Deep Learning

Dr. Mehran Safayani

safayani@iut.ac.ir

safayani.iut.ac.ir

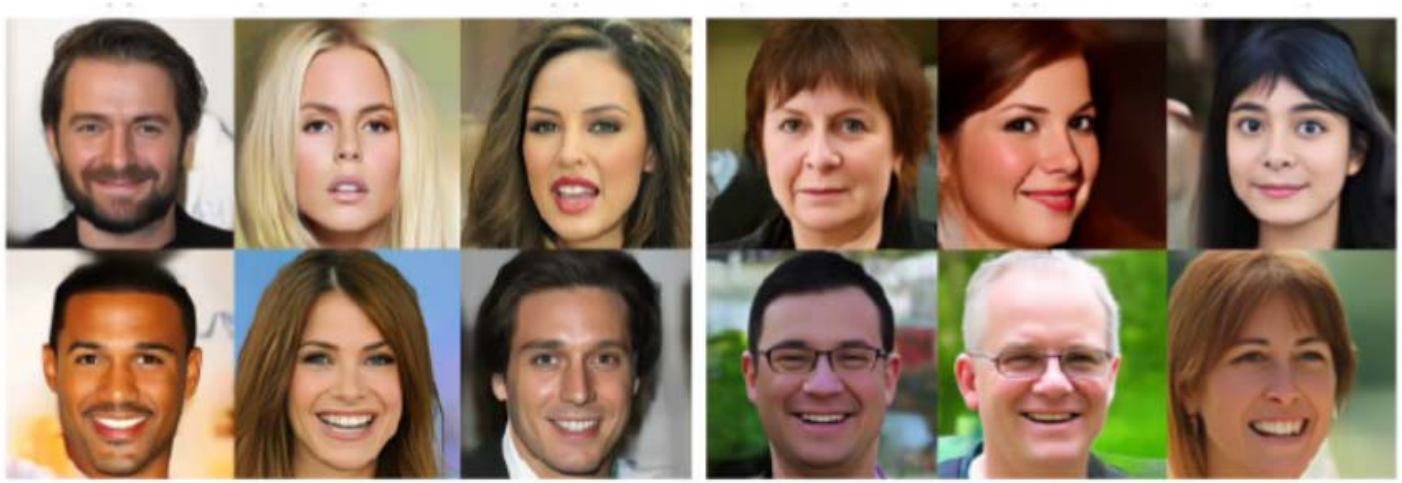
https://www.aparat.com/mehran.safayani

https://github.com/safayani/deep_learning_course

Department of Electrical and computer engineering, Isfahan university of technology, Isfahan, Iran

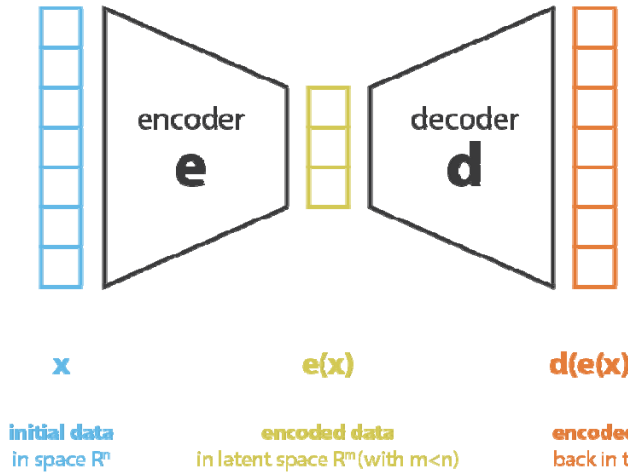
Variational Autoencoder

Face images generated with a Variational Autoencoder



Dimensionality reduction, PCA and autoencoders

Illustration of the dimensionality reduction principle with encoder and decoder.



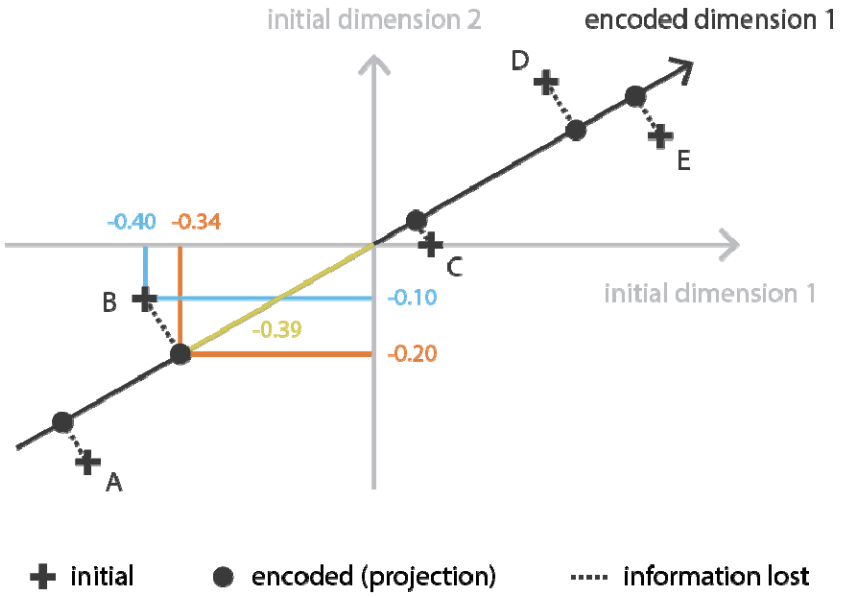
$x = d(e(x))$ ➡ **lossless encoding**
no information is lost when reducing the number of dimensions

$x \neq d(e(x))$ ➡ **lossy encoding**
some information is lost when reducing the number of dimensions and can't be recovered later

$$(e^*, d^*) = \underset{(e,d) \in E \times D}{\arg\min}\ \epsilon(x, d(e(x)))$$

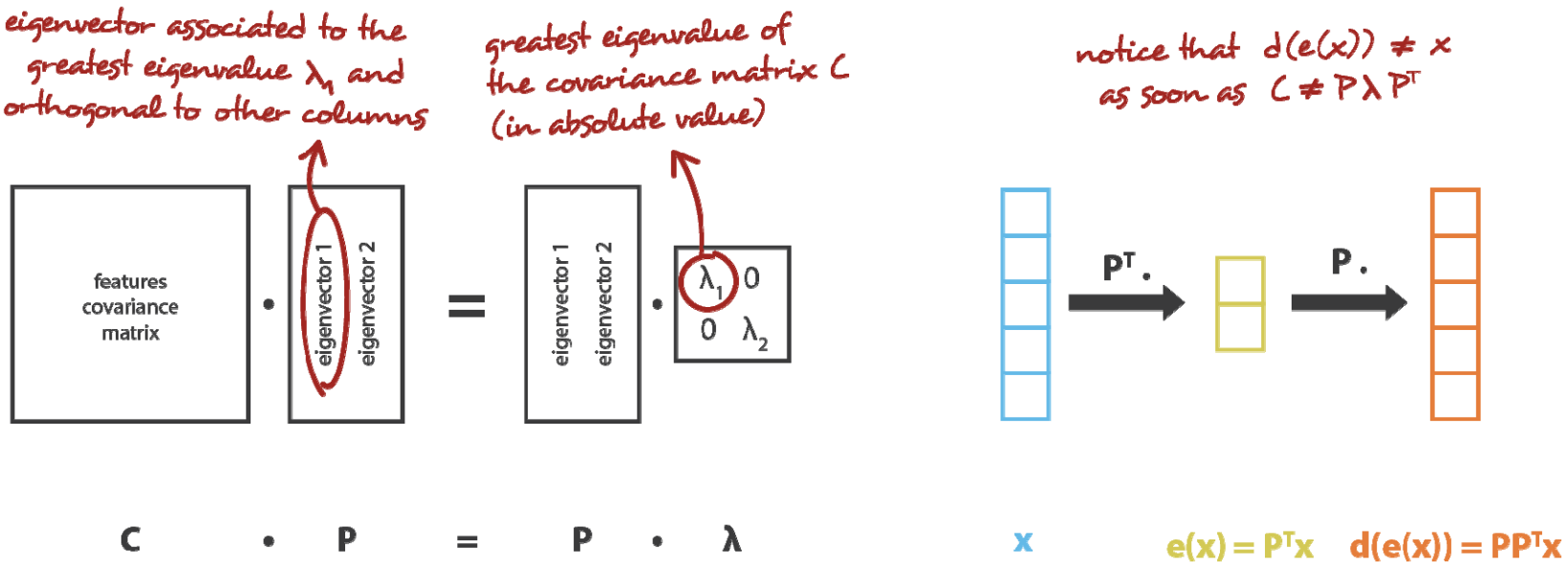
$$\epsilon(x, d(e(x)))$$

Principal components analysis (PCA)

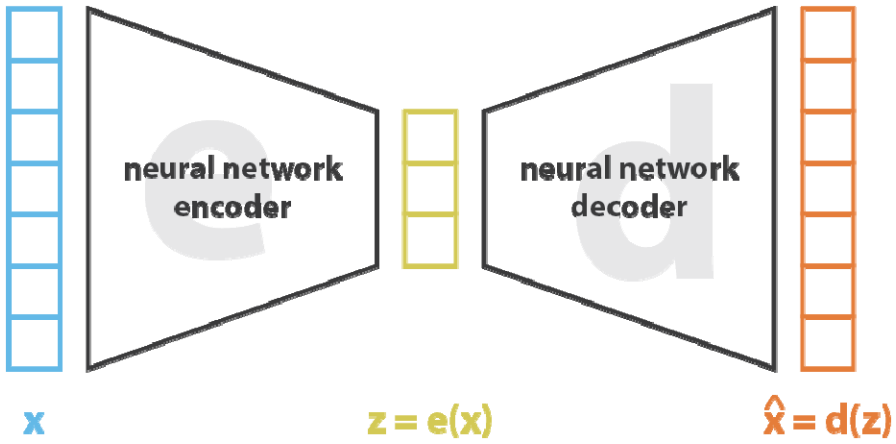


Point	Initial	Encoded	Decoded
A	(-0.50, -0.40)	-0.63	(-0.54, -0.33)
B	(-0.40, -0.10)	-0.39	(-0.34, -0.20)
C	(0.10, 0.00)	0.09	(0.07 0.04)
D	(0.30, 0.30)	0.41	(0.35, 0.21)
E	(0.50, 0.20)	0.53	(0.46, 0.27)

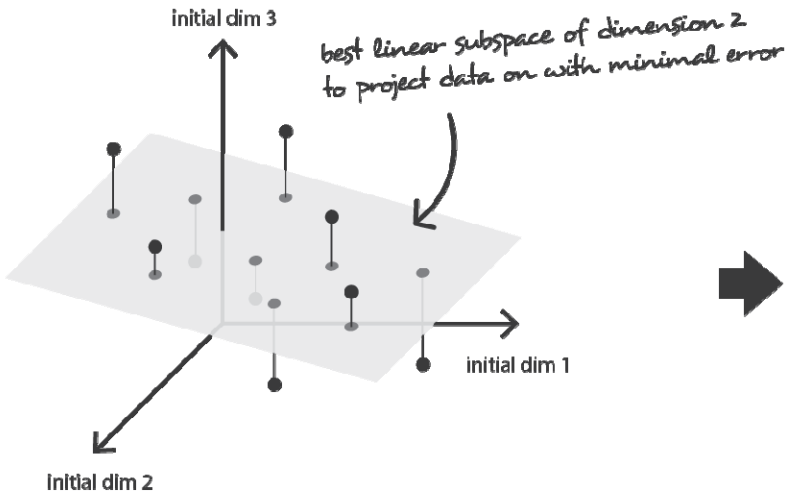
Principal components analysis (PCA)



Autoencoders

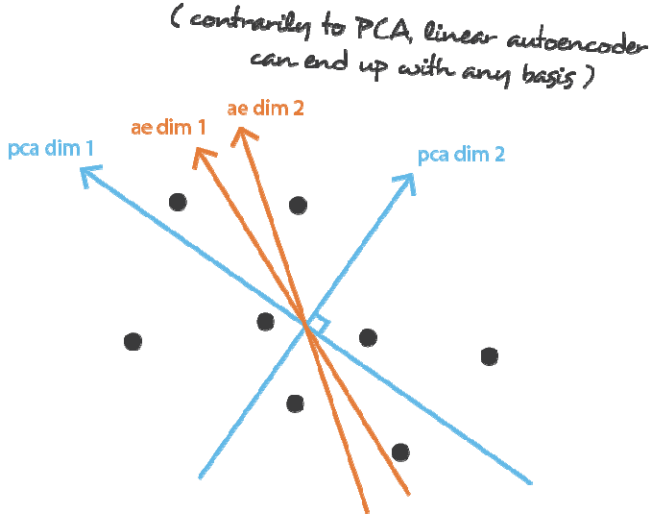


$$\text{loss} = \| x - \hat{x} \|^2 = \| x - d(z) \|^2 = \| x - d(e(x)) \|^2$$



Data in the full initial space

In order to reduce dimensionality, PCA and linear autoencoder target, in theory, the same optimal subspace to project data on...

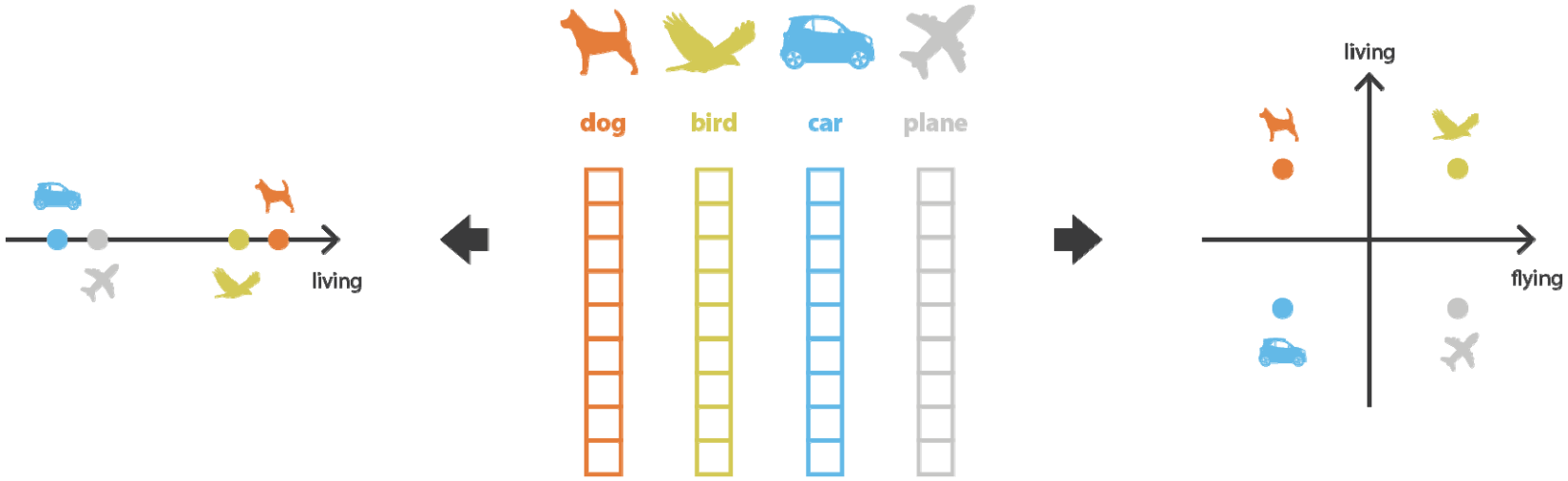


Data projected on the best linear subspace

... but not necessarily with the same basis due to different constraints (in PCA the first component is the one that explains the maximum of variance and components are orthogonal)

Autoencoder Limitations

- The lack of interpretable and exploitable structures in the latent space (**lack of regularity**)
- most of the time the final purpose of dimensionality reduction is not to only reduce the number of dimensions of the data but to reduce this number of dimensions **while keeping the major part of the data structure information in the reduced representations**.



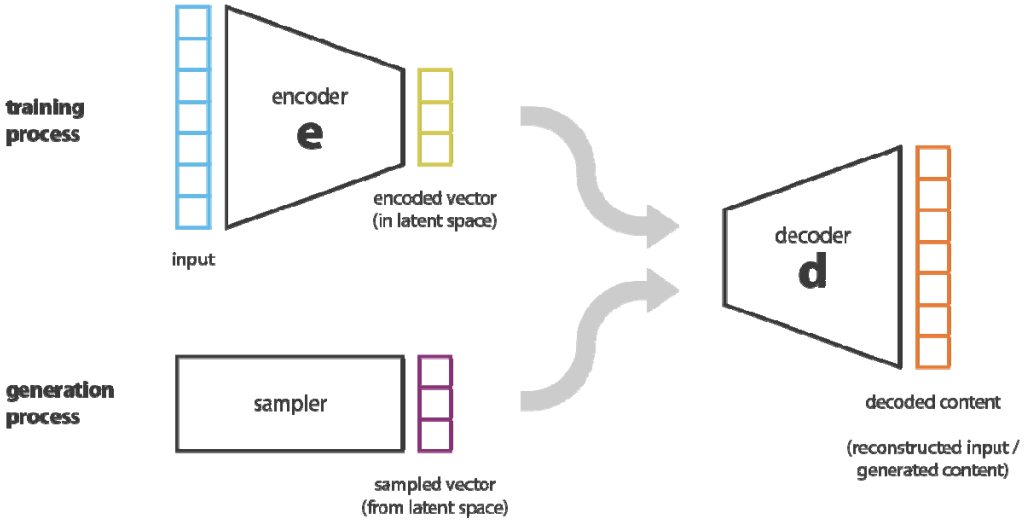
near optimal encoding
in one dimension
(too much information lost)

initial data with many features

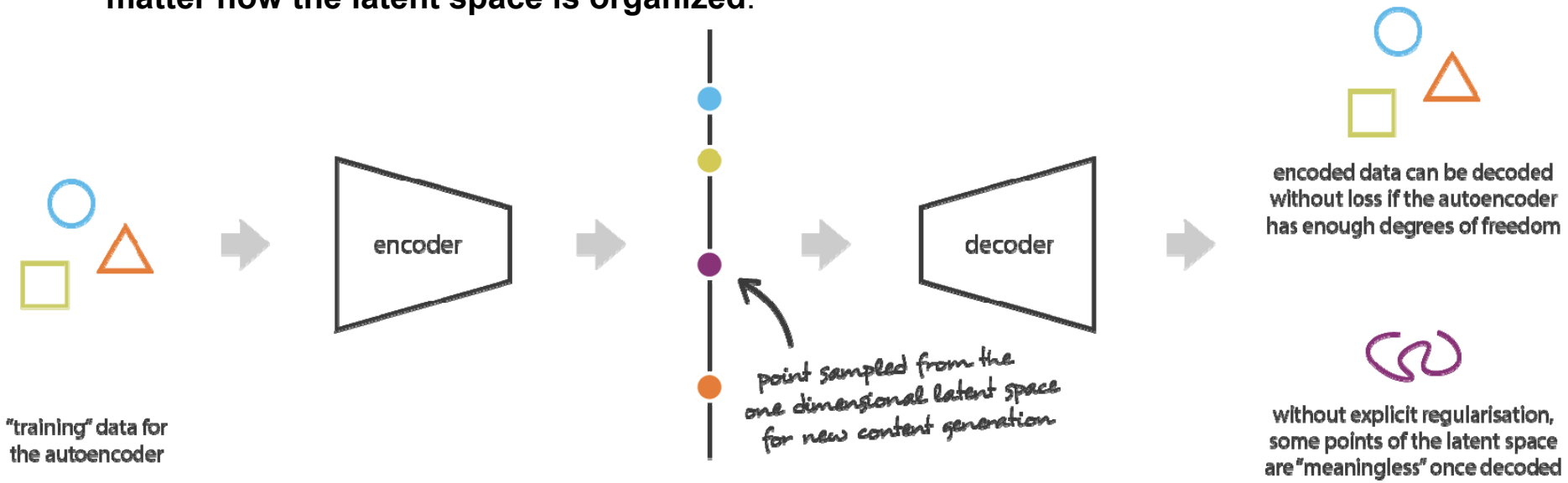
near optimal encoding
in two dimensions
(less information lost)

Variational Autoencoders

- **Limitations of autoencoders for content generation**



the autoencoder is solely trained to encode and decode with as few loss as possible, no matter how the latent space is organized.

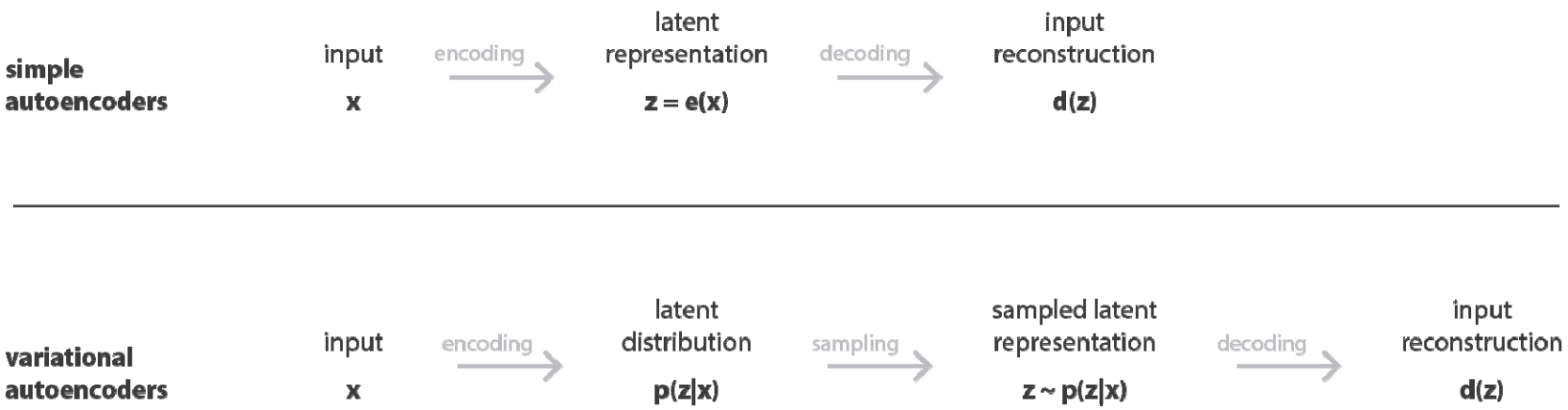


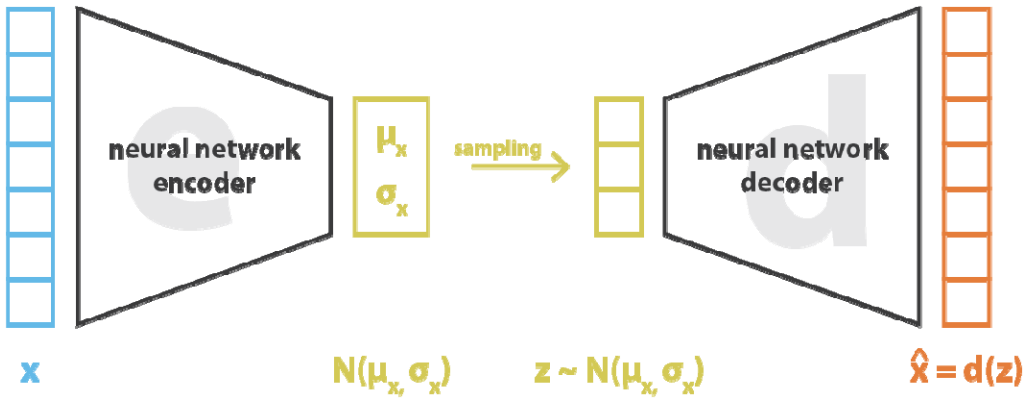
Definition of variational autoencoders

- **a variational autoencoder can be defined as being an autoencoder whose training is regularized to avoid overfitting and ensure that the latent space has good properties that enable generative process**
- **instead of encoding an input as a single point, we encode it as a distribution over the latent space**.

The model is then trained as follows:

- first, the input is encoded as distribution over the latent space
- second, a point from the latent space is sampled from that distribution
- third, the sampled point is decoded and the reconstruction error can be computed
- finally, the reconstruction error is back propagated through the network


simple autoencoders
input
x
encoding
latent representation
z = e(x)
decoding
input reconstruction
d(z)
variational autoencoders
input
x
encoding
latent distribution
p(z|x)
sampling
sampled latent representation
z ~ p(z|x)
decoding
input reconstruction
d(z)



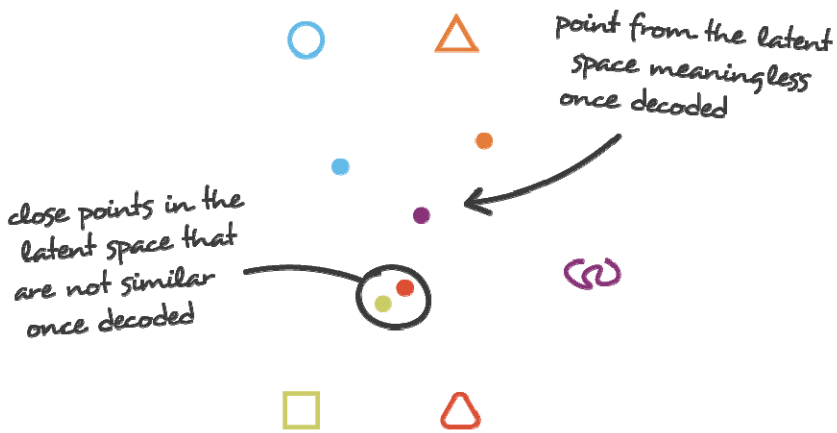
$$\text{loss} = \|x - \hat{x}\|^2 + KL[N(\mu_x, \sigma_x), N(0, I)] = \|x - d(z)\|^2 + KL[N(\mu_x, \sigma_x), N(0, I)]$$

Kullback-Leibler divergence

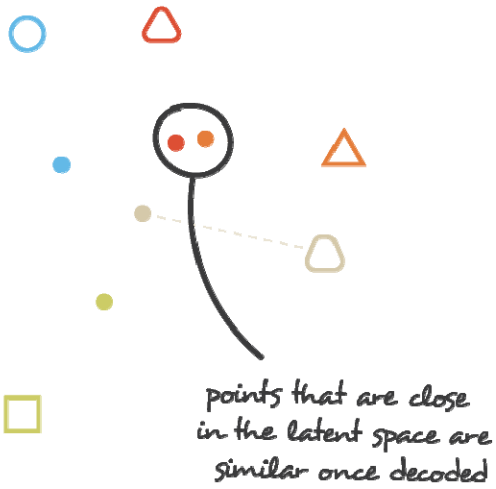
$$D_{\mathrm{KL}}(P \parallel Q) = \int_{-\infty}^{\infty} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

Intuitions about the regularization

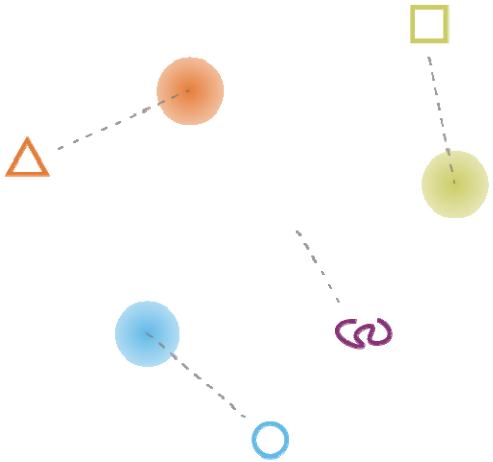
- The regularity that is expected from the latent space in order to make generative process possible can be expressed through two main properties: **continuity** (two close points in the latent space should not give two completely different contents once decoded)
  and **completeness** (for a chosen distribution, a point sampled from the latent space should give “meaningful” content once decoded).



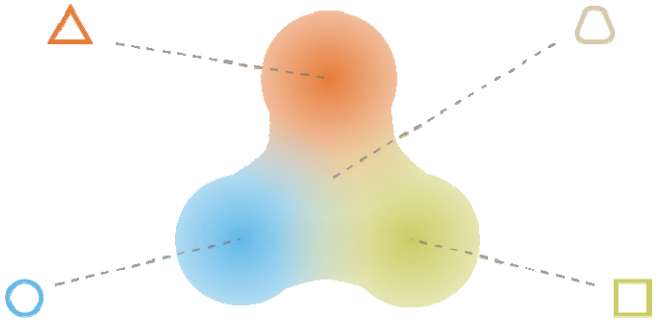
irregular latent space



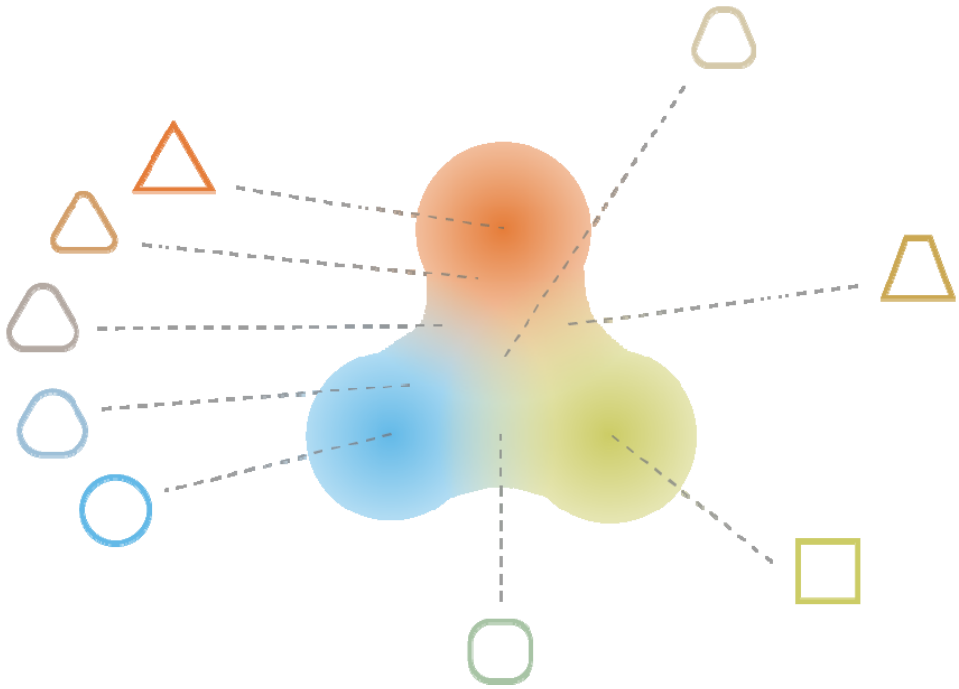
regular latent space



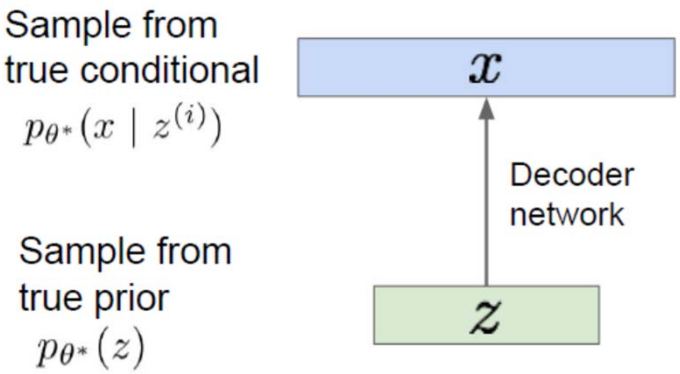
what can happen without regularisation



what we want to obtain with regularisation



How to train the model?



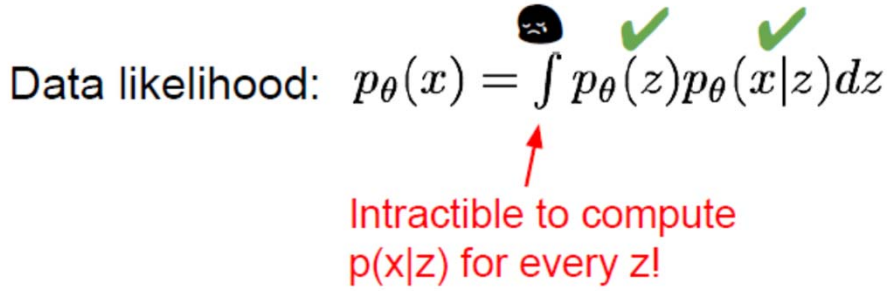
$$p_\theta(x) = \int p_\theta(z) p_\theta(x|z) dz$$

Q: What is the problem with this?

Intractable!

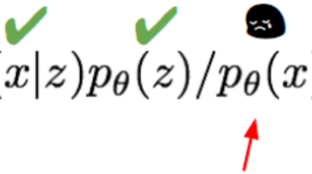
Intractability

Data likelihood: $p_\theta(x) = \int p_\theta(z) p_\theta(x|z) dz$



Intractability

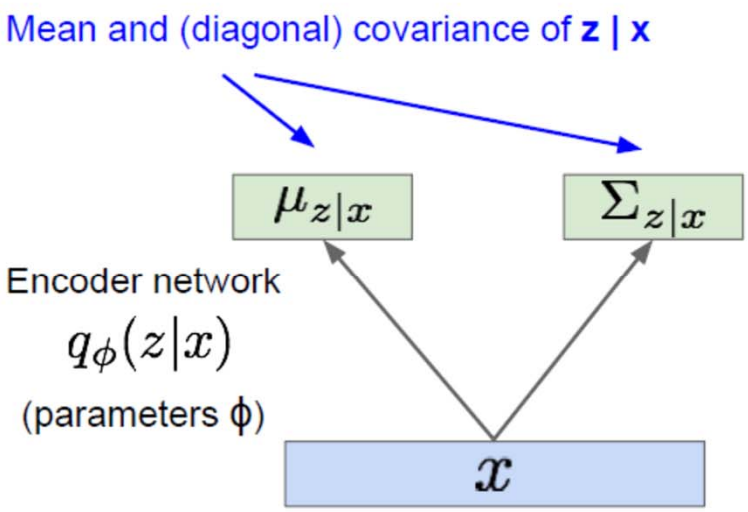
Posterior density also intractable: $p_\theta(z|x) = p_\theta(x|z)p_\theta(z)/p_\theta(x)$

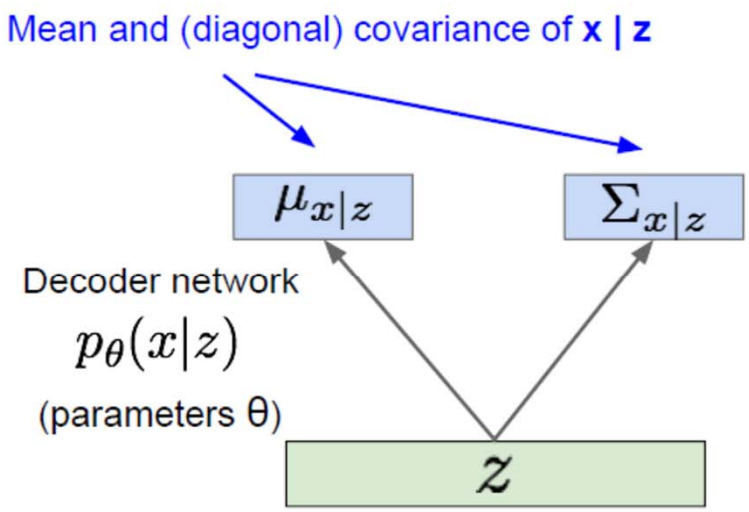


Solution: In addition to decoder network modeling $p_\theta(x|z)$, define additional encoder network $q_\phi(z|x)$ that approximates $p_\theta(z|x)$

Will see that this allows us to derive a lower bound on the data likelihood that is tractable, which we can optimize

Variational Autoencoder





$$\log p_\theta(x^{(i)}) = \mathbf{E}_{z\sim q_\phi(z|x^{(i)})}\left[\log p_\theta(x^{(i)})\right]$$

$$= \mathbf{E}_z\left[\log \frac{p_\theta(x^{(i)} \mid z)p_\theta(z)}{p_\theta(z \mid x^{(i)})}\right] \quad \text{(Bayes' Rule)}$$

$$= \mathbf{E}_z\left[\log \frac{p_\theta(x^{(i)} \mid z)p_\theta(z)}{p_\theta(z \mid x^{(i)})}\frac{q_\phi(z \mid x^{(i)})}{q_\phi(z \mid x^{(i)})}\right] \quad \text{(Multiply by constant)}$$

$$= \mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - \mathbf{E}_z\left[\log \frac{q_\phi(z \mid x^{(i)})}{p_\theta(z)}\right] + \mathbf{E}_z\left[\log \frac{q_\phi(z \mid x^{(i)})}{p_\theta(z \mid x^{(i)})}\right] \quad \text{(Logarithms)}$$

$$= \mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z)) + D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z \mid x^{(i)}))$$

$$\log p_\theta(x^{(i)}) = \mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z)) + D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z \mid x^{(i)}))$$

Decoder network gives $p_\theta(x|z)$, can compute estimate of this term through sampling. (Sampling differentiable through reparam. trick, see paper.)

This KL term (between Gaussians for encoder and z prior) has nice closed-form solution!

$p_\theta(z|x)$ intractable (saw earlier), can't compute this KL term :(But we know KL divergence always >= 0.

$$= \underbrace{\mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z))}_{\mathcal{L}(x^{(i)}, \theta, \phi)} + \underbrace{D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z \mid x^{(i)}))}_{\geq 0}$$

Tractable lower bound which we can take gradient of and optimize! ($p_\theta(x|z)$ differentiable, KL term differentiable)

$$\log p_\theta(x^{(i)}) \geq \mathcal{L}(x^{(i)}, \theta, \phi)$$

Variational lower bound ("ELBO")

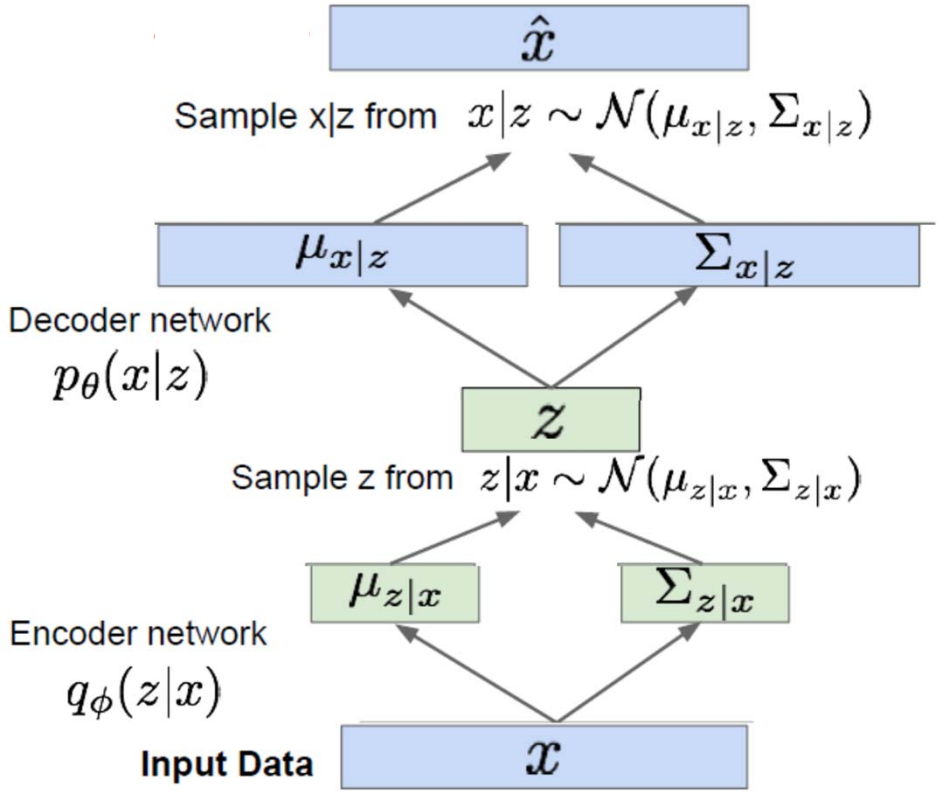
$$\theta^*, \phi^* = \arg\max_{\theta,\phi} \sum_{i=1}^{N} \mathcal{L}(x^{(i)}, \theta, \phi)$$

Training: Maximize lower bound

Putting it all together: maximizing the likelihood lower bound

$$\underbrace{\mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z))}_{\mathcal{L}(x^{(i)}, \theta, \phi)}$$

Let's look at computing the bound (forward pass) for a given minibatch of input data

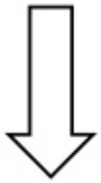


Objective Function

$$\tilde{L}(\theta, \phi, \mathbf{x}^{(i)}) = -D_{KL}[q_\phi(\mathbf{z}|\mathbf{x}^{(i)})||p_\theta(\mathbf{z})] + \frac{1}{L}\sum_{l=1}^{L}(\log p_\theta(\mathbf{x}^{(i)}|\mathbf{z}^{(i,l)})$$

$$q_\phi(\mathbf{z}|\mathbf{x}^{(i)}) = \mathcal{N}(\mathbf{z}, \mu^{(i)}, \sigma^{2(i)}\mathbf{I})$$

$$p_\theta(\mathbf{z}) = \mathcal{N}(\mathbf{z}, 0, \mathbf{I})$$



$$\tilde{L}(\theta, \phi, \mathbf{x}^{(i)}) = \frac{1}{2}\sum_{j=1}^{J}(1+\log((\sigma_j^{(i)})^2)-(\mu_j^{(i)})^2-(\sigma_j^{(i)})^2)+\frac{1}{L}\sum_{l=1}^{L}(\log p_\theta(\mathbf{x}^{(i)}|\mathbf{z}^{(i,l)})$$

Regularization

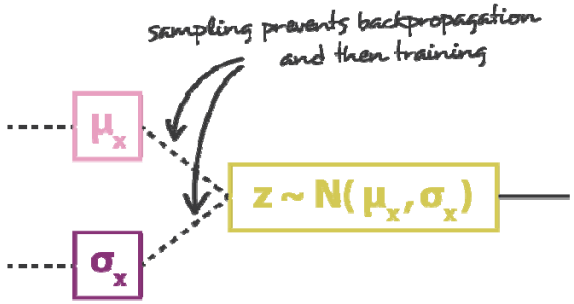
Reconstruction Error

reparametrization trick

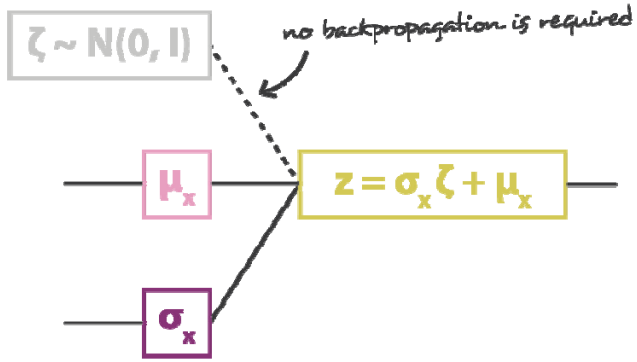
$$z = h(x)\zeta + g(x) \qquad \zeta \sim \mathcal{N}(0, I)$$

—— no problem for backpropagation

- - - - - backpropagation is not possible due to sampling



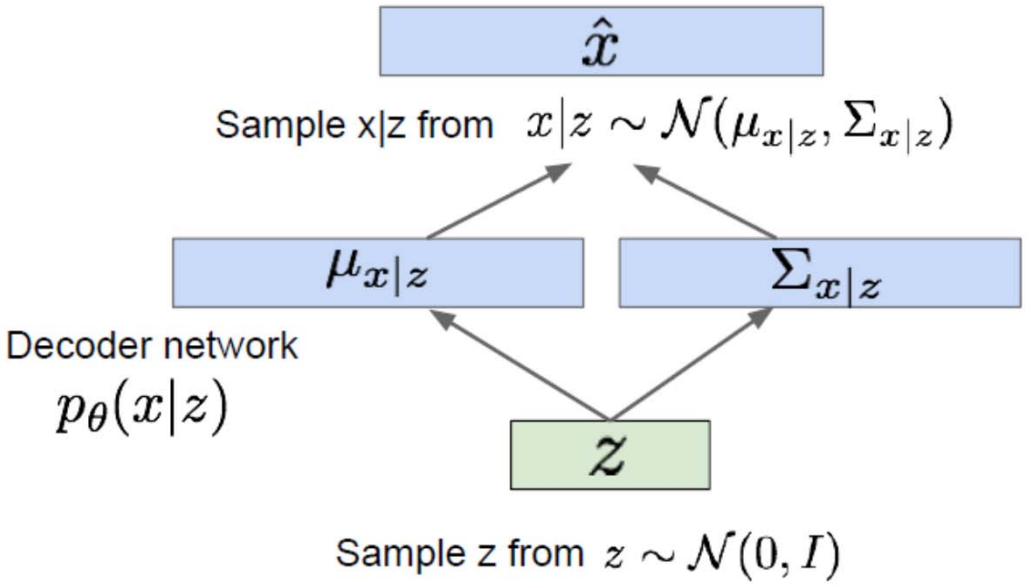
sampling without reparametrisation trick



sampling with reparametrisation trick

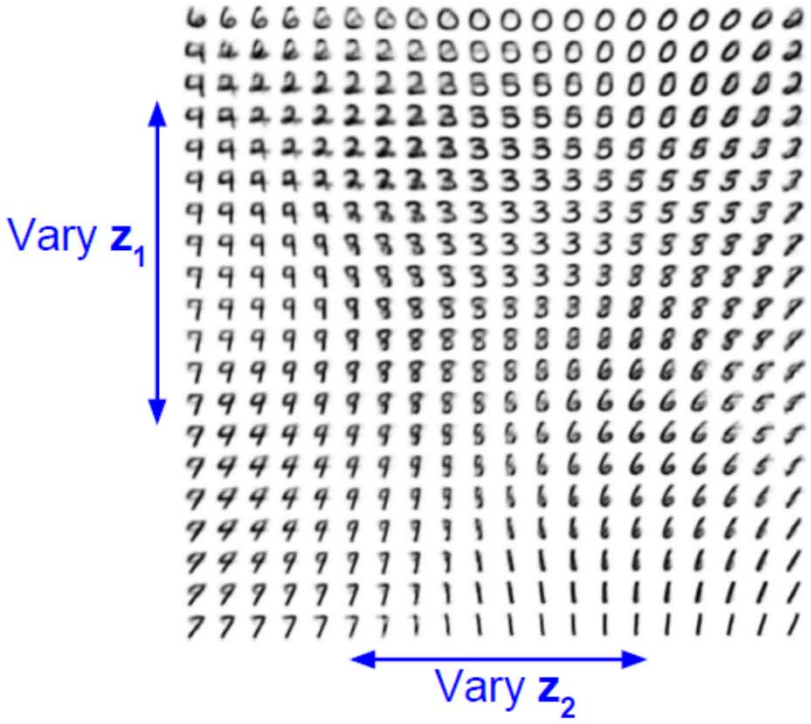
Generating data

Use decoder network. Now sample z from prior!



Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

Data manifold for 2-d **z**



Generating data

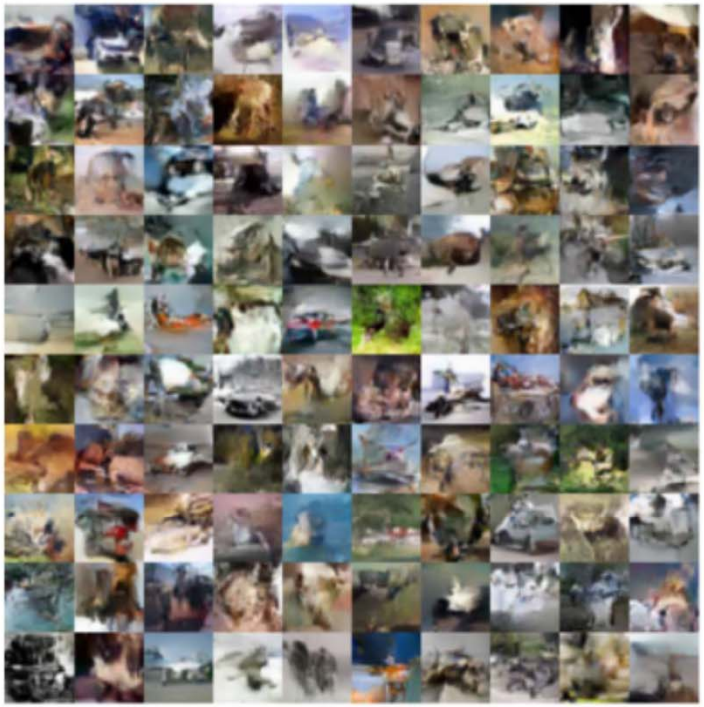
Diagonal prior on **z**
=> independent
latent variables

Different
dimensions of **z**
encode
interpretable factors
of variation

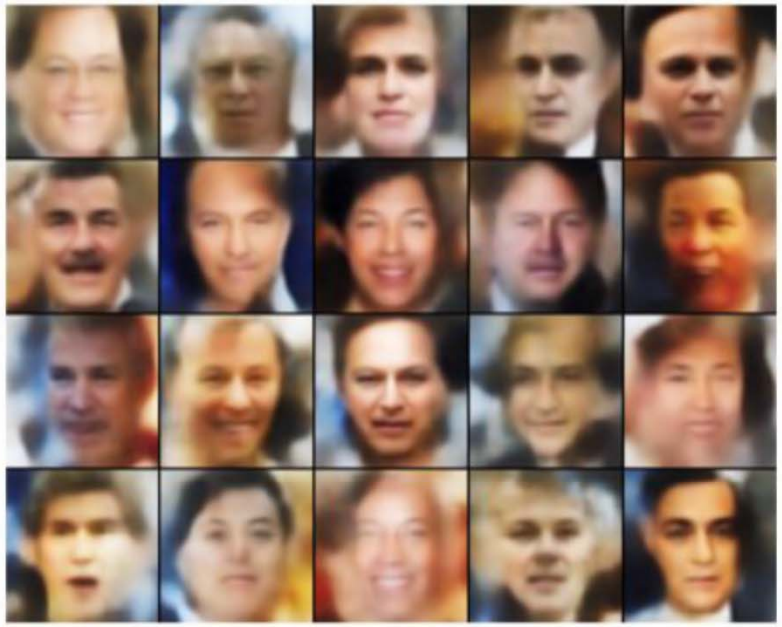
Also good feature representation that can be computed using $q_\phi(z|x)$!

Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

Generating data



32x32 CIFAR-10



Labeled Faces in the Wild

Variational Autoencoder

Probabilistic spin to traditional autoencoders => allows generating data
Defines an intractable density => derive and optimize a (variational) lower bound

Pros:

- Principled approach to generative models
- Allows inference of q(z|x), can be useful feature representation for other tasks

Cons:

- Samples blurrier and lower quality compared to state-of-the-art (GANs)

Active areas of research:

- More flexible approximations, e.g. richer approximate posterior instead of diagonal Gaussian
- Incorporating structure in latent variables

References

https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73

https://www.slideshare.net/ckmarkohchang/variational-autoencoder

Fei-Fei Li & Justin Johnson & Serena Yeung Convolutional Neural Networks for Visual Recognition Lecture 12, Generative models, May 15, 2018