Linear Programming

Unit 4

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- A Linear Programming Problem
- Formulating a LP Problem
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A Linear Programming Problem

- Many problems take the form of maximizing or minimizing an objective, given limited resources and competing constraints.
- If we can specify the objective as a linear function of certain variables, and if we can specify the constraints on resources as equalities or inequalities on those variables, then we have a ***linear programming problem.***
- Linear programming:
  - Technique of *optimizing* a *linear objective function* expressed in terms of certain variables subject to some *linear constraints* imposed on these variables.

A Linear Programming Problem

- Linear programming:
  - Technique of *optimizing* a *linear objective function* expressed in terms of certain variables subject to some *linear constraints* imposed on these variables.

Given a set of real numbers $c_1, c_2, \ldots, c_n$ and a set of variables $x_1, x_2, \ldots, x_n$, A linear function f can be formulated as optimizing :

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = \sum_{j=1}^{n} c_j x_j$$

subject to set of linear constraints such as:

$$f(x_1, x_2, \ldots, x_n) \le b; \;\; f(x_1, x_2, \ldots, x_n) \ge b;$$

$$f(x_1, x_2, \ldots, x_n) = b$$

Linear Programming

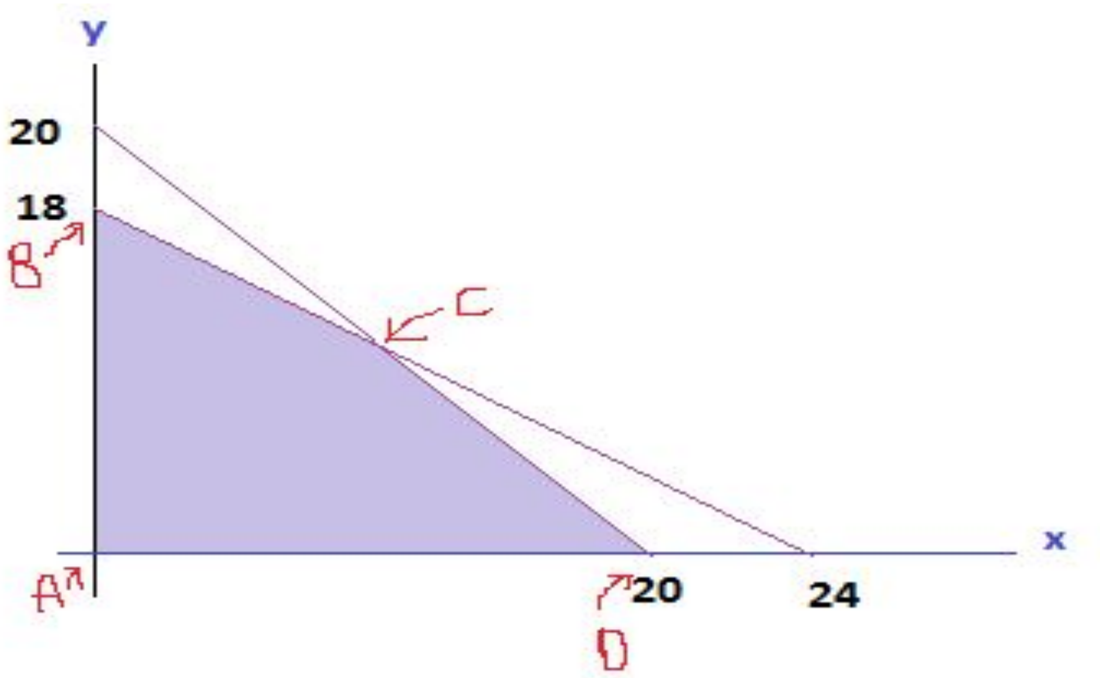
- Constraints:
  - $x+y<=20$
  - $3x+4y<=72$
- Objective:
  - $Z=4x+5y$
- Our goal is to maximize the objective function Z

Linear Programming

- Step 1: Given a problem determine the constraints and objective.
- Constraints:
  - $x+y<=20$
  - $3x+4y<=72$
- Objective:
  - $Z=4x+5y$
- Our goal is to maximize the objective function Z

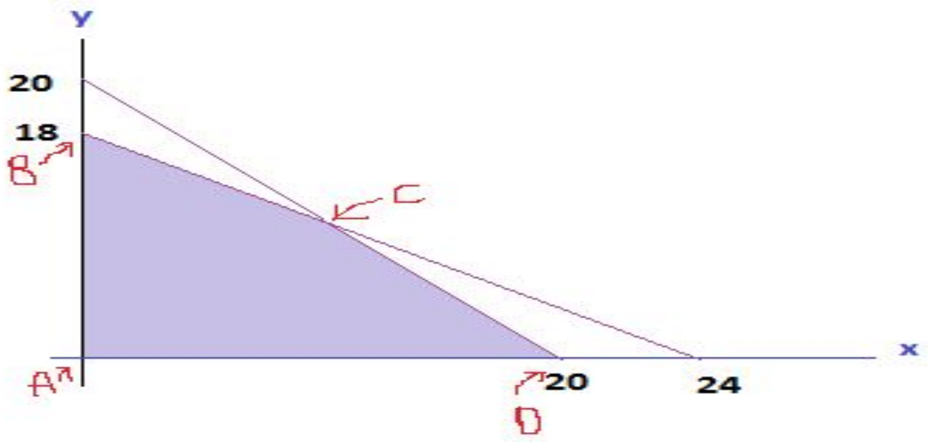
Linear Programming

- Step 2:Plot the constraints:
    - x+y<=20
    - 3x+4y<=72



Linear Programming

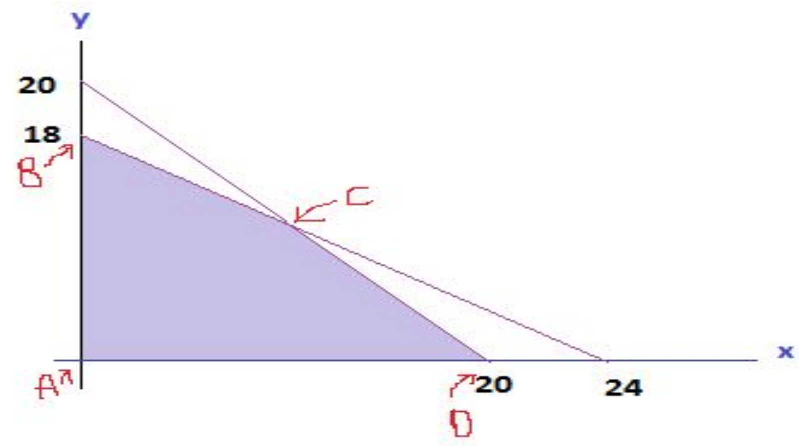
- Step 3: After plotting the graph, identify the corner points A, B, C and D. We don’tneed the corner point A i.e. (0,0) as we want to maximize Z.
- Z=4x+5y



Points	x	y	Z
B	0	18	
C			
D	20	0	

Linear Programming

- x+y<=20
- 3x+4y<=72
  - Multiplying eqn 1 by 3 and subtracting



- **3x+4y<=72**

  -(3x3y<=60)

  y <= 12
- **Substituting for y =12 in eqn 1**

  x+12<=20

  Therefore, x<=8

Points	x	y	Z
B	0	18	
C	8	12	
D	20	0	

Linear Programming

- Step 4: Calculate Z and determine which of these three value (B, C and D) gives the maximum value for Z.
- $x+y<=20$
- $3x+4y<=72$
- $Z=4x+5y$

Points	x	y	Z
B	0	18	90
C	8	12	92
D	20	0	80

- The maximum Z = 92 and it occurs hen x=8 & y=12

- A company receives in sales Rs20/- per book and Rs18/- per calculator. The cost per unit to manufacture each book and calculator are Rs5/- and Rs4/- respectively. The monthly (30 days) cost must not exceed Rs27,000/- per month. If the manufacturing equipment used by the company takes 5 minutes to produce a book and 15 minutes to produce a calculator, how many books and calculators should the company make to maximize profit? Determine the max profit the company earns in a 30 day period.

	B	C
Sales	20	18
Cost	5	4
Time	5	15

Objective: S= 20B+18C

Constraints:

5B+4C<=27000

30*24*60=43200

5B+15C<=43200

B	C	Sales
5400	0	108000
0	2880	51840
4221	1473	110,934

Objective: S= 20B+18C

Profit=Sales-Cost=110934-27000=83934/-

Formulating a LP Problem

- Consider a political problem:

Suppose that you are a politician trying to win an election. Your district has three different types of areas—urban, suburban, and rural. These areas have, respectively, 100,000, 200,000, and 50,000 registered voters.
Although not all the registered voters actually go to the polls, you decide that to govern effectively, you would like at least half the registered voters in each of the three regions to vote for you. You are honorable and would never consider supporting policies in which you do not believe. You realize, however, that certain issues may be more effective in winning votes in certain places.
Your primary issues are building more roads, gun control, farm subsidies, and a gasoline tax dedicated to improved public transit.
According to your campaign staff's research, you can estimate how many votes you win or lose from each population segment by spending $1,000 on advertising on each issue. This information appears in the table on the next slide. In this table, each entry indicates the number of thousands of either urban, suburban, or rural voters who would be won over by spending $1,000 on advertising in support of a particular issue. Negative entries denote votes that would be lost. Your task is to figure out the minimum amount of money that you need to spend in order to win 50,000 urban votes, 100,000 suburban votes, and 25,000 rural votes.

Formulating a LP Problem

- Consider a political problem:

	Total No. of voters in each region		
	100,000 voters	200,000 voters	50,000 voters
Policy	Urban	Suburban	Rural
Build Road	-2	5	3
Gun Control	8	2	-5
Agriculture	0	0	10
LPG subsidy	10	0	-2

Thousands of voters who could be won with $1,000 of advertisement

By trial and error method, one possible strategy to win the election is:

- Spent $20,000 of advertising to building roads, $0 to gun control, $4,000 to agriculture and $9,000 to a LPG subsidy.

- But, Is the strategy gives us an optimum advertizing investment?

Formulating a LP Problem

- Our **Objective** is to "Win election by winning at least half the registered votes in each region while minimizing the advertizing cost".

- Representing the problem as a Linear Program:
  - Introduce 4 variables x_1, x_2, x_3, x_4 denoting the number of thousands of dollars spent on advertising on building roads, gun control, agriculture and LPG subsidy respectively.
  - The problem can be formulated as:

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
\text{urban} & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
\text{suburban} & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
\text{rural} & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & & & x_1, x_2, x_3, x_4 & & & \geq & 0 .
\end{array}
$$

Thousands of voters representing majority.

Some LP Terminologies

Objective function

Minimization linear Program

Linear constraints

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
& -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
& 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
& 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
& \multicolumn{7}{l}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}
$$

Nonnegativity Constraint

Some LP Terminologies

$$
\begin{array}{lrcrcr}
\text{maximize} & \multicolumn{3}{c}{\overbrace{x_1 \;+\; x_2}^{\text{Linear function}}} & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

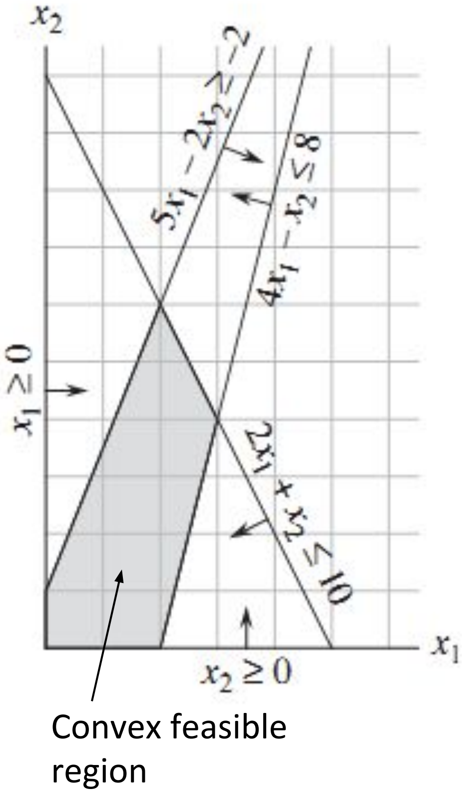
Linear inequalities

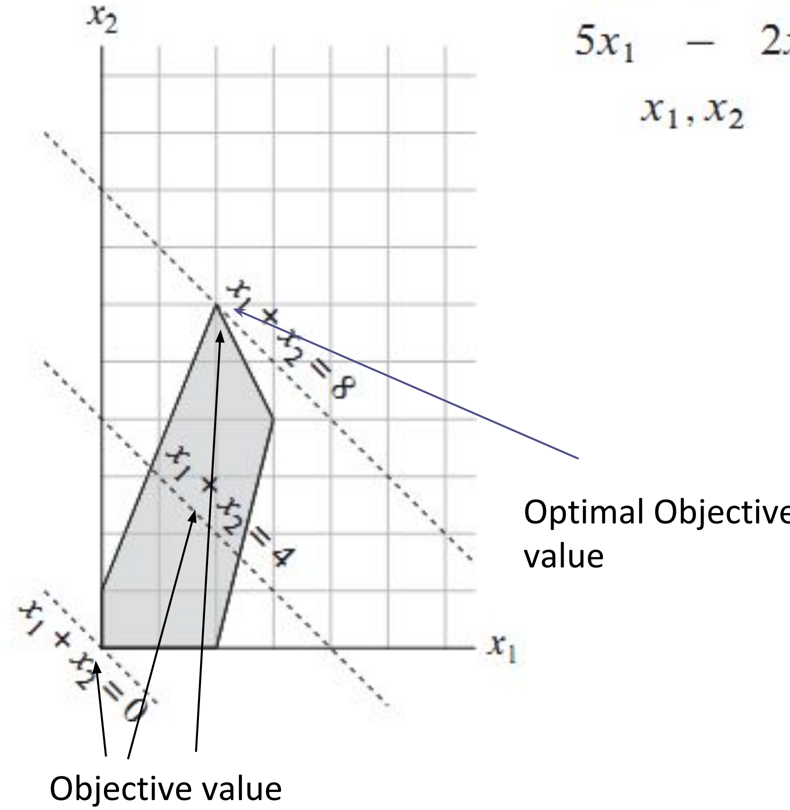
Maximization linear Program

- Feasible solution
- Optimal solution
- Objective Value

Solution to a LP problem (a geometric overview)

$$
\begin{array}{llcl}
\text{maximize} & x_1 + x_2 & & \\
\text{subject to} & & & \\
& 4x_1 - x_2 & \leq & 8 \\
& 2x_1 + x_2 & \leq & 10 \\
& 5x_1 - 2x_2 & \geq & -2 \\
& x_1, x_2 & \geq & 0
\end{array}
$$





Standard and Slack form

- Canonical forms useful for specifying and working with linear programs.

- Standard form:
  - Informally, in standard form, a linear program is the **maximization** of a linear function subject to linear inequalities.

- Slack form:
  - In slack form, a linear program is the **maximization** of a linear function subject to linear equalities.

Standard and Slack form

Converting linear program in Standard form:

- A linear program in standard form is represented as:

$$maximize \quad \sum_{j=1}^{n} c_j x_j$$

$$subject \ \text{to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i$$

$$x_j \ge 0$$

Where a_{ij}, b_i, c_j ; for i=1,2,...m and j=1,2,...,n are the set of real numbers.

Standard and Slack form

Converting a LP in Standard form: cont…

- A linear program may not be in standard form because of any these four possible reasons:

1. The objective function may be a minimization rather than a maximization.
2. There may be variables without non-negativity constraints.
3. There may be equality constraints having an equal sign instead of a less-than-or-equal-to sign.
4. There may be inequality constraints, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign.

$$
\begin{array}{lrcrcl}
\text{minimize} & -2x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& x_1 & & & \geq & 0
\end{array}
$$

$\downarrow$ Negate coefficients (1)

$$
\begin{array}{lrcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& x_1 & & & \geq & 0
\end{array}
$$

Converting a LP in Standard form

2. If x_i has no non-negativity constraints:
   - Replace each occurrence of x_i with x_i'-x_i''.
   - Add constraints: $x_i'\geq 0$ and $x_i''\geq 0$.

$$\begin{array}{lrcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& x_1 & & & \geq & 0
\end{array}
\xrightarrow{\quad\quad\quad\quad}
\begin{array}{lrcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & = & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}$$

Removing
non-negative
Constraints(2)

Converting a LP in Standard form:

3. There may be equality constraints having an equal sign instead of a less-than-or-equal-to sign.

$$
\begin{array}{llllllll}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & = & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

Replace equality constraints by pair of inequality constraints

$$
\begin{array}{llllllll}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0 .
\end{array}
$$

Converting a LP in Standard form:

4. There may be inequality constraints, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0 .
\end{array}
$$

Convert >= to <= by multiplying these constraints by -1

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}
$$

For consistency in variable names, we rename x'_2 to x_2 and x''_2 to x_3, obtaining the standard form

Standard and Slack form

Converting linear program in Slack form:

Recall: In slack form, a linear program is the **maximization** of a linear function subject to linear equalities.

– Convert it into a form in which the nonnegativity constraints are the only inequality constraints.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \longrightarrow x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \,, \quad x_{n+i} \geq 0.$$

x_{n+i} are called slack variables as they measure the difference between left and right hand side of the equations.

Converting a LP in Slack form:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

After introducing slack variables x_4 , x_5, x_6

non-basic variables

$$
\begin{array}{lrrrrrrrr}
\text{maximize} & & & \multicolumn{6}{c}{\overbrace{2x_1 - 3x_2 + 3x_3}^{\text{non-basic variables}}} \\
\text{subject to} & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + \; x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - \; x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - \; 2x_3 \\
& \multicolumn{6}{r}{x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 .} & &
\end{array}
$$

basic variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Slack Form

Using the variable z to denote the value of the objective function

Recall: Slack form

As an example, in the slack form given as:

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} ,
\end{aligned}
$$

$$
\begin{aligned}
z &= v + \sum_{j \in N} c_j x_j \\
x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B ,
\end{aligned}
$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$
A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix} ,
$$

$$
b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} ,
$$

$$
c = \begin{pmatrix} c_3 & c_5 & c_6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1/6 & -1/6 & -2/3 \end{pmatrix}^{\mathrm{T}}, \text{ and } v = 28.
$$

Solution to LP problems

- Solution to a LP problem (a geometric overview)
- Simplex Method
  - Pivoting
  - Formal Simplex algorithm
  - Finding Initial solution
  - Duality

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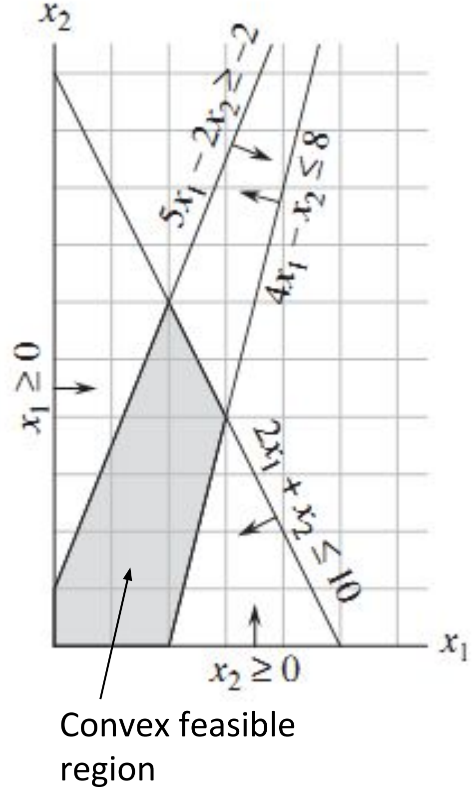
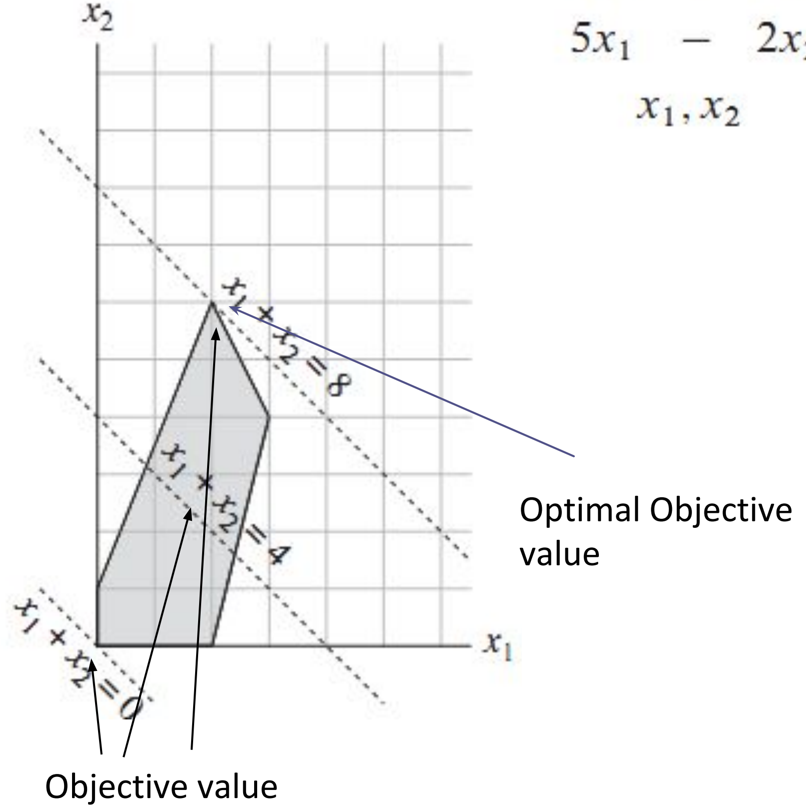
Solution to a LP problem (a geometric overview)

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$





Simplex Method

- As in two dimensions, because the feasible region is convex, the set of points that achieve the optimal objective value must include a vertex of the feasible region.
- Similarly, if we have n variables, each constraint defines a half-space in n-dimensional space. We call the feasible region formed by the intersection of these half-spaces a ***simplex.***
- ***The objective function is now a hyperplane and, because of convexity, an* optimal solution still occurs at a vertex of the simplex**.
- Simplex:in mathematics, the simplest convex polyhedron of some given

dimension n.

- When n = 3, we have a threedimensional simplex, which is a tetrahedron; the tetrahedron may be irregular.
- A twodimensional simplex is a triangle, a onedimensional simplex is a line segment, and a zerodimensional simplex is a point.

Simplex Method

- Move from vertex to vertex, looking for an optimal solution.
- To start, simply find a vertex on the polytope.
- Of the nearest vertices, find one which increases z.
- If each neighboring vertex either decreases z or does not increase z and you've checked it, the algorithm finishes.

Simplex Method

- Based on Geometric View:
  - Starts from some vertex of the simplex and performs a sequence of iterations.
  - In each iteration, moves along an edge of the simplex from a current vertex to a neighboring vertex having objective value is no smaller than that of the current vertex.
  - The simplex algorithm terminates when it reaches a local maximum.
    - Since, feasible region is convex and the objective function is linear, this local optimum is actually a global optimum.

Simplex Method (Algebraic View)

- Basic Steps:

1. Write the given linear program into the slack form.
2. Find the initial basic solution by setting all non-basic variables to 0 and compute all basic variables.
3. Iterate from one slack form to another by making a basic variable as nonbasic and a nonbasic variable as basic in such a way that
   - Objective value of the function does not decrease.
     - Select a nonbasic variable x_e having positive coefficient in objective function
     - Increase value of x_e as much as possible without violating any of constraints, and identify tightest constraint. Find basic variable x_l corresponding to the tightest constraint.
     - Swap the role of x_l and x_e
   - Underlying LP problem does not change
   - The feasible solutions keep the same.

Example

Consider a Linear program given in standard form

Maximize $3x_1+x_2+2x_3$

Subject to: $x_1+x_2+3x_3 \leq 30$

$2x_1+2x_2+5x_3 \leq 24$

$4x_1+x_2+2x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

- Step 1: Change to slack form:

$z= 3x_1+x_2+2x_3$

$x_4=30- x_1-x_2-3x_3$

$x_5=24- 2x_1-2x_2-5x_3$

$x_6=36- 4x_1-x_2-2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Step 2: Initial Basic solution:

  $(x_1,x_2,x_3,x_4,x_5,x_6) =(0,0,0,30,24,36)$.

Recall: To get basic sol., set each non-basic variable to 0.

  – The result is $z = 3 \cdot 0 + 0 + 2 \cdot 0 = 0$. Not maximum.

- The result is $z = 3 \cdot 0 + 0 + 2 \cdot 0 = 0$. Not maximum.

- Our goal, in each iteration, is to reformulate the linear program so that the **basic solution has a greater objective value.**
- We select a nonbasic variable xe whose coefficient in the objective function is positive, and we increase the value of xe as much as possible without violating any of the constraints.

$$z = 3x_1 + x_2 + 2x_3$$

- The variable xe becomes basic, and some other variable xl becomes nonbasic. The values of other basic variables and of the objective function may also change.

-To continue the example, let’s think about increasing the value of x1.

- **As we increase x1, the values of x4, x5, and x6 all decrease**. Because we have a nonnegativity constraint for each variable, we cannot allow any of them to become negative.
  - **If x1 increases above 30, then x4 becomes negative, and x5 and x6 become negative when x1 increases above 12 and 9, respectively**.
  - The third constraint is the tightest constraint, and it limits how much we can increase x1. Therefore, we switch the roles of x1 and x6.

$x_4=30- x_1-x_2-3x_3$

$x_5=24- 2x_1-2x_2-5x_3$

$x_6=36- 4x_1-x_2-2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

•Solving $x_6=36- 4x_1-x_2-2x_3$ for $x_1,$ we get

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} .$$

Example cont…

- Step 3: Iteration **(Recall)**
  1. Select a nonbasic variable x_e having positive coefficient in objective function
  2. Increase value of x_e as much as possible without violating any of constraints, and identify tightest constraint. Find basic variable x_l corresponding to the tightest constraint.
  3. Swap the role of x_l and x_e .

2. In objective function $3x_1+x_2+2x_3$, each variable x_1, x_2, x_3 has positive value. Let us try to increase the value of x_1.
3. 30: x_4 will be OK; 12: x_5; 9: x_6. So only to 9 corresponding to x_6. It act as x_l
4. Change x_1 to basic variable and change the equations accordingly.

entering variable

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4}
\end{array}
$$

x4 = 30 - x1 - x2 - 3.x3

= 30 - (9 - x2/4 - x3/2 - x6/4) - x2- 3.x3

= 30 - 9 + x2/4 + x3/2 + x6/4 - x2- 3.x3

= 21 - 3.x2/4 + 5.x3/2 + x6/4

entering variable

$$
\begin{aligned}
z &= & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 &= 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 &= 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 &= 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{aligned}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Pivoting

new objective value

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

Basic Solution (9,0,0,21,6,0)

We call this operation a ***pivot. As demonstrated , a pivot chooses a nonbasic variable xe*** (here x1), called the ***entering variable, and a basic variable xl*** *(here x6)* ***, called the leaving variable,*** *and exchanges their roles.*

entering variable

$$
\begin{aligned}
z &= & & 3x_1 &+& x_2 &+& 2x_3 \\
x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
x_6 &= 36 &-& 4x_1 &-& x_2 &-& 2x_3
\end{aligned}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Pivoting

new objective value

$$
\begin{aligned}
z &= 27 &+& \frac{x_2}{4} &+& \frac{x_3}{2} &-& \frac{3x_6}{4} \\
x_1 &= 9 &-& \frac{x_2}{4} &-& \frac{x_3}{2} &-& \frac{x_6}{4} \\
x_4 &= 21 &-& \frac{3x_2}{4} &-& \frac{5x_3}{2} &+& \frac{x_6}{4} \\
x_5 &= 6 &-& \frac{3x_2}{2} &-& 4x_3 &+& \frac{x_6}{2}
\end{aligned}
$$

Basic Solution (9,0,0,21,6,0)

Continuing, we wish to find a new variable whose value we wish to increase. We do not want to increase x6, since as its value increases, the objective value decreases. (pick a positive variable) We can attempt to increase either x2 or x3; let us choose x3.
How far can we increase x3 without violating any of the constraints?

entering variable

$$
\begin{aligned}
z &= & & 3x_1 &+& x_2 &+& 2x_3 \\
x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
x_6 &= 36 &-& 4x_1 &-& x_2 &-& 2x_3
\end{aligned}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Pivoting

new objective value

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

Basic Solution (9,0,0,21,6,0)

How far can we increase x3 without violating any of the constraints? Constraint 1 limits x3 to 18, constraint 2 limits it to 42/5 and constraint 3 limits it x3 to 3/2.
The third constraint is again the tightest one, and therefore we rewrite
the third constraint so that x3 is on the left-hand side and x5 is on the right-hand side. We then substitute this new equation,
x3 =3/2 – 3.x2/8 – x5/4 +x6/8 in all these equations.

entering variable

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Pivoting

new objective value

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

Basic Solution (9,0,0,21,6,0)

Pivoting

entering variable

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

Basic Solution (33/4, 0,3/2,69/4,0,0)

leaving variable

Entering variable

Pivoting

entering variable

$$
\begin{aligned}
z &= && 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 &-& x_1 - x_2 - 3x_3 \\
x_5 &= 24 &-& 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 &-& 4x_1 - x_2 - 2x_3
\end{aligned}
$$

Basic Solution (0,0,0,30,24,36)

leaving variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Pivoting

new objective value

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

Basic Solution (9,0,0,21,6,0)

Pivoting

entering variable

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

Basic Solution (33/4, 0,3/2,69/4,0,0)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} .
\end{aligned}
$$

leaving variable

Entering variable

Pivoting

Basic Solution (8, 4,0,18, 0,0)

- At this point, all coefficients in the objective function are negative.

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$

- This situation occurs only when we have rewritten the linear program so that the basic solution is an optimal solution.
- **Thus, for this problem, the solution (8, 4, 0, 18, 0, 0), with objective value 28, is optimal.**
- The only variables in the original linear program are x1, x2, and x3, and so our solution is x1 = 8, x2 = 4, and x3 = 0, with objective value

  $z = 3x_1 + x_2 + x_3 = (3 * 8)+(1*4)+(2*0) = 28$

- Note that the values of the slack variables in the final solution measure how much slack remains in each inequality.

- Slack variable x4 is 18, and in inequality $x_1+x_2+3x_3 \le 30$, the left-hand side, with value 8+4+0 = 12, is 18 less than the right-hand side of 30.

- Slack variables x5 and x6 are 0 and indeed, in inequalities $2x_1+2x_2+5x_3 \le 24$ and $4x_1+x_2+2x_3 \le 36$, the left-hand and right-hand sides are equal.

Recall: Slack form

- A slack form can also be represented as a tuple *(N, B, A, b, c, v)* denoting the slack form:

$$
\begin{aligned}
z &= \nu + \sum_{j \in N} c_j x_j \\
x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B ,
\end{aligned}
$$

Where :

all variables *x are non-negatives*

N, B denotes the set of non-basic and basic variables resp.

and A is a matrix having elements a_{ij} .

Recall: Slack form (cont...)

As an example, in the slack form given as:

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} ,
\end{aligned}
$$

$$
\begin{aligned}
z &= v + \sum_{j \in N} c_j x_j \\
x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B ,
\end{aligned}
$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$
A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix} ,
$$

$$
b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} ,
$$

$$
c = \begin{pmatrix} c_3 & c_5 & c_6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1/6 & -1/6 & -2/3 \end{pmatrix}^{\mathrm{T}}, \text{ and } v = 28.
$$

Date:

N: Non Basic variables $\{x_1, x_2 \,\&\, x_3\}$ $n = 1, 2, 3 \leftarrow j$

B: Basic variables $\{x_4, x_5 \,\&\, x_6\}$ $m = 4, 5, 6 \leftarrow i$

Slack Form

$$z = 3x_1 + x_2 + 2x_3 \quad \text{— (6)}$$

$$x_4 = 30 - x_1 - x_2 - 3x_3 \quad \text{— (7)}$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \quad \text{— (8)}$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3 \quad \text{— (9)}$$

a_{ij} $i \to 4, 5, 6$
$j \to 1, 2, 3$

$$A = \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 4 & 1 & 2 \end{pmatrix}$$

b_i

$$b = \begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 36 \end{pmatrix}$$

From eqn (6)

$$C = (c_1 \; c_2 \; c_3)^T = (3 \; 1 \; 2)^T$$

$$v = 0$$

Formal Simplex Algorithm cont...

initial basic feasible solution

```
SIMPLEX(A, b, c)
(N, B, A, b, c, v) ← INITIALIZE-SIMPLEX(A, b, c)
while some index j ∈ N has c_j > 0
    do choose an index e ∈ N for which c_e > 0
       for each index i ∈ B
           do if a_ie > 0
                then Δ_i ← b_i / a_ie
                else Δ_i ← ∞
       choose an index l ∈ B that minimizes Δ_i
       if Δ_l = ∞
         then return "unbounded"
         else (N, B, A, b, c, v) ← PIVOT(N, B, A, b, c, v, l, e)
for i ← 1 to n
    do if i ∈ B
         then x̄_i ← b_i
         else x̄_i ← 0
return (x̄_1, x̄_2, ..., x̄_n)
```

Return initial basic feasible solution.

detects unbounded-ness

optimal solution

Abbreviations

N: indices set of nonbasic variables
B: indices set of basic variables
A: a_{ij}
b: b_i
c: c_i
v: constant coefficient.
e: index of entering variable
l: index of leaving variable

$z = v + \sum_{j \in N} c_j x_j$
$x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$

Formal Simplex Algorithm cont…

initial basic feasible solution

Return initial basic feasible solution.

SIMPLEX(A, b, c)

```
(N, B, A, b, c, v) ← INITIALIZE-SIMPLEX(A, b, c)
while some index j ∈ N has c_j > 0
    do choose an index e ∈ N for which c_e > 0
        for each index i ∈ B
            do if a_ie > 0
                then Δ_i ← b_i / a_ie
                else Δ_i ← ∞
        choose an index l ∈ B that minimizes Δ_i
```

Abbreviations

N: indices set of nonbasic variables
B: indices set of basic variables
A: a_{ij}
b: b_i
c: c_i
v: constant coefficient.
e: index of entering variable
l: index of leaving variable
$z=v+\sum_{j\in N}c_jx_j$
$x_i=b_i-\sum_{j\in N}a_{ij}x_j$ for $i\in B$

SIMPLEX(A, b, c)

```
(N, B, A, b, c, v) ← INITIALIZE-SIMPLEX(A, b, c)
while some index j ∈ N has c_j > 0
    do choose an index e ∈ N for which c_e > 0
        for each index i ∈ B
            do if a_ie > 0
                then Δ_i ← b_i / a_ie
                else Δ_i ← ∞
        choose an index l ∈ B that minimizes Δ_i
        if Δ_l = ∞
          then return "unbounded"
          else (N, B, A, b, c, v) ← PIVOT(N, B, A, b, c, v, l, e)
```

a_{ij} $i \to 4,5,6$ $j \to 1,2,3$

$$A = \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 4 & 1 & 2 \end{pmatrix}$$

b_i

$$b = \begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 36 \end{pmatrix}$$

from eqn (6)

$C = (c_1\ c_2\ c_3)^T = (3\ 1\ 2)^T$

$v = 0$

$i=4,5,6$
$e=1$
$b_i/a_{ie} = b_4/a_{41} = 30/1 = 30$

$l = 6$
$e = 1$

*

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

t

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

PIVOT(N, B, A, b, c, v, l, e)

Rewrite the equation that has x_l on LHS to have x_e on LHS

▷ Compute the coefficients of the equation for new basic variable x_e.
$\widehat{b}_e \leftarrow b_l / a_{le}$
for each $j \in N - \{e\}$
  do $\widehat{a}_{ej} \leftarrow a_{lj} / a_{le}$
$\widehat{a}_{el} \leftarrow 1/a_{le}$

Update remaining equations by substituting RHS of new equation for each occurrence of x_e.

▷ Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
  do $\widehat{b}_i \leftarrow b_i - a_{ie}\widehat{b}_e$
    for each $j \in N - \{e\}$
      do $\widehat{a}_{ij} \leftarrow a_{ij} - a_{ie}\widehat{a}_{ej}$
    $\widehat{a}_{il} \leftarrow -a_{ie}\widehat{a}_{el}$

Do the same for objective function.

▷ Compute the objective function.
$\widehat{v} \leftarrow v + c_e\widehat{b}_e$
for each $j \in N - \{e\}$
  do $\widehat{c}_j \leftarrow c_j - c_e\widehat{a}_{ej}$
$\widehat{c}_l \leftarrow -c_e\widehat{a}_{el}$

Update sets of nonbasic, basic variables.

▷ Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Abbreviations

N: indices set of nonbasic variables
B: indices set of basic variables
A: a_{ij}
b: b_i
c: c_i
v: constant coefficient.
e: index of entering variable
l: index of leaving variable

$z = v + \sum_{j \in N} c_j x_j$
$x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
\quad \xrightarrow{t} \quad
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

```
PIVOT(N, B, A, b, c, v, l, e)
  ▷ Compute the coefficients of the equation for new basic variable x_e.
  b̂_e ← b_l / a_le
  for each j ∈ N − {e}
        do â_ej ← a_lj / a_le
  â_el ← 1 / a_le
  ▷ Compute the coefficients of the remaining constraints.
  for each i ∈ B − {l}
        do b̂_i ← b_i − a_ie b̂_e
           for each j ∈ N − {e}
                do â_ij ← a_ij − a_ie â_ej
           â_il ← −a_ie â_el
  ▷ Compute the objective function.
  v̂ ← v + c_e b̂_e
  for each j ∈ N − {e}
        do ĉ_j ← c_j − c_e â_ej
  ĉ_l ← −c_e â_el
  ▷ Compute new sets of basic and nonbasic variabl
  N̂ = N − {e} ∪ {l}
  B̂ = B − {l} ∪ {e}
  return (N̂, B̂, Â, b̂, ĉ, v̂)
```

Rewrite the equation that has x_l on LHS to have x_e on LHS

Update remaining equations by substituting RHS of new equation for each occurrence of x_e.

Do the same for objective function.

Update sets of nonbasic, basic variables.

$$\hat{b} = \begin{pmatrix} b_1 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} 9 \\ 21 \\ 6 \end{pmatrix}$$

$$
\begin{aligned}
z &= & & 3x_1 + x_2 + 2x_3\\
x_4 &= 30 &-& x_1 - x_2 - 3x_3\\
x_5 &= 24 &-& 2x_1 - 2x_2 - 5x_3\\
x_6 &= 36 &-& 4x_1 - x_2 - 2x_3
\end{aligned}
$$

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}\\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}\\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

PIVOT(N, B, A, b, c, v, l, e)

```
▷ Compute the coefficients of the equation for new basic variable x_e.
b̂_e ← b_l / a_le
for each j ∈ N − {e}
      do â_ej ← a_lj / a_le
â_el ← 1 / a_le
▷ Compute the coefficients of the remaining cons
for each i ∈ B − {l}
      do b̂_i ← b_i − a_ie b̂_e
         for each j ∈ N − {e}
             do â_ij ← a_ij − a_ie â_ej
         â_il ← −a_ie â_el
▷ Compute the objective function.
v̂ ← v + c_e b̂_e
for each j ∈ N − {e}
      do ĉ_j ← c_j − c_e â_ej
ĉ_l ← −c_e â_el
▷ Compute new sets of basic and nonbasic variable
N̂ = N − {e} ∪ {l}
B̂ = B − {l} ∪ {e}
return (N̂, B̂, Â, b̂, ĉ, v̂)
```

Rewrite the equation that has x_l on LHS to have x_e on LHS

Update remaining equations by substituting RHS of new equation for each occurrence of x_e.

Do the same for objective function

Update sets of nonbasic, basic variables.

$$
\hat{A} = \begin{pmatrix} a_{12} & a_{13} & a_{16}\\ a_{42} & a_{43} & a_{46}\\ a_{52} & a_{53} & a_{56}\end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{3}{4} & \frac{5}{2} & -\frac{1}{4}\\ \frac{3}{2} & 4 & -\frac{1}{2}\end{pmatrix}
$$

$$
\hat{c} = (c_1\ c_2\ c_3)^T = \left(\frac{1}{4}\ \ \frac{1}{2}\ \ -\frac{3}{4}\right)^T
$$

$$
\hat{v} = 27
$$

Formal Simplex algorithm

The various issues to be considered during solving a linear programming problem are :

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Procedure INITIALIZE-SIMPLEX (A,b,c) is called *by* ***SIMPLEX (A,b,c)*** to determine whether a problem is feasible.

And if it is, INITIALIZE-SIMPLEX (A,b,c) returns a slack form in which the initial basic solution is feasible.

These two issues are considered by ***SIMPLEX (A,b,c)*** procedure during its execution

Formal Simplex algorithm

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

Recall: Procedure INITIALIZE-SIMPLEX determines that whether a linear program has any feasible solutions, and if it does, gives a slack form for which the basic solution is feasible.

- But, it may be the case that a linear program can be feasible, yet the initial basic solution may not be feasible.

$$
\begin{array}{llrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

Corresponding Slack form $\longrightarrow$

$$
\begin{aligned}
z &= 2x_1 - x_2 \\
x_3 &= 2 - 2x_1 + x_2 \\
x_4 &= -4 - x_1 + 5x_2 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Example of an LP problem whose initial basic sol. is not feasible

Initial basic solution (0,0,2,-4)
-violating the constraints

By Inspection, even it is not clear whether this LP has any feasible solution

Formal Simplex algorithm

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*] cont…

- In order to determine whether a linear program *(L)* has any feasible solutions, formulate an auxiliary linear program *(L_{aux})* satisfying Lemma 1.

- *Lemma 1: Let L be a linear program in standard form, Let x_0 be a new variable, and let L_{aux} be the following linear program with n-1 variables:*

$$
\begin{array}{lrcll}
\text{maximize} & -x_0 & & & \\
\text{subject to} & & & & \\
& \displaystyle\sum_{j=1}^{n} a_{ij}x_j - x_0 & \leq & b_i & \text{for } i = 1, 2, \ldots, m \\
& x_j & \geq & 0 & \text{for } j = 0, 1, \ldots, n
\end{array}
$$

Then L is feasible if and only if the optimal objective value of L_{aux} is 0.

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

Lemma 1cont..

- Suppose that L has a feasible solution given by: $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
  i.e. , we have

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

- Then the solution $\bar{x}_0 = 0$ combined with $\bar{x}$ is a feasible solution to L_{aux} with objective value 0.
- L_{aux} has the objective function to maximize $-x_0$, this solution must be optimal for L_{aux} .
- Conversely,
  - suppose that the optimal objective value of L_{aux} is 0. Then $\bar{x}_0 = 0$ and thus remaining solution values of $\bar{x}$ satisfy the constraints of L.

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

- Recall the LP problem (given on top right).
  - We know that this problem has not initial feasible solution.
- Let us find whether it has any feasible solution.
  - To do so, formulate it as auxiliary linear program (given on bottom right)

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

$$\Big\downarrow \text{ Formulating } L_{aux}$$

$$
\begin{array}{lrcrcrcr}
\text{maximize} & & & & & -x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& & x_1, x_2, x_0 & & & & \geq & 0
\end{array}
$$

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

- Recall: From Lemma 1, If the optimal objective value of L_{aux} is 0, then the original linear program has a feasible solution otherwise solution is infeasible.
- To find optimum objective value of L_{aux} , convert it into slack form and solve the linear program.

entering

$$
\begin{aligned}
z &= && && && - x_0 \\
x_3 &= 2 &&- 2x_1 &&+ x_2 &&+ x_0 \\
x_4 &= -4 &&- x_1 &&+ 5x_2 &&+ x_0
\end{aligned}
$$

leaving

most negative (restrictive)

Corresponding slack form of L_{aux}
associated basic solution (0,0,0,2,-4)
which is infeasible

Pivoting

entering

$$
\begin{aligned}
z &= -4 - x_1 + 5x_2 - x_4 \\
x_0 &= 4 + x_1 - 5x_2 + x_4 \\
x_3 &= 6 - x_1 - 4x_2 + x_4
\end{aligned}
$$

leaving

associated basic solution (4,0,0,6,0)
which is feasible

Pivoting

$$
\begin{aligned}
z &= - x_0 \\
x_2 &= \frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
x_3 &= \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{aligned}
$$

Final slack form to L_{aux} having optimum objective value '0'. Thus our initial problem was feasible!

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

- Restoring the original objective function, (which is $2x_1$-x_2 in our example) with appropriate substitutions, we get the objective function:

$$2x_1 - x_2 = 2x_1 - \left(\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5}\right)$$

- Setting x_0=0 and simplifying, we get the slack form

$$\begin{aligned} z &= -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\ x_2 &= \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\ x_3 &= \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \end{aligned}$$

This slack form has a feasible basic solution, and is returned by procedure *INITIALIZE-SIMPLEX* to procedure SIMPLEX.

Feasible Basic Solution: (0, 4/5, 14/5, 0)

Finding Initial solution [*INITIALIZE-SIMPLEX (A,b,c)*]

INITIALIZE-SIMPLEX(A, b, c)

1 let k be the index of the minimum b_i
2 **if** $b_k \geq 0$ // is the initial basic solution feasible?
3   **return** $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
4 form L_{aux} by adding $-x_0$ to the left-hand side of each constraint and setting the objective function to $-x_0$
5 let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
6 $l = n + k$
7 // L_{aux} has $n + 1$ nonbasic variables and m basic variables.
8 $(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
9 // The basic solution is now feasible for L_{aux}.
10 iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution to L_{aux} is found
11 **if** the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
12   **if** $\bar{x}_0$ is basic
13   perform one (degenerate) pivot to make it nonbasic
14   from the final slack form of L_{aux}, remove x_0 from the constraints and restore the original objective function of L, but replace each basic variable in this objective function by the right-hand side of its associated constraint
15   **return** the modified final slack form
16 **else return** "infeasible"

If the initial basic solution is feasible, algo. terminate here and return the corresponding slack form

If the initial basic solution is not feasible, algo. Perform step 4 to 16 and return the corresponding slack form if exist or return infeasible otherwise

ReCall: Simplex Algorithm

initial basic feasible solution

SIMPLEX(A, b, c)

$(N, B, A, b, c, v) \leftarrow$ INITIALIZE-SIMPLEX(A, b, c) Return initial basic feasible solution.
while some index $j \in N$ has $c_j > 0$
  do choose an index $e \in N$ for which $c_e > 0$
    for each index $i \in B$
      do if $a_{ie} > 0$
        then $\Delta_i \leftarrow b_i / a_{ie}$
        else $\Delta_i \leftarrow \infty$
    choose an index $l \in B$ that minimizes Δ_i
    if $\Delta_l = \infty$
      then return "unbounded" detects unbounded-ness
      else $(N, B, A, b, c, v) \leftarrow$ PIVOT(N, B, A, b, c, v, l, e)
for $i \leftarrow 1$ **to** n
  do if $i \in B$
    then $\bar{x}_i \leftarrow b_i$
    else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

optimal solution

Abbreviations

N: indices set of nonbasic variables
B: indices set of basic variables
A: a_{ij}
b: b_i
c: c_i
v: constant coefficient.
e: index of entering variable
l: index of leaving variable

$z = v + \sum_{j \in N} c_j x_j$
$x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$

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