Q1 a)

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<< Notation`

Symbolize[ x_{t+Δt} ]

Symbolize[ ẋ_{t+Δt} ]

Symbolize[ ẍ_{t+Δt} ]

Symbolize[ x_{t+γΔt} ]

Symbolize[ ẋ_{t+γΔt} ]

Symbolize[ ẍ_{t+γΔt} ]

Symbolize[ x_t ]

Symbolize[ ẋ_t ]

Symbolize[ ẍ_t ]

Symbolize[ r_{t+Δt} ]

Symbolize[ r_{t+γΔt} ]

Symbolize[ Ω_o ]

Symbolize[ Ω̄_d ]

Symbolize[ ξ̄ ]

Symbolize[ β_1 ]

Symbolize[ β_2 ]

Symbolize[ X_t ]

ClearAll["Global`*"]

(*
Writing in the modal form
*)

ξ = 0;
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```
Solve[
  ẍ_{t+γΔt} + 2 ξ ω ẋ_{t+γΔt} + ω² x_{t+γΔt} == r_{t+γΔt} &&
  x_{t+γΔt} == x_t + (γ Δt)/2 (ẋ_t + ẋ_{t+γΔt}) &&
  ẋ_{t+γΔt} == ẋ_t + (γ Δt)/2 (ẍ_t + ẍ_{t+γΔt}),
  {ẍ_{t+γΔt}, ẋ_{t+γΔt}, x_{t+γΔt}}]
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$$\{\{\ddot{x}_{t+\gamma\Delta t} \to -\left(\left(-4\, r_{t+\gamma\Delta t} + 8\, \dot{x}_t\, \xi\, \omega + 4\, x_t\, \gamma\, \Delta t\, \xi\, \omega + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + x_t\, \gamma^2\, \Delta t^2\, \omega^2\right) /\right.$$
$$\left(4 + 4\, \gamma\, \Delta t\, \xi\, \omega + \gamma^2\, \Delta t^2\, \omega^2\right)\right),$$
$$\dot{x}_{t+\gamma\Delta t} \to -\left(\left(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 2\, x_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 + \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\right) /\right.$$
$$\left(4 + 4\, \gamma\, \Delta t\, \xi\, \omega + \gamma^2\, \Delta t^2\, \omega^2\right)\right),$$
$$x_{t+\gamma\Delta t} \to -\left(\left(-4\, x_t - 4\, \dot{x}_t\, \gamma\, \Delta t - r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2 - x_t\, \gamma^2\, \Delta t^2 - 4\, x_t\, \gamma\, \Delta t\, \xi\, \omega - 2\, \dot{x}_t\, \gamma^2\, \Delta t^2\, \xi\, \omega\right) /\right.$$
$$\left(4 + 4\, \gamma\, \Delta t\, \xi\, \omega + \gamma^2\, \Delta t^2\, \omega^2\right)\right)\}\}$$

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ClearAll["Global`*"]
```

$$\ddot{x}_{t+\gamma\Delta t} = -\frac{-4\, r_{t+\gamma\Delta t} + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + \ddot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2}{4 + \gamma^2\, \Delta t^2\, \omega^2}$$

$$-\frac{-4\, r_{t+\gamma\Delta t} + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + x_t\, \gamma^2\, \Delta t^2\, \omega^2}{4 + \gamma^2\, \Delta t^2\, \omega^2}$$

$$\dot{x}_{t+\gamma\Delta t} = -\frac{1}{4 + \gamma^2\, \Delta t^2\, \omega^2}\left(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 2\, \ddot{x}_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 + \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\right)$$

$$-\frac{1}{4 + \gamma^2\, \Delta t^2\, \omega^2}\left(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 2\, x_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 + \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\right)$$

$$x_{t+\gamma\Delta t} = -\frac{-4\, x_t - 4\, \dot{x}_t\, \gamma\, \Delta t - r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2 - \ddot{x}_t\, \gamma^2\, \Delta t^2}{4 + \gamma^2\, \Delta t^2\, \omega^2}$$

$$-\frac{-4\, x_t - 4\, \dot{x}_t\, \gamma\, \Delta t - r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2 - x_t\, \gamma^2\, \Delta t^2}{4 + \gamma^2\, \Delta t^2\, \omega^2}$$

```
Solve[ ẍ_{t+Δt} + 2 ξ ω ẋ_{t+Δt} + ω² x_{t+Δt} == r_{t+Δt} &&
  x_{t+Δt} == x_t + γ Δt ((1 - β₁) ẋ_t + β₁ ẋ_{t+γΔt}) + (1 - γ) Δt ((1 - β₂) ẋ_{t+γΔt} + β₂ ẋ_{t+Δt}) &&
  ẋ_{t+Δt} == ẋ_t + γ Δt ((1 - β₁) ẍ_t + β₁ ẍ_{t+γΔt}) + (1 - γ) Δt ((1 - β₂) ẍ_{t+γΔt} + β₂ ẍ_{t+Δt}),
  {ẍ_{t+Δt}, ẋ_{t+Δt}, x_{t+Δt}}]
```

$$\{\{x_{t+\Delta t} \to$$
$$-\left(\left(r_{t+\Delta t} - \left(-2\, \xi\, \omega - \beta_2\, \Delta t\, \omega^2 + \beta_2\, \gamma\, \Delta t\, \omega^2\right)\left(-\dot{x}_t + \left((1 - \beta_2)\, (1 - \gamma)\, \Delta t\, \left(-4\, r_{t+\gamma\Delta t} + 4\, x_t\, \omega^2 +\right.\right.\right.\right.\right.$$
$$\left.\left.\left.4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + x_t\, \gamma^2\, \Delta t^2\, \omega^2\right)\right) / \left(4 + \gamma^2\, \Delta t^2\, \omega^2\right) - \gamma\, \Delta t\, \left(x_t\, (1 - \beta_1) -\right.\right.$$
$$\left.\left.\left(\beta_1\, \left(-4\, r_{t+\gamma\Delta t} + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + x_t\, \gamma^2\, \Delta t^2\, \omega^2\right)\right) / \left(4 + \gamma^2\, \Delta t^2\, \omega^2\right)\right)\right) +$$
$$\omega^2\, \left(-x_t + \left((1 - \beta_2)\, (1 - \gamma)\, \Delta t\, \left(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 2\, x_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 +\right.\right.\right.$$
$$\left.\dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\right)\right) / \left(4 + \gamma^2\, \Delta t^2\, \omega^2\right) - \gamma\, \Delta t\, \left(\dot{x}_t\, (1 - \beta_1) - \left(\beta_1\, \left(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\gamma\, \Delta t - 2\, x_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 + \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\right)\right) / \left(4 + \gamma^2\, \Delta t^2\, \omega^2\right)\right)\right)\right) /$$
$$\left(-1 + \beta_2\, (1 - \gamma)\, \Delta t\, \left(-2\, \xi\, \omega - \beta_2\, \Delta t\, \omega^2 + \beta_2\, \gamma\, \Delta t\, \omega^2\right)\right)\right),\, \dot{x}_{t+\Delta t} \to$$
$$-\left(\left(-4\, \dot{x}_t - 4\, r_{t+\gamma\Delta t}\, \Delta t + 4\, r_{t+\gamma\Delta t}\, \beta_2\, \Delta t - 4\, r_{t+\Delta t}\, \beta_2\, \Delta t + 4\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 4\, x_t\, \gamma\, \Delta t -\right.\right.$$
$$4\, r_{t+\gamma\Delta t}\, \beta_1\, \gamma\, \Delta t + 4\, x_t\, \beta_1\, \gamma\, \Delta t - 4\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma\, \Delta t + 4\, r_{t+\Delta t}\, \beta_2\, \gamma\, \Delta t +$$
$$4\, x_t\, \Delta t\, \omega^2 - 4\, x_t\, \gamma\, \Delta t\, \omega^2 + 4\, x_t\, \beta_1\, \gamma\, \Delta t\, \omega^2 + 4\, \dot{x}_t\, \beta_2\, \Delta t^2\, \omega^2 - 4\, \dot{x}_t\, \beta_2^2\, \Delta t^2\, \omega^2 +$$
$$4\, \dot{x}_t\, \gamma\, \Delta t^2\, \omega^2 - 8\, \dot{x}_t\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 8\, \dot{x}_t\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 - 5\, \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2 +$$

$$4 \dot{x}_t \beta_1 \gamma^2 \Delta t^2 \omega^2 + 4 \dot{x}_t \beta_2 \gamma^2 \Delta t^2 \omega^2 - 4 \dot{x}_t \beta_2^2 \gamma^2 \Delta t^2 \omega^2 + 2 r_{t+\gamma\Delta t} \beta_2 \gamma \Delta t^3 \omega^2 +$$
$$2 x_t \beta_2 \gamma \Delta t^3 \omega^2 - 2 r_{t+\gamma\Delta t} \beta_2^2 \gamma \Delta t^3 \omega^2 - 2 x_t \beta_2^2 \gamma \Delta t^3 \omega^2 + x_t \gamma^2 \Delta t^3 \omega^2 -$$
$$4 r_{t+\gamma\Delta t} \beta_2 \gamma^2 \Delta t^3 \omega^2 - r_{t+\Delta t} \beta_2 \gamma^2 \Delta t^3 \omega^2 - 5 x_t \beta_2 \gamma^2 \Delta t^3 \omega^2 +$$
$$2 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^2 + 2 x_t \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^2 + 4 r_{t+\gamma\Delta t} \beta_2^2 \gamma^2 \Delta t^3 \omega^2 +$$
$$4 x_t \beta_2^2 \gamma^2 \Delta t^3 \omega^2 - 2 x_t \gamma^3 \Delta t^3 \omega^2 + 2 x_t \beta_1 \gamma^3 \Delta t^3 \omega^2 + 2 r_{t+\gamma\Delta t} \beta_2 \gamma^3 \Delta t^3 \omega^2 +$$
$$r_{t+\Delta t} \beta_2 \gamma^3 \Delta t^3 \omega^2 + 3 x_t \beta_2 \gamma^3 \Delta t^3 \omega^2 - 2 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^2 -$$
$$2 x_t \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^2 - 2 r_{t+\gamma\Delta t} \beta_2^2 \gamma^3 \Delta t^3 \omega^2 - 2 x_t \beta_2^2 \gamma^3 \Delta t^3 \omega^2 - 2 x_t \beta_2 \gamma \Delta t^3 \omega^4 +$$
$$2 x_t \beta_2^2 \gamma \Delta t^3 \omega^4 + 5 x_t \beta_2 \gamma^2 \Delta t^3 \omega^4 - 2 x_t \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^4 - 4 x_t \beta_2^2 \gamma^2 \Delta t^3 \omega^4 -$$
$$3 x_t \beta_2 \gamma^3 \Delta t^3 \omega^4 + 2 x_t \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^4 + 2 x_t \beta_2^2 \gamma^3 \Delta t^3 \omega^4 - \dot{x}_t \beta_2 \gamma^2 \Delta t^4 \omega^4 +$$
$$\dot{x}_t \beta_2^2 \gamma^2 \Delta t^4 \omega^4 + 3 \dot{x}_t \beta_2 \gamma^3 \Delta t^4 \omega^4 - 2 \dot{x}_t \beta_1 \beta_2 \gamma^3 \Delta t^4 \omega^4 - 2 \dot{x}_t \beta_2^2 \gamma^3 \Delta t^4 \omega^4 -$$
$$2 \dot{x}_t \beta_2 \gamma^4 \Delta t^4 \omega^4 + 2 \dot{x}_t \beta_1 \beta_2 \gamma^4 \Delta t^4 \omega^4 + \dot{x}_t \beta_2^2 \gamma^4 \Delta t^4 \omega^4 \Big) \Big/ \Big(\big(4 + \gamma^2 \Delta t^2 \omega^2\big)$$
$$\big(1 + 2 \beta_2 \Delta t \xi \omega - 2 \beta_2 \gamma \Delta t \xi \omega + \beta_2^2 \Delta t^2 \omega^2 - 2 \beta_2^2 \gamma \Delta t^2 \omega^2 + \beta_2^2 \gamma^2 \Delta t^2 \omega^2 \big) \Big) \Big),$$

$$x_{t+\Delta t} \to -\Big(\Big(-4 x_t - 4 \dot{x}_t \Delta t - 4 r_{t+\gamma\Delta t} \beta_2 \Delta t^2 + 4 r_{t+\gamma\Delta t} \beta_2^2 \Delta t^2 - 4 r_{t+\Delta t} \beta_2^2 \Delta t^2 -$$
$$2 r_{t+\gamma\Delta t} \gamma \Delta t^2 - 2 x_t \gamma \Delta t^2 + 10 r_{t+\gamma\Delta t} \beta_2 \gamma \Delta t^2 - 2 x_t \beta_2 \gamma \Delta t^2 - 4 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma \Delta t^2 +$$
$$4 x_t \beta_1 \beta_2 \gamma \Delta t^2 - 8 r_{t+\gamma\Delta t} \beta_2^2 \gamma \Delta t^2 + 8 r_{t+\Delta t} \beta_2^2 \gamma \Delta t^2 + 2 r_{t+\gamma\Delta t} \gamma^2 \Delta t^2 + 2 x_t \gamma^2 \Delta t^2 -$$
$$2 r_{t+\gamma\Delta t} \beta_1 \gamma^2 \Delta t^2 - 2 x_t \beta_1 \gamma^2 \Delta t^2 - 6 r_{t+\gamma\Delta t} \beta_2 \gamma^2 \Delta t^2 + 2 x_t \beta_2 \gamma^2 \Delta t^2 +$$
$$4 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma^2 \Delta t^2 - 4 x_t \beta_1 \beta_2 \gamma^2 \Delta t^2 + 4 r_{t+\gamma\Delta t} \beta_2^2 \gamma^2 \Delta t^2 - 4 r_{t+\Delta t} \beta_2^2 \gamma^2 \Delta t^2 -$$
$$8 x_t \beta_2 \Delta t \xi \omega + 8 x_t \beta_2 \gamma \Delta t \xi \omega - 8 \dot{x}_t \beta_2 \Delta t^2 \xi \omega + 8 \dot{x}_t \beta_2^2 \Delta t^2 \xi \omega + 8 \dot{x}_t \beta_2 \gamma \Delta t^2 \xi \omega -$$
$$16 \dot{x}_t \beta_2^2 \gamma \Delta t^2 \xi \omega + 8 \dot{x}_t \beta_2^2 \gamma^2 \Delta t^2 \xi \omega - 4 r_{t+\gamma\Delta t} \beta_2 \gamma \Delta t^3 \xi \omega - 4 x_t \beta_2 \gamma \Delta t^3 \xi \omega +$$
$$4 r_{t+\gamma\Delta t} \beta_2^2 \gamma \Delta t^3 \xi \omega + 4 x_t \beta_2^2 \gamma \Delta t^3 \xi \omega + 8 r_{t+\gamma\Delta t} \beta_2 \gamma^2 \Delta t^3 \xi \omega + 8 x_t \beta_2 \gamma^2 \Delta t^3 \xi \omega -$$
$$4 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma^2 \Delta t^3 \xi \omega - 4 x_t \beta_1 \beta_2 \gamma^2 \Delta t^3 \xi \omega - 8 r_{t+\gamma\Delta t} \beta_2^2 \gamma^2 \Delta t^3 \xi \omega -$$
$$8 x_t \beta_2^2 \gamma^2 \Delta t^3 \xi \omega - 4 r_{t+\gamma\Delta t} \beta_2 \gamma^3 \Delta t^3 \xi \omega - 4 x_t \beta_2 \gamma^3 \Delta t^3 \xi \omega + 4 r_{t+\gamma\Delta t} \beta_1 \beta_2 \gamma^3 \Delta t^3 \xi \omega +$$
$$4 x_t \beta_1 \beta_2 \gamma^3 \Delta t^3 \xi \omega + 4 r_{t+\gamma\Delta t} \beta_2^2 \gamma^3 \Delta t^3 \xi \omega + 4 x_t \beta_2^2 \gamma^3 \Delta t^3 \xi \omega + 4 x_t \beta_2 \Delta t^2 \omega^2 -$$
$$4 x_t \beta_2^2 \Delta t^2 \omega^2 + 2 x_t \gamma \Delta t^2 \omega^2 - 10 x_t \beta_2 \gamma \Delta t^2 \omega^2 + 4 x_t \beta_1 \beta_2 \gamma \Delta t^2 \omega^2 + 8 x_t \beta_2^2 \gamma \Delta t^2 \omega^2 -$$
$$3 x_t \gamma^2 \Delta t^2 \omega^2 + 2 x_t \beta_1 \gamma^2 \Delta t^2 \omega^2 + 6 x_t \beta_2 \gamma^2 \Delta t^2 \omega^2 - 4 x_t \beta_1 \beta_2 \gamma^2 \Delta t^2 \omega^2 -$$
$$4 x_t \beta_2^2 \gamma^2 \Delta t^2 \omega^2 + 4 \dot{x}_t \beta_2 \gamma \Delta t^3 \omega^2 - 4 \dot{x}_t \beta_2^2 \gamma \Delta t^3 \omega^2 + \dot{x}_t \gamma^2 \Delta t^3 \omega^2 -$$
$$10 \dot{x}_t \beta_2 \gamma^2 \Delta t^3 \omega^2 + 4 \dot{x}_t \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^2 + 8 \dot{x}_t \beta_2^2 \gamma^2 \Delta t^3 \omega^2 - 2 \dot{x}_t \gamma^3 \Delta t^3 \omega^2 +$$
$$2 \dot{x}_t \beta_1 \gamma^3 \Delta t^3 \omega^2 + 6 \dot{x}_t \beta_2 \gamma^3 \Delta t^3 \omega^2 - 4 \dot{x}_t \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^2 - 4 \dot{x}_t \beta_2^2 \gamma^3 \Delta t^3 \omega^2 +$$
$$x_t \beta_2 \gamma^2 \Delta t^4 \omega^2 - r_{t+\Delta t} \beta_2^2 \gamma^2 \Delta t^4 \omega^2 - x_t \beta_2^2 \gamma^2 \Delta t^4 \omega^2 - 3 x_t \beta_2 \gamma^3 \Delta t^4 \omega^2 +$$
$$2 x_t \beta_1 \beta_2 \gamma^3 \Delta t^4 \omega^2 + 2 r_{t+\Delta t} \beta_2^2 \gamma^3 \Delta t^4 \omega^2 + 2 x_t \beta_2^2 \gamma^3 \Delta t^4 \omega^2 + 2 x_t \beta_2 \gamma^4 \Delta t^4 \omega^2 -$$
$$2 x_t \beta_1 \beta_2 \gamma^4 \Delta t^4 \omega^2 - r_{t+\Delta t} \beta_2^2 \gamma^4 \Delta t^4 \omega^2 - x_t \beta_2^2 \gamma^4 \Delta t^4 \omega^2 + 4 x_t \beta_2 \gamma \Delta t^3 \xi \omega^3 -$$
$$4 x_t \beta_2^2 \gamma \Delta t^3 \xi \omega^3 - 10 x_t \beta_2 \gamma^2 \Delta t^3 \xi \omega^3 + 4 x_t \beta_1 \beta_2 \gamma^2 \Delta t^3 \xi \omega^3 + 8 x_t \beta_2^2 \gamma^2 \Delta t^3 \xi \omega^3 +$$
$$6 x_t \beta_2 \gamma^3 \Delta t^3 \xi \omega^3 - 4 x_t \beta_1 \beta_2 \gamma^3 \Delta t^3 \xi \omega^3 - 4 x_t \beta_2^2 \gamma^3 \Delta t^3 \xi \omega^3 + 2 \dot{x}_t \beta_2 \gamma^2 \Delta t^4 \xi \omega^3 -$$
$$2 \dot{x}_t \beta_2^2 \gamma^2 \Delta t^4 \xi \omega^3 - 6 \dot{x}_t \beta_2 \gamma^3 \Delta t^4 \xi \omega^3 + 4 \dot{x}_t \beta_1 \beta_2 \gamma^3 \Delta t^4 \xi \omega^3 + 4 \dot{x}_t \beta_2^2 \gamma^3 \Delta t^4 \xi \omega^3 +$$
$$4 \dot{x}_t \beta_2 \gamma^4 \Delta t^4 \xi \omega^3 - 4 \dot{x}_t \beta_1 \beta_2 \gamma^4 \Delta t^4 \xi \omega^3 - 2 \dot{x}_t \beta_2^2 \gamma^4 \Delta t^4 \xi \omega^3\Big) \Big/ \Big(\big(4 + \gamma^2 \Delta t^2 \omega^2\big)$$
$$\big(1 + 2 \beta_2 \Delta t \xi \omega - 2 \beta_2 \gamma \Delta t \xi \omega + \beta_2^2 \Delta t^2 \omega^2 - 2 \beta_2^2 \gamma \Delta t^2 \omega^2 + \beta_2^2 \gamma^2 \Delta t^2 \omega^2\big)\Big)\Big)\Big\}\Big\}$$

```
recursive =
  {
   Collect[ẍ_{t+Δt} -
      (- 1/(-1 - β₂² (1 - γ)² Δt² ω²) (r_{t+Δt} + β₂ (1 - γ) Δt ω² (-ẋ_t + ((1 - β₂) (1 - γ) Δt (-4 r_{t+γΔt} +

            4 x_t ω² + 4 ẋ_t γ Δt ω² + ẍ_t γ² Δt² ω²)) / (4 + γ² Δt² ω²) - γ Δt (ẍ_t (1 - β₁) -
            (β₁ (-4 r_{t+γΔt} + 4 x_t ω² + 4 ẋ_t γ Δt ω² + ẍ_t γ² Δt² ω²)) / (4 + γ² Δt² ω²))) +
         ω² (-x_t + ((1 - β₂) (1 - γ) Δt (-4 ẋ_t - 2 r_{t+γΔt} γ Δt - 2 ẍ_t γ Δt +
            2 x_t γ Δt ω² + ẋ_t γ² Δt² ω²)) / (4 + γ² Δt² ω²) -
         γ Δt (ẋ_t (1 - β₁) - (β₁ (-4 ẋ_t - 2 r_{t+γΔt} γ Δt - 2 ẍ_t γ Δt + 2 x_t γ Δt ω² +

            ẋ_t γ² Δt² ω²)) / (4 + γ² Δt² ω²)))))), {ẍ_t, ẋ_t, x_t, r_{t+Δt}, r_{t+γΔt}}],

   Collect[ẋ_{t+Δt} - (- (((-4 ẋ_t - 4 r_{t+γΔt} Δt + 4 r_{t+γΔt} β₂ Δt - 4 r_{t+Δt} β₂ Δt + 4 r_{t+γΔt} γ Δt -
         4 ẍ_t γ Δt - 4 r_{t+γΔt} β₁ γ Δt + 4 ẍ_t β₁ γ Δt - 4 r_{t+γΔt} β₂ γ Δt + 4 r_{t+Δt} β₂ γ Δt +
         4 x_t Δt ω² - 4 x_t γ Δt ω² + 4 x_t β₁ γ Δt ω² + 4 ẋ_t β₂ Δt² ω² - 4 ẋ_t β₂² Δt² ω² +
         4 ẋ_t γ Δt² ω² - 8 ẋ_t β₂ γ Δt² ω² + 8 ẋ_t β₂² γ Δt² ω² - 5 ẋ_t γ² Δt² ω² +
         4 ẋ_t β₁ γ² Δt² ω² + 4 ẋ_t β₂ γ² Δt² ω² - 4 ẋ_t β₂² γ² Δt² ω² + 2 r_{t+γΔt} β₂ γ Δt³ ω² +
         2 ẍ_t β₂ γ Δt³ ω² - 2 r_{t+γΔt} β₂² γ Δt³ ω² - 2 ẍ_t β₂² γ Δt³ ω² + ẍ_t γ² Δt³ ω² -
         4 r_{t+γΔt} β₂ γ² Δt³ ω² - r_{t+Δt} β₂ γ² Δt³ ω² - 5 ẍ_t β₂ γ² Δt³ ω² +
         2 r_{t+γΔt} β₁ β₂ γ² Δt³ ω² + 2 ẍ_t β₁ β₂ γ² Δt³ ω² + 4 r_{t+γΔt} β₂² γ² Δt³ ω² +
         4 ẍ_t β₂² γ² Δt³ ω² - 2 ẍ_t γ³ Δt³ ω² + 2 ẍ_t β₁ γ³ Δt³ ω² + 2 r_{t+γΔt} β₂ γ³ Δt³ ω² +
         r_{t+Δt} β₂ γ³ Δt³ ω² + 3 ẍ_t β₂ γ³ Δt³ ω² - 2 r_{t+γΔt} β₁ β₂ γ³ Δt³ ω² -
         2 ẍ_t β₁ β₂ γ³ Δt³ ω² - 2 r_{t+γΔt} β₂² γ³ Δt³ ω² - 2 ẍ_t β₂² γ³ Δt³ ω² - 2 x_t β₂ γ Δt³ ω⁴ +
         2 x_t β₂² γ Δt³ ω⁴ + 5 x_t β₂ γ² Δt³ ω⁴ - 2 x_t β₁ β₂ γ² Δt³ ω⁴ - 4 x_t β₂² γ² Δt³ ω⁴ -
         3 x_t β₂ γ³ Δt³ ω⁴ + 2 x_t β₁ β₂ γ³ Δt³ ω⁴ + 2 x_t β₂² γ³ Δt³ ω⁴ - ẋ_t β₂ γ² Δt⁴ ω⁴ +
         ẋ_t β₂² γ² Δt⁴ ω⁴ + 3 ẋ_t β₂ γ³ Δt⁴ ω⁴ - 2 ẋ_t β₁ β₂ γ³ Δt⁴ ω⁴ - 2 ẋ_t β₂² γ³ Δt⁴ ω⁴ -
         2 ẋ_t β₂ γ⁴ Δt⁴ ω⁴ + 2 ẋ_t β₁ β₂ γ⁴ Δt⁴ ω⁴ + ẋ_t β₂² γ⁴ Δt⁴ ω⁴) / ((4 + γ² Δt² ω²)
         (1 + β₂² Δt² ω² - 2 β₂² γ Δt² ω² + β₂² γ² Δt² ω²))))), {ẍ_t, ẋ_t, x_t, r_{t+Δt}, r_{t+γΔt}}],
   Collect[x_{t+Δt} - (- (((-4 x_t - 4 ẋ_t Δt - 4 r_{t+γΔt} β₂ Δt² + 4 r_{t+γΔt} β₂² Δt² - 4 r_{t+Δt} β₂² Δt² -
         2 r_{t+γΔt} γ Δt² - 2 ẍ_t γ Δt² + 10 r_{t+γΔt} β₂ γ Δt² - 2 ẍ_t β₂ γ Δt² -
         4 r_{t+γΔt} β₁ β₂ γ Δt² + 4 ẍ_t β₁ β₂ γ Δt² - 8 r_{t+γΔt} β₂² γ Δt² + 8 r_{t+Δt} β₂² γ Δt² +
         2 r_{t+γΔt} γ² Δt² + 2 ẍ_t γ² Δt² - 2 r_{t+γΔt} β₁ γ² Δt² - 2 ẍ_t β₁ γ² Δt² -
         6 r_{t+γΔt} β₂ γ² Δt² + 2 ẍ_t β₂ γ² Δt² + 4 r_{t+γΔt} β₁ β₂ γ² Δt² - 4 ẍ_t β₁ β₂ γ² Δt² +
         4 r_{t+γΔt} β₂² γ² Δt² - 4 r_{t+Δt} β₂² γ² Δt² + 4 x_t β₂ Δt² ω² - 4 x_t β₂² Δt² ω² +
         2 x_t γ Δt² ω² - 10 x_t β₂ γ Δt² ω² + 4 x_t β₁ β₂ γ Δt² ω² + 8 x_t β₂² γ Δt² ω² -
         3 x_t γ² Δt² ω² + 2 x_t β₁ γ² Δt² ω² + 6 x_t β₂ γ² Δt² ω² - 4 x_t β₁ β₂ γ² Δt² ω² -
         4 x_t β₂² γ² Δt² ω² + 4 ẋ_t β₂ γ Δt³ ω² - 4 ẋ_t β₂² γ Δt³ ω² + ẋ_t γ² Δt³ ω² -
         10 ẋ_t β₂ γ² Δt³ ω² + 4 ẋ_t β₁ β₂ γ² Δt³ ω² + 8 ẋ_t β₂² γ² Δt³ ω² - 2 ẋ_t γ³ Δt³ ω² +
         2 ẋ_t β₁ γ³ Δt³ ω² + 6 ẋ_t β₂ γ³ Δt³ ω² - 4 ẋ_t β₁ β₂ γ³ Δt³ ω² - 4 ẋ_t β₂² γ³ Δt³ ω² +
         ẍ_t β₂ γ² Δt⁴ ω² - r_{t+Δt} β₂² γ² Δt⁴ ω² - ẍ_t β₂² γ² Δt⁴ ω² - 3 ẍ_t β₂ γ³ Δt⁴ ω² +
         2 ẍ_t β₁ β₂ γ³ Δt⁴ ω² + 2 r_{t+Δt} β₂² γ³ Δt⁴ ω² + 2 ẍ_t β₂² γ³ Δt⁴ ω² + 2 ẍ_t β₂ γ⁴ Δt⁴ ω² -
         2 ẍ_t β₁ β₂ γ⁴ Δt⁴ ω² - r_{t+Δt} β₂² γ⁴ Δt⁴ ω² - ẍ_t β₂² γ⁴ Δt⁴ ω²) / ((4 + γ² Δt² ω²)
         (1 + β₂² Δt² ω² - 2 β₂² γ Δt² ω² + β₂² γ² Δt² ω²))))), {ẍ_t, ẋ_t, x_t, r_{t+γΔt}, r_{t+Δt}}]
   } /. {-1 - β₂² (1 - γ)² Δt² ω² → η₁ , 4 + γ² Δt² ω² → η₂,
     1 + β₂² Δt² ω² - 2 β₂² γ Δt² ω² + β₂² γ² Δt² ω² → η₃}
```

$$\Big\{ x_{t+\Delta t} + \frac{r_{t+\Delta t}}{\eta_1} + r_{t+\gamma\Delta t} \Big(-\frac{4 (1-\beta_2) \beta_2 (1-\gamma)^2 \Delta t^2 \omega^2}{\eta_1 \eta_2} -$$

$$\frac{2 (1-\beta_2) (1-\gamma) \gamma \Delta t^2 \omega^2}{\eta_1 \eta_2} - \frac{4 \beta_1 \beta_2 (1-\gamma) \gamma \Delta t^2 \omega^2}{\eta_1 \eta_2} - \frac{2 \beta_1 \gamma^2 \Delta t^2 \omega^2}{\eta_1 \eta_2} \Big) +$$

$$x_t \Big(-\frac{\omega^2}{\eta_1} + \frac{4 (1-\beta_2) \beta_2 (1-\gamma)^2 \Delta t^2 \omega^4}{\eta_1 \eta_2} + \frac{2 (1-\beta_2) (1-\gamma) \gamma \Delta t^2 \omega^4}{\eta_1 \eta_2} +$$

$$\frac{4 \beta_1 \beta_2 (1-\gamma) \gamma \Delta t^2 \omega^4}{\eta_1 \eta_2} + \frac{2 \beta_1 \gamma^2 \Delta t^2 \omega^4}{\eta_1 \eta_2} \Big) +$$

$$\dot{x}_t \Big(-\frac{\beta_2 (1-\gamma) \Delta t \omega^2}{\eta_1} + \frac{(-1+\beta_1) \gamma \Delta t \omega^2}{\eta_1} - \frac{4 (1-\beta_2) (1-\gamma) \Delta t \omega^2}{\eta_1 \eta_2} -$$

$$\frac{4 \beta_1 \gamma \Delta t \omega^2}{\eta_1 \eta_2} + \frac{4 (1-\beta_2) \beta_2 (1-\gamma)^2 \gamma \Delta t^3 \omega^4}{\eta_1 \eta_2} +$$

$$\frac{(1-\beta_2) (1-\gamma) \gamma^2 \Delta t^3 \omega^4}{\eta_1 \eta_2} + \frac{4 \beta_1 \beta_2 (1-\gamma) \gamma^2 \Delta t^3 \omega^4}{\eta_1 \eta_2} + \frac{\beta_1 \gamma^3 \Delta t^3 \omega^4}{\eta_1 \eta_2} \Big) +$$

$$x_t \Big(\frac{(-1+\beta_1) \beta_2 (1-\gamma) \gamma \Delta t^2 \omega^2}{\eta_1} - \frac{2 (1-\beta_2) (1-\gamma) \gamma \Delta t^2 \omega^2}{\eta_1 \eta_2} - \frac{2 \beta_1 \gamma^2 \Delta t^2 \omega^2}{\eta_1 \eta_2} +$$

$$\frac{(1-\beta_2) \beta_2 (1-\gamma)^2 \gamma^2 \Delta t^4 \omega^4}{\eta_1 \eta_2} + \frac{\beta_1 \beta_2 (1-\gamma) \gamma^3 \Delta t^4 \omega^4}{\eta_1 \eta_2} \Big),$$

$$\dot{x}_{t+\Delta t} + r_{t+\Delta t} \Big(-\frac{4 \beta_2 \Delta t}{\eta_2 \eta_3} + \frac{4 \beta_2 \gamma \Delta t}{\eta_2 \eta_3} - \frac{\beta_2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{\beta_2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} \Big) +$$

$$r_{t+\gamma\Delta t} \Big(-\frac{4 \Delta t}{\eta_2 \eta_3} + \frac{4 \beta_2 \Delta t}{\eta_2 \eta_3} + \frac{4 \gamma \Delta t}{\eta_2 \eta_3} - \frac{4 \beta_1 \gamma \Delta t}{\eta_2 \eta_3} - \frac{4 \beta_2 \gamma \Delta t}{\eta_2 \eta_3} +$$

$$\frac{2 \beta_2 \gamma \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_2^2 \gamma \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{4 \beta_2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{2 \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} +$$

$$\frac{4 \beta_2^2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{2 \beta_2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_2^2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} \Big) +$$

$$x_t \Big(-\frac{4 \gamma \Delta t}{\eta_2 \eta_3} + \frac{4 \beta_1 \gamma \Delta t}{\eta_2 \eta_3} + \frac{2 \beta_2 \gamma \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_2^2 \gamma \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{\gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} -$$

$$\frac{5 \beta_2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{2 \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{4 \beta_2^2 \gamma^2 \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} +$$

$$\frac{2 \beta_1 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} + \frac{3 \beta_2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_2^2 \gamma^3 \Delta t^3 \omega^2}{\eta_2 \eta_3} \Big) +$$

$$x_t \Big(\frac{4 \Delta t \omega^2}{\eta_2 \eta_3} - \frac{4 \gamma \Delta t \omega^2}{\eta_2 \eta_3} + \frac{4 \beta_1 \gamma \Delta t \omega^2}{\eta_2 \eta_3} - \frac{2 \beta_2 \gamma \Delta t^3 \omega^4}{\eta_2 \eta_3} + \frac{2 \beta_2^2 \gamma \Delta t^3 \omega^4}{\eta_2 \eta_3} + \frac{5 \beta_2 \gamma^2 \Delta t^3 \omega^4}{\eta_2 \eta_3} -$$

$$\frac{2 \beta_1 \beta_2 \gamma^2 \Delta t^3 \omega^4}{\eta_2 \eta_3} - \frac{4 \beta_2^2 \gamma^2 \Delta t^3 \omega^4}{\eta_2 \eta_3} - \frac{3 \beta_2 \gamma^3 \Delta t^3 \omega^4}{\eta_2 \eta_3} + \frac{2 \beta_1 \beta_2 \gamma^3 \Delta t^3 \omega^4}{\eta_2 \eta_3} + \frac{2 \beta_2^2 \gamma^3 \Delta t^3 \omega^4}{\eta_2 \eta_3} \Big) +$$

$$\dot{x}_t \Big(-\frac{4}{\eta_2 \eta_3} + \frac{4 \beta_2 \Delta t^2 \omega^2}{\eta_2 \eta_3} - \frac{4 \beta_2^2 \Delta t^2 \omega^2}{\eta_2 \eta_3} + \frac{4 \gamma \Delta t^2 \omega^2}{\eta_2 \eta_3} - \frac{8 \beta_2 \gamma \Delta t^2 \omega^2}{\eta_2 \eta_3} + \frac{8 \beta_2^2 \gamma \Delta t^2 \omega^2}{\eta_2 \eta_3} -$$

$$\frac{5 \gamma^2 \Delta t^2 \omega^2}{\eta_2 \eta_3} + \frac{4 \beta_1 \gamma^2 \Delta t^2 \omega^2}{\eta_2 \eta_3} + \frac{4 \beta_2 \gamma^2 \Delta t^2 \omega^2}{\eta_2 \eta_3} - \frac{4 \beta_2^2 \gamma^2 \Delta t^2 \omega^2}{\eta_2 \eta_3} -$$

$$\frac{\beta_2 \gamma^2 \Delta t^4 \omega^4}{\eta_2 \eta_3} + \frac{\beta_2^2 \gamma^2 \Delta t^4 \omega^4}{\eta_2 \eta_3} + \frac{3 \beta_2 \gamma^3 \Delta t^4 \omega^4}{\eta_2 \eta_3} - \frac{2 \beta_1 \beta_2 \gamma^3 \Delta t^4 \omega^4}{\eta_2 \eta_3} -$$

$$\frac{2 \beta_2^2 \gamma^3 \Delta t^4 \omega^4}{\eta_2 \eta_3} - \frac{2 \beta_2 \gamma^4 \Delta t^4 \omega^4}{\eta_2 \eta_3} + \frac{2 \beta_1 \beta_2 \gamma^4 \Delta t^4 \omega^4}{\eta_2 \eta_3} + \frac{\beta_2^2 \gamma^4 \Delta t^4 \omega^4}{\eta_2 \eta_3} \Big),$$

$$x_{t+\Delta t} + r_{t+\gamma\Delta t} \Big(-\frac{4 \beta_2 \Delta t^2}{\eta_2 \eta_3} + \frac{4 \beta_2^2 \Delta t^2}{\eta_2 \eta_3} - \frac{2 \gamma \Delta t^2}{\eta_2 \eta_3} + \frac{10 \beta_2 \gamma \Delta t^2}{\eta_2 \eta_3} - \frac{4 \beta_1 \beta_2 \gamma \Delta t^2}{\eta_2 \eta_3} -$$

$$\frac{8\,\beta_2^2\,\gamma\,\Delta t^2}{\eta_2\,\eta_3} + \frac{2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} - \frac{2\,\beta_1\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} - \frac{6\,\beta_2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} + \frac{4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} + \frac{4\,\beta_2^2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3}\Bigg) +$$

$$x_t\Bigg(-\frac{4}{\eta_2\,\eta_3} + \frac{4\,\beta_2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_2^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\gamma\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} - \frac{10\,\beta_2\,\gamma\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} +$$

$$\frac{4\,\beta_1\,\beta_2\,\gamma\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} + \frac{8\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} - \frac{3\,\gamma^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\beta_1\,\gamma^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} +$$

$$\frac{6\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3}\Bigg) +$$

$$\dot{x}_t\Bigg(-\frac{4\,\Delta t}{\eta_2\,\eta_3} + \frac{4\,\beta_2\,\gamma\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_2^2\,\gamma\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} + \frac{\gamma^2\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} - \frac{10\,\beta_2\,\gamma^2\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} +$$

$$\frac{4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} + \frac{8\,\beta_2^2\,\gamma^2\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} - \frac{2\,\gamma^3\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\beta_1\,\gamma^3\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} +$$

$$\frac{6\,\beta_2\,\gamma^3\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_1\,\beta_2\,\gamma^3\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3} - \frac{4\,\beta_2^2\,\gamma^3\,\Delta t^3\,\omega^2}{\eta_2\,\eta_3}\Bigg) +$$

$$r_{t+\Delta t}\Bigg(-\frac{4\,\beta_2^2\,\Delta t^2}{\eta_2\,\eta_3} + \frac{8\,\beta_2^2\,\gamma\,\Delta t^2}{\eta_2\,\eta_3} - \frac{4\,\beta_2^2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} - \frac{\beta_2^2\,\gamma^2\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} - \frac{\beta_2^2\,\gamma^4\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3}\Bigg) +$$

$$x_t\Bigg(-\frac{2\,\gamma\,\Delta t^2}{\eta_2\,\eta_3} - \frac{2\,\beta_2\,\gamma\,\Delta t^2}{\eta_2\,\eta_3} + \frac{4\,\beta_1\,\beta_2\,\gamma\,\Delta t^2}{\eta_2\,\eta_3} + \frac{2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} - \frac{2\,\beta_1\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} + \frac{2\,\beta_2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} -$$

$$\frac{4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2}{\eta_2\,\eta_3} + \frac{\beta_2\,\gamma^2\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} - \frac{\beta_2^2\,\gamma^2\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} - \frac{3\,\beta_2\,\gamma^3\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\beta_1\,\beta_2\,\gamma^3\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} +$$

$$\frac{2\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} + \frac{2\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} - \frac{2\,\beta_1\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3} - \frac{\beta_2^2\,\gamma^4\,\Delta t^4\,\omega^2}{\eta_2\,\eta_3}\Bigg)\Bigg\}$$

```
a =
    ⎛ Coefficient[Part[recursive, 1], ẍₜ]  Coefficient[Part[recursive, 1], ẋₜ]  Coeff
  - ⎜ Coefficient[Part[recursive, 2], ẍₜ]  Coefficient[Part[recursive, 2], ẋₜ]  Coeff
    ⎝ Coefficient[Part[recursive, 3], ẍₜ]  Coefficient[Part[recursive, 3], ẋₜ]  Coeff

    ⎞
    ⎟ ;
    ⎠
```

```
a // MatrixForm // Simplify
```

$$\left(\begin{array}{}
\frac{\gamma\,\Delta t^2\,\omega^2\,((1+\beta_2\,(-1+\gamma)+(-1+\beta_1)\,\gamma)\,(2+\beta_2\,(-1+\gamma)\,\gamma\,\Delta t^2\,\omega^2)+(-1+\beta_1)\,\beta_2\,(-1+\gamma)\,\eta_2)}{\eta_1\,\eta_2} & \\
\frac{\gamma\,\Delta t\,(4+2\,\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2 - \gamma\,\Delta t^2\,\omega^2 + 2\,\gamma^2\,\Delta t^2\,\omega^2 + \beta_2\,(-2+5\,\gamma-3\,\gamma^2)\,\Delta t^2\,\omega^2 + 2\,\beta_1\,(-2+\beta_2\,(-1+\gamma)\,\gamma\,\Delta t^2\,\omega^2 - \gamma^2\,\Delta t^2\,\omega^2))}{\eta_2\,\eta_3} & \frac{4-4\,\gamma\,\Delta t^2\,\omega^2+(5-}{} \\
\frac{\gamma\,\Delta t^2\,(2+2\,(-1+\beta_1)\,\gamma+\beta_2^2\,(-1+\gamma)^2\,\gamma\,\Delta t^2\,\omega^2+\beta_2\,(-1+\gamma)\,(-2+\gamma\,\Delta t^2\,\omega^2-2\,\gamma^2\,\Delta t^2\,\omega^2+2\,\beta_1\,(2+\gamma^2\,\Delta t^2\,\omega^2)))}{\eta_2\,\eta_3} & \frac{\Delta t}{}
\end{array}\right.$$

```
(*
Truncating the acceleration terms, we get the 2x2 matrix a0 as follows:
*)
```

```
a0 = - ⎛ Coefficient[Part[recursive, 2], ẋₜ]  Coefficient[Part[recursive, 2], xₜ] ⎞ ;
       ⎝ Coefficient[Part[recursive, 3], ẋₜ]  Coefficient[Part[recursive, 3], xₜ] ⎠
```

```
a0 // MatrixForm // Simplify
```

$$\left(\begin{array}{}
\frac{4-4\,\gamma\,\Delta t^2\,\omega^2+(5-4\,\beta_1)\,\gamma^2\,\Delta t^2\,\omega^2-\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2\,(-4+\gamma^2\,\Delta t^2\,\omega^2)-\beta_2\,(-1+\gamma)\,\Delta t^2\,\omega^2\,(-4+4\,\gamma+\gamma^2\,\Delta t^2\,\omega^2+2\,(-1+\beta_1)\,\gamma^3\,\Delta t^2\,\omega^2)}{\eta_2\,\eta_3} & \frac{\Delta t\,\omega^2\,(-}{} \\
\frac{\Delta t\,(4+4\,\beta_2^2\,(-1+\gamma)^2\,\gamma\,\Delta t^2\,\omega^2-\gamma^2\,\Delta t^2\,\omega^2-2\,(-1+\beta_1)\,\gamma^3\,\Delta t^2\,\omega^2+2\,\beta_2\,(-1+\gamma)\,\gamma\,(2+(-3+2\,\beta_1)\,\gamma)\,\Delta t^2\,\omega^2)}{\eta_2\,\eta_3} & 4
\end{array}\right.$$

(*

So answer for Q1 (a) is: $A_0 = \begin{pmatrix} A_{0,11} & A_{0,12} \\ A_{0,21} & A_{0,22} \end{pmatrix}$

where:

$$A_{0,11} = \frac{1}{\eta_2\,\eta_3}\left(4 - 4\,\gamma\,\Delta t^2\,\omega^2 + (5 - 4\,\beta_1)\,\gamma^2\,\Delta t^2\,\omega^2 - \beta_2^2\,\left(-1+\gamma\right)^2\,\Delta t^2\,\omega^2\,\left(-4+\gamma^2\,\Delta t^2\,\omega^2\right) - \beta_2\,\left(-1+\gamma\right)\,\Delta t^2\,\omega^2\,\left(-4 + 4\,\gamma + \gamma^2\,\Delta t^2\,\omega^2 + 2\,\left(-1+\beta_1\right)\,\gamma^3\,\Delta t^2\,\omega^2\right)\right)$$

$$A_{0,12} = \frac{\Delta t\,\omega^2\,\left(-4 + \beta_2\,\left(-5 + 2\,\beta_1 + 4\,\beta_2\right)\,\gamma^2\,\Delta t^2\,\omega^2 + (3 - 2\,\beta_1 - 2\,\beta_2)\,\beta_2\,\gamma^3\,\Delta t^2\,\omega^2 + \gamma\,\left(4 - 4\,\beta_1 + 2\,\beta_2\,\Delta t^2\,\omega^2 - 2\,\beta_2^2\,\Delta t^2\,\omega^2\right)\right)}{\eta_2\,\eta_3}$$

$$A_{0,21} = \frac{\Delta t\,\left(4 + 4\,\beta_2^2\,\left(-1+\gamma\right)^2\,\gamma\,\Delta t^2\,\omega^2 - \gamma^2\,\Delta t^2\,\omega^2 - 2\,\left(-1+\beta_1\right)\,\gamma^3\,\Delta t^2\,\omega^2 + 2\,\beta_2\,\left(-1+\gamma\right)\,\gamma\,\left(2 + (-3 + 2\,\beta_1)\,\gamma\right)\,\Delta t^2\,\omega^2\right)}{\eta_2\,\eta_3}$$

$$A_{0,22} = \frac{4 + 4\,\beta_2^2\,\left(-1+\gamma\right)^2\,\Delta t^2\,\omega^2 - 2\,\gamma\,\Delta t^2\,\omega^2 + (3 - 2\,\beta_1)\,\gamma^2\,\Delta t^2\,\omega^2 + 2\,\beta_2\,\left(-1+\gamma\right)\,\left(2 + (-3 + 2\,\beta_1)\,\gamma\right)\,\Delta t^2\,\omega^2}{\eta_2\,\eta_3}$$

and:

$\rightarrow\eta_1 = -1 - \beta_2^2\,\left(1-\gamma\right)^2\,\Delta t^2\,\omega^2$,

$\rightarrow\eta_2 = 4 + \gamma^2\,\Delta t^2\,\omega^2$,

$\rightarrow\eta_3 = 1 + \beta_2^2\,\Delta t^2\,\omega^2 - 2\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2 + \beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2$

*)