Motivation

We have discussed serial algorithms that are suitable for running on a uniprocessor computer. We will now extend our model to parallel algorithms that can run on a multiprocessor computer.

Computational Model

- There exist many competing models of parallel computation that are essentially different. For example, one can have shared or distributed memory.
- Since multicore processors are ubiquitous, we focus on a parallel computing model with shared memory.

Dynamic Multithreading

Programming a shared-memory parallel computer can be difficult and error-prone. In particular, it is difficult to partition the work among several threads so that each thread approximately has the same load.

A concurrency platform is a software layer that coordinates, schedules, and manages parallel-computing resources. We will use a simple extension of the serial programming model that uses the concurrency instructions parallel, spawn, and sync.

Spawn

Spawn: If spawn proceeds a procedure call, then the procedure instance that executes the spawn (the parent) may continue to execute in parallel with the spawned subroutine (the child), instead of waiting for the child to complete.

The keyword spawn does not say that a procedure must execute concurrently, but simply that it may.

At runtime, it is up to the scheduler to decide which subcomputations should run concurrently.

Sync

The keyword sync indicates that the procedure must wait for all its spawned children to complete.

Parallel

Many algorithms contain loops, where all iterations can operate in parallel. If the parallel keyword proceeds a for loop, then this indicates that the loop body can be executed in parallel.

Fibonacci Numbers

Definition

The Fibonacci numbers are defined by the recurrence:

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i-1} + F_{i-2}$

for $i > 1$.

Naive Algorithm

Computing the Fibonacci numbers can be done with the following algorithm:

```
Fibonacci(n)
if n < 2 then return n;
x = Fibonacci(n-1);
y = Fibonacci(n-2) ;
return x + y;
```

Caveat: Running Time

Let $T(n)$ denote the running time of Fibonacci(n). Since this procedure contains two recursive calls and a constant amount of extra work, we get

$T(n) = T(n-1) + T(n-2) + \theta(1)$

which yields $T(n) = \theta(F_n)= \theta(((1+sqrt(5))/2)^n)$

Since this is an exponential growth, this is a particularly bad way to calculate Fibonacci numbers.

How would you calculate the Fibonacci numbers?

Fibonacci Numbers

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

This allows you to calculate F_n in O(log n) steps by repeated squaring of the matrix. This is how you can calculate the Fibonacci numbers with a serial algorithm.

To illustrate the principles of parallel programming, we will use the naive (bad) algorithm, though.

Fibonacci Example

Parallel algorithm to compute Fibonacci numbers:

```
Fibonacci(n)
if n < 2 then return n;
x = spawn Fibonacci(n-1);   // parallel execution
y = Fibonacci(n-2) ;
sync;  // wait for results of x and y
return x + y;
```

Computation DAG

Multithreaded computation can be better understood with the help of a computation directed acyclic graph $G=(V,E)$.

The vertices V in the graph are the instructions.

The edges E represent dependencies between instructions.

An edge (u,v) is in E means that the instruction u must execute before instruction v.

[Problem: Somewhat too detailed. We will group the instructions into threads.]

Strand and Threads

A sequence of instructions containing no parallel control (spawn, sync, return from spawn, parallel) can be grouped into a single strand.

A strand of maximal length will be called a thread.

Computation DAG

A computation directed acyclic graph $G=(V,E)$ consists a vertex set V that comprises the threads of the program.

The edge set E contains an edge (u,v) if and only if the thread u need to execute before thread v.

If there is an edge between thread u and v, then they are said to be (logically) in series. If there is no thread, then they are said to be (logically) in parallel.

Edge Classification

A continuation edge (u,v) connects a thread u to its successor v within the same procedure instance.

When a thread u spawns a new thread v, then (u,v) is called a spawn edge.

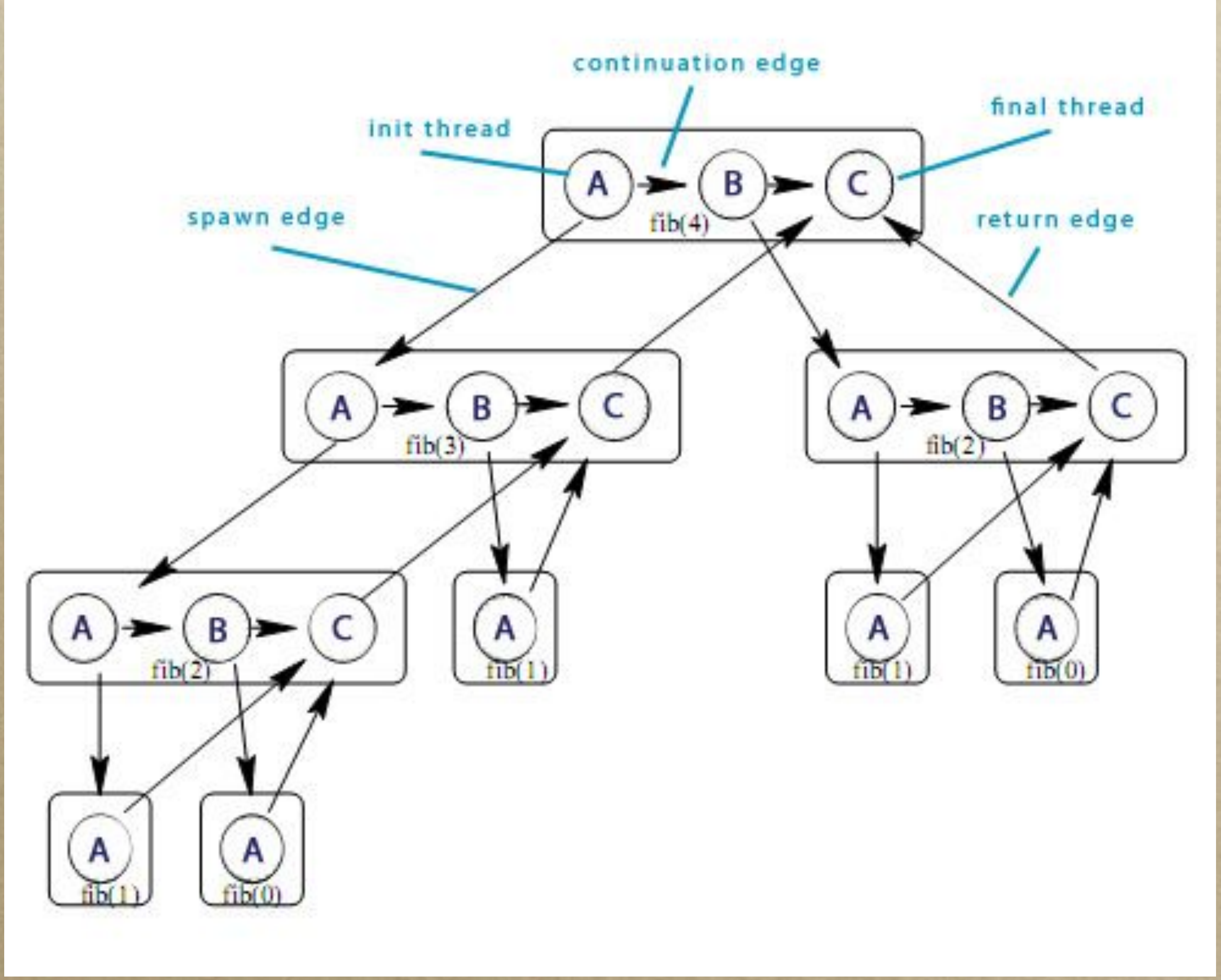
When a thread v returns to its calling procedure and x is the thread following the parallel control, then the return edge (v,x) is included in the graph.

Fibonacci Example

Parallel algorithm to compute Fibonacci numbers:

```
Fibonacci(n)
if n < 2 then return n;          // thread A
x = spawn Fibonacci(n-1);
y = Fibonacci(n-2) ;
sync;
return x + y; // thread C
```

Fibonacci(4)



continuation edge
init thread
final thread
spawn edge
return edge
A
B
C
fib(4)
A
B
C
fib(3)
A
B
C
fib(2)
A
B
C
fib(2)
A
fib(1)
A
fib(1)
A
fib(0)
A
fib(1)
A
fib(0)

Performance Measures

The work of a multithreaded computation is the total time to execute the entire computation on one processor.

Work = sum of the times taken by each thread

Performance Measures

The span is the longest time to execute the threads along any path of the computational directed acyclic graph.

Performance Measure Example

In Fibonacci(4), we have

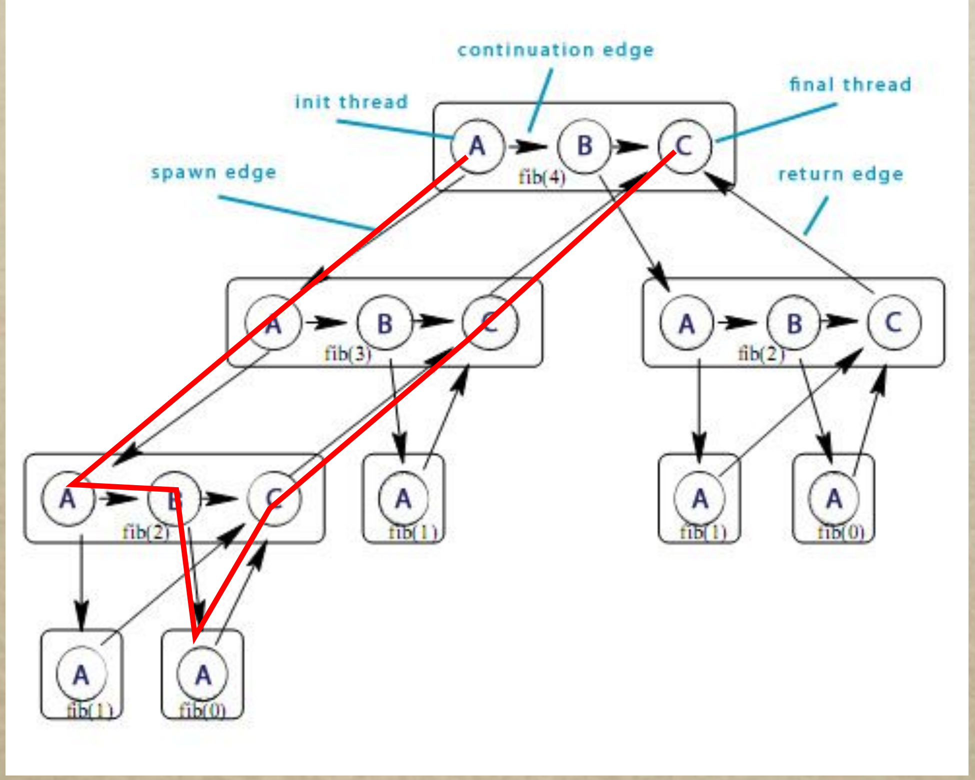
17 vertices = 17 threads.

8 vertices on longest path.

Assuming unit time for each thread, we get

work = 17 time units

span = 8 time units



The actual running time of a multithreaded computation depends not just on its work and span, but also on how many processors (cores) are available, and how the scheduler allocates strands to processors.

Running time on P processors is indicated by subscript P

- T_1 running time on a single processor

- T_P running time on P processors

- T_∞ running time on unlimited processors

Work Law

An ideal parallel computer with P processors can do at most P units of work. Total work to do is T_1.

Thus, $PT_p >= T_1$

The work law is

$T_p >= T_1/P$

Span Law

A P-processor ideal parallel computer cannot run faster than a machine with unlimited number of processors.

However, a computer with unlimited number of processors can emulate a P-processor machine by using simply P of its processors. Therefore,

$T_p >= T_\infty$

which is called the span law.

Speedup and Parallelism

The speed up of a computation on P processors is defined as T_1 / T_p

The parallelism of a multithreaded computation is given by T_1 / T_∞

Scheduling

The performance depends not just on the work and span. Additionally, the strands must be scheduled efficiently.

The strands must be mapped to static threads, and the operating system schedules the threads on the processors themselves.

The scheduler must schedule the computation with no advance knowledge of when the strands will be spawned or when they will complete; it must operate online.

Greedy Scheduler

We will assume a greedy scheduler in our analysis, since this keeps things simple. A greedy scheduler assigns as many strands to processors as possible in each time step.

On P processors, if at least P strands are ready to execute during a time step, then we say that the step is a complete step; otherwise we say that it is an incomplete step.

Greedy Scheduler Theorem

On an ideal parallel computer with P processors, a greedy scheduler executes a multithreaded computation with work T_1 and span T_∞ in time

$T_P <= T_1 / P + T_\infty$

[Given the fact the best we can hope for on P processors is $T_P = T_1 / P$ by the work law, and $T_P = T_\infty$ by the span law, the sum of these two lower bounds]

Proof (1/3)

Let's consider the complete steps. In each complete step, the P processors perform a total of P work.

Seeking a contradiction, we assume that the number of complete steps exceeds T_1/P. Then the total work of the complete steps is at least

$$\begin{aligned} P(\lfloor T_1/P \rfloor + 1) &= P\lfloor T_1/P \rfloor + P \\ &= T_1 - (T_1 \mod P) + P \\ &> T_1 \end{aligned}$$

as this exceeds the total work required by the computation, this is impossible.

Proof (2/3)

Now consider an incomplete step. Let G be the DAG representing the entire computation. W.l.o.g. assume that each strand takes unit time (otherwise replace longer strands by a chain of unit-time strands).

Let G' be the subgraph of G that has yet to be executed at the start of the incomplete step, and let G'' be the subgraph remaining to be executed after the completion of the incomplete step.

Proof (3/3)

A longest path in a DAG must necessarily start at a vertex with in-degree 0. Since an incomplete step of a greedy scheduler executes all strands with in-degree 0 in G', the length of the longest path in G'' must be 1 less than the length of the longest path in G'.

Put differently, an incomplete step decreases the span of the unexecuted DAG by 1. Thus, the number of incomplete steps is at most T_∞.

Since each step is either complete or incomplete, the theorem follows. q.e.d.

Corollary

The running time of any multithreaded computation scheduled by a greedy scheduler on an ideal parallel computer with P processors is within a factor of 2 of optimal.

Proof.: The T_P^* be the running time produced by an optimal scheduler. Let T_1 be the work and T_∞ be the span of the computation. Then $T_P^* >= \max(T_1/P, T_\infty)$. By the theorem,

$T_P <= T_1/P + T_\infty <= 2\max(T_1/P, T_\infty) <= 2\, T_P^*$

Slackness

The parallel slackness of a multithreaded computation executed on an ideal parallel computer with P processors is the ratio of parallelism by P.

Slackness = $(T_1 / T_\infty) / P$

If the slackness is less than 1, we cannot hope to achieve a linear speedup.

Speedup

Let T_P be the running time of a multithreaded computation produced by a greedy scheduler on an ideal computer with P processors. Let T_1 be the work and T_∞ be the span of the computation. If the slackness is big, $P \ll (T_1 / T_\infty)$, then

T_P is approximately T_1 / P.

Proof: If $P \ll (T_1 / T_\infty)$, then $T_\infty \ll T_1 / P$. Thus, by the theorem, $T_P \le T_1 /P + T_\infty \approx T_1 /P$. By the work law, $T_P \ge T_1 /P$. Hence,

$T_P \approx T_1 /P$, as claimed.

Back to Fibonacci

Parallel Fibonacci Computation

Parallel algorithm to compute Fibonacci numbers:

```
Fibonacci(n)
if n < 2 then return n;
x = spawn Fibonacci(n-1);   // parallel execution
y = spawn Fibonacci(n-2) ;  // parallel execution
sync;  // wait for results of x and y
return x + y;
```

Work of Fibonacci

We want to know the work and span of the Fibonacci computation, so that we can compute the parallelism (work/span) of the computation.

The work T_1 is straightforward, since it amounts to compute the running time of the serialized algorithm.

$T_1 = \theta(((1+sqrt(5))/2)^n)$

Span of Fibonacci

Recall that the span T_∞ in the longest path in the computational DAG. Since Fibonacci(n) spawns

- Fibonacci(n-1)
- Fibonacci(n-2)

we have

$T_\infty(n) = \max(\ T_\infty(n-1)\ ,\ T_\infty(n-2)\ ) + \theta(1) = T_\infty(n-1) + \theta(1)$

which yields $T_\infty(n) = \theta(n)$.

Parallelism of Fibonacci

The parallelism of the Fibonacci computation is

$T_1(n)/T_\infty(n) = \theta(((1+sqrt(5))/2)^n / n)$

which grows dramatically as n gets large.

Therefore, even on the largest parallel computers, a modest value of n suffices to achieve near perfect linear speedup, since we have considerable parallel slackness.

Race Conditions

Race Conditions

A multithreaded algorithm is deterministic if and only if does the same thing on the same input, no matter how the instructions are scheduled.

A multithreaded algorithm is nondeterministic if its behavior might vary from run to run.

Often, a multithreaded algorithm that is intended to be deterministic fails to be.

Determinacy Race

A determinacy race occurs when two logically parallel instructions access the same memory location and at least one of the instructions performs a write.

```
Race-Example()
x = 0
parallel for i = 1 to 2 do
  x = x+1
print x
```

Determinacy Race

When a processor increments x, the operation is not indivisible, but composed of a sequence of instructions.

1) Read x from memory into one of the processor's registers

2) Increment the value of the register

3) Write the value in the register back into x in memory

Determinacy Race

x = 0

assign r1 = 0

incr r1, so r1=1

assign r2 = 0

incr r2, so r2 = 1

write back x = r1

write back x = r2

print x // now prints 1 instead of 2

Matrix Multiplication

Matrix Multiplication

Recall that one can multiply nxn matrices serially in time $\Theta(n^{\log 7}) = O(n^{2.81})$ using Strassen's divide-and-conquer method.

We will use multithreading for a simpler divide-and-conquer algorithm.

Simple Divide-and-Conquer

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

To multiply two nxn matrices, we perform 8 matrix multiplications of n/2 x n/2 matrices and one addition of n x n matrices.

Addition of Matrices

```
Matrix-Add(C, T, n):
   // Adds matrices C and T in-place, producing C = C + T
   // n is power of 2 (for simplicity).
   if  n == 1:
     C[1, 1] = C[1, 1] + T[1, 1]
   else:
     partition C and T into (n/2)x(n/2) submatrices
     spawn Matrix-Add(C11, T11, n/2)
     spawn Matrix-Add(C12, T12, n/2)
     spawn Matrix-Add(C21, T21, n/2)
     spawn Matrix-Add(C22, T22, n/2)
     sync
```

Matrix Multiplication

Matrix-Multiply(C, A, B, n):

// Multiplies matrices A and B, storing the result in C.

// n is power of 2 (for simplicity).

if n == 1:

C[1, 1] = A[1, 1] · B[1, 1]

else:

allocate a temporary matrix T[1...n, 1...n]

partition A, B, C, and T into (n/2)x(n/2) submatrices

spawn Matrix-Multiply(C_{11},A_{11},B_{11}, n/2)

spawn Matrix-Multiply(C_{12},A_{11},B_{12}, n/2)

spawn Matrix-Multiply(C_{21},A_{21},B_{11}, n/2)

spawn Matrix-Multiply(C_{22},A_{21},B_{12}, n/2)

spawn Matrix-Multiply(T_{11},A_{12},B_{21}, n/2)

spawn Matrix-Multiply(T_{12},A_{12},B_{22}, n/2)

spawn Matrix-Multiply(T_{21},

spawn Matrix-Multiply(T_{22},

sync

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Work of Matrix Multiplication

The work $T_1(n)$ of matrix multiplication satisfies the recurrence

$T_1(n) = 8\ T_1(n/2) + \theta(n^2) = \theta(n^3)$

by case 1 of the Master theorem.

Span of Matrix Multiplication

The span $T_\infty(n)$ of matrix multiplication is determined by

- the span for partitioning $\theta(1)$
- the span of the parallel nested for loops at the end $\theta(\log n)$
- the maximum span of the 8 matrix multiplications

$T_\infty(n) = T_\infty(n/2) + \theta(\log n)$

This recurrence does not fall under any of the cases of the Master theorem. One can show that $T_\infty(n) = \theta((\log n)^2)$

Parallelism of Matrix Mult

The parallelism of matrix multiplication is given by

$T_1(n) / T_\infty(n) = \Theta(n^3 / (\log n)^2)$

which is very high.

Multithreaded Algorithms

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