MATH 412 - Project 1

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Computer Graphics - Project I

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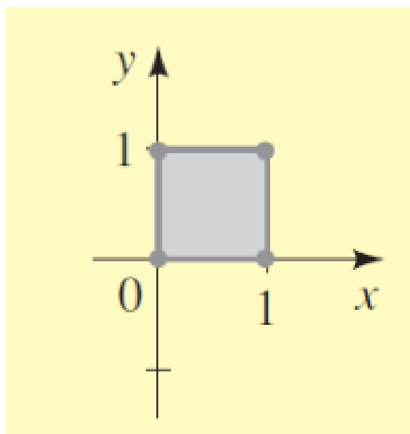
Projects 1 - Computer Graphics I

Problem 1

Apply each of the three transformations given in Table 1 to these vertices and sketch the result, to verify that each transformation has the indicated effect. Use $c = 2$ in the expansion matrix and in the shear matrix.

Reflection Matrix

The points of the grey square as depicted below:



Can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The reflection matrix is:

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

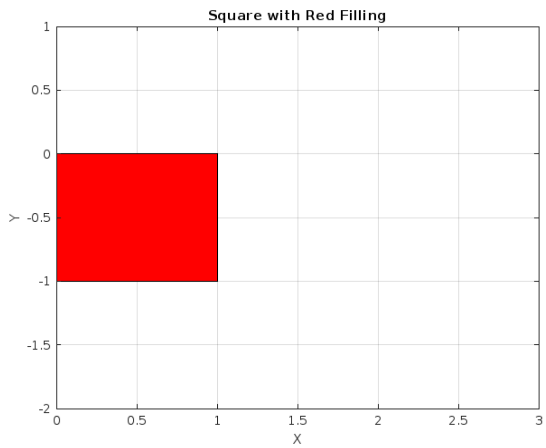
Acting with matrix T_1 on matrix P yields:

$$P' = T1 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

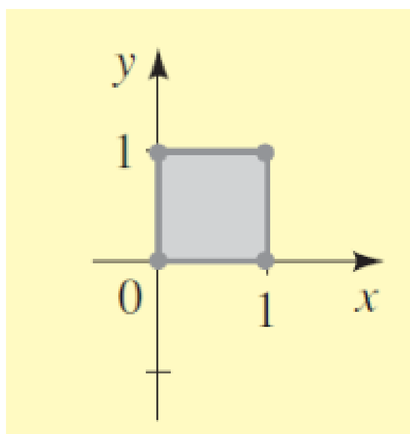
$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Which corresponds to a reflection across the x-axis. This is depicted in graph:



Expansion/Contraction Matrix

The points of the grey square as depicted below:



Can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The expansion matrix is:

$$T_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

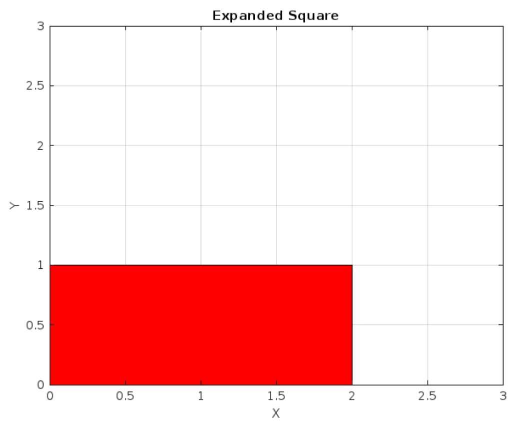
Acting with matrix T_2 on matrix P yields:

$$P' = T2 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

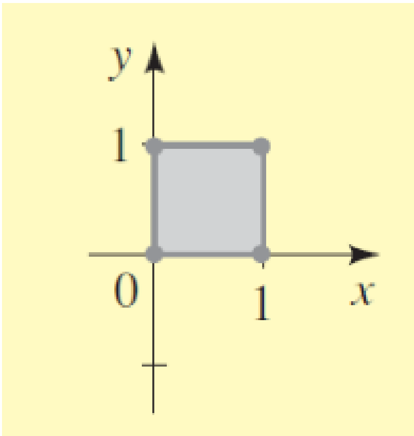
$$P' = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Which corresponds to increasing the horizontal dimension of the square by a factor of 2. This is depicted in graph:



Shear Matrix

The points of the grey square as depicted below:



Can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The shear matrix is:

$$T_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

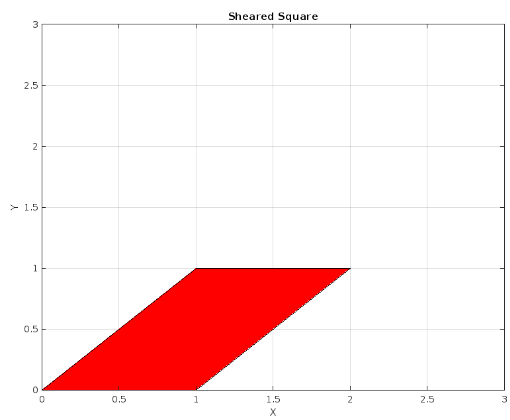
Acting with matrix T_3 on matrix P yields:

$$P' = T3 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Which corresponds to a shear in the x-direction by a factor of 1. This is depicted in the graph below:



Problem 2

Verify that multiplication by the given matrix has the indicated effect when applied to the gray square in the table. Use $c = 3$ in the expansion matrix and $c = 1$ in the shear matrix.

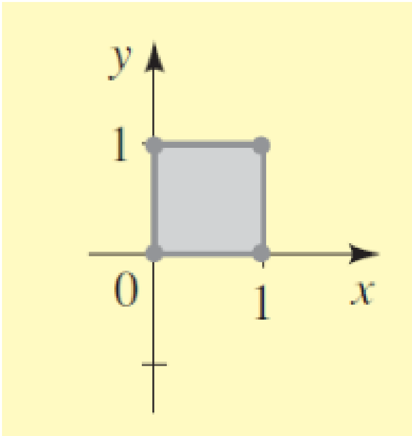
Reflection Matrix

The points of the grey square as depicted below:

Can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The reflection matrix is:



$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

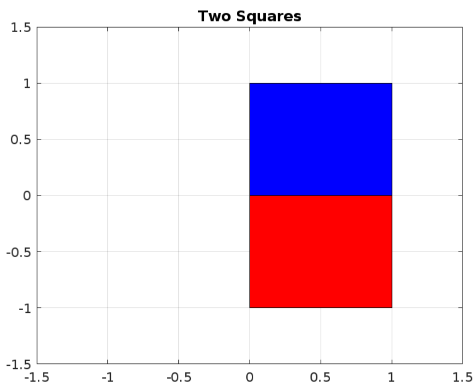
Acting with matrix T_1 on matrix P yields:

$$P' = T1 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

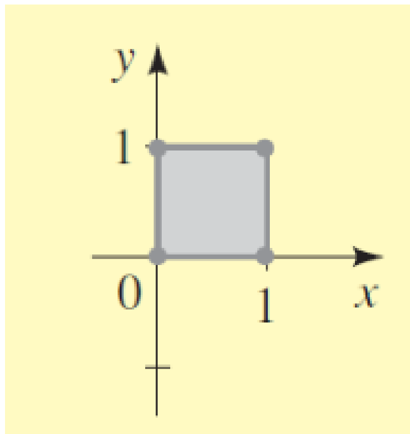
$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Which corresponds to a reflection across the y-axis. This is depicted in graph as the red square (the blue square is the original square):



Expansion/Contraction Matrix

The points of the grey square as depicted below:



Can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The expansion matrix is:

$$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

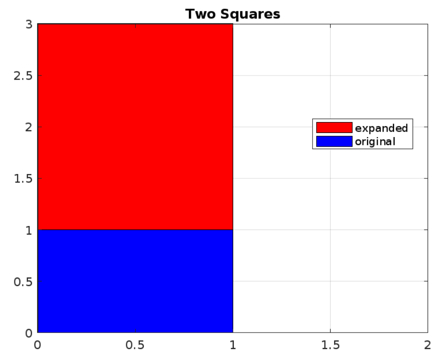
Acting with matrix T_2 on matrix P yields:

$$P' = T1 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

Which corresponds to increasing the height of the square (now rectangle) by a factor of 3. This is represented below:



Shear Matrix

The points of the grey square can be expressed as:

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The shear matrix is:

$$T_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Acting with matrix T_3 on matrix P yields:

$$P' = T1 \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

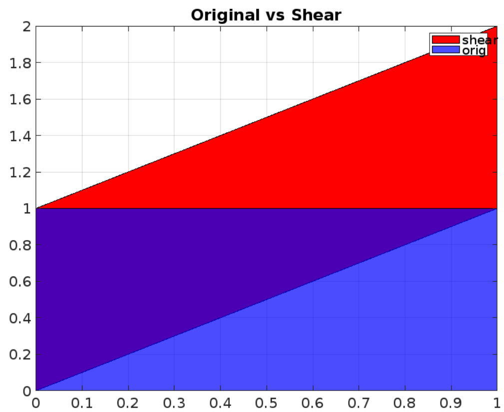
$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Which corresponds to a shear in the y-direction by 1. This is represented below:

Problem 3

Let

$$T = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$$



Part a

T does an extreme shear on the grey square, P, - shearing by a factor of 0.5.

$$P' = T \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 2.5 & 1.5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Part b

Given a matrix T, the formula for finding its inverse T^{-1} is:

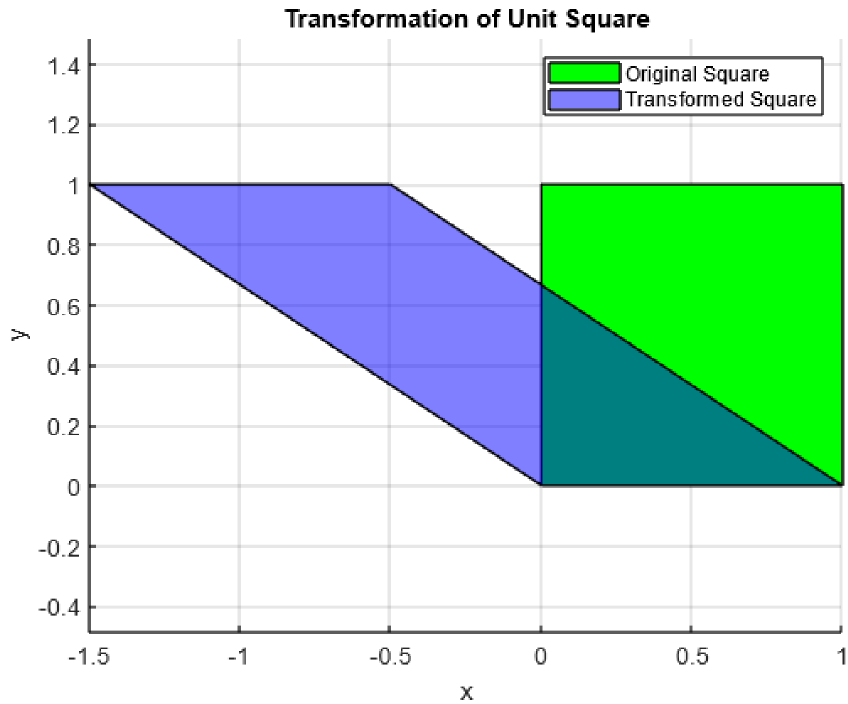
$$T^{-1} = \frac{1}{\det(T)} \cdot \text{adj}(T)$$

where $\det(T)$ is the determinant of matrix T, and $\text{adj}(T)$ is the adjugate (or adjoint) of T. The $\det(T)$ for a 2 x 2 matrix is $(ad - bc)$. For the matrix T that is $(1 * 1 - (1.5 * 0)) = 1$. The adjoint, $\text{adj}(T)$, is:

$$adj(T) = \begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix}$$

Therefore:

$$T^{-1} = \begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix}$$



Part c

T^{-1} does an extreme shear on the grey square, but shearing to the negative direction for x.

$$P' = T^{-1} \cdot P = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & -0.5 & -1.5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Part d

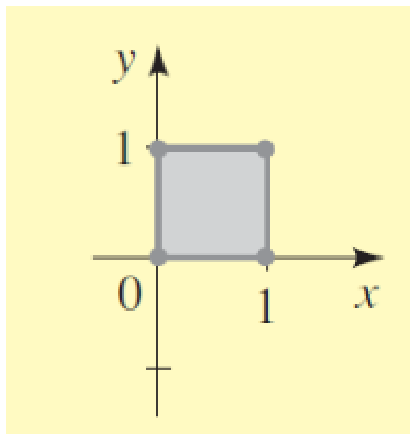
Multiplying the matrix T with the matrix T^{-1} should yield the identity matrix:

$$T = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix}$$

$$T^{-1}T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $T^{-1}(T(S))$ should reproduce the same unit square we started with. That is:



Which expressed as a matrix:

$$S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Problem 4

Part a

Let T be an expansion matrix such that:

Let

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Acting with this expansion matrix on the unit square, S, yields:

$$P' = T{\cdot}S = \begin{bmatrix} t_{11}s_{11} + t_{12}s_{21} & t_{11}s_{12} + t_{12}s_{22} & t_{11}s_{13} + t_{12}s_{23} & t_{11}s_{14} + t_{12}s_{24} \\ t_{21}s_{11} + t_{22}s_{21} & t_{21}s_{12} + t_{22}s_{22} & t_{21}s_{13} + t_{22}s_{23} & t_{21}s_{14} + t_{22}s_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This represents an expansion along the horizontal axis for the square by a factor of 3.

Part b

Let S be an expansion matrix such that:

Let

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Acting with this expansion matrix on the unit square, G, yields:

$$P' = S{\cdot}G = \begin{bmatrix} s_{11}g_{11} + s_{12}g_{21} & s_{11}g_{12} + s_{12}g_{22} & s_{11}g_{13} + s_{12}g_{23} & s_{11}g_{14} + s_{12}g_{24} \\ s_{21}g_{11} + s_{22}g_{21} & s_{21}g_{12} + s_{22}g_{22} & s_{21}g_{13} + s_{22}g_{23} & s_{21}g_{14} + s_{22}g_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

This represents an expansion along the vertical axis for the square by a factor of 2.

Part c

We must act on the unit square (represented by matrix G) with matrix S, then matrix T.

Acting with S on G we have proven yields:

$$P = S{\cdot}G = \begin{bmatrix} s_{11}g_{11} + s_{12}g_{21} & s_{11}g_{12} + s_{12}g_{22} & s_{11}g_{13} + s_{12}g_{23} & s_{11}g_{14} + s_{12}g_{24} \\ s_{21}g_{11} + s_{22}g_{21} & s_{21}g_{12} + s_{22}g_{22} & s_{21}g_{13} + s_{22}g_{23} & s_{21}g_{14} + s_{22}g_{24} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Next, if we act with T on P:

$$P' = T{\cdot}P = \begin{bmatrix} t_{11}p_{11} + t_{12}p_{21} & t_{11}p_{12} + t_{12}p_{22} & t_{11}p_{13} + t_{12}p_{23} & t_{11}p_{14} + t_{12}p_{24} \\ t_{21}p_{11} + t_{22}p_{21} & t_{21}p_{12} + t_{22}p_{22} & t_{21}p_{13} + t_{22}p_{23} & t_{21}p_{14} + t_{22}p_{24} \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

The final result is that the unit square has been expanded by three along the x-axis and expanded by two along the y-axis as depicted.

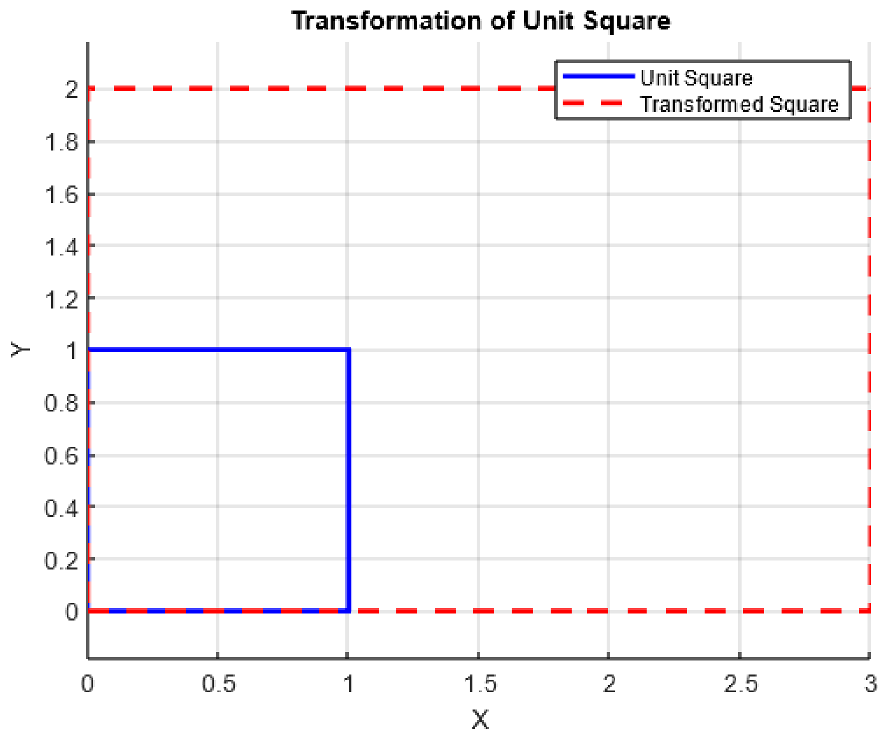
Part d

Multiplying matrix T and S to produce matrix W yields:

$$W = T{\cdot}S = \begin{bmatrix} t_{11}s_{11} + t_{12}s_{21} & t_{11}s_{12} + t_{12}s_{22} & t_{11}s_{13} + t_{12}s_{23} & t_{11}s_{14} + t_{12}s_{24} \\ t_{21}s_{11} + t_{22}s_{21} & t_{21}s_{12} + t_{22}s_{22} & t_{21}s_{13} + t_{22}s_{23} & t_{21}s_{14} + t_{22}s_{24} \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Depicted below:



Part e

Operating with matrix W as calculated above on the unit square G yields:

$$W' = W{\cdot}G = \begin{bmatrix} w_{11}g_{11} + w_{12}g_{21} & w_{11}g_{12} + w_{12}g_{22} & w_{11}g_{13} + w_{12}g_{23} & w_{11}g_{14} + w_{12}g_{24} \\ w_{21}g_{11} + w_{22}g_{21} & w_{21}g_{12} + w_{22}g_{22} & w_{21}g_{13} + w_{22}g_{23} & w_{21}g_{14} + w_{22}g_{24} \end{bmatrix}$$

$$W' = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Since G was a unit square this will have the same form as the matrix in problem 4d.

Problem 5

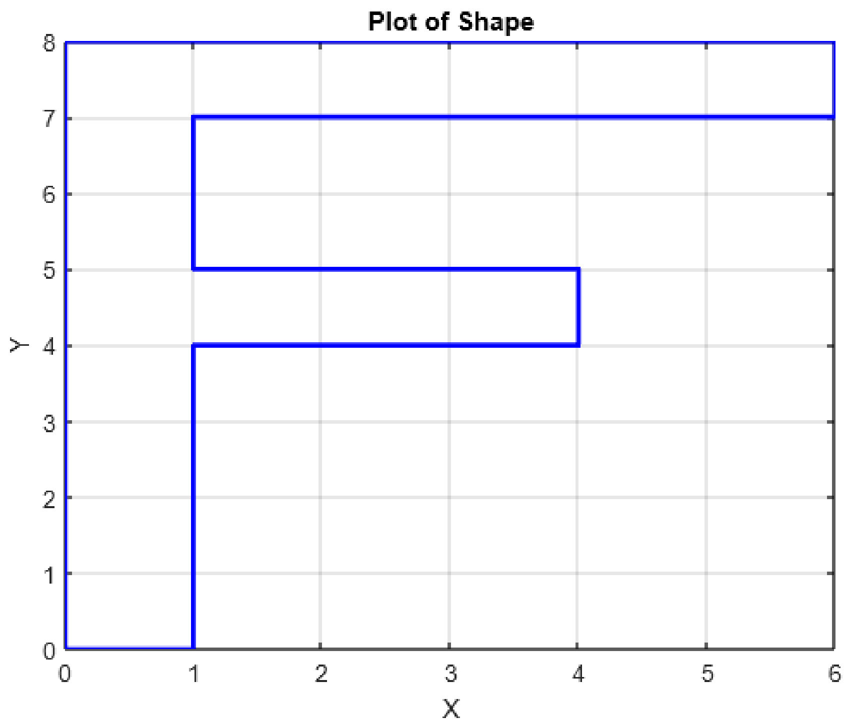
The figure shows three outline versions of the letter F. The second one is obtained from the first by shrinking horizontally by a factor of 0.75, and the third is obtained from the first by shearing horizontally by a factor of 0.25

Part a

The data matrix for the letter F can be expressed using the matrix D:

$$D = \begin{bmatrix} 0 & 1 & 1 & 4 & 4 & 1 & 1 & 6 & 6 & 0 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$$

Depicted below:



Part b

The transformed matrix is F'. This matrix is the result of horizontally shrinking the letter F by a factor of 0.75. This corresponds to the transformed matrix:

$$F' = \begin{bmatrix} 0 & 0.75 & 0.75 & 3 & 3 & 0.75 & 0.75 & 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$$

This matrix is obtained by multiplying D by a matrix T. This matrix is:

$$T = \begin{bmatrix} 0.75 & 0 \\ 0 & 1 \end{bmatrix}$$

Part c

The transformed matrix is F'. This matrix is the result by shearing the letter F by a factor of 0.25. This shear matrix is:

Shear matrix with a factor of 0.25:

$$S = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$$

Resultant matrix after shear transformation:

$$DS = \begin{bmatrix} 0 & 1 & 2 & 5 & 5.25 & 2.25 & 2.75 & 7.75 & 8 & 2 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$$

Problem 6

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & 4 & 0 \end{bmatrix}$$

Part a

Part b

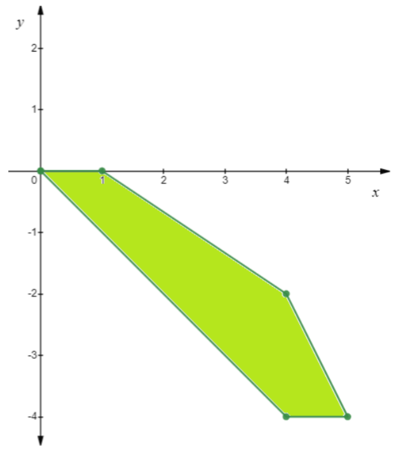
T is the matrix:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Acting with this matrix on matrix D produces:

$$TD = \begin{bmatrix} 0 & 1 & 4 & 5 & 4 & 0 \\ 0 & 0 & -2 & -4 & -4 & 0 \end{bmatrix}$$

This produces a reflection across the x-axis in addition to a shear by 1 towards the positive x-direction as depicted below:



Part c

To express matrix T as the product of a shear matrix and a reflection matrix, we recall that a reflection matrix across the x-axis can be expressed as:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

While a shear matrix can be expressed for this problem as:

$$S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Multiplying these two matrices yields the matrix T:

$$T = SR = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Computer Graphics - Project II

Erica Cain, Takiya Eastmond, Makayla Greer, Twymun Safford

February 20th, 2024

Projects 1 - Computer Graphics II

Problem 1

Use a rotation matrix to find the new coordinates of the given point when it is rotated through the given angle.

Part a

The point P can be represented in vector form as:

$$P = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{\pi}{2}$, this rotation matrix becomes:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Operating with this rotation matrix on the vector yields:

$$P' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Which yields:

$$P' = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Part b

The point P can be represented in vector form as:

$$P = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{\pi}{3}$, this rotation matrix becomes:

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Operating with this rotation matrix on the vector yields:

$$P' = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Which yields:

$$P' = \begin{bmatrix} -1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \sqrt{3} \end{bmatrix}$$

Part c

The point P can be represented in vector form as:

$$P = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{5\pi}{4}$, this rotation matrix becomes:

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Operating with this rotation matrix on the vector yields:

$$P' = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Which yields:

$$P' = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

Part d

The point P can be represented in vector form as:

$$P = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When the angle is negative, this matrix becomes:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

When θ = -$\frac{\pi}{3}$, this rotation matrix becomes:

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Operating with this rotation matrix on the vector yields:

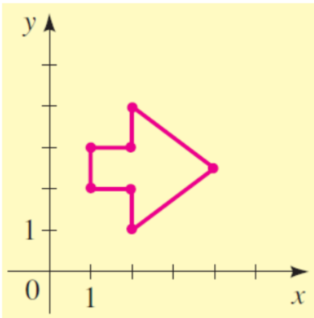
$$P' = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Which yields:

$$P' = \begin{bmatrix} \frac{7+3\sqrt{3}}{2} \\ \frac{3-7\sqrt{3}}{2} \end{bmatrix}$$

Problem 2

The original shape (arrow) is depicted below:



Part a

The data matrix for this arrow is:

$$D = \begin{bmatrix} 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2.5 & 4 & 3 & 3 & 2 \end{bmatrix}$$

The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{2\pi}{3}$ like in this problem, this rotation matrix becomes:

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

If we act with this rotation matrix on the arrow (based on points), this yields:

$$D' = RD = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \left[\begin{array}{cccccccc} 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2.5 & 4 & 3 & 3 & 2 \end{array} \right]$$

Which yields the final shape of our matrix (and our arrow's points after rotation) as:

$$D' = \left[\begin{array}{cccccccc} -2.23 & -2.73 & -1.87 & -3.67 & -4.46 & -3.69 & -3.10 & -2.23 \\ -0.13 & 0.73 & 1.23 & 1.35 & -0.27 & 0.23 & -0.63 & -0.13 \end{array} \right]$$

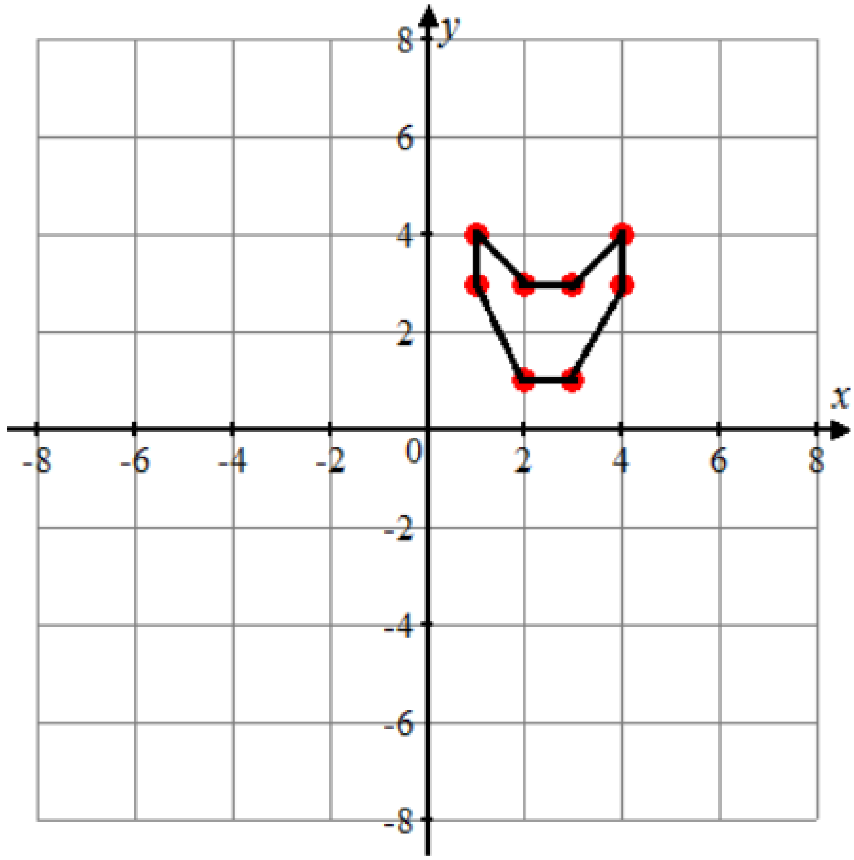
This now looks like:

Problem 3

Sketch the image represented by the data matrix D:

$$D = \left[\begin{array}{ccccccccc} 2 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 & 4 & 4 & 3 & 3 & 1 \end{array} \right]$$

Sketched image is:



The 2D rotation matrix for an angle θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{\pi}{4}$, this rotation matrix becomes:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

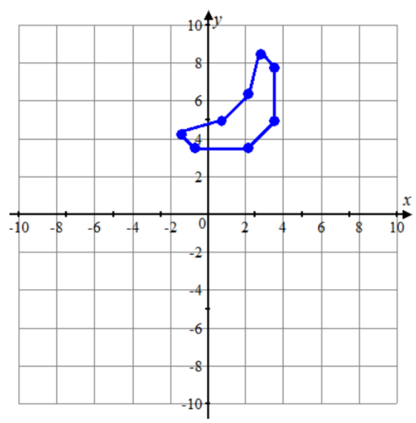
The transformation matrix that is an expansion by a factor of 2 in the x-direction is:

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix RT which results from multiplying matrix T by matrix R yields:

$$RT = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \times 2 + (-\frac{\sqrt{2}}{2}) \times 0 & \frac{\sqrt{2}}{2} \times 0 + (-\frac{\sqrt{2}}{2}) \times 1 \\ \frac{\sqrt{2}}{2} \times 2 + \frac{\sqrt{2}}{2} \times 0 & \frac{\sqrt{2}}{2} \times 0 + \frac{\sqrt{2}}{2} \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ \sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Sketch of RT:



While matrix TR which results from multiplying matrix R by matrix T yields:

$$TR = \begin{pmatrix} 2 \times \frac{\sqrt{2}}{2} + 0 \times \frac{\sqrt{2}}{2} & 2 \times (-\frac{\sqrt{2}}{2}) + 0 \times \frac{\sqrt{2}}{2} \\ 0 \times \frac{\sqrt{2}}{2} + 1 \times \frac{\sqrt{2}}{2} & 0 \times (-\frac{\sqrt{2}}{2}) + 1 \times \frac{\sqrt{2}}{2} \end{pmatrix}$$

This simplifies to

$$TR = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

The matrix product TRD will be:

$$TRD = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 & 4 & 4 & 3 & 3 & 1 \end{pmatrix}$$

$$TRD = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 & 4 & 4 & 3 & 3 & 1 \end{pmatrix}$$

While the matrix product RTD yields:

$$RTD = \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ \sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{pmatrix} 2 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 & 4 & 4 & 3 & 3 & 1 \end{pmatrix}$$

Problem 4

Let R be the rotation matrix for the angle θ. We want to show that R^{-1} is the rotation matrix for the angle $-\theta$.

The inverse of a rotation matrix is the transpose of the matrix, thus:

$$R^{-1} = R^T$$

To show that R^{-1} is the rotation matrix for the angle $-\theta$, we need to show that $(R^{-1})^T$ represents a rotation of $-\theta$.

$$(R^{-1})^T = (R^T)^T = R$$

Since R is the rotation matrix for the angle θ, R represents a rotation of θ. Therefore, $(R^{-1})^T = R$ represents a rotation of θ, which is the same as the original matrix R.

Thus, R^{-1} is the rotation matrix for the angle $-\theta$.

This can be proven explicitly. Using trig. identities, we know that $cos(-\theta)$ = $cos(\theta)$ and that $sin(-\theta)$ = -$sin(\theta)$ as depicted below:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Problem 5

$$(x, y) \leftrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

,

$$T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5a

Show that matrix T translates the point (x,y) to the point (x+h, y+h) by verifying the following matrix multiplication.

$$P' = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The result is:

$$\begin{bmatrix} 1 \cdot x + 0 \cdot y + h \cdot 1 \\ 0 \cdot x + 1 \cdot y + k \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}$$

Problem 5b

The inverse of the matrix T is given by:

$$T^{-1} = \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

By multiplying this matrix on the original point:

$$\begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y + (-h) \cdot 1 \\ 0 \cdot x + 1 \cdot y + (-k) \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} x - h \\ y - k \\ 1 \end{bmatrix}$$

We see that this will translate the point from (x,y) to (x-h, y-k).

Problem 5c

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection in x-axis produces:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 1 \end{bmatrix}$$

Expansion (or contraction) in x-direction produces:

$$\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} cx \\ y \\ 1 \end{bmatrix}$$

Shear in x-direction produces:

$$\begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + cy \\ y \\ 1 \end{bmatrix}$$

Rotation about the origin by angle θ:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \cdot \cos(\theta) - y \cdot \sin(\theta) \\ x \cdot \sin(\theta) + y \cdot \cos(\theta) \\ 1 \end{bmatrix}$$

Problem 5d

Sketch the image represented (in homogeneous coordinates) by this data matrix:

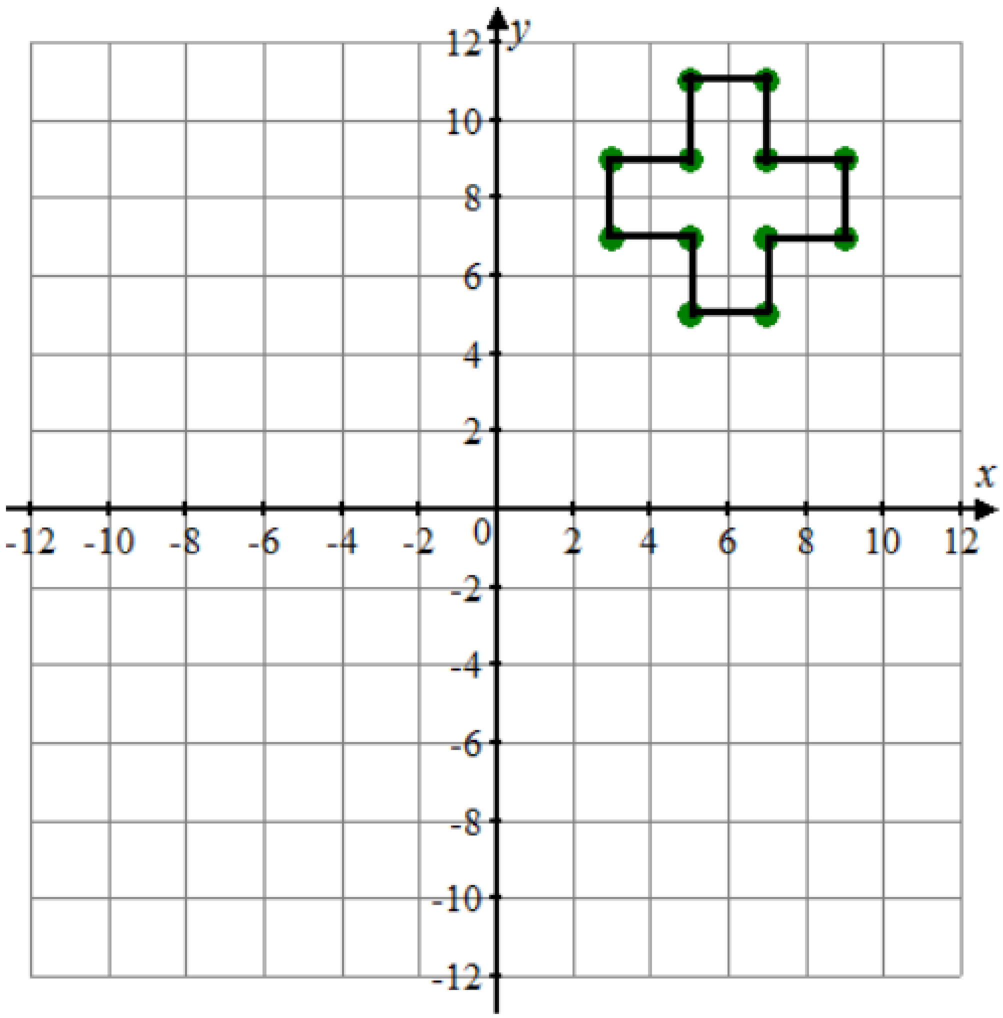
$$D = \begin{bmatrix} 3 & 5 & 5 & 7 & 7 & 9 & 9 & 7 & 7 & 5 & 5 & 3 & 3 \\ 7 & 7 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 & 9 & 9 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Find a matrix T that translates the image by $(-6, -8)$ and a matrix R that rotates the image by 45°. Sketch the images represented by the data matrices TD, RTD, and $T^{-1}RTD$. Describe how an image is changed when its data matrix is multiplied by T, by RT, and by $T^{-1}RT$

.

The data matrix for the letter F can be expressed using the matrix D:

$$D = \begin{bmatrix} 3 & 5 & 5 & 7 & 7 & 9 & 9 & 7 & 7 & 5 & 5 & 3 & 3 \\ 7 & 7 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 & 9 & 9 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



The matrix that translates the image (based on the data matrix) 6 units left and 8 units down is:

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the data matrix by the translation matrix produces:

$$TD = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 5 & 7 & 7 & 9 & 9 & 7 & 7 & 5 & 5 & 3 & 3 \\ 7 & 7 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 & 9 & 9 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Which produces:

$$\begin{bmatrix} -3 & -1 & -1 & 1 & 1 & 3 & 3 & 1 & 1 & -1 & -1 & -3 & -3 \\ -1 & -1 & -3 & -3 & -1 & -1 & 1 & 1 & 3 & 3 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The rotation matrix is:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And for 45 degrees this matrix is:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product of this rotation matrix with the prior matrix TD yields:

$$\begin{bmatrix} -1.41 & 0 & 1.41 & 2.83 & 1.41 & 2.83 & 1.41 & 0 & -1.41 & -2.83 & -1.41 & -2.83 & -1.41 \\ -2.83 & -1.41 & -2.83 & -2.83 & 0 & 1.41 & 2\sqrt{2} & 1.41 & 2.83 & 1.41 & 0 & -1.41 & -2.83 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

When multiplied by matrix T, an image will be translated 6 points left and 8 points down.

When multiplied by RT, an image will be translated 6 points left and 8 points down and rotated 45 degrees.

When multiplied by $T^{-1}RT$, an image will be translated 6 points left and 8 points down and rotated 45 degrees, but will then be moved back to the original place it started at.

Image Manipulations

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Original Image

Code for the image manipulations are in the project subfolder "Matlab codes". Corresponding matrices for manipulation are saved in the subfolder "Matrices".

Image Manipulation - Reflection of Image (x- and y-axis)

y-axis

Y-Reflected Image

x-axis

Image Manipulation - Expansion

Image Manipulation - Shear of Image (Factor of 0.75)

Image Manipulation - Rotation of Image (67.36 degrees counterclockwise)