Midterm: Finite Element Method 2 (Oct. 25, 2018 – Nov. 1, 2018) - Prof. Gunwoo Noh

This is a take-home, open-book test. You may use any materials (including software), but no collaborations are allowed for the answers. You may get help from other people only for the indirect subjects (e.g. usage of the software, etc.). Submit your answers including the related codes **to E6-319 by 11:59 pm Nov. 1, 2018.** Describe the procedures in detail.

1. (20) points,

Consider the following composite method:

<First substep>

$$\mathbf{M}\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}}+\mathbf{C}\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}}+\mathbf{K}\,{}^{t+\gamma\Delta t}\mathbf{U}={}^{t+\gamma\Delta t}\mathbf{R}$$

$${}^{t+\gamma\Delta t}\mathbf{U}={}^{t}\mathbf{U}+\frac{\gamma\Delta t}{2}({}^{t}\dot{\mathbf{U}}+{}^{t+\gamma\Delta t}\dot{\mathbf{U}})$$

$${}^{t+\gamma\Delta t}\dot{\mathbf{U}}={}^{t}\dot{\mathbf{U}}+\frac{\gamma\Delta t}{2}({}^{t}\ddot{\mathbf{U}}+{}^{t+\gamma\Delta t}\ddot{\mathbf{U}})$$

<Second substep>

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}+\mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}+\mathbf{K}\,{}^{t+\Delta t}\mathbf{U}={}^{t+\Delta t}\mathbf{R}$$

$${}^{t+\Delta t}\mathbf{U}={}^{t}\mathbf{U}+\gamma\Delta t\,((1-\beta_1){}^{t}\dot{\mathbf{U}}+\beta_1{}^{t+\gamma\Delta t}\dot{\mathbf{U}})+(1-\gamma)\Delta t((1-\beta_2){}^{t+\gamma\Delta t}\dot{\mathbf{U}}+\beta_2\,{}^{t+\Delta t}\dot{\mathbf{U}})$$

$${}^{t+\Delta t}\dot{\mathbf{U}}={}^{t}\dot{\mathbf{U}}+\gamma\Delta t\,((1-\beta_1){}^{t}\ddot{\mathbf{U}}+\beta_1{}^{t+\gamma\Delta t}\ddot{\mathbf{U}})+(1-\gamma)\Delta t((1-\beta_2){}^{t+\gamma\Delta t}\ddot{\mathbf{U}}+\beta_2\,{}^{t+\Delta t}\ddot{\mathbf{U}})$$

Consider the case of no physical damping.

a) **(3 points)** Construct the 2 by 2 amplification matrix $\mathbf{A}$ where

$$\begin{bmatrix} {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix}=\mathbf{A}\begin{bmatrix} {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix}+\mathbf{L_a}\,{}^{t+\gamma\Delta t}r+\mathbf{L_b}\,{}^{t+\Delta t}r$$

b) **(7 points)** Derive the required condition for the second order accuracy.

c) **(5 points)** Check whether the conditions for the complex conjugate eigenvalues and the unconditional stability are satisfied for the cases of $\beta_2 = 2\beta_1$ and $\beta_2 = 1 - \beta_1$.

d) **(5 points)** Draw the curves of the spectral radius, percentage period elongations and percentage amplitude decays for $\beta_2 = 2\beta_1$ (with β_1 = 0.5, 0.6) and $\beta_2 = 1 - \beta_1$ (with β_1 = 0.4, 0.5) . Discuss the characteristics of the method shortly.

2. (30 points)

Consider the following composite method:

<First substep>

$$\mathbf{M}\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\gamma\Delta t}\mathbf{U} = {}^{t+\gamma\Delta t}\mathbf{R}$$

$${}^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \gamma\Delta t\,{}^{t}\dot{\mathbf{U}} + \gamma^2\Delta t^2\left((\frac{1}{2} - \alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} \right)$$

$${}^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \gamma\Delta t\left((1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} \right)$$

<Second substep>

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\mathbf{R}$$

$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \Delta t\,(q_0\,{}^{t}\dot{\mathbf{U}} + q_1\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + q_2\,{}^{t+\Delta t}\dot{\mathbf{U}})$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \Delta t\,(s_0\,{}^{t}\ddot{\mathbf{U}} + s_1\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + s_2\,{}^{t+\Delta t}\ddot{\mathbf{U}})$$

where

$$q_0 = \frac{-2\gamma^2\alpha + (2\alpha + 2\delta)\gamma - \delta}{2\delta\gamma}; \quad q_1 = \frac{-2\alpha\gamma + \delta}{2\delta\gamma} \ ; \quad q_2 = \frac{\alpha\gamma}{\delta}$$

$$s_0 = \frac{-1-2\delta\gamma^2+(2\delta+2)\gamma}{2\gamma}; \quad s_1 = \frac{-2\delta\gamma+1}{2\gamma}; \quad s_2 = \delta\gamma$$

and $\alpha = \dfrac{-\delta(6\delta\gamma^2-12\delta\gamma-3\gamma+4)}{6\gamma(2\delta\gamma-1)}$.

After we express the method in modal space as

$$\begin{bmatrix} {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix} = \mathbf{A}\begin{bmatrix} {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix} + \mathbf{L_a}\,{}^{t+\gamma\Delta t}r + \mathbf{L_b}\,{}^{t+\Delta t}r$$

we have, considering no physical damping, the characteristic polynomial of $\mathbf{A}$:

$$p(\lambda) = \lambda^2 - 2A_1\lambda + A_2 \tag{1}$$

where

$$A_2 = \frac{1}{\beta}\Big(\big((1/4)\delta^3 + \big(-\alpha\gamma+(1/8)\big)\delta^2 + \alpha\big(\alpha\gamma^2-(\gamma/2)-(1/4)\big)\delta + (\alpha/8)\big)\gamma^2\Omega_0^4$$
$$+ (1/8)\big(16\alpha\gamma^2-4\big)\delta\Omega_0^2 + \delta\Big)$$

$$A_2 = \frac{1}{\beta}\Big((2\delta^3\gamma^2 + \big(-8\alpha\gamma^3+3\gamma^2-4\gamma\big)\delta^2 + \big(4\alpha^2\gamma^4-4\alpha\gamma^3+10\alpha\gamma^2-\gamma+1\big)\delta + \gamma\alpha\big(\gamma-2\big))\Omega_0^4$$
$$+ 2\Omega_0^2\alpha\gamma^2\delta + \delta)\Big)$$

$\beta = \delta(\Omega_0^2\alpha\gamma^2+1)^2$;

where ω_0 is the modal natural frequency and $\Omega_0 = \omega_0\Delta t$. We finally have two free parameters: γ and δ.

a) **(8 points)** Draw the four lines (or the boundaries) for the unconditional stability on $\gamma - \delta$ plane. Indicate the regions of the unconditional stability, if there is any.

b) **(7 points)** Indicate the region for the complex conjugate eigenvalues for all possible Ω_0.

c) **(5 points)** Draw the curves of spectral radius, percentage period elongations and amplitude decays when $\delta = 0.3$ and $\gamma = 2.78$. Discuss the properties of the method shortly.

d) **(10 points)** Now we consider $\gamma = 0.5$, and we treat α and δ as free parameters. Note that we now do **NOT** use the relation, $\alpha = \dfrac{-\delta(6\delta\gamma^2 - 12\delta\gamma - 3\gamma + 4)}{6\gamma(2\delta\gamma - 1)}$ for α, but still use the relations for the other parameters. Suggest the useful ranges of the parameters, α and δ, and discuss/show its usefulness.