 3} and

  B = {a, b}



- R = {(1, a), (2, a), (3, b)} is **a function** from A to B



- R = {(1, a), (1, b), (2, a), (3, b)} is **not a function** from A to B

Notations and concepts

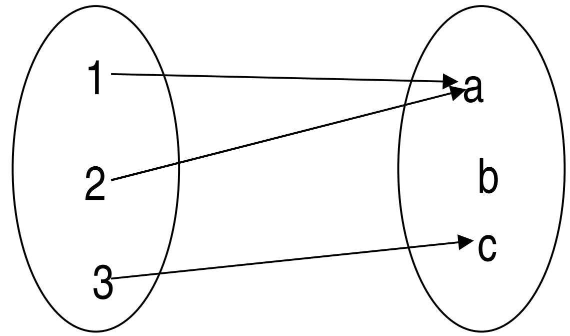
- Let A and B be sets, f is a function from A to B. We denote the function by:

  f: A $\rightarrow$ B

- *A* is the **domain**, and *B* is the **codomain** of the function.
- If $(a, b) \in f$, then *b* is denoted by $f(a)$; *b* is the **image** of *a* under *f*, *a* is a **preimage** of *b* under *f*.
- The range of *f* is the set of images of *f*.
  - The range of *f* is the set $f(A)$.

An example

- Let the function f be



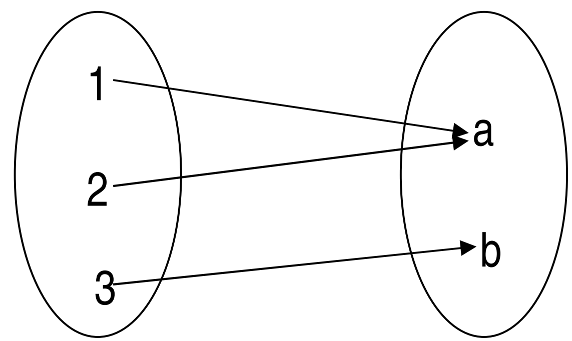
- Domain is {1, 2, 3}
- Codomain is {a, b, c}
- Range is {a, c}

Equality of functions

- Let f: $A \to B$ and g: $C \to D$.
- We denote function f = function g
  - iff set f = set g
- Note that this force A = C, but not B = D
  - Some require B = D as well.

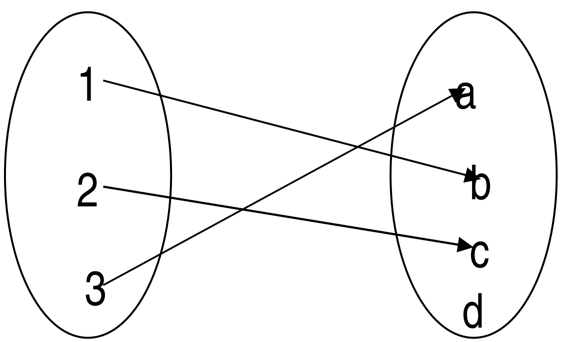
Properties of functions: **onto**

- Let f: $A \rightarrow B$
  - The function f is an **onto** or **surjective** function iff the range of f equals to the codomain of f.
  - Or for any $y \in B$, there exists some $x \in A$, such that $f(x) = y$.
- The function on the right is onto.
- $f: Z \rightarrow Z$ with $f(x) = x^2$ is not onto



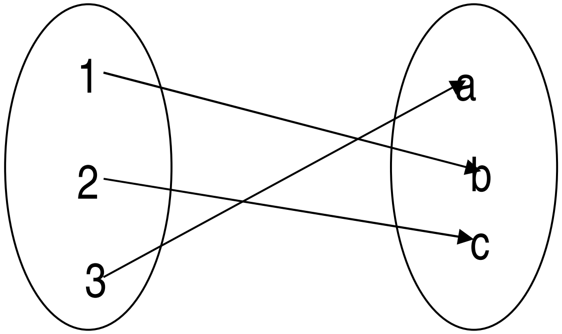
One-to-one functions

- A function f: $A \to B$ is **one-to-one**, or **injective** if no member of B is the image under f of two distinct elements of A.
- Let A = {1, 2, 3}
- Let B = {a, b, c, d}
- Let f = {(1, b), (2, c), (3, a)}
- The function f is one-to-one
- f: $Z \to Z$ with $f(x) = x^2$ is not one-to-one because $f(2) = f(-2) = 4$.



Bijections (one-to-one correspondences)

- A function $f: A \rightarrow B$ is **bijective** if f is both one-to-one and onto.
- Let $A = \{1, 2, 3\}$
- Let $B = \{a, b, c\}$
- Let $f = \{(1, b), (2, c), (3, a)\}$
- The function f is one-to-one
- $f: Z \rightarrow Z$ with $f(x) = x^2$ is not bijective because it is not one-to-one.



Composition of functions

- Let $f: A \rightarrow B$ and $g: B \rightarrow C$. Then the composition function , $g \circ f$, is a function from A to C defined by $(g \circ f)(a) = g(f(a))$.
- Note that the function f is applied first and then g.
- Let $f: R \rightarrow R$ be defined by $f(x) = x^2$.
- Let $g: R \rightarrow R$ be defined by $g(x) = \lfloor x \rfloor$.

$$(g \circ f)(2.3) = g(f(2.3)) = g((2.3)^2) = g(5.29)$$
$$= \lfloor 5.29 \rfloor = 5.$$

Inverse functions

- **Identity function**: the function that maps each element of a set A to itself, denoted by i_A. We have $i_A: A \rightarrow A$.
- Let f: $A \rightarrow B$. If there exists a function g: $B \rightarrow A$ such that $g \circ f = i_a$ and $f \circ g = i_b$, then g is called **the inverse function** of f, denoted by f^{-1}
- **Theorem**: Let f: $A \rightarrow B$. f is a bijection iff f^{-1} exists.
- Example:
  – f: $R \rightarrow R$ given by $f(x) = 3x+4$. $f^{-1} = (x - 4)/3$
  – $(f \circ f^{-1})(x) = 3(x-4)/3 + 4 = x$ identity function

Summary

- We have introduced many concepts,
  - Function
  - Domain, codomain
  - Image, preimpage
  - Range
  - Onto (surjective)
  - One-to-one (injective)
  - Bijection (one-to-one correspondence)
  - Function composition
  - Identity function
  - Inverse function