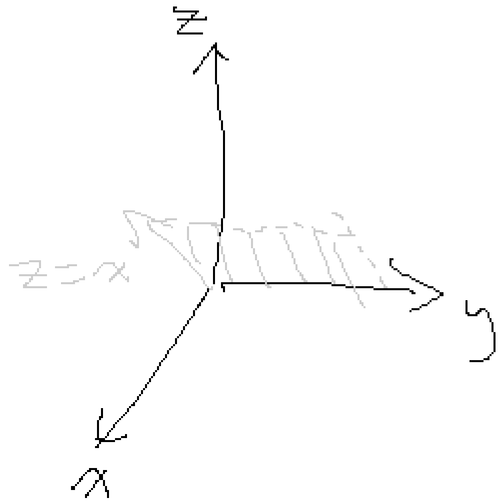
# Calc in 3d Notes

saffron_

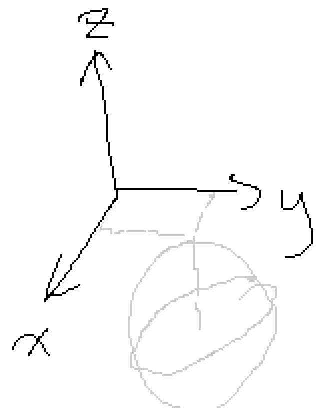
## Contents

# Chapter 2: Vectors in Space

Graphing

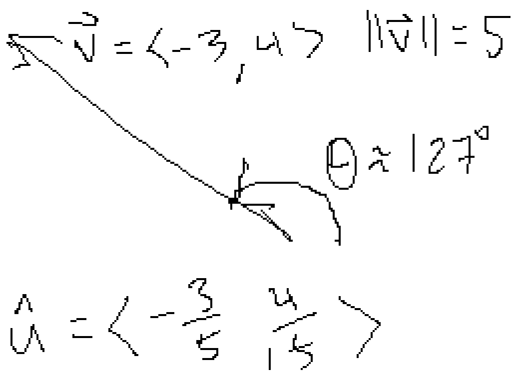



*convention*




*sphere:* $(x-1)^2 + (y-2)^2 + (z+3)^2 = 9$

The Vector



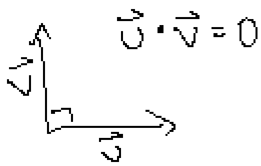

- a quantity with a magnitude and direction
- unit vector $\hat{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Vector Operations

- Addition
  - $\vec{a} = \langle 1, 2\rangle$, $\vec{b} = \langle 3, 4\rangle$
  - $\vec{a} + \vec{b} = \langle 4, 6\rangle$
- Scalar Multiplication
  - $\vec{v} = \langle 1, 3\rangle$, $c = 2$
  - $c\vec{v} = \langle 2, 6\rangle$

simple inverses for subtraction and scalar division exist.

- Dot Product (also 2d!)
  - geom.:



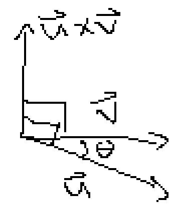

    $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$
  - alg.:
    $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$
    $\vec{u} \cdot \vec{v} = \sum u_i v_i$
  - $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
  - two vectors are orthogonal aka $\perp$ iff their dot product is 0
  - $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}$ wtf is equation 2.5 "unique over this range" on abt
  - work $= \vec{F} \cdot \vec{D}$
  - comp (scalar projection)

$$\begin{aligned} \operatorname{comp}_{\vec{u}} \vec{v} &= \|\vec{v}\| \cos\theta \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \end{aligned}$$

  - proj (vector projection)

$$\begin{aligned} \operatorname{proj}_{\vec{u}} \vec{v} &= \operatorname{comp}_{\vec{u}} \vec{v} \cdot \frac{\vec{u}}{\|\vec{u}\|} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \end{aligned}$$

- Cross Product (3d)
  - geom.:




    $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$
  - alg.:
  - $\vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - result is $\perp$ to both input vectors, direction by right hand rule
  - triple scalar product:
    $\vec{u} \cdot (\vec{v} \times \vec{w})$
    $= (\vec{u} \times \vec{v}) \cdot \vec{w}$
    $= \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
  - if $\vec{u}$ and $\vec{v}$ are the sides of a parallelogram, then its area is $\|\vec{u} \times \vec{v}\|$
  - if a parallelepiped has edges $\vec{u}, \vec{v}, \vec{w}$, its volume is the absolute value of its triple scalar product
  - torque $= \vec{\tau} = \vec{r} \times \vec{F}$

Lines

(1) Vector Equation Form

$\vec{r} = \vec{r_0} + t\vec{v}$
eg $\langle x, y, z\rangle = \langle 1, 2, -5\rangle + t\,\langle 3, -4, -1\rangle$

(2) Parametric Equation Form

$$x = 1 + 3t$$
$$y = 2 - 4t$$
$$z = -5 - t$$

(3) Symmetric Equation Form

solve for $t$

$$\frac{x-1}{3} = \frac{y-2}{-4} = \frac{z+5}{-1}$$

(4) Edge Case: 0-component

Let a line be defined by the point and vector $(1, -2, 6)$ and $\langle 3, 7, 0\rangle$. We say the line is:

$$\frac{x-1}{3} = \frac{y+2}{7}, z = 6$$

- point to line distance: use paralleogram area trick

Planes

(1) Vector Equation Form

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$
eg $(\langle x, y, z\rangle - \langle 1, 0, 1\rangle) \cdot \langle 1, 2, -3\rangle = 0$

(2) Scalar Equation, "General" Form

$ax + by + cz + d = 0$

- point to plane distance: use $\mathrm{comp}_{\vec{u}}\,\vec{v}$ trick
- angle between planes: same as angle between their normal vectors

Quadric Surfaces

- cylinder: 3d shape consisting of all parallel lines (eg $y = 3x^2$)
- see `quadric_surfaces.pdf`

# Chapter 3: Vector-Valued Functions

Vector-Valued Function

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle, i < t < j$$

Unit Circle Parameterization
temp

Limits of VVFs

- pass them into the vec. pretty intuitive
- a VVF $\vec{r}(a)$ is cont. at $a$ iff $\lim\limits_{t\to a} \vec{r}(t) = \vec{r}(a)$ (and both are defined )

Calc with VVFs

- derivatives are intuitive. use the corresponding dot/cross/scalar in deriving $\vec{u} \cdot \vec{v}$
- unit tangent vector is the derivative's unit vector
- integrals are intuitive. consider constant vector $C$ instead of scalar constant

Consult exam 1 cheatsheet for notes on the rest of the chapter

# Chapter 4: Differentiation

Functions of Multiple Variables

- domain: analyze what values are invalid
- range: image of domain
- level planes/level surfaces/contour maps: setting the function to some constant and drawing out the resulting shape

Limits and Continuity

- limit rules are identical to 2d, including
- limits must be unique. ie, the $\delta$ disk around a point must only contain one value. (disprove limit by finding different values through different "paths" to the limit)
- as in 2d, $f(x,y)$ is continuous at $(a,b)$ iff

  $$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

  with both defined.
- sum, product, comp of cont functions: cont

Partial Derivatives

- slope of line in a direction, at a point
- Limit definition:

  $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

- four second order partials exist

  $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}$$
  $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{yx}$$
  $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{xy}$$
  $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}$$

- Clairaut's Thrm: if $f_{xy}$ and $f_{yx}$ are cont near a point, they are equal

Tangent Planes

- if all tangent lines to a point are in the same plane, call that the tangent plane
  - (not true if there is a point)
  - maybe theres a correlation somewhere with differentiability? (p393)
- the tangent plane to $z = f(x,y)$ at $(a,b)$ is

  $$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- find an approx at $(a,b)$ via the linear approximation plane $L(x,y) = \text{RHS}$ (above)
- a function is differentiable at $(a,b)$ iff

  $$f(x,y) = \text{RHS} + E(x,y)$$

  where the error term $E$ satisfies

  $$\lim_{(x,y)\to(a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

  alternatively, if $f$, $f_x$, and $f_y$ exist near $(a,b)$ and are cont. at $(a,b)$, then $f$ is differentiable there.
- let $z = f(x,y)$ with $(a,b)$ in the domain of $f$, and let $\Delta x$ and $\Delta y$ be chosen such that $(a+\Delta x, b+\Delta y)$ is also in the domain of $f$.

  then

  $$dx = \Delta x$$
  $$dy = \Delta y$$
  $$dz = f_x(a,b)dx + f_y(a,b)dy.$$

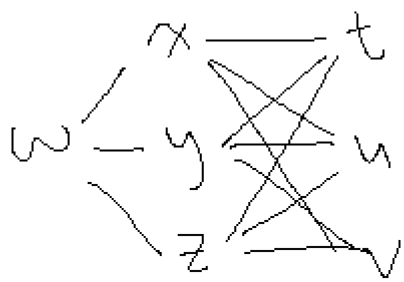
  $dx$ and $dy$ are differentials, $dz$ the "total differential," and we estimate error with it.

  notice the similarity to the tangent plane equation.

The Chain Rule

- consider

$$
\begin{aligned}
w &= f(x, y, z) \\
x &= x(t, u, v) \\
y &= y(t, u, v) \\
z &= z(t, u, v)
\end{aligned}
$$




then,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial w}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

others are left as an exercise to the reader.

Implicit Differentiation

- consider $x^2 + 3y^2 + 4y - 4 = 0$.

  to find $\frac{dy}{dx}$, we may implicitly differentiate this by taking $\frac{d}{dx}$ of both sides and solving.

  but we may also define

$$f(x, y) = x^2 + 3y^2 + 4y - 4, f(x, y) = 0.$$

  with this in mind, suppose $f(x, y) = 0$. then,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

  and if $f(x, y, z) = 0$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}, \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

  these can be derived from the chain rule.

Directional Derivatives and the Gradient

- the directional derivative of $f(x, y)$ in the direction $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ is

$$D_{\hat{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}$$

  alternatively, if the partials exist,

$$
\begin{aligned}
D_{\hat{u}} f(x, y) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\
&= \langle\, f_x(x, y), f_y(x, y) \,\rangle \cdot \hat{u} \\
&= \nabla f(x, y) \cdot \hat{u}.
\end{aligned}
$$

- $\nabla f(x, y)$ is called the gradient and points toward the greatest increase of a function. it's perpendicular to the graph's level curves (if the partials are cont. near the points)

- suppose $z = f(x, y)$ diffbl at $(a, b)$.
  - if $\nabla f(a, b) = \vec{0}$, then $D_{\hat{u}} f(a, b) = 0$ for any $\hat{u}$
  - if $\nabla f(a, b) \neq \vec{0}$, then $D_{\hat{u}} f(a, b)$ is max when $\hat{u}$ is in the same direction as $\nabla f(a, b)$.
    max of $D_{\hat{u}} f(a, b)$ is $\|\nabla f(a, b)\|$
  - if $\nabla f(a, b) \neq \vec{0}$, then $D_{\hat{u}} f(a, b)$ is min when $\hat{u}$ is in the opposite direction as $\nabla f(a, b)$.
    min of $D_{\hat{u}} f(a, b)$ is $-\|\nabla f(a, b)\|$

- yes, these work for multivar funcs.

Finding Maxima/Minima

- like 2d, critical points $(a, b)$ exist iff
  - $f_x(a, b) = f_y(a, b) = 0$
  - or the partials there don't exist
- local extrema are crit points
- Second Derivative Test (for 3d calc)

  let $z = f(x, y)$ where the first and second order partials are cont near a point $(a, b)$

  $$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

  - $D > 0$ and $f_{xx}(a, b) > 0$
    $\Rightarrow f$ has a local min at $(a, b)$
  - $D > 0$ and $f_{xx}(a, b) < 0$
    $\Rightarrow f$ has a local max at $(a, b)$
  - $D < 0$
    $\Rightarrow f$ has a saddle point at $(a, b)$
  - $D = 0$
    $\Rightarrow$ the test is inconclusive
- to find local extrema:
  1. find crit points. discard if a partial DNE
  2. find discriminant $D$ for each point
  3. apply 2nd derivative test
- Extreme Value Theorem

  a cont function on a closed and bounded set has an abs min and abs max in the set
- the abs max and abs min will be either at a critical point or on a boundry
- to analyze the boundry, consider parameterizing line segments/ellipses or using Lagrange multipliers

Lagrange Multipliers

- Theory: consider an objective function and a constraint function. when they are tangent, their gradients must be in the same direction. so, we say they are different by $\lambda$, the Lagrange multiplier.
- Let $f(x, y)$ and $g(x, y)$ have cont partials along $g(x, y) = 0$. if $f$ has a local extrema on $g(x, y) = 0$ at $(a, b)$ and $\nabla g(a, b) \neq 0$, then
  $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
- to use $\lambda$ (yes, $f$ and $g$ may be fn of 3 vars):
  1. find the objective function $f(x, y)$, constraint function $g(x, y)$.
  2. solve for $a$ and $b$ using
     $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
     $$g(a, b) = 0$$
  3. largest value of $f$ will be largest among all $f(a, b)$ found. similar for smallest.
- alternatively, with two constraints:

  let obj fn be $w = f(x, y, z)$, and constraint fns $g(x, y, z) = 0$, $h(x, y, z) = 0$. solve for

  $$\nabla f(a, b, c) = \lambda_1 \nabla g(a, b, c) + \lambda_2 \nabla h(a, b, c)$$
  $$g(a, b, c) = 0$$
  $$h(a, b, c) = 0.$$

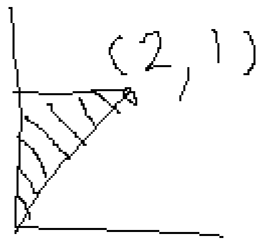
TODO CLOSED/OPEN SETS AND STUFF

- $S$ is called an open set if every point of S is an interior point.
- $S$ is called a closed set if it contains all its boundary points.

# Chapter 5: Multiple Integration

Double Integration

- Consider




.

- its area via a double integral:

$$\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{2} 1 dy dx.$$

- notice the order of bounds variables.
- to change order of integration, graph the region to get the bounds ($x=$, $y=$) correct
- for improper integrals that come from unbounded functions, adapting the above is fine.
- but for improper integrals that come from unbounded regions, consider $\iint xye^{-x^2-y^2} dA$ for the first quadrant. then we set up and solve

$$\begin{aligned} &\lim_{(b,d)\to(\infty,\infty)} \int_0^b \int_0^d xye^{-x^2-y^2} dy dx \\ =& \lim_{(b,d)\to(\infty,\infty)} \int_0^b \int_0^d xe^{-x^2} ye^{-y^2} dy dx \\ =& \lim_{(b,d)\to(\infty,\infty)} \int_0^b xe^{-x^2} dx \int_0^b ye^{-y^2} dy \\ =& \dots \\ =& \frac{1}{4} \end{aligned}$$

- converting to polar from rectangular? use

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dA &= r dr d\theta \end{aligned}$$

Triple Integration

- six orderings exist for bounds. one ex:

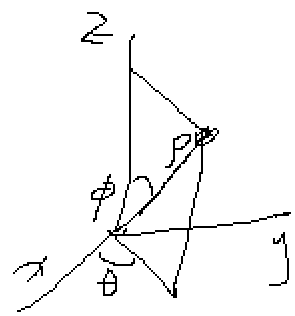
$$\iiint_E f(x,y,z) dV$$
$$= \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z) dz dy dx.$$

- think: fix $x$ with first bound. now we can use $x$ in the second bound. etc.
- avg value of $f$ on region $E$:

$$\text{Avg}(f) = \frac{1}{\text{Vol}(E)} \iiint_E f dV$$

- rect to cylindrical? just polar, but keep $z$.

  $dV = r dz dr d\theta$ is usually best.
- cyl to rect? $r = \sqrt{x^2+y^2}, \tan\theta = y/x$, keep $z$. careful: $\forall\theta, \tan^{-1}(\theta) \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- consider spherical:




- rect to spherical: $\rho^2 = x^2+y^2+z^2$, $\tan\theta = \frac{y}{x}, \phi = \cos^{-1}(\frac{z}{\sqrt{x^2+y^2+z^2}})$

  $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ usually best
- cyl to spherical: $\rho^2 = r^2+z^2$, $\theta = \theta, \phi = \cos^{-1}(\frac{z}{\sqrt{r^2+z^2}})$
- spherical to rect: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$
- spherical to cyl: $r = \rho\sin\phi$, $\theta = \theta, z = \rho\cos\phi$

Centers of Mass, Moments of Inertia

- consider a lamina (2d object). its center of mass $P(\overline{x}, \overline{y})$ is defined by

$$\overline{x} = \frac{M_y}{m} \text{ and } \overline{y} = \frac{M_x}{m}.$$

  (if density is constant, $P(\overline{x}, \overline{y})$ is the centroid.)

- the moments about the $x$ and $y$ axes are

$$M_x = \iint_R y\rho(x,y)dA$$
$$M_y = \iint_R x\rho(x,y)dA$$

- the moments of inertia about the axes are

$$I_x = \iint_R y^2\rho(x,y)dA$$
$$I_y = \iint_R x^2\rho(x,y)dA$$

  and about the origin (the polar m. of i.):

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x,y)dA$$

- also consider:

### Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object $Q$ with a density function $\rho(x, y, z)$ at any point $(x, y, z)$ in space, then its mass is

$$m = \iiint_Q \rho(x, y, z)dV.$$

Its moments about the $xy$-plane, the $xz$-plane, and the $yz$-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV, \quad M_{xz} = \iiint_Q y\rho(x, y, z)dV,$$
$$M_{yz} = \iiint_Q x\rho(x, y, z)dV.$$

If the center of mass of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, then

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the $yz$-plane, the $xz$-plane, and the $xy$-plane are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)dV,$$
$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)dV,$$
$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)dV.$$

# Chapter 6: Vector Calc

Conservative Vector Fields

- a vector field $\vec{F}$ is a gradient/conservative vec field iff $\exists f, \nabla f = \vec{F}$
  - $f$ is called a potential function
- potential functions are unique to a constant (because they're found sorta with antiderivs??). ie, if $\nabla f = \nabla g = \vec{F}$, then $f = g + C$.
- to prove a field is not conservative:
  - if $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
  - if $\vec{F} = \langle P, Q, R\rangle$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \wedge \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \wedge \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$
  - use the contrapos.
- to prove it is
  - use the converse of above, only if $\vec{F}$ is on a open, simply connected region through which the converse holds

Line Integrals

- scalar line integral (along curve in space)
  - let $f$ be cont with a domain that includes the smooth curve $C$ with parameterization $\vec{r}(t), a \leq t \leq b$.

$$\int_C f\, ds = \int_a^b f(\vec{r}(t)) \left\|\vec{r}\,'(t)\right\| dt.$$

  - idc about reparameterization
  - $f = 1$ finds arclen
- vector line integral (along oriented curve)
- find work to move object along curve in a vec field
- flux and circulation of a vec field

# Appendix

Sorted newest first

## centers of mass, moments of inertia

think of moments about axes like how much a lamina wants to rotate about the axis. then the moment / the mass is simply the center of mass. uhh, trust.

and then moments of inertia measure how much objects can resist rotation about axes

## double integral properties

for most properties, just feel it out from single integrals.

but some more esoteric ones (even if in notation only) exist:

- if $m \le f(x, y) \le M$, then
$$m \times A(R) \le \iint\limits_R f(x, y)dA \le M \times A(R).$$
- if $f(x, y)$ is factorable as a product of $g(x)$ (of $x$ only) and $h(y)$ (of $y$ only), then over $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$,
$$\iint\limits_R f(x, y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$
I think this is because of that property where you can factor out numbers, applied creatively several times?

## cross/dot products

- see:
  - `media/cross_prod_area_ex.png`
  - `media/scalar_projection_composition_ex.png`
  - `media/scalar_triple_prod_parallelepiped_ex.png`

## matrices, 3x3 determinants

In depth lesson:

- `https://www.mathcentre.ac.uk/resources/uploaded/sigma-matrices9-2009-1.pdf`

Recall that a elements of a matrix are enumerated $a_{ij}$ where $i$ is column and $j$ is row, both 1-indexed.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Recall that the minor of the matrix element here is the 2 by 2 determinant when you take away the row and column of the element in a 3 by 3 matrix. Picking an arbitrary row (or even column), with $a_{ij}$ being an element and $M_{ij}$ being a minor, a 3 by 3 determinant is calculated by

$$\sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij}$$

In the figure `media/3x3determinant.png`, $(-1)^{i+j}$ is in green, $a_{ij}$ in orange, and $M_{ij}$ in blue.