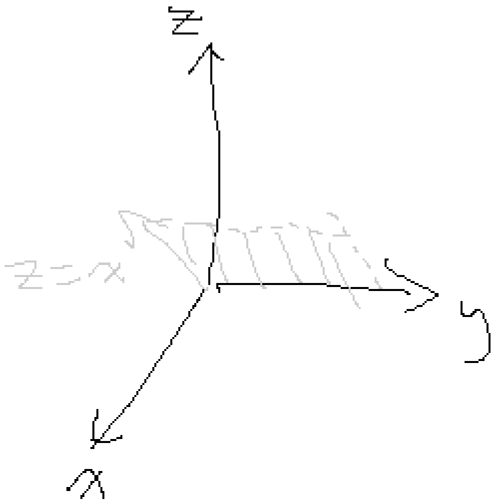
Calc in 3d Notes

saffron_

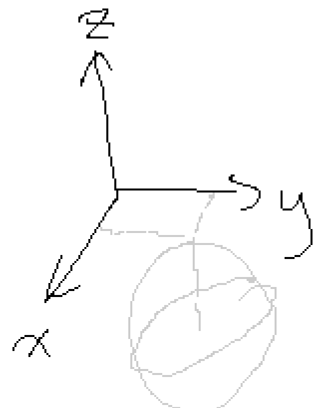
Contents

Chapter 2: Vectors in Space

Graphing

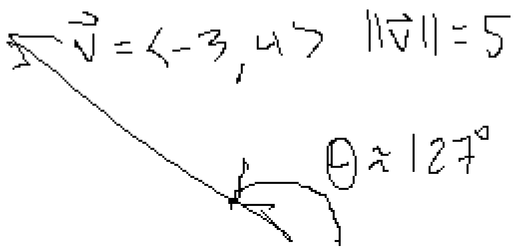


convention



sphere: $(x-1)^2 + (y-2)^2 + (z+3)^2 = 9$

The Vector



$\hat{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

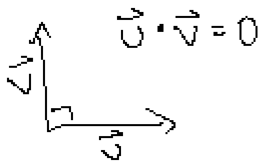
- a quantity with a magnitude and direction
- unit vector $\hat{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Vector Operations

- Addition
  - $\vec{a} = \langle 1, 2\rangle$, $\vec{b} = \langle 3, 4\rangle$
  - $\vec{a} + \vec{b} = \langle 4, 6\rangle$
- Scalar Multiplication
  - $\vec{v} = \langle 1, 3\rangle$, $c = 2$
  - $c\vec{v} = \langle 2, 6\rangle$

simple inverses for subtraction and scalar division exist.

- Dot Product (also 2d!)
  - geom.:



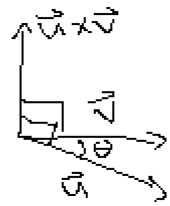
    $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$
  - alg.:
    $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$
    $\vec{u} \cdot \vec{v} = \sum u_i v_i$
  - $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
  - two vectors are orthogonal aka $\perp$ iff their dot product is 0
  - $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}$ wtf is equation 2.5 "unique over this range" on abt
  - work $= \vec{F} \cdot \vec{D}$
  - comp (scalar projection)

$$\begin{aligned} \text{comp}_{\vec{u}}\, \vec{v} &= \|\vec{v}\| \cos\theta \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \end{aligned}$$

  - proj (vector projection)

$$\begin{aligned} \text{proj}_{\vec{u}}\, \vec{v} &= \text{comp}_{\vec{u}}\, \vec{v} \cdot \frac{\vec{u}}{\|\vec{u}\|} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \end{aligned}$$

- Cross Product (3d)
  - geom.:



    $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$
  - alg.:
  - $\vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - result is $\perp$ to both input vectors, direction by right hand rule
  - triple scalar product:
    $\vec{u} \cdot (\vec{v} \times \vec{w})$
    $= (\vec{u} \times \vec{v}) \cdot \vec{w}$
    $= \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
  - if $\vec{u}$ and $\vec{v}$ are the sides of a parallelogram, then its area is $\|\vec{u} \times \vec{v}\|$
  - if a parallelepiped has edges $\vec{u}, \vec{v}, \vec{w}$, its volume is the absolute value of its triple scalar product
  - torque $= \vec{\tau} = \vec{r} \times \vec{F}$

Lines

intentionally blank

(1) Vector Equation Form

$\vec{r} = \vec{r_0} + t\vec{v}$
eg $\langle x, y, z\rangle = \langle 1, 2, -5\rangle + t\,\langle 3, -4, -1\rangle$

(2) Parametric Equation Form

$$x = 1 + 3t$$
$$y = 2 - 4t$$
$$z = -5 - t$$

(3) Symmetric Equation Form

solve for t

$$\frac{x-1}{3} = \frac{y-2}{-4} = \frac{z+5}{-1}$$

(4) Edge Case: 0-component

Let a line be defined by the point and vector $(1, -2, 6)$ and $\langle 3, 7, 0\rangle$. We say the line is:

$$\frac{x-1}{3} = \frac{y+2}{7}, z = 6$$

- point to line distance: use paralleogram area trick

Planes

(1) Vector Equation Form

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$
eg $(\langle x, y, z\rangle - \langle 1, 0, 1\rangle) \cdot \langle 1, 2, -3\rangle = 0$

(2) Scalar Equation, "General" Form

$ax + by + cz + d = 0$

- point to plane distance: use $\text{comp}_{\vec{u}}\,\vec{v}$ trick
- angle between planes: same as angle between their normal vectors

Quadric Surfaces

- cylinder: 3d shape consisting of all parallel lines (eg $y = 3x^2$)
- see `quadric_surfaces.pdf`

Chapter 3: Vector-Valued Functions

Vector-Valued Function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, i < t < j$$

Unit Circle Parameterization
temp

Limits of VVFs

- pass them into the vec. pretty intuitive
- a VVF $\vec{r}(a)$ is cont. at a iff $\lim\limits_{t \to a} \vec{r}(t) = \vec{r}(a)$ (and both are defined)

Calc with VVFs

- derivatives are intuitive. use the corresponding dot/cross/scalar in deriving $\vec{u} \cdot \vec{v}$
- unit tangent vector is the derivative's unit vector
- integrals are intuitive. consider constant vector C instead of scalar constant

Appendix

matrices, 3x3 determinants

In depth lesson:

- `https://www.mathcentre.ac.uk/resources/uploaded/sigma-matrices9-2009-1.pdf`

Recall that a elements of a matrix are enumerated a_{ij} where i is column and j is row, both 1-indexed.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Recall that the minor of the matrix element here is the 2 by 2 determinant when you take away the row and column of the element in a 3 by 3 matrix. Picking an arbitrary row (or even column), with a_{ij} being an element and M_{ij} being a minor, a 3 by 3 determinant is calculated by

$$\sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij}$$

In the figure `media/3x3determinant.png`, $(-1)^{i+j}$ is in green, a_{ij} in orange, and M_{ij} in blue.

cross/dot products

- see:
  - `media/cross_prod_area_ex.png`
  - `media/scalar_projection_composition_ex.png`
  - `media/scalar_triple_prod_parallelepiped_ex.png`