29 Linear Programming

Many problems take the form of maximizing or minimizing an objective, given limited resources and competing constraints. If we can specify the objective as a linear function of certain variables, and if we can specify the constraints on resources as equalities or inequalities on those variables, then we have a ***linear-programming problem***. Linear programs arise in a variety of practical applications. We begin by studying an application in electoral politics.

A political problem

Suppose that you are a politician trying to win an election. Your district has three different types of areas—urban, suburban, and rural. These areas have, respectively, 100,000, 200,000, and 50,000 registered voters. Although not all the registered voters actually go to the polls, you decide that to govern effectively, you would like at least half the registered voters in each of the three regions to vote for you. You are honorable and would never consider supporting policies in which you do not believe. You realize, however, that certain issues may be more effective in winning votes in certain places. Your primary issues are building more roads, gun control, farm subsidies, and a gasoline tax dedicated to improved public transit. According to your campaign staff's research, you can estimate how many votes you win or lose from each population segment by spending \$1,000 on advertising on each issue. This information appears in the table of Figure 29.1. In this table, each entry indicates the number of thousands of either urban, suburban, or rural voters who would be won over by spending \$1,000 on advertising in support of a particular issue. Negative entries denote votes that would be lost. Your task is to figure out the minimum amount of money that you need to spend in order to win 50,000 urban votes, 100,000 suburban votes, and 25,000 rural votes.

You could, by trial and error, devise a strategy that wins the required number of votes, but the strategy you come up with might not be the least expensive one. For example, you could devote \$20,000 of advertising to building roads, \$0 to gun control, \$4,000 to farm subsidies, and \$9,000 to a gasoline tax. In this case, you

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

Figure 29.1 The effects of policies on voters. Each entry describes the number of thousands of urban, suburban, or rural voters who could be won over by spending \$1,000 on advertising support of a policy on a particular issue. Negative entries denote votes that would be lost.

would win $20(-2)+0(8)+4(0)+9(10) = 50$ thousand urban votes, $20(5)+0(2)+4(0)+9(0) = 100$ thousand suburban votes, and $20(3)+0(-5)+4(10)+9(-2) = 82$ thousand rural votes. You would win the exact number of votes desired in the urban and suburban areas and more than enough votes in the rural area. (In fact, in the rural area, you would receive more votes than there are voters.) In order to garner these votes, you would have paid for $20 + 0 + 4 + 9 = 33$ thousand dollars of advertising.

Naturally, you may wonder whether this strategy is the best possible. That is, could you achieve your goals while spending less on advertising? Additional trial and error might help you to answer this question, but wouldn't you rather have a systematic method for answering such questions? In order to develop one, we shall formulate this question mathematically. We introduce 4 variables:

- x_1 is the number of thousands of dollars spent on advertising on building roads,
- x_2 is the number of thousands of dollars spent on advertising on gun control,
- x_3 is the number of thousands of dollars spent on advertising on farm subsidies, and
- x_4 is the number of thousands of dollars spent on advertising on a gasoline tax.

We can write the requirement that we win at least 50,000 urban votes as

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50 . \tag{29.1}$$

Similarly, we can write the requirements that we win at least 100,000 suburban votes and 25,000 rural votes as

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100 \tag{29.2}$$

and

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25 . \tag{29.3}$$

Any setting of the variables x_1, x_2, x_3, x_4 that satisfies inequalities (29.1)–(29.3) yields a strategy that wins a sufficient number of each type of vote. In order to keep costs as small as possible, you would like to minimize the amount spent on advertising. That is, you want to minimize the expression

$$x_1 + x_2 + x_3 + x_4 . \tag{29.4}$$

Although negative advertising often occurs in political campaigns, there is no such thing as negative-cost advertising. Consequently, we require that

$$x_1 \geq 0,\; x_2 \geq 0,\; x_3 \geq 0, \text{ and } x_4 \geq 0 . \tag{29.5}$$

Combining inequalities (29.1)–(29.3) and (29.5) with the objective of minimizing (29.4), we obtain what is known as a "linear program." We format this problem as

$$\begin{array}{llllllllllll}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & & & (29.6) \\
\text{subject to} & & & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 & & (29.7) \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 & & (29.8) \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 & & (29.9) \\
 & & & x_1, x_2, x_3, x_4 & & & & & \geq & 0 & . & (29.10)
\end{array}$$

The solution of this linear program yields your optimal strategy.

General linear programs

In the general linear-programming problem, we wish to optimize a linear function subject to a set of linear inequalities. Given a set of real numbers $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, we define a ***linear function*** f on those variables by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{j=1}^{n} a_j x_j .$$

If b is a real number and f is a linear function, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is a ***linear equality*** and the inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b$$

and

$$f(x_1, x_2, \ldots, x_n) \geq b$$

are ***linear inequalities***. We use the general term ***linear constraints*** to denote either linear equalities or linear inequalities. In linear programming, we do not allow strict inequalities. Formally, a ***linear-programming problem*** is the problem of either minimizing or maximizing a linear function subject to a finite set of linear constraints. If we are to minimize, then we call the linear program a ***minimization linear program***, and if we are to maximize, then we call the linear program a ***maximization linear program***.

The remainder of this chapter covers how to formulate and solve linear programs. Although several polynomial-time algorithms for linear programming have been developed, we will not study them in this chapter. Instead, we shall study the simplex algorithm, which is the oldest linear-programming algorithm. The simplex algorithm does not run in polynomial time in the worst case, but it is fairly efficient and widely used in practice.

An overview of linear programming

In order to describe properties of and algorithms for linear programs, we find it convenient to express them in canonical forms. We shall use two forms, ***standard*** and ***slack***, in this chapter. We will define them precisely in Section 29.1. Informally, a linear program in standard form is the maximization of a linear function subject to linear *inequalities*, whereas a linear program in slack form is the maximization of a linear function subject to linear *equalities*. We shall typically use standard form for expressing linear programs, but we find it more convenient to use slack form when we describe the details of the simplex algorithm. For now, we restrict our attention to maximizing a linear function on n variables subject to a set of m linear inequalities.

Let us first consider the following linear program with two variables:

$$\begin{array}{llrcrcrl}
\text{maximize} & & x_1 & + & x_2 & & & (29.11) \\
\text{subject to} & & & & & & & \\
 & & 4x_1 & - & x_2 & \leq & 8 & (29.12) \\
 & & 2x_1 & + & x_2 & \leq & 10 & (29.13) \\
 & & 5x_1 & - & 2x_2 & \geq & -2 & (29.14) \\
 & & & x_1, x_2 & & \geq & 0\ . & (29.15)
\end{array}$$

We call any setting of the variables x_1 and x_2 that satisfies all the constraints (29.12)–(29.15) a ***feasible solution*** to the linear program. If we graph the constraints in the (x_1, x_2)-Cartesian coordinate system, as in Figure 29.2(a), we see

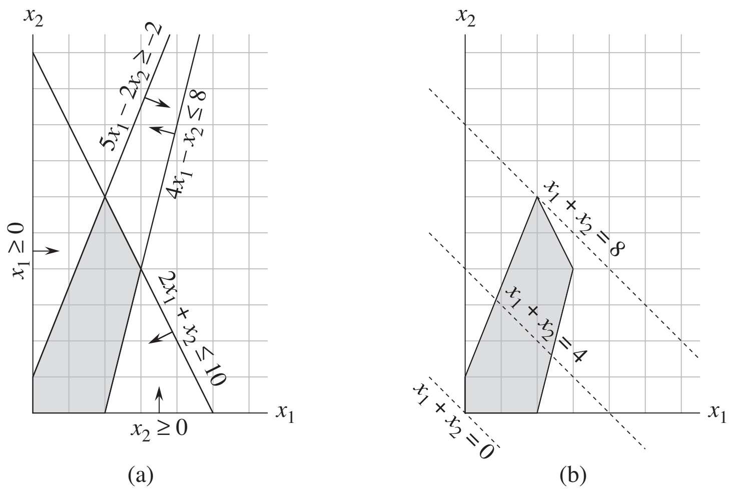


Figure 29.2 **(a)** The linear program given in (29.12)–(29.15). Each constraint is represented by a line and a direction. The intersection of the constraints, which is the feasible region, is shaded. **(b)** The dotted lines show, respectively, the points for which the objective value is 0, 4, and 8. The optimal solution to the linear program is $x_1 = 2$ and $x_2 = 6$ with objective value 8.

that the set of feasible solutions (shaded in the figure) forms a convex region[1] in the two-dimensional space. We call this convex region the ***feasible region*** and the function we wish to maximize the ***objective function***. Conceptually, we could evaluate the objective function $x_1 + x_2$ at each point in the feasible region; we call the value of the objective function at a particular point the ***objective value***. We could then identify a point that has the maximum objective value as an optimal solution. For this example (and for most linear programs), the feasible region contains an infinite number of points, and so we need to determine an efficient way to find a point that achieves the maximum objective value without explicitly evaluating the objective function at every point in the feasible region.

In two dimensions, we can optimize via a graphical procedure. The set of points for which $x_1+x_2 = z$, for any z, is a line with a slope of -1. If we plot $x_1+x_2 = 0$, we obtain the line with slope -1 through the origin, as in Figure 29.2(b). The intersection of this line and the feasible region is the set of feasible solutions that have an objective value of 0. In this case, that intersection of the line with the feasible region is the single point $(0, 0)$. More generally, for any z, the intersection

[1]An intuitive definition of a convex region is that it fulfills the requirement that for any two points in the region, all points on a line segment between them are also in the region.

of the line $x_1 + x_2 = z$ and the feasible region is the set of feasible solutions that have objective value z. Figure 29.2(b) shows the lines $x_1 + x_2 = 0$, $x_1 + x_2 = 4$, and $x_1 + x_2 = 8$. Because the feasible region in Figure 29.2 is bounded, there must be some maximum value z for which the intersection of the line $x_1 + x_2 = z$ and the feasible region is nonempty. Any point at which this occurs is an optimal solution to the linear program, which in this case is the point $x_1 = 2$ and $x_2 = 6$ with objective value 8.

It is no accident that an optimal solution to the linear program occurs at a vertex of the feasible region. The maximum value of z for which the line $x_1 + x_2 = z$ intersects the feasible region must be on the boundary of the feasible region, and thus the intersection of this line with the boundary of the feasible region is either a single vertex or a line segment. If the intersection is a single vertex, then there is just one optimal solution, and it is that vertex. If the intersection is a line segment, every point on that line segment must have the same objective value; in particular, both endpoints of the line segment are optimal solutions. Since each endpoint of a line segment is a vertex, there is an optimal solution at a vertex in this case as well.

Although we cannot easily graph linear programs with more than two variables, the same intuition holds. If we have three variables, then each constraint corresponds to a half-space in three-dimensional space. The intersection of these half-spaces forms the feasible region. The set of points for which the objective function obtains a given value z is now a plane (assuming no degenerate conditions). If all coefficients of the objective function are nonnegative, and if the origin is a feasible solution to the linear program, then as we move this plane away from the origin, in a direction normal to the objective function, we find points of increasing objective value. (If the origin is not feasible or if some coefficients in the objective function are negative, the intuitive picture becomes slightly more complicated.) As in two dimensions, because the feasible region is convex, the set of points that achieve the optimal objective value must include a vertex of the feasible region. Similarly, if we have n variables, each constraint defines a half-space in n-dimensional space. We call the feasible region formed by the intersection of these half-spaces a ***simplex***. The objective function is now a hyperplane and, because of convexity, an optimal solution still occurs at a vertex of the simplex.

The ***simplex algorithm*** takes as input a linear program and returns an optimal solution. It starts at some vertex of the simplex and performs a sequence of iterations. In each iteration, it moves along an edge of the simplex from a current vertex to a neighboring vertex whose objective value is no smaller than that of the current vertex (and usually is larger.) The simplex algorithm terminates when it reaches a local maximum, which is a vertex from which all neighboring vertices have a smaller objective value. Because the feasible region is convex and the objective function is linear, this local optimum is actually a global optimum. In Section 29.4, we shall use a concept called "duality" to show that the solution returned by the simplex algorithm is indeed optimal.

Although the geometric view gives a good intuitive view of the operations of the simplex algorithm, we shall not refer to it explicitly when developing the details of the simplex algorithm in Section 29.3. Instead, we take an algebraic view. We first write the given linear program in slack form, which is a set of linear equalities. These linear equalities express some of the variables, called "basic variables," in terms of other variables, called "nonbasic variables." We move from one vertex to another by making a basic variable become nonbasic and making a nonbasic variable become basic. We call this operation a "pivot" and, viewed algebraically, it is nothing more than rewriting the linear program in an equivalent slack form.

The two-variable example described above was particularly simple. We shall need to address several more details in this chapter. These issues include identifying linear programs that have no solutions, linear programs that have no finite optimal solution, and linear programs for which the origin is not a feasible solution.

Applications of linear programming

Linear programming has a large number of applications. Any textbook on operations research is filled with examples of linear programming, and linear programming has become a standard tool taught to students in most business schools. The election scenario is one typical example. Two more examples of linear programming are the following:

- An airline wishes to schedule its flight crews. The Federal Aviation Administration imposes many constraints, such as limiting the number of consecutive hours that each crew member can work and insisting that a particular crew work only on one model of aircraft during each month. The airline wants to schedule crews on all of its flights using as few crew members as possible.
- An oil company wants to decide where to drill for oil. Siting a drill at a particular location has an associated cost and, based on geological surveys, an expected payoff of some number of barrels of oil. The company has a limited budget for locating new drills and wants to maximize the amount of oil it expects to find, given this budget.

With linear programs, we also model and solve graph and combinatorial problems, such as those appearing in this textbook. We have already seen a special case of linear programming used to solve systems of difference constraints in Section 24.4. In Section 29.2, we shall study how to formulate several graph and network-flow problems as linear programs. In Section 35.4, we shall use linear programming as a tool to find an approximate solution to another graph problem.

Algorithms for linear programming

This chapter studies the simplex algorithm. This algorithm, when implemented carefully, often solves general linear programs quickly in practice. With some carefully contrived inputs, however, the simplex algorithm can require exponential time. The first polynomial-time algorithm for linear programming was the ***ellipsoid algorithm***, which runs slowly in practice. A second class of polynomial-time algorithms are known as ***interior-point methods***. In contrast to the simplex algorithm, which moves along the exterior of the feasible region and maintains a feasible solution that is a vertex of the simplex at each iteration, these algorithms move through the interior of the feasible region. The intermediate solutions, while feasible, are not necessarily vertices of the simplex, but the final solution is a vertex. For large inputs, interior-point algorithms can run as fast as, and sometimes faster than, the simplex algorithm. The chapter notes point you to more information about these algorithms.

If we add to a linear program the additional requirement that all variables take on integer values, we have an ***integer linear program***. Exercise 34.5-3 asks you to show that just finding a feasible solution to this problem is NP-hard; since no polynomial-time algorithms are known for any NP-hard problems, there is no known polynomial-time algorithm for integer linear programming. In contrast, we can solve a general linear-programming problem in polynomial time.

In this chapter, if we have a linear program with variables $x = (x_1, x_2, \ldots, x_n)$ and wish to refer to a particular setting of the variables, we shall use the notation $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$.

29.1 Standard and slack forms

This section describes two formats, standard form and slack form, that are useful when we specify and work with linear programs. In standard form, all the constraints are inequalities, whereas in slack form, all constraints are equalities (except for those that require the variables to be nonnegative).

Standard form

In ***standard form***, we are given n real numbers $c_1, c_2, \ldots, c_n$; m real numbers $b_1, b_2, \ldots, b_m$; and mn real numbers a_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We wish to find n real numbers $x_1, x_2, \ldots, x_n$ that

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j \tag{29.16}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \tag{29.17}$$

$$x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n \,. \tag{29.18}$$

Generalizing the terminology we introduced for the two-variable linear program, we call expression (29.16) the ***objective function*** and the $n + m$ inequalities in lines (29.17) and (29.18) the ***constraints***. The n constraints in line (29.18) are the ***nonnegativity constraints***. An arbitrary linear program need not have nonnegativity constraints, but standard form requires them. Sometimes we find it convenient to express a linear program in a more compact form. If we create an $m \times n$ matrix $A = (a_{ij})$, an m-vector $b = (b_i)$, an n-vector $c = (c_j)$, and an n-vector $x = (x_j)$, then we can rewrite the linear program defined in (29.16)–(29.18) as

$$\text{maximize} \quad c^{\mathrm{T}} x \tag{29.19}$$

subject to

$$Ax \le b \tag{29.20}$$

$$x \ge 0 \,. \tag{29.21}$$

In line (29.19), $c^{\mathrm{T}}x$ is the inner product of two vectors. In inequality (29.20), Ax is a matrix-vector product, and in inequality (29.21), $x \ge 0$ means that each entry of the vector x must be nonnegative. We see that we can specify a linear program in standard form by a tuple (A, b, c), and we shall adopt the convention that A, b, and c always have the dimensions given above.

We now introduce terminology to describe solutions to linear programs. We used some of this terminology in the earlier example of a two-variable linear program. We call a setting of the variables $\bar{x}$ that satisfies all the constraints a ***feasible solution***, whereas a setting of the variables $\bar{x}$ that fails to satisfy at least one constraint is an ***infeasible solution***. We say that a solution $\bar{x}$ has ***objective value*** $c^{\mathrm{T}}\bar{x}$. A feasible solution $\bar{x}$ whose objective value is maximum over all feasible solutions is an ***optimal solution***, and we call its objective value $c^{\mathrm{T}}\bar{x}$ the ***optimal objective value***. If a linear program has no feasible solutions, we say that the linear program is ***infeasible***; otherwise it is ***feasible***. If a linear program has some feasible solutions but does not have a finite optimal objective value, we say that the linear program is ***unbounded***. Exercise 29.1-9 asks you to show that a linear program can have a finite optimal objective value even if the feasible region is not bounded.

Converting linear programs into standard form

It is always possible to convert a linear program, given as minimizing or maximizing a linear function subject to linear constraints, into standard form. A linear program might not be in standard form for any of four possible reasons:

1. The objective function might be a minimization rather than a maximization.
2. There might be variables without nonnegativity constraints.
3. There might be ***equality constraints***, which have an equal sign rather than a less-than-or-equal-to sign.
4. There might be ***inequality constraints***, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign.

When converting one linear program L into another linear program L', we would like the property that an optimal solution to L' yields an optimal solution to L. To capture this idea, we say that two maximization linear programs L and L' are ***equivalent*** if for each feasible solution $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value z, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value z. (This definition does not imply a one-to-one correspondence between feasible solutions.) A minimization linear program L and a maximization linear program L' are equivalent if for each feasible solution $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}'$ to L' with objective value $-z$, and for each feasible solution $\bar{x}'$ to L' with objective value z, there is a corresponding feasible solution $\bar{x}$ to L with objective value $-z$.

We now show how to remove, one by one, each of the possible problems in the list above. After removing each one, we shall argue that the new linear program is equivalent to the old one.

To convert a minimization linear program L into an equivalent maximization linear program L', we simply negate the coefficients in the objective function. Since L and L' have identical sets of feasible solutions and, for any feasible solution, the objective value in L is the negative of the objective value in L', these two linear programs are equivalent. For example, if we have the linear program

$$\begin{array}{lrcrcl}
\text{minimize} & -2x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0 \;,
\end{array}$$

and we negate the coefficients of the objective function, we obtain

$$\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2 \\
\text{subject to} \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& x_1 & & & \geq & 0 \; .
\end{array}$$

Next, we show how to convert a linear program in which some of the variables do not have nonnegativity constraints into one in which each variable has a nonnegativity constraint. Suppose that some variable x_j does not have a nonnegativity constraint. Then, we replace each occurrence of x_j by $x'_j - x''_j$, and add the nonnegativity constraints $x'_j \geq 0$ and $x''_j \geq 0$. Thus, if the objective function has a term $c_j x_j$, we replace it by $c_j x'_j - c_j x''_j$, and if constraint i has a term $a_{ij} x_j$, we replace it by $a_{ij} x'_j - a_{ij} x''_j$. Any feasible solution $\widehat{x}$ to the new linear program corresponds to a feasible solution $\bar{x}$ to the original linear program with $\bar{x}_j = \widehat{x}'_j - \widehat{x}''_j$ and with the same objective value. Also, any feasible solution $\bar{x}$ to the original linear program corresponds to a feasible solution $\widehat{x}$ to the new linear program with $\widehat{x}'_j = \bar{x}_j$ and $\widehat{x}''_j = 0$ if $\bar{x}_j \geq 0$, or with $\widehat{x}''_j = \bar{x}_j$ and $\widehat{x}'_j = 0$ if $\bar{x}_j < 0$. The two linear programs have the same objective value regardless of the sign of $\bar{x}_j$. Thus, the two linear programs are equivalent. We apply this conversion scheme to each variable that does not have a nonnegativity constraint to yield an equivalent linear program in which all variables have nonnegativity constraints.

Continuing the example, we want to ensure that each variable has a corresponding nonnegativity constraint. Variable x_1 has such a constraint, but variable x_2 does not. Therefore, we replace x_2 by two variables x'_2 and x''_2, and we modify the linear program to obtain

$$\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x'_2 & + & 3x''_2 \\
\text{subject to} \\
& x_1 & + & x'_2 & - & x''_2 & = & 7 \\
& x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
& \multicolumn{3}{c}{x_1, x'_2, x''_2} & & & \geq & 0 \; .
\end{array} \tag{29.22}$$

Next, we convert equality constraints into inequality constraints. Suppose that a linear program has an equality constraint $f(x_1, x_2, \ldots, x_n) = b$. Since $x = y$ if and only if both $x \geq y$ and $x \leq y$, we can replace this equality constraint by the pair of inequality constraints $f(x_1, x_2, \ldots, x_n) \leq b$ and $f(x_1, x_2, \ldots, x_n) \geq b$. Repeating this conversion for each equality constraint yields a linear program in which all constraints are inequalities.

Finally, we can convert the greater-than-or-equal-to constraints to less-than-or-equal-to constraints by multiplying these constraints through by -1. That is, any inequality of the form

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$

is equivalent to

$$\sum_{j=1}^{n} -a_{ij} x_j \leq -b_i \ .$$

Thus, by replacing each coefficient a_{ij} by $-a_{ij}$ and each value b_i by $-b_i$, we obtain an equivalent less-than-or-equal-to constraint.

Finishing our example, we replace the equality in constraint (29.22) by two inequalities, obtaining

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2', x_2''} & \geq & 0 \ .
\end{array} \tag{29.23}$$

Finally, we negate constraint (29.23). For consistency in variable names, we rename x_2' to x_2 and x_2'' to x_3, obtaining the standard form

$$\text{maximize} \quad 2x_1 - 3x_2 + 3x_3 \tag{29.24}$$

subject to

$$x_1 + x_2 - x_3 \leq 7 \tag{29.25}$$

$$-x_1 - x_2 + x_3 \leq -7 \tag{29.26}$$

$$x_1 - 2x_2 + 2x_3 \leq 4 \tag{29.27}$$

$$x_1, x_2, x_3 \geq 0 \ . \tag{29.28}$$

Converting linear programs into slack form

To efficiently solve a linear program with the simplex algorithm, we prefer to express it in a form in which some of the constraints are equality constraints. More precisely, we shall convert it into a form in which the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities. Let

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \tag{29.29}$$

be an inequality constraint. We introduce a new variable s and rewrite inequality (29.29) as the two constraints

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j \; , \tag{29.30}$$

$$s \geq 0 \; . \tag{29.31}$$

We call s a ***slack variable*** because it measures the ***slack***, or difference, between the left-hand and right-hand sides of equation (29.29). (We shall soon see why we find it convenient to write the constraint with only the slack variable on the left-hand side.) Because inequality (29.29) is true if and only if both equation (29.30) and inequality (29.31) are true, we can convert each inequality constraint of a linear program in this way to obtain an equivalent linear program in which the only inequality constraints are the nonnegativity constraints. When converting from standard to slack form, we shall use x_{n+i} (instead of s) to denote the slack variable associated with the ith inequality. The ith constraint is therefore

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \; , \tag{29.32}$$

along with the nonnegativity constraint $x_{n+i} \geq 0$.

By converting each constraint of a linear program in standard form, we obtain a linear program in a different form. For example, for the linear program described in (29.24)–(29.28), we introduce slack variables x_4, x_5, and x_6, obtaining

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 & & (29.33) \\
\text{subject to} & & & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 & & (29.34) \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 & & (29.35) \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 & & (29.36) \\
 & \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & . & & & (29.37)
\end{array}$$

In this linear program, all the constraints except for the nonnegativity constraints are equalities, and each variable is subject to a nonnegativity constraint. We write each equality constraint with one of the variables on the left-hand side of the equality and all others on the right-hand side. Furthermore, each equation has the same set of variables on the right-hand side, and these variables are also the only ones that appear in the objective function. We call the variables on the left-hand side of the equalities ***basic variables*** and those on the right-hand side ***nonbasic variables***.

For linear programs that satisfy these conditions, we shall sometimes omit the words "maximize" and "subject to," as well as the explicit nonnegativity constraints. We shall also use the variable z to denote the value of the objective function. We call the resulting format ***slack form***. If we write the linear program given in (29.33)–(29.37) in slack form, we obtain

$$
\begin{aligned}
z &= & & 2x_1 &-& 3x_2 &+& 3x_3 & \quad (29.38)\\
x_4 &= 7 &-& x_1 &-& x_2 &+& x_3 & \quad (29.39)\\
x_5 &= -7 &+& x_1 &+& x_2 &-& x_3 & \quad (29.40)\\
x_6 &= 4 &-& x_1 &+& 2x_2 &-& 2x_3 \; . & \quad (29.41)
\end{aligned}
$$

As with standard form, we find it convenient to have a more concise notation for describing a slack form. As we shall see in Section 29.3, the sets of basic and nonbasic variables will change as the simplex algorithm runs. We use N to denote the set of indices of the nonbasic variables and B to denote the set of indices of the basic variables. We always have that $|N| = n$, $|B| = m$, and $N \cup B = \{1, 2, \ldots, n + m\}$. The equations are indexed by the entries of B, and the variables on the right-hand sides are indexed by the entries of N. As in standard form, we use b_i, c_j, and a_{ij} to denote constant terms and coefficients. We also use ν to denote an optional constant term in the objective function. (We shall see a little later that including the constant term in the objective function makes it easy to determine the value of the objective function.) Thus we can concisely define a slack form by a tuple (N, B, A, b, c, ν), denoting the slack form

$$z = \nu + \sum_{j \in N} c_j x_j \qquad (29.42)$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B \, , \qquad (29.43)$$

in which all variables x are constrained to be nonnegative. Because we subtract the sum $\sum_{j \in N} a_{ij} x_j$ in (29.43), the values a_{ij} are actually the negatives of the coefficients as they "appear" in the slack form.

For example, in the slack form

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}\\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}\\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}\\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} \; ,
\end{aligned}
$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

$c = \begin{pmatrix} c_3 & c_5 & c_6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1/6 & -1/6 & -2/3 \end{pmatrix}^{\mathrm{T}}$, and $v = 28$. Note that the indices into A, b, and c are not necessarily sets of contiguous integers; they depend on the index sets B and N. As an example of the entries of A being the negatives of the coefficients as they appear in the slack form, observe that the equation for x_1 includes the term $x_3/6$, yet the coefficient a_{13} is actually $-1/6$ rather than $+1/6$.

Exercises

29.1-1
If we express the linear program in (29.24)–(29.28) in the compact notation of (29.19)–(29.21), what are n, m, A, b, and c?

29.1-2
Give three feasible solutions to the linear program in (29.24)–(29.28). What is the objective value of each one?

29.1-3
For the slack form in (29.38)–(29.41), what are N, B, A, b, c, and v?

29.1-4
Convert the following linear program into standard form:

$$\begin{array}{lrcrcrcr}
\text{minimize} & 2x_1 & + & 7x_2 & + & x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & & & - & x_3 & = & 7 \\
& 3x_1 & + & x_2 & & & \geq & 24 \\
& & & x_2 & & & \geq & 0 \\
& & & & & x_3 & \leq & 0 \ .
\end{array}$$

29.1-5

Convert the following linear program into slack form:

$$\begin{array}{lrcrcrcl}
\text{maximize} & 2x_1 & & & - & 6x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& 3x_1 & - & x_2 & & & \geq & 8 \\
& -x_1 & + & 2x_2 & + & 2x_3 & \geq & 0 \\
& & x_1, x_2, x_3 & & & & \geq & 0 \; .
\end{array}$$

What are the basic and nonbasic variables?

29.1-6

Show that the following linear program is infeasible:

$$\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & - & 2x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & \leq & 2 \\
& -2x_1 & - & 2x_2 & \leq & -10 \\
& & x_1, x_2 & & \geq & 0 \; .
\end{array}$$

29.1-7

Show that the following linear program is unbounded:

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& -2x_1 & + & x_2 & \leq & -1 \\
& -x_1 & - & 2x_2 & \leq & -2 \\
& & x_1, x_2 & & \geq & 0 \; .
\end{array}$$

29.1-8

Suppose that we have a general linear program with n variables and m constraints, and suppose that we convert it into standard form. Give an upper bound on the number of variables and constraints in the resulting linear program.

29.1-9

Give an example of a linear program for which the feasible region is not bounded, but the optimal objective value is finite.

29.2 Formulating problems as linear programs

Although we shall focus on the simplex algorithm in this chapter, it is also important to be able to recognize when we can formulate a problem as a linear program. Once we cast a problem as a polynomial-sized linear program, we can solve it in polynomial time by the ellipsoid algorithm or interior-point methods. Several linear-programming software packages can solve problems efficiently, so that once the problem is in the form of a linear program, such a package can solve it.

We shall look at several concrete examples of linear-programming problems. We start with two problems that we have already studied: the single-source shortest-paths problem (see Chapter 24) and the maximum-flow problem (see Chapter 26). We then describe the minimum-cost-flow problem. Although the minimum-cost-flow problem has a polynomial-time algorithm that is not based on linear programming, we won't describe the algorithm. Finally, we describe the multicommodity-flow problem, for which the only known polynomial-time algorithm is based on linear programming.

When we solved graph problems in Part VI, we used attribute notation, such as $v.d$ and $(u,v).f$. Linear programs typically use subscripted variables rather than objects with attached attributes, however. Therefore, when we express variables in linear programs, we shall indicate vertices and edges through subscripts. For example, we denote the shortest-path weight for vertex v not by $v.d$ but by d_v. Similarly, we denote the flow from vertex u to vertex v not by $(u,v).f$ but by f_{uv}. For quantities that are given as inputs to problems, such as edge weights or capacities, we shall continue to use notations such as $w(u,v)$ and $c(u.v)$.

Shortest paths

We can formulate the single-source shortest-paths problem as a linear program. In this section, we shall focus on how to formulate the single-pair shortest-path problem, leaving the extension to the more general single-source shortest-paths problem as Exercise 29.2-3.

In the single-pair shortest-path problem, we are given a weighted, directed graph $G = (V,E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the value d_t, which is the weight of a shortest path from s to t. To express this problem as a linear program, we need to determine a set of variables and constraints that define when we have a shortest path from s to t. Fortunately, the Bellman-Ford algorithm does exactly this. When the Bellman-Ford algorithm terminates, it has computed, for each vertex v, a value d_v (using subscript notation here rather than attribute notation) such that for each edge $(u,v) \in E$, we have $d_v \le d_u + w(u,v)$.

The source vertex initially receives a value $d_s = 0$, which never changes. Thus we obtain the following linear program to compute the shortest-path weight from s to t:

$$\begin{array}{llll} \text{maximize} & d_t & & (29.44) \\ \text{subject to} & & & \\ & d_v \le d_u + w(u,v) & \text{for each edge } (u,v) \in E\ , & (29.45) \\ & d_s = 0\ . & & (29.46) \end{array}$$

You might be surprised that this linear program maximizes an objective function when it is supposed to compute shortest paths. We do not want to minimize the objective function, since then setting $\bar{d}_v = 0$ for all $v \in V$ would yield an optimal solution to the linear program without solving the shortest-paths problem. We maximize because an optimal solution to the shortest-paths problem sets each $\bar{d}_v$ to $\min_{u:(u,v)\in E} \left\{\bar{d}_u + w(u,v)\right\}$, so that $\bar{d}_v$ is the largest value that is less than or equal to all of the values in the set $\left\{\bar{d}_u + w(u,v)\right\}$. We want to maximize d_v for all vertices v on a shortest path from s to t subject to these constraints on all vertices v, and maximizing d_t achieves this goal.

This linear program has $|V|$ variables d_v, one for each vertex $v \in V$. It also has $|E| + 1$ constraints: one for each edge, plus the additional constraint that the source vertex's shortest-path weight always has the value 0.

Maximum flow

Next, we express the maximum-flow problem as a linear program. Recall that we are given a directed graph $G = (V, E)$ in which each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v) \ge 0$, and two distinguished vertices: a source s and a sink t. As defined in Section 26.1, a flow is a nonnegative real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the capacity constraint and flow conservation. A maximum flow is a flow that satisfies these constraints and maximizes the flow value, which is the total flow coming out of the source minus the total flow into the source. A flow, therefore, satisfies linear constraints, and the value of a flow is a linear function. Recalling also that we assume that $c(u,v) = 0$ if $(u,v) \notin E$ and that there are no antiparallel edges, we can express the maximum-flow problem as a linear program:

$$\begin{array}{llll} \text{maximize} & \displaystyle\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & & (29.47) \\ \text{subject to} & & & \\ & f_{uv} \le c(u,v) & \text{for each } u,v \in V\ , & (29.48) \\ & \displaystyle\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv} & \text{for each } u \in V - \{s,t\}\ , & (29.49) \\ & f_{uv} \ge 0 & \text{for each } u,v \in V\ . & (29.50) \end{array}$$

This linear program has $|V|^2$ variables, corresponding to the flow between each pair of vertices, and it has $2|V|^2 + |V| - 2$ constraints.

It is usually more efficient to solve a smaller-sized linear program. The linear program in (29.47)–(29.50) has, for ease of notation, a flow and capacity of 0 for each pair of vertices u, v with $(u, v) \notin E$. It would be more efficient to rewrite the linear program so that it has $O(V + E)$ constraints. Exercise 29.2-5 asks you to do so.

Minimum-cost flow

In this section, we have used linear programming to solve problems for which we already knew efficient algorithms. In fact, an efficient algorithm designed specifically for a problem, such as Dijkstra's algorithm for the single-source shortest-paths problem, or the push-relabel method for maximum flow, will often be more efficient than linear programming, both in theory and in practice.

The real power of linear programming comes from the ability to solve new problems. Recall the problem faced by the politician in the beginning of this chapter. The problem of obtaining a sufficient number of votes, while not spending too much money, is not solved by any of the algorithms that we have studied in this book, yet we can solve it by linear programming. Books abound with such real-world problems that linear programming can solve. Linear programming is also particularly useful for solving variants of problems for which we may not already know of an efficient algorithm.

Consider, for example, the following generalization of the maximum-flow problem. Suppose that, in addition to a capacity $c(u, v)$ for each edge (u, v), we are given a real-valued cost $a(u, v)$. As in the maximum-flow problem, we assume that $c(u, v) = 0$ if $(u, v) \notin E$, and that there are no antiparallel edges. If we send f_{uv} units of flow over edge (u, v), we incur a cost of $a(u, v) f_{uv}$. We are also given a flow demand d. We wish to send d units of flow from s to t while minimizing the total cost $\sum_{(u,v)\in E} a(u, v) f_{uv}$ incurred by the flow. This problem is known as the ***minimum-cost-flow problem***.

Figure 29.3(a) shows an example of the minimum-cost-flow problem. We wish to send 4 units of flow from s to t while incurring the minimum total cost. Any particular legal flow, that is, a function f satisfying constraints (29.48)–(29.49), incurs a total cost of $\sum_{(u,v)\in E} a(u, v) f_{uv}$. We wish to find the particular 4-unit flow that minimizes this cost. Figure 29.3(b) shows an optimal solution, with total cost $\sum_{(u,v)\in E} a(u, v) f_{uv} = (2 \cdot 2) + (5 \cdot 2) + (3 \cdot 1) + (7 \cdot 1) + (1 \cdot 3) = 27$.

There are polynomial-time algorithms specifically designed for the minimum-cost-flow problem, but they are beyond the scope of this book. We can, however, express the minimum-cost-flow problem as a linear program. The linear program looks similar to the one for the maximum-flow problem with the additional con-

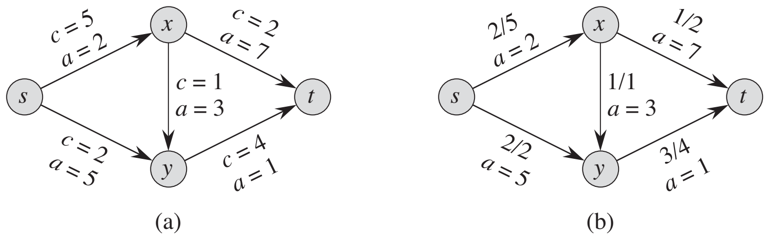


Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

straint that the value of the flow be exactly d units, and with the new objective function of minimizing the cost:

$$\text{minimize} \quad \sum_{(u,v)\in E} a(u,v) f_{uv} \tag{29.51}$$

subject to

$$\begin{aligned}
f_{uv} &\le c(u,v) && \text{for each } u,v \in V\ , \\
\sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} &= 0 && \text{for each } u \in V - \{s,t\}\ , \\
\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\ , \\
f_{uv} &\ge 0 && \text{for each } u,v \in V\ .
\end{aligned} \tag{29.52}$$

Multicommodity flow

As a final example, we consider another flow problem. Suppose that the Lucky Puck company from Section 26.1 decides to diversify its product line and ship not only hockey pucks, but also hockey sticks and hockey helmets. Each piece of equipment is manufactured in its own factory, has its own warehouse, and must be shipped, each day, from factory to warehouse. The sticks are manufactured in Vancouver and must be shipped to Saskatoon, and the helmets are manufactured in Edmonton and must be shipped to Regina. The capacity of the shipping network does not change, however, and the different items, or ***commodities***, must share the same network.

This example is an instance of a ***multicommodity-flow problem***. In this problem, we are again given a directed graph $G = (V, E)$ in which each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v) \ge 0$. As in the maximum-flow problem, we implicitly assume that $c(u,v) = 0$ for $(u,v) \notin E$, and that the graph has no antiparallel edges. In addition, we are given k different commodities, $K_1, K_2, \ldots, K_k$, where we specify commodity i by the triple $K_i = (s_i, t_i, d_i)$. Here, vertex s_i is the source of commodity i, vertex t_i is the sink of commodity i, and d_i is the demand for commodity i, which is the desired flow value for the commodity from s_i to t_i. We define a flow for commodity i, denoted by f_i, (so that f_{iuv} is the flow of commodity i from vertex u to vertex v) to be a real-valued function that satisfies the flow-conservation and capacity constraints. We now define f_{uv}, the ***aggregate flow***, to be the sum of the various commodity flows, so that $f_{uv} = \sum_{i=1}^{k} f_{iuv}$. The aggregate flow on edge (u, v) must be no more than the capacity of edge (u, v). We are not trying to minimize any objective function in this problem; we need only determine whether such a flow exists. Thus, we write a linear program with a "null" objective function:

$$\begin{array}{lrcll}
\text{minimize} & & & 0 & \\
\text{subject to} & & & & \\
& \displaystyle\sum_{i=1}^{k} f_{iuv} & \le & c(u,v) & \text{for each } u, v \in V \, , \\
& \displaystyle\sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} & = & 0 & \text{for each } i = 1, 2, \ldots, k \text{ and} \\
& & & & \text{for each } u \in V - \{s_i, t_i\} \, , \\
& \displaystyle\sum_{v \in V} f_{i,s_i,v} - \sum_{v \in V} f_{i,v,s_i} & = & d_i & \text{for each } i = 1, 2, \ldots, k \, , \\
& f_{iuv} & \ge & 0 & \text{for each } u, v \in V \text{ and} \\
& & & & \text{for each } i = 1, 2, \ldots, k \, .
\end{array}$$

The only known polynomial-time algorithm for this problem expresses it as a linear program and then solves it with a polynomial-time linear-programming algorithm.

Exercises

29.2-1
Put the single-pair shortest-path linear program from (29.44)–(29.46) into standard form.

29.2-2
Write out explicitly the linear program corresponding to finding the shortest path from node s to node y in Figure 24.2(a).

29.2-3
In the single-source shortest-paths problem, we want to find the shortest-path weights from a source vertex s to all vertices $v \in V$. Given a graph G, write a

linear program for which the solution has the property that d_v is the shortest-path weight from s to v for each vertex $v \in V$.

29.2-4
Write out explicitly the linear program corresponding to finding the maximum flow in Figure 26.1(a).

29.2-5
Rewrite the linear program for maximum flow (29.47)–(29.50) so that it uses only $O(V + E)$ constraints.

29.2-6
Write a linear program that, given a bipartite graph $G = (V, E)$, solves the maximum-bipartite-matching problem.

29.2-7
In the ***minimum-cost multicommodity-flow problem***, we are given directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$ and a cost $a(u, v)$. As in the multicommodity-flow problem, we are given k different commodities, $K_1, K_2, \ldots, K_k$, where we specify commodity i by the triple $K_i = (s_i, t_i, d_i)$. We define the flow f_i for commodity i and the aggregate flow f_{uv} on edge (u, v) as in the multicommodity-flow problem. A feasible flow is one in which the aggregate flow on each edge (u, v) is no more than the capacity of edge (u, v). The cost of a flow is $\sum_{u,v \in V} a(u, v) f_{uv}$, and the goal is to find the feasible flow of minimum cost. Express this problem as a linear program.

29.3 The simplex algorithm

The simplex algorithm is the classical method for solving linear programs. In contrast to most of the other algorithms in this book, its running time is not polynomial in the worst case. It does yield insight into linear programs, however, and is often remarkably fast in practice.

In addition to having a geometric interpretation, described earlier in this chapter, the simplex algorithm bears some similarity to Gaussian elimination, discussed in Section 28.1. Gaussian elimination begins with a system of linear equalities whose solution is unknown. In each iteration, we rewrite this system in an equivalent form that has some additional structure. After some number of iterations, we have rewritten the system so that the solution is simple to obtain. The simplex algorithm proceeds in a similar manner, and we can view it as Gaussian elimination for inequalities.

We now describe the main idea behind an iteration of the simplex algorithm. Associated with each iteration will be a "basic solution" that we can easily obtain from the slack form of the linear program: set each nonbasic variable to 0 and compute the values of the basic variables from the equality constraints. An iteration converts one slack form into an equivalent slack form. The objective value of the associated basic feasible solution will be no less than that at the previous iteration, and usually greater. To achieve this increase in the objective value, we choose a nonbasic variable such that if we were to increase that variable's value from 0, then the objective value would increase, too. The amount by which we can increase the variable is limited by the other constraints. In particular, we raise it until some basic variable becomes 0. We then rewrite the slack form, exchanging the roles of that basic variable and the chosen nonbasic variable. Although we have used a particular setting of the variables to guide the algorithm, and we shall use it in our proofs, the algorithm does not explicitly maintain this solution. It simply rewrites the linear program until an optimal solution becomes "obvious."

An example of the simplex algorithm

We begin with an extended example. Consider the following linear program in standard form:

$$\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0\ .
\end{array}
\qquad
\begin{array}{r}
(29.53) \\ \\ (29.54) \\ (29.55) \\ (29.56) \\ (29.57)
\end{array}$$

In order to use the simplex algorithm, we must convert the linear program into slack form; we saw how to do so in Section 29.1. In addition to being an algebraic manipulation, slack is a useful algorithmic concept. Recalling from Section 29.1 that each variable has a corresponding nonnegativity constraint, we say that an equality constraint is ***tight*** for a particular setting of its nonbasic variables if they cause the constraint's basic variable to become 0. Similarly, a setting of the nonbasic variables that would make a basic variable become negative ***violates*** that constraint. Thus, the slack variables explicitly maintain how far each constraint is from being tight, and so they help to determine how much we can increase values of nonbasic variables without violating any constraints.

Associating the slack variables x_4, x_5, and x_6 with inequalities (29.54)–(29.56), respectively, and putting the linear program into slack form, we obtain

$$\begin{array}{rcrcrcrcrl} z & = & & & 3x_1 & + & x_2 & + & 2x_3 & \qquad (29.58) \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & \qquad (29.59) \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 & \qquad (29.60) \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \; . & \qquad (29.61) \end{array}$$

The system of constraints (29.59)–(29.61) has 3 equations and 6 variables. Any setting of the variables x_1, x_2, and x_3 defines values for x_4, x_5, and x_6; therefore, we have an infinite number of solutions to this system of equations. A solution is feasible if all of $x_1, x_2, \ldots, x_6$ are nonnegative, and there can be an infinite number of feasible solutions as well. The infinite number of possible solutions to a system such as this one will be useful in later proofs. We focus on the ***basic solution***: set all the (nonbasic) variables on the right-hand side to 0 and then compute the values of the (basic) variables on the left-hand side. In this example, the basic solution is $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6) = (0, 0, 0, 30, 24, 36)$ and it has objective value $z = (3 \cdot 0) + (1 \cdot 0) + (2 \cdot 0) = 0$. Observe that this basic solution sets $\bar{x}_i = b_i$ for each $i \in B$. An iteration of the simplex algorithm rewrites the set of equations and the objective function so as to put a different set of variables on the right-hand side. Thus, a different basic solution is associated with the rewritten problem. We emphasize that the rewrite does not in any way change the underlying linear-programming problem; the problem at one iteration has the identical set of feasible solutions as the problem at the previous iteration. The problem does, however, have a different basic solution than that of the previous iteration.

If a basic solution is also feasible, we call it a ***basic feasible solution***. As we run the simplex algorithm, the basic solution is almost always a basic feasible solution. We shall see in Section 29.5, however, that for the first few iterations of the simplex algorithm, the basic solution might not be feasible.

Our goal, in each iteration, is to reformulate the linear program so that the basic solution has a greater objective value. We select a nonbasic variable x_e whose coefficient in the objective function is positive, and we increase the value of x_e as much as possible without violating any of the constraints. The variable x_e becomes basic, and some other variable x_l becomes nonbasic. The values of other basic variables and of the objective function may also change.

To continue the example, let's think about increasing the value of x_1. As we increase x_1, the values of x_4, x_5, and x_6 all decrease. Because we have a nonnegativity constraint for each variable, we cannot allow any of them to become negative. If x_1 increases above 30, then x_4 becomes negative, and x_5 and x_6 become negative when x_1 increases above 12 and 9, respectively. The third constraint (29.61) is the tightest constraint, and it limits how much we can increase x_1. Therefore, we switch the roles of x_1 and x_6. We solve equation (29.61) for x_1 and obtain

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \; . \qquad (29.62)$$

To rewrite the other equations with x_6 on the right-hand side, we substitute for x_1 using equation (29.62). Doing so for equation (29.59), we obtain

$$\begin{aligned} x_4 &= 30 - x_1 - x_2 - 3x_3 \\ &= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\right) - x_2 - 3x_3 \\ &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} . \end{aligned} \tag{29.63}$$

Similarly, we combine equation (29.62) with constraint (29.60) and with objective function (29.58) to rewrite our linear program in the following form:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \tag{29.64}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \tag{29.65}$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \tag{29.66}$$

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} . \tag{29.67}$$

We call this operation a ***pivot***. As demonstrated above, a pivot chooses a nonbasic variable x_e, called the ***entering variable***, and a basic variable x_l, called the ***leaving variable***, and exchanges their roles.

The linear program described in equations (29.64)–(29.67) is equivalent to the linear program described in equations (29.58)–(29.61). We perform two operations in the simplex algorithm: rewrite equations so that variables move between the left-hand side and the right-hand side, and substitute one equation into another. The first operation trivially creates an equivalent problem, and the second, by elementary linear algebra, also creates an equivalent problem. (See Exercise 29.3-3.)

To demonstrate this equivalence, observe that our original basic solution $(0, 0, 0, 30, 24, 36)$ satisfies the new equations (29.65)–(29.67) and has objective value $27 + (1/4) \cdot 0 + (1/2) \cdot 0 - (3/4) \cdot 36 = 0$. The basic solution associated with the new linear program sets the nonbasic values to 0 and is $(9, 0, 0, 21, 6, 0)$, with objective value $z = 27$. Simple arithmetic verifies that this solution also satisfies equations (29.59)–(29.61) and, when plugged into objective function (29.58), has objective value $(3 \cdot 9) + (1 \cdot 0) + (2 \cdot 0) = 27$.

Continuing the example, we wish to find a new variable whose value we wish to increase. We do not want to increase x_6, since as its value increases, the objective value decreases. We can attempt to increase either x_2 or x_3; let us choose x_3. How far can we increase x_3 without violating any of the constraints? Constraint (29.65) limits it to 18, constraint (29.66) limits it to $42/5$, and constraint (29.67) limits it to $3/2$. The third constraint is again the tightest one, and therefore we rewrite the third constraint so that x_3 is on the left-hand side and x_5 is on the right-hand side. We then substitute this new equation, $x_3 = 3/2 - 3x_2/8 - x_5/4 + x_6/8$, into equations (29.64)–(29.66) and obtain the new, but equivalent, system

$$\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} && (29.68)\\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} && (29.69)\\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} && (29.70)\\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} . && (29.71)
\end{aligned}$$

This system has the associated basic solution $(33/4, 0, 3/2, 69/4, 0, 0)$, with objective value $111/4$. Now the only way to increase the objective value is to increase x_2. The three constraints give upper bounds of 132, 4, and ∞, respectively. (We get an upper bound of ∞ from constraint (29.71) because, as we increase x_2, the value of the basic variable x_4 increases also. This constraint, therefore, places no restriction on how much we can increase x_2.) We increase x_2 to 4, and it becomes nonbasic. Then we solve equation (29.70) for x_2 and substitute in the other equations to obtain

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} && (29.72)\\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} && (29.73)\\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} && (29.74)\\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} . && (29.75)
\end{aligned}$$

At this point, all coefficients in the objective function are negative. As we shall see later in this chapter, this situation occurs only when we have rewritten the linear program so that the basic solution is an optimal solution. Thus, for this problem, the solution $(8, 4, 0, 18, 0, 0)$, with objective value 28, is optimal. We can now return to our original linear program given in (29.53)–(29.57). The only variables in the original linear program are x_1, x_2, and x_3, and so our solution is $x_1 = 8$, $x_2 = 4$, and $x_3 = 0$, with objective value $(3 \cdot 8) + (1 \cdot 4) + (2 \cdot 0) = 28$. Note that the values of the slack variables in the final solution measure how much slack remains in each inequality. Slack variable x_4 is 18, and in inequality (29.54), the left-hand side, with value $8 + 4 + 0 = 12$, is 18 less than the right-hand side of 30. Slack variables x_5 and x_6 are 0 and indeed, in inequalities (29.55) and (29.56), the left-hand and right-hand sides are equal. Observe also that even though the coefficients in the original slack form are integral, the coefficients in the other linear programs are not necessarily integral, and the intermediate solutions are not necessarily integral. Furthermore, the final solution to a linear program need not be integral; it is purely coincidental that this example has an integral solution.

Pivoting

We now formalize the procedure for pivoting. The procedure PIVOT takes as input a slack form, given by the tuple (N, B, A, b, c, ν), the index l of the leaving variable x_l, and the index e of the entering variable x_e. It returns the tuple $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{\nu})$ describing the new slack form. (Recall again that the entries of the $m \times n$ matrices A and $\widehat{A}$ are actually the negatives of the coefficients that appear in the slack form.)

```
PIVOT$(N, B, A, b, c, \nu, l, e)$
1   // Compute the coefficients of the equation for new basic variable $x_e$.
2   let $\widehat{A}$ be a new $m \times n$ matrix
3   $\widehat{b}_e = b_l / a_{le}$
4   for each $j \in N - \{e\}$
5       $\widehat{a}_{ej} = a_{lj} / a_{le}$
6   $\widehat{a}_{el} = 1 / a_{le}$
7   // Compute the coefficients of the remaining constraints.
8   for each $i \in B - \{l\}$
9       $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
10      for each $j \in N - \{e\}$
11          $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
12      $\widehat{a}_{il} = -a_{ie} \widehat{a}_{el}$
13  // Compute the objective function.
14  $\widehat{\nu} = \nu + c_e \widehat{b}_e$
15  for each $j \in N - \{e\}$
16      $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
17  $\widehat{c}_l = -c_e \widehat{a}_{el}$
18  // Compute new sets of basic and nonbasic variables.
19  $\widehat{N} = N - \{e\} \cup \{l\}$
20  $\widehat{B} = B - \{l\} \cup \{e\}$
21  return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{\nu})$
```

PIVOT works as follows. Lines 3–6 compute the coefficients in the new equation for x_e by rewriting the equation that has x_l on the left-hand side to instead have x_e on the left-hand side. Lines 8–12 update the remaining equations by substituting the right-hand side of this new equation for each occurrence of x_e. Lines 14–17 do the same substitution for the objective function, and lines 19 and 20 update the sets of nonbasic and basic variables. Line 21 returns the new slack form. As given, if $a_{le} = 0$, PIVOT would cause an error by dividing by 0, but as we shall see in the proofs of Lemmas 29.2 and 29.12, we call PIVOT only when $a_{le} \neq 0$.

We now summarize the effect that PIVOT has on the values of the variables in the basic solution.

Lemma 29.1
Consider a call to $\text{PIVOT}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\bar{x}$ denote the basic solution after the call. Then

1. $\bar{x}_j = 0$ for each $j \in \widehat{N}$.
2. $\bar{x}_e = b_l / a_{le}$.
3. $\bar{x}_i = b_i - a_{ie}\widehat{b}_e$ for each $i \in \widehat{B} - \{e\}$.

Proof The first statement is true because the basic solution always sets all nonbasic variables to 0. When we set each nonbasic variable to 0 in a constraint

$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j ,$$

we have that $\bar{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Since $e \in \widehat{B}$, line 3 of PIVOT gives

$$\bar{x}_e = \widehat{b}_e = b_l / a_{le} ,$$

which proves the second statement. Similarly, using line 9 for each $i \in \widehat{B} - \{e\}$, we have

$$\bar{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e ,$$

which proves the third statement. ∎

The formal simplex algorithm

We are now ready to formalize the simplex algorithm, which we demonstrated by example. That example was a particularly nice one, and we could have had several other issues to address:

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

In Section 29.5, we shall show how to determine whether a problem is feasible, and if so, how to find a slack form in which the initial basic solution is feasible. Therefore, let us assume that we have a procedure INITIALIZE-SIMPLEX(A, b, c) that takes as input a linear program in standard form, that is, an $m \times n$ matrix $A = (a_{ij})$, an m-vector $b = (b_i)$, and an n-vector $c = (c_j)$. If the problem is infeasible, the procedure returns a message that the program is infeasible and then terminates. Otherwise, the procedure returns a slack form for which the initial basic solution is feasible.

The procedure SIMPLEX takes as input a linear program in standard form, as just described. It returns an n-vector $\bar{x} = (\bar{x}_j)$ that is an optimal solution to the linear program described in (29.19)–(29.21).

```
SIMPLEX$(A, b, c)$
 1  $(N, B, A, b, c, v)$ = INITIALIZE-SIMPLEX$(A, b, c)$
 2  let $\Delta$ be a new vector of length $n$
 3  while some index $j \in N$ has $c_j > 0$
 4      choose an index $e \in N$ for which $c_e > 0$
 5      for each index $i \in B$
 6          if $a_{ie} > 0$
 7              $\Delta_i = b_i / a_{ie}$
 8          else $\Delta_i = \infty$
 9      choose an index $l \in B$ that minimizes $\Delta_i$
10      if $\Delta_l == \infty$
11          return "unbounded"
12      else $(N, B, A, b, c, v)$ = PIVOT$(N, B, A, b, c, v, l, e)$
13  for $i = 1$ to $n$
14      if $i \in B$
15          $\bar{x}_i = b_i$
16      else $\bar{x}_i = 0$
17  return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

The SIMPLEX procedure works as follows. In line 1, it calls the procedure INITIALIZE-SIMPLEX(A, b, c), described above, which either determines that the linear program is infeasible or returns a slack form for which the basic solution is feasible. The **while** loop of lines 3–12 forms the main part of the algorithm. If all coefficients in the objective function are negative, then the **while** loop terminates. Otherwise, line 4 selects a variable x_e, whose coefficient in the objective function is positive, as the entering variable. Although we may choose any such variable as the entering variable, we assume that we use some prespecified deterministic rule. Next, lines 5–9 check each constraint and pick the one that most severely limits the amount by which we can increase x_e without violating any of the nonnegativity constraints; the basic variable associated with this constraint is x_l. Again, we are free to choose one of several variables as the leaving variable, but we assume that we use some prespecified deterministic rule. If none of the constraints limits the amount by which the entering variable can increase, the algorithm returns "unbounded" in line 11. Otherwise, line 12 exchanges the roles of the entering and leaving variables by calling PIVOT$(N, B, A, b, c, \nu, l, e)$, as described above. Lines 13–16 compute a solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ for the original linear-programming variables by setting all the nonbasic variables to 0 and each basic variable $\bar{x}_i$ to b_i, and line 17 returns these values.

To show that SIMPLEX is correct, we first show that if SIMPLEX has an initial feasible solution and eventually terminates, then it either returns a feasible solution or determines that the linear program is unbounded. Then, we show that SIMPLEX terminates. Finally, in Section 29.4 (Theorem 29.10) we show that the solution returned is optimal.

Lemma 29.2
Given a linear program (A, b, c), suppose that the call to INITIALIZE-SIMPLEX in line 1 of SIMPLEX returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution in line 17, that solution is a feasible solution to the linear program. If SIMPLEX returns "unbounded" in line 11, the linear program is unbounded.

Proof We use the following three-part loop invariant:

At the start of each iteration of the **while** loop of lines 3–12,

1. the slack form is equivalent to the slack form returned by the call of INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$, and
3. the basic solution associated with the slack form is feasible.

Initialization: The equivalence of the slack forms is trivial for the first iteration. We assume, in the statement of the lemma, that the call to INITIALIZE-SIMPLEX in line 1 of SIMPLEX returns a slack form for which the basic solution is feasible. Thus, the third part of the invariant is true. Because the basic solution is feasible, each basic variable x_i is nonnegative. Furthermore, since the basic solution sets each basic variable x_i to b_i, we have that $b_i \geq 0$ for all $i \in B$. Thus, the second part of the invariant holds.

Maintenance: We shall show that each iteration of the **while** loop maintains the loop invariant, assuming that the **return** statement in line 11 does not execute. We shall handle the case in which line 11 executes when we discuss termination.

An iteration of the **while** loop exchanges the role of a basic and a nonbasic variable by calling the PIVOT procedure. By Exercise 29.3-3, the slack form is equivalent to the one from the previous iteration which, by the loop invariant, is equivalent to the initial slack form.

We now demonstrate the second part of the loop invariant. We assume that at the start of each iteration of the **while** loop, $b_i \geq 0$ for each $i \in B$, and we shall show that these inequalities remain true after the call to PIVOT in line 12. Since the only changes to the variables b_i and the set B of basic variables occur in this assignment, it suffices to show that line 12 maintains this part of the invariant. We let b_i, a_{ij}, and B refer to values before the call of PIVOT, and $\widehat{b}_i$ refer to values returned from PIVOT.

First, we observe that $\widehat{b}_e \geq 0$ because $b_l \geq 0$ by the loop invariant, $a_{le} > 0$ by lines 6 and 9 of SIMPLEX, and $\widehat{b}_e = b_l/a_{le}$ by line 3 of PIVOT.

For the remaining indices $i \in B - \{l\}$, we have that

$$\begin{aligned} \widehat{b}_i &= b_i - a_{ie}\widehat{b}_e & \text{(by line 9 of PIVOT)} \\ &= b_i - a_{ie}(b_l/a_{le}) & \text{(by line 3 of PIVOT)} \; . \end{aligned} \tag{29.76}$$

We have two cases to consider, depending on whether $a_{ie} > 0$ or $a_{ie} \leq 0$. If $a_{ie} > 0$, then since we chose l such that

$$b_l/a_{le} \leq b_i/a_{ie} \quad \text{for all } i \in B \; , \tag{29.77}$$

we have

$$\begin{aligned} \widehat{b}_i &= b_i - a_{ie}(b_l/a_{le}) & \text{(by equation (29.76))} \\ &\geq b_i - a_{ie}(b_i/a_{ie}) & \text{(by inequality (29.77))} \\ &= b_i - b_i \\ &= 0 \; , \end{aligned}$$

and thus $\widehat{b}_i \geq 0$. If $a_{ie} \leq 0$, then because a_{le}, b_i, and b_l are all nonnegative, equation (29.76) implies that $\widehat{b}_i$ must be nonnegative, too.

We now argue that the basic solution is feasible, i.e., that all variables have nonnegative values. The nonbasic variables are set to 0 and thus are nonnegative. Each basic variable x_i is defined by the equation

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \; .$$

The basic solution sets $\bar{x}_i = b_i$. Using the second part of the loop invariant, we conclude that each basic variable $\bar{x}_i$ is nonnegative.

Termination: The **while** loop can terminate in one of two ways. If it terminates because of the condition in line 3, then the current basic solution is feasible and line 17 returns this solution. The other way it terminates is by returning "unbounded" in line 11. In this case, for each iteration of the **for** loop in lines 5–8, when line 6 is executed, we find that $a_{ie} \leq 0$. Consider the solution $\bar{x}$ defined as

$$\bar{x}_i = \begin{cases} \infty & \text{if } i = e \text{ ,} \\ 0 & \text{if } i \in N - \{e\} \text{ ,} \\ b_i - \sum_{j \in N} a_{ij}\bar{x}_j & \text{if } i \in B \text{ .} \end{cases}$$

We now show that this solution is feasible, i.e., that all variables are nonnegative. The nonbasic variables other than $\bar{x}_e$ are 0, and $\bar{x}_e = \infty > 0$; thus all nonbasic variables are nonnegative. For each basic variable $\bar{x}_i$, we have

$$\begin{aligned} \bar{x}_i &= b_i - \sum_{j \in N} a_{ij}\bar{x}_j \\ &= b_i - a_{ie}\bar{x}_e \text{ .} \end{aligned}$$

The loop invariant implies that $b_i \geq 0$, and we have $a_{ie} \leq 0$ and $\bar{x}_e = \infty > 0$. Thus, $\bar{x}_i \geq 0$.

Now we show that the objective value for the solution $\bar{x}$ is unbounded. From equation (29.42), the objective value is

$$\begin{aligned} z &= \nu + \sum_{j \in N} c_j \bar{x}_j \\ &= \nu + c_e \bar{x}_e \text{ .} \end{aligned}$$

Since $c_e > 0$ (by line 4 of SIMPLEX) and $\bar{x}_e = \infty$, the objective value is ∞, and thus the linear program is unbounded. ■

It remains to show that SIMPLEX terminates, and when it does terminate, the solution it returns is optimal. Section 29.4 will address optimality. We now discuss termination.

Termination

In the example given in the beginning of this section, each iteration of the simplex algorithm increased the objective value associated with the basic solution. As Exercise 29.3-2 asks you to show, no iteration of SIMPLEX can decrease the objective value associated with the basic solution. Unfortunately, it is possible that an iteration leaves the objective value unchanged. This phenomenon is called ***degeneracy***, and we shall now study it in greater detail.

The assignment in line 14 of PIVOT, $\hat{\nu} = \nu + c_e\hat{b}_e$, changes the objective value. Since SIMPLEX calls PIVOT only when $c_e > 0$, the only way for the objective value to remain unchanged (i.e., $\hat{\nu} = \nu$) is for $\hat{b}_e$ to be 0. This value is assigned as $\hat{b}_e = b_l/a_{le}$ in line 3 of PIVOT. Since we always call PIVOT with $a_{le} \neq 0$, we see that for $\hat{b}_e$ to equal 0, and hence the objective value to be unchanged, we must have $b_l = 0$.

Indeed, this situation can occur. Consider the linear program

$$\begin{array}{rcrcrcrcr} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \; . \end{array}$$

Suppose that we choose x_1 as the entering variable and x_4 as the leaving variable. After pivoting, we obtain

$$\begin{array}{rcrcrcrcr} z & = & 8 & & & + & x_3 & - & x_4 \\ x_1 & = & 8 & - & x_2 & & & - & x_4 \\ x_5 & = & & & x_2 & - & x_3 \; . & & \end{array}$$

At this point, our only choice is to pivot with x_3 entering and x_5 leaving. Since $b_5 = 0$, the objective value of 8 remains unchanged after pivoting:

$$\begin{array}{rcrcrcrcr} z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\ x_1 & = & 8 & - & x_2 & - & x_4 & & \\ x_3 & = & & & x_2 & & & - & x_5 \; . \end{array}$$

The objective value has not changed, but our slack form has. Fortunately, if we pivot again, with x_2 entering and x_1 leaving, the objective value increases (to 16), and the simplex algorithm can continue.

Degeneracy can prevent the simplex algorithm from terminating, because it can lead to a phenomenon known as ***cycling***: the slack forms at two different iterations of SIMPLEX are identical. Because of degeneracy, SIMPLEX could choose a sequence of pivot operations that leave the objective value unchanged but repeat a slack form within the sequence. Since SIMPLEX is a deterministic algorithm, if it cycles, then it will cycle through the same series of slack forms forever, never terminating.

Cycling is the only reason that SIMPLEX might not terminate. To show this fact, we must first develop some additional machinery.

At each iteration, SIMPLEX maintains A, b, c, and ν in addition to the sets N and B. Although we need to explicitly maintain A, b, c, and ν in order to implement the simplex algorithm efficiently, we can get by without maintaining them. In other words, the sets of basic and nonbasic variables suffice to uniquely determine the slack form. Before proving this fact, we prove a useful algebraic lemma.

Lemma 29.3

Let I be a set of indices. For each $j \in I$, let α_j and β_j be real numbers, and let x_j be a real-valued variable. Let γ be any real number. Suppose that for any settings of the x_j, we have

$$\sum_{j \in I} \alpha_j x_j = \gamma + \sum_{j \in I} \beta_j x_j \ . \tag{29.78}$$

Then $\alpha_j = \beta_j$ for each $j \in I$, and $\gamma = 0$.

Proof Since equation (29.78) holds for any values of the x_j, we can use particular values to draw conclusions about α, β, and γ. If we let $x_j = 0$ for each $j \in I$, we conclude that $\gamma = 0$. Now pick an arbitrary index $j \in I$, and set $x_j = 1$ and $x_k = 0$ for all $k \neq j$. Then we must have $\alpha_j = \beta_j$. Since we picked j as any index in I, we conclude that $\alpha_j = \beta_j$ for each $j \in I$. ■

A particular linear program has many different slack forms; recall that each slack form has the same set of feasible and optimal solutions as the original linear program. We now show that the slack form of a linear program is uniquely determined by the set of basic variables. That is, given the set of basic variables, a unique slack form (unique set of coefficients and right-hand sides) is associated with those basic variables.

Lemma 29.4

Let (A, b, c) be a linear program in standard form. Given a set B of basic variables, the associated slack form is uniquely determined.

Proof Assume for the purpose of contradiction that there are two different slack forms with the same set B of basic variables. The slack forms must also have identical sets $N = \{1, 2, \ldots, n + m\} - B$ of nonbasic variables. We write the first slack form as

$$z = v + \sum_{j \in N} c_j x_j \tag{29.79}$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B \ , \tag{29.80}$$

and the second as

$$z = v' + \sum_{j \in N} c'_j x_j \tag{29.81}$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B \ . \tag{29.82}$$

Consider the system of equations formed by subtracting each equation in line (29.82) from the corresponding equation in line (29.80). The resulting system is

$$0 = (b_i - b'_i) - \sum_{j \in N} (a_{ij} - a'_{ij}) x_j \quad \text{for } i \in B$$

or, equivalently,

$$\sum_{j \in N} a_{ij} x_j = (b_i - b'_i) + \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B \; .$$

Now, for each $i \in B$, apply Lemma 29.3 with $\alpha_j = a_{ij}$, $\beta_j = a'_{ij}$, $\gamma = b_i - b'_i$, and $I = N$. Since $\alpha_i = \beta_i$, we have that $a_{ij} = a'_{ij}$ for each $j \in N$, and since $\gamma = 0$, we have that $b_i = b'_i$. Thus, for the two slack forms, A and b are identical to A' and b'. Using a similar argument, Exercise 29.3-1 shows that it must also be the case that $c = c'$ and $\nu = \nu'$, and hence that the slack forms must be identical. ∎

We can now show that cycling is the only possible reason that SIMPLEX might not terminate.

Lemma 29.5
If SIMPLEX fails to terminate in at most $\binom{n+m}{m}$ iterations, then it cycles.

Proof By Lemma 29.4, the set B of basic variables uniquely determines a slack form. There are $n + m$ variables and $|B| = m$, and therefore, there are at most $\binom{n+m}{m}$ ways to choose B. Thus, there are only at most $\binom{n+m}{m}$ unique slack forms. Therefore, if SIMPLEX runs for more than $\binom{n+m}{m}$ iterations, it must cycle. ∎

Cycling is theoretically possible, but extremely rare. We can prevent it by choosing the entering and leaving variables somewhat more carefully. One option is to perturb the input slightly so that it is impossible to have two solutions with the same objective value. Another option is to break ties by always choosing the variable with the smallest index, a strategy known as ***Bland's rule***. We omit the proof that these strategies avoid cycling.

Lemma 29.6
If lines 4 and 9 of SIMPLEX always break ties by choosing the variable with the smallest index, then SIMPLEX must terminate. ∎

We conclude this section with the following lemma.

Lemma 29.7

Assuming that INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that a linear program is unbounded, or it terminates with a feasible solution in at most $\binom{n+m}{m}$ iterations.

Proof Lemmas 29.2 and 29.6 show that if INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that a linear program is unbounded, or it terminates with a feasible solution. By the contrapositive of Lemma 29.5, if SIMPLEX terminates with a feasible solution, then it terminates in at most $\binom{n+m}{m}$ iterations. ■

Exercises

29.3-1

Complete the proof of Lemma 29.4 by showing that it must be the case that $c = c'$ and $\nu = \nu'$.

29.3-2

Show that the call to PIVOT in line 12 of SIMPLEX never decreases the value of ν.

29.3-3

Prove that the slack form given to the PIVOT procedure and the slack form that the procedure returns are equivalent.

29.3-4

Suppose we convert a linear program (A, b, c) in standard form to slack form. Show that the basic solution is feasible if and only if $b_i \geq 0$ for $i = 1, 2, \ldots, m$.

29.3-5

Solve the following linear program using SIMPLEX:

$$\begin{array}{lrcrcr}
\text{maximize} & 18x_1 & + & 12.5x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & \leq & 20 \\
& x_1 & & & \leq & 12 \\
& & & x_2 & \leq & 16 \\
& & x_1, x_2 & & \geq & 0\ .
\end{array}$$

29.3-6

Solve the following linear program using SIMPLEX:

$$
\begin{array}{lrcrcl}
\text{maximize} & 5x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & - & x_2 & \leq & 1 \\
& 2x_1 & + & x_2 & \leq & 2 \\
& & x_1, x_2 & & \geq & 0 \; .
\end{array}
$$

29.3-7

Solve the following linear program using SIMPLEX:

$$
\begin{array}{lrcrcrcl}
\text{minimize} & x_1 & + & x_2 & + & x_3 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & + & 7.5x_2 & + & 3x_3 & \geq & 10000 \\
& 20x_1 & + & 5x_2 & + & 10x_3 & \geq & 30000 \\
& & x_1, x_2, x_3 & & & & \geq & 0 \; .
\end{array}
$$

29.3-8

In the proof of Lemma 29.5, we argued that there are at most $\binom{m+n}{n}$ ways to choose a set B of basic variables. Give an example of a linear program in which there are strictly fewer than $\binom{m+n}{n}$ ways to choose the set B.

29.4 Duality

We have proven that, under certain assumptions, SIMPLEX terminates. We have not yet shown that it actually finds an optimal solution to a linear program, however. In order to do so, we introduce a powerful concept called ***linear-programming duality***.

Duality enables us to prove that a solution is indeed optimal. We saw an example of duality in Chapter 26 with Theorem 26.6, the max-flow min-cut theorem. Suppose that, given an instance of a maximum-flow problem, we find a flow f with value $|f|$. How do we know whether f is a maximum flow? By the max-flow min-cut theorem, if we can find a cut whose value is also $|f|$, then we have verified that f is indeed a maximum flow. This relationship provides an example of duality: given a maximization problem, we define a related minimization problem such that the two problems have the same optimal objective values.

Given a linear program in which the objective is to maximize, we shall describe how to formulate a ***dual*** linear program in which the objective is to minimize and

33 Computational Geometry

Computational geometry is the branch of computer science that studies algorithms for solving geometric problems. In modern engineering and mathematics, computational geometry has applications in such diverse fields as computer graphics, robotics, VLSI design, computer-aided design, molecular modeling, metallurgy, manufacturing, textile layout, forestry, and statistics. The input to a computational-geometry problem is typically a description of a set of geometric objects, such as a set of points, a set of line segments, or the vertices of a polygon in counterclockwise order. The output is often a response to a query about the objects, such as whether any of the lines intersect, or perhaps a new geometric object, such as the convex hull (smallest enclosing convex polygon) of the set of points.

In this chapter, we look at a few computational-geometry algorithms in two dimensions, that is, in the plane. We represent each input object by a set of points $\{p_1, p_2, p_3, \ldots\}$, where each $p_i = (x_i, y_i)$ and $x_i, y_i \in \mathbb{R}$. For example, we represent an n-vertex polygon P by a sequence $\langle p_0, p_1, p_2, \ldots, p_{n-1} \rangle$ of its vertices in order of their appearance on the boundary of P. Computational geometry can also apply to three dimensions, and even higher-dimensional spaces, but such problems and their solutions can be very difficult to visualize. Even in two dimensions, however, we can see a good sample of computational-geometry techniques.

Section 33.1 shows how to answer basic questions about line segments efficiently and accurately: whether one segment is clockwise or counterclockwise from another that shares an endpoint, which way we turn when traversing two adjoining line segments, and whether two line segments intersect. Section 33.2 presents a technique called "sweeping" that we use to develop an $O(n \lg n)$-time algorithm for determining whether a set of n line segments contains any intersections. Section 33.3 gives two "rotational-sweep" algorithms that compute the convex hull (smallest enclosing convex polygon) of a set of n points: Graham's scan, which runs in time $O(n \lg n)$, and Jarvis's march, which takes $O(nh)$ time, where h is the number of vertices of the convex hull. Finally, Section 33.4 gives an $O(n \lg n)$-time divide-and-conquer algorithm for finding the closest pair of points in a set of n points in the plane.

33.1 Line-segment properties

Several of the computational-geometry algorithms in this chapter require answers to questions about the properties of line segments. A ***convex combination*** of two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is any point $p_3 = (x_3, y_3)$ such that for some α in the range $0 \leq \alpha \leq 1$, we have $x_3 = \alpha x_1 + (1 - \alpha)x_2$ and $y_3 = \alpha y_1 + (1 - \alpha)y_2$. We also write that $p_3 = \alpha p_1 + (1 - \alpha)p_2$. Intuitively, p_3 is any point that is on the line passing through p_1 and p_2 and is on or between p_1 and p_2 on the line. Given two distinct points p_1 and p_2, the ***line segment*** $\overline{p_1 p_2}$ is the set of convex combinations of p_1 and p_2. We call p_1 and p_2 the ***endpoints*** of segment $\overline{p_1 p_2}$. Sometimes the ordering of p_1 and p_2 matters, and we speak of the ***directed segment*** $\overrightarrow{p_1 p_2}$. If p_1 is the ***origin*** $(0, 0)$, then we can treat the directed segment $\overrightarrow{p_1 p_2}$ as the ***vector*** p_2.

In this section, we shall explore the following questions:

1. Given two directed segments $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$, is $\overrightarrow{p_0 p_1}$ clockwise from $\overrightarrow{p_0 p_2}$ with respect to their common endpoint p_0?
2. Given two line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$, if we traverse $\overline{p_0 p_1}$ and then $\overline{p_1 p_2}$, do we make a left turn at point p_1?
3. Do line segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect?

There are no restrictions on the given points.

We can answer each question in $O(1)$ time, which should come as no surprise since the input size of each question is $O(1)$. Moreover, our methods use only additions, subtractions, multiplications, and comparisons. We need neither division nor trigonometric functions, both of which can be computationally expensive and prone to problems with round-off error. For example, the "straightforward" method of determining whether two segments intersect—compute the line equation of the form $y = mx + b$ for each segment (m is the slope and b is the y-intercept), find the point of intersection of the lines, and check whether this point is on both segments—uses division to find the point of intersection. When the segments are nearly parallel, this method is very sensitive to the precision of the division operation on real computers. The method in this section, which avoids division, is much more accurate.

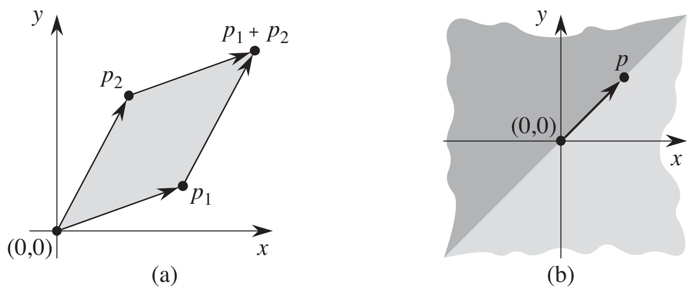


Figure 33.1 **(a)** The cross product of vectors p_1 and p_2 is the signed area of the parallelogram. **(b)** The lightly shaded region contains vectors that are clockwise from p. The darkly shaded region contains vectors that are counterclockwise from p.

Cross products

Computing cross products lies at the heart of our line-segment methods. Consider vectors p_1 and p_2, shown in Figure 33.1(a). We can interpret the ***cross product*** $p_1 \times p_2$ as the signed area of the parallelogram formed by the points $(0,0)$, p_1, p_2, and $p_1 + p_2 = (x_1 + x_2, y_1 + y_2)$. An equivalent, but more useful, definition gives the cross product as the determinant of a matrix:[1]

$$\begin{aligned} p_1 \times p_2 &= \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \\ &= x_1 y_2 - x_2 y_1 \\ &= -p_2 \times p_1 . \end{aligned}$$

If $p_1 \times p_2$ is positive, then p_1 is clockwise from p_2 with respect to the origin $(0,0)$; if this cross product is negative, then p_1 is counterclockwise from p_2. (See Exercise 33.1-1.) Figure 33.1(b) shows the clockwise and counterclockwise regions relative to a vector p. A boundary condition arises if the cross product is 0; in this case, the vectors are ***colinear***, pointing in either the same or opposite directions.

To determine whether a directed segment $\overrightarrow{p_0 p_1}$ is closer to a directed segment $\overrightarrow{p_0 p_2}$ in a clockwise direction or in a counterclockwise direction with respect to their common endpoint p_0, we simply translate to use p_0 as the origin. That is, we let $p_1 - p_0$ denote the vector $p_1' = (x_1', y_1')$, where $x_1' = x_1 - x_0$ and $y_1' = y_1 - y_0$, and we define $p_2 - p_0$ similarly. We then compute the cross product

[1] Actually, the cross product is a three-dimensional concept. It is a vector that is perpendicular to both p_1 and p_2 according to the "right-hand rule" and whose magnitude is $|x_1 y_2 - x_2 y_1|$. In this chapter, however, we find it convenient to treat the cross product simply as the value $x_1 y_2 - x_2 y_1$.

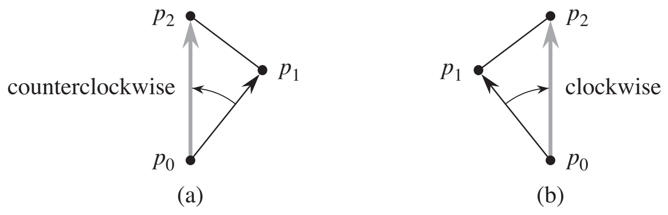


Figure 33.2 Using the cross product to determine how consecutive line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ turn at point p_1. We check whether the directed segment $\overrightarrow{p_0 p_2}$ is clockwise or counterclockwise relative to the directed segment $\overrightarrow{p_0 p_1}$. **(a)** If counterclockwise, the points make a left turn. **(b)** If clockwise, they make a right turn.

$$(p_1 - p_0) \times (p_2 - p_0) = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0) \ .$$

If this cross product is positive, then $\overrightarrow{p_0 p_1}$ is clockwise from $\overrightarrow{p_0 p_2}$; if negative, it is counterclockwise.

Determining whether consecutive segments turn left or right

Our next question is whether two consecutive line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ turn left or right at point p_1. Equivalently, we want a method to determine which way a given angle $\angle p_0 p_1 p_2$ turns. Cross products allow us to answer this question without computing the angle. As Figure 33.2 shows, we simply check whether directed segment $\overrightarrow{p_0 p_2}$ is clockwise or counterclockwise relative to directed segment $\overrightarrow{p_0 p_1}$. To do so, we compute the cross product $(p_2 - p_0) \times (p_1 - p_0)$. If the sign of this cross product is negative, then $\overrightarrow{p_0 p_2}$ is counterclockwise with respect to $\overrightarrow{p_0 p_1}$, and thus we make a left turn at p_1. A positive cross product indicates a clockwise orientation and a right turn. A cross product of 0 means that points p_0, p_1, and p_2 are colinear.

Determining whether two line segments intersect

To determine whether two line segments intersect, we check whether each segment straddles the line containing the other. A segment $\overline{p_1 p_2}$ ***straddles*** a line if point p_1 lies on one side of the line and point p_2 lies on the other side. A boundary case arises if p_1 or p_2 lies directly on the line. Two line segments intersect if and only if either (or both) of the following conditions holds:

1. Each segment straddles the line containing the other.
2. An endpoint of one segment lies on the other segment. (This condition comes from the boundary case.)

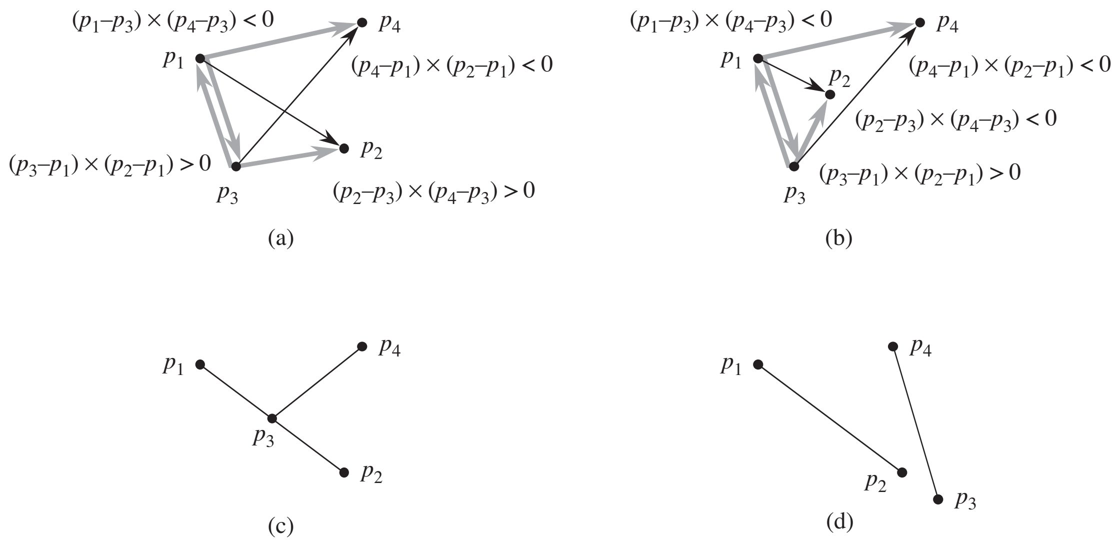
The following procedures implement this idea. SEGMENTS-INTERSECT returns TRUE if segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect and FALSE if they do not. It calls the subroutines DIRECTION, which computes relative orientations using the cross-product method above, and ON-SEGMENT, which determines whether a point known to be colinear with a segment lies on that segment.

```
SEGMENTS-INTERSECT($p_1, p_2, p_3, p_4$)
 1  $d_1$ = DIRECTION($p_3, p_4, p_1$)
 2  $d_2$ = DIRECTION($p_3, p_4, p_2$)
 3  $d_3$ = DIRECTION($p_1, p_2, p_3$)
 4  $d_4$ = DIRECTION($p_1, p_2, p_4$)
 5  if (($d_1 > 0$ and $d_2 < 0$) or ($d_1 < 0$ and $d_2 > 0$)) and
        (($d_3 > 0$ and $d_4 < 0$) or ($d_3 < 0$ and $d_4 > 0$))
 6      return TRUE
 7  elseif $d_1 == 0$ and ON-SEGMENT($p_3, p_4, p_1$)
 8      return TRUE
 9  elseif $d_2 == 0$ and ON-SEGMENT($p_3, p_4, p_2$)
10      return TRUE
11  elseif $d_3 == 0$ and ON-SEGMENT($p_1, p_2, p_3$)
12      return TRUE
13  elseif $d_4 == 0$ and ON-SEGMENT($p_1, p_2, p_4$)
14      return TRUE
15  else return FALSE
```

```
DIRECTION($p_i, p_j, p_k$)
1  return $(p_k - p_i) \times (p_j - p_i)$
```

```
ON-SEGMENT($p_i, p_j, p_k$)
1  if $\min(x_i, x_j) \le x_k \le \max(x_i, x_j)$ and $\min(y_i, y_j) \le y_k \le \max(y_i, y_j)$
2      return TRUE
3  else return FALSE
```

SEGMENTS-INTERSECT works as follows. Lines 1–4 compute the relative orientation d_i of each endpoint p_i with respect to the other segment. If all the relative orientations are nonzero, then we can easily determine whether segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect, as follows. Segment $\overline{p_1 p_2}$ straddles the line containing segment $\overline{p_3 p_4}$ if directed segments $\overrightarrow{p_3 p_1}$ and $\overrightarrow{p_3 p_2}$ have opposite orientations relative to $\overrightarrow{p_3 p_4}$. In this case, the signs of d_1 and d_2 differ. Similarly, $\overline{p_3 p_4}$ straddles the line containing $\overline{p_1 p_2}$ if the signs of d_3 and d_4 differ. If the test of line 5 is true, then the segments straddle each other, and SEGMENTS-INTERSECT returns TRUE. Figure 33.3(a) shows this case. Otherwise, the segments do not straddle



$(p_1-p_3) \times (p_4-p_3) < 0$
p_4
p_1
$(p_4-p_1) \times (p_2-p_1) < 0$
$(p_3-p_1) \times (p_2-p_1) > 0$
p_2
p_3
$(p_2-p_3) \times (p_4-p_3) > 0$
(a)
$(p_1-p_3) \times (p_4-p_3) < 0$
p_4
p_1
p_2
$(p_4-p_1) \times (p_2-p_1) < 0$
$(p_2-p_3) \times (p_4-p_3) < 0$
$(p_3-p_1) \times (p_2-p_1) > 0$
p_3
(b)
p_1
p_4
p_3
p_2
(c)
p_1
p_4
p_2
p_3
(d)

Figure 33.3 Cases in the procedure SEGMENTS-INTERSECT. **(a)** The segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ straddle each other's lines. Because $\overline{p_3 p_4}$ straddles the line containing $\overline{p_1 p_2}$, the signs of the cross products $(p_3 - p_1) \times (p_2 - p_1)$ and $(p_4 - p_1) \times (p_2 - p_1)$ differ. Because $\overline{p_1 p_2}$ straddles the line containing $\overline{p_3 p_4}$, the signs of the cross products $(p_1 - p_3) \times (p_4 - p_3)$ and $(p_2 - p_3) \times (p_4 - p_3)$ differ. **(b)** Segment $\overline{p_3 p_4}$ straddles the line containing $\overline{p_1 p_2}$, but $\overline{p_1 p_2}$ does not straddle the line containing $\overline{p_3 p_4}$. The signs of the cross products $(p_1 - p_3) \times (p_4 - p_3)$ and $(p_2 - p_3) \times (p_4 - p_3)$ are the same. **(c)** Point p_3 is colinear with $\overline{p_1 p_2}$ and is between p_1 and p_2. **(d)** Point p_3 is colinear with $\overline{p_1 p_2}$, but it is not between p_1 and p_2. The segments do not intersect.

each other's lines, although a boundary case may apply. If all the relative orientations are nonzero, no boundary case applies. All the tests against 0 in lines 7–13 then fail, and SEGMENTS-INTERSECT returns FALSE in line 15. Figure 33.3(b) shows this case.

A boundary case occurs if any relative orientation d_k is 0. Here, we know that p_k is colinear with the other segment. It is directly on the other segment if and only if it is between the endpoints of the other segment. The procedure ON-SEGMENT returns whether p_k is between the endpoints of segment $\overline{p_i p_j}$, which will be the other segment when called in lines 7–13; the procedure assumes that p_k is colinear with segment $\overline{p_i p_j}$. Figures 33.3(c) and (d) show cases with colinear points. In Figure 33.3(c), p_3 is on $\overline{p_1 p_2}$, and so SEGMENTS-INTERSECT returns TRUE in line 12. No endpoints are on other segments in Figure 33.3(d), and so SEGMENTS-INTERSECT returns FALSE in line 15.

Other applications of cross products

Later sections of this chapter introduce additional uses for cross products. In Section 33.3, we shall need to sort a set of points according to their polar angles with respect to a given origin. As Exercise 33.1-3 asks you to show, we can use cross products to perform the comparisons in the sorting procedure. In Section 33.2, we shall use red-black trees to maintain the vertical ordering of a set of line segments. Rather than keeping explicit key values which we compare to each other in the red-black tree code, we shall compute a cross-product to determine which of two segments that intersect a given vertical line is above the other.

Exercises

33.1-1
Prove that if $p_1 \times p_2$ is positive, then vector p_1 is clockwise from vector p_2 with respect to the origin $(0, 0)$ and that if this cross product is negative, then p_1 is counterclockwise from p_2.

33.1-2
Professor van Pelt proposes that only the x-dimension needs to be tested in line 1 of ON-SEGMENT. Show why the professor is wrong.

33.1-3
The ***polar angle*** of a point p_1 with respect to an origin point p_0 is the angle of the vector $p_1 - p_0$ in the usual polar coordinate system. For example, the polar angle of $(3, 5)$ with respect to $(2, 4)$ is the angle of the vector $(1, 1)$, which is 45 degrees or $\pi/4$ radians. The polar angle of $(3, 3)$ with respect to $(2, 4)$ is the angle of the vector $(1, -1)$, which is 315 degrees or $7\pi/4$ radians. Write pseudocode to sort a sequence $\langle p_1, p_2, \ldots, p_n \rangle$ of n points according to their polar angles with respect to a given origin point p_0. Your procedure should take $O(n \lg n)$ time and use cross products to compare angles.

33.1-4
Show how to determine in $O(n^2 \lg n)$ time whether any three points in a set of n points are colinear.

33.1-5
A ***polygon*** is a piecewise-linear, closed curve in the plane. That is, it is a curve ending on itself that is formed by a sequence of straight-line segments, called the ***sides*** of the polygon. A point joining two consecutive sides is a ***vertex*** of the polygon. If the polygon is ***simple***, as we shall generally assume, it does not cross itself. The set of points in the plane enclosed by a simple polygon forms the ***interior*** of

the polygon, the set of points on the polygon itself forms its ***boundary***, and the set of points surrounding the polygon forms its ***exterior***. A simple polygon is ***convex*** if, given any two points on its boundary or in its interior, all points on the line segment drawn between them are contained in the polygon's boundary or interior. A vertex of a convex polygon cannot be expressed as a convex combination of any two distinct points on the boundary or in the interior of the polygon.

Professor Amundsen proposes the following method to determine whether a sequence $\langle p_0, p_1, \ldots, p_{n-1}\rangle$ of n points forms the consecutive vertices of a convex polygon. Output "yes" if the set $\{\angle p_i p_{i+1} p_{i+2} : i = 0, 1, \ldots, n-1\}$, where subscript addition is performed modulo n, does not contain both left turns and right turns; otherwise, output "no." Show that although this method runs in linear time, it does not always produce the correct answer. Modify the professor's method so that it always produces the correct answer in linear time.

33.1-6

Given a point $p_0 = (x_0, y_0)$, the ***right horizontal ray*** from p_0 is the set of points $\{p_i = (x_i, y_i) : x_i \geq x_0 \text{ and } y_i = y_0\}$, that is, it is the set of points due right of p_0 along with p_0 itself. Show how to determine whether a given right horizontal ray from p_0 intersects a line segment $\overline{p_1 p_2}$ in $O(1)$ time by reducing the problem to that of determining whether two line segments intersect.

33.1-7

One way to determine whether a point p_0 is in the interior of a simple, but not necessarily convex, polygon P is to look at any ray from p_0 and check that the ray intersects the boundary of P an odd number of times but that p_0 itself is not on the boundary of P. Show how to compute in $\Theta(n)$ time whether a point p_0 is in the interior of an n-vertex polygon P. (*Hint:* Use Exercise 33.1-6. Make sure your algorithm is correct when the ray intersects the polygon boundary at a vertex and when the ray overlaps a side of the polygon.)

33.1-8

Show how to compute the area of an n-vertex simple, but not necessarily convex, polygon in $\Theta(n)$ time. (See Exercise 33.1-5 for definitions pertaining to polygons.)

33.2 Determining whether any pair of segments intersects

This section presents an algorithm for determining whether any two line segments in a set of segments intersect. The algorithm uses a technique known as "sweeping," which is common to many computational-geometry algorithms. Moreover, as the exercises at the end of this section show, this algorithm, or simple variations of it, can help solve other computational-geometry problems.

The algorithm runs in $O(n \lg n)$ time, where n is the number of segments we are given. It determines only whether or not any intersection exists; it does not print all the intersections. (By Exercise 33.2-1, it takes $\Omega(n^2)$ time in the worst case to find *all* the intersections in a set of n line segments.)

In ***sweeping***, an imaginary vertical ***sweep line*** passes through the given set of geometric objects, usually from left to right. We treat the spatial dimension that the sweep line moves across, in this case the x-dimension, as a dimension of time. Sweeping provides a method for ordering geometric objects, usually by placing them into a dynamic data structure, and for taking advantage of relationships among them. The line-segment-intersection algorithm in this section considers all the line-segment endpoints in left-to-right order and checks for an intersection each time it encounters an endpoint.

To describe and prove correct our algorithm for determining whether any two of n line segments intersect, we shall make two simplifying assumptions. First, we assume that no input segment is vertical. Second, we assume that no three input segments intersect at a single point. Exercises 33.2-8 and 33.2-9 ask you to show that the algorithm is robust enough that it needs only a slight modification to work even when these assumptions do not hold. Indeed, removing such simplifying assumptions and dealing with boundary conditions often present the most difficult challenges when programming computational-geometry algorithms and proving their correctness.

Ordering segments

Because we assume that there are no vertical segments, we know that any input segment intersecting a given vertical sweep line intersects it at a single point. Thus, we can order the segments that intersect a vertical sweep line according to the y-coordinates of the points of intersection.

To be more precise, consider two segments s_1 and s_2. We say that these segments are ***comparable*** at x if the vertical sweep line with x-coordinate x intersects both of them. We say that s_1 is ***above*** s_2 at x, written $s_1 \succeq_x s_2$, if s_1 and s_2 are comparable at x and the intersection of s_1 with the sweep line at x is higher than the intersection of s_2 with the same sweep line, or if s_1 and s_2 intersect at the sweep line. In Figure 33.4(a), for example, we have the relationships $a \succeq_r c$, $a \succeq_t b$, $b \succeq_t c$, $a \succeq_t c$, and $b \succeq_u c$. Segment d is not comparable with any other segment.

For any given x, the relation "$\succeq_x$" is a total preorder (see Section B.2) for all segments that intersect the sweep line at x. That is, the relation is transitive, and if segments s_1 and s_2 each intersect the sweep line at x, then either $s_1 \succeq_x s_2$ or $s_2 \succeq_x s_1$, or both (if s_1 and s_2 intersect at the sweep line). (The relation $\succeq_x$ is

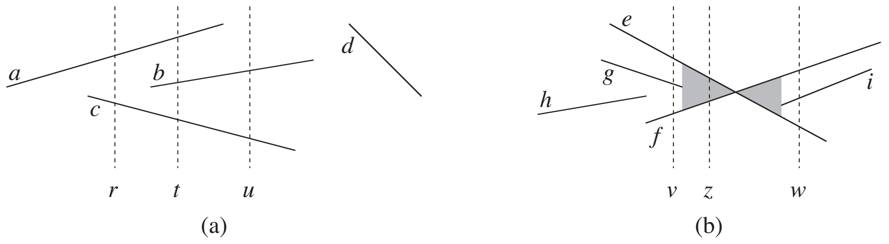


Figure 33.4 The ordering among line segments at various vertical sweep lines. **(a)** We have $a \succsim_r c$, $a \succsim_t b$, $b \succsim_t c$, $a \succsim_t c$, and $b \succsim_u c$. Segment d is comparable with no other segment shown. **(b)** When segments e and f intersect, they reverse their orders: we have $e \succsim_v f$ but $f \succsim_w e$. Any sweep line (such as z) that passes through the shaded region has e and f consecutive in the ordering given by the relation $\succsim_z$.

also reflexive, but neither symmetric nor antisymmetric.) The total preorder may differ for differing values of x, however, as segments enter and leave the ordering. A segment enters the ordering when its left endpoint is encountered by the sweep, and it leaves the ordering when its right endpoint is encountered.

What happens when the sweep line passes through the intersection of two segments? As Figure 33.4(b) shows, the segments reverse their positions in the total preorder. Sweep lines v and w are to the left and right, respectively, of the point of intersection of segments e and f, and we have $e \succsim_v f$ and $f \succsim_w e$. Note that because we assume that no three segments intersect at the same point, there must be some vertical sweep line x for which intersecting segments e and f are *consecutive* in the total preorder $\succsim_x$. Any sweep line that passes through the shaded region of Figure 33.4(b), such as z, has e and f consecutive in its total preorder.

Moving the sweep line

Sweeping algorithms typically manage two sets of data:

1. The ***sweep-line status*** gives the relationships among the objects that the sweep line intersects.
2. The ***event-point schedule*** is a sequence of points, called ***event points***, which we order from left to right according to their x-coordinates. As the sweep progresses from left to right, whenever the sweep line reaches the x-coordinate of an event point, the sweep halts, processes the event point, and then resumes. Changes to the sweep-line status occur only at event points.

For some algorithms (the algorithm asked for in Exercise 33.2-7, for example), the event-point schedule develops dynamically as the algorithm progresses. The algorithm at hand, however, determines all the event points before the sweep, based solely on simple properties of the input data. In particular, each segment endpoint is an event point. We sort the segment endpoints by increasing x-coordinate and proceed from left to right. (If two or more endpoints are ***covertical***, i.e., they have the same x-coordinate, we break the tie by putting all the covertical left endpoints before the covertical right endpoints. Within a set of covertical left endpoints, we put those with lower y-coordinates first, and we do the same within a set of covertical right endpoints.) When we encounter a segment's left endpoint, we insert the segment into the sweep-line status, and we delete the segment from the sweep-line status upon encountering its right endpoint. Whenever two segments first become consecutive in the total preorder, we check whether they intersect.

The sweep-line status is a total preorder T, for which we require the following operations:

- INSERT(T, s): insert segment s into T.
- DELETE(T, s): delete segment s from T.
- ABOVE(T, s): return the segment immediately above segment s in T.
- BELOW(T, s): return the segment immediately below segment s in T.

It is possible for segments s_1 and s_2 to be mutually above each other in the total preorder T; this situation can occur if s_1 and s_2 intersect at the sweep line whose total preorder is given by T. In this case, the two segments may appear in either order in T.

If the input contains n segments, we can perform each of the operations INSERT, DELETE, ABOVE, and BELOW in $O(\lg n)$ time using red-black trees. Recall that the red-black-tree operations in Chapter 13 involve comparing keys. We can replace the key comparisons by comparisons that use cross products to determine the relative ordering of two segments (see Exercise 33.2-2).

Segment-intersection pseudocode

The following algorithm takes as input a set S of n line segments, returning the boolean value TRUE if any pair of segments in S intersects, and FALSE otherwise. A red-black tree maintains the total preorder T.

```
ANY-SEGMENTS-INTERSECT($S$)
1  $T = \emptyset$
2  sort the endpoints of the segments in $S$ from left to right,
       breaking ties by putting left endpoints before right endpoints
       and breaking further ties by putting points with lower
       $y$-coordinates first
3  for each point $p$ in the sorted list of endpoints
4      if $p$ is the left endpoint of a segment $s$
5          INSERT($T, s$)
6          if (ABOVE($T, s$) exists and intersects $s$)
               or (BELOW($T, s$) exists and intersects $s$)
7              return TRUE
8      if $p$ is the right endpoint of a segment $s$
9          if both ABOVE($T, s$) and BELOW($T, s$) exist
               and ABOVE($T, s$) intersects BELOW($T, s$)
10             return TRUE
11         DELETE($T, s$)
12 return FALSE
```

Figure 33.5 illustrates how the algorithm works. Line 1 initializes the total preorder to be empty. Line 2 determines the event-point schedule by sorting the $2n$ segment endpoints from left to right, breaking ties as described above. One way to perform line 2 is by lexicographically sorting the endpoints on (x, e, y), where x and y are the usual coordinates, $e = 0$ for a left endpoint, and $e = 1$ for a right endpoint.

Each iteration of the **for** loop of lines 3–11 processes one event point p. If p is the left endpoint of a segment s, line 5 adds s to the total preorder, and lines 6–7 return TRUE if s intersects either of the segments it is consecutive with in the total preorder defined by the sweep line passing through p. (A boundary condition occurs if p lies on another segment s'. In this case, we require only that s and s' be placed consecutively into T.) If p is the right endpoint of a segment s, then we need to delete s from the total preorder. But first, lines 9–10 return TRUE if there is an intersection between the segments surrounding s in the total preorder defined by the sweep line passing through p. If these segments do not intersect, line 11 deletes segment s from the total preorder. If the segments surrounding segment s intersect, they would have become consecutive after deleting s had the **return** statement in line 10 not prevented line 11 from executing. The correctness argument, which follows, will make it clear why it suffices to check the segments surrounding s. Finally, if we never find any intersections after having processed all $2n$ event points, line 12 returns FALSE.

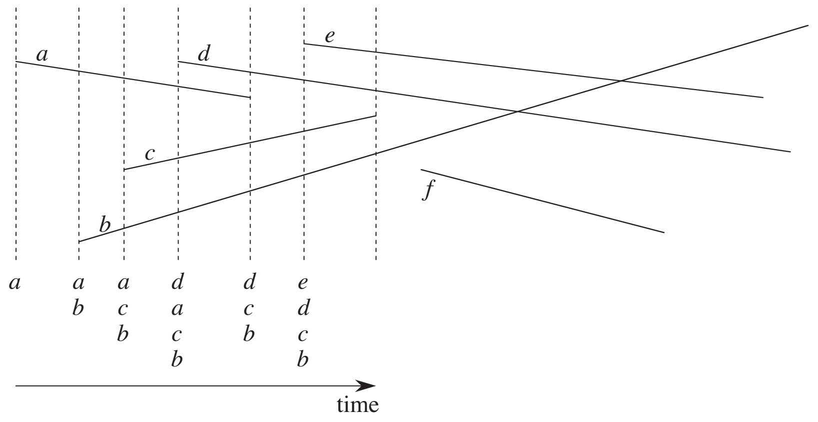


Figure 33.5 The execution of ANY-SEGMENTS-INTERSECT. Each dashed line is the sweep line at an event point. Except for the rightmost sweep line, the ordering of segment names below each sweep line corresponds to the total preorder T at the end of the **for** loop processing the corresponding event point. The rightmost sweep line occurs when processing the right endpoint of segment c; because segments d and b surround c and intersect each other, the procedure returns TRUE.

Correctness

To show that ANY-SEGMENTS-INTERSECT is correct, we will prove that the call ANY-SEGMENTS-INTERSECT(S) returns TRUE if and only if there is an intersection among the segments in S.

It is easy to see that ANY-SEGMENTS-INTERSECT returns TRUE (on lines 7 and 10) only if it finds an intersection between two of the input segments. Hence, if it returns TRUE, there is an intersection.

We also need to show the converse: that if there is an intersection, then ANY-SEGMENTS-INTERSECT returns TRUE. Let us suppose that there is at least one intersection. Let p be the leftmost intersection point, breaking ties by choosing the point with the lowest y-coordinate, and let a and b be the segments that intersect at p. Since no intersections occur to the left of p, the order given by T is correct at all points to the left of p. Because no three segments intersect at the same point, a and b become consecutive in the total preorder at some sweep line z.[2] Moreover, z is to the left of p or goes through p. Some segment endpoint q on sweep line z

[2]If we allow three segments to intersect at the same point, there may be an intervening segment c that intersects both a and b at point p. That is, we may have $a \succcurlyeq_w c$ and $c \succcurlyeq_w b$ for all sweep lines w to the left of p for which $a \succcurlyeq_w b$. Exercise 33.2-8 asks you to show that ANY-SEGMENTS-INTERSECT is correct even if three segments do intersect at the same point.

is the event point at which a and b become consecutive in the total preorder. If p is on sweep line z, then $q = p$. If p is not on sweep line z, then q is to the left of p. In either case, the order given by T is correct just before encountering q. (Here is where we use the lexicographic order in which the algorithm processes event points. Because p is the lowest of the leftmost intersection points, even if p is on sweep line z and some other intersection point p' is on z, event point $q = p$ is processed before the other intersection p' can interfere with the total preorder T. Moreover, even if p is the left endpoint of one segment, say a, and the right endpoint of the other segment, say b, because left endpoint events occur before right endpoint events, segment b is in T upon first encountering segment a.) Either event point q is processed by ANY-SEGMENTS-INTERSECT or it is not processed.

If q is processed by ANY-SEGMENTS-INTERSECT, only two possible actions may occur:

1. Either a or b is inserted into T, and the other segment is above or below it in the total preorder. Lines 4–7 detect this case.
2. Segments a and b are already in T, and a segment between them in the total preorder is deleted, making a and b become consecutive. Lines 8–11 detect this case.

In either case, we find the intersection p and ANY-SEGMENTS-INTERSECT returns TRUE.

If event point q is not processed by ANY-SEGMENTS-INTERSECT, the procedure must have returned before processing all event points. This situation could have occurred only if ANY-SEGMENTS-INTERSECT had already found an intersection and returned TRUE.

Thus, if there is an intersection, ANY-SEGMENTS-INTERSECT returns TRUE. As we have already seen, if ANY-SEGMENTS-INTERSECT returns TRUE, there is an intersection. Therefore, ANY-SEGMENTS-INTERSECT always returns a correct answer.

Running time

If set S contains n segments, then ANY-SEGMENTS-INTERSECT runs in time $O(n \lg n)$. Line 1 takes $O(1)$ time. Line 2 takes $O(n \lg n)$ time, using merge sort or heapsort. The **for** loop of lines 3–11 iterates at most once per event point, and so with $2n$ event points, the loop iterates at most $2n$ times. Each iteration takes $O(\lg n)$ time, since each red-black-tree operation takes $O(\lg n)$ time and, using the method of Section 33.1, each intersection test takes $O(1)$ time. The total time is thus $O(n \lg n)$.

Exercises

33.2-1

Show that a set of n line segments may contain $\Theta(n^2)$ intersections.

33.2-2

Given two segments a and b that are comparable at x, show how to determine in $O(1)$ time which of $a \succcurlyeq_x b$ or $b \succcurlyeq_x a$ holds. Assume that neither segment is vertical. (*Hint:* If a and b do not intersect, you can just use cross products. If a and b intersect—which you can of course determine using only cross products—you can still use only addition, subtraction, and multiplication, avoiding division. Of course, in the application of the $\succcurlyeq_x$ relation used here, if a and b intersect, we can just stop and declare that we have found an intersection.)

33.2-3

Professor Mason suggests that we modify ANY-SEGMENTS-INTERSECT so that instead of returning upon finding an intersection, it prints the segments that intersect and continues on to the next iteration of the **for** loop. The professor calls the resulting procedure PRINT-INTERSECTING-SEGMENTS and claims that it prints all intersections, from left to right, as they occur in the set of line segments. Professor Dixon disagrees, claiming that Professor Mason's idea is incorrect. Which professor is right? Will PRINT-INTERSECTING-SEGMENTS always find the leftmost intersection first? Will it always find all the intersections?

33.2-4

Give an $O(n \lg n)$-time algorithm to determine whether an n-vertex polygon is simple.

33.2-5

Give an $O(n \lg n)$-time algorithm to determine whether two simple polygons with a total of n vertices intersect.

33.2-6

A ***disk*** consists of a circle plus its interior and is represented by its center point and radius. Two disks intersect if they have any point in common. Give an $O(n \lg n)$-time algorithm to determine whether any two disks in a set of n intersect.

33.2-7

Given a set of n line segments containing a total of k intersections, show how to output all k intersections in $O((n + k) \lg n)$ time.

33.2-8

Argue that ANY-SEGMENTS-INTERSECT works correctly even if three or more segments intersect at the same point.

33.2-9

Show that ANY-SEGMENTS-INTERSECT works correctly in the presence of vertical segments if we treat the bottom endpoint of a vertical segment as if it were a left endpoint and the top endpoint as if it were a right endpoint. How does your answer to Exercise 33.2-2 change if we allow vertical segments?

33.3 Finding the convex hull

The ***convex hull*** of a set Q of points, denoted by $\text{CH}(Q)$, is the smallest convex polygon P for which each point in Q is either on the boundary of P or in its interior. (See Exercise 33.1-5 for a precise definition of a convex polygon.) We implicitly assume that all points in the set Q are unique and that Q contains at least three points which are not colinear. Intuitively, we can think of each point in Q as being a nail sticking out from a board. The convex hull is then the shape formed by a tight rubber band that surrounds all the nails. Figure 33.6 shows a set of points and its convex hull.

In this section, we shall present two algorithms that compute the convex hull of a set of n points. Both algorithms output the vertices of the convex hull in counterclockwise order. The first, known as Graham's scan, runs in $O(n \lg n)$ time. The second, called Jarvis's march, runs in $O(nh)$ time, where h is the number of vertices of the convex hull. As Figure 33.6 illustrates, every vertex of $\text{CH}(Q)$ is a

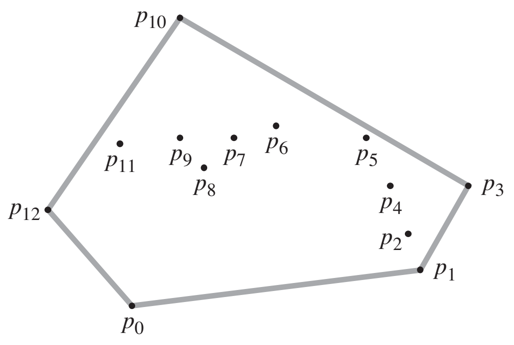


Figure 33.6 A set of points $Q = \{p_0, p_1, \ldots, p_{12}\}$ with its convex hull $\text{CH}(Q)$ in gray.

point in Q. Both algorithms exploit this property, deciding which vertices in Q to keep as vertices of the convex hull and which vertices in Q to reject.

We can compute convex hulls in $O(n \lg n)$ time by any one of several methods. Both Graham's scan and Jarvis's march use a technique called "rotational sweep," processing vertices in the order of the polar angles they form with a reference vertex. Other methods include the following:

- In the ***incremental method***, we first sort the points from left to right, yielding a sequence $\langle p_1, p_2, \ldots, p_n \rangle$. At the ith stage, we update the convex hull of the $i-1$ leftmost points, $\text{CH}(\{p_1, p_2, \ldots, p_{i-1}\})$, according to the ith point from the left, thus forming $\text{CH}(\{p_1, p_2, \ldots, p_i\})$. Exercise 33.3-6 asks you how to implement this method to take a total of $O(n \lg n)$ time.
- In the ***divide-and-conquer method***, we divide the set of n points in $\Theta(n)$ time into two subsets, one containing the leftmost $\lceil n/2 \rceil$ points and one containing the rightmost $\lfloor n/2 \rfloor$ points, recursively compute the convex hulls of the subsets, and then, by means of a clever method, combine the hulls in $O(n)$ time. The running time is described by the familiar recurrence $T(n) = 2T(n/2) + O(n)$, and so the divide-and-conquer method runs in $O(n \lg n)$ time.
- The ***prune-and-search method*** is similar to the worst-case linear-time median algorithm of Section 9.3. With this method, we find the upper portion (or "upper chain") of the convex hull by repeatedly throwing out a constant fraction of the remaining points until only the upper chain of the convex hull remains. We then do the same for the lower chain. This method is asymptotically the fastest: if the convex hull contains h vertices, it runs in only $O(n \lg h)$ time.

Computing the convex hull of a set of points is an interesting problem in its own right. Moreover, algorithms for some other computational-geometry problems start by computing a convex hull. Consider, for example, the two-dimensional ***farthest-pair problem***: we are given a set of n points in the plane and wish to find the two points whose distance from each other is maximum. As Exercise 33.3-3 asks you to prove, these two points must be vertices of the convex hull. Although we won't prove it here, we can find the farthest pair of vertices of an n-vertex convex polygon in $O(n)$ time. Thus, by computing the convex hull of the n input points in $O(n \lg n)$ time and then finding the farthest pair of the resulting convex-polygon vertices, we can find the farthest pair of points in any set of n points in $O(n \lg n)$ time.

Graham's scan

Graham's scan solves the convex-hull problem by maintaining a stack S of candidate points. It pushes each point of the input set Q onto the stack one time, and it eventually pops from the stack each point that is not a vertex of CH(Q). When the algorithm terminates, stack S contains exactly the vertices of CH(Q), in counterclockwise order of their appearance on the boundary.

The procedure GRAHAM-SCAN takes as input a set Q of points, where $|Q| \geq 3$. It calls the functions TOP(S), which returns the point on top of stack S without changing S, and NEXT-TO-TOP(S), which returns the point one entry below the top of stack S without changing S. As we shall prove in a moment, the stack S returned by GRAHAM-SCAN contains, from bottom to top, exactly the vertices of CH(Q) in counterclockwise order.

```
GRAHAM-SCAN($Q$)
 1  let $p_0$ be the point in $Q$ with the minimum $y$-coordinate,
        or the leftmost such point in case of a tie
 2  let $\langle p_1, p_2, \ldots, p_m \rangle$ be the remaining points in $Q$,
        sorted by polar angle in counterclockwise order around $p_0$
        (if more than one point has the same angle, remove all but
        the one that is farthest from $p_0$)
 3  let $S$ be an empty stack
 4  PUSH($p_0, S$)
 5  PUSH($p_1, S$)
 6  PUSH($p_2, S$)
 7  for $i = 3$ to $m$
 8      while the angle formed by points NEXT-TO-TOP($S$), TOP($S$),
                and $p_i$ makes a nonleft turn
 9          POP($S$)
10      PUSH($p_i, S$)
11  return $S$
```

Figure 33.7 illustrates the progress of GRAHAM-SCAN. Line 1 chooses point p_0 as the point with the lowest y-coordinate, picking the leftmost such point in case of a tie. Since there is no point in Q that is below p_0 and any other points with the same y-coordinate are to its right, p_0 must be a vertex of CH(Q). Line 2 sorts the remaining points of Q by polar angle relative to p_0, using the same method—comparing cross products—as in Exercise 33.1-3. If two or more points have the same polar angle relative to p_0, all but the farthest such point are convex combinations of p_0 and the farthest point, and so we remove them entirely from consideration. We let m denote the number of points other than p_0 that remain. The polar angle, measured in radians, of each point in Q relative to p_0 is in the half-open interval $[0, \pi)$. Since the points are sorted according to polar angles, they are sorted in counterclockwise order relative to p_0. We designate this sorted sequence of points by $\langle p_1, p_2, \ldots, p_m \rangle$. Note that points p_1 and p_m are vertices

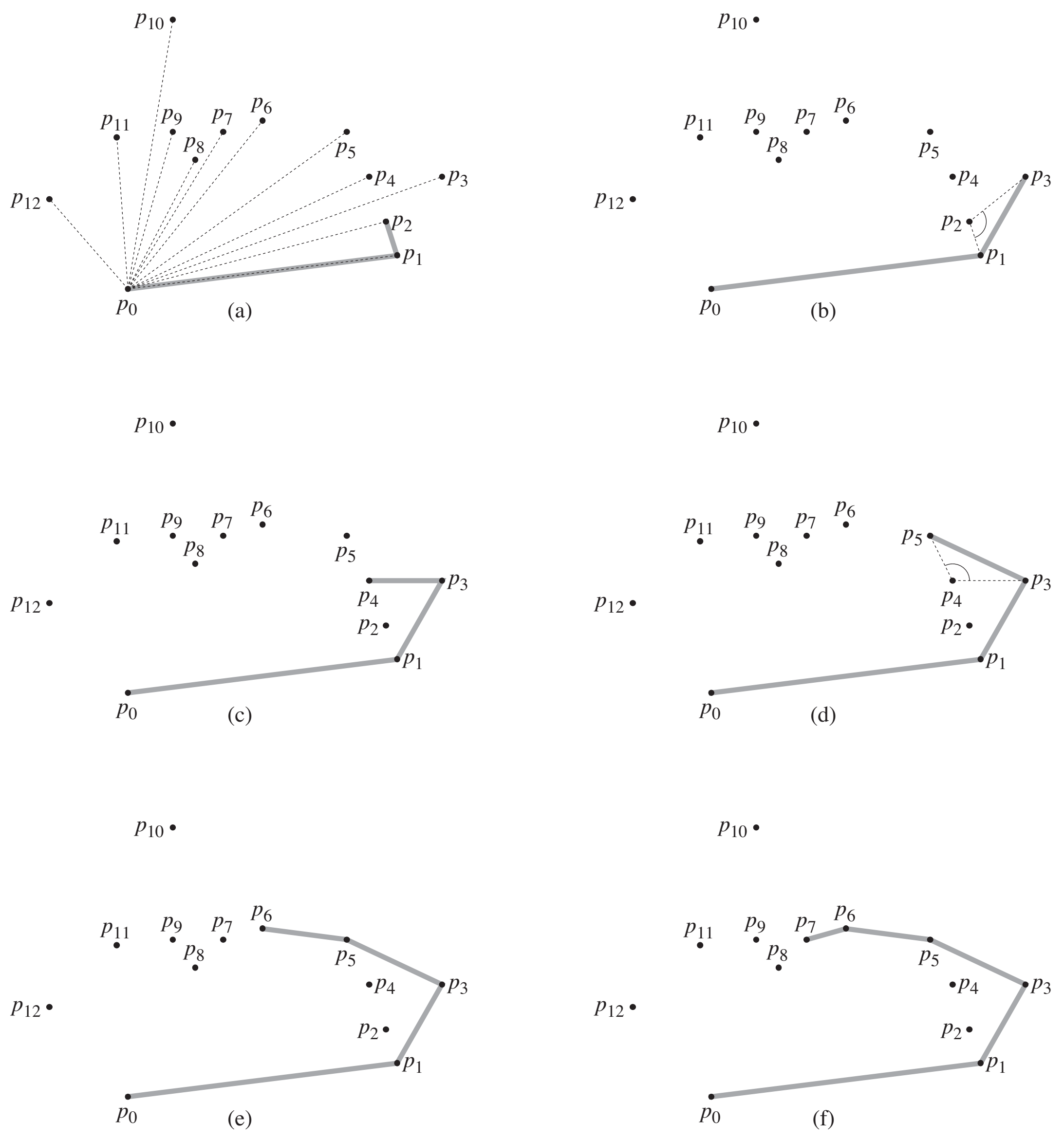


Figure 33.7 The execution of GRAHAM-SCAN on the set Q of Figure 33.6. The current convex hull contained in stack S is shown in gray at each step. **(a)** The sequence $\langle p_1, p_2, \ldots, p_{12} \rangle$ of points numbered in order of increasing polar angle relative to p_0, and the initial stack S containing p_0, p_1, and p_2. **(b)–(k)** Stack S after each iteration of the **for** loop of lines 7–10. Dashed lines show nonleft turns, which cause points to be popped from the stack. In part (h), for example, the right turn at angle $\angle p_7 p_8 p_9$ causes p_8 to be popped, and then the right turn at angle $\angle p_6 p_7 p_9$ causes p_7 to be popped.

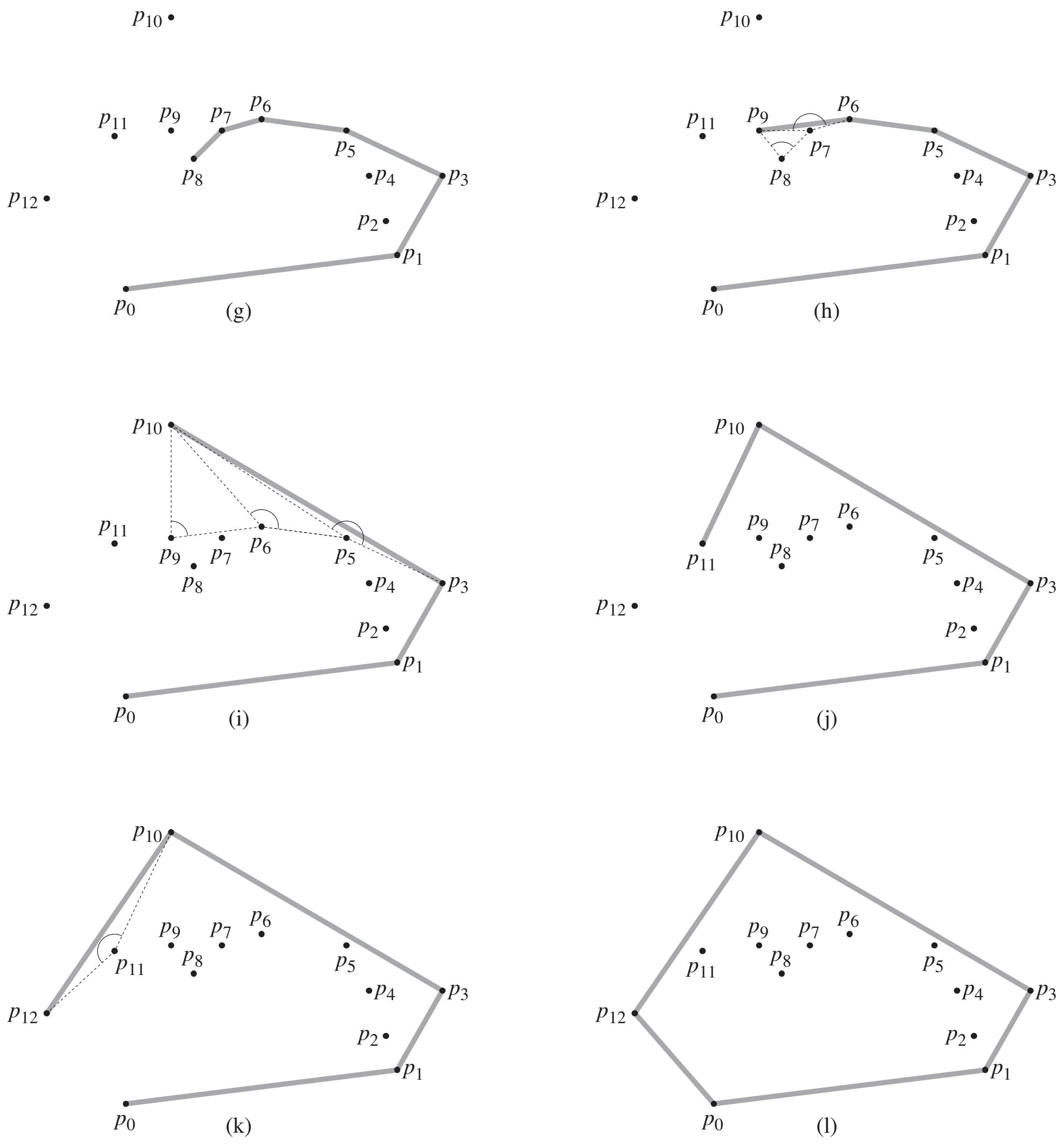

p_{10} p_{11} p_9 p_7 p_6 p_5 p_8 p_4 p_3 p_{12} p_2 p_1 p_0
(g)
(h)
(i)
(j)
(k)
(l)

Figure 33.7, continued **(l)** The convex hull returned by the procedure, which matches that of Figure 33.6.

of CH(Q) (see Exercise 33.3-1). Figure 33.7(a) shows the points of Figure 33.6 sequentially numbered in order of increasing polar angle relative to p_0.

The remainder of the procedure uses the stack S. Lines 3–6 initialize the stack to contain, from bottom to top, the first three points p_0, p_1, and p_2. Figure 33.7(a) shows the initial stack S. The **for** loop of lines 7–10 iterates once for each point in the subsequence $\langle p_3, p_4, \ldots, p_m \rangle$. We shall see that after processing point p_i, stack S contains, from bottom to top, the vertices of CH($\{p_0, p_1, \ldots, p_i\}$) in counterclockwise order. The **while** loop of lines 8–9 removes points from the stack if we find them not to be vertices of the convex hull. When we traverse the convex hull counterclockwise, we should make a left turn at each vertex. Thus, each time the **while** loop finds a vertex at which we make a nonleft turn, we pop the vertex from the stack. (By checking for a nonleft turn, rather than just a right turn, this test precludes the possibility of a straight angle at a vertex of the resulting convex hull. We want no straight angles, since no vertex of a convex polygon may be a convex combination of other vertices of the polygon.) After we pop all vertices that have nonleft turns when heading toward point p_i, we push p_i onto the stack. Figures 33.7(b)–(k) show the state of the stack S after each iteration of the **for** loop. Finally, GRAHAM-SCAN returns the stack S in line 11. Figure 33.7(l) shows the corresponding convex hull.

The following theorem formally proves the correctness of GRAHAM-SCAN.

Theorem 33.1 (Correctness of Graham's scan)
If GRAHAM-SCAN executes on a set Q of points, where $|Q| \geq 3$, then at termination, the stack S consists of, from bottom to top, exactly the vertices of CH(Q) in counterclockwise order.

Proof After line 2, we have the sequence of points $\langle p_1, p_2, \ldots, p_m \rangle$. Let us define, for $i = 2, 3, \ldots, m$, the subset of points $Q_i = \{p_0, p_1, \ldots, p_i\}$. The points in $Q - Q_m$ are those that were removed because they had the same polar angle relative to p_0 as some point in Q_m; these points are not in CH(Q), and so CH(Q_m) $=$ CH(Q). Thus, it suffices to show that when GRAHAM-SCAN terminates, the stack S consists of the vertices of CH(Q_m) in counterclockwise order, when listed from bottom to top. Note that just as p_0, p_1, and p_m are vertices of CH(Q), the points p_0, p_1, and p_i are all vertices of CH(Q_i).

The proof uses the following loop invariant:

> At the start of each iteration of the **for** loop of lines 7–10, stack S consists of, from bottom to top, exactly the vertices of CH(Q_{i-1}) in counterclockwise order.

Initialization: The invariant holds the first time we execute line 7, since at that time, stack S consists of exactly the vertices of $Q_2 = Q_{i-1}$, and this set of three

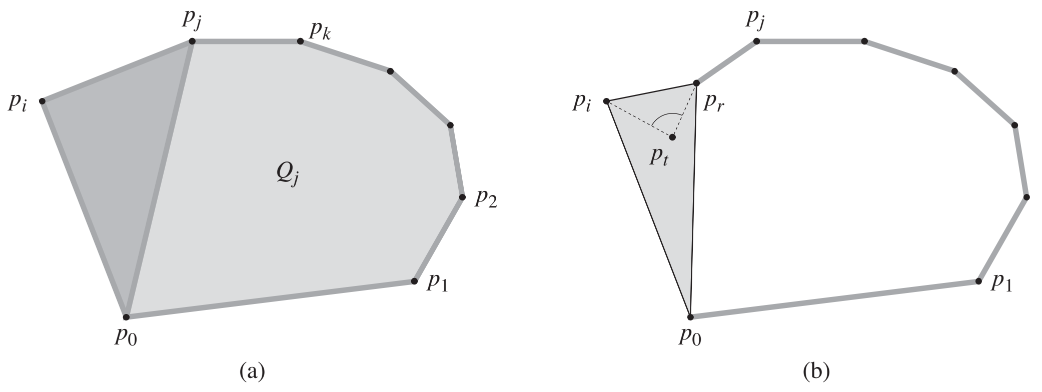


Figure 33.8 The proof of correctness of GRAHAM-SCAN. **(a)** Because p_i's polar angle relative to p_0 is greater than p_j's polar angle, and because the angle $\angle p_k p_j p_i$ makes a left turn, adding p_i to $\text{CH}(Q_j)$ gives exactly the vertices of $\text{CH}(Q_j \cup \{p_i\})$. **(b)** If the angle $\angle p_r p_t p_i$ makes a nonleft turn, then p_t is either in the interior of the triangle formed by p_0, p_r, and p_i or on a side of the triangle, which means that it cannot be a vertex of $\text{CH}(Q_i)$.

vertices forms its own convex hull. Moreover, they appear in counterclockwise order from bottom to top.

Maintenance: Entering an iteration of the **for** loop, the top point on stack S is p_{i-1}, which was pushed at the end of the previous iteration (or before the first iteration, when $i = 3$). Let p_j be the top point on S after executing the while loop of lines 8–9 but before line 10 pushes p_i, and let p_k be the point just below p_j on S. At the moment that p_j is the top point on S and we have not yet pushed p_i, stack S contains exactly the same points it contained after iteration j of the **for** loop. By the loop invariant, therefore, S contains exactly the vertices of $\text{CH}(Q_j)$ at that moment, and they appear in counterclockwise order from bottom to top.

Let us continue to focus on this moment just before pushing p_i. We know that p_i's polar angle relative to p_0 is greater than p_j's polar angle and that the angle $\angle p_k p_j p_i$ makes a left turn (otherwise we would have popped p_j). Therefore, because S contains exactly the vertices of $\text{CH}(Q_j)$, we see from Figure 33.8(a) that once we push p_i, stack S will contain exactly the vertices of $\text{CH}(Q_j \cup \{p_i\})$, still in counterclockwise order from bottom to top.

We now show that $\text{CH}(Q_j \cup \{p_i\})$ is the same set of points as $\text{CH}(Q_i)$. Consider any point p_t that was popped during iteration i of the **for** loop, and let p_r be the point just below p_t on stack S at the time p_t was popped (p_r might be p_j). The angle $\angle p_r p_t p_i$ makes a nonleft turn, and the polar angle of p_t relative to p_0 is greater than the polar angle of p_r. As Figure 33.8(b) shows, p_t must be either in the interior of the triangle formed by p_0, p_r, and p_i or on a side of this triangle (but it is not a vertex of the triangle). Clearly, since p_t is within a triangle formed by three other points of Q_i, it cannot be a vertex of $\text{CH}(Q_i)$. Since p_t is not a vertex of $\text{CH}(Q_i)$, we have that

$$\text{CH}(Q_i - \{p_t\}) = \text{CH}(Q_i) . \tag{33.1}$$

Let P_i be the set of points that were popped during iteration i of the **for** loop. Since the equality (33.1) applies for all points in P_i, we can apply it repeatedly to show that $\text{CH}(Q_i - P_i) = \text{CH}(Q_i)$. But $Q_i - P_i = Q_j \cup \{p_i\}$, and so we conclude that $\text{CH}(Q_j \cup \{p_i\}) = \text{CH}(Q_i - P_i) = \text{CH}(Q_i)$.

We have shown that once we push p_i, stack S contains exactly the vertices of $\text{CH}(Q_i)$ in counterclockwise order from bottom to top. Incrementing i will then cause the loop invariant to hold for the next iteration.

Termination: When the loop terminates, we have $i = m + 1$, and so the loop invariant implies that stack S consists of exactly the vertices of $\text{CH}(Q_m)$, which is $\text{CH}(Q)$, in counterclockwise order from bottom to top. This completes the proof. ■

We now show that the running time of GRAHAM-SCAN is $O(n \lg n)$, where $n = |Q|$. Line 1 takes $\Theta(n)$ time. Line 2 takes $O(n \lg n)$ time, using merge sort or heapsort to sort the polar angles and the cross-product method of Section 33.1 to compare angles. (We can remove all but the farthest point with the same polar angle in total of $O(n)$ time over all n points.) Lines 3–6 take $O(1)$ time. Because $m \leq n - 1$, the **for** loop of lines 7–10 executes at most $n - 3$ times. Since PUSH takes $O(1)$ time, each iteration takes $O(1)$ time exclusive of the time spent in the **while** loop of lines 8–9, and thus overall the **for** loop takes $O(n)$ time exclusive of the nested **while** loop.

We use aggregate analysis to show that the **while** loop takes $O(n)$ time overall. For $i = 0, 1, \ldots, m$, we push each point p_i onto stack S exactly once. As in the analysis of the MULTIPOP procedure of Section 17.1, we observe that we can pop at most the number of items that we push. At least three points—p_0, p_1, and p_m—are never popped from the stack, so that in fact at most $m - 2$ POP operations are performed in total. Each iteration of the **while** loop performs one POP, and so there are at most $m - 2$ iterations of the **while** loop altogether. Since the test in line 8 takes $O(1)$ time, each call of POP takes $O(1)$ time, and $m \leq n - 1$, the total time taken by the **while** loop is $O(n)$. Thus, the running time of GRAHAM-SCAN is $O(n \lg n)$.

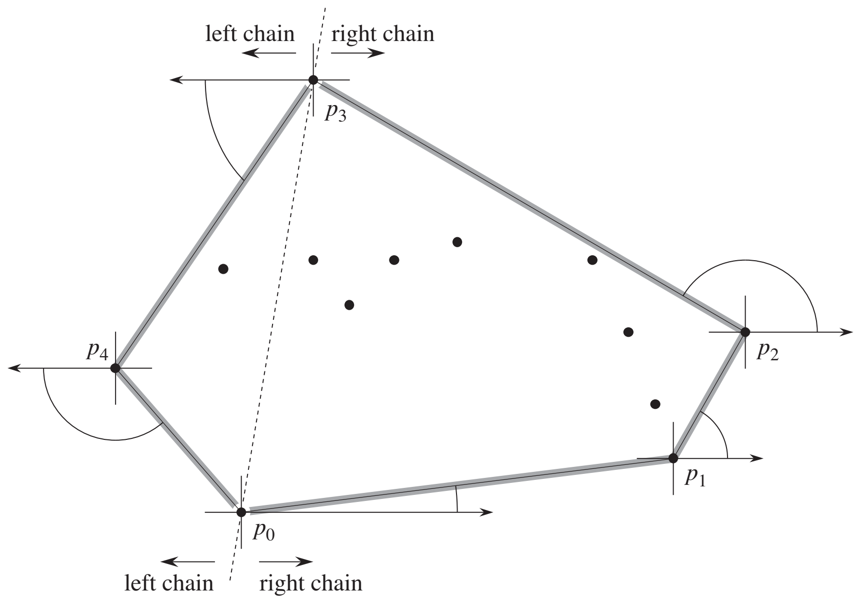


Figure 33.9 The operation of Jarvis's march. We choose the first vertex as the lowest point p_0. The next vertex, p_1, has the smallest polar angle of any point with respect to p_0. Then, p_2 has the smallest polar angle with respect to p_1. The right chain goes as high as the highest point p_3. Then, we construct the left chain by finding smallest polar angles with respect to the negative x-axis.

Jarvis's march

Jarvis's march computes the convex hull of a set Q of points by a technique known as ***package wrapping*** (or ***gift wrapping***). The algorithm runs in time $O(nh)$, where h is the number of vertices of $\text{CH}(Q)$. When h is $o(\lg n)$, Jarvis's march is asymptotically faster than Graham's scan.

Intuitively, Jarvis's march simulates wrapping a taut piece of paper around the set Q. We start by taping the end of the paper to the lowest point in the set, that is, to the same point p_0 with which we start Graham's scan. We know that this point must be a vertex of the convex hull. We pull the paper to the right to make it taut, and then we pull it higher until it touches a point. This point must also be a vertex of the convex hull. Keeping the paper taut, we continue in this way around the set of vertices until we come back to our original point p_0.

More formally, Jarvis's march builds a sequence $H = \langle p_0, p_1, \ldots, p_{h-1} \rangle$ of the vertices of $\text{CH}(Q)$. We start with p_0. As Figure 33.9 shows, the next vertex p_1 in the convex hull has the smallest polar angle with respect to p_0. (In case of ties, we choose the point farthest from p_0.) Similarly, p_2 has the smallest polar angle with respect to p_1, and so on. When we reach the highest vertex, say p_k (breaking ties by choosing the farthest such vertex), we have constructed, as Figure 33.9 shows, the ***right chain*** of CH(Q). To construct the ***left chain***, we start at p_k and choose p_{k+1} as the point with the smallest polar angle with respect to p_k, but *from the negative x-axis*. We continue on, forming the left chain by taking polar angles from the negative x-axis, until we come back to our original vertex p_0.

We could implement Jarvis's march in one conceptual sweep around the convex hull, that is, without separately constructing the right and left chains. Such implementations typically keep track of the angle of the last convex-hull side chosen and require the sequence of angles of hull sides to be strictly increasing (in the range of 0 to 2π radians). The advantage of constructing separate chains is that we need not explicitly compute angles; the techniques of Section 33.1 suffice to compare angles.

If implemented properly, Jarvis's march has a running time of $O(nh)$. For each of the h vertices of CH(Q), we find the vertex with the minimum polar angle. Each comparison between polar angles takes $O(1)$ time, using the techniques of Section 33.1. As Section 9.1 shows, we can compute the minimum of n values in $O(n)$ time if each comparison takes $O(1)$ time. Thus, Jarvis's march takes $O(nh)$ time.

Exercises

33.3-1
Prove that in the procedure GRAHAM-SCAN, points p_1 and p_m must be vertices of CH(Q).

33.3-2
Consider a model of computation that supports addition, comparison, and multiplication and for which there is a lower bound of $\Omega(n \lg n)$ to sort n numbers. Prove that $\Omega(n \lg n)$ is a lower bound for computing, in order, the vertices of the convex hull of a set of n points in such a model.

33.3-3
Given a set of points Q, prove that the pair of points farthest from each other must be vertices of CH(Q).

33.3-4
For a given polygon P and a point q on its boundary, the ***shadow*** of q is the set of points r such that the segment $\overline{qr}$ is entirely on the boundary or in the interior of P. As Figure 33.10 illustrates, a polygon P is ***star-shaped*** if there exists a point p in the interior of P that is in the shadow of every point on the boundary of P. The set of all such points p is called the ***kernel*** of P. Given an n-vertex,

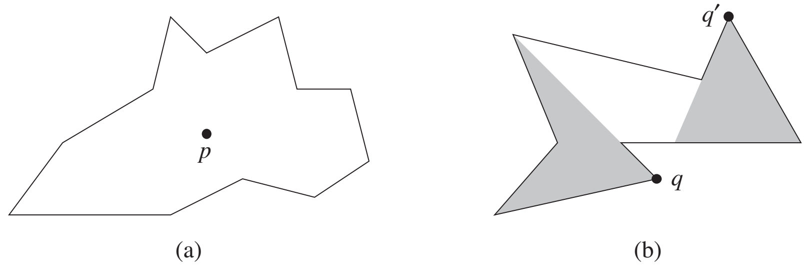


Figure 33.10 The definition of a star-shaped polygon, for use in Exercise 33.3-4. **(a)** A star-shaped polygon. The segment from point p to any point q on the boundary intersects the boundary only at q. **(b)** A non-star-shaped polygon. The shaded region on the left is the shadow of q, and the shaded region on the right is the shadow of q'. Since these regions are disjoint, the kernel is empty.

star-shaped polygon P specified by its vertices in counterclockwise order, show how to compute $\text{CH}(P)$ in $O(n)$ time.

33.3-5
In the ***on-line convex-hull problem***, we are given the set Q of n points one point at a time. After receiving each point, we compute the convex hull of the points seen so far. Obviously, we could run Graham's scan once for each point, with a total running time of $O(n^2 \lg n)$. Show how to solve the on-line convex-hull problem in a total of $O(n^2)$ time.

33.3-6 $\star$
Show how to implement the incremental method for computing the convex hull of n points so that it runs in $O(n \lg n)$ time.

33.4 Finding the closest pair of points

We now consider the problem of finding the closest pair of points in a set Q of $n \geq 2$ points. "Closest" refers to the usual euclidean distance: the distance between points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Two points in set Q may be coincident, in which case the distance between them is zero. This problem has applications in, for example, traffic-control systems. A system for controlling air or sea traffic might need to identify the two closest vehicles in order to detect potential collisions.

A brute-force closest-pair algorithm simply looks at all the $\binom{n}{2} = \Theta(n^2)$ pairs of points. In this section, we shall describe a divide-and-conquer algorithm for this problem, whose running time is described by the familiar recurrence $T(n) = 2T(n/2) + O(n)$. Thus, this algorithm uses only $O(n \lg n)$ time.

The divide-and-conquer algorithm

Each recursive invocation of the algorithm takes as input a subset $P \subseteq Q$ and arrays X and Y, each of which contains all the points of the input subset P. The points in array X are sorted so that their x-coordinates are monotonically increasing. Similarly, array Y is sorted by monotonically increasing y-coordinate. Note that in order to attain the $O(n \lg n)$ time bound, we cannot afford to sort in each recursive call; if we did, the recurrence for the running time would be $T(n) = 2T(n/2) + O(n \lg n)$, whose solution is $T(n) = O(n \lg^2 n)$. (Use the version of the master method given in Exercise 4.6-2.) We shall see a little later how to use "presorting" to maintain this sorted property without actually sorting in each recursive call.

A given recursive invocation with inputs P, X, and Y first checks whether $|P| \le 3$. If so, the invocation simply performs the brute-force method described above: try all $\binom{|P|}{2}$ pairs of points and return the closest pair. If $|P| > 3$, the recursive invocation carries out the divide-and-conquer paradigm as follows.

Divide: Find a vertical line l that bisects the point set P into two sets P_L and P_R such that $|P_L| = \lceil |P|/2 \rceil$, $|P_R| = \lfloor |P|/2 \rfloor$, all points in P_L are on or to the left of line l, and all points in P_R are on or to the right of l. Divide the array X into arrays X_L and X_R, which contain the points of P_L and P_R respectively, sorted by monotonically increasing x-coordinate. Similarly, divide the array Y into arrays Y_L and Y_R, which contain the points of P_L and P_R respectively, sorted by monotonically increasing y-coordinate.

Conquer: Having divided P into P_L and P_R, make two recursive calls, one to find the closest pair of points in P_L and the other to find the closest pair of points in P_R. The inputs to the first call are the subset P_L and arrays X_L and Y_L; the second call receives the inputs P_R, X_R, and Y_R. Let the closest-pair distances returned for P_L and P_R be δ_L and δ_R, respectively, and let $\delta = \min(\delta_L, \delta_R)$.

Combine: The closest pair is either the pair with distance δ found by one of the recursive calls, or it is a pair of points with one point in P_L and the other in P_R. The algorithm determines whether there is a pair with one point in P_L and the other point in P_R and whose distance is less than δ. Observe that if a pair of points has distance less than δ, both points of the pair must be within δ units of line l. Thus, as Figure 33.11(a) shows, they both must reside in the 2δ-wide vertical strip centered at line l. To find such a pair, if one exists, we do the following:

1. Create an array Y', which is the array Y with all points not in the 2δ-wide vertical strip removed. The array Y' is sorted by y-coordinate, just as Y is.
2. For each point p in the array Y', try to find points in Y' that are within δ units of p. As we shall see shortly, only the 7 points in Y' that follow p need be considered. Compute the distance from p to each of these 7 points, and keep track of the closest-pair distance δ' found over all pairs of points in Y'.
3. If $\delta' < \delta$, then the vertical strip does indeed contain a closer pair than the recursive calls found. Return this pair and its distance δ'. Otherwise, return the closest pair and its distance δ found by the recursive calls.

The above description omits some implementation details that are necessary to achieve the $O(n \lg n)$ running time. After proving the correctness of the algorithm, we shall show how to implement the algorithm to achieve the desired time bound.

Correctness

The correctness of this closest-pair algorithm is obvious, except for two aspects. First, by bottoming out the recursion when $|P| \le 3$, we ensure that we never try to solve a subproblem consisting of only one point. The second aspect is that we need only check the 7 points following each point p in array Y'; we shall now prove this property.

Suppose that at some level of the recursion, the closest pair of points is $p_L \in P_L$ and $p_R \in P_R$. Thus, the distance δ' between p_L and p_R is strictly less than δ. Point p_L must be on or to the left of line l and less than δ units away. Similarly, p_R is on or to the right of l and less than δ units away. Moreover, p_L and p_R are within δ units of each other vertically. Thus, as Figure 33.11(a) shows, p_L and p_R are within a $\delta \times 2\delta$ rectangle centered at line l. (There may be other points within this rectangle as well.)

We next show that at most 8 points of P can reside within this $\delta \times 2\delta$ rectangle. Consider the $\delta \times \delta$ square forming the left half of this rectangle. Since all points within P_L are at least δ units apart, at most 4 points can reside within this square; Figure 33.11(b) shows how. Similarly, at most 4 points in P_R can reside within the $\delta \times \delta$ square forming the right half of the rectangle. Thus, at most 8 points of P can reside within the $\delta \times 2\delta$ rectangle. (Note that since points on line l may be in either P_L or P_R, there may be up to 4 points on l. This limit is achieved if there are two pairs of coincident points such that each pair consists of one point from P_L and one point from P_R, one pair is at the intersection of l and the top of the rectangle, and the other pair is where l intersects the bottom of the rectangle.)

Having shown that at most 8 points of P can reside within the rectangle, we can easily see why we need to check only the 7 points following each point in the array Y'. Still assuming that the closest pair is p_L and p_R, let us assume without

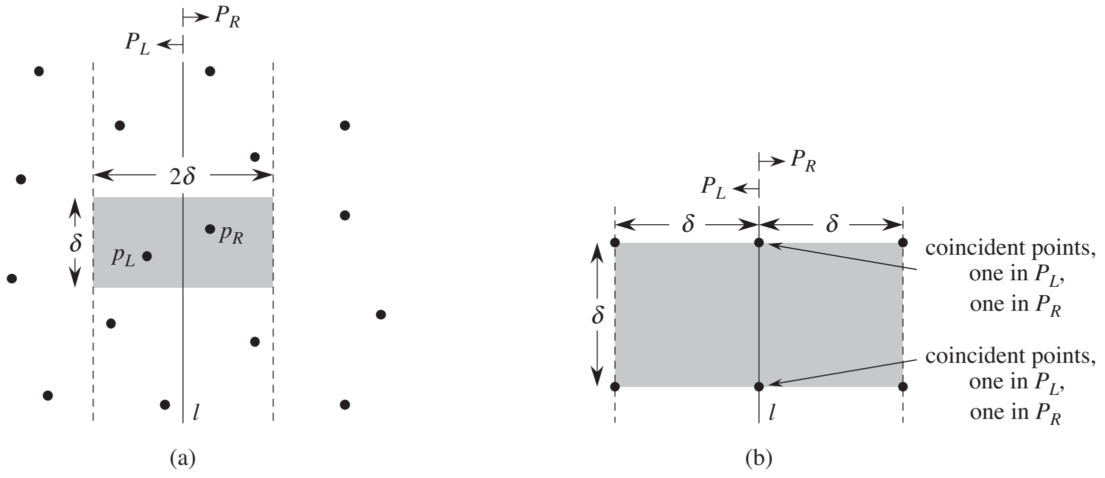

P_R
P_L
2δ
δ
p_R
p_L
l
(a)
P_R
P_L
δ
δ
δ
coincident points, one in P_L, one in P_R
coincident points, one in P_L, one in P_R
l
(b)

Figure 33.11 Key concepts in the proof that the closest-pair algorithm needs to check only 7 points following each point in the array Y'. **(a)** If $p_L \in P_L$ and $p_R \in P_R$ are less than δ units apart, they must reside within a $\delta \times 2\delta$ rectangle centered at line l. **(b)** How 4 points that are pairwise at least δ units apart can all reside within a $\delta \times \delta$ square. On the left are 4 points in P_L, and on the right are 4 points in P_R. The $\delta \times 2\delta$ rectangle can contain 8 points if the points shown on line l are actually pairs of coincident points with one point in P_L and one in P_R.

loss of generality that p_L precedes p_R in array Y'. Then, even if p_L occurs as early as possible in Y' and p_R occurs as late as possible, p_R is in one of the 7 positions following p_L. Thus, we have shown the correctness of the closest-pair algorithm.

Implementation and running time

As we have noted, our goal is to have the recurrence for the running time be $T(n) = 2T(n/2) + O(n)$, where $T(n)$ is the running time for a set of n points. The main difficulty comes from ensuring that the arrays X_L, X_R, Y_L, and Y_R, which are passed to recursive calls, are sorted by the proper coordinate and also that the array Y' is sorted by y-coordinate. (Note that if the array X that is received by a recursive call is already sorted, then we can easily divide set P into P_L and P_R in linear time.)

The key observation is that in each call, we wish to form a sorted subset of a sorted array. For example, a particular invocation receives the subset P and the array Y, sorted by y-coordinate. Having partitioned P into P_L and P_R, it needs to form the arrays Y_L and Y_R, which are sorted by y-coordinate, in linear time. We can view the method as the opposite of the MERGE procedure from merge sort in

Section 2.3.1: we are splitting a sorted array into two sorted arrays. The following pseudocode gives the idea.

```
let $Y_L[1 \, . . \, Y.length]$ and $Y_R[1 \, . . \, Y.length]$ be new arrays
$Y_L.length = Y_R.length = 0$
for $i = 1$ to $Y.length$
    if $Y[i] \in P_L$
        $Y_L.length = Y_L.length + 1$
        $Y_L[Y_L.length] = Y[i]$
    else $Y_R.length = Y_R.length + 1$
        $Y_R[Y_R.length] = Y[i]$
```

We simply examine the points in array Y in order. If a point $Y[i]$ is in P_L, we append it to the end of array Y_L; otherwise, we append it to the end of array Y_R. Similar pseudocode works for forming arrays X_L, X_R, and Y'.

The only remaining question is how to get the points sorted in the first place. We ***presort*** them; that is, we sort them once and for all *before* the first recursive call. We pass these sorted arrays into the first recursive call, and from there we whittle them down through the recursive calls as necessary. Presorting adds an additional $O(n \lg n)$ term to the running time, but now each step of the recursion takes linear time exclusive of the recursive calls. Thus, if we let $T(n)$ be the running time of each recursive step and $T'(n)$ be the running time of the entire algorithm, we get $T'(n) = T(n) + O(n \lg n)$ and

$$T(n) = \begin{cases} 2T(n/2) + O(n) & \text{if } n > 3 \, , \\ O(1) & \text{if } n \le 3 \, . \end{cases}$$

Thus, $T(n) = O(n \lg n)$ and $T'(n) = O(n \lg n)$.

Exercises

33.4-1

Professor Williams comes up with a scheme that allows the closest-pair algorithm to check only 5 points following each point in array Y'. The idea is always to place points on line l into set P_L. Then, there cannot be pairs of coincident points on line l with one point in P_L and one in P_R. Thus, at most 6 points can reside in the $\delta \times 2\delta$ rectangle. What is the flaw in the professor's scheme?

33.4-2

Show that it actually suffices to check only the points in the 5 array positions following each point in the array Y'.

33.4-3

We can define the distance between two points in ways other than euclidean. In the plane, the ***L_m-distance*** between points p_1 and p_2 is given by the expression $(|x_1 - x_2|^m + |y_1 - y_2|^m)^{1/m}$. Euclidean distance, therefore, is L_2-distance. Modify the closest-pair algorithm to use the L_1-distance, which is also known as the ***Manhattan distance***.

33.4-4

Given two points p_1 and p_2 in the plane, the L_∞-distance between them is given by $\max(|x_1 - x_2|, |y_1 - y_2|)$. Modify the closest-pair algorithm to use the L_∞-distance.

33.4-5

Suppose that $\Omega(n)$ of the points given to the closest-pair algorithm are covertical. Show how to determine the sets P_L and P_R and how to determine whether each point of Y is in P_L or P_R so that the running time for the closest-pair algorithm remains $O(n \lg n)$.

33.4-6

Suggest a change to the closest-pair algorithm that avoids presorting the Y array but leaves the running time as $O(n \lg n)$. (*Hint:* Merge sorted arrays Y_L and Y_R to form the sorted array Y.)

Problems

33-1 Convex layers

Given a set Q of points in the plane, we define the ***convex layers*** of Q inductively. The first convex layer of Q consists of those points in Q that are vertices of $\text{CH}(Q)$. For $i > 1$, define Q_i to consist of the points of Q with all points in convex layers $1, 2, \ldots, i-1$ removed. Then, the ith convex layer of Q is $\text{CH}(Q_i)$ if $Q_i \neq \emptyset$ and is undefined otherwise.

a. Give an $O(n^2)$-time algorithm to find the convex layers of a set of n points.

b. Prove that $\Omega(n \lg n)$ time is required to compute the convex layers of a set of n points with any model of computation that requires $\Omega(n \lg n)$ time to sort n real numbers.

33-2 Maximal layers

Let Q be a set of n points in the plane. We say that point (x, y) ***dominates*** point (x', y') if $x \geq x'$ and $y \geq y'$. A point in Q that is dominated by no other points in Q is said to be ***maximal***. Note that Q may contain many maximal points, which can be organized into ***maximal layers*** as follows. The first maximal layer L_1 is the set of maximal points of Q. For $i > 1$, the ith maximal layer L_i is the set of maximal points in $Q - \bigcup_{j=1}^{i-1} L_j$.

Suppose that Q has k nonempty maximal layers, and let y_i be the y-coordinate of the leftmost point in L_i for $i = 1, 2, \ldots, k$. For now, assume that no two points in Q have the same x- or y-coordinate.

a. Show that $y_1 > y_2 > \cdots > y_k$.

Consider a point (x, y) that is to the left of any point in Q and for which y is distinct from the y-coordinate of any point in Q. Let $Q' = Q \cup \{(x, y)\}$.

b. Let j be the minimum index such that $y_j < y$, unless $y < y_k$, in which case we let $j = k + 1$. Show that the maximal layers of Q' are as follows:

- If $j \leq k$, then the maximal layers of Q' are the same as the maximal layers of Q, except that L_j also includes (x, y) as its new leftmost point.
- If $j = k+1$, then the first k maximal layers of Q' are the same as for Q, but in addition, Q' has a nonempty $(k + 1)$st maximal layer: $L_{k+1} = \{(x, y)\}$.

c. Describe an $O(n \lg n)$-time algorithm to compute the maximal layers of a set Q of n points. (*Hint:* Move a sweep line from right to left.)

d. Do any difficulties arise if we now allow input points to have the same x- or y-coordinate? Suggest a way to resolve such problems.

33-3 Ghostbusters and ghosts

A group of n Ghostbusters is battling n ghosts. Each Ghostbuster carries a proton pack, which shoots a stream at a ghost, eradicating it. A stream goes in a straight line and terminates when it hits the ghost. The Ghostbusters decide upon the following strategy. They will pair off with the ghosts, forming n Ghostbuster-ghost pairs, and then simultaneously each Ghostbuster will shoot a stream at his chosen ghost. As we all know, it is *very* dangerous to let streams cross, and so the Ghostbusters must choose pairings for which no streams will cross.

Assume that the position of each Ghostbuster and each ghost is a fixed point in the plane and that no three positions are colinear.

a. Argue that there exists a line passing through one Ghostbuster and one ghost such that the number of Ghostbusters on one side of the line equals the number of ghosts on the same side. Describe how to find such a line in $O(n \lg n)$ time.

b. Give an $O(n^2 \lg n)$-time algorithm to pair Ghostbusters with ghosts in such a way that no streams cross.

33-4 Picking up sticks

Professor Charon has a set of n sticks, which are piled up in some configuration. Each stick is specified by its endpoints, and each endpoint is an ordered triple giving its (x, y, z) coordinates. No stick is vertical. He wishes to pick up all the sticks, one at a time, subject to the condition that he may pick up a stick only if there is no other stick on top of it.

a. Give a procedure that takes two sticks a and b and reports whether a is above, below, or unrelated to b.

b. Describe an efficient algorithm that determines whether it is possible to pick up all the sticks, and if so, provides a legal order in which to pick them up.

33-5 Sparse-hulled distributions

Consider the problem of computing the convex hull of a set of points in the plane that have been drawn according to some known random distribution. Sometimes, the number of points, or size, of the convex hull of n points drawn from such a distribution has expectation $O(n^{1-\epsilon})$ for some constant $\epsilon > 0$. We call such a distribution ***sparse-hulled***. Sparse-hulled distributions include the following:

- Points drawn uniformly from a unit-radius disk. The convex hull has expected size $\Theta(n^{1/3})$.
- Points drawn uniformly from the interior of a convex polygon with k sides, for any constant k. The convex hull has expected size $\Theta(\lg n)$.
- Points drawn according to a two-dimensional normal distribution. The convex hull has expected size $\Theta(\sqrt{\lg n})$.

a. Given two convex polygons with n_1 and n_2 vertices respectively, show how to compute the convex hull of all $n_1 + n_2$ points in $O(n_1 + n_2)$ time. (The polygons may overlap.)

b. Show how to compute the convex hull of a set of n points drawn independently according to a sparse-hulled distribution in $O(n)$ average-case time. (*Hint:* Recursively find the convex hulls of the first $n/2$ points and the second $n/2$ points, and then combine the results.)

Chapter notes

This chapter barely scratches the surface of computational-geometry algorithms and techniques. Books on computational geometry include those by Preparata and Shamos [282], Edelsbrunner [99], and O'Rourke [269].

Although geometry has been studied since antiquity, the development of algorithms for geometric problems is relatively new. Preparata and Shamos note that the earliest notion of the complexity of a problem was given by E. Lemoine in 1902. He was studying euclidean constructions—those using a compass and a ruler—and devised a set of five primitives: placing one leg of the compass on a given point, placing one leg of the compass on a given line, drawing a circle, passing the ruler's edge through a given point, and drawing a line. Lemoine was interested in the number of primitives needed to effect a given construction; he called this amount the "simplicity" of the construction.

The algorithm of Section 33.2, which determines whether any segments intersect, is due to Shamos and Hoey [313].

The original version of Graham's scan is given by Graham [150]. The package-wrapping algorithm is due to Jarvis [189]. Using a decision-tree model of computation, Yao [359] proved a worst-case lower bound of $\Omega(n \lg n)$ for the running time of any convex-hull algorithm. When the number of vertices h of the convex hull is taken into account, the prune-and-search algorithm of Kirkpatrick and Seidel [206], which takes $O(n \lg h)$ time, is asymptotically optimal.

The $O(n \lg n)$-time divide-and-conquer algorithm for finding the closest pair of points is by Shamos and appears in Preparata and Shamos [282]. Preparata and Shamos also show that the algorithm is asymptotically optimal in a decision-tree model.