



# Machine Learning

## MAP Estimation

Dr. Mehran Safayani

safayani@iut.ac.ir

safayani.iut.ac.ir



<https://www.aparat.com/mehran.safayani>



[https://github.com/safayani/machine\\_learning\\_course](https://github.com/safayani/machine_learning_course)



# Maximum a posteriori (MAP) Estimation

- MLE Recall:

In Maximum Likelihood Estimation (MLE), we used iid samples  $\mathbf{x} = (x_1, \dots, x_n)$  from some distribution with unknown parameter(s)  $\theta$ , in order to estimate  $\theta$ .

$$\hat{\theta}_{MLE} = \arg \max_{\theta} L(\mathbf{x} \mid \theta) = \arg \max_{\theta} \prod_{i=1}^n f_X(x_i \mid \theta)$$

You might have been thinking: shouldn't we be trying to maximize " $\mathbb{P}(\theta \mid x)$ " instead? Well, this doesn't make sense **unless  $\Theta$  is a R.V.!** And this is where Maximum A Posteriori (MAP) Estimation comes in.

- Now, we are in the Bayesian framework, meaning that our unknown parameter is a random variable  $\theta$
- This means, we will have some belief distribution  $p(\theta)$ (think of this as a density function over all possible values of the parameter), and after observing data  $x$ , we will have a new/updated belief distribution  $p(\theta|x)$ .
- Using Bayes theorem:

# Bayes theorem

$$\underbrace{P(\theta|X)}_{\text{a posterior probability}} = \frac{\overbrace{P(X|\theta)}^{\text{Likelihood}} \overbrace{P(\theta)}^{\text{a prior probability}}}{\underbrace{P(X)}_{\text{Marginal probability}}} \approx P(X|\theta)P(\theta)$$

# MAP estimation Example

- We have observed data  $X = \{x_1, x_2, \dots, x_n\}$  and we want to estimate the mean  $\mu$  of a normal distribution. We assume a normal prior on  $\mu$  with mean  $\mu_0$  and variance  $\tau^2$ , and we assume the data  $X$  is drawn from a normal distribution with mean  $\mu$  and known variance  $\sigma^2$ .

$$P(X|\mu) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$P(\mu) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(\mu - \mu_0)^2}{2\tau^2}\right)$$

$$P(\mu|X) \propto P(X|\mu) \cdot P(\mu)$$

# MAP estimation

$$P(\mu|X) \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right) \cdot \exp\left(-\frac{(\mu - \mu_0)^2}{2\tau^2}\right)$$

$$\ln P(\mu|X) = -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{(\mu - \mu_0)^2}{2\tau^2} + \text{constant}$$

- Set the derivative to zero

$$\frac{d}{d\mu} \ln P(\mu | X) = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) + \frac{\mu_0 - \mu}{\tau^2} = 0$$

# MAP estimation

$$\sum_{i=1}^n x_i - n\mu + \frac{\sigma^2}{\tau^2}(\mu_0 - \mu) = 0$$

$$n\mu + \frac{\sigma^2}{\tau^2}\mu = \sum_{i=1}^n x_i + \frac{\sigma^2}{\tau^2}\mu_0$$

$$\mu\left(n + \frac{\sigma^2}{\tau^2}\right) = \sum_{i=1}^n x_i + \frac{\sigma^2}{\tau^2}\mu_0$$

$$\mu = \frac{\sum_{i=1}^n x_i + \frac{\sigma^2}{\tau^2}\mu_0}{n + \frac{\sigma^2}{\tau^2}}$$

# Regularization

$L_2$ \_regularization:

$$\text{Min}_{\theta} \frac{1}{2N} \sum_{n=1}^N [y_n - x_n^T \theta]^2 + \lambda \|\theta\|_2$$

Prove the solution for w:

$$\begin{aligned} \nabla L(\theta) &= \frac{-1}{N} x^T (y - x\theta) \\ &+ \\ \nabla \Omega(\theta) &= 2 \lambda \theta \end{aligned}$$

$$\theta_{ridge}^* = (x^T x + \lambda' I)^{-1} x^T y, \quad \frac{\lambda'}{2N} = \lambda$$

نکته: ماتریس  $x^T x + \lambda' I$  معکوس پذیر است.



# Regression as MLE estimator: Recall

$$\boldsymbol{\theta}_{\text{lse}} \stackrel{(a)}{=} \arg \min_{\boldsymbol{\theta}} -\log p(\mathbf{y}, \mathbf{X} \mid \boldsymbol{\theta})$$

$$\stackrel{(b)}{=} \arg \min_{\boldsymbol{\theta}} -\log p(\mathbf{X} \mid \mathbf{w})p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\theta})$$

$$\stackrel{(c)}{=} \arg \min_{\boldsymbol{\theta}} -\log p(\mathbf{X})p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\theta})$$

$$\stackrel{(d)}{=} \arg \min_{\boldsymbol{\theta}} -\log p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\theta})$$

$$\stackrel{(e)}{=} \arg \min_{\boldsymbol{\theta}} -\log \left[ \prod_{n=1}^N p(y_n \mid \mathbf{x}_n, \boldsymbol{\theta}) \right]$$

# Regression as MLE estimator: Recall

$$\begin{aligned} &\stackrel{(f)}{=} \arg \min_{\boldsymbol{\theta}} -\log \left[ \prod_{n=1}^N \mathcal{N}(y_n \mid \mathbf{x}_n^{\top} \boldsymbol{\theta}, \sigma^2) \right] \\ &= \arg \min_{\boldsymbol{\theta}} -\log \left[ \prod_{n=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_n - \mathbf{x}_n^{\top} \boldsymbol{\theta})^2} \right] \\ &= \arg \min_{\boldsymbol{\theta}} -N \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right) + \sum_{n=1}^N \frac{1}{2\sigma^2} (y_n - \mathbf{x}_n^{\top} \boldsymbol{\theta})^2 \\ &= \arg \min_{\boldsymbol{\theta}} \frac{1}{2\sigma^2} \sum_{n=1}^N (y_n - \mathbf{x}_n^{\top} \boldsymbol{\theta})^2 \end{aligned}$$

## Ridge Regression as MAP estimator

فرض می کنیم که اجزای  $\mathbf{w}$  دارای توزیع گاوسین مستقل و یکنواخت با میانگین صفر و واریانس  $\frac{1}{\lambda}$  است.

$$\theta_{\text{ridge}} = \arg \min_{\theta} -\log p(\theta \mid \mathbf{X}, \mathbf{y})$$

$$\stackrel{(a)}{=} \arg \min_{\theta} -\log \frac{p(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta)}{p(\mathbf{y}, \mathbf{X})}$$

$$\stackrel{(b)}{=} \arg \min_{\theta} -\log p(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta)$$

$$\stackrel{(c)}{=} \arg \min_{\theta} -\log p(\mathbf{y} \mid \mathbf{X}, \theta) p(\theta) \quad \bullet$$

# Ridge Regression as MAP estimator

$$\begin{aligned} &= \arg \min_{\boldsymbol{\theta}} -\log[p(\boldsymbol{\theta}) \prod_{n=1}^N p(y_n \mid \mathbf{x}_n, \boldsymbol{\theta})] \\ \bullet &= \arg \min_{\boldsymbol{\theta}} -\log \left[ \mathcal{N} \left( \boldsymbol{\theta} \mid 0, \frac{1}{\lambda} \mathbf{I} \right) \prod_{n=1}^N \mathcal{N}(y_n \mid \mathbf{x}_n^\top \boldsymbol{\theta}, \sigma^2) \right] \\ &= \arg \min_{\boldsymbol{\theta}} -\log \left[ \frac{1}{\left(2\pi\frac{1}{\lambda}\right)^{D/2}} e^{-\frac{\lambda}{2}\|\boldsymbol{\theta}\|^2} \prod_{n=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_n - \mathbf{x}_n^\top \boldsymbol{\theta})^2} \right] \\ &= \arg \min_{\boldsymbol{\theta}} \sum_{n=1}^N \frac{1}{2\sigma^2} (y_n - \mathbf{x}_n^\top \boldsymbol{\theta})^2 + \frac{\lambda}{2} \|\boldsymbol{\theta}\|^2. \end{aligned}$$