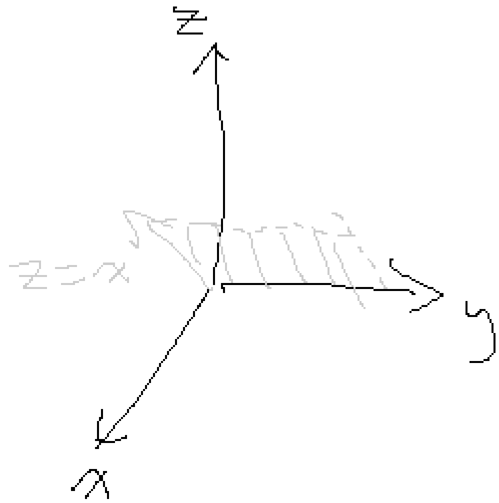
Calc in 3d Notes

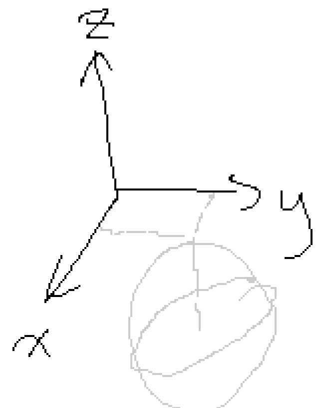
saffron_

Contents

Chapter 2: Vectors in Space

Graphing

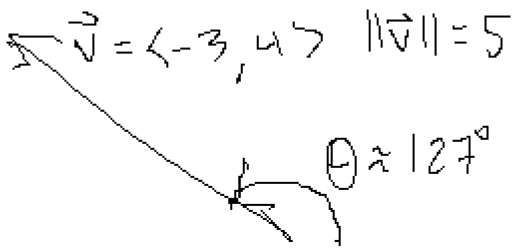




convention

sphere: $(x-1)^2 + (y-2)^2 + (z+3)^2 = 9$

The Vector



$\hat{u} = \langle -\frac{3}{5} \; \frac{4}{5} \rangle$

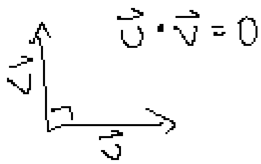
- a quantity with a magnitude and direction
- unit vector $\hat{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Vector Operations

- Addition
  - $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 3, 4 \rangle$
  - $\vec{a} + \vec{b} = \langle 4, 6 \rangle$
- Scalar Multiplication
  - $\vec{v} = \langle 1, 3 \rangle$, $c = 2$
  - $c\vec{v} = \langle 2, 6 \rangle$

simple inverses for subtraction and scalar division exist.

- Dot Product (also 2d!)
  - geom.:



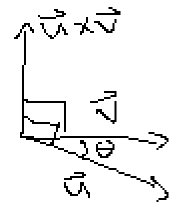
    $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$
  - alg.:
    $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$
    $\vec{u} \cdot \vec{v} = \sum u_i v_i$
  - $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
  - two vectors are orthogonal aka $\perp$ iff their dot product is 0
  - $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}$ wtf is equation 2.5 "unique over this range" on abt
  - work $= \vec{F} \cdot \vec{D}$
  - comp (scalar projection)

$$\begin{aligned} \operatorname{comp}_{\vec{u}} \vec{v} &= \|\vec{v}\| \cos\theta \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \end{aligned}$$

  - proj (vector projection)

$$\begin{aligned} \operatorname{proj}_{\vec{u}} \vec{v} &= \operatorname{comp}_{\vec{u}} \vec{v} \cdot \frac{\vec{u}}{\|\vec{u}\|} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \end{aligned}$$

- Cross Product (3d)
  - geom.:



    $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$
  - alg.:
  - $\vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - result is $\perp$ to both input vectors, direction by right hand rule
  - triple scalar product:
    $\vec{u} \cdot (\vec{v} \times \vec{w})$
    $= (\vec{u} \times \vec{v}) \cdot \vec{w}$
    $= \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
  - if $\vec{u}$ and $\vec{v}$ are the sides of a parallelogram, then its area is $\|\vec{u} \times \vec{v}\|$
  - if a parallelepiped has edges $\vec{u}, \vec{v}, \vec{w}$, its volume is the absolute value of its triple scalar product
  - torque $= \vec{\tau} = \vec{r} \times \vec{F}$

Lines

(1) Vector Equation Form

$\vec{r} = \vec{r_0} + t\vec{v}$
eg $\langle x, y, z\rangle = \langle 1, 2, -5\rangle + t\,\langle 3, -4, -1\rangle$

(2) Parametric Equation Form

$$x = 1 + 3t$$
$$y = 2 - 4t$$
$$z = -5 - t$$

(3) Symmetric Equation Form

solve for t

$$\frac{x-1}{3} = \frac{y-2}{-4} = \frac{z+5}{-1}$$

(4) Edge Case: 0-component

Let a line be defined by the point and vector $(1, -2, 6)$ and $\langle 3, 7, 0\rangle$. We say the line is:

$$\frac{x-1}{3} = \frac{y+2}{7}, z = 6$$

- point to line distance: use paralleogram area trick

Planes

(1) Vector Equation Form

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$
eg $(\langle x, y, z\rangle - \langle 1, 0, 1\rangle) \cdot \langle 1, 2, -3\rangle = 0$

(2) Scalar Equation, "General" Form

$ax + by + cz + d = 0$

- point to plane distance: use $\text{comp}_{\vec{u}}\,\vec{v}$ trick
- angle between planes: same as angle between their normal vectors

Quadric Surfaces

- cylinder: 3d shape consisting of all parallel lines (eg $y = 3x^2$)
- see `quadric_surfaces.pdf`

Chapter 3: Vector-Valued Functions

Vector-Valued Function

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle, i < t < j$$

Unit Circle Parameterization
temp

Limits of VVFs

- pass them into the vec. pretty intuitive
- a VVF $\vec{r}(a)$ is cont. at a iff $\lim\limits_{t\to a} \vec{r}(t) = \vec{r}(a)$ (and both are defined)

Calc with VVFs

- derivatives are intuitive. use the corresponding dot/cross/scalar in deriving $\vec{u} \cdot \vec{v}$
- unit tangent vector is the derivative's unit vector
- integrals are intuitive. consider constant vector C instead of scalar constant

Consult exam 1 cheatsheet for notes on the rest of the chapter

Chapter 4: Differentiation

Functions of Multiple Variables

- domain: analyze what values are invalid
- range: image of domain
- level planes/level surfaces/contour maps: setting the function to some constant and drawing out the resulting shape

Limits and Continuity

- limit rules are identical to 2d, including
- limits must be unique. ie, the δ disk around a point must only contain one value. (disprove limit by finding different values through different "paths" to the limit)
- as in 2d, $f(x,y)$ is continuous at (a,b) iff

  $$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

  with both defined.
- sum, product, comp of cont functions: cont

Partial Derivatives

- slope of line in a direction, at a point
- Limit definition:

  $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

- four second order partials exist

  $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}$$
  $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{yx}$$
  $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{xy}$$
  $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}$$

- Clairaut's Thrm: if f_{xy} and f_{yx} are cont near a point, they are equal

Tangent Planes

- if all tangent lines to a point are in the same plane, call that the tangent plane
  - (not true if there is a point)
  - maybe theres a correlation somewhere with differentiability? (p393)
- the tangent plane to $z = f(x,y)$ at (a,b) is

  $$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- find an approx at (a,b) via the linear approximation plane $L(x,y) = \text{RHS}$ (above)
- a function is differentiable at (a,b) iff

  $$f(x,y) = \text{RHS} + E(x,y)$$

  where the error term E satisfies

  $$\lim_{(x,y)\to(a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

  alternatively, if f, f_x, and f_y exist near (a,b) and are cont. at (a,b), then f is differentiable there.
- let $z = f(x,y)$ with (a,b) in the domain of f, and let Δx and Δy be chosen such that $(a+\Delta x, b+\Delta y)$ is also in the domain of f.

  then

  $$\begin{aligned} dx &= \Delta x \\ dy &= \Delta y \\ dz &= f_x(a,b)dx + f_y(a,b)dy. \end{aligned}$$

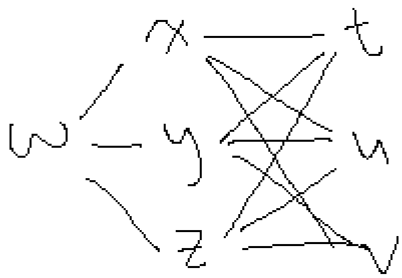
  dx and dy are differentials, dz the "total differential," and we estimate error with it.

  notice the similarity to the tangent plane equation.

The Chain Rule

- consider

$$\begin{aligned} w &= f(x,y,z) \\ x &= x(t,u,v) \\ y &= y(t,u,v) \\ z &= z(t,u,v) \end{aligned}$$



then,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial w}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

others are left as an exercise to the reader.

Implicit Differentiation

- consider $x^2 + 3y^2 + 4y - 4 = 0$.

  to find $\frac{dy}{dx}$, we may implicitly differentiate this by taking $\frac{d}{dx}$ of both sides and solving.

  but we may also define

$$f(x,y) = x^2 + 3y^2 + 4y - 4, f(x,y) = 0.$$

  with this in mind, suppose $f(x,y) = 0$. then,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

  and if $f(x,y,z) = 0$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}, \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

  these can be derived from the chain rule.

Directional Derivatives and the Gradient

- the directional derivative of $f(x,y)$ in the direction $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ is

$$D_{\hat{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

  alternatively, if the partials exist,

$$\begin{aligned} D_{\hat{u}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= \langle\, f_x(x,y), f_y(x,y)\,\rangle \cdot \hat{u} \\ &= \nabla f(x,y) \cdot \hat{u}. \end{aligned}$$

- $\nabla f(x,y)$ is called the gradient and points toward the greatest increase of a function. it's perpendicular to the graph's level curves (if the partials are cont. near the points)

- suppose $z = f(x,y)$ diffbl at (a,b).
  - if $\nabla f(a,b) = \vec{0}$, then $D_{\hat{u}} f(a,b) = 0$ for any $\hat{u}$
  - if $\nabla f(a,b) \neq \vec{0}$, then $D_{\hat{u}} f(a,b)$ is max when $\hat{u}$ is in the same direction as $\nabla f(a,b)$.
    max of $D_{\hat{u}} f(a,b)$ is $\|\nabla f(a,b)\|$
  - if $\nabla f(a,b) \neq \vec{0}$, then $D_{\hat{u}} f(a,b)$ is min when $\hat{u}$ is in the opposite direction as $\nabla f(a,b)$.
    min of $D_{\hat{u}} f(a,b)$ is $-\|\nabla f(a,b)\|$

- yes, these work for multivar funcs.

TODO do example 4.34-5 on page 429 and onwards

Finding Maxima/Minima

- like 2d, critical points (a, b) exist iff
  - $f_x(a, b) = f_y(a, b) = 0$
  - or the partials there don't exist
- local extrema are crit points
- Second Derivative Test (for 3d calc)

  let $z = f(x, y)$ where the first and second order partials are cont near a point (a, b)

  $$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

  - $D > 0$ and $f_{xx}(a, b) > 0$
    $\Rightarrow f$ has a local min at (a, b)
  - $D > 0$ and $f_{xx}(a, b) < 0$
    $\Rightarrow f$ has a local max at (a, b)
  - $D < 0$
    $\Rightarrow f$ has a saddle point at (a, b)
  - $D = 0$
    $\Rightarrow$ the test is inconclusive
- to find local extrema:
  1. find crit points. discard if a partial DNE
  2. find discriminant D for each point
  3. apply 2nd derivative test
- Extreme Value Theorem

  a cont function on a closed and bounded set has an abs min and abs max in the set
- the abs max and abs min will be either at a critical point or on a boundry
- to analyze the boundry, consider parameterizing line segments/ellipses or using Lagrange multipliers

Lagrange Multipliers

- Theory: consider an objective function and a constraint function. when they are tangent, their gradients must be in the same direction. so, we say they are different by λ, the Lagrange multiplier.
- Let $f(x, y)$ and $g(x, y)$ have cont partials along $g(x, y) = 0$. if f has a local extrema on $g(x, y) = 0$ at (a, b) and $\nabla g(a, b) \neq 0$, then
  $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
- to use λ (yes, f and g may be fn of 3 vars):
  1. find the objective function $f(x, y)$, constraint function $g(x, y)$.
  2. solve for a and b using
     $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
     $$g(a, b) = 0$$
  3. largest value of f will be largest among all $f(a, b)$ found. similar for smallest.
- alternatively, with two constraints:

  let obj fn be $w = f(x, y, z)$, and constraint fns $g(x, y, z) = 0$, $h(x, y, z) = 0$. solve for

  $$\nabla f(a, b, c) = \lambda_1 \nabla g(a, b, c) + \lambda_2 \nabla h(a, b, c)$$
  $$g(a, b, c) = 0$$
  $$h(a, b, c) = 0.$$

Appendix

matrices, 3x3 determinants

In depth lesson:

- `https://www.mathcentre.ac.uk/resources/uploaded/sigma-matrices9-2009-1.pdf`

Recall that a elements of a matrix are enumerated a_{ij} where i is column and j is row, both 1-indexed.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Recall that the minor of the matrix element here is the 2 by 2 determinant when you take away the row and column of the element in a 3 by 3 matrix. Picking an arbitrary row (or even column), with a_{ij} being an element and M_{ij} being a minor, a 3 by 3 determinant is calculated by

$$\sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij}$$

In the figure `media/3x3determinant.png`, $(-1)^{i+j}$ is in green, a_{ij} in orange, and M_{ij} in blue.

cross/dot products

- see:
  - `media/cross_prod_area_ex.png`
  - `media/scalar_projection_composition_ex.png`
  - `media/scalar_triple_prod_parallelepiped_ex.png`