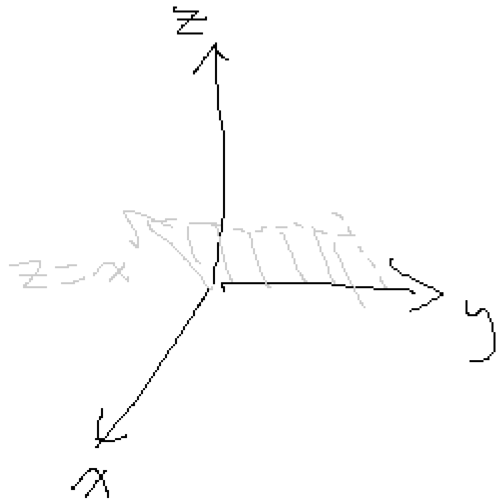
Calc in 3d Notes

saffron_

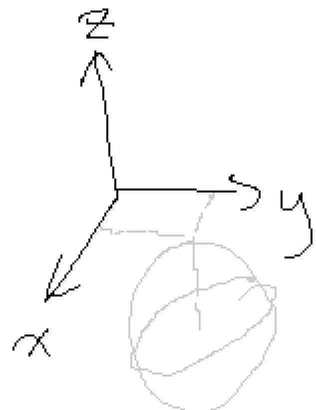
Contents

Chapter 2: Vectors in Space

Graphing

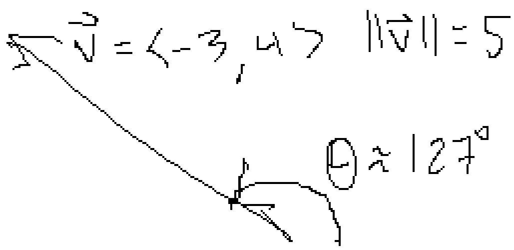


convention



sphere: $(x-1)^2 + (y-2)^2 + (z+3)^2 = 9$

The Vector



$\hat{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

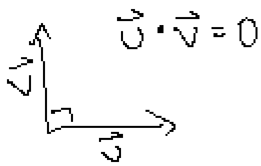
- a quantity with a magnitude and direction
- unit vector $\hat{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Vector Operations

- Addition
  - $\vec{a} = \langle 1, 2\rangle$, $\vec{b} = \langle 3, 4\rangle$
  - $\vec{a} + \vec{b} = \langle 4, 6\rangle$
- Scalar Multiplication
  - $\vec{v} = \langle 1, 3\rangle$, $c = 2$
  - $c\vec{v} = \langle 2, 6\rangle$

simple inverses for subtraction and scalar division exist.

- Dot Product (also 2d!)
  - geom.:



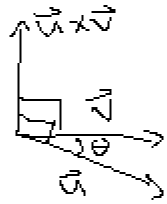
    $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$
  - alg.:
    $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$
    $\vec{u} \cdot \vec{v} = \sum u_i v_i$
  - $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
  - two vectors are orthogonal aka $\perp$ iff their dot product is 0
  - $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}$ wtf is equation 2.5 "unique over this range" on abt
  - work $= \vec{F} \cdot \vec{D}$
  - comp (scalar projection)

$$\begin{aligned} \operatorname{comp}_{\vec{u}} \vec{v} &= \|\vec{v}\| \cos\theta \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \end{aligned}$$

  - proj (vector projection)

$$\begin{aligned} \operatorname{proj}_{\vec{u}} \vec{v} &= \operatorname{comp}_{\vec{u}} \vec{v} \cdot \frac{\vec{u}}{\|\vec{u}\|} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \end{aligned}$$

- Cross Product (3d)
  - geom.:



    $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$
  - alg.:
  - $\vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - result is $\perp$ to both input vectors, direction by right hand rule
  - triple scalar product:
    $\vec{u} \cdot (\vec{v} \times \vec{w})$
    $= (\vec{u} \times \vec{v}) \cdot \vec{w}$
    $= \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
  - if $\vec{u}$ and $\vec{v}$ are the sides of a parallelogram, then its area is $\|\vec{u} \times \vec{v}\|$
  - if a parallelepiped has edges $\vec{u}, \vec{v}, \vec{w}$, its volume is the absolute value of its triple scalar product
  - torque $= \vec{\tau} = \vec{r} \times \vec{F}$

Lines

(1) Vector Equation Form

$\vec{r} = \vec{r_0} + t\vec{v}$
eg $\langle x, y, z \rangle = \langle 1, 2, -5 \rangle + t \langle 3, -4, -1 \rangle$

(2) Parametric Equation Form

$$\begin{aligned} x &= 1 + 3t \\ y &= 2 - 4t \\ z &= -5 - t \end{aligned}$$

(3) Symmetric Equation Form

solve for t

$$\frac{x-1}{3} = \frac{y-2}{-4} = \frac{z+5}{-1}$$

(4) Edge Case: 0-component

Let a line be defined by the point and vector $(1, -2, 6)$ and $\langle 3, 7, 0 \rangle$. We say the line is:

$$\frac{x-1}{3} = \frac{y+2}{7}, z = 6$$

- point to line distance: use parallelogram area trick

Planes

(1) Vector Equation Form

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$
eg $(\langle x, y, z \rangle - \langle 1, 0, 1 \rangle) \cdot \langle 1, 2, -3 \rangle = 0$

(2) Scalar Equation, "General" Form

$ax + by + cz + d = 0$

- point to plane distance: use $\text{comp}_{\vec{u}} \vec{v}$ trick
- angle between planes: same as angle between their normal vectors

Quadric Surfaces

- cylinder: 3d shape consisting of all parallel lines (eg $y = 3x^2$)
- see `quadric_surfaces.pdf`

Chapter 3: Vector-Valued Functions

Vector-Valued Function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, i < t < j$$

Unit Circle Parameterization
temp

Limits of VVFs

- pass them into the vec. pretty intuitive
- a VVF $\vec{r}(a)$ is cont. at a iff $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$ (and both are defined)

Calc with VVFs

- derivatives are intuitive. use the corresponding dot/cross/scalar in deriving $\vec{u} \cdot \vec{v}$
- unit tangent vector is the derivative's unit vector
- integrals are intuitive. consider constant vector C instead of scalar constant

Consult exam 1 cheatsheet for notes on the rest of the chapter

Chapter 4: Differentiation

Functions of Multiple Variables

- domain: analyze what values are invalid
- range: image of domain
- level planes/level surfaces/contour maps: setting the function to some constant and drawing out the resulting shape

Limits and Continuity

- limit rules are identical to 2d, including
- limits must be unique. ie, the δ disk around a point must only contain one value. (disprove limit by finding different values through different "paths" to the limit)
- as in 2d, $f(x, y)$ is continuous at (a, b) iff

  $$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

  with both defined.
- sum, product, comp of cont functions: cont

Partial Derivatives

- slope of line in a direction, at a point
- Limit definition:

  $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

- four second order partials exist

  $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}$$
  $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{yx}$$
  $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{xy}$$
  $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}$$

- Clairaut's Thrm: if f_{xy} and f_{yx} are cont near a point, they are equal

Tangent Planes

- if all tangent lines to a point are in the same plane, call that the tangent plane
  - (not true if there is a point)
  - maybe theres a correlation somewhere with differentiability? (p393)
- the tangent plane to $z = f(x, y)$ at (a, b) is

  $$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- find an approx at (a, b) via the linear approximation plane $L(x, y) = \text{RHS (above)}$
- a function is differentiable at (a, b) iff

  $$f(x, y) = \text{RHS} + E(x, y)$$

  where the error term E satisfies

  $$\lim_{(x,y)\to(a,b)} \frac{E(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

  alternatively, if f, f_x, and f_y exist near (a, b) and are cont. at (a, b), then f is differentiable there.
- let $z = f(x, y)$ with (a, b) in the domain of f, and let Δx and Δy be chosen such that $(a+\Delta x, b+\Delta y)$ is also in the domain of f.

  then

  $$\begin{aligned} dx &= \Delta x \\ dy &= \Delta y \\ dz &= f_x(a, b)dx + f_y(a, b)dy. \end{aligned}$$

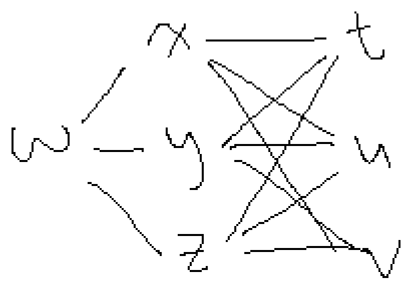
  dx and dy are differentials, dz the "total differential," and we estimate error with it.

  notice the similarity to the tangent plane equation.

The Chain Rule

- consider

$$
\begin{aligned}
w &= f(x,y,z)\\
x &= x(t,u,v)\\
y &= y(t,u,v)\\
z &= z(t,u,v)
\end{aligned}
$$



then,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial w}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

others are left as an exercise to the reader.

Implicit Differentiation

- consider $x^2 + 3y^2 + 4y - 4 = 0$.

  to find $\frac{dy}{dx}$, we may implicitly differentiate this by taking $\frac{d}{dx}$ of both sides and solving.

  but we may also define

$$f(x,y) = x^2 + 3y^2 + 4y - 4, f(x,y) = 0.$$

  with this in mind, suppose $f(x,y) = 0$. then,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

  and if $f(x,y,z) = 0$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}, \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

  these can be derived from the chain rule.

Directional Derivatives and the Gradient

- the directional derivative of $f(x,y)$ in the direction $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ is

$$D_{\hat{u}} f(a,b) = \lim_{h\to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

  alternatively, if the partials exist,

$$
\begin{aligned}
D_{\hat{u}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta\\
&= \langle\, f_x(x,y), f_y(x,y)\,\rangle \cdot \hat{u}\\
&= \nabla f(x,y)\cdot \hat{u}.
\end{aligned}
$$

- $\nabla f(x,y)$ is called the gradient and points toward the greatest increase of a function. it's perpendicular to the graph's level curves (if the partials are cont. near the points)

- suppose $z = f(x,y)$ diffbl at (a,b).
  - if $\nabla f(a,b) = \vec{0}$, then $D_{\hat{u}} f(a,b) = 0$ for any $\hat{u}$
  - if $\nabla f(a,b) \neq \vec{0}$, then $D_{\hat{u}} f(a,b)$ is max when $\hat{u}$ is in the same direction as $\nabla f(a,b)$.
    max of $D_{\hat{u}} f(a,b)$ is $\|\nabla f(a,b)\|$
  - if $\nabla f(a,b) \neq \vec{0}$, then $D_{\hat{u}} f(a,b)$ is min when $\hat{u}$ is in the opposite direction as $\nabla f(a,b)$.
    min of $D_{\hat{u}} f(a,b)$ is $-\|\nabla f(a,b)\|$

- yes, these work for multivar funcs.

Finding Maxima/Minima

- like 2d, critical points (a, b) exist iff
  - $f_x(a, b) = f_y(a, b) = 0$
  - or the partials there don't exist
- local extrema are crit points
- Second Derivative Test (for 3d calc)

  let $z = f(x, y)$ where the first and second order partials are cont near a point (a, b)

  $$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

  - $D > 0$ and $f_{xx}(a, b) > 0$
    $\Rightarrow f$ has a local min at (a, b)
  - $D > 0$ and $f_{xx}(a, b) < 0$
    $\Rightarrow f$ has a local max at (a, b)
  - $D < 0$
    $\Rightarrow f$ has a saddle point at (a, b)
  - $D = 0$
    $\Rightarrow$ the test is inconclusive
- to find local extrema:
  1. find crit points. discard if a partial DNE
  2. find discriminant D for each point
  3. apply 2nd derivative test
- Extreme Value Theorem

  a cont function on a closed and bounded set has an abs min and abs max in the set
- the abs max and abs min will be either at a critical point or on a boundry
- to analyze the boundry, consider parameterizing line segments/ellipses or using Lagrange multipliers

Lagrange Multipliers

- Theory: consider an objective function and a constraint function. when they are tangent, their gradients must be in the same direction. so, we say they are different by λ, the Lagrange multiplier.
- Let $f(x, y)$ and $g(x, y)$ have cont partials along $g(x, y) = 0$. if f has a local extrema on $g(x, y) = 0$ at (a, b) and $\nabla g(a, b) \neq 0$, then
  $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
- to use λ (yes, f and g may be fn of 3 vars):
  1. find the objective function $f(x, y)$, constraint function $g(x, y)$.
  2. solve for a and b using
     $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
     $$g(a, b) = 0$$
  3. largest value of f will be largest among all $f(a, b)$ found. similar for smallest.
- alternatively, with two constraints:

  let obj fn be $w = f(x, y, z)$, and constraint fns $g(x, y, z) = 0$, $h(x, y, z) = 0$. solve for

  $$\nabla f(a, b, c) = \lambda_1 \nabla g(a, b, c) + \lambda_2 \nabla h(a, b, c)$$
  $$g(a, b, c) = 0$$
  $$h(a, b, c) = 0.$$

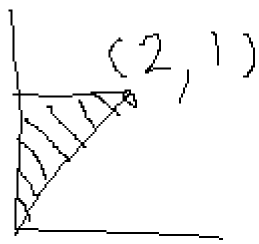
TODO CLOSED/OPEN SETS AND STUFF

- S is called an open set if every point of S is an interior point.
- S is called a closed set if it contains all its boundary points.

Chapter 5: Multiple Integration

Double Integration

- Consider



- its area via a double integral:

$$\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{2} 1 dydx.$$

- notice the order of bounds variables.
- to change order of integration, graph the region to get the bounds ($x=$, $y=$) correct
- for improper integrals that come from unbounded functions, adapting the above is fine.
- but for improper integrals that come from unbounded regions, consider $\iint xye^{-x^2-y^2}dA$ for the first quadrant. then we set up and solve

$$\begin{aligned}
&\lim_{(b,d)\to(\infty,\infty)} \int_0^b \int_0^d xye^{-x^2-y^2} dydx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_0^b \int_0^d xe^{-x^2} ye^{-y^2} dydx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_0^b xe^{-x^2} dx \int_0^b ye^{-y^2} dy \\
&= \dots \\
&= \frac{1}{4}
\end{aligned}$$

- converting to polar from rectangular? use

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
dA &= rdrd\theta
\end{aligned}$$

Triple Integration

- six orderings exist for bounds. one ex:

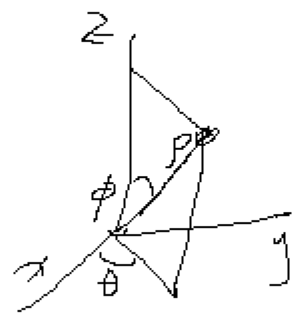
$$\iiint_E f(x,y,z)dV$$
$$= \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)dzdydx.$$

- think: fix x with first bound. now we can use x in the second bound. etc.
- avg value of f on region E:

$$\text{Avg}(f) = \frac{1}{\text{Vol}(E)} \iiint_E f dV$$

- rect to cylindrical? just polar, but keep z.

  $dV = rdzdrd\theta$ is usually best.
- cyl to rect? $r = \sqrt{x^2+y^2}, \tan\theta = y/x$, keep z. careful: $\forall\theta, \tan^{-1}(\theta) \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- consider spherical:



- rect to spherical: $\rho^2 = x^2+y^2+z^2$, $\tan\theta = \frac{y}{x}, \phi = \cos^{-1}(\frac{z}{\sqrt{x^2+y^2+z^2}})$

  $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ usually best
- cyl to spherical: $\rho^2 = r^2 + z^2$, $\theta = \theta, \phi = \cos^{-1}(\frac{z}{\sqrt{r^2+z^2}})$
- spherical to rect: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$
- spherical to cyl: $r = \rho\sin\phi$, $\theta = \theta, z = \rho\cos\phi$

Centers of Mass, Moments of Inertia

- consider a lamina (2d object). its center of mass $P(\overline{x}, \overline{y})$ is defined by

$$\overline{x} = \frac{M_y}{m} \text{ and } \overline{y} = \frac{M_x}{m}.$$

where mass is obtained by integrating ρ over the domain, and moments about the x and y axes are

$$M_x = \iint\limits_R y\rho(x,y)dA, M_y = \iint\limits_R x\rho(x,y)dA.$$

(constant density? $P(\overline{x}, \overline{y})$ is the centroid.)

- the moments of inertia about the axes are

$$I_x = \iint\limits_R y^2\rho(x,y)dA, I_y = \iint\limits_R x^2\rho(x,y)dA$$

and about the origin (the polar m. of i.):

$$I_0 = I_x + I_y = \iint\limits_R (x^2+y^2)\rho(x,y)dA.$$

(transforming to polar works for all these)

- radius of gyration about axes (anal rot?):

$$R_x = \sqrt{\frac{I_x}{m}}, R_y = \sqrt{\frac{I_y}{m}}, R_0 = \sqrt{\frac{I_0}{m}}$$

- also consider 3d:

  for center mass, $\overline{x} = \frac{M_x}{m}, \ldots$

  mass m is similar.

  moments about the *axes*:

$$M_x = \iiint\limits_Q x\rho(x,y,z)dV, \ldots$$

  moments of inertia about the *axes*

$$I_x = \iiint\limits_Q (y^2+z^2)\rho(x,y,z)dV, \ldots$$

Multivar Change of Variables (like "u-sub")

- planar transformations
  - let $G, R \subseteq \mathbb{R}^2$. a pl. trans. is a fn $T : G \to R$, transes G region to R, via $x = g(u,v), y = h(u,v)$.

    (typically assume/require first partials $\exists$ and are cont. ie, C^1 trans.)

    a trans, $T : G \to R, T(u,v) = (x,y)$ is one to one iff 2 distinct points cannot map to the same output. ie, $(\forall G_1, G_2 \in G)(T(G_1) = T(G_2) \Rightarrow G_1 = G_2)$

    if T is one to one, $(\exists T^{-1})(T^{-1} \circ T, T \circ T^{-1})$ are id. fns.

    finding $\mathrm{Im}_T G$? can try eval. each section of G's boundry thru T.

- Jacobians:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

  The same pattern holds for 3d.

- changing variables strategy:
  1. sketch integration region (xy plane)
  2. examine region/bounds and integrand, choose $u = \ldots, v = \ldots$
  3. find bounds in uv plane
  4. evaluate jacobian
  5. substitute bounds, replace dA with $J(u,v)\,du\,dv$.

Chapter 6: Vector Calc (Exhaustive)

Conservative Vector Fields

- defns: simple curve: no crossings. simply connected region: no holes.
- a vector field $\vec{F}$ is a gradient/conservative vec field iff $\exists f, \nabla f = \vec{F}$
  - f is called a potential function
- pot. fn. uniqueness is like antiderivs: $\nabla f = \nabla g = \vec{F} \Rightarrow f = g + C$.
- cross partial property (prove not cons.):

$$\vec{F} = \langle P, Q, R \rangle \text{ is cons.}$$
$$\Rightarrow P_y = Q_x \wedge Q_z = R_y \wedge R_x = P_z.$$

- to prove it is: use the converse of above, only if $\vec{F}$ is on a open, simply connected region through which the converse holds

Finding a Potential Function f

- given $\vec{F}(x, y) = \langle P, Q \rangle$,

$$f = \int P\,\partial x + g(y) \text{ (g, “a constant”)} \tag{1}$$
$$f_y = \frac{\partial}{\partial y}\left(\int P\,\partial x\right) + g'(y) \text{ taking derivative} \tag{2}$$
$$= Q \text{ by defn of } \vec{F}\text{, if it has a pot. fn.} \tag{3}$$

solve for g, which finishes f, via (2) and (3), usually choosing 0 for the constant after antideriv.

Scalar Line Integrals

- let f be cont with a domain that includes the smooth curve C with parameterization $\vec{r}(t), a \le t \le b$.

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt.$$

- idc about reparameterization
- $f = 1$ finds arclen

Vector Line Integrals

- recall unit tang. vec $\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$. since $ds = \|\vec{r}\,'(t)\|\,dt$, we may derive

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt.$$

thus, VLIs are often denoted

$$\int_C \vec{F} \cdot d\vec{r}.$$

also consider that $\vec{F} = \langle P, Q, R \rangle$ and $d\vec{r} = \langle dx, dy, dz \rangle$. so, yet another form:

$$\int_C P dx + Q dy + R dz$$
$$= \int P(\vec{r}(t))\frac{dx}{dt} + Q(\vec{r}(t))\frac{dy}{dt} + R(\vec{r}(t))\frac{dz}{dt} dt$$

- vector LIs are reversible

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

- if an object moves along C in force field $\vec{F}$, the work req'd to move it is $\int_C \vec{F} \cdot d\vec{r}$.

Flux

- flux of $\vec{F}$ across C measures at what rate fluid crosses a curve.
- let C be $\vec{r}(t) = \langle x(t), y(t) \rangle, a \le t \le b$.

  let $\vec{n}(t) = \langle y'(t), -x'(t) \rangle$ (the normal pointing right as we traverse the curve).

  let unit normal vec $\vec{N}(t) = \frac{\vec{n}(t)}{\|\vec{n}(t)\|}$.

  flux is

$$\int_C \vec{F} \cdot \vec{N} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) dt$$

- notice the similarity:

$$\text{circ/LI} \int_C \vec{F} \cdot \vec{T} ds = \int_C P dx + Q dy$$
$$\text{flux} \int_C \vec{F} \cdot \vec{N} ds = \int_C -Q dx + P dy$$

Circulation

- a vector LI along an oriented closed curve is called the circulation of $\vec{F}$ along C:

$$\oint_C \vec{F} \cdot \vec{T} ds \text{ or } \oint_C \vec{F} \cdot d\vec{r}$$

Fundamental Thrm of Line Integrals

- similarly to the FTC,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

- so, to find $\int_C \vec{F} \cdot d\vec{r}$,
  1. find potential fn ("antiderivative")
  2. evaluate.
- it follows that if $\vec{F}$ is conserv. (has pot. fn.) and C is closed, $\oint_C \vec{F} \cdot d\vec{r} = 0$.
- we also discover path independence:
  - $\vec{F}$ is conserv. $\Rightarrow$ $\vec{F}$ path indep.
  - ie, work by grav is same for 3 hikers who take diff paths but start/end similarly.
  - converse is true if domain of $\vec{F}$ is open and connected.

Green's Theorem

- connects a $\iint_D$ to a $\int_C$ around the boundry of D.

Circulation Form

- let an open, simply connected region D, with piecewise smooth, closed, simple, c.clockwise boundry C, with $\vec{F} = \langle P, Q \rangle$. (only works with 2D field)

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$$

- circulation eq. can be expressed with $\vec{T}$. so, this is also called the "tangential form"
- notice: if $Q_x - P_y = 1$, we may use this form to find area. as such, let $\vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$. so,

$$\text{Area}(D) = \iint_D 1 dA = \frac{1}{2} \oint_C -y dx + x dy$$

Flux Form

- let an open, simply connected region D, with piecewise smooth, closed, simple, c.clockwise boundry C, with $\vec{F} = \langle P, Q \rangle$.

$$\oint_C \vec{F} \cdot \vec{N} ds = \oint_C -Q dx + P dy = \iint_D P_x + Q_y dA.$$

- also called the "normal form"

General Form (holed regions)

- if you have a region with finitely many holes, you may convert the line integral around its boundry into the double integral anyways lmao.
- subtract out any contributions of the empty regions

Source-Free Fields

- notice the similarities with conservative fields: $\vec{F} = \langle P, Q \rangle$ is a SFF iff

  1. flux $\oint_C \vec{F} \cdot \vec{N} ds = 0$

  or 2. flux indep. of path

  or 3. $\exists$ a stream fn $g(x, y)$ for $\vec{F}$

  or 4. $P_x + Q_y = 0$

- a stream fn for $\vec{F}$ is a fn g st

$$\vec{F} = \langle P, Q \rangle = \langle g_y, -g_x \rangle .$$

  geometrically, $\vec{F} = \langle a, b \rangle$ is tangent to the level curve of the stream fn g at (a, b). since grad g is $\perp$ to the level curve of g, $\vec{F}(a, b) \cdot \nabla g(a, b) = 0$ on the domain of g.

  it's like a pot. fn. but for SFF

- Laplace Equation: $f_{xx} + f_{yy}(+ f_{zz}) = 0$.

- harmonic fns are fns that satisfy Laplace.

  the pot fn of $\vec{F}$, f, satisfies the Laplace Eq. $\iff \exists \vec{F}, \vec{F}$ is BOTH conserv. and src free $\iff f$ is harmonic

  proof: if f is both conserv and src free, $\langle P, Q \rangle = \langle f_x, f_y \rangle$ (by conserv), $f_{xx} + f_{yy} = P_x + Q_y = 0$. (by src free)

Divergence (the scalar)

- measures "outgoingness."

- or: imagine dropping elastic band into water flow. its change in area is divergence.

- defn. div$\vec{F} = P_x + Q_y + R_z$.

- mnemonic: div$\vec{F} = \nabla \cdot \vec{F}$ since ∇ can be thought as $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

- $\vec{F}$ is src free $\Rightarrow$ div$\vec{F} = 0$.

  converse is true on simply connected $\vec{F}$.

- by defn of Green's (flux) and diver,

$$\oint_C \vec{F} \cdot \vec{N} ds = \iint_D P_x + Q_y dA = \iint_D \text{div}\vec{F} dA.$$

  notice: if we think of div$\vec{F}$ as a kind of deriv, this looks much like the FTC

Curl (the vector field)

- measure of rotation about a point

- if curl is a vec, it measures the tendency of water near a point to rotate about the axis in the vec's dir

- $\|$curl$\|$ would be how quick the water rotation is about the axis

- imagine paddlewheel. axis is curl vec dir, curl magn is rotation speed

- curl$\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

- mnemonic: curl$\vec{F} = \nabla \times \vec{F}$

  intuitively, 2D means curl is only $\hat{k}$ comp

- $\vec{F}$ is conserv $\Rightarrow$ curl$\vec{F} = 0$.

  converse is true on simply connected $\vec{F}$.

- sim. with div, notice that with Green's (circ) and 2d curl$\vec{F} \cdot \hat{k} = (Q_x - P_y)$,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA = \iint_D \text{curl}\vec{F} \cdot \hat{k} \, dA$$

Div/Curl Applications

- defn: div curl$\vec{F} = 0$. (3d)

- consider curl test: let curl$\vec{F}$ = a field $\vec{G}$.

$$\exists \vec{F} \Rightarrow \text{ div curl}\vec{F} = 0$$
$$\iff \text{div}\vec{G} \neq 0 \Rightarrow \vec{G} \text{ cannot be curl}\vec{F}$$

  - converse true on simpl conn domain

- what about div of a conserv (a gradient)?

  - div$(\nabla f) = \nabla \cdot (\nabla f)$ abbreviated $\nabla^2 f$
  - so, harmonic iff $\nabla^2 f = 0$.
  - fun fact: potfn of electrostatic field in a region with no static charge is harmonic

Surface Integrals

- you can parameterize a 2d surface with height $z = f(x,y)$ with $\vec{r}(x,y) = \langle x, y, z(x,y)\rangle$.
- a surface parameterization is regular (actually a 3d surface and not, like, a line) and smooth if $\vec{r}_u \times \vec{r}_v \neq 0$
- define a scalar surface integral as follows:

$$\iint_S f(x,y,z)dS = \iint_D f(\vec{r}(u,v)) \left\|\vec{r}_u \times \vec{r}_v\right\| dA$$

- a surface is orientable if you can separate an "outer" and "inner" side. some surfaces (like mobius strip) are not.
  - recall: any curve can have forward/backward orientation
  - the choice of unit normal vec $\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ gives an orientation.
- let $\vec{F}$ by a cont vec field with a dom that contains oriented surface S with unit norm vec N. the vector surface integral of F over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} dS$$
$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA$$

  This is sort of like flux.

Stokes' Theorem

- a higher-dimensional general. of Green's theorem. it's also like FTC.
- relates vector surface integral over surface of a line integral around its boundry.
- let S be an oriented smooth surface with un norm vec $\vec{N}$. suppose boundry of S is a simple closed curve C. orientation of S induces positive orientation of C if, walking in positive dir of C with head pointed $\vec{N}$, surface is on left.
- Stokes': let S be a piecewise smooth oriented surface with a boundary, a simple closed curve C with positive orientation.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

- surface independence

Divergence Theorem

- relates flux integral of field over surface S to a triple integral of the divergence of the field over the solid enclosed by S.
- Let S be a piecewise smooth close surface enclosing solid E. Assume S oriented outward, let F be a field. Then

$$\iiint_E \text{div}\vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$$

Chapter 6: Vector Calc (Formula sheet)

- simple curve: no crossings. simply connected region: no holes.
- field $\vec{F}$ is a gradient iff $\exists$ potential function $f, \nabla f = \vec{F}$. be able to find pot. fns.
- cross partial prop.: $\vec{F} = \langle P, Q, R \rangle$ is cons. $\Rightarrow P_y = Q_x \wedge Q_z = R_y \wedge R_x = P_z$
  - converse true on open, simply connected field
- scalar line integral: $\int_C f\, ds = \int_a^b f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$
- vector line integral: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt = \int_C P dx + Q dy$
  - reversible: $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$
  - measures work along C in $\vec{F}$
  - circulation is this, on a closed curve. denoted with $\oint$
- flux, rate of fluid crossing 2d curve: $\int_C \vec{F} \cdot \vec{N} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) dt = \int_C -Q dx + P dy$
- FTLI: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ (notice: $\vec{F}$ is conserv. $\Rightarrow \vec{F}$ path indep ($\Leftarrow$ if open&conn))
- Green's (2D): open D and piecewise smooth closed simple c.clockwise C
  - $\oint_C \vec{F} \cdot \vec{T} ds = \oint_C P dx + Q dy = \iint_D Q_x - P_y dA = \iint_D \text{curl}\vec{F} \cdot \hat{k} dA$ (circulation form)
  - $\oint_C \vec{F} \cdot \vec{N} ds = \oint_C -Q dx + P dy = \iint_D P_x + Q_y dA = \iint_D \text{div}\vec{F} dA$ (flux form)
  - consider $\vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$. Area$(D) = \iint_D 1 dA = \frac{1}{2} \oint_C -y dx + x dy$
- $\vec{F}$ is source free field iff
  - flux $\oint_C \vec{F} \cdot \vec{N} ds = 0 \vee$ flux indep. of path $\vee \exists$ a stream fn $g(x,y)$ for $\vec{F} \vee P_x + Q_y = 0$
  - define stream fn g s.t. $\vec{F} = \langle P, Q \rangle = \langle g_y, -g_x \rangle$
  - pot fn of $\vec{F}$ satisfies Laplace ($f_{xx} + f_{yy} (+ f_{zz}) = 0$) iff $\vec{F}$ conserv. & src free iff f harmonic
- divergence, outgoingness: $\text{div}\vec{F} = \nabla \cdot \vec{F}$
- $\vec{F}$ is src free $\Rightarrow \text{div}\vec{F} = 0$ ($\Leftarrow$ if simpl conn $\vec{F}$)
- curl, paddlewheel: $\text{curl}\vec{F} = \nabla \times \vec{F}$
- defn: div curl$\vec{F} = 0$. test for curl.
- $\vec{F}$ conserv $\Rightarrow \text{curl}\vec{F} = 0$
- harmonic iff $\nabla^2 f = \text{div}(\nabla f) = 0$
- scalar surface integral: $\iint_S f(x,y,z) dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$
- vector surface integral: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA$ (like flux)
- Stokes': $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$ what the FUCK is surface independence (p787)
  - also what the FUCK is positive orientation for stokes surfaces
- divergence thrm: $\iiint_E \text{div}\vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$

Appendix

Sorted newest first

evaulating limits

factor and try to naiively solve for the limit. doesn't work? double path test. for limit approaching (a, b), try $x = a$, $y = b$, $x = y$ (works more often than I think), matching exponents, and factoring even harder. don't forget normal limit rules like how $\lim_{x\to 0} \frac{x^2 - x^6}{x^4}$ tends toward ∞.

centers of mass, moments of inertia

think of moments about axes like how much a lamina wants to rotate about the axis. then the moment / the mass is simply the center of mass. uhh, trust.

and then moments of inertia measure how much objects can resist rotation about axes

notice how for both formulas, the integrals sum over how far away each point is from the corresponding axis.

double integral properties

for most properties, just feel it out from single integrals.

but some more esoteric ones (even if in notation only) exist:

- if $m \le f(x, y) \le M$, then

$$m \times A(R) \le \iint\limits_R f(x, y) dA \le M \times A(R).$$

- if $f(x, y)$ is factorable as a product of $g(x)$ (of x only) and $h(y)$ (of y only), then over $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$,

$$\iint\limits_R f(x, y) dA = \left(\int_a^b g(x) dx\right) \left(\int_c^d h(y) dy\right).$$

  I think this is because of that property where you can factor out numbers, applied creatively several times?

cross/dot products

- see:

- media/cross_prod_area_ex.png
- media/scalar_projection_composition_ex.png
- media/scalar_triple_prod_parallelepiped_ex.png

matrices, 3x3 determinants

In depth lesson:

- https://www.mathcentre.ac.uk/resources/uploaded/sigma-matrices9-2009-1.pdf

Recall that a elements of a matrix are enumerated a_{ij} where i is column and j is row, both 1-indexed.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Recall that the minor of the matrix element here is the 2 by 2 determinant when you take away the row and column of the element in a 3 by 3 matrix. Picking an arbitrary row (or even column), with a_{ij} being an element and M_{ij} being a minor, a 3 by 3 determinant is calculated by

$$\sum_{j=1}^{3}(-1)^{i+j}a_{ij}M_{ij}$$

In the figure `media/3x3determinant.png`, $(-1)^{i+j}$ is in green, a_{ij} in orange, and M_{ij} in blue.