Impossible Walks and Planar Graphs

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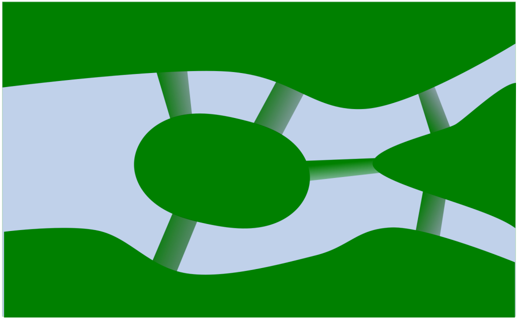
Reed College

St Mary's Academy, April 24th 2018

Bridges of Königsberg

At first, there were 6 bridges in the town of Königsberg



and you could take a stroll across each bridge exactly once.

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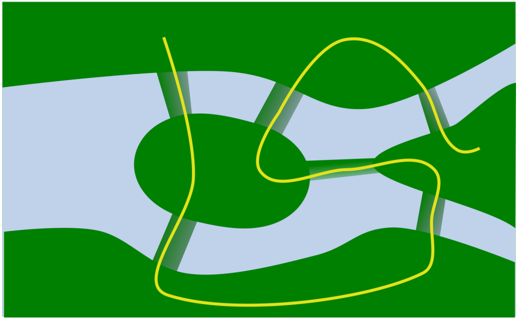

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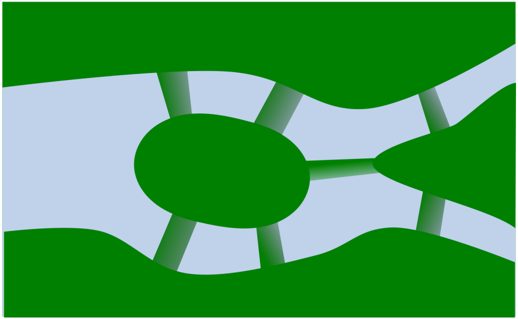
Bridges of Königsberg

But when they built another bridge, no one could figure out how to do it anymore.



It seemed like no matter which way they went, they would always miss a bridge or get stuck.

Bridges of Königsberg

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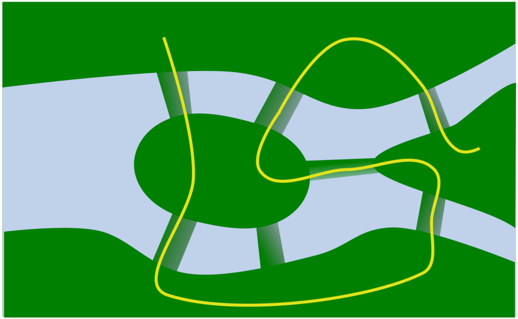

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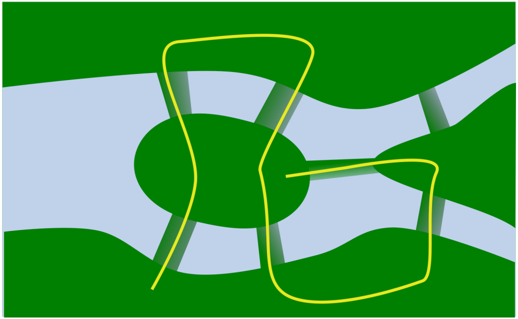

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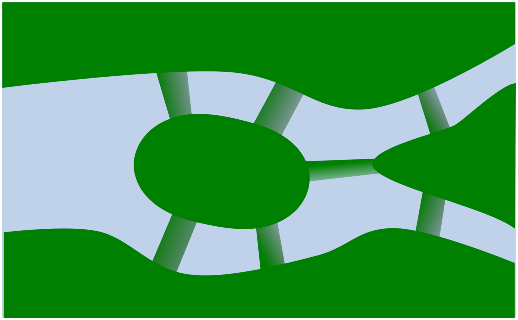
Bridges of Königsberg

Does a walk that passes through each bridge exactly once exist?



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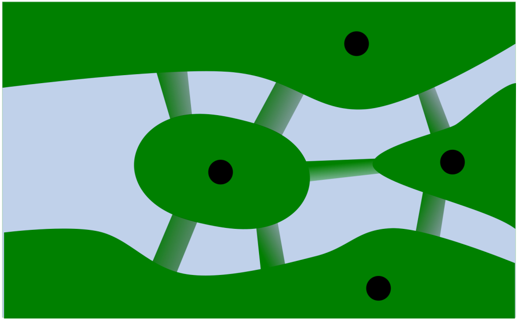

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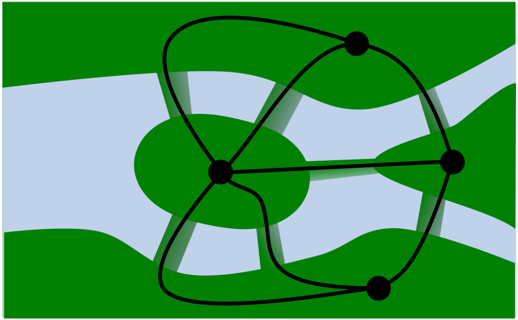
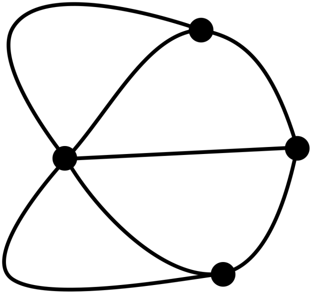


Graphs

Is there a walk on the graph that passes through each edge exactly once?



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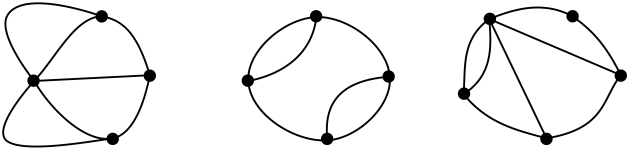
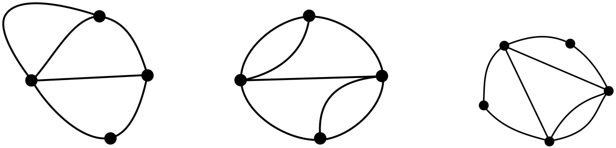


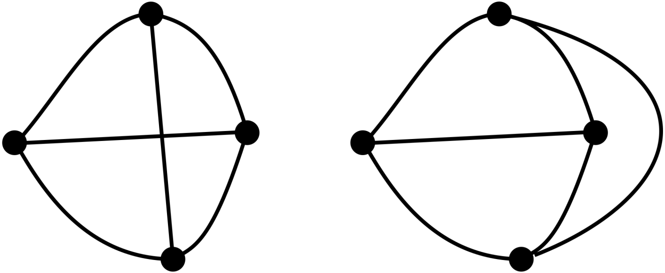
NO



YES

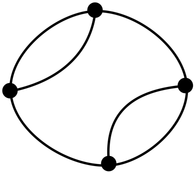
Definitions

A **graph** is a collection of vertices and edges between them.
How we draw a graph is not unique.

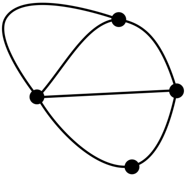


Definitions

The **degree** of a vertex is the number of edges attached to it.

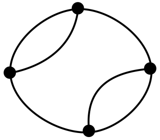


Every vertex in this graph has degree 3.

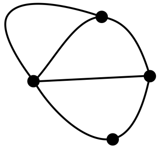


The degrees of the vertices in this graph are 4, 3, 3, 2.

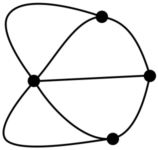
How does degree relate to the number of edges?



deg: 3, 3, 3, 3
# of edges: 6



deg: 4, 3, 3, 2
# of edges: 6



deg: 5, 3, 3, 3
# of edges: 7

Multiple Choice

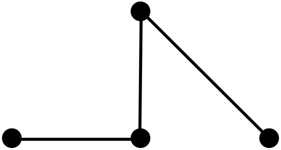
Suppose I have a graph whose vertices have degrees 2, 2, 1, 1. How many edges does it have?

- 2
- 3
- 4
- 5

Multiple Choice

Suppose I have a graph whose vertices have degrees 2, 2, 1, 1. How many edges does it have?

- 2
- $\boxed{3}$
- 4
- 5



Why? If we sum all the degrees, each edge gets counted twice.

Degree Theorem

Theorem

The sum of the degrees of all vertices in a graph is twice the number of edges.

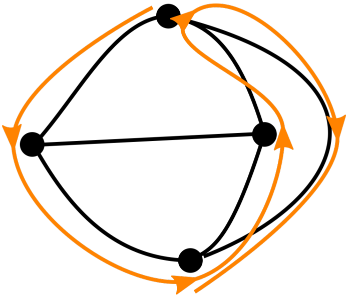
$$\sum_{v \in V(G)} \deg v = 2E$$

Corollary. The number of vertices with odd degree is even.

It is impossible to draw a graph with an odd number of vertices where all of them have odd degree!

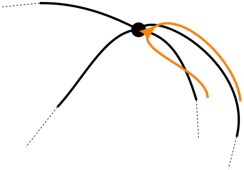
Definitions

Two vertices are **adjacent** if there is an edge between them.
A **walk** is a sequence of adjacent vertices and the edges between them.

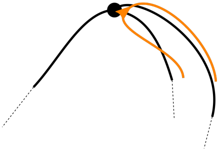


How do walks relate to degree?

If a walk enters a vertex with even degree, it exits as well.



If a walk enters a vertex with odd degree, it may get stuck or it may exit.

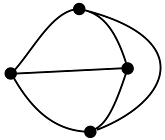


Walks that pass through vertices leave untouched edges with the same parity (even or odd) as the degree of the vertex.

Eulerian Walks

A walk is **Eulerian** (pronounced “oilerian”) if it passes through each edge of the graph exactly once.

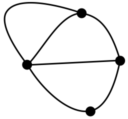
4 or more vertices of odd degree: there is no Eulerian walk



exactly 3 vertice of odd degree: impossible by corollary,
because vertices with odd degree occur in pairs

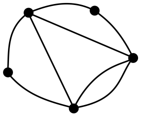
Eulerian Walks

exactly 2 vertices of odd degree: any Eulerian walk starts at one and ends at the other



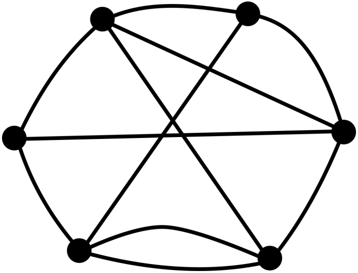
exactly 1 vertex of odd degree: impossible by corollary

no vertices of odd degree: any Eulerian walk must start and end in the same place



Finding Eulerian Walks

Look for the vertices of odd degree!



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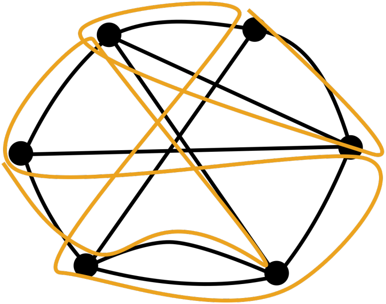
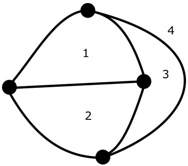


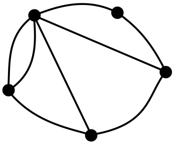
Definitions

A **connected** graph has at least one walk between any pair of vertices.
If we draw a connected graph so that no edges cross, we can count the number of faces of the graph by counting the 'holes', including the empty space around the graph.

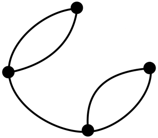


$F = 4, E = 6, V = 4$

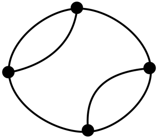
Euler's Formula



$$F = 5$$
$$E = 8$$
$$V = 5$$



$$F = 3$$
$$E = 5$$
$$V = 4$$



$$F = 4$$
$$E = 6$$
$$V = 4$$

Multiple Choice

Suppose I have a connected graph with 3 faces and 3 vertices. How many edges does the graph have, if it can be drawn so that none of the edges are crossing?

- 3
- 4
- 5
- 6

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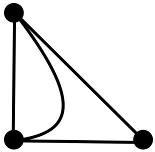
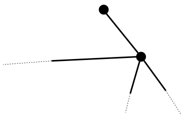


$F + V = E + 2$. Why?

Proof by Induction

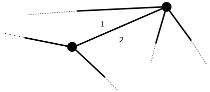
If we have the most simple graph possible, we can check that it satisfies $F + V = E + 2$:

If we add a vertex, we have to connect it with an edge, $F + (V + 1) = (E + 1) + 2$:



Euler's Formula

If we add an edge between two existing vertices, we divide a face into two, $(F+1)+V=(E+1)+2$:



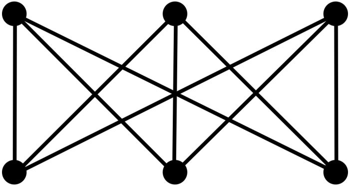
A **planar** graph is a graph that can be drawn so that there are no edges crossing.

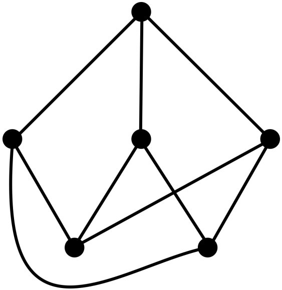
Theorem (Euler)

For any connected, planar graph, $F+V=E+2$.

Nonplanar Graphs

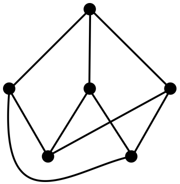
Not every graph is planar!





This graph violates Euler's Formula.

Nonplanar Graphs



Any face must have at least 4 edges on its boundary, but each edge is counted twice, so $E \geq \frac{4F}{2} = 2F$.
But $V = 6$, $E = 9$ so Euler's formula gives

$$F + 6 = 9 + 2 = 11$$
$$F = 5$$

But $E = 9 \not\geq 2F = 2 \cdot 5 = 10$, so the graph can't possibly be planar.

Summary

- To know whether a graph has a walk that passes through each edge exactly once, count the number of vertices with an odd number of edges coming out. If there are 4 or more, then it's impossible.
- If you can draw a graph with no edges crossing, then it satisfies the formula $F + V = E + 2$.
- If it doesn't satisfy the formula, then every drawing will have at least one crossing.

Thank you!

My email: `safia@reed.edu`

R. Trudeau.
Introduction to Graph Theory (Dover Books on Mathematics).

A. Benjamin, G. Chartrand, and P. Zhang.
The Fascinating World of Graph Theory.

https://brilliant.org/wiki/graph-theory/