CS201: Data Structures and Discrete Mathematics I

Predicate Logic

Predicate Logic

- Propositional logic is rather limited in its expressive power.
- E.g., “For every x, x > 0” is true if x is a positive integer.
- We cannot say it in propositional logic.
- Nor can we show the following logical equivalences:

"Not all birds fly" is equivalent to "Some birds don't fly".

"Not all integers are even" is equivalent to "Some integers are not even".

"Not all cars are expensive" is equivalent to "Some cars are not expensive“.

Predicate and Quantifiers

- To deal with deficiencies of propositional logic.
- Two new features are added:
  - predicates and quantifiers
- Predicate: A **predicate** is a property about of some objects or a relationship among objects represented by the variables.
- Quantifier: Tell how many objects have a certain property.

Example predicates

- "The sky is blue”
- "The cover of this book is blue"

For the above two sentences, we can use is_Blue as the predicate. We can also simply use B.

- B(x), where x represents an arbitrary object.
- B(x) reads as "x is blue".

Another example

- "John gives the book to Mary",
- "Jim gives a loaf of bread to Tom", and
- "Jane give a lecture to Mary"

All can be expressed using this predicate:

give(x, y, z)

which reads, x gives y to z.

E.g., give(john, book, Mary), …

Quantification

- A predicate with variables is not a proposition.
- E.g., the statement $x > 1$ with variable x over the universe of real numbers is neither true nor false since we don't know what x is.
- It can be true or false depending on the value of x.
- For $x > 1$ to be a proposition either we substitute a specific number for x or change it to something like "There is a number x for which $x > 1$ holds", or "For every number x, $x > 1$ holds". We are using quantifiers …

From predicate to propositions

- A predicate with variables (called an *atomic formula*) can be made a proposition by applying one of the following two operations to each of its variables:
  – assign a value to the variable
  – quantify the variable using a quantifier

universal quantifier: “for every object *x* in the universe, *x* > 1” written as: $\forall x\ x > 1$

existential quantifier: "for some object *x* in the universe, *x* > 1“ written as: $\exists x\ x > 1$

Universe of Discourse

- The universe of discourse, also called universe (also called domain), is the set of objects of interest.
- Universal Quantifier ($\forall$): The expression: $\forall x\, P(x)$, denotes the universal quantification of $P(x)$.
  - In English: "*For all x, P(x)* holds" or "*for every x, P(x)* holds".
  - $P(x)$ is true for every object x in the universe
- Existential Quantifier($\exists$): The expression: $\exists x P(x)$, denotes the existential quantification of $P(x)$.
  - In English: "There exists an x such that $P(x)$" or "There is at least one x such that $P(x)$“.

    $\exists x$ means at least one object x in the universe.

Application of Quantifiers

- When more than one variables are quantified in a wff such as $\exists y \forall x P(x, y)$, they are applied from the inside
- I.e., the one closest to the atomic formula is applied first. Thus $\exists y \forall x P(x, y)$, reads $\exists y [\forall x P(x, y)]$
- Note that the positions of different types of quantifiers **cannot** be switched.
  - For example $\exists x \forall y P(x, y)$ is **not** equivalent to $\forall y \exists x P(x, y)$.

Examples for (1) $\exists x \forall y\, P(x, y)$ and (2) $\forall y \exists x\, P(x, y)$.

- Let P(X, Y) stands for “X likes Y”.

(1). There is a person who likes every person.

(2). For any person y, there is a person x who likes y.

- Let P(X, Y) stands for “X < Y”.

(1). There is a number x that is smaller than any number y.

(1). For any number, there is a smaller number

How to read quantified formulas

- Let F(x, y) stands for "***x*** flies faster than ***y***". let the universe be the set of airplanes.
- $\forall x \forall y\, F(x, y)$: "For every airplane ***x*** the following holds: ***x*** is faster than any airplane ***y***". Or simply: "Every airplane is faster than every airplane (including itself!)".
- $\forall x \exists y\, F(x, y)$: "For every airplane ***x*** the following holds: for some airplane ***y***, ***x*** is faster than ***y***". Or simply: "Every airplane is faster than some airplane".
- $\exists x \forall y\, F(x, y)$: "There exist an airplane ***x*** which satisfies the following: (or such that) for every airplane ***y***, ***x*** is faster than ***y***". Or simply: "some airplane is faster than every airplane".
- $\exists x \exists y\, F(x, y)$: "For some airplane ***x*** there exists an airplane ***y*** such that ***x*** is faster than ***y***". Or simply: "Some airplane is faster than some airplane".

Well-Formed Formula of Predicate Logic

Wffs are constructed using the following rules:

- *True* and *False* are wffs.
- Each propositional constant (i.e., specific proposition), and each propositional variable (i.e., a variable representing propositions) are wffs.
- Each atomic formula (i.e. a specific predicate with variables) is a wff.
- If A and B are wffs, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.
- If x is a variable (representing objects of the universe of discourse), and A is a wff, then so are $\exists x\, A$ and $\forall x\, A$.

Bound and free variables

- A variable in a wff is said to be **bound** if either a specific value is assigned to it or it is quantified.
- If an appearance of a variable is not bound, it is called **free**.
- The extent of the application (effect) of a quantifier, called the **scope** of the quantifier, is indicated by square brackets **[ ]**.
- If there are no square brackets, then the scope is understood to be the smallest wff following the quantification.
- For example, in $\exists x\, P(x, y)$,

  The variable x is bound while y is free.

  In $\forall x\, [\exists y\, P(x, y) \vee Q(x, y)]$,

  x and y in $P(x, y)$ are bound, while y in $Q(x, y)$ is free, because the scope of $\exists y$ is $P(x, y)$. The scope of $\forall x$ is $[\exists y\, P(x, y) \vee Q(x, y)]$.

From Wff to Proposition

- A wff is, in general, not a proposition.
- For example, consider the wff $\forall x\, P(x)$, where $P(x)$ means $x \geq 0$.
  - If the universe is {1, 2, 3, 4, 5, 6} or any subset of natural numbers, the wff is true.
  - But if the universe is {-2, -3, 5}, then it is not true.
- Also, $\forall x\, Q(x, y)$, where $Q(x, y)$ means $x > y$ for the universe $\{1, 3, 5\}$ may be true or false depending on the value of y.

- The specification of the universe and an assignment of a value to each free variable in a wff is called an **interpretation** for the wff.

Satisfiable

- A wff is said to be **satisfiable** if there exists an interpretation that makes it true,
  - I.e., if there are a universe and an assignment of values to its free variables that make the wff true.
- A wff is called **unsatisfiable**, if there is no interpretation that makes it true.
- E.g.: $\forall x\, P(x)$, where $P(x)$ means $x \geq 0$, is satisfiable, if our universe is {1, 2, 3, 4, 5, 6}.
- $\forall x\, [P(x) \wedge \neg P(x)]$ is not satisfiable.

Validity

- A wff is **valid** if it is true for every interpretation.
- E.g., the wff $\forall x\, P(x) \vee \exists x\, \neg P(x)$ is valid for any predicate name P, because $\exists x\, \neg P(x)$ is the negation of $\forall x\, P(x)$.
- Validity is analogue to tautology.
- E.g., the wff $\forall x\, P(x)$ is satisfiable but not valid.

Translate English to wffs

- Transcribing English sentences into wffs is sometimes a non-trivial task. We just give a few simple examples.
- E stands for even, and O stands for odd.
- "Some integers are even and some are odd" can be translated as
  $\exists x\, E(x) \wedge \exists x\, O(x)$ =? $\exists x\, E(x) \wedge \exists y\, O(y)$ =? $\exists x\, [E(x) \wedge O(y)]$
- "No integer is even"
  $\forall x\, \neg E(x)$
- "If an integer is not even, then it is odd"
  $\forall x\, [\neg E(x) \rightarrow O(x)]$
- We assume that the universe is the set of integers. However, if we do not assume that, then we need to narrow it down using $\wedge$, $\rightarrow$, and I (predicate for integer).
- E.g.,: $\exists x\, [I(x) \wedge E(x)] \wedge \exists x\, [I(x) \wedge O(x)]]$
  $\forall x\, [I(x) \rightarrow \neg E(x)]$ =? $\forall x\, [I(x) \wedge \neg E(x)]$

Reasoning with predicate logic

- Predicate logic is more powerful than propositional logic as it allows one to reason about properties and relationships of individual objects.
- In predicate logic, one can use some additional **inference rules**, as well as those for propositional logic.

Inference rules of predicate logic

- universal instantiation
- universal generalization
- existential instantiation
- existential generalization

Universal Instantiation

- $\forall x\, P(x)$

  $P(c)$
  where *c* is some arbitrary element of the universe.

- For example, the following argument can be proven correct using the Universal Instantiation: "No humans can fly. Tom is human. Therefore Tom cannot fly."

  The argument is
  $[\forall x\, [Human(x) \rightarrow \neg Fly(x)] \wedge Human(tom)\,] \rightarrow \neg Fly(tom)$.

  The proof is

  1. $\forall x\, [Human(x) \rightarrow \neg Fly(x)]$ Hypothesis
  2. $Human(tom)$ Hypothesis
  3. $Human(tom) \rightarrow \neg Fly(tom)$ Universal instantiation on 1.
  4. $\neg Fly(tom)$ Modus ponens on 2 and 3.

Universal Generalization

- P(c)

  $\forall x\, P(x)$
  where *P(c)* holds for every element c of the universe of discourse.
- For every number x if x > 1, then x - 1 > 0. Also for every number x, x > 1. We conclude that for every number x, x - 1 >0.

- Then the argument above is represented by (P(x): x > 1; Q(x): x - 1> 0)
  $[\forall x[P(x) \rightarrow Q(x)] \wedge \forall x\, P(x)] \rightarrow \forall x\, Q(x)$

- To prove it we proceed as follows:
  1. $\forall x\, [P(x) \rightarrow Q(x)]$ Hypothesis
  2. $\forall x\, P(x)$ Hypothesis
  3. $[P(x) \rightarrow Q(x)]$ Universal Instantiation on 1.
  4. P(x) for the same x as in 3. Universal Instantiation on 2.
  5. Q(x) Modus ponens on 3 and 4.
  6. $\forall x\, Q(x)$ Universal Generalization on 5

Existential Instantiation

$\forall$ $\exists x\, P(x)$

P(c)

where *c* is some element of the universe of discourse. It is not arbitrary but must be one for which P(c) is true.

- If you get 95 on the final exam for CS201, then you get an A for the course. Someone, say s, gets 95 on the final exam. Therefore, s gets an A for CS201.
- Let the universe be the set of all people in the world, let M(x) mean that x gets 95 on the final exam of CS201, and let A(x) represent that x gets an A for CS201.
- Then the proof proceeds as follows:

1. $\forall x\, [M(x) \rightarrow A(x)]$	Hypothesis
2. $\exists x\, M(x)$	Hypothesis
3. $M(s)$	Existential instantiation on 3.
4. $M(s) \rightarrow A(s)$	Universal instantiation on 1.
5. $A(s)$	Modus ponens on 3 and 4.

Existential Generalization

- $P(c)$

  $\exists x\, P(x)$
  where *c* is an element of the universe.

- "if everyone is happy then someone is happy"
- To prove it, first let the universe be the set of all people and let Happy(x) mean that x is happy.

  Then the argument is
  $\forall x\, Happy(x) \rightarrow \exists x\, Happy(x)$

- The proof is
  1. $\forall x\, Happy(x)$ Hypothesis
  2. $Happy(c)$ Universal instantiation
  3. $\exists x\, Happy(x)$ Existential generalization.