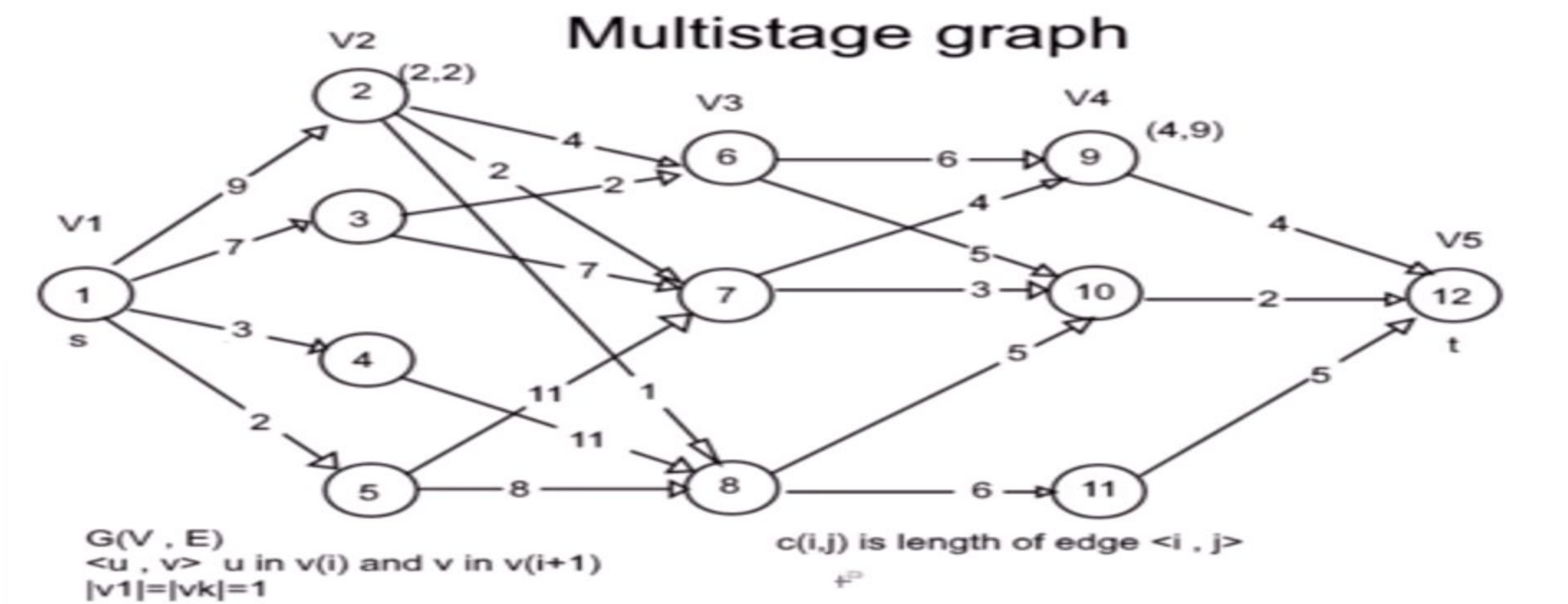
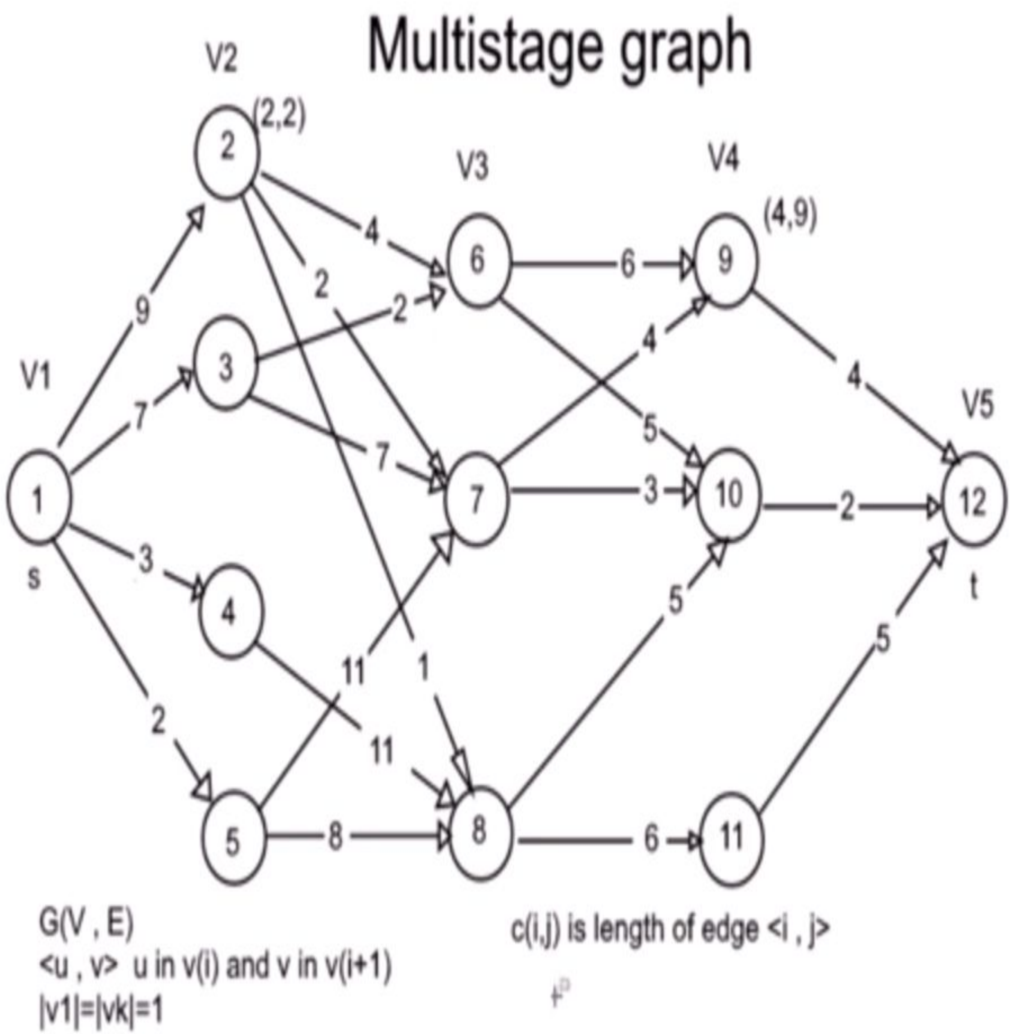
Finding Shortest Path in Multistage Graph using Dynamic Programming

Multistage Graph

- To find the shortest path between the source vertex s and the destination vertex t.
- A multistage graph is a directed graph which is divided into stagesV1, V2, ….
- Vertices from one stage are connected to vertices of next stage (no edges between vertices of the same stage and from a vertex of current stage to previous stage).
- The first and the last stage have single vertex.



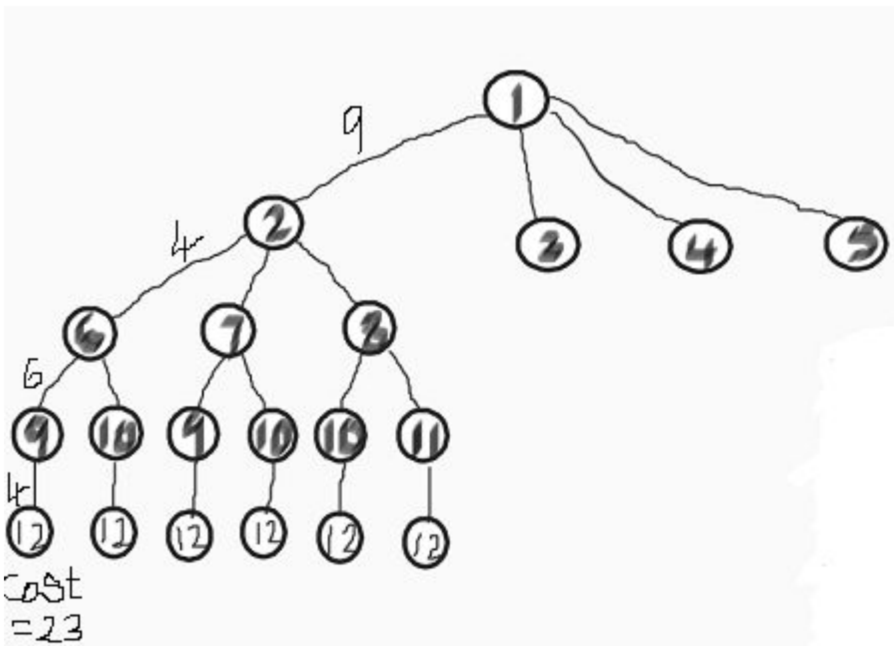
Applying Greedy approach to solve

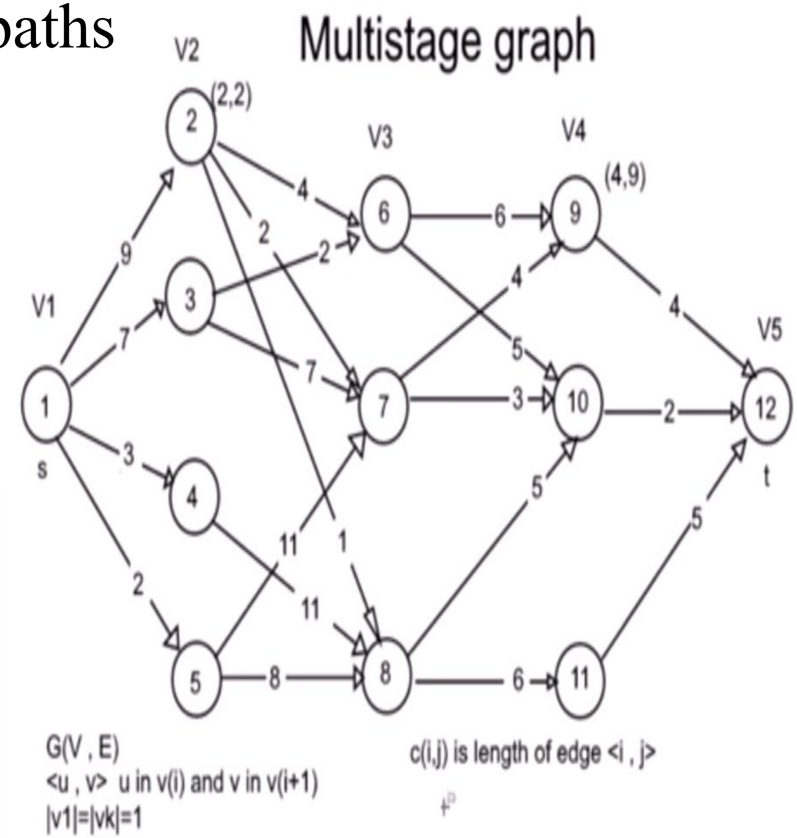


- Greedy Choice 1:
- Edge:(1,5) (5,8) (8,10) (10,12)
- Cost: 2 + 8 + 5 + 2 =17
- Choice 2:
- Edge:(1,2) (2,7) (7,10) (10,12)
- Cost: 9 + 2 + 3 + 2 =16
- Greedy choice fails

Applying Brute force to solve

Brute Force: Enumerate all possible paths
And find the minimum cost path.

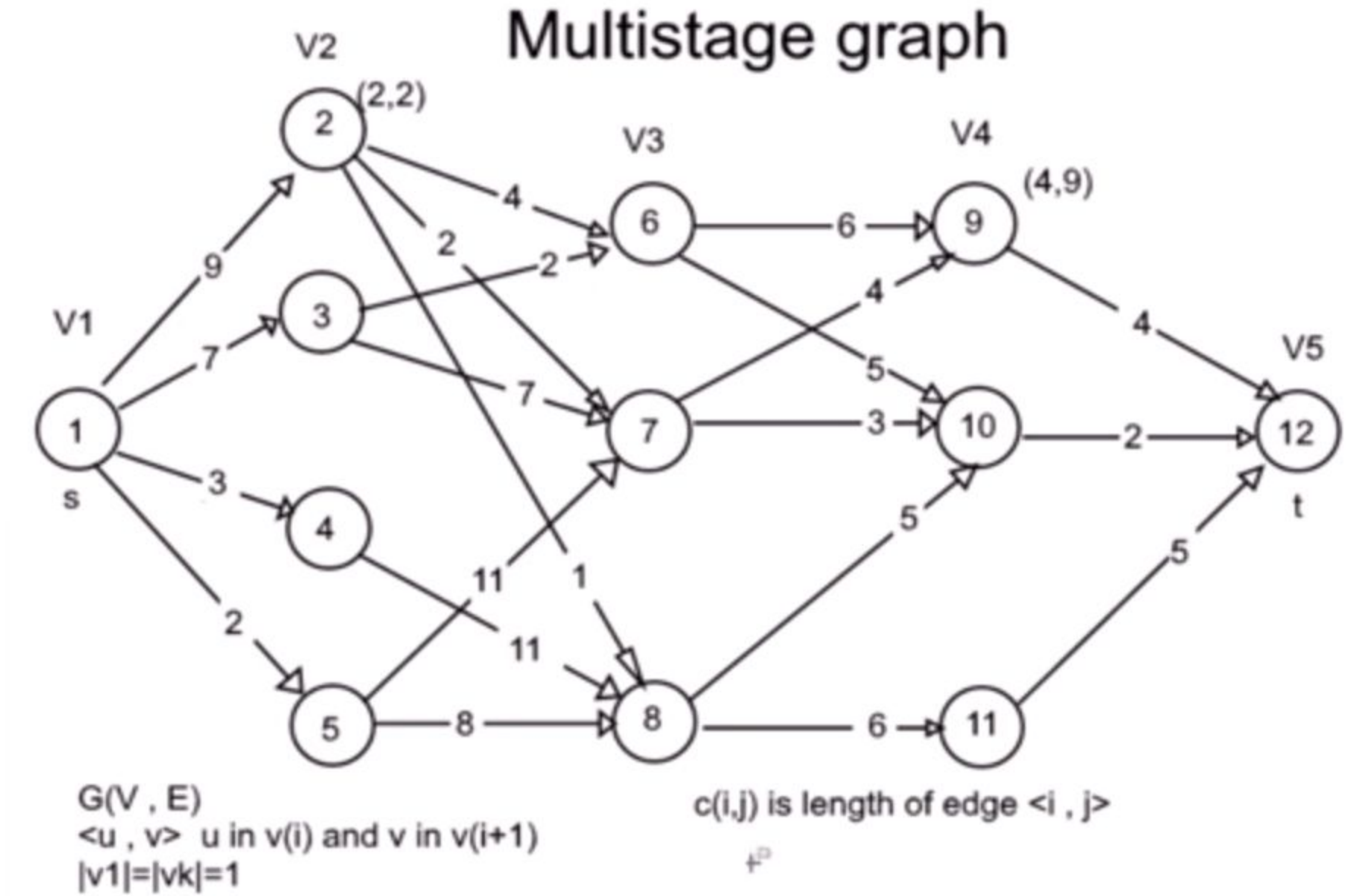




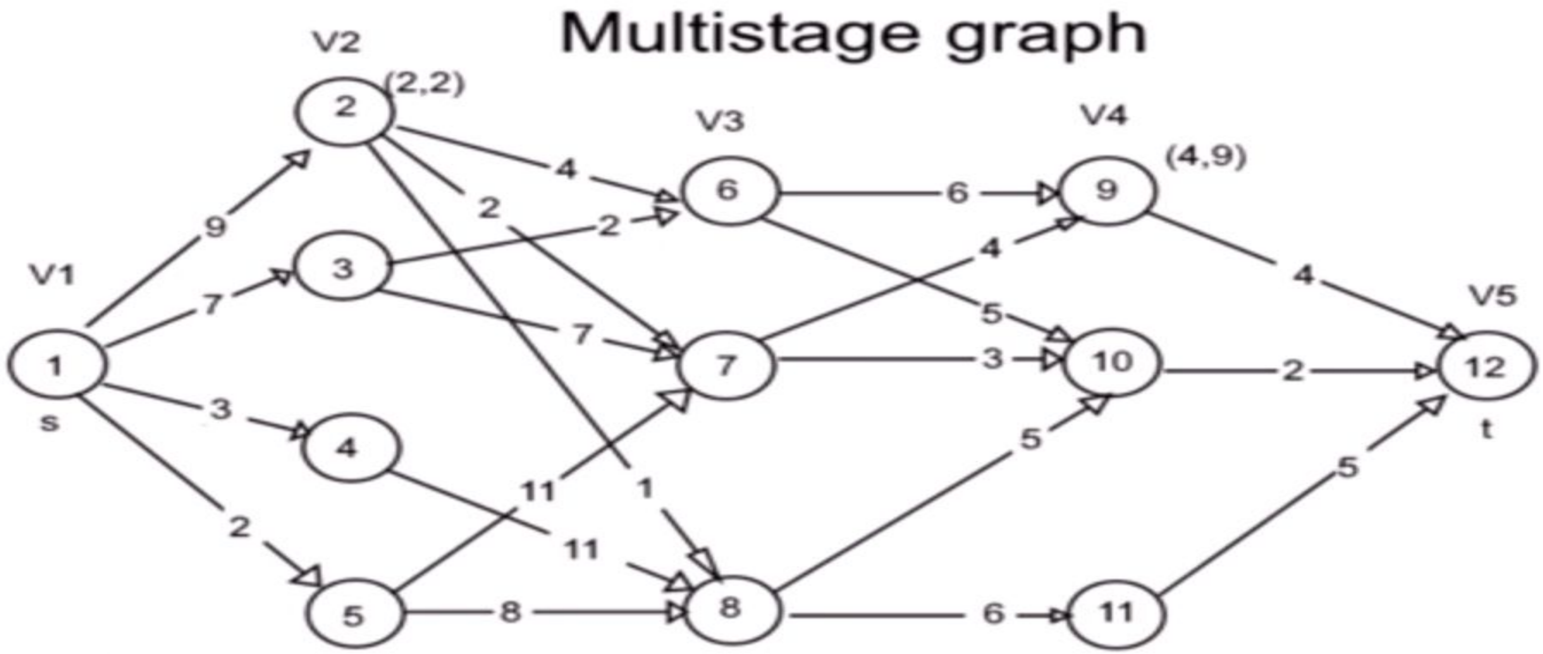
Solving using Dynamic Programming

- Forward approach
- Backward approach

Solving using Dynamic Programming



Solving using Dynamic Programming



G(V , E)
<u , v> u in v(i) and v in v(i+1)
|v1|=|vk|=1

c(i,j) is length of edge <i , j>

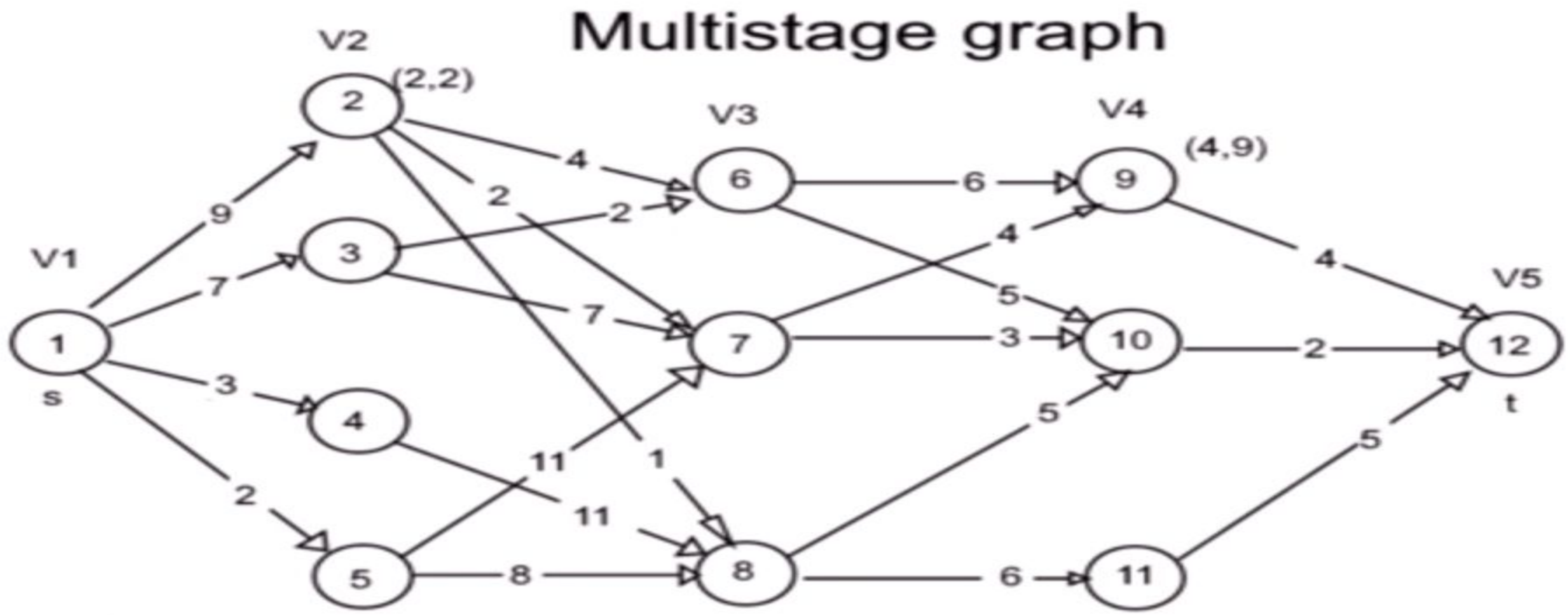
cost(i , j) = min{ c(j , l) + cost(i+1, l) }

stage

node

legth of edge <j , l>
if there is no edge infinity

Solving using Dynamic Programming: Forward approach



path : 1 -> 2 -> 7 -> 10 -> 12

1-> 3 -> 6 ->10 -> 12

cost(4,9) = c(9,12) = 4

cost(4,10) = c(10,12) = 2

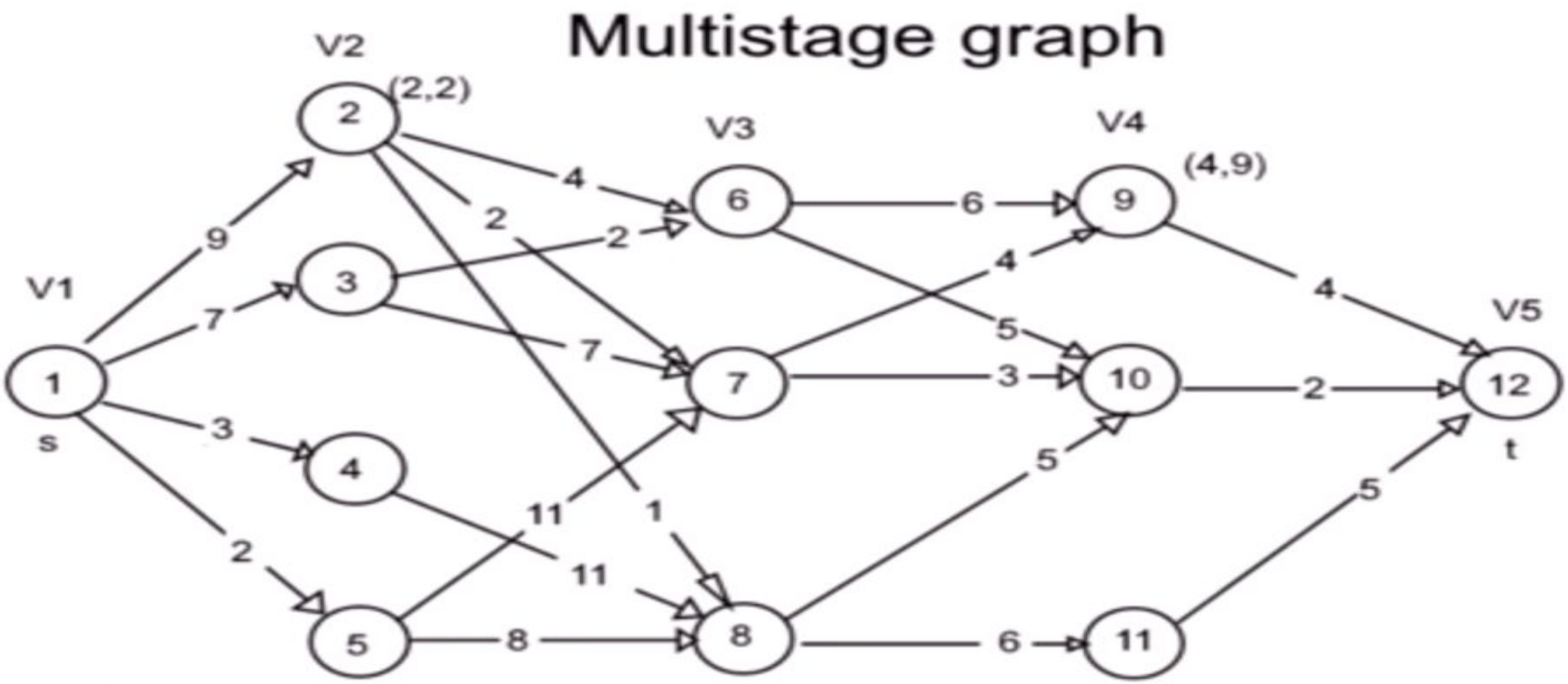
cost(4,11) = c(11,12) = 5

cost(3,6) = min {c(6,9) + cost(4,9),c(6,10) + cost(4,10)} = min { 6+ 4, 5+2} = 7

cost(3,7) = min{c(7,9)+cost(4,9), c(7,10) + cost(4,10)} = min{4 + 4, 3 + 2} = 5

cost(3,8) = min{c(8,10)+ cost(4,10), c(8,11)+cost(4,11)} = min{5+2, 6+5}=7

Solving using Dynamic Programming: Forward approach



$cost(2,2) = \min\{c(2,6)+cost(3,6)\ ,\ c(2,7)+cost(3,7),\ c(2,8)+cost(3,8)\} = \min\{4+7, 2+5, 1+7\} = 7$

$cost(2,3) = \min\{c(3,6)+cost(3,6),\ c(3,7)+cost(3,7)\} = \min\{2+7,\ 7+5\} = 9$

$cost(2,4) = \min\{c(4,8)+cost(3,8)\} = 11+7 = 18$

$cost(2,5) = \min\{c(5,7)+cost(3,7),\ c(5,8)+cost(3,8)\} = \min\{11+5, 8+7\} = 15$

cost(4,9) = c(9,12) = 4
cost(4,10) = c(10,12) = 2
cost(4,11) = c(11,12) = 5

path : 1 -> 2 -> 7 -> 10 -> 12
1-> 3 -> 6 ->10 -> 12

cost(3,6) = min {c(6,9) + cost(4,9),c(6,10) + cost(4,10)} = min { 6+ 4, 5+2} = 7
cost(3,7) = min{c(7,9)+cost(4,9), c(7,10) + cost(4,10)} = min{4 + 4, 3 + 2} = 5
cost(3,8) = min{c(8,10)+ cost(4,10), c(8,11)+cost(4,11)} = min{5+2, 6+5}=7

cost(2,2) =min{c(2,6)+cost(3,6) , c(2,7) +cost(3,7), c(2,8)+cost(3,8)} = min{4+7,2+5,1+7} = 7
cost(2,3) = min{c(3,6)+cost(3,6), c(3,7) +cost(3,7)} = min{2+7, 7+5} = 9
cost(2,4)= min{c(4,8) +cost(3,8)} = 11+7 = 18
cost(2,5) = min{c(5,7)+cost(3,7), c(5,8) +cost(3,8)} = min{11+5,8+7} =15

vertex	1	2	3	4	5	6	7	8	9	10	11	12
Cost	**16**	**7**	**9**	**18**	**15**	**7**	**5**	**7**	**4**	**2**	**5**	**0**
d-destination	**2/3**	**7**	**6**	**8**	**8**	**10**	**10**	**10**	**12**	**12**	**12**	**12**

Solving using Dynamic Programming: Forward approach

$$cost(1,1) = \min\{ c(1,2) + cost(2,2)\ ,\ c(1,3) + cost(2,3)\ ,\ c(1,4) + cost(2,4),\ c(1,5) + cost(2,5)\}$$

$$= \min\{9 + 7\ ,7 + 9,\ 3 + 18,\ 2 + 15\} = 16 \qquad d(1,1) = 2,3$$

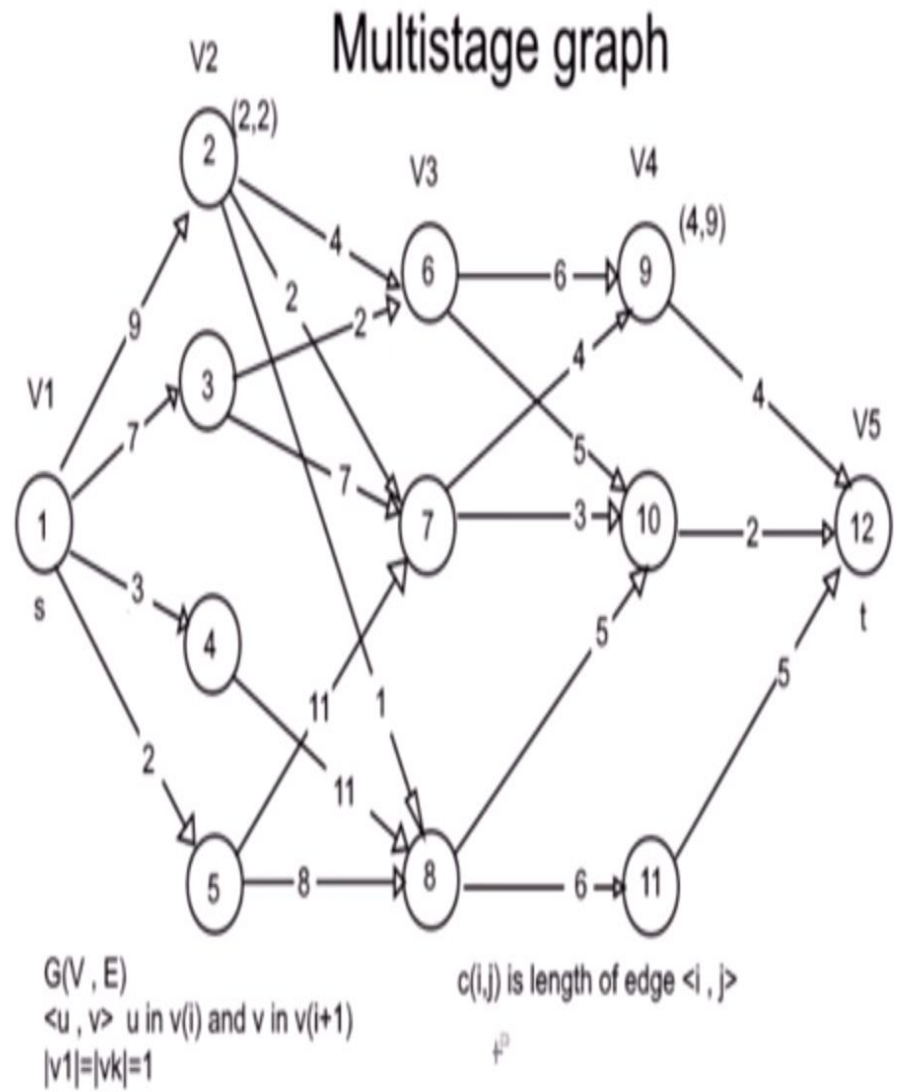
vertex	1	2	3	4	5	6	7	8	9	10	11	12
Cost	**16**	**7**	**9**	**18**	**15**	**7**	**5**	**7**	**4**	**2**	**5**	**0**
d-destination	**2/3**	**7**	**6**	**8**	**8**	**10**	**10**	**10**	**12**	**12**	**12**	**12**

Multistage Graph pseudo code : forward approach

```
Algorithm FGraph(G, k, n, p)
// The input is a k-stage graph G = (V, E) with n vertices
// indexed in order of stages. E is a set of edges and c[i, j]
// is the cost of ⟨i, j⟩. p[1 : k] is a minimum-cost path.
{
    cost[n] := 0.0;
    for j := n − 1 to 1 step −1 do
    { // Compute cost[j].
        Let r be a vertex such that ⟨j, r⟩ is an edge
        of G and c[j, r] + cost[r] is minimum;
        cost[j] := c[j, r] + cost[r];
        d[j] := r;
    }
    // Find a minimum-cost path.
    p[1] := 1; p[k] := n;
    for j := 2 to k − 1 do p[j] := d[p[j − 1]];
}
```

Solving using Dynamic Programming: Backward approach

- bcost(i,j) : Minimum cost path from vertex s to vertex j in Vi.
- bcost(i,j) = min{bcost(i-1, k) + c(k,j)}
- k ϵVi-1
- bcost(2,2) = 9
- bcost(2,3) = 7
- bcost(2,4) = 3
- bcost(2,5) = 2
- bcost(3,6) =min(bcost(2,2)+c(2,6),
- bcost(2,3)+c(3,6)}
- =min{9+4, 7+2}=9
- bcost(3,7) =min(bcost(2,2)+c(2,7),
- bcost(2,3)+c(3,7)
- bcost(2,5)+c(2,7)}
- =min{9+2, 7+7, 2+11}=11



Solving using Dynamic Programming: Backward approach

$$
\begin{aligned}
bcost(3,7) &= 11\\
bcost(3,8) &= 10\\
bcost(4,9) &= 15\\
bcost(4,10) &= 14\\
bcost(4,11) &= 16\\
bcost(5,12) &= 16
\end{aligned}
$$

Multistage Graph pseudo code: backward approach

```
Algorithm BGraph(G, k, n, p)
// Same function as FGraph
{
    bcost[1] := 0.0;
    for j := 2 to n do
    { // Compute bcost[j].
        Let r be such that ⟨r, j⟩ is an edge of
        G and bcost[r] + c[r, j] is minimum;
        bcost[j] := bcost[r] + c[r, j];
        d[j] := r;
    }
    // Find a minimum-cost path.
    p[1] := 1; p[k] := n;
    for j := k - 1 to 2 do p[j] := d[p[j + 1]];
}
```

Solve

Find minimum cost path from s to t in the multistage graph given below using:

a.Forward approach

b.Backward approach