Q1 a)

```
<< Notation`
```

```
Symbolize[ x_{t+Δt} ]
```

```
Symbolize[ ẋ_{t+Δt} ]
```

```
Symbolize[ ẍ_{t+Δt} ]
```

```
Symbolize[ x_{t+γΔt} ]
```

```
Symbolize[ ẋ_{t+γΔt} ]
```

```
Symbolize[ ẍ_{t+γΔt} ]
```

```
Symbolize[ x_t ]
```

```
Symbolize[ ẋ_t ]
```

```
Symbolize[ ẍ_t ]
```

```
Symbolize[ r_{t+Δt} ]
```

```
Symbolize[ r_{t+γΔt} ]
```

```
Symbolize[ Ω_o ]
```

```
Symbolize[ Ω̄_d ]
```

```
Symbolize[ ξ̄ ]
```

```
Symbolize[ β_1 ]
```

```
Symbolize[ β_2 ]
```

```
ClearAll["Global`*"]
```

```
ξ = 0;
```

```
Column[Collect[{ẍ_{t+γΔt} + 2 ξ ω ẋ_{t+γΔt} + ω² x_{t+γΔt} == r_{t+γΔt} ,
```

$$x_{t+\gamma\Delta t} == x_t + \frac{\gamma \Delta t}{2} \left(\dot{x}_t + \dot{x}_{t+\gamma\Delta t}\right),$$

$$\dot{x}_{t+\gamma\Delta t} == \dot{x}_t + \frac{\gamma \Delta t}{2} \left(\ddot{x}_t + \ddot{x}_{t+\gamma\Delta t}\right),$$

$$\ddot{x}_{t+\Delta t} + 2 \xi \omega \dot{x}_{t+\Delta t} + \omega^2 x_{t+\Delta t} == r_{t+\Delta t},$$

$$x_{t+\Delta t} == x_t + \gamma \Delta t \left(\left(1 - \beta_1\right) \dot{x}_t + \beta_1 \dot{x}_{t+\gamma\Delta t}\right) + \left(1 - \gamma\right) \Delta t \left(\left(1 - \beta_2\right) \dot{x}_{t+\gamma\Delta t} + \beta_2 \dot{x}_{t+\Delta t}\right),$$

$$\dot{x}_{t+\Delta t} == \dot{x}_t + \gamma \Delta t \left(\left(1 - \beta_1\right) \ddot{x}_t + \beta_1 \ddot{x}_{t+\gamma\Delta t}\right) + \left(1 - \gamma\right) \Delta t \left(\left(1 - \beta_2\right) \ddot{x}_{t+\gamma\Delta t} + \beta_2 \ddot{x}_{t+\Delta t}\right)\},$$

$$\{\ddot{x}_{t+\gamma\Delta t}, \dot{x}_{t+\gamma\Delta t}, x_{t+\gamma\Delta t}, \ddot{x}_{t+\Delta t}, \dot{x}_{t+\Delta t}, x_{t+\Delta t}\}]]$$

$x_{t+\gamma\Delta t} + 2 \dot{x}_{t+\gamma\Delta t} \xi \omega + x_{t+\gamma\Delta t} \omega^2 == r_{t+\gamma\Delta t}$

$x_{t+\gamma\Delta t} == x_t + \frac{1}{2} \dot{x}_t \gamma \Delta t + \frac{1}{2} \dot{x}_{t+\gamma\Delta t} \gamma \Delta t$

$\dot{x}_{t+\gamma\Delta t} == \dot{x}_t + \frac{1}{2} x_t \gamma \Delta t + \frac{1}{2} x_{t+\gamma\Delta t} \gamma \Delta t$

$x_{t+\Delta t} + 2 \dot{x}_{t+\Delta t} \xi \omega + x_{t+\Delta t} \omega^2 == r_{t+\Delta t}$

$x_{t+\Delta t} == x_t + \dot{x}_{t+\Delta t} \beta_2 \left(1 - \gamma\right) \Delta t + \dot{x}_t \left(1 - \beta_1\right) \gamma \Delta t + \dot{x}_{t+\gamma\Delta t} \left(\left(1 - \beta_2\right) \left(1 - \gamma\right) \Delta t + \beta_1 \gamma \Delta t\right)$

$\dot{x}_{t+\Delta t} == \dot{x}_t + x_{t+\Delta t} \beta_2 \left(1 - \gamma\right) \Delta t + x_t \left(1 - \beta_1\right) \gamma \Delta t + x_{t+\gamma\Delta t} \left(\left(1 - \beta_2\right) \left(1 - \gamma\right) \Delta t + \beta_1 \gamma \Delta t\right)$

```
Collect[x_{t+Δt} - ẋ_{t+Δt} β₂ (1 - γ) Δt - ẋ_{t+γΔt} ((1 - β₂) (1 - γ) Δt + β₁ γ Δt) == x_t + ẋ_t (1 - β₁) γ Δt,
    {ẍ_{t+γΔt}, ẋ_{t+γΔt}, x_{t+γΔt}, ẍ_{t+Δt}, ẋ_{t+Δt}, x_{t+Δt}}]
```

$x_{t+\Delta t} - \dot{x}_{t+\Delta t} \beta_2 \left(1 - \gamma\right) \Delta t + \dot{x}_{t+\gamma\Delta t} \left(- \left(1 - \beta_2\right) \left(1 - \gamma\right) \Delta t - \beta_1 \gamma \Delta t\right) == x_t + \dot{x}_t \left(1 - \beta_1\right) \gamma \Delta t$

```
Collect[ẋ_{t+Δt} - ẍ_{t+Δt} β₂ (1 - γ) Δt - ẍ_{t+γΔt} ((1 - β₂) (1 - γ) Δt + β₁ γ Δt) ==
    ẋ_t + ẍ_t (1 - β₁) γ Δt, {ẍ_{t+γΔt}, ẋ_{t+γΔt}, x_{t+γΔt}, ẍ_{t+Δt}, ẋ_{t+Δt}, x_{t+Δt}}]
```

$\dot{x}_{t+\Delta t} - x_{t+\Delta t} \beta_2 \left(1 - \gamma\right) \Delta t + x_{t+\gamma\Delta t} \left(- \left(1 - \beta_2\right) \left(1 - \gamma\right) \Delta t - \beta_1 \gamma \Delta t\right) == \dot{x}_t + x_t \left(1 - \beta_1\right) \gamma \Delta t$

```
Solve[
```

$$\ddot{x}_{t+\gamma\Delta t} + 2 \xi \omega \dot{x}_{t+\gamma\Delta t} + \omega^2 x_{t+\gamma\Delta t} == r_{t+\gamma\Delta t} \;\&\&$$

$$x_{t+\gamma\Delta t} == x_t + \frac{\gamma \Delta t}{2} \left(\dot{x}_t + \dot{x}_{t+\gamma\Delta t}\right) \;\&\&$$

$$\dot{x}_{t+\gamma\Delta t} == \dot{x}_t + \frac{\gamma \Delta t}{2} \left(\ddot{x}_t + \ddot{x}_{t+\gamma\Delta t}\right) \;\&\&$$

$$\ddot{x}_{t+\Delta t} + 2 \xi \omega \dot{x}_{t+\Delta t} + \omega^2 x_{t+\Delta t} == r_{t+\Delta t} \;\&\&$$

$$x_{t+\Delta t} == x_t + \gamma \Delta t \left(\left(1 - \beta_1\right) \dot{x}_t + \beta_1 \dot{x}_{t+\gamma\Delta t}\right) + \left(1 - \gamma\right) \Delta t \left(\left(1 - \beta_2\right) \dot{x}_{t+\gamma\Delta t} + \beta_2 \dot{x}_{t+\Delta t}\right) \;\&\&$$

$$\dot{x}_{t+\Delta t} == \dot{x}_t + \gamma \Delta t \left(\left(1 - \beta_1\right) \ddot{x}_t + \beta_1 \ddot{x}_{t+\gamma\Delta t}\right) + \left(1 - \gamma\right) \Delta t \left(\left(1 - \beta_2\right) \ddot{x}_{t+\gamma\Delta t} + \beta_2 \ddot{x}_{t+\Delta t}\right),$$

$$\{\ddot{x}_{t+\gamma\Delta t}, \dot{x}_{t+\gamma\Delta t}, x_{t+\gamma\Delta t}, \ddot{x}_{t+\Delta t}, \dot{x}_{t+\Delta t}, x_{t+\Delta t}\}]$$

$$
\begin{aligned}
&\Big\{\Big\{x_{t+\gamma\Delta t} \to -\Big(\big(-4\, r_{t+\gamma\Delta t} + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t\, \omega^2 + x_t\, \gamma^2\, \Delta t^2\, \omega^2\big)\big/\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\Big),\\
&\dot{x}_{t+\gamma\Delta t} \to -\Big(\big(-4\, \dot{x}_t - 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 2\, x_t\, \gamma\, \Delta t + 2\, x_t\, \gamma\, \Delta t\, \omega^2 + \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2\big)\big/\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\Big),\\
&\ddot{x}_{t+\gamma\Delta t} \to -\Big(\big(-4\, x_t - 4\, \dot{x}_t\, \gamma\, \Delta t - r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2 - x_t\, \gamma^2\, \Delta t^2\big)\big/\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\Big),\ x_{t+\Delta t} \to\\
&\quad -\Big(\big(-4\, r_{t+\Delta t} + 4\, x_t\, \omega^2 + 4\, \dot{x}_t\, \Delta t\, \omega^2 + 4\, r_{t+\gamma\Delta t}\, \beta_2\, \Delta t^2\, \omega^2 - 4\, r_{t+\gamma\Delta t}\, \beta_2^2\, \Delta t^2\, \omega^2 + 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t^2\, \omega^2 +\\
&\qquad 2\, x_t\, \gamma\, \Delta t^2\, \omega^2 - 10\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 2\, x_t\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 4\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 -\\
&\qquad 4\, x_t\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 8\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 - 2\, r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2\, \omega^2 - r_{t+\Delta t}\, \gamma^2\, \Delta t^2\, \omega^2 -\\
&\qquad 2\, x_t\, \gamma^2\, \Delta t^2\, \omega^2 + 2\, r_{t+\gamma\Delta t}\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2 + 2\, x_t\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2 + 6\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 -\\
&\qquad 2\, x_t\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 - 4\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 + 4\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 - 4\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 -\\
&\qquad 4\, x_t\, \beta_2\, \Delta t^2\, \omega^4 + 4\, x_t\, \beta_2^2\, \Delta t^2\, \omega^4 - 2\, x_t\, \gamma\, \Delta t^2\, \omega^4 + 10\, x_t\, \beta_2\, \gamma\, \Delta t^2\, \omega^4 - 4\, x_t\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^4 -\\
&\qquad 8\, x_t\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^4 + 3\, x_t\, \gamma^2\, \Delta t^2\, \omega^4 - 2\, x_t\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^4 - 6\, x_t\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^4 +\\
&\qquad 4\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^4 + 4\, x_t\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^4 - 4\, \dot{x}_t\, \beta_2\, \gamma\, \Delta t^3\, \omega^4 + 4\, \dot{x}_t\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^4 - \dot{x}_t\, \gamma^2\, \Delta t^3\, \omega^4 +\\
&\qquad 10\, \dot{x}_t\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 - 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 - 8\, \dot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^4 + 2\, \dot{x}_t\, \gamma^3\, \Delta t^3\, \omega^4 -\\
&\qquad 2\, \dot{x}_t\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^4 - 6\, \dot{x}_t\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 4\, \dot{x}_t\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^4 -\\
&\qquad x_t\, \beta_2\, \gamma^2\, \Delta t^4\, \omega^4 + x_t\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^4 + 3\, x_t\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4 - 2\, x_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4 -\\
&\qquad 2\, x_t\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^4 - 2\, x_t\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4 + 2\, x_t\, \beta_1\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4 + x_t\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^4\big)\Big/\\
&\qquad \Big(\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\big(1 + \beta_2^2\, \Delta t^2\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 + \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2\big)\Big)\Big),\\
&\dot{x}_{t+\Delta t} \to -\Big(\big(-4\, \dot{x}_t - 4\, r_{t+\gamma\Delta t}\, \Delta t + 4\, r_{t+\gamma\Delta t}\, \beta_2\, \Delta t - 4\, r_{t+\Delta t}\, \beta_2\, \Delta t + 4\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t - 4\, x_t\, \gamma\, \Delta t -\\
&\qquad 4\, r_{t+\gamma\Delta t}\, \beta_1\, \gamma\, \Delta t + 4\, x_t\, \beta_1\, \gamma\, \Delta t - 4\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma\, \Delta t + 4\, r_{t+\Delta t}\, \beta_2\, \gamma\, \Delta t + 4\, x_t\, \Delta t\, \omega^2 -\\
&\qquad 4\, x_t\, \gamma\, \Delta t\, \omega^2 + 4\, x_t\, \beta_1\, \gamma\, \Delta t\, \omega^2 + 4\, \dot{x}_t\, \beta_2\, \Delta t^2\, \omega^2 - 4\, \dot{x}_t\, \beta_2^2\, \Delta t^2\, \omega^2 + 4\, \dot{x}_t\, \gamma\, \Delta t^2\, \omega^2 -\\
&\qquad 8\, \dot{x}_t\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 8\, \dot{x}_t\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 - 5\, \dot{x}_t\, \gamma^2\, \Delta t^2\, \omega^2 + 4\, \dot{x}_t\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2 +\\
&\qquad 4\, \dot{x}_t\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 - 4\, \dot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 + 2\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma\, \Delta t^3\, \omega^2 + 2\, x_t\, \beta_2\, \gamma\, \Delta t^3\, \omega^2 -\\
&\qquad 2\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^2 - 2\, x_t\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^2 + x_t\, \gamma^2\, \Delta t^3\, \omega^2 - 4\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 -\\
&\qquad r_{t+\Delta t}\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 - 5\, x_t\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 2\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 +\\
&\qquad 2\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, x_t\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 - 2\, x_t\, \gamma^3\, \Delta t^3\, \omega^2 +\\
&\qquad 2\, x_t\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^2 + 2\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 + r_{t+\Delta t}\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 + 3\, x_t\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 -\\
&\qquad 2\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, x_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, x_t\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 -\\
&\qquad 2\, x_t\, \beta_2\, \gamma\, \Delta t^3\, \omega^4 + 2\, x_t\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^4 + 5\, x_t\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 - 2\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 -\\
&\qquad 4\, x_t\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^4 - 3\, x_t\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 2\, x_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 2\, x_t\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^4 -\\
&\qquad \dot{x}_t\, \beta_2\, \gamma^2\, \Delta t^4\, \omega^4 + \dot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^4 + 3\, \dot{x}_t\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4 - 2\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4 -\\
&\qquad 2\, \dot{x}_t\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^4 - 2\, \dot{x}_t\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4 + 2\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4 + \dot{x}_t\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^4\big)\Big/\\
&\qquad \Big(\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\big(1 + \beta_2^2\, \Delta t^2\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 + \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2\big)\Big)\Big),\\
&\ddot{x}_{t+\Delta t} \to -\Big(\big(-4\, x_t - 4\, \dot{x}_t\, \Delta t - 4\, r_{t+\gamma\Delta t}\, \beta_2\, \Delta t^2 + 4\, r_{t+\gamma\Delta t}\, \beta_2^2\, \Delta t^2 - 4\, r_{t+\Delta t}\, \beta_2^2\, \Delta t^2 -\\
&\qquad 2\, r_{t+\gamma\Delta t}\, \gamma\, \Delta t^2 - 2\, x_t\, \gamma\, \Delta t^2 + 10\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma\, \Delta t^2 - 2\, x_t\, \beta_2\, \gamma\, \Delta t^2 - 4\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma\, \Delta t^2 +\\
&\qquad 4\, x_t\, \beta_1\, \beta_2\, \gamma\, \Delta t^2 - 8\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma\, \Delta t^2 + 8\, r_{t+\Delta t}\, \beta_2^2\, \gamma\, \Delta t^2 + 2\, r_{t+\gamma\Delta t}\, \gamma^2\, \Delta t^2 + 2\, x_t\, \gamma^2\, \Delta t^2 -\\
&\qquad 2\, r_{t+\gamma\Delta t}\, \beta_1\, \gamma^2\, \Delta t^2 - 2\, x_t\, \beta_1\, \gamma^2\, \Delta t^2 - 6\, r_{t+\gamma\Delta t}\, \beta_2\, \gamma^2\, \Delta t^2 + 2\, x_t\, \beta_2\, \gamma^2\, \Delta t^2 +\\
&\qquad 4\, r_{t+\gamma\Delta t}\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2 - 4\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2 + 4\, r_{t+\gamma\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^2 - 4\, r_{t+\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^2 +\\
&\qquad 4\, x_t\, \beta_2\, \Delta t^2\, \omega^2 - 4\, x_t\, \beta_2^2\, \Delta t^2\, \omega^2 + 2\, x_t\, \gamma\, \Delta t^2\, \omega^2 - 10\, x_t\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 4\, x_t\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 +\\
&\qquad 8\, x_t\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 - 3\, x_t\, \gamma^2\, \Delta t^2\, \omega^2 + 2\, x_t\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2 + 6\, x_t\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 -\\
&\qquad 4\, x_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 - 4\, x_t\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 + 4\, \dot{x}_t\, \beta_2\, \gamma\, \Delta t^3\, \omega^2 - 4\, \dot{x}_t\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^2 +\\
&\qquad \dot{x}_t\, \gamma^2\, \Delta t^3\, \omega^2 - 10\, \dot{x}_t\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 8\, \dot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 -\\
&\qquad 2\, \dot{x}_t\, \gamma^3\, \Delta t^3\, \omega^2 + 2\, \dot{x}_t\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^2 + 6\, \dot{x}_t\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 -\\
&\qquad 4\, \dot{x}_t\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 + x_t\, \beta_2\, \gamma^2\, \Delta t^4\, \omega^2 - r_{t+\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^2 - x_t\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^2 -\\
&\qquad 3\, x_t\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^2 + 2\, x_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^2 + 2\, r_{t+\Delta t}\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^2 + 2\, x_t\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^2 +\\
&\qquad 2\, x_t\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^2 - 2\, x_t\, \beta_1\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^2 - r_{t+\Delta t}\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^2 - x_t\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^2\big)\Big/\\
&\qquad \Big(\big(4 + \gamma^2\, \Delta t^2\, \omega^2\big)\big(1 + \beta_2^2\, \Delta t^2\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 + \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2\big)\Big)\Big)\Big\}\Big\}
\end{aligned}
$$

```mathematica
ClearAll["Global`*"]

recursive =
 {
   Collect[
    ẍₜ₊Δₜ - (- ((-4 rₜ₊Δₜ + 4 xₜ ω² + 4 ẋₜ Δt ω² + 4 rₜ₊γΔₜ β₂ Δt² ω² - 4 rₜ₊γΔₜ β₂² Δt² ω² + 2 rₜ₊γΔₜ γ
                 Δt² ω² + 2 ẍₜ γ Δt² ω² - 10 rₜ₊γΔₜ β₂ γ Δt² ω² + 2 ẍₜ β₂ γ Δt² ω² + 4 rₜ₊γΔₜ β₁
                 β₂ γ Δt² ω² - 4 ẍₜ β₁ β₂ γ Δt² ω² + 8 rₜ₊γΔₜ β₂² γ Δt² ω² - 2 rₜ₊γΔₜ γ² Δt² ω² -
                 rₜ₊Δₜ γ² Δt² ω² - 2 ẍₜ γ² Δt² ω² + 2 rₜ₊γΔₜ β₁ γ² Δt² ω² + 2 ẍₜ β₁ γ² Δt² ω² +
                 6 rₜ₊γΔₜ β₂ γ² Δt² ω² - 2 ẍₜ β₂ γ² Δt² ω² - 4 rₜ₊γΔₜ β₁ β₂ γ² Δt² ω² +
                 4 ẍₜ β₁ β₂ γ² Δt² ω² - 4 rₜ₊γΔₜ β₂² γ² Δt² ω² - 4 xₜ β₂ Δt² ω⁴ + 4 xₜ β₂² Δt² ω⁴ -
                 2 xₜ γ Δt² ω⁴ + 10 xₜ β₂ γ Δt² ω⁴ - 4 xₜ β₁ β₂ γ Δt² ω⁴ - 8 xₜ β₂² γ Δt² ω⁴ +
                 3 xₜ γ² Δt² ω⁴ - 2 xₜ β₁ γ² Δt² ω⁴ - 6 xₜ β₂ γ² Δt² ω⁴ + 4 xₜ β₁ β₂ γ² Δt² ω⁴ +
                 4 xₜ β₂² γ² Δt² ω⁴ - 4 ẋₜ β₂ γ Δt³ ω⁴ + 4 ẋₜ β₂² γ Δt³ ω⁴ - ẋₜ γ² Δt³ ω⁴ +
                 10 ẋₜ β₂ γ² Δt³ ω⁴ - 4 ẋₜ β₁ β₂ γ² Δt³ ω⁴ - 8 ẋₜ β₂² γ² Δt³ ω⁴ + 2 ẋₜ γ³ Δt³ ω⁴ -
                 2 ẋₜ β₁ γ³ Δt³ ω⁴ - 6 ẋₜ β₂ γ³ Δt³ ω⁴ + 4 ẋₜ β₁ β₂ γ³ Δt³ ω⁴ + 4 ẋₜ β₂² γ³ Δt³ ω⁴ -
                 ẍₜ β₂ γ² Δt⁴ ω⁴ + ẍₜ β₂² γ² Δt⁴ ω⁴ + 3 ẍₜ β₂ γ³ Δt⁴ ω⁴ - 2 ẍₜ β₁ β₂ γ³ Δt⁴ ω⁴ -
                 2 ẍₜ β₂² γ³ Δt⁴ ω⁴ - 2 ẍₜ β₂ γ⁴ Δt⁴ ω⁴ + 2 ẍₜ β₁ β₂ γ⁴ Δt⁴ ω⁴ + ẍₜ β₂² γ⁴ Δt⁴ ω⁴) /
                ((4 + γ² Δt² ω²) (1 + β₂² Δt² ω² - 2 β₂² γ Δt² ω² + β₂² γ² Δt² ω²)))),
    {ẍₜ, ẋₜ, xₜ, rₜ₊Δₜ, rₜ₊γΔₜ}],
   Collect[ẋₜ₊Δₜ - (- (((-4 ẋₜ - 4 rₜ₊γΔₜ Δt + 4 rₜ₊γΔₜ β₂ Δt - 4 rₜ₊Δₜ β₂ Δt +
                 4 rₜ₊γΔₜ γ Δt - 4 ẍₜ γ Δt - 4 rₜ₊γΔₜ β₁ γ Δt + 4 ẍₜ β₁ γ Δt - 4 rₜ₊γΔₜ β₂ γ Δt +
                 4 rₜ₊Δₜ β₂ γ Δt + 4 xₜ Δt ω² - 4 xₜ γ Δt ω² + 4 xₜ β₁ γ Δt ω² + 4 ẋₜ β₂ Δt² ω² -
                 4 ẋₜ β₂² Δt² ω² + 4 ẋₜ γ Δt² ω² - 8 ẋₜ β₂ γ Δt² ω² + 8 ẋₜ β₂² γ Δt² ω² -
                 5 ẋₜ γ² Δt² ω² + 4 ẋₜ β₁ γ² Δt² ω² + 4 ẋₜ β₂ γ² Δt² ω² - 4 ẋₜ β₂² γ² Δt² ω² +
                 2 rₜ₊γΔₜ β₂ γ Δt³ ω² + 2 ẍₜ β₂ γ Δt³ ω² - 2 rₜ₊γΔₜ β₂² γ Δt³ ω² - 2 ẍₜ β₂² γ Δt³ ω² +
                 ẍₜ γ² Δt³ ω² - 4 rₜ₊γΔₜ β₂ γ² Δt³ ω² - rₜ₊Δₜ β₂ γ² Δt³ ω² - 5 ẍₜ β₂ γ² Δt³ ω² +
                 2 rₜ₊γΔₜ β₁ β₂ γ² Δt³ ω² + 2 ẍₜ β₁ β₂ γ² Δt³ ω² + 4 rₜ₊γΔₜ β₂² γ² Δt³ ω² +
                 4 ẍₜ β₂² γ² Δt³ ω² - 2 ẍₜ γ³ Δt³ ω² + 2 ẍₜ β₁ γ³ Δt³ ω² + 2 rₜ₊γΔₜ β₂ γ³ Δt³ ω² +
                 rₜ₊Δₜ β₂ γ³ Δt³ ω² + 3 ẍₜ β₂ γ³ Δt³ ω² - 2 rₜ₊γΔₜ β₁ β₂ γ³ Δt³ ω² -
                 2 ẍₜ β₁ β₂ γ³ Δt³ ω² - 2 rₜ₊γΔₜ β₂² γ³ Δt³ ω² - 2 ẍₜ β₂² γ³ Δt³ ω² -
                 2 xₜ β₂ γ Δt³ ω⁴ + 2 xₜ β₂² γ Δt³ ω⁴ + 5 xₜ β₂ γ² Δt³ ω⁴ - 2 xₜ β₁ β₂ γ² Δt³ ω⁴ -
                 4 xₜ β₂² γ² Δt³ ω⁴ - 3 xₜ β₂ γ³ Δt³ ω⁴ + 2 xₜ β₁ β₂ γ³ Δt³ ω⁴ + 2 xₜ β₂² γ³ Δt³ ω⁴ -
                 ẋₜ β₂ γ² Δt⁴ ω⁴ + ẋₜ β₂² γ² Δt⁴ ω⁴ + 3 ẋₜ β₂ γ³ Δt⁴ ω⁴ - 2 ẋₜ β₁ β₂ γ³ Δt⁴ ω⁴ -
                 2 ẋₜ β₂² γ³ Δt⁴ ω⁴ - 2 ẋₜ β₂ γ⁴ Δt⁴ ω⁴ + 2 ẋₜ β₁ β₂ γ⁴ Δt⁴ ω⁴ + ẋₜ β₂² γ⁴ Δt⁴ ω⁴) /
                ((4 + γ² Δt² ω²) (1 + β₂² Δt² ω² - 2 β₂² γ Δt² ω² + β₂² γ² Δt² ω²))))) //
       Factor, {ẍₜ, ẋₜ, xₜ, rₜ₊Δₜ, rₜ₊γΔₜ}],
   Collect[xₜ₊Δₜ - (- (((-4 xₜ - 4 ẋₜ Δt - 4 rₜ₊γΔₜ β₂ Δt² + 4 rₜ₊γΔₜ β₂² Δt² - 4 rₜ₊Δₜ β₂² Δt² -
                 2 rₜ₊γΔₜ γ Δt² - 2 ẍₜ γ Δt² + 10 rₜ₊γΔₜ β₂ γ Δt² - 2 ẍₜ β₂ γ Δt² -
                 4 rₜ₊γΔₜ β₁ β₂ γ Δt² + 4 ẍₜ β₁ β₂ γ Δt² - 8 rₜ₊γΔₜ β₂² γ Δt² + 8 rₜ₊Δₜ β₂² γ Δt² +
                 2 rₜ₊γΔₜ γ² Δt² + 2 ẍₜ γ² Δt² - 2 rₜ₊γΔₜ β₁ γ² Δt² - 2 ẍₜ β₁ γ² Δt² -
                 6 rₜ₊γΔₜ β₂ γ² Δt² + 2 ẍₜ β₂ γ² Δt² + 4 rₜ₊γΔₜ β₁ β₂ γ² Δt² - 4 ẍₜ β₁ β₂ γ² Δt² +
                 4 rₜ₊γΔₜ β₂² γ² Δt² - 4 rₜ₊Δₜ β₂² γ² Δt² + 4 xₜ β₂ Δt² ω² - 4 xₜ β₂² Δt² ω² +
                 2 xₜ γ Δt² ω² - 10 xₜ β₂ γ Δt² ω² + 4 xₜ β₁ β₂ γ Δt² ω² + 8 xₜ β₂² γ Δt² ω² -
                 3 xₜ γ² Δt² ω² + 2 xₜ β₁ γ² Δt² ω² + 6 xₜ β₂ γ² Δt² ω² - 4 xₜ β₁ β₂ γ² Δt² ω² -
                 4 xₜ β₂² γ² Δt² ω² + 4 ẋₜ β₂ γ Δt³ ω² - 4 ẋₜ β₂² γ Δt³ ω² + ẋₜ γ² Δt³ ω² -
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$$10\, \dot{x}_t\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 8\, \dot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 - 2\, \dot{x}_t\, \gamma^3\, \Delta t^3\, \omega^2 +$$

$$2\, \dot{x}_t\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^2 + 6\, \dot{x}_t\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 4\, \dot{x}_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 4\, \dot{x}_t\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 +$$

$$\ddot{x}_t\, \beta_2\, \gamma^2\, \Delta t^4\, \omega^2 - r_{t+\Delta t}\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^2 - \ddot{x}_t\, \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^2 - 3\, \ddot{x}_t\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^2 +$$

$$2\, \ddot{x}_t\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^2 + 2\, r_{t+\Delta t}\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^2 + 2\, \ddot{x}_t\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^2 +$$

$$2\, \ddot{x}_t\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^2 - 2\, \ddot{x}_t\, \beta_1\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^2 - r_{t+\Delta t}\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^2 - \ddot{x}_t\, \beta_2^2\, \gamma^4\, \Delta t^4\, \omega^2 \Big) \Big/$$

$$\Big(\big(4 + \gamma^2\, \Delta t^2\, \omega^2 \big)\, \big(1 + \beta_2^2\, \Delta t^2\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 + \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 \big) \Big) \Big) \Big) //$$

Factor, $\{ \ddot{x}_t,\, \dot{x}_t,\, x_t,\, r_{t+\gamma\Delta t},\, r_{t+\Delta t} \}]$
$\} \, / . \, \{ -1 - \beta_2^2\, (1 - \gamma)^2\, \Delta t^2\, \omega^2 \to \eta_1 \,,\; 4 + \gamma^2\, \Delta t^2\, \omega^2 \to \eta_2 ,$
$\; 1 + \beta_2^2\, \Delta t^2\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 + \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 \to \eta_3 \}$

$$\Big\{ x_{t+\Delta t} + r_{t+\Delta t} \left(-\frac{4}{\eta_2\, \eta_3} - \frac{\gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} \right) + r_{t+\gamma\Delta t}$$

$$\left(\frac{4\, \beta_2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_2^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{10\, \beta_2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{4\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{8\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \right.$$

$$\left. \frac{2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{2\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{6\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} \right) +$$

$$x_t \left(\frac{4\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_2^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} - \frac{2\, \gamma\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} + \frac{10\, \beta_2\, \gamma\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} - \right.$$

$$\frac{4\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} - \frac{8\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} + \frac{3\, \gamma^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} - \frac{2\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} - $$

$$\left. \frac{6\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^4}{\eta_2\, \eta_3} \right) +$$

$$\dot{x}_t \left(\frac{4\, \Delta t\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_2\, \gamma\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} - \frac{\gamma^2\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} + \frac{10\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} - \right.$$

$$\frac{4\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} - \frac{8\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} + \frac{2\, \gamma^3\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} - \frac{2\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} - $$

$$\left. \frac{6\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} + \frac{4\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^4}{\eta_2\, \eta_3} \right) +$$

$$x_t \left(\frac{2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{2\, \beta_2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{4\, \beta_1\, \beta_2\, \gamma\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{2\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \right.$$

$$\frac{2\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} + \frac{4\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2}{\eta_2\, \eta_3} - \frac{\beta_2\, \gamma^2\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} + \frac{\beta_2^2\, \gamma^2\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} + \frac{3\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} - $$

$$\left. \frac{2\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} - \frac{2\, \beta_2^2\, \gamma^3\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} - \frac{2\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} + \frac{2\, \beta_1\, \beta_2\, \gamma^4\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} + \frac{\beta_2^2\, \gamma^4\, \Delta t^4\, \omega^4}{\eta_2\, \eta_3} \right),$$

$$\frac{1}{\eta_2\, \eta_3} r_{t+\Delta t} \left(-4\, \beta_2\, \Delta t + 4\, \beta_2\, \gamma\, \Delta t - \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 \right) + \frac{1}{\eta_2\, \eta_3}\, r_{t+\gamma\Delta t}$$

$$\left(-4\, \Delta t + 4\, \beta_2\, \Delta t + 4\, \gamma\, \Delta t - 4\, \beta_1\, \gamma\, \Delta t - 4\, \beta_2\, \gamma\, \Delta t + 2\, \beta_2\, \gamma\, \Delta t^3\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^2 - 4\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + \right.$$

$$\left. 2\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 + 2\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 \right) +$$

$$\frac{1}{\eta_2\, \eta_3}\, x_t \left(-4\, \gamma\, \Delta t + 4\, \beta_1\, \gamma\, \Delta t + 2\, \beta_2\, \gamma\, \Delta t^3\, \omega^2 - 2\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^2 + \gamma^2\, \Delta t^3\, \omega^2 - \right.$$

$$5\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 2\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^2 + 4\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^2 - 2\, \gamma^3\, \Delta t^3\, \omega^2 + $$

$$\left. 2\, \beta_1\, \gamma^3\, \Delta t^3\, \omega^2 + 3\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^2 - 2\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^2 \right) +$$

$$\frac{1}{\eta_2\, \eta_3}\, x_t \left(4\, \Delta t\, \omega^2 - 4\, \gamma\, \Delta t\, \omega^2 + 4\, \beta_1\, \gamma\, \Delta t\, \omega^2 - 2\, \beta_2\, \gamma\, \Delta t^3\, \omega^4 + 2\, \beta_2^2\, \gamma\, \Delta t^3\, \omega^4 + 5\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 - \right.$$

$$\left. 2\, \beta_1\, \beta_2\, \gamma^2\, \Delta t^3\, \omega^4 - 4\, \beta_2^2\, \gamma^2\, \Delta t^3\, \omega^4 - 3\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 2\, \beta_1\, \beta_2\, \gamma^3\, \Delta t^3\, \omega^4 + 2\, \beta_2^2\, \gamma^3\, \Delta t^3\, \omega^4 \right) +$$

$$\frac{1}{\eta_2\, \eta_3}\, \dot{x}_t \left(-4 + 4\, \beta_2\, \Delta t^2\, \omega^2 - 4\, \beta_2^2\, \Delta t^2\, \omega^2 + 4\, \gamma\, \Delta t^2\, \omega^2 - 8\, \beta_2\, \gamma\, \Delta t^2\, \omega^2 + 8\, \beta_2^2\, \gamma\, \Delta t^2\, \omega^2 - 5\, \gamma^2\, \Delta t^2\, \omega^2 + \right.$$

$$4\, \beta_1\, \gamma^2\, \Delta t^2\, \omega^2 + 4\, \beta_2\, \gamma^2\, \Delta t^2\, \omega^2 - 4\, \beta_2^2\, \gamma^2\, \Delta t^2\, \omega^2 - \beta_2\, \gamma^2\, \Delta t^4\, \omega^4 + \beta_2^2\, \gamma^2\, \Delta t^4\, \omega^4 + 3\, \beta_2\, \gamma^3\, \Delta t^4\, \omega^4 - $$

$$2\,\beta_1\,\beta_2\,\gamma^3\,\Delta t^4\,\omega^4 - 2\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^4 - 2\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^4 + 2\,\beta_1\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^4 + \beta_2^2\,\gamma^4\,\Delta t^4\,\omega^4\Big) +$$

$$\frac{1}{\eta_2\,\eta_3}\Big(4\,\dot{x}_{t+\Delta t} + 4\,\dot{x}_{t+\Delta t}\,\beta_2^2\,\Delta t^2\,\omega^2 - 8\,\dot{x}_{t+\Delta t}\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2 + \dot{x}_{t+\Delta t}\,\gamma^2\,\Delta t^2\,\omega^2 +$$

$$4\,\dot{x}_{t+\Delta t}\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2 + \dot{x}_{t+\Delta t}\,\beta_2^2\,\gamma^2\,\Delta t^4\,\omega^4 - 2\,\dot{x}_{t+\Delta t}\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^4 + \dot{x}_{t+\Delta t}\,\beta_2^2\,\gamma^4\,\Delta t^4\,\omega^4\Big),$$

$$\frac{1}{\eta_2\,\eta_3}r_{t+\gamma\Delta t}\Big(-4\,\beta_2\,\Delta t^2 + 4\,\beta_2^2\,\Delta t^2 - 2\,\gamma\,\Delta t^2 + 10\,\beta_2\,\gamma\,\Delta t^2 - 4\,\beta_1\,\beta_2\,\gamma\,\Delta t^2 - 8\,\beta_2^2\,\gamma\,\Delta t^2 +$$

$$2\,\gamma^2\,\Delta t^2 - 2\,\beta_1\,\gamma^2\,\Delta t^2 - 6\,\beta_2\,\gamma^2\,\Delta t^2 + 4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2 + 4\,\beta_2^2\,\gamma^2\,\Delta t^2\Big) + \frac{1}{\eta_2\,\eta_3}$$

$$x_t\Big(-4 + 4\,\beta_2\,\Delta t^2\,\omega^2 - 4\,\beta_2^2\,\Delta t^2\,\omega^2 + 2\,\gamma\,\Delta t^2\,\omega^2 - 10\,\beta_2\,\gamma\,\Delta t^2\,\omega^2 + 4\,\beta_1\,\beta_2\,\gamma\,\Delta t^2\,\omega^2 + 8\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2 -$$

$$3\,\gamma^2\,\Delta t^2\,\omega^2 + 2\,\beta_1\,\gamma^2\,\Delta t^2\,\omega^2 + 6\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2 - 4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2 - 4\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2\Big) +$$

$$\frac{1}{\eta_2\,\eta_3}\,\dot{x}_t\Big(-4\,\Delta t + 4\,\beta_2\,\gamma\,\Delta t^3\,\omega^2 - 4\,\beta_2^2\,\gamma\,\Delta t^3\,\omega^2 + \gamma^2\,\Delta t^3\,\omega^2 - 10\,\beta_2\,\gamma^2\,\Delta t^3\,\omega^2 +$$

$$4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^3\,\omega^2 + 8\,\beta_2^2\,\gamma^2\,\Delta t^3\,\omega^2 - 2\,\gamma^3\,\Delta t^3\,\omega^2 + 2\,\beta_1\,\gamma^3\,\Delta t^3\,\omega^2 +$$

$$6\,\beta_2\,\gamma^3\,\Delta t^3\,\omega^2 - 4\,\beta_1\,\beta_2\,\gamma^3\,\Delta t^3\,\omega^2 - 4\,\beta_2^2\,\gamma^3\,\Delta t^3\,\omega^2\Big) + \frac{1}{\eta_2\,\eta_3}$$

$$r_{t+\Delta t}\Big(-4\,\beta_2^2\,\Delta t^2 + 8\,\beta_2^2\,\gamma\,\Delta t^2 - 4\,\beta_2^2\,\gamma^2\,\Delta t^2 - \beta_2^2\,\gamma^2\,\Delta t^4\,\omega^2 + 2\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^2 - \beta_2^2\,\gamma^4\,\Delta t^4\,\omega^2\Big) +$$

$$\frac{1}{\eta_2\,\eta_3}\,x_t\Big(-2\,\gamma\,\Delta t^2 - 2\,\beta_2\,\gamma\,\Delta t^2 + 4\,\beta_1\,\beta_2\,\gamma\,\Delta t^2 + 2\,\gamma^2\,\Delta t^2 - 2\,\beta_1\,\gamma^2\,\Delta t^2 + 2\,\beta_2\,\gamma^2\,\Delta t^2 -$$

$$4\,\beta_1\,\beta_2\,\gamma^2\,\Delta t^2 + \beta_2\,\gamma^2\,\Delta t^4\,\omega^2 - \beta_2^2\,\gamma^2\,\Delta t^4\,\omega^2 - 3\,\beta_2\,\gamma^3\,\Delta t^4\,\omega^2 + 2\,\beta_1\,\beta_2\,\gamma^3\,\Delta t^4\,\omega^2 +$$

$$2\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^2 + 2\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^2 - 2\,\beta_1\,\beta_2\,\gamma^4\,\Delta t^4\,\omega^2 - \beta_2^2\,\gamma^4\,\Delta t^4\,\omega^2\Big) + \frac{1}{\eta_2\,\eta_3}$$

$$\Big(4\,x_{t+\Delta t} + 4\,x_{t+\Delta t}\,\beta_2^2\,\Delta t^2\,\omega^2 - 8\,x_{t+\Delta t}\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2 + x_{t+\Delta t}\,\gamma^2\,\Delta t^2\,\omega^2 + 4\,x_{t+\Delta t}\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2 +$$

$$x_{t+\Delta t}\,\beta_2^2\,\gamma^2\,\Delta t^4\,\omega^4 - 2\,x_{t+\Delta t}\,\beta_2^2\,\gamma^3\,\Delta t^4\,\omega^4 + x_{t+\Delta t}\,\beta_2^2\,\gamma^4\,\Delta t^4\,\omega^4\Big)\Big\}$$

```
a =
  -( Coefficient[Part[recursive, 1], ẍₜ]  Coefficient[Part[recursive, 1], ẋₜ]  Coeff
     Coefficient[Part[recursive, 2], ẍₜ]  Coefficient[Part[recursive, 2], ẋₜ]  Coeff
     Coefficient[Part[recursive, 3], ẍₜ]  Coefficient[Part[recursive, 3], ẋₜ]  Coeff

  );
```

```
a // MatrixForm // Simplify
```

$$\left(\begin{array}{cc} -\frac{\gamma\,\Delta t^2\,\omega^2\,\big(2 + 2\,(-1+\beta_1)\,\gamma + \beta_2^2\,(-1+\gamma)^2\,\gamma\,\Delta t^2\,\omega^2 + \beta_2\,(-1+\gamma)\,(-2+\gamma\,\Delta t^2\,\omega^2 - 2\,\gamma^2\,\Delta t^2\,\omega^2 + 2\,\beta_1\,(2+\gamma^2\,\Delta t^2\,\omega^2))\big)}{\eta_2\,\eta_3} & \\ \frac{\gamma\,\Delta t\,\big(4 + 2\,\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2 - \gamma\,\Delta t^2\,\omega^2 + 2\,\gamma^2\,\Delta t^2\,\omega^2 + \beta_2\,(-2+5\,\gamma-3\,\gamma^2)\,\Delta t^2\,\omega^2 + 2\,\beta_1\,(-2+\beta_2\,(-1+\gamma)\,\gamma\,\Delta t^2\,\omega^2 - \gamma^2\,\Delta t^2\,\omega^2)\big)}{\eta_2\,\eta_3} & \frac{4 - 4\,\gamma\,\Delta t^2\,\omega^2 + (5- \\ \frac{\gamma\,\Delta t^2\,\big(2 + 2\,(-1+\beta_1)\,\gamma + \beta_2^2\,(-1+\gamma)^2\,\gamma\,\Delta t^2\,\omega^2 + \beta_2\,(-1+\gamma)\,(-2+\gamma\,\Delta t^2\,\omega^2 - 2\,\gamma^2\,\Delta t^2\,\omega^2 + 2\,\beta_1\,(2+\gamma^2\,\Delta t^2\,\omega^2))\big)}{\eta_2\,\eta_3} & \Delta t \end{array}\right.$$

```
la = -( Coefficient[Part[recursive, 1], rₜ₊ᵧ∆t]
        Coefficient[Part[recursive, 2], rₜ₊ᵧ∆t] );
        Coefficient[Part[recursive, 3], rₜ₊ᵧ∆t]
```

```
la // MatrixForm // Simplify
```

$$\begin{pmatrix} \frac{2\,(2\,\beta_2\,(-1+\gamma)-\gamma)\,(1+\beta_2\,(-1+\gamma)+(-1+\beta_1)\,\gamma)\,\Delta t^2\,\omega^2}{\left(1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2\right)\,\left(4+\gamma^2\,\Delta t^2\,\omega^2\right)} \\ \frac{2\,(1+\beta_2\,(-1+\gamma)+(-1+\beta_1)\,\gamma)\,\Delta t\,(2+\beta_2\,(-1+\gamma)\,\gamma\,\Delta t^2\,\omega^2)}{\left(1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2\right)\,\left(4+\gamma^2\,\Delta t^2\,\omega^2\right)} \\ -\frac{2\,(2\,\beta_2\,(-1+\gamma)-\gamma)\,(1+\beta_2\,(-1+\gamma)+(-1+\beta_1)\,\gamma)\,\Delta t^2}{\left(1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2\right)\,\left(4+\gamma^2\,\Delta t^2\,\omega^2\right)} \end{pmatrix}$$

```
lb = - ( Coefficient[Part[recursive, 1], r_{t+Δt}]
         Coefficient[Part[recursive, 2], r_{t+Δt}] );
         Coefficient[Part[recursive, 3], r_{t+Δt}]
```

```
lb // MatrixForm // Simplify
```

$$\begin{pmatrix} \frac{1}{1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2} \\ -\frac{\beta_2\,(-1+\gamma)\,\Delta t}{1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2} \\ \frac{\beta_2^2\,(-1+\gamma)^2\,\Delta t^2}{1+\beta_2^2\,(-1+\gamma)^2\,\Delta t^2\,\omega^2} \end{pmatrix}$$

```
eigA = Eigenvalues[a];
```

```
eigA // MatrixForm // Simplify
```

$$\begin{pmatrix} -\frac{-4+4\,\beta_2\,\Delta t^2\,\omega^2-4\,\beta_2^2\,\Delta t^2\,\omega^2+4\,\gamma\,\Delta t^2\,\omega^2-8\,\beta_2\,\gamma\,\Delta t^2\,\omega^2+8\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2-5\,\gamma^2\,\Delta t^2\,\omega^2+4\,\beta_1\,\gamma^2\,\Delta t^2\,\omega^2+4\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2-4\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2-\beta_2\,\gamma^2\,\Delta t^4\,\omega^4\cdots}{\cdots} \\ \frac{4-4\,\beta_2\,\Delta t^2\,\omega^2+4\,\beta_2^2\,\Delta t^2\,\omega^2-4\,\gamma\,\Delta t^2\,\omega^2+8\,\beta_2\,\gamma\,\Delta t^2\,\omega^2-8\,\beta_2^2\,\gamma\,\Delta t^2\,\omega^2+5\,\gamma^2\,\Delta t^2\,\omega^2-4\,\beta_1\,\gamma^2\,\Delta t^2\,\omega^2-4\,\beta_2\,\gamma^2\,\Delta t^2\,\omega^2+4\,\beta_2^2\,\gamma^2\,\Delta t^2\,\omega^2+\beta_2\,\gamma^2\,\Delta t^4\,\omega^4-\beta\cdots}{\cdots} \end{pmatrix}$$

```
ξ = 0;
```

```
ω = 2 Pi / T;
```

```
λ1 = Part[eigA, 2] /. T → 1 // Simplify // Apart
```

$$-\frac{1-\beta_2-2\,\gamma+2\,\beta_1\,\gamma+\beta_2\,\gamma}{\beta_2\,(-1+\gamma)}+\frac{4\,(1-\beta_2-\gamma+\beta_1\,\gamma+\beta_2\,\gamma)}{(-2\,\beta_2+\gamma+2\,\beta_2\,\gamma)\,\left(1+p^2\,\pi^2\,\gamma^2\right)}+$$

$$\left(2\,\beta_2+\gamma-4\,\beta_2\,\gamma-2\,\gamma^2+2\,\beta_1\,\gamma^2+2\,\beta_2\,\gamma^2\right)\Big/$$

$$\left(\beta_2\,(-1+\gamma)\,(-2\,\beta_2+\gamma+2\,\beta_2\,\gamma)\,\left(1+4\,p^2\,\pi^2\,\beta_2^2-8\,p^2\,\pi^2\,\beta_2^2\,\gamma+4\,p^2\,\pi^2\,\beta_2^2\,\gamma^2\right)\right)+$$

$$\left(2\,\pi\,\gamma^2\,\sqrt{\left(-\left(p+p^3\,\pi^2\,\gamma\right.\right.}\right.$$

$$\left.\left.\left(4\,\beta_2^2\,(-1+\gamma)^2+\gamma\,(-1-2\,(-1+\beta_1)\,\gamma)+2\,\beta_2\,(-1+\gamma)\,(2+(-3+2\,\beta_1)\,\gamma))\right)^2\right)\right)\Big/$$

$$\left(\left(1+p^2\,\pi^2\,\gamma^2\right)\,\left(4\,\beta_2^2-8\,\beta_2^2\,\gamma-\gamma^2+4\,\beta_2^2\,\gamma^2\right)\right)-\left(8\,\pi\,\beta_2^2\,\sqrt{\left(-\left(p+p^3\,\pi^2\,\gamma\right.\right.}\right.$$

$$\left.\left.\left(4\,\beta_2^2\,(-1+\gamma)^2+\gamma\,(-1-2\,(-1+\beta_1)\,\gamma)+2\,\beta_2\,(-1+\gamma)\,(2+(-3+2\,\beta_1)\,\gamma))\right)^2\right)\right)\Big/$$

$$\left(\left(4\,\beta_2^2-8\,\beta_2^2\,\gamma-\gamma^2+4\,\beta_2^2\,\gamma^2\right)\,\left(1+4\,p^2\,\pi^2\,\beta_2^2-8\,p^2\,\pi^2\,\beta_2^2\,\gamma+4\,p^2\,\pi^2\,\beta_2^2\,\gamma^2\right)\right)+$$

$$\left(16\,\pi\,\beta_2^2\,\gamma\,\sqrt{\left(-\left(p+p^3\,\pi^2\,\gamma\right.\right.}\right.$$

$$\left.\left.\left(4\,\beta_2^2\,(-1+\gamma)^2+\gamma\,(-1-2\,(-1+\beta_1)\,\gamma)+2\,\beta_2\,(-1+\gamma)\,(2+(-3+2\,\beta_1)\,\gamma))\right)^2\right)\right)\Big/$$

$$\left(\left(4\,\beta_2^2-8\,\beta_2^2\,\gamma-\gamma^2+4\,\beta_2^2\,\gamma^2\right)\,\left(1+4\,p^2\,\pi^2\,\beta_2^2-8\,p^2\,\pi^2\,\beta_2^2\,\gamma+4\,p^2\,\pi^2\,\beta_2^2\,\gamma^2\right)\right)-$$

$$\left(8\,\pi\,\beta_2^2\,\gamma^2\,\sqrt{\left(-\left(p+p^3\,\pi^2\,\gamma\right.\right.}\right.$$

$$\left.\left.\left(4\,\beta_2^2\,(-1+\gamma)^2+\gamma\,(-1-2\,(-1+\beta_1)\,\gamma)+2\,\beta_2\,(-1+\gamma)\,(2+(-3+2\,\beta_1)\,\gamma))\right)^2\right)\right)\Big/$$

$$\left(\left(4\,\beta_2^2-8\,\beta_2^2\,\gamma-\gamma^2+4\,\beta_2^2\,\gamma^2\right)\,\left(1+4\,p^2\,\pi^2\,\beta_2^2-8\,p^2\,\pi^2\,\beta_2^2\,\gamma+4\,p^2\,\pi^2\,\beta_2^2\,\gamma^2\right)\right)$$

$\text{pe} = \dfrac{\Omega_o}{\overline{\Omega}_d} - 1$

$-1 + \dfrac{\Omega_o}{\overline{\Omega}_d}$

$\Omega_o = \omega \, \Delta t \, /. \, \dfrac{\Delta t}{T} \to p$

$2 \, p \, \pi$

$\overline{\Omega}_d = \text{ArcTan}\left[\dfrac{2 \, \pi \, \sqrt{p^2 \left(-36 + 5 \, p^2 \, \pi^2\right)^2}}{36 - 47 \, p^2 \, \pi^2}\right]$ (*simply copy paste this from above*)

$\text{ArcTan}\left[\dfrac{2 \, \pi \, \sqrt{p^2 \left(-36 + 5 \, p^2 \, \pi^2\right)^2}}{36 - 47 \, p^2 \, \pi^2}\right]$

`Plot[(pe) * 100, {p, 0, .2}, PlotRange → {0, 20}]`
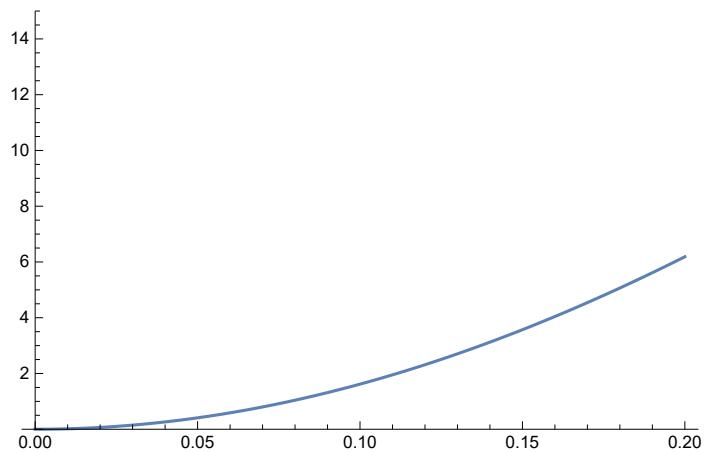
$\text{AD} = 1 - \text{Exp}\left[-2 \, \text{Pi} \, \overline{\xi} \, \dfrac{\Omega_o}{\overline{\Omega}_d}\right]$

$1 - \text{Abs}\left[\dfrac{36 - 47 \, p^2 \, \pi^2 - 2 \, \pi \, \sqrt{-p^2 \left(-36 + 5 \, p^2 \, \pi^2\right)^2}}{\left(4 + p^2 \, \pi^2\right) \left(9 + 4 \, p^2 \, \pi^2\right)}\right]^{\frac{2\pi}{\text{ArcTan}\left[\frac{2\pi\sqrt{p^2\left(-36+5\,p^2\,\pi^2\right)^2}}{36-47\,p^2\,\pi^2}\right]}}$

$\overline{\xi} = -\dfrac{1}{\Omega_o} \, \text{Log}\left[\text{Abs}\left[\lambda 1\right]\right]$

$-\dfrac{\text{Log}\left[\text{Abs}\left[\dfrac{36 - 47 \, p^2 \, \pi^2 - 2 \, \pi \, \sqrt{-p^2 \left(-36 + 5 \, p^2 \, \pi^2\right)^2}}{\left(4 + p^2 \, \pi^2\right) \left(9 + 4 \, p^2 \, \pi^2\right)}\right]\right]}{2 \, p \, \pi}$

```
Plot[(AD) * 100, {p, 0, .2}, PlotRange → {0, 20}]
```
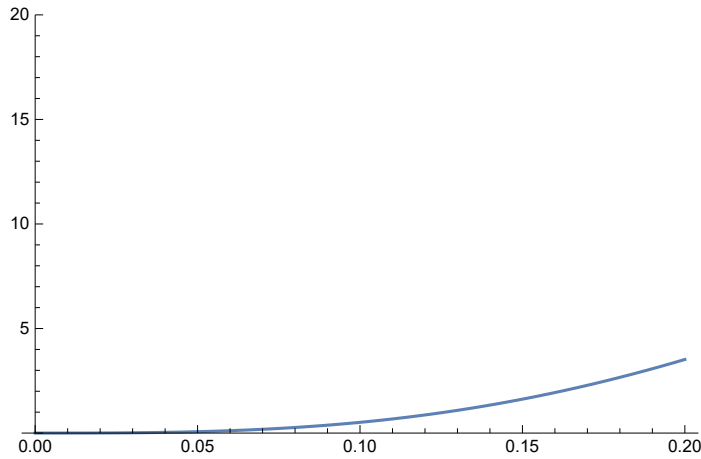


```
aNot = - ( Coefficient[Part[recursive, 2], ẋ_t]   Coefficient[Part[recursive, 2], x_t] )
          ( Coefficient[Part[recursive, 3], ẋ_t]   Coefficient[Part[recursive, 3], x_t] )
```

$$\left\{\left\{ \frac{144}{\left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)} - \frac{188 \pi^2 \Delta t^2}{T^2 \left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)}, \right.\right.$$

$$\left. -\frac{384 \pi^2 \Delta t}{T^2 \left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)} + \frac{16 \pi^4 \Delta t^3}{T^4 \left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)}\right\},$$

$$\left\{ \frac{144 \Delta t}{\left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)} - \frac{20 \pi^2 \Delta t^3}{T^2 \left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)}, \right.$$

$$\left.\left. \frac{144}{\left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)} - \frac{76 \pi^2 \Delta t^2}{T^2 \left(9 + \frac{4 \pi^2 \Delta t^2}{T^2}\right) \left(16 + \frac{4 \pi^2 \Delta t^2}{T^2}\right)}\right\}\right\}$$

```
eigaNot = Eigenvalues[aNot] /. Δt → p T // Expand // Simplify // Cancel;
```

```
ppp = {0, Part[eigaNot, 1], Part[eigaNot, 2]}
```

$$\left\{0, \frac{36 T^4 - 33 p^2 \pi^2 T^4 - 2 \pi \sqrt{-p^2 \left(864 - 205 p^2 \pi^2 + 5 p^4 \pi^4\right) T^8}}{\left(4 + p^2 \pi^2\right) \left(9 + 4 p^2 \pi^2\right) T^4}, \right.$$

$$\left. \frac{36 T^4 - 33 p^2 \pi^2 T^4 + 2 \pi \sqrt{-p^2 \left(864 - 205 p^2 \pi^2 + 5 p^4 \pi^4\right) T^8}}{\left(4 + p^2 \pi^2\right) \left(9 + 4 p^2 \pi^2\right) T^4}\right\}$$

```
qqq = eigA
```

$$\left\{0, \frac{36 T^4 - 47 p^2 \pi^2 T^4 - 2 \pi \sqrt{-p^2 \left(-36 + 5 p^2 \pi^2\right)^2 T^8}}{\left(4 + p^2 \pi^2\right) \left(9 + 4 p^2 \pi^2\right) T^4}, \right.$$

$$\left. \frac{36 T^4 - 47 p^2 \pi^2 T^4 + 2 \pi \sqrt{-p^2 \left(-36 + 5 p^2 \pi^2\right)^2 T^8}}{\left(4 + p^2 \pi^2\right) \left(9 + 4 p^2 \pi^2\right) T^4}\right\}$$

ppp - qqq // Simplify

$$\left\{0, \frac{2\pi\left(7\,p^2\,\pi\,T^4 + \sqrt{-p^2\left(-36+5\,p^2\,\pi^2\right)^2\,T^8} - \sqrt{-p^2\left(864-205\,p^2\,\pi^2+5\,p^4\,\pi^4\right)\,T^8}\right)}{\left(4+p^2\,\pi^2\right)\left(9+4\,p^2\,\pi^2\right)\,T^4},\right.$$

$$\left.\frac{2\pi\left(7\,p^2\,\pi\,T^4 - \sqrt{-p^2\left(-36+5\,p^2\,\pi^2\right)^2\,T^8} + \sqrt{-p^2\left(864-205\,p^2\,\pi^2+5\,p^4\,\pi^4\right)\,T^8}\right)}{\left(4+p^2\,\pi^2\right)\left(9+4\,p^2\,\pi^2\right)\,T^4}\right\}$$