Gram-Schmidt Orthogonalization Process

Theorem Gram–Schmidt Orthogonalization Process

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m \le n$, be a basis for a subspace W_m of R^n. Then $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_m = \mathbf{u}_m - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}}\right)\mathbf{v}_{m-1},$$

is an orthogonal basis for W_m. An orthonormal basis for W_m is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\} = \left\{\frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2, \ldots, \frac{1}{\|\mathbf{v}_m\|}\mathbf{v}_m\right\}.$$

1- Use the Gram–Schmidt orthogonalization process to transform the given basis B = {u1, u2, u3} for R^3 into an orthogonal basis B'={v1, v2, v3}. Then form an orthonormal basis B''= {w1, w2, w3}.
B= {<1, 1, 1> , <9, -1, 1> , <1, 4, 2>}

2- an inner product defined on the vector space P of all polynomials of degree less than or equal to 2, is given by $(p, q) = \int_{-1}^{1} p(x)(q(x)dx$

Use the Gram–Schmidt orthogonalization process to transform the given basis B for P_2 into an orthogonal basis B'.

a) B= $\{1, x, x^2\}$

b) B= $\{x^2 - x, x^2 + 1, 1 - x^2\}$

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Homework 4

Gram-Schmidt Orthogonalization Process

Let $B=\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_m\}$, $m\le n$ be a basis for a subspace W_m of R^n. Then $B'=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m\}$, where:

$$\vec{v}_1=\vec{u}_1$$
$$\vec{v}_2=\vec{u}_2-\left(\frac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1$$
$$\vec{v}_3=\vec{u}_3-\left(\frac{\vec{u}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1-\left(\frac{\vec{u}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\cdot\vec{v}_2$$
$$\vdots$$
$$\vec{v}_m=\vec{u}_m-\left(\frac{\vec{u}_m\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1-\left(\frac{\vec{u}_m\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2-\dots-\left(\frac{\vec{u}_m\cdot\vec{v}_{m-1}}{\vec{v}_{m-1}\cdot\vec{v}_{m-1}}\right)\cdot\vec{v}_{m-1},$$

is an orthogonal basis for W_m. An orthonormal basis for W_m is:

$$B''=\{\vec{w}_1,\vec{w}_2,\dots,\vec{w}_m\}=\left\{\frac{1}{\|\vec{v}_1\|}\vec{v}_1,\frac{1}{\|\vec{v}_2\|}\vec{v}_2,\dots,\frac{1}{\|\vec{v}_m\|}\vec{v}_m\right\}.$$

Homework

1) Use the Gram-Schmidt orthogonalization process to transform the given basis $B=\{u_1,u_2,u_3\}$ for $\mathbb{R}^3$ into an orthogonal basis $B'=\{V_1,V_2,V_3\}$. Then form an orthonormal basis $B''=\{w_1,w_2,w_3\}$.

$B=\{\langle1,1,1\rangle,\langle9,-1,1\rangle,\langle1,4,2\rangle\}$ is a basis of $\mathbb{R}^3$ and we need to find $B'=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

The Gram-Schmidt orthogonalization process implies:

- $\vec{v}_1=\vec{u}_1=\langle1,1,1\rangle$
- $\vec{v}_2=\vec{u}_2-\left(\frac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1=\langle9,-1,1\rangle-\left(\frac{\langle9,-1,1\rangle\cdot\langle1,1,1\rangle}{\langle1,1,1\rangle\cdot\langle1,1,1\rangle}\right)\cdot\langle1,1,1\rangle$

$$=\langle9,-1,1\rangle-\left(\frac{(9\cdot1+(-1)\cdot1+1\cdot1)}{(1\cdot1)+(1\cdot1)+(1\cdot1)}\right)\cdot\langle1,1,1\rangle$$
$$=\langle9,-1,1\rangle-(9/3)\cdot\langle1,1,1\rangle$$
$$=\langle9,-1,1\rangle-\langle3,3,3\rangle=\langle6,-4,-2\rangle$$

- $\vec{v}_3=\vec{u}_3-\left(\frac{\vec{u}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1-\left(\frac{\vec{u}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$

$\vec{u}_3\cdot\vec{v}_1=\langle1,4,2\rangle\cdot\langle1,1,1\rangle=1+4+2=7$

$\vec{u}_3\cdot\vec{v}_2=\langle1,4,2\rangle\cdot\langle6,-4,-2\rangle=6-16-4=-14$

$\vec{v}_1\cdot\vec{v}_1=3$; $\vec{v}_2\cdot\vec{v}_2=\langle6,-4,-2\rangle\cdot\langle6,-4,-2\rangle$
$=36+16+4=56$

$$\vec{v}_3=\langle1,4,2\rangle-\frac{7}{3}\cdot\langle1,1,1\rangle+\frac{14}{56}\cdot\langle6,-4,-2\rangle$$
$$=\langle1,4,2\rangle-\left\langle\frac{7}{3},\frac{7}{3},\frac{7}{3}\right\rangle+\left\langle\frac{3}{2},-1,-\frac{1}{2}\right\rangle$$
$$=\left\langle1-\frac{7}{3}+\frac{3}{2},\ 4-\frac{7}{3}-1,\ 2-\frac{7}{3}-\frac{1}{2}\right\rangle=\left\langle\frac{1}{6},\frac{2}{3},-\frac{5}{6}\right\rangle$$

So that:

$$\boxed{B'=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}=\{\langle1,1,1\rangle,\langle6,-4,-2\rangle,\langle 1/6,2/3,-5/6\rangle\}}$$

Now, for $B'' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\frac{1}{\|\vec{v}_1\|}\vec{v}_1, \frac{1}{\|\vec{v}_2\|}\vec{v}_2, \frac{1}{\|\vec{v}_3\|}\vec{v}_3\right\}$

$w_1 = \frac{1}{\|\vec{v}_1\|} \cdot \vec{v}_1 = \frac{1}{\sqrt{1\cdot 1 + 1\cdot 1 + 1\cdot 1}} \langle 1,1,1 \rangle = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$ $\quad * \ \|\alpha\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$

$w_2 = \frac{1}{\|\vec{v}_2\|} \cdot \vec{v}_2 = \frac{1}{\sqrt{(6)^2 + (-4)^2 + 2^2}} \cdot \langle 6,-4,-2 \rangle = \frac{1}{\sqrt{56}} \cdot \langle 6,-4,-2 \rangle = \frac{1}{\sqrt{4\cdot 14}} \cdot \langle 6,-4,-2 \rangle$

$= \frac{1}{2\sqrt{14}} \cdot \langle 6,-4,-2 \rangle = \frac{1}{\sqrt{14}} \cdot \langle 3,-2,-1 \rangle$

$w_3 = \frac{1}{\|\vec{v}_3\|} \cdot \vec{v}_3 = \frac{1}{\sqrt{(1/6)^2 + (2/3)^2 + (-5/6)^2}} \cdot \langle 1/6, 2/3, -5/6 \rangle = \frac{1}{\sqrt{1/36 + 4/9 + 25/36}} \cdot \langle 1/6, 2/3, -5/6 \rangle$

$= \frac{1}{\sqrt{1/36 + 16/36 + 25/36}} \cdot \langle 1/6, 2/3, -5/6 \rangle = \frac{1}{\sqrt{42/36}} \cdot \langle 1/6, 2/3, -5/6 \rangle$

$= \frac{\sqrt{36}}{\sqrt{42}} \cdot \langle 1/6, 2/3, -5/6 \rangle = \frac{6}{\sqrt{42}} \cdot \langle 1/6, 2/3, -5/6 \rangle$

$= \frac{1}{\sqrt{42}} \cdot \langle 1, 4, -5 \rangle$

$$\boxed{\therefore B'' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle, \left\langle \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}} \right\rangle, \left\langle \frac{1}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{-5}{\sqrt{42}} \right\rangle\right\}}$$

Orthogonal Basis

2) Inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\,q(x)\,dx$$

2a) $B = \{1, x, x^2\}$

Orthogonal Basis

$u_1 = 1$
$u_2 = x$
$u_3 = x^2$

- $v_1 = u_1 = 1$
- $v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$\langle v_1, v_1\rangle = \int_{-1}^{1} 1 \cdot 1\,dx = \int_{-1}^{1} dx = x\Big|_{-1}^{1} = 2$

$\langle u_2, v_1\rangle = \int_{-1}^{1} x \cdot 1\,dx = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0$

$v_2 = x - \left(\frac{0}{2}\right) \cdot 1 = x$

- $v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$\langle u_3, v_1\rangle = \int_{-1}^{1} x^2 \cdot 1\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$

$\langle u_3, v_2\rangle = \int_{-1}^{1} x^2 \cdot x\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

$\langle v_2, v_2\rangle = \int_{-1}^{1} x^2\,dx = x^3/3\Big|_{-1}^{1} = 2/3$

$v_3 = x^2 - \frac{1}{2} \cdot \frac{2}{3} \cdot 1 - 0 = x^2 - \frac{1}{3}$

$$\boxed{B' = \left\{1, x, \left(x^2 - \frac{1}{3}\right)\right\}}$$

Orthonormal Basis

$$\boxed{\begin{aligned} B'' &= \{w_1, w_2, w_3\} = \left\{\frac{1}{\|v_1\|} v_1, \frac{1}{\|v_2\|} v_2, \frac{1}{\|v_3\|} v_3\right\} \\ &= \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2/3}} x, \frac{1}{\sqrt{28/45}} \left(x^2 - \frac{1}{3}\right)\right\} \end{aligned}}$$

$\langle v_3, v_3\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right) dx$

$= \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{3}\,dx$

$= \frac{x^5}{5}\Big|_{-1}^{1} - \frac{2}{3}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{1}{3}[x]_{-1}^{1}$

$= \frac{1}{5}(1 - (-1)) - \frac{2}{3}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) + \frac{1}{3}($

$= 2/5 - 2/9\,(2) + 2/3$

$= 2/5 - 4/9 + 2/3$

$= 18/45 - 20/45 + \frac{30}{45}$

$= 28/45$

2b) $B = \{x^2-x,\ x^2+1,\ 1-x^2\}$

Orthogonal Basis

$u_1 = x^2 - x$

$u_2 = x^2 + 1$

$u_3 = 1 - x^2$

$\therefore$ • $v_1 = u_1 = x^2 - x$

• $v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$

$\langle v_1, v_1 \rangle = \int_{-1}^{1} v_1 \cdot v_1 \, dx = \int_{-1}^{1} (x^2-x)^2 \, dx = \int_{-1}^{1} x^4 - 2x^3 + x^2 \, dx$

$= \int_{-1}^{1} x^4 \, dx - 2\int_{-1}^{1} x^3 \, dx + \int_{-1}^{1} x^2 \, dx$

$= \frac{x^5}{5}\Big|_{-1}^{1} - 2\,\frac{x^4}{4}\Big|_{-1}^{1} + \frac{x^3}{3}\Big|_{-1}^{1}$

$= \frac{1}{5}(1-(-1)) - 2(1-1) + \frac{1}{3}(1-(-1)) = \frac{2}{5} + \frac{2}{3} = \frac{6}{15} + \frac{10}{15} = \frac{16}{15}$

$\langle u_2, v_1 \rangle = \int_{-1}^{1} (x^2+1) \cdot (x^2-x) \, dx = \int_{-1}^{1} x^4 - x^3 + x^2 - x \, dx$

$= \frac{1}{5}\left[x^5\right]_{-1}^{1} - \frac{1}{4}\left[x^4\right]_{-1}^{1} + \frac{1}{3}\left[x^3\right]_{-1}^{1} - \frac{1}{2}\left[x^2\right]_{-1}^{1}$

$= \frac{1}{5}(2) - \frac{1}{4}(0) + \frac{1}{3}(2) - \frac{1}{2}(0) = \frac{16}{15}$

$\therefore v_2 = x^2 + 1 - \left(\frac{16/15}{16/15}\right) \cdot x^2 - x = x^2 + 1 - x^2 - x = 1 - x$

• $v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$\langle u_3, v_1 \rangle = \int_{-1}^{1} u_3 \cdot v_1 \, dx = \int_{-1}^{1} (1-x^2) \cdot (x^2-x) \, dx = \int_{-1}^{1} x^2 - x - x^4 + x^3 \, dx$

$= \left[\frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^5}{5}\right]_{-1}^{1} + \left[\frac{x^4}{4}\right]_{-1}^{1}$

$= 2/3 - 2/5 = \frac{10}{15} - \frac{6}{15} = 4/15$

$\langle u_3, v_2 \rangle = \int_{-1}^{1} u_3 \cdot v_2 \, dx = \int_{-1}^{1} (1-x^2) \cdot (1-x) \, dx = \int_{-1}^{1} 1 - x - x^2 + x^3 \, dx$

$= x\Big|_{-1}^{1} - \frac{1}{2}\left[x^2\right]_{-1}^{1} - \frac{1}{3}\left[x^3\right]_{-1}^{1} + \frac{1}{4}\left[x^4\right]_{-1}^{1}$

$= 2 - \frac{1}{2}(1-1) - \frac{1}{3}(1-(-1)) + \frac{1}{4}(1-1)$

$= 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = 4/3$

$2\rangle = \int_{-1}^{1} (1-x)^2\,dx = \int_{-1}^{1} 1-2x+x^2\,dx$

$= [x]_{-1}^{1} - [x^2]_{-1}^{1} + \frac{1}{3}[x^3]_{-1}^{1}$

$= 2 + 2/3 = 8/3$

$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right)v_1 - \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right)v_2$

$= (1-x^2) - \left(\frac{4/15}{16/15}\right)(x^2-x) - \left(\frac{4/3}{8/3}\right)(1-x)$

$= (1-x^2) - \left(\frac{4}{\cancel{15}} \cdot \frac{\cancel{15}}{16}\right)(x^2-x) - \left(\frac{4}{\cancel{3}}\right)\left(\frac{\cancel{3}}{8}\right)(1-x)$

$= (1-x^2) - \left(\frac{4}{16}\right)(x^2-x) - \left(\frac{4}{8}\right)(1-x)$

$= (1-x^2) - \frac{1}{4}(x^2-x) - \frac{1}{2}(1-x)$

$\boxed{B' = \left\{x^2-x,\ 1-x,\ (1-x^2) - \frac{1}{4}(x^2-x) - \frac{1}{2}(1-x)\right\}}$ ← Final Orthogonal Basis

Orthonormal Basis

$B'' = \{w_1, w_2, w_3\} = \left\{\frac{1}{\|v_1\|}v_1, \frac{1}{\|v_2\|}v_2, \frac{1}{\|v_3\|}v_3\right\}$

$\langle v_3, v_3\rangle = \int_{-1}^{1}\left((1-x^2) - \frac{1}{4}(x^2-x) - \frac{1}{2}(1-x)\right)^2 dx$

$= \int_{-1}^{1} \frac{(-5x^2+3x+2)^2}{16}\,dx$

$= \int_{-1}^{1} \frac{(-5x^2+3x+2)(-5x^2+3x+2)}{16}\,dx$

$= \int_{-1}^{1} \frac{25x^4 - 15x^3 - 10x^2 - 15x^3 + 9x^2 + 6x - 10x^2 + 6x + 4}{16}\,dx$

$= \frac{1}{16}\int_{-1}^{1} 25x^4 - 30x^3 - 11x^2 + 12x + 4$

$= \frac{1}{16}\left(5[x^5]_{-1}^{1} - \frac{30}{4}\overset{0}{\cancel{[x^4]_{-1}^{1}}} - \frac{11}{3}[x^3]_{-1}^{1} + 6\cancel{[x^2]_{-1}^{1}} + 4[x]_{-1}^{1}\right)$

$= \frac{1}{16}\left(5(2) - \frac{11}{3}(2) + 4\right) = \frac{10}{16} - \frac{1}{16}\cdot\frac{22}{3} + \frac{4}{16} = \frac{10}{16} - \frac{22}{48} + \frac{4}{16} = \frac{30}{48} - \frac{22}{48} + \frac{12}{48}$

$= \frac{20}{48}$

Orthonormal

$\boxed{B'' = \left\{\frac{1}{\sqrt{16/15}}x^2-x,\ \frac{1}{\sqrt{8/3}}(1-x),\ \frac{1}{\sqrt{20/48}}\left((1-x^2) - \frac{1}{4}(x^2-x) - \frac{1}{2}(1-x)\right)\right\}}$