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SIMPLIFYING QUANTUM GRAVITY CALCULATIONS

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Overview

- Introduction
- Deriving Lagrangian
- Manipulating Lagrangian
- Loop Integral
- Our Strategies
- Our Results
- Scattering at Tree Level
- Scattering at One-loop Level
- Conclusions

What are the main reasons behind this thesis?

- The Feynman rules are very complicated although the resulting amplitudes are often simple.
- A better understanding of the math in this theory when fewer terms actually contribute.
- Reducing the running time in FORM program.
- Following the belief that nature should be described in a beautiful and simple mathematical way.

Spin-2 Graviton

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- Spin 0:
  $\Rightarrow$ a Newtonian gravitational potential which considers that the mass, fixed Yukawa coupling, is the only source for gravity.
- Spin 1:
  $\Rightarrow$ an attractive and repulsive gravitational potential.
- Spin 2:
  $\Rightarrow$ a gravitational potential which considers that the EMT is the source for gravity.
- The higher spin is not consistent with QFT.

Analogy With Yang-Mills Theory

Considering the real scalar field Lagrangian:

$$\mathcal{L}_{\text{Matter}} = \frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - \frac{1}{2}m^2\phi^2$$

Invariant under the global translational symmetry: $y^a = y^a + d^a$

Gauging the global symmetry $\Rightarrow$ The general coordinate transformations.

$$\boxed{y^a = y^a + d^a \quad \Rightarrow \quad x^\mu = x^\mu + d^\mu(x)}$$

(A)

$a, b, c, \cdots \rightarrow \mu, \nu, \alpha, \cdots$

Interval invariant: $ds^2 = \eta_{ab}dy^a dy^b = g_{\mu\nu}dx^\mu dx^\nu$

(B)

$d^a \rightarrow d^a(x)$

Measure correction: $d^4y = \sqrt{-\det(g_{\mu\nu})}\, d^4x = \sqrt{-g}\, d^4x$

Analogy With Yang-Mills Theory

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The Lagrangian for matter becomes: (invariant under GCT)

$$\boxed{\mathcal{L}_{\text{Matter}} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2}$$

The commutator of covariant derivatives: ($\Rightarrow$ Analogy with the field strengh tensor)

$$[D_\mu, D_\nu] V^\beta = R_{\mu\nu\alpha}{}^{\beta} V^\alpha \qquad \Rightarrow \qquad (R_{\mu\nu\alpha\beta}, R_{\nu\alpha}, R)$$

The Lagrangian for gravity: (invariant under GCT)

$$\boxed{\mathcal{L}_{\text{Gravity}} = -\frac{2}{\kappa^2} R}$$

Equation of motion: (Einstein's equation)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{4} \mathcal{T}_{\mu\nu}$$

Where $\frac{\kappa^2}{4} = 8\pi G$ $\qquad$ $\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_{\text{Matter}}}{\delta g^{\mu\nu}} = \mathcal{T}_{\mu\nu}$

Effective Field Theory (EFT)

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- EFT is to study the physics in particular ranges of energy while neglecting the physics at higher energy.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \cdots$$

- Weinberg's power counting theorem: $\mathcal{D} = 2 + \sum_n V_n(n-2) + 2L$

- The most general effective Lagrangian for gravity in energy expansion:

$$\boxed{\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + \cdots \\ &= -\Lambda - \frac{2}{\kappa^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \cdots\end{aligned}}$$

  - Λ is Cosmological constant: this term is $\mathcal{O}(E^0)$
  - κ is Newtonian strength of gravitational interactions: this term is $\mathcal{O}(E^2) \Rightarrow$ (tree level)
  - c_i are higher order corrections: these terms are $\mathcal{O}(E^4) \Rightarrow$ (one-loop level)

Where $R_{\mu\nu\alpha\beta}, R_{\nu\alpha}, R \sim (\partial\Gamma, \Gamma\Gamma) \sim (\partial g \partial g, \partial\partial g)$ n the # derivatives in vertex, L the # loops

Why do we need a gauge condition?

- Considering an infinitesimal coordinate transformation: $\left(x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x)\right)$

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + h_{\mu\sigma}\partial_\nu \xi^\sigma(x) + h_{\nu\sigma}\partial_\mu \xi^\sigma(x) + \xi^\sigma(x)\partial_\sigma h_{\mu\nu}$$

  - $\mathcal{L}$ invariant under GCT $\Rightarrow$ a redundancy of the description.
  - Insert $\mathcal{L}_{\text{FG}}, \mathcal{L}_{\text{GH}}$ $\Rightarrow$ break this gauge symmetry $\Rightarrow$ remove this redundancy.

- In the path integral formalism:

$$\mathcal{Z} = \int D[h]\ \exp(i\,S(h)) = \int D[h]\ \exp\left(i \int d^4x \mathcal{L}(h)\right)$$

  - The measure $\int[h]$ $\Rightarrow$ over all configurations of h.
  - Insert $\mathcal{L}_{\text{FG}}, \mathcal{L}_{\text{GH}}$ $\Rightarrow$ over the correct configurations of h.

- Degrees of freedom:
  - $\mathcal{L}$ has more degrees of freedom than its gauge boson.
  - Insert $\mathcal{L}_{\text{FG}}, \mathcal{L}_{\text{GH}}$ $\Rightarrow$ get rid of the extra degrees of freedom.

Where $g_{\mu\nu} \rightarrow g'_{\mu\nu}(x') = g_{\alpha\beta}(x)(\frac{\partial x^\alpha}{\partial x'^\mu})(\frac{\partial x^\beta}{\partial x'^\nu})$ $\qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa\ h_{\mu\nu}$

How do we derive $\mathcal{L}_{\mathrm{FG}}, \mathcal{L}_{\mathrm{GH}}$ using Faddeev-Popov Method?

- Using the identities:

$$\boxed{1 = \int D[\xi_\nu]\delta\Big(\mathcal{C}_\mu(h) - F_\mu(x)\Big)\ \Delta(h)} \qquad \boxed{1 = N(\epsilon)\int D[F]\exp\Big(-\frac{i}{2\epsilon}\int d^4xF_\mu(x)F^\mu(x)\Big)}$$

Where $\mathcal{C}_\mu(h) = F_\mu(x)$ is the gauge condition and $\Delta(h)$ is Faddeev-Popov determinant.

- Inserting these identities into the generating functional:

$$\mathcal{Z} = \int D[h]\ \exp(i\,S(h)) = \int D[h]\ \exp\Big(i\int d^4x\,\mathcal{L}(h)\Big)$$

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How do we derive $\mathcal{L}_{\mathrm{FG}}, \mathcal{L}_{\mathrm{GH}}$ using Faddeev-Popov Method?

- Using the identities:

$$\boxed{1 = \int D[\xi_\nu]\delta\Big(\mathcal{C}_\mu(h) - F_\mu(x)\Big)\ \Delta(h)} \qquad \boxed{1 = N(\epsilon)\int D[F]\exp\Big(-\tfrac{i}{2\epsilon}\int d^4x F_\mu(x)F^\mu(x)\Big)}$$

Where $\mathcal{C}_\mu(h) = F_\mu(x)$ is the gauge condition and $\Delta(h)$ is Faddeev-Popov determinant.

- It yields:

$$\mathcal{Z} = N(\epsilon)\ N'^{-1} \int D[F]\ D[h]\ D[\xi_\nu]\ \delta\Big(\mathcal{C}_\mu(h) - F_\mu(x)\Big)\ \Delta(h)\ \exp\left(iS - \frac{i}{2\epsilon}\int d^4x F_\mu(x)F^\mu(x)\right)$$

- Integrating over $\xi_\nu, F(x)$:

$$\mathcal{Z} = N^{-1}\int D[h]\ D[\bar{\chi}_\mu]\ D[\chi_\nu]\ \exp\left(iS - \frac{i}{2\epsilon}\int d^4x\ \mathcal{C}_\mu(h)\mathcal{C}^\mu(h) + i\int d^4x\ \bar{\chi}_\mu \frac{\partial \mathcal{C}_\mu(h)}{\partial \xi_\nu}\ \chi_\nu\right)$$

(A)

$$\boxed{\mathcal{L}_{\mathrm{FG}}(h) = \frac{1}{2\epsilon}\mathcal{C}_\mu(h)\mathcal{C}^\mu(h)}$$

(B)

$$\boxed{\mathcal{L}_{\mathrm{GH}}(\bar{\chi}_\mu, \chi_\mu, h) = \bar{\chi}_\mu\ \frac{\partial \mathcal{C}_\mu(h)}{\partial \xi_\nu}\ \chi_\nu}$$

The Faddeev-Popov determinant $\Delta(h) = \det\left(\frac{\partial \mathcal{C}_\mu(h)}{\partial \xi_\nu}\right) = \int D[\bar{\chi}_\mu]D[\chi_\nu]\exp\left(i\int d^4x\ \bar{\chi}_\mu\ \frac{\partial \mathcal{C}_\mu(h)}{\partial \xi_\nu}\ \chi_\nu\right)$

What gauge condition did we use?

- The de Donder (harmonic) gauge condition:

$$\mathcal{C}_\mu(h) = \partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h_\lambda{}^\lambda$$

- The general parameterized gauge condition:

$$\begin{aligned}
\mathcal{C}_\mu(h) = \; & \kappa\Big[b_1\partial^\nu h_{\nu\mu} + b_2\partial_\mu h_\nu{}^\nu\Big] \\
& + \kappa^2\Big[b_3\partial_\mu h_\nu{}^\nu h_\alpha{}^\alpha + b_4\partial_\mu h^{\nu\alpha} h_{\nu\alpha} + b_5\partial^\nu h_{\mu\nu} h_\alpha{}^\alpha + b_6\partial_\nu h_{\mu\alpha} h^{\nu\alpha} \\
& \quad + b_7\partial_\nu h^{\nu\alpha} h_{\mu\alpha} + b_8\partial^\nu h_\alpha{}^\alpha h_{\mu\nu}\Big] \\
& + \kappa^3\Big[b_9\partial_\mu h_\nu{}^\nu h_\alpha{}^\alpha h_\beta{}^\beta + b_{10}\partial_\mu h_\nu{}^\nu h^{\alpha\beta} h_{\alpha\beta} + b_{11}\partial_\mu h^{\nu\alpha} h_{\nu\alpha} h_\beta{}^\beta + b_{12}\partial_\mu h^{\nu\alpha} h_\alpha{}^\beta h_{\beta\nu} \\
& \quad + b_{13}\partial^\nu h_{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta + b_{14}\partial^\nu h_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} + b_{15}\partial_\nu h_{\mu\alpha} h^{\nu\alpha} h_\beta{}^\beta + b_{16}\partial^\nu h_{\mu\alpha} h^{\alpha\beta} h_{\beta\nu} \\
& \quad + b_{17}\partial_\nu h^{\nu\alpha} h_{\mu\alpha} h_\beta{}^\beta + b_{18}\partial^\nu h^{\alpha\beta} h_{\mu\alpha} h_{\nu\beta} + b_{19}\partial^\nu h_{\nu\alpha} h_{\mu\beta} h^{\alpha\beta} + b_{20}\partial_\alpha h_\nu{}^\nu h_{\mu\beta} h^{\alpha\beta} \\
& \quad + b_{21}\partial^\nu h_\alpha{}^\alpha h_{\mu\nu} h_\beta{}^\beta + b_{22}\partial^\nu h^{\alpha\beta} h_{\mu\nu} h_{\alpha\beta}\Big] + \cdots
\end{aligned}$$

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Why is it allowed to add total derivative terms?

- Principle of least action:

$$\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu F^\mu(h) \quad \Rightarrow$$

$$\delta\tilde{\mathcal{S}} = \delta \int d^4x \tilde{\mathcal{L}} = \delta \int d^4x \left(\mathcal{L} + \partial_\mu F^\mu(h)\right) = \delta \int d^4x \mathcal{L} + \delta \int d^4x \partial_\mu F^\mu(h) = 0$$

Where $\delta\mathcal{S} = \delta \int d^4x \mathcal{L} = 0$, and the infinitesimal variation let the total derivative part to vanish at the boundary of the integration.

- Integration by parts:

$$\int d^4x \, (\phi \, \partial_\mu \partial^\mu \phi) = \phi \, \partial^\mu \phi \Big|_\delta - \int d^4x \, \partial_\mu \phi \, \partial^\mu \phi = -\int d^4x \, \partial_\mu \phi \, \partial^\mu \phi$$

$$\int d^4x \, (\phi \, \partial_\mu \partial^\mu \phi - \partial_\mu(\phi \, \partial^\mu \phi)) = \int d^4x \, (\phi \, \partial_\mu \partial^\mu \phi - \partial_\mu \phi \, \partial^\mu \phi - \phi \, \partial_\mu \partial^\mu \phi) = -\int d^4x \, \partial_\mu \phi \, \partial^\mu \phi$$

The transformation of fields will be a symmetry transformation if the Lagrangian changes by a total derivative.
The momentum conservation in a vertex.

Total Derivative Lagrangian For h

$$
\begin{aligned}
\mathcal{L}_{\mathrm{TD}}(h) = \frac{1}{\kappa^2}\partial^\mu \Big[& \kappa\big[a_1\partial_\mu h_\nu{}^\nu + a_2\partial^\nu h_{\mu\nu}\big] + \kappa^2\big[a_3\partial_\mu h_\alpha{}^\alpha h_\beta{}^\beta + a_4\partial_\mu h^{\alpha\nu}h_{\alpha\nu} + a_5\partial^\alpha h_{\mu\alpha}h_\nu{}^\nu \\
& + a_6\partial_\alpha h_{\mu\nu}h^{\alpha\nu} + a_7 h_{\mu\nu}\partial_\alpha h^{\alpha\nu} + a_8 h_{\mu\alpha}\partial^\alpha h_\nu{}^\nu\big] + \kappa^3\big[a_9\partial_\mu h_\nu{}^\nu h_\alpha{}^\alpha h_\beta{}^\beta \\
& + a_{10}\partial_\mu h_\nu{}^\nu h^{\alpha\beta}h_{\alpha\beta} + a_{11}\partial_\mu h_{\nu\alpha}h^{\nu\alpha}h_\beta{}^\beta + a_{12}\partial_\mu h^{\nu\alpha}h_\nu{}^\beta h_{\alpha\beta} + a_{13}\partial^\nu h_{\mu\nu}h_\alpha{}^\alpha h_\beta{}^\beta \\
& + a_{14}\partial^\nu h_{\mu\nu}h^{\alpha\beta}h_{\alpha\beta} + a_{15}\partial_\nu h_{\mu\alpha}h^{\nu\alpha}h_\beta{}^\beta + a_{16}\partial^\nu h_{\mu\alpha}h_{\nu\beta}h^{\alpha\beta} + a_{17}\partial_\nu h^{\nu\alpha}h_{\mu\alpha}h_\beta{}^\beta \\
& + a_{18}\partial_\nu h^{\nu\alpha}h_{\mu\beta}h_\alpha{}^\beta + a_{19}\partial^\nu h_\alpha{}^\alpha h_{\mu\nu}h_\beta{}^\beta + a_{20}\partial^\nu h^{\alpha\beta}h_{\mu\nu}h_{\alpha\beta} + a_{21}\partial_\nu h_\alpha{}^\alpha h_{\mu\beta}h^{\nu\beta} \\
& + a_{22}\partial^\nu h^{\alpha\beta}h_{\mu\alpha}h_{\nu\beta}\big] + \kappa^4\big[a_{23}\partial_\mu h_\alpha{}^\alpha h_\beta{}^\beta h_\gamma{}^\gamma h_\delta{}^\delta + a_{24}\partial_\mu h_\alpha{}^\alpha h_\beta{}^\beta h^{\gamma\delta}h_{\gamma\delta} \\
& + a_{25}\partial_\mu h_\alpha{}^\alpha h^{\beta\gamma}h_\gamma{}^\delta h_{\delta\beta} + a_{26}\partial^\alpha h_{\mu\alpha}h_\beta{}^\beta h_\gamma{}^\gamma h_\delta{}^\delta + a_{27}\partial^\alpha h_{\mu\alpha}h_\beta{}^\beta h^{\gamma\delta}h_{\gamma\delta} \\
& + a_{28}\partial^\alpha h_{\mu\alpha}h^{\beta\gamma}h_\gamma{}^\delta h_{\delta\beta} + a_{29}\partial_\mu h_{\alpha\beta}h^{\alpha\beta}h_\gamma{}^\gamma h_\delta{}^\delta + a_{30}\partial_\mu h_{\alpha\beta}h^{\alpha\beta}h_{\gamma\delta}h^{\gamma\delta} \\
& + a_{31}\partial_\alpha h_{\mu\beta}h^{\alpha\beta}h_\gamma{}^\gamma h_\delta{}^\delta + a_{32}\partial_\alpha h_{\mu\beta}h^{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta} + a_{33}\partial^\beta h_\alpha{}^\alpha h_{\mu\beta}h_\gamma{}^\gamma h_\delta{}^\delta \\
& + \cdots \\
& + a_{48}\partial_\alpha h_{\mu\beta}h^{\alpha\gamma}h^{\beta\delta}h_{\gamma\delta} + a_{49}\partial_\mu h^{\alpha\beta}h_\alpha{}^\gamma h_{\beta\gamma}h_\delta{}^\delta + a_{50}\partial_\mu h^{\alpha\beta}h_\alpha{}^\gamma h_\beta{}^\delta h_{\gamma\delta}\big]\Big] + \cdots
\end{aligned}
$$

Total Derivative Lagrangian For ϕ, h

$$\begin{aligned}\mathcal{L}_{\mathrm{TD}}(\phi, h) = \partial^\mu \Big[& d_1 \phi \partial_\mu \phi + \kappa \big[d_2 \phi^2 \partial_\mu h_\nu{}^\nu + d_3 \phi^2 \partial^\nu h_{\nu\mu} + d_4 \phi \partial_\mu \phi h_\nu{}^\nu + d_5 \phi \partial^\nu \phi h_{\nu\mu} \big] \\ & + \kappa^2 \big[d_6 \phi^2 \partial_\mu h_\nu{}^\nu h_\alpha{}^\alpha + d_7 \phi^2 \partial_\mu h^{\nu\alpha} h_{\nu\alpha} + d_8 \phi^2 \partial^\nu h_{\mu\nu} h_\alpha{}^\alpha + d_9 \phi^2 \partial_\nu h_{\mu\alpha} h^{\nu\alpha} \\ & + d_{10} \phi^2 \partial_\nu h^{\nu\alpha} h_{\mu\alpha} + d_{11} \phi^2 \partial^\nu h_\alpha{}^\alpha h_{\mu\nu} + d_{12} \phi \partial_\mu \phi h_\nu{}^\nu h_\alpha{}^\alpha + d_{13} \phi \partial_\mu \phi h^{\nu\alpha} h_{\nu\alpha} \\ & + d_{14} \phi \partial^\nu \phi h_{\mu\nu} h_\alpha{}^\alpha + d_{15} \phi \partial_\nu \phi h_{\mu\alpha} h^{\nu\alpha} \big] + \kappa^3 \big[d_{16} \phi^2 \partial_\mu h^{\nu\alpha} h_{\nu\alpha} h_\beta{}^\beta \\ & + d_{17} \phi^2 \partial^\nu h_{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta + d_{18} \phi^2 \partial_\nu h_{\mu\alpha} h^{\nu\alpha} h_\beta{}^\beta + d_{19} \phi^2 \partial_\nu h^{\nu\alpha} h_{\mu\alpha} h_\beta{}^\beta \\ & + d_{20} \phi^2 \partial^\nu h_\alpha{}^\alpha h_{\mu\nu} h_\beta{}^\beta + d_{21} \phi \partial_\mu \phi h_\nu{}^\nu h_\alpha{}^\alpha h_\beta{}^\beta + d_{22} \phi \partial_\mu \phi h^{\nu\alpha} h_{\nu\alpha} h_\beta{}^\beta \\ & + d_{23} \phi \partial^\nu \phi h_{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta + d_{24} \phi \partial_\nu \phi h_{\mu\alpha} h^{\nu\alpha} h_\beta{}^\beta + d_{25} \phi^2 \partial_\mu h_\nu{}^\nu h_\alpha{}^\alpha h_\beta{}^\beta \big] \Big] + \cdots\end{aligned}$$

Total Derivative Lagrangian For $\chi, \bar{\chi}, h$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{TD}}(\chi,\bar{\chi},h) = \partial^{\mu}\Bigg[& h_1\bar{\chi}^{\nu}\partial_{\mu}\chi_{\nu} + \kappa\Big[h_2 h^{\nu\alpha}\bar{\chi}_{\nu}\partial_{\mu}\chi_{\alpha} + h_3\bar{\chi}_{\nu}\partial_{\alpha}\chi_{\mu}h^{\nu\alpha} + h_4\bar{\chi}^{\nu}\partial_{\mu}\chi_{\nu}h_{\alpha}{}^{\alpha} \\
& + h_5\bar{\chi}^{\nu}\partial_{\nu}\chi_{\mu}h_{\alpha}{}^{\alpha} + h_6\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\alpha}h_{\mu\nu} + h_7\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\nu}h_{\mu\alpha} + h_8\bar{\chi}_{\nu}\partial^{\nu}\chi^{\alpha}h_{\mu\alpha} \\
& + h_9\bar{\chi}_{\mu}\partial^{\nu}\chi_{\nu}h_{\alpha}{}^{\alpha} + h_{10}\bar{\chi}_{\mu}\partial^{\nu}\chi^{\alpha}h_{\nu\alpha} + h_{11}\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\nu}h_{\mu\alpha} + h_{12}\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\alpha}h_{\nu\mu} \\
& + h_{13}\bar{\chi}_{\nu}\partial_{\mu}\chi_{\alpha}h^{\nu\alpha} + h_{14}\bar{\chi}_{\nu}\partial_{\mu}\chi^{\nu}h_{\alpha}{}^{\alpha} + h_{15}\bar{\chi}_{\mu}\partial_{\nu}\chi_{\alpha}h^{\nu\alpha}\Big] \\
& + \kappa^2\Big[h_{20}\bar{\chi}_{\mu}\partial^{\nu}\chi_{\nu}h_{\alpha}{}^{\alpha}h_{\beta}{}^{\beta} + h_{21}\bar{\chi}_{\mu}\partial_{\nu}\chi_{\alpha}h^{\nu\alpha}h_{\beta}{}^{\beta} + h_{22}\bar{\chi}_{\mu}\partial^{\nu}\chi_{\alpha}h_{\nu\beta}h^{\alpha\beta} \\
& + h_{23}\bar{\chi}_{\nu}\partial_{\mu}\chi^{\nu}h_{\alpha}{}^{\alpha}h_{\beta}{}^{\beta} + h_{24}\bar{\chi}_{\nu}\partial_{\mu}\chi^{\nu}h^{\alpha\beta}h_{\alpha\beta} + h_{25}\bar{\chi}_{\nu}\partial_{\mu}\chi_{\alpha}h^{\nu\alpha}h_{\beta}{}^{\beta} \\
& + h_{26}\bar{\chi}^{\nu}\partial_{\mu}\chi_{\alpha}h_{\nu\beta}h^{\alpha\beta} + h_{27}\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\nu}h_{\mu\alpha}h_{\beta}{}^{\beta} + h_{28}\bar{\chi}^{\nu}\partial_{\alpha}\chi_{\nu}h_{\mu\beta}h^{\alpha\beta} \\
& + h_{29}\bar{\chi}_{\nu}\partial_{\alpha}\chi_{\mu}h^{\nu\alpha}h_{\beta}{}^{\beta} + h_{30}\bar{\chi}^{\nu}\partial_{\alpha}\chi_{\mu}h_{\nu\beta}h^{\alpha\beta} + h_{31}\bar{\chi}^{\nu}\partial^{\alpha}\chi_{\alpha}h_{\nu\mu}h_{\beta}{}^{\beta} \\
& + h_{32}\bar{\chi}_{\nu}\partial^{\alpha}\chi_{\alpha}h^{\nu\beta}h_{\mu\beta} + h_{33}\bar{\chi}^{\nu}\partial_{\alpha}\chi_{\beta}h_{\nu\mu}h^{\alpha\beta} + h_{34}\bar{\chi}^{\nu}\partial^{\alpha}\chi^{\beta}h_{\nu\alpha}h_{\mu\beta} \\
& + h_{35}\bar{\chi}_{\nu}\partial^{\alpha}\chi_{\beta}h^{\nu\beta}h_{\mu\alpha} + h_{36}\bar{\chi}_{\mu}\partial^{\nu}\chi_{\nu}h^{\alpha\beta}h_{\alpha\beta}\Big]\Bigg] + \cdots
\end{aligned}
$$

The Equivalence Theorem

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The S-matrix in quantum field theory remains unchanged under reparameterization of the field operators.

For scalar field ϕ, the generating functional is given by:

$$\mathcal{Z} = \int D[\phi]\ \exp\left(i \int d^4x\ \mathcal{L}(\phi, \partial_\mu \phi)\right)$$

If we redefine the scalar field:

$$\phi = \tilde{\phi} \qquad\qquad \text{Where, } \tilde{\phi} = a_1\phi + a_2\phi^2 + \cdots$$

We get:

$$\mathcal{Z} = \int D[\tilde{\phi}]\ \exp\left(i \int d^4x\ \mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi})\right)$$

This redefinition is allowed as long as the Jacobian of the integral is essentially one.

How can field redefinition simplify Lagrangian?

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \kappa \left[a_1 h_{\mu\gamma} h_{\nu}{}^{\gamma} + a_2 h_{\mu\nu} h_{\gamma}{}^{\gamma} \right] + \cdots$$

A term of the triple graviton vertex:

$$h_{\mu\nu}\, \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} \qquad \to \qquad h_{\mu\nu}\, \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} + a_1\, \kappa\, h_{\mu\gamma} h_{\nu}{}^{\gamma}\, \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} + a_2\, \kappa\, h_{\mu\nu} h_{\gamma}{}^{\gamma}\, \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} + \cdots$$

Schematically:

Field Redefinition For h and ϕ

$$\begin{aligned}
h_{\mu\nu} = h_{\mu\nu} &+ \kappa\Big[c_1 h_{\mu\alpha} h_\nu{}^\alpha + c_2 h_{\mu\nu} h_\alpha{}^\alpha\Big] \\
&+ \kappa^2\Big[c_3 h_{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta + c_4 h_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} + c_5 h_{\mu\alpha} h_\nu{}^\alpha h_\beta{}^\beta + c_6 h_{\mu\alpha} h_{\nu\beta} h^{\alpha\beta}\Big] \\
&+ \kappa^3\Big[c_7 h_{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta h_\gamma{}^\gamma + c_8 h_{\mu\nu} h_\alpha{}^\alpha h^{\beta\gamma} h_{\beta\gamma} + c_9 h_{\mu\nu} h^{\alpha\beta} h_\beta{}^\gamma h_{\gamma\alpha} \\
&\qquad + c_{10} h_{\mu\alpha} h_\nu{}^\alpha h_\beta{}^\beta h_\gamma{}^\gamma + c_{11} h_{\mu\alpha} h_\nu{}^\alpha h_{\beta\gamma} h^{\beta\gamma} + c_{12} h_{\mu\alpha} h_{\nu\beta} h^{\alpha\beta} h_\gamma{}^\gamma \\
&\qquad + c_{13} h_{\mu\alpha} h_\nu{}^\beta h^{\alpha\gamma} h_{\beta\gamma}\Big] + \cdots
\end{aligned}$$

$$\begin{aligned}
\phi = \phi &+ \kappa\Big[e_1 h_\alpha{}^\alpha \phi\Big] \\
&+ \kappa^2\Big[e_2 h_\alpha{}^\alpha h_\beta{}^\beta \phi + e_3 h_{\alpha\beta} h^{\alpha\beta} \phi\Big] \\
&+ \kappa^3\Big[e_4 h_\alpha{}^\alpha h_\beta{}^\beta h_\gamma{}^\gamma \phi + e_5 h_{\alpha\beta} h^{\alpha\beta} h_\gamma{}^\gamma \phi + e_6 h^{\alpha\beta} h_\beta{}^\gamma h_{\gamma\alpha} \phi\Big] + \cdots
\end{aligned}$$

For the lowest order vertices.

Field Redefinition For χ and $\bar{\chi}$

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$$\chi_\mu = \chi_\mu + \kappa\left[g_1 h_\alpha{}^\alpha \chi_\mu + g_2 h_{\alpha\mu}\chi^\alpha\right] + \kappa^2\left[g_3 h_\alpha{}^\alpha h_\beta{}^\beta \chi_\mu + g_4 h^{\alpha\beta} h_{\alpha\beta}\chi_\mu + g_5 h_\alpha{}^\alpha h_{\beta\mu}\chi_\beta + g_6 h^{\alpha\beta} h_{\alpha\mu}\chi_\beta\right] + \cdots$$

$$\bar{\chi}_\mu = \bar{\chi}_\mu + \kappa\left[f_1 h_\alpha{}^\alpha \bar{\chi}_\mu + f_2 h_{\alpha\mu}\bar{\chi}^\alpha\right] + \kappa^2\left[f_3 h_\alpha{}^\alpha h_\beta{}^\beta \bar{\chi}_\mu + f_4 h^{\alpha\beta} h_{\alpha\beta}\bar{\chi}_\mu + f_5 h_\alpha{}^\alpha h_{\beta\mu}\bar{\chi}^\beta + f_6 h^{\alpha\beta} h_{\alpha\mu}\bar{\chi}_\beta\right] + \cdots$$

For the lowest order vertices.

Dimensional Regularization

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$$\text{Dimension 4} \underset{\text{analytic continuation}}{\overset{d=4-2\epsilon}{\rightleftarrows}} \text{Dimension } d$$

The loop integral diverges — analytic continuation — The loop integral converges

- Some considerations:

$$g_4^{\mu\nu} \to g_d^{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} g_{\mu\nu} = \delta_\mu^\mu = d = 4 - 2\epsilon$$

$$\int \frac{d^4 p}{(2\pi)^4} \to \int \frac{(\mu)^{2\epsilon}\, d^d p}{(2\pi)^d}$$

Where μ is a regulator parameter of dimensional regularization with dimension $[\mu] = M^\epsilon$.

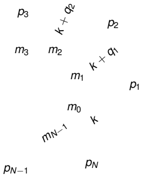
- Mathematically:
Transfer to Euclidean space, do the Wick rotation, apply Feynman parameters, shift the integration variable, perform the integral, go back to Minkowski space.

This scheme preserves the gauge and Lorentz invariances.

Classifying the loop Integrals

$$I^N(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1}) \sim$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_0^2+i\epsilon)((k+q_1)^2-m_1^2+i\epsilon)\cdots((k+q_{N-1})^2-m_{N-1}^2+i\epsilon)}$$



Where $q_i = p_1 + \ldots + p_i = \sum_{k=1}^{i} p_k$

$$A_0(m_0) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_0^2+i\epsilon)}$$

$$B_0(p_1,m_0,m_1) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_0^2+i\epsilon)((k+q_1)^2-m_1^2+i\epsilon)}$$

$$C_0(p_1,p_2,m_0,m_1,m_2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_0^2+i\epsilon)((k+q_1)^2-m_1^2+i\epsilon)((k+q_2)^2-m_2^2+i\epsilon)}$$

$$D_0(p_1,p_2,p_3,m_0,m_1,m_2,m_3) =$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_0^2+i\epsilon)((k+q_1)^2-m_1^2+i\epsilon)((k+q_2)^2-m_2^2+i\epsilon)((k+q_3)^2-m_3^2+i\epsilon)}$$

Classifying the loop Integrals

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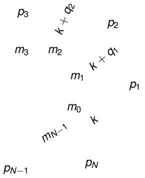
$$I^N_{\mu_1,\dots,\mu_M}(p_1,\dots,p_{N-1},m_0,\dots,m_{N-1}) \sim$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{k_{\mu_1}\cdots k_{\mu_M}}{(k^2-m_0^2+i\epsilon)((k+q_1)^2-m_1^2+i\epsilon)\cdots((k+q_{N-1})^2-m_{N-1}^2+i\epsilon)}$$



Where $q_i = p_1 + \dots + p_i = \sum_{k=1}^{i} p_k$

A_μ ,	$A_{\mu\nu}$,	$A_{\mu\nu\alpha}$,	$A_{\mu\nu\alpha\beta}$		
B_μ ,	$B_{\mu\nu}$,	$B_{\mu\nu\alpha}$,	$B_{\mu\nu\alpha\beta}$,	$B_{\mu\nu\alpha\beta\rho}$	
C_μ ,	$C_{\mu\nu}$,	$C_{\mu\nu\alpha}$,	$C_{\mu\nu\alpha\beta}$,	$C_{\mu\nu\alpha\beta\rho}$,	$C_{\mu\nu\alpha\beta\rho\sigma}$
D_μ ,	$D_{\mu\nu}$,	$D_{\mu\nu\alpha}$,	$D_{\mu\nu\alpha\beta}$		

The Standard Calculations Vs. The Simplified Calculations

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$$
\begin{aligned}
\mathcal{L}_{\text{Total}}(h,\phi,\chi,\bar{\chi}) =& \mathcal{L}_{\text{Gravity}}(h) + \mathcal{L}_{\text{Matter}}(h,\phi) + \mathcal{L}_{\text{FG}}(h) + \mathcal{L}_{\text{Ghost}}(\chi,\bar{\chi},h) \\
&+ \mathcal{L}_{\text{TD}}(h) + \mathcal{L}_{\text{TD}}(\phi,h) + \mathcal{L}_{\text{TD}}(\chi,\bar{\chi},h) \\
=& -\sqrt{-g}\frac{2}{\kappa^2}R + \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) + \frac{1}{2\epsilon}\mathcal{C}_\mu(h)\mathcal{C}^\mu(h) \\
&+ \bar{\chi}_\mu \frac{\partial\mathcal{C}_\mu(h)}{\partial\xi_\nu}\chi_\nu + \mathcal{L}_{\text{TD}}(h) + \mathcal{L}_{\text{TD}}(\phi,h) + \mathcal{L}_{\text{TD}}(\chi,\bar{\chi},h)
\end{aligned}
$$

The Standard Calculations Vs. The Simplified Calculations

$$\mathcal{L}_{\text{Total}}(h,\phi,\chi,\bar{\chi}) = -\sqrt{-g}\frac{2}{\kappa^2}R + \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) + \frac{1}{2\epsilon}\mathcal{C}_\mu(h)\mathcal{C}^\mu(h)$$
$$+\bar{\chi}_\mu \frac{\partial \mathcal{C}_\mu(h)}{\partial \xi_\nu}\chi_\nu + \mathcal{L}_{\text{TD}}(h) + \mathcal{L}_{\text{TD}}(\phi,h) + \mathcal{L}_{\text{TD}}(\chi,\bar{\chi},h)$$

Standard

- Weak-gravitational field expansion.
  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$
- The de Donder gauge.
  $\mathcal{C}_\mu(h) = \partial_\nu {h_\mu}^\nu - \frac{1}{2}\partial_\mu {h_\lambda}^\lambda$

Simplified

- Weak-gravitational field expansion.
  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$
- The general parameterized gauge.
  $\mathcal{C}_\mu(h) = \kappa\left[b_1\partial^\nu h_{\nu\mu} + b_2\partial_\mu {h_\nu}^\nu\right] + \cdots$
- The total derivative Lagrangians.
  $\mathcal{L}_{\text{TD}}(h) + \mathcal{L}_{\text{TD}}(\phi,h) + \mathcal{L}_{\text{TD}}(\chi,\bar{\chi},h)$
- The redefinition of the fields $h, \phi, \chi\bar{\chi}$.
  $h_{\mu\nu} = h_{\mu\nu} + \kappa\left[c_1 h_{\mu\alpha}{h_\nu}^\alpha + c_2 h_{\mu\nu}{h_\alpha}^\alpha\right] + \cdots$

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$$\mathcal{L}_{\textbf{Gravity}}(h) + \mathcal{L}_{\textbf{Matter}}(h,\phi) = -\sqrt{-g}\,\frac{2}{\kappa^2}R + \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

$$\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = \Big(-\det(\eta_{\mu\lambda})\det\big(\delta^\lambda_\nu + \kappa h^\lambda{}_\nu + \cdots\big)\Big)^{1/2}$$

$$= \Big(e^{\mathrm{tr}\left(\ln\left(\delta^\lambda_\nu + \kappa h^\lambda{}_\nu + \cdots\right)\right)}\Big)^{1/2} = \sum_i \frac{1}{i!}\Big(\frac{1}{2}\sum_j \frac{(-1)^{j+1}}{j}(\kappa h^\lambda{}_\lambda + \cdots)^j\Big)^i$$

$$R = g^{\nu\alpha}R_{\nu\alpha}$$

$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}$$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu\Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu\Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\mu\sigma}{}^{\beta}\Gamma_{\nu\alpha}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\beta}\Gamma_{\mu\alpha}{}^{\sigma}$$

$$\Gamma_{\nu\alpha}{}^{\beta} = \frac{1}{2}g^{\beta\rho}(\partial_\nu g_{\rho\alpha} + \partial_\alpha g_{\rho\nu} - \partial_\rho g_{\nu\alpha})$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$$

$$g^{\mu\nu} = \eta_{\mu\nu} - \kappa\, h^{\mu\nu} + \mathcal{O}(h^2)$$

$g_{\mu\nu} = \eta_{\mu\lambda}(\delta^\lambda_\nu + \kappa h^\lambda{}_\nu + \cdots)$, and the expansion up to 4 h.

$$\mathcal{L}_{\mathbf{FG}}(h) + \mathcal{L}_{\mathbf{Ghost}}(\chi, \bar{\chi}, h) = \frac{1}{2\epsilon}\mathcal{C}_\mu(h)\mathcal{C}^\mu(h) + \bar{\chi}_\mu \frac{\partial \mathcal{C}'_\mu(h)}{\partial \xi_\nu} \chi_\nu$$

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- The gauge condition:

$$\mathcal{C}_\mu(h) = \partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h_\lambda{}^\lambda$$

$$\mathcal{C}^\mu(h) = \partial_\alpha h^{\mu\alpha} - \frac{1}{2}\partial^\mu h_\beta{}^\beta$$

- The gauge transformation:

$$x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x)$$

$$g_{\mu\nu} \rightarrow g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right)$$

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + h_{\mu\sigma}\partial_\nu \xi^\sigma(x) + h_{\nu\sigma}\partial_\mu \xi^\sigma(x) + \xi^\sigma(x)\partial_\sigma h_{\mu\nu}$$

$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$, and the expansion up to 2 h with neglecting ξ^2 for $\mathcal{L}_{\mathrm{Ghost}}$.

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The Standard Results:

$$\mathcal{L}_{\phi\phi} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\phi^2 m^2 \tag{2}$$

$$\mathcal{L}_{\phi\phi h} = -\frac{1}{4}\phi^2 h_\mu{}^\mu m^2 + \frac{1}{4}\partial^\mu\phi\partial_\mu\phi h_\nu{}^\nu - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi h^{\mu\nu} \tag{3}$$

$$\begin{aligned}\mathcal{L}_{\phi\phi hh} = &-\frac{1}{16}\phi^2 h_\mu{}^\mu h_\nu{}^\nu m^2 + \frac{1}{8}\phi^2 h^{\mu\nu}h_{\mu\nu}m^2 + \frac{1}{16}\partial^\mu\phi\partial_\mu\phi h_\nu{}^\nu h_\alpha{}^\alpha - \frac{1}{8}\partial^\mu\phi\partial_\mu\phi h^{\nu\alpha}h_{\nu\alpha}\\ &-\frac{1}{4}\partial_\mu\phi\partial_\nu\phi h^{\mu\nu}h_\alpha{}^\alpha + \frac{1}{2}\partial^\mu\phi\partial_\nu\phi h_{\mu\alpha}h^{\nu\alpha}\end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{L}_{\phi\phi hhh} = &-\frac{1}{96}h_\mu{}^\mu h_\nu{}^\nu h_\alpha{}^\alpha\phi^2 m^2 + \frac{1}{96}h_\mu{}^\mu h_\nu{}^\nu h_\alpha{}^\alpha\partial^\beta\phi\partial_\beta\phi - \frac{1}{16}h_\mu{}^\mu h_\nu{}^\nu h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\\ &+\frac{1}{16}h_\mu{}^\mu h^{\nu\alpha}h_{\nu\alpha}\phi^2 m^2 - \frac{1}{16}h_\mu{}^\mu h^{\nu\alpha}h_{\nu\alpha}\partial^\beta\phi\partial_\beta\phi + \frac{1}{4}h_\mu{}^\mu h^{\nu\alpha}h_{\alpha\beta}\partial_\nu\phi\partial^\beta\phi\\ &+\frac{1}{8}h^{\mu\nu}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{12}h^{\mu\nu}h_\mu{}^\alpha h_{\nu\alpha}\phi^2 m^2 + \frac{1}{12}h^{\mu\nu}h_\mu{}^\alpha h_{\nu\alpha}\partial^\beta\phi\partial_\beta\phi\\ &-\frac{1}{2}h^{\mu\nu}h_{\nu\alpha}h^{\alpha\beta}\partial_\mu\phi\partial_\beta\phi\end{aligned} \tag{10}$$

[3] M. T. Grisaru, P. van Nieuwenhuizen, and C. C. Wu, “Background-field method versus normal field theory.

The Standard Results:

$$\mathcal{L}_{hh} = -\frac{1}{4}\partial^\mu {h_\nu}^\nu \partial_\mu {h_\alpha}^\alpha + \frac{1}{2}\partial^\mu h^{\nu\alpha}\partial_\mu h_{\nu\alpha} \quad (2)$$

$$\begin{aligned}
\mathcal{L}_{hhh} =& \frac{1}{4}{h_\mu}^\mu {h_\nu}^\nu \partial^\alpha\partial_\alpha {h_\beta}^\beta - \frac{1}{4}{h_\mu}^\mu {h_\nu}^\nu \partial^\alpha\partial^\beta h_{\alpha\beta} - {h_\mu}^\mu \partial^\nu h_{\nu\alpha}\partial^\alpha {h_\beta}^\beta + {h_\mu}^\mu \partial_\nu h^{\nu\alpha}\partial^\beta h_{\alpha\beta} \\
& - {h_\mu}^\mu \partial^\nu\partial_\nu h^{\alpha\beta} h_{\alpha\beta} - {h_\mu}^\mu h^{\nu\alpha}\partial_\nu\partial_\alpha {h_\beta}^\beta + {h_\mu}^\mu h^{\nu\alpha}\partial_\nu\partial^\beta h_{\alpha\beta} + {h_\mu}^\mu h^{\nu\alpha}\partial_\alpha\partial^\beta h_{\nu\beta} \\
& + \frac{1}{4}{h_\mu}^\mu \partial^\nu {h_\alpha}^\alpha \partial_\nu {h_\beta}^\beta - \frac{3}{4}{h_\mu}^\mu \partial^\nu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \frac{1}{2}{h_\mu}^\mu \partial^\nu h^{\alpha\beta}\partial_\alpha h_{\nu\beta} + \partial^\mu h_{\mu\nu} h^{\nu\alpha}\partial_\alpha {h_\beta}^\beta \\
& - \frac{3}{2}\partial_\mu h^{\mu\nu} h_{\nu\alpha}\partial_\beta h^{\alpha\beta} + \partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta} h^{\alpha\beta} - 2\partial_\mu h^{\mu\nu}\partial_\alpha h_{\nu\beta} h^{\alpha\beta} - \frac{1}{2}\partial^\mu\partial_\mu {h_\nu}^\nu h^{\alpha\beta} h_{\alpha\beta} \\
& + 2\partial^\mu\partial_\mu h^{\nu\alpha} h_{\nu\beta} {h_\alpha}^\beta + \frac{1}{2}h^{\mu\nu} h_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + h^{\mu\nu}\partial_\mu h_{\nu\alpha}\partial^\alpha {h_\beta}^\beta - h^{\mu\nu}\partial_\mu h_{\nu\alpha}\partial_\beta h^{\alpha\beta} \\
& + 2h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{2}h^{\mu\nu} h_{\mu\alpha}\partial_\nu\partial^\alpha {h_\beta}^\beta - h^{\mu\nu} h_{\mu\alpha}\partial_\nu\partial_\beta h^{\alpha\beta} - h^{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\beta} h^{\alpha\beta} \\
& - \frac{1}{2}h^{\mu\nu}\partial_\mu {h_\alpha}^\alpha \partial_\nu {h_\beta}^\beta + \frac{1}{2}h^{\mu\nu}\partial_\mu {h_\alpha}^\alpha \partial^\beta h_{\nu\beta} - 2h^{\mu\nu}\partial_\mu\partial_\alpha h^{\alpha\beta} h_{\nu\beta} + \frac{3}{2}h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} \\
& - 2h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\alpha h_{\nu\beta} + \frac{3}{2}h^{\mu\nu}\partial_\mu\partial^\alpha {h_\beta}^\beta h_{\nu\alpha} + h^{\mu\nu}\partial_\nu h_{\mu\alpha}\partial^\alpha {h_\beta}^\beta - h^{\mu\nu}\partial_\nu h_{\mu\alpha}\partial_\beta h^{\alpha\beta} \\
& + \cdots
\end{aligned} \quad (40)$$

The Standard Results:

$$
\begin{aligned}
\mathcal{L}_{hhhh} = &+\frac{1}{2}4h_{\mu\mu}h_{\nu\nu}h_{\alpha\alpha}\partial_\beta\partial_\beta h_{\gamma\gamma} - \frac{1}{2}4h_{\mu\mu}h_{\nu\nu}h_{\alpha\alpha}\partial_\beta\partial_\gamma h_{\beta\gamma} - \frac{1}{4}h_{\mu\mu}h_{\nu\nu}h_{\alpha\beta}\partial_\alpha\partial_\beta h_{\gamma\gamma} \\
&+\frac{1}{4}h_{\mu\mu}h_{\nu\nu}h_{\alpha\beta}\partial_\alpha\partial_\gamma h_{\beta\gamma} + \frac{1}{4}h_{\mu\mu}h_{\nu\nu}h_{\alpha\beta}\partial_\beta\partial_\gamma h_{\alpha\gamma} - \frac{1}{4}h_{\mu\mu}h_{\nu\nu}\partial_\alpha h_{\alpha\beta}\partial_\beta h_{\gamma\gamma} \\
&+\frac{1}{4}h_{\mu\mu}h_{\nu\nu}\partial_\alpha h_{\alpha\beta}\partial_\gamma h_{\beta\gamma} + \frac{1}{16}h_{\mu\mu}h_{\nu\nu}\partial_\alpha h_{\beta\beta}\partial_\alpha h_{\gamma\gamma} - \frac{3}{16}h_{\mu\mu}h_{\nu\nu}\partial_\alpha h_{\beta\gamma}\partial_\alpha h_{\beta\gamma} \\
&+\frac{1}{8}h_{\mu\mu}h_{\nu\nu}\partial_\alpha h_{\beta\gamma}\partial_\beta h_{\alpha\gamma} - \frac{1}{4}h_{\mu\mu}h_{\nu\nu}\partial_\alpha\partial_\alpha h_{\beta\gamma}h_{\beta\gamma} + \frac{1}{4}h_{\mu\mu}h_{\nu\alpha}h_{\nu\alpha}\partial_\beta\partial_\gamma h_{\beta\gamma} \\
&+\frac{1}{4}h_{\mu\mu}h_{\nu\alpha}h_{\nu\beta}\partial_\alpha\partial_\beta h_{\gamma\gamma} - \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}h_{\nu\beta}\partial_\alpha\partial_\gamma h_{\beta\gamma} - \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}h_{\alpha\beta}\partial_\beta\partial_\gamma h_{\nu\gamma} \\
&+\frac{1}{2}h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\alpha\beta}\partial_\beta h_{\gamma\gamma} - \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\alpha\beta}\partial_\gamma h_{\beta\gamma} - \frac{1}{4}h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\beta\beta}\partial_\alpha h_{\gamma\gamma} \\
&+\frac{1}{4}h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\beta\beta}\partial_\gamma h_{\alpha\gamma} + \frac{3}{4}h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\beta\gamma}\partial_\alpha h_{\beta\gamma} - h_{\mu\mu}h_{\nu\alpha}\partial_\nu h_{\beta\gamma}\partial_\beta h_{\alpha\gamma} \\
&+h_{\mu\mu}h_{\nu\alpha}\partial_\nu\partial_\alpha h_{\beta\gamma}h_{\beta\gamma} - \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}\partial_\nu\partial_\beta h_{\alpha\gamma}h_{\beta\gamma} - h_{\mu\mu}h_{\nu\alpha}\partial_\nu\partial_\beta h_{\beta\gamma}h_{\alpha\gamma} \\
&+\frac{3}{4}h_{\mu\mu}h_{\nu\alpha}\partial_\nu\partial_\beta h_{\gamma\gamma}h_{\alpha\beta} + \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}\partial_\alpha h_{\nu\beta}\partial_\beta h_{\gamma\gamma} - \frac{1}{2}h_{\mu\mu}h_{\nu\alpha}\partial_\alpha h_{\nu\beta}\partial_\gamma h_{\beta\gamma} \\
&+\cdots
\end{aligned}
\tag{113}
$$

The Standard Results:

$$\mathcal{L}_{\bar{\chi}\chi} = \bar{\chi}^\mu \partial^\nu \partial_\nu \chi_\mu \tag{1}$$

$$\begin{aligned}\mathcal{L}_{\bar{\chi}\chi h} = &-\frac{1}{2}\bar{\chi}^\mu \partial_\mu \chi^\nu \partial_\nu h_\alpha{}^\alpha + \bar{\chi}^\mu \partial_\mu \chi^\nu \partial^\alpha h_{\nu\alpha} - \frac{1}{2}\bar{\chi}^\mu \chi^\nu \partial_\mu \partial_\nu h_\alpha{}^\alpha \\ &+ \bar{\chi}^\mu \chi^\nu \partial_\nu \partial^\alpha h_{\mu\alpha} + \bar{\chi}^\mu \partial^\nu \partial_\nu \chi^\alpha h_{\mu\alpha} - \bar{\chi}^\mu \partial^\nu \chi^\alpha \partial_\mu h_{\nu\alpha} \\ &+ \bar{\chi}^\mu \partial^\nu \chi^\alpha \partial_\nu h_{\mu\alpha} + \bar{\chi}^\mu \partial^\nu \chi^\alpha \partial_\alpha h_{\mu\nu}\end{aligned} \tag{8}$$

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The Simplified Calculations

$$\mathcal{L}_{\text{Gravity}}(h) + \mathcal{L}_{\text{Matter}}(h,\phi) = -\sqrt{-g}\,\frac{2}{\kappa^2}R + \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

$$\mathcal{L}_{\text{FG}}(h) + \mathcal{L}_{\text{Ghost}}(\chi,\bar{\chi},h) = \frac{1}{2\epsilon}\mathcal{C}_\mu(h)\mathcal{C}^\mu(h) + \bar{\chi}_\mu\,\frac{\partial\mathcal{C}'_\mu(h)}{\partial\xi_\nu}\,\chi_\nu$$

$$\mathcal{L}_{\text{TD}}(h) \quad + \quad \mathcal{L}_{\text{TD}}(\phi,h) \quad + \quad \mathcal{L}_{\text{TD}}(\chi,\bar{\chi},h)$$

$$h_{\mu\nu} \to h'_{\mu\nu} \quad , \quad \phi \to \phi' \quad , \quad \chi \to \chi' \quad , \quad \bar{\chi} \to \bar{\chi}'$$

The Total Lagrangian $\Rightarrow$ Choosing The Parameters $\Rightarrow$ The Result

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- Eight sets of parameters:

	h	ϕ	$\chi, \bar{\chi}$
$\mathcal{L}_{\text{TD}}$	$(a_1, \ldots, a_{50})$	$(d_1, \ldots, d_{25})$	$(h_1, \ldots, h_{36})$
Field Redefinition	$(c_1, \ldots, c_{13})$	$(e_1, \ldots, e_6)$	$(f_1, \ldots, f_6), (g_1, \ldots, g_6)$
$\mathcal{C}_\mu(h)$	$(b_1, \ldots, b_{22})$		

- Three norms:
  - Cancelling all terms that have second order derivative.
  - Minimizing number of the terms as much as possible.
  - Trying to keep the terms that have the same indices for ∂.

$$\partial_\mu h_{\nu\alpha} \partial^\mu h^{\nu\alpha} \qquad \cancel{\partial^\nu h^{\mu\alpha} \partial_\mu h_{\nu\alpha}}$$

In particular, (V_{hhh}, V_{hhhh}).

The Total Lagrangian ⇒ Choosing The Parameters ⇒ The Result

The Total Lagrangian For Triple Graviton Vertex

```
LagT3 =
 + H(mu,mu)*H(nu,nu)*H(al,al,be,be)*Fact(1 + 4*a9,4)
 + H(mu,mu)*H(nu,nu)*H(al,be,al,be)*Fact( - 1 + 4*a13,4)
 + H(mu,mu)*H(nu,nu,al)*H(al,be,be)*Fact( - 1 - b5 + 2*b3 + a19 + 2*a13,1)
 + H(mu,mu)*H(nu,nu,al)*H(be,al,be)*Fact(1 + 2*b5 + a17,1)
 + H(mu,mu)*H(nu,nu,al,be)*H(al,be)*Fact( - 1 + a11,1)
 + H(mu,mu)*H(nu,al)*H(nu,al,be,be)*Fact( - 1 + a19,1)
 + H(mu,mu)*H(nu,al)*H(nu,be,al,be)*Fact(2 + a17 + a15,2)
 + H(mu,mu)*H(nu,al)*H(al,be,nu,be)*Fact(2 + a17 + a15,2)
 + H(mu,mu)*H(nu,al,al)*H(nu,be,be)*Fact(1 - 4*c2 - 4*b3 + 8*a9,4)
 + H(mu,mu)*H(nu,al,be)^2*Fact( - 3 + 4*c2 + 4*a11,4)
 + H(mu,mu)*H(nu,al,be)*H(al,nu,be)*Fact(1 + 2*a15,2)
 + H(mu,mu,nu)*H(nu,al)*H(al,be,be)*Fact(2 + 2*b8 - b7 + a21 + a17,2)
 + H(mu,mu,nu)*H(nu,al,be)*H(al,be)*Fact(2 + 2*b4 + a20 + 2*a14,2)
 + H(mu,mu,nu)*H(al,nu,be)*H(al,be)*Fact( - 4 + 2*b6 + a22 + a18,2)
 + H(mu,mu,nu,nu)*H(al,be)^2*Fact( - 1 + 2*a10,2)
 + H(mu,mu,nu,al)*H(nu,be)*H(al,be)*Fact(2 + a12,1)
 + H(mu,nu)^2*H(al,be,al,be)*Fact(1 + 2*a14,2)
 + H(mu,nu)*H(mu,nu,al)*H(al,be,be)*Fact(2 - b6 + a21 + a15,2)
 + H(mu,nu)*H(mu,nu,al)*H(be,al,be)*Fact( - 4 + 2*b6 + a22 + a18,4)
 + H(mu,nu)*H(mu,nu,al,be)*H(al,be)*Fact(2 + a20,1)
 + H(mu,nu)*H(mu,al)*H(nu,al,be,be)*Fact(2 + a21,4)
 + H(mu,nu)*H(mu,al)*H(nu,be,al,be)*Fact( - 4 + a18 + a16,1)
 + H(mu,nu)*H(mu,al,nu,be)*H(al,be)*Fact( - 2 + a22,2)
 + H(mu,nu)*H(mu,al,al)*H(nu,be,be)*Fact( - 1 - 2*b8 + 2*a19,2)
 + H(mu,nu)*H(mu,al,al)*H(be,nu,be)*Fact(2 + 2*b8 - b7 + a21 + a17,4)
 + H(mu,nu)*H(mu,al,be)*H(nu,al,be)*Fact(3 + 2*a20,2)
 + H(mu,nu)*H(mu,al,be)*H(al,nu,be)*Fact( - 2 + a22 + a16,1)
 + H(mu,nu)*H(mu,al,be,be)*H(nu,al)*Fact(6 + 3*a21,4)
 + H(mu,nu)*H(nu,mu,al)*H(al,be,be)*Fact(2 - b6 + a21 + a15,2)
 + H(mu,nu)*H(nu,mu,al)*H(be,al,be)*Fact( - 4 + 2*b6 + a22 + a18,4)
 + H(mu,nu)*H(nu,al,mu,be)*H(al,be)*Fact( - 2 + a22,2)
 + H(mu,nu)*H(nu,al,al)*H(be,mu,be)*Fact(2 + 2*b8 - b7 + a21 + a17,4)
 + H(mu,nu)*H(al,mu,nu)*H(al,be,be)*Fact( - 1 + c2 - c1 - b4 + a11 + 2*a10,1)
 + H(mu,nu)*H(al,mu,nu)*H(be,al,be)*Fact(2 + 2*b4 + a20 + 2*a14,2)
 + H(mu,nu)*H(al,mu,al)*H(be,nu,be)*Fact( - 2 + 2*b7 + a18,1)
 + H(mu,nu)*H(al,mu,be)*H(al,nu,be)*Fact(3 + 2*c1 + 2*a12,1)
 + H(mu,nu)*H(al,mu,be)*H(be,nu,al)*Fact( - 1 + a16,1);
```

The Total Lagrangian For Triple Graviton Vertex

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The parameters of the total derivative Lagrangian that remove all second order derivative terms for the vertex V_{hhh}.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
V_{hhh}	a_9	-1/4	a_{10}	1/2	a_{11}	1
	a_{12}	-2	a_{13}	1/4	a_{14}	-1/2
	a_{15}	-1/2	a_{16}	1	a_{17}	-3/2
	a_{18}	3	a_{19}	1	a_{20}	-2
	a_{21}	-2	a_{22}	2		

The parameters of the gauge condition that reduce the number of Largrangian terms for the vertex V_{hhh}.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
V_{hhh}	b_3	-1/8	b_4	1/2	b_5	1/4
	b_6	-1/2	b_7	-1/2	b_8	1/2

The parameters of gravitational field redefinition that reduce the number of Lagrangian terms for the vertex V_{hhh}.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
V_{hhh}	c_1	1/2	c_2	-1/4		

The Total Lagrangian ⇒ Choosing The Parameters ⇒ The Result

Result For Triple Graviton Vertex

```
Time =          5.66 sec     Generated terms =          4
               LagT3         Terms in output =          4
                             Bytes used      =        444

LagT3 =
     + H(mu,mu)*H(nu,al,al)*H(nu,be,be)*Fact(1,8)
     + H(mu,nu)*H(mu,al,be)*H(nu,al,be)*Fact(-1,2)
     + H(mu,nu)*H(mu,al,be)*H(al,nu,be)*Fact(1,1)
     + H(mu,nu)*H(al,mu,nu)*H(al,be,be)*Fact(-1,4)
    ;
```

Result For Triple Graviton Vertex

Our Choice Of The Parameters

The parameters of the total derivative Lagrangian.

Propagator/Vertex	Parameter	Value	Parameter	Value	Parameter	Value
	a_1	-2	a_2	2		
S_{hh}	a_3	-1	a_4	2	a_5	1
	a_6	-1	a_7	-3	a_8	2
V_{hhh}	a_9	-1/4	a_{10}	1/2	a_{11}	1
	a_{12}	-2	a_{13}	1/4	a_{14}	-1/2
	a_{15}	-1/2	a_{16}	1	a_{17}	-3/2
	a_{18}	3	a_{19}	1	a_{20}	-2
	a_{21}	-2	a_{22}	2		
V_{hhhh}	a_{23}	-1/24	a_{24}	1/4	a_{25}	-1/3
	a_{26}	1/24	a_{27}	-1/4	a_{28}	1/3
	a_{29}	1/4	a_{30}	-1/2	a_{31}	-1/8
	a_{32}	1/4	a_{33}	1/4	a_{34}	-1/2
	a_{35}	-1	a_{36}	2	a_{37}	-3/8
	a_{38}	3/4	a_{39}	1	a_{40}	-2
	a_{41}	-1	a_{42}	2	a_{43}	3/2
	a_{44}	-2	a_{45}	-3	a_{46}	2
	a_{47}	1/2	a_{48}	-1	a_{49}	-1
	a_{50}	2				

Our Choice Of The Parameters

The parameters of the gauge condition, (b_1 , b_2) ensure the same de Donder propagator.

Propagator/Vertex	Parameter	Value	Parameter	Value	Parameter	Value
S_{hh}	b_1	1	b_2	-1/2		
V_{hhh}	b_3	-1/8	b_4	1/2	b_5	1/4
	b_6	-1/2	b_7	-1/2	b_8	1/2
V_{hhhh}	b_9	-1/64	b_{10}	1/16	b_{11}	1/8
	b_{12}	-1/2	b_{13}	1/32	b_{14}	-1/8
	b_{15}	-1/8	b_{16}	1/4	b_{17}	-1/8
	b_{18}	1/4	b_{19}	3/8	b_{20}	-1/4
	b_{21}	1/8	b_{22}	-1/4		

The parameters of gravitational field redefinition.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
V_{hhh}	c_1	1/2	c_2	-1/4		
V_{hhhh}	c_3	3/32	c_4	0	c_5	-1/8
	c_6	1/4				

Our Choice Of The Parameters

The parameters of the total derivative Lagrangian.

Propagator/Vertex	Parameter	Value
$S_{\phi\phi}$	d_1	0
$V_{\phi\phi h}$	$d_2, \ldots, d_5$	0
$V_{\phi\phi hh}$	$d_6, \ldots, d_{14}$	0
$V_{\phi\phi hhh}$	$d_{15}, \ldots, d_{22}$	0

The parameters of the scalar field redefinition.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
$V_{\phi\phi h}$	e_1	0				
$V_{\phi\phi hh}$	e_2	0	e_3	0		
$V_{\phi\phi hhh}$	e_4	-1/384	e_5	0	e_6	-1/48

Our Choice Of The Parameters

The parameters of the total derivative Lagrangian.

Propagator/Vertex	Parameter	Value	Parameter	Value	Parameter	Value
$S_{\bar{\chi}\chi}$	h_1	-1				
$V_{\bar{\chi}\chi h}$	$h_2, \dots, h_{10}$	0	h_{11}	1/2	h_{12}	-1/2
	h_{13}	-1/2	h_{14}	-1/4	h_{15}	-1/2
$V_{\bar{\chi}\chi hh}$	h_{20}	0	h_{21}	-1/8	h_{22}	1/4
	h_{23}	-1/32	h_{24}	1/8	h_{25}	-1/8
	h_{26}	1/8	h_{27}	1/8	h_{28}	-1/4
	h_{29}	0	h_{30}	0	h_{31}	-1/8
	h_{32}	1/8	h_{33}	-1/2	h_{34}	1/4
	h_{35}	1/4	h_{36}	0		

The parameters of the ghost and antighost field redefinition.

Vertex	Parameter	Value	Parameter	Value	Parameter	Value
$V_{\bar{\chi}\chi h}$	f_1	1/4	f_2	0	g_1, g_2	0
$V_{\bar{\chi}\chi hh}$	f_3	-1/32	f_4	1/8	f_5	1/8
	f_6	-1/8	$g_3, \dots, g_6$	0		

The Propagators $S_{\{\phi\phi\}}(Q,m)$, $S^{\alpha\beta}_{\{\bar{\chi}\chi\}}(Q)$, $S^{\alpha\beta\gamma\delta}_{\{hh\}}(Q)$:

$\phi(Q)$ $\phi(Q)$

$$\mathcal{L}_{\phi\phi} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \Rightarrow S_{\{\phi\phi\}}(Q,m) = \frac{i}{Q^2 - m^2 + i\epsilon}$$

$\chi^\alpha(Q)$ $\bar{\chi}^\beta(Q)$

$$\mathcal{L}_{\bar{\chi}\chi} = -\eta_{\mu\nu}\partial_\lambda\bar{\chi}^\mu\partial^\lambda\chi^\nu \quad \Rightarrow \quad S^{\alpha\beta}_{\{\bar{\chi}\chi\}}(Q) = -\frac{i}{Q^2}\eta^{\alpha\beta}$$

$h^{\alpha\beta}(Q)$ $h^{\gamma\delta}(Q)$

$$\begin{aligned}\mathcal{L}_{hh} &= \frac{1}{2}\partial_\mu h_{\nu\lambda}\partial^\mu h^{\nu\lambda} - \frac{1}{4}\partial_\mu h_\nu{}^\nu\partial^\mu h_\lambda{}^\lambda \\ &= \frac{1}{2}h_{\mu\nu}\partial^\lambda\partial_\lambda\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta} \qquad \Rightarrow\end{aligned}$$

$$S^{\alpha\beta\gamma\delta}_{\{hh\}}(Q) = \frac{i}{Q^2}\left[\frac{1}{2}(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta})\right] = \frac{i}{Q^2}P^{\alpha\beta\gamma\delta}$$

Green's equation: $\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\partial^\lambda\partial_\lambda S_{\alpha\beta\gamma\delta}(x-y) = -I^{\mu\nu}{}_{\gamma\delta}\,\delta^{(d)}(x-y)$.

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The Triple Graviton Vertex $V^{\{hhh\}}_{\gamma\delta\rho\sigma\eta\lambda}(q_1, q_2)$:

$h_{\gamma\delta}(q_1)$

(40) $\xrightarrow{\text{Reduced to}}$ **(4)**

$h_{\rho\sigma}(q_2)$ $h_{\eta\lambda}(q_3)$

$$\mathcal{L}_{hhh} = \frac{\kappa}{2}\left[\frac{1}{4} h_\mu{}^\mu \partial_\nu h_\alpha{}^\alpha \partial^\nu h_\beta{}^\beta - h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + 2 h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\alpha h_{\nu\beta} - \frac{1}{2} h^{\mu\nu}\partial_\alpha h_{\mu\nu}\partial^\alpha h_\beta{}^\beta\right]$$

The Quadruple Graviton Vertex $V^{\{hhhh\}}_{\gamma\delta\rho\sigma\eta\lambda\kappa\epsilon}(q_1, q_2, q_3)$:

$h_{\gamma\delta}(q_1)$ $h_{\eta\lambda}(q_3)$

$h_{\rho\sigma}(q_2)$ $h_{\kappa\epsilon}(q_4)$

(113) $\xrightarrow{\text{Reduced to}}$ **(12)**

$$\begin{aligned}\mathcal{L}_{hhhh} = \frac{\kappa^2}{2}\Big[&-\frac{5}{32} h_\mu{}^\mu h_\nu{}^\nu \partial_\alpha h_\beta{}^\beta \partial^\alpha h_\tau{}^\tau + \frac{1}{4} h_\mu{}^\mu h^{\nu\alpha} \partial_\nu h_{\beta\tau} \partial_\alpha h^{\beta\tau} - \frac{1}{2} h_\mu{}^\mu h^{\nu\alpha} \partial_\nu h^{\beta\tau} \partial_\beta h_{\alpha\tau} \\ &+ \frac{1}{2} h_\mu{}^\mu h^{\nu\alpha} \partial_\beta h_{\nu\tau} \partial^\beta h_\alpha{}^\tau - \frac{1}{16} h_{\mu\nu} h^{\mu\nu} \partial_\alpha h_\beta{}^\beta \partial^\alpha h_\tau{}^\tau + \frac{1}{2} h^{\mu\nu} \partial_\mu h_{\nu\alpha} \partial^\beta h^{\alpha\tau} h_{\beta\tau} \\ &+ \frac{1}{8} h^{\mu\nu} \partial_\mu h_\alpha{}^\alpha h_{\nu\beta} \partial^\beta h_\tau{}^\tau - h^{\mu\nu} \partial_\mu h^{\alpha\beta} h_{\nu\alpha} \partial^\tau h_{\beta\tau} + \frac{1}{2} h^{\mu\nu} \partial_\mu h_{\alpha\beta} h_{\nu\tau} \partial^\tau h^{\alpha\beta} \\ &- h^{\mu\nu} \partial_\mu h_{\alpha\beta} h^{\alpha\tau} \partial_\tau h_\nu{}^\beta + \frac{1}{2} h^{\mu\nu} h_{\nu\alpha} \partial_\beta h_{\mu\tau} \partial^\beta h^{\alpha\tau} + h^{\mu\nu} \partial_\nu h_{\alpha\beta} h^{\alpha\beta} \partial^\tau h_{\mu\tau} \Big]\end{aligned}$$

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The Scalar-Scalar-Graviton Vertex $V_{\alpha\beta}^{\{\phi\phi h\}}(p_1, p_2, m)$:

$h_{\alpha\beta}(q_1)$

(3) The same **(3)**

$\phi(p_1)$ $\phi(p_2)$

$$\mathcal{L}_{\phi\phi h} = \frac{\kappa}{2}\Big[-\frac{1}{2} h_\mu{}^\mu \phi^2 m^2 + \frac{1}{2} h_\mu{}^\mu \partial_\nu\phi\partial^\nu\phi - h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \Big]$$

The Scalar-Scalar-Graviton-Graviton Vertex $V^{\{\phi\phi hh\}}_{\gamma\delta\rho\sigma}(p_1, p_2)$:

$\phi(p_1)$ $\phi(p_2)$

(6) $\xrightarrow{\text{Reduced to}}$ **(2)**

$h_{\gamma\delta}(q_1)$ $h_{\rho\sigma}(q_2)$

$$\mathcal{L}_{\phi\phi hh} = \frac{\kappa^2}{4}\left[h^{\mu\nu}h_{\nu}{}^{\alpha}\partial_\mu\phi\partial_\alpha\phi - \frac{1}{2}h_{\mu}{}^{\mu}h^{\nu\alpha}\partial_\nu\phi\partial_\alpha\phi\right]$$

The Scalar-Scalar-Graviton-Graviton-Graviton Vertex $V^{\{\phi\phi hhh\}}_{\gamma\delta\rho\sigma\lambda\epsilon}$:

(10) $\xRightarrow{\text{Reduced to}}$ **(7)**

$h_{\rho\sigma}(q_2)$

$h_{\gamma\delta}(q_1)$ $h_{\lambda\epsilon}(q_3)$

$\phi(p_1)$ $\phi(p_2)$

$$\mathcal{L}_{\phi\phi hhh} = \frac{\kappa^3}{8}\Bigg[-\frac{1}{4} m^2 \phi^2 h_\mu{}^\mu h^{\nu\alpha} h_{\nu\alpha} - \frac{1}{16}\phi\, \partial_\mu\phi\partial^\mu h_\nu{}^\nu h_\alpha{}^\alpha h_\beta{}^\beta - \frac{1}{2}\phi\, \partial_\mu\phi\partial^\mu h^{\nu\alpha} h_\nu{}^\beta h_{\alpha\beta}$$
$$+ \frac{1}{4}\partial_\mu\phi\partial^\mu\phi h_\nu{}^\nu h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{8}\partial_\mu\phi\partial_\nu\phi h^{\mu\nu} h_\alpha{}^\alpha h_\beta{}^\beta - \frac{1}{2}\partial_\mu\phi\partial^\nu\phi h^{\mu\alpha} h_{\nu\alpha} h_\beta{}^\beta$$
$$- \partial^\mu\phi\partial^\nu\phi h_{\mu\alpha} h_{\nu\beta} h^{\alpha\beta}\Bigg]$$

The Ghosts Vertices $V^{\{\bar{\chi}\chi h\}}_{\rho\sigma\gamma\delta}$, $V^{\{\bar{\chi}\chi hh\}}_{\gamma\delta\rho\sigma\lambda\epsilon}$:

$h_{\rho\sigma}(q_1)$

$$\mathcal{L}_{\bar{\chi}\chi h} = \frac{\kappa}{2}\Big[\ \bar{\chi}_\mu\partial^\mu\chi_\nu\partial_\alpha h^{\nu\alpha} - \bar{\chi}^\mu\chi^\nu\partial_\mu\partial_\nu h_\alpha{}^\alpha + 2\,\bar{\chi}_\mu\chi^\nu\partial_\nu\partial_\alpha h^{\mu\alpha} - \frac{1}{2}\bar{\chi}_\mu\partial_\nu\chi^\nu\partial^\mu h_\alpha{}^\alpha$$
$$- \bar{\chi}_\mu\partial_\nu\chi_\alpha\partial^\mu h^{\nu\alpha} + \bar{\chi}_\mu\partial_\nu\chi_\alpha\partial^\alpha h^{\mu\nu} - \partial^\mu\bar{\chi}_\mu\partial_\nu\chi_\alpha h^{\nu\alpha} - \partial^\mu\bar{\chi}^\nu\partial_\mu\chi_\nu h_\alpha{}^\alpha$$
$$- \partial_\mu\bar{\chi}_\nu\partial^\mu\chi_\alpha h^{\nu\alpha} + \partial_\mu\bar{\chi}_\nu\partial_\alpha\chi^\nu h^{\mu\alpha} - \partial_\mu\bar{\chi}_\nu\partial_\alpha\chi^\alpha h^{\mu\nu}\Big]$$

$\chi_\gamma(p_1)$ $\chi_\delta(p_2)$

$$\mathcal{L}_{\bar{\chi}\chi hh} = \kappa^2\Big[\ \frac{1}{8}\bar{\chi}_\mu\partial^\mu\chi_\nu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta + \frac{1}{8}\bar{\chi}_\mu\partial^\mu\chi^\nu h_{\nu\alpha}\partial_\beta h^{\alpha\beta} + \frac{1}{4}\bar{\chi}_\mu\partial^\mu\chi^\nu\partial_\nu h^{\alpha\beta}h_{\alpha\beta}$$
$$+ \frac{1}{8}\bar{\chi}_\mu\partial^\mu\chi^\nu\partial^\alpha h_{\nu\alpha}h_\beta{}^\beta + \frac{1}{4}\bar{\chi}_\mu\chi^\nu h^{\mu\alpha}\partial_\nu\partial_\alpha h_\beta{}^\beta + \frac{1}{8}\bar{\chi}^\mu\chi^\nu\partial_\mu h_\alpha{}^\alpha\partial_\nu h_\beta{}^\beta$$
$$+ \frac{1}{4}\bar{\chi}^\mu\chi^\nu\partial_\nu h_{\mu\alpha}\partial^\alpha h_\beta{}^\beta + \frac{1}{4}\bar{\chi}^\mu\chi^\nu\partial_\nu\partial^\alpha h_{\mu\alpha}h_\beta{}^\beta + \frac{1}{8}\bar{\chi}^\mu\partial^\nu\chi_\mu h_{\nu\alpha}\partial_\beta h^{\alpha\beta}$$
$$+ \cdots$$
$$- \frac{1}{2}\partial^\mu\bar{\chi}^\nu\partial^\alpha\chi^\beta h_{\mu\nu}h_{\alpha\beta} + \frac{1}{4}\partial^\mu\bar{\chi}^\nu\partial^\alpha\chi^\beta h_{\mu\alpha}h_{\nu\beta} + \frac{1}{4}\partial^\mu\bar{\chi}^\nu\partial^\alpha\chi^\beta h_{\mu\beta}h_{\nu\alpha}\Big]$$

$\chi_\lambda(p_1)$ $\chi_\epsilon(p_2)$

$h_{\gamma\delta}(q_1)$ $h_{\rho\sigma}(q_2)$

$V^{\{\bar{\chi}\chi h\}}$ 11 terms "appears in three diagrams". $V^{\{\bar{\chi}\chi hh\}}$ 29 terms "appears in one diagram".

Scalar-Graviton Scattering $\phi(p_1)\, h_{\mu\nu}(p_2) \to \phi(p_3)\, h_{\alpha\beta}(p_4)$:

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$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

$\xRightarrow[\text{helicity amplitudes}]{\text{Two independent}}$ $\mathcal{M}_{(0,+2;0,+2)}$ & $\mathcal{M}_{(0,+2;0,-2)}$

$\xRightarrow{\text{Parity}}$ $\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(-\lambda_3,-\lambda_4;-\lambda_1,-\lambda_2)}$

$\xRightarrow{\text{Time-reversal}}$ $\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(\lambda_1,\lambda_2;\lambda_3,\lambda_4)}$

$\xRightarrow{\text{Charge conjugation}}$ $\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(\lambda_4,\lambda_3;\lambda_2,\lambda_1)}$

$\xRightarrow{\text{Exchanging bosons}}$ $\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)}(s,t,u) = (-1)^{\lambda-2s_1}\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_2,\lambda_1)}(s,u,t)$

Where $\lambda = \lambda_1 - \lambda_2,\ \mu = \lambda_3 - \lambda_4,\ s_1, s_2, s_3, s_4$ are the spin of the particles.

Scalar-Graviton Scattering $\phi(p_1)\, h_{\mu\nu}(p_2) \to \phi(p_3)\, h_{\alpha\beta}(p_4)$:

$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

$$p_1 + p_2 = p_3 + p_4 \qquad s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \qquad \frac{i}{(p_1+p_2)^2 - m^2} = \frac{i}{s - m^2}$$

$$p_1^2 = p_3^2 = m^2 \qquad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \qquad \frac{i}{(p_1-p_3)^2} P^{\sigma\eta\kappa\epsilon} = \frac{i}{t} P^{\sigma\eta\kappa\epsilon}$$

$$p_2^2 = p_4^2 = 0 \qquad u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \qquad \frac{i}{(p_1-p_4)^2 - m^2} = \frac{i}{u - m^2}$$

$$s + t + u = \sum_i M_i^2 = 2m^2$$

The dispersion relation — The Mandelstam variables — The propagators

Scalar-Graviton Scattering $\phi(p_1)\, h_{\mu\nu}(p_2) \to \phi(p_3)\, h_{\alpha\beta}(p_4)$:

$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

$$p_1.p_1 = p_3.p_3 = m^2 \qquad p_1.p_4 = p_2.p_3 = \frac{m^2 - u}{2} \qquad p^\mu \epsilon^{\pm 2}_{\mu\nu}(p) = p^\nu \epsilon^{\pm 2}_{\mu\nu}(p) = 0$$

$$p_2.p_2 = p_4.p_4 = 0 \qquad p_1.p_3 = \frac{2m^2 - t}{2} \qquad \eta^{\mu\nu} \epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\nu\,\pm 2}_{\ \nu}(p) \quad = 0$$

$$p_1.p_2 = p_3.p_4 = \frac{s - m^2}{2} \qquad p_2.p_4 = -\frac{t}{2} \qquad \epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\pm 2}_{\nu\mu}(p)$$

$$\epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\pm 1}_{\mu}(p)\, \epsilon^{\pm 1}_{\nu}(p)$$

The dot products of the external momenta

Transverse, traceless, symmetric

Scalar-Graviton Scattering $\phi(p_1)\, h_{\mu\nu}(p_2) \to \phi(p_3)\, h_{\alpha\beta}(p_4)$:

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$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

In the CM frame

$$p_1 = (E, 0, 0, -k) \qquad p_3 = (E, 0, -k\sin(\theta), -k\cos(\theta))$$

$$p_2 = (k, 0, 0, k) \qquad p_4 = (k, 0, k\sin(\theta), k\cos(\theta))$$

$$\epsilon_\mu^{\pm1}(p_2) = \left(0, \frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}, 0\right) \qquad \epsilon_\alpha^{*\pm1}(p_4) = \left(0, \frac{1}{\sqrt{2}}, \frac{\mp i\cos(\theta)}{\sqrt{2}}, \frac{\pm i\sin(\theta)}{\sqrt{2}}\right)$$

The incoming particles along the z-axis

The outgoing particles scattered by angle θ

Scalar-Graviton Scattering $\phi(p_1)\, h_{\mu\nu}(p_2) \to \phi(p_3)\, h_{\alpha\beta}(p_4)$:

$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

$$\mathcal{M}_{(0,+2;0,+2)} = \kappa^2 \frac{k^4\, s^2}{(s-m^2)(u-m^2)t}\left(1+\cos(\theta)\right)^2$$
$$\mathcal{M}_{(0,+2;0,-2)} = \kappa^2 \frac{k^4\, m^4}{(s-m^2)(u-m^2)t}\left(1-\cos(\theta)\right)^2$$

Where k is the momentum of incoming particles on z-axis in CM frame.

Scalar-Graviton Scattering $\phi(p_1)\,h_{\mu\nu}(p_2) \to \phi(p_3)\,h_{\alpha\beta}(p_4)$:

$h_{\mu\nu}(p_2)$ $h_{\alpha\beta}(p_4)$

$\phi(p_1)$ (1) $\phi(p_3)$ (2) (3) (4)

$$\mathcal{M}_{(0,+2;0,+2)} = \kappa^2 \frac{k^4\, s^2}{(s-m^2)(u-m^2)t}\big(1+\cos(\theta)\big)^2$$

$$\mathcal{M}_{(0,+2;0,-2)} = \kappa^2 \frac{k^4\, m^4}{(s-m^2)(u-m^2)t}\big(1-\cos(\theta)\big)^2$$

Agree with the standard calculations.

Agree with the results from the paper "Gravitational born amplitudes and kinematical constraints" [4].

[4] M. T. Grisaru, P. van Nieuwenhuizen, and C. C. Wu, PRD12, 397.

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

$$\xrightarrow[\text{helicity amplitudes}]{\text{Four independent}} \mathcal{M}_{(+2,+2;+2,+2)} \ \& \ \mathcal{M}_{(+2,-2;+2,-2)} \ \& \ \mathcal{M}_{(+2,+2;+2,-2)} \ \& \ \mathcal{M}_{(+2,+2;-2,-2)}$$

$$\xrightarrow{\text{Parity}} \mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(-\lambda_3,-\lambda_4;-\lambda_1,-\lambda_2)}$$

$$\xrightarrow{\text{Time-reversal}} \mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(\lambda_1,\lambda_2;\lambda_3,\lambda_4)}$$

$$\xrightarrow{\text{Charge conjugation}} \mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)} = (-1)^{\lambda-\mu}\mathcal{M}_{(\lambda_4,\lambda_3;\lambda_2,\lambda_1)}$$

$$\xrightarrow{\text{Exchanging bosons}} \mathcal{M}_{(\lambda_3,\lambda_4;\lambda_1,\lambda_2)}(s,t,u) = (-1)^{\lambda-2s_1}\mathcal{M}_{(\lambda_3,\lambda_4;\lambda_2,\lambda_1)}(s,u,t)$$

Where $\lambda = \lambda_1 - \lambda_2,\ \mu = \lambda_3 - \lambda_4,\ s_1, s_2, s_3, s_4$ are the spin of the particles.

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

$$p_1 + p_2 = p_3 + p_4$$

$$p_1^2 = p_2^2 = 0$$

$$p_3^2 = p_4^2 = 0$$

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

$$s + t + u = \sum_i M_i^2 = 0$$

$$\frac{i}{(p_1 + p_2)^2} P^{\sigma\eta\kappa\epsilon} = \frac{i}{s} P^{\sigma\eta\kappa\epsilon}$$

$$\frac{i}{(p_1 - p_3)^2} P^{\sigma\eta\kappa\epsilon} = \frac{i}{t} P^{\sigma\eta\kappa\epsilon}$$

$$\frac{i}{(p_1 - p_4)^2} P^{\sigma\eta\kappa\epsilon} = \frac{i}{u} P^{\sigma\eta\kappa\epsilon}$$

The dispersion relation | The Mandelstam variables | The propagators

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

$$p_1.p_1 = p_3.p_3 = 0 \qquad p_1.p_4 = p_2.p_3 = -\frac{1}{2}u \qquad p^{\mu}\epsilon^{\pm 2}_{\mu\nu}(p) = p^{\nu}\epsilon^{\pm 2}_{\mu\nu}(p) = 0$$

$$p_2.p_2 = p_4.p_4 = 0 \qquad p_1.p_3 = -\frac{1}{2}t \qquad \eta^{\mu\nu}\epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\nu\,\pm 2}_{\ \nu}(p) = 0$$

$$p_1.p_2 = p_3.p_4 = \frac{1}{2}s \qquad p_2.p_4 = -\frac{1}{2}t \qquad \epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\pm 2}_{\nu\mu}(p)$$

$$\epsilon^{\pm 2}_{\mu\nu}(p) = \epsilon^{\pm 1}_{\mu}(p)\,\epsilon^{\pm 1}_{\nu}(p)$$

The dot products of the external momenta

Transverse, traceless, symmetric

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

In the CM frame

$$p_1 = (k, 0, 0, k) \qquad p_3 = (k, 0, k\sin(\theta), k\cos(\theta))$$

$$p_2 = (k, 0, 0, -k) \qquad p_4 = (k, 0, -k\sin(\theta), -k\cos(\theta))$$

$$\epsilon_\mu^{\pm 1}(p_1) = \left(0, \frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}, 0\right) \qquad \epsilon_\gamma^{*\pm 1}(p_3) = \left(0, \frac{1}{\sqrt{2}}, \frac{\mp i\cos(\theta)}{\sqrt{2}}, \frac{\pm i\sin(\theta)}{\sqrt{2}}\right)$$

$$\epsilon_\alpha^{\pm 1}(p_2) = \left(0, \frac{1}{\sqrt{2}}, \mp\frac{i}{\sqrt{2}}, 0\right) \qquad \epsilon_\lambda^{*\pm 1}(p_4) = \left(0, \frac{1}{\sqrt{2}}, \frac{\pm i\cos(\theta)}{\sqrt{2}}, \frac{\mp i\sin(\theta)}{\sqrt{2}}\right)$$

The incoming particles along the z-axis

The outgoing particles scattered by angle θ

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

$$\mathcal{M}_{(+2,+2;+2,+2)} = \kappa^2 \frac{1}{4}\frac{s^3}{t\,u} \qquad \mathcal{M}_{(+2,-2;+2,-2)} = \kappa^2 \frac{1}{4}\frac{u^3}{s\,t}$$

$$\mathcal{M}_{(+2,+2;+2,-2)} = 0 \qquad \mathcal{M}_{(+2,+2;-2,-2)} = 0$$

Where k is the momentum of incoming particles on z-axis in CM frame.

Graviton-Graviton Scattering $h^{\mu\nu}(p_1)h^{\alpha\beta}(p_2) \to h^{\gamma\delta}(p_3)h^{\lambda\rho}(p_4)$:

$h_{\alpha\beta}(p_2)$ $h_{\lambda\rho}(p_4)$

$h_{\mu\nu}(p_1)$ (1) $h_{\gamma\delta}(p_3)$ (2) (3) (4)

$$\mathcal{M}_{(+2,+2;+2,+2)} = \kappa^2 \frac{1}{4} \frac{s^3}{t\,u} \qquad \mathcal{M}_{(+2,-2;+2,-2)} = \kappa^2 \frac{1}{4} \frac{u^3}{s\,t}$$

$$\mathcal{M}_{(+2,+2;+2,-2)} = 0 \qquad \mathcal{M}_{(+2,+2;-2,-2)} = 0$$

Agree with the standard calculations.

Agree with the results from the paper “Gravitational born amplitudes and kinematical constraints” [4].

Agree with the results from the paper “Infrared behavior of graviton-graviton scattering” [10].

[4] M. T. Grisaru, P. van Nieuwenhuizen, and C. C. Wu, PRD12,397.
[10] J. F. Donoghue and T. Torma, PRD60, 024003.

The Graviton Self-Energy Correction:

k

$h_{\mu\nu}(P_1)$ $h_{\alpha\beta}(P_1)$

$$\mathcal{M}_{\mu\nu\alpha\beta} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} V^{\{hhh\}}_{\mu\nu\rho\sigma\eta\lambda}(k, Q_1)\ S^{\rho\sigma\gamma\delta}_{\{hh\}}(k)\ S^{\eta\lambda\epsilon\kappa}_{\{hh\}}(Q_1)\ V^{\{hhh\}}_{\gamma\delta\epsilon\kappa\alpha\beta}(k, Q_1)$$

40 × 40 = 1600 $\xRightarrow{\text{Reduced to}}$ **4 × 4 = 16**

The Graviton Self-Energy Correction:

k

$h_{\mu\nu}(P_1)$ $h_{\alpha\beta}(P_1)$

$$
\begin{aligned}
\mathcal{M}_{\mu\nu\alpha\beta} = & \frac{\kappa^2\, B_0(0,0,P_1)}{64\, d^4 - 256\, d^3 + 192\, d^2 + 256\, d - 256} \Big[\\
& + P_{1\mu} P_{1\nu} P_{1\alpha} P_{1\beta} \left(d^6 - 2\, d^4 - 116\, d^3 + 312\, d^2 + 144\, d - 256 \right) \\
& + \eta_{\mu\nu}\eta_{\alpha\beta} P_1 . P_1^2 \left(d^6 - 5\, d^5 + 31\, d^3 + 6\, d^2 - 36\, d - 8 \right) \\
& + P_1 . P_1 \left(\eta_{\mu\nu} P_{1\alpha} P_{1\beta} + \eta_{\alpha\beta} P_{1\mu} P_{1\nu} \right) \left(64 - d^6 + 3\, d^5 + 13\, d^4 - 34\, d^3 - 76\, d^2 + 40\, d \right) \Big] \\
& + \frac{\kappa^2\, B_0(0,0,P_1)}{64\, d^3 - 128\, d^2 - 64\, d + 128} \Big[\\
& + P_1 . P_1^2 \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} \right) \left(3\, d^3 - 24\, d^2 - 8\, d + 16 \right) \\
& + P_1 . P_1 \left(\eta_{\mu\alpha} P_{1\nu} P_{1\beta} + \eta_{\mu\beta} P_{1\nu} P_{1\alpha} + \eta_{\nu\alpha} P_{1\mu} P_{1\beta} + \eta_{\nu\beta} P_{1\mu} P_{1\alpha} \right) \left(32\, d^2 - 7\, d^3 + 20\, d - 16 \right) \Big]
\end{aligned}
$$

(26.38 sec) $\xRightarrow{\text{Reduced to}}$ **(0.58 sec)**

The Triple Graviton Vertex Correction:

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$\boxed{40 \times 113 = 4\,520}$ $\xRightarrow{\text{Reduced to}}$ $\boxed{4 \times 12 = 48}$

$\boxed{803.35 \text{ sec} \approx 13 \text{ min}}$ $\xRightarrow{\text{Reduced to}}$ $\boxed{20.15 \text{ sec}}$

The Triple Graviton Vertex Correction:

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$$\boxed{40 \times 40 \times 40 = 64\,000} \xRightarrow{\text{Reduced to}} \boxed{4 \times 4 \times 4 = 64}$$

$$\boxed{T_{\text{ST}} > 2 \text{ hr}} \xRightarrow{\text{Reduced to}} \boxed{T_{\text{SM}} \approx 7 \text{ min}}$$

The Triple Graviton Vertex Correction:

(1) (2) (3) (4) (5) (6)

Diagram	The number of Lagrangian terms in the standard way[1]	The number of Lagrangian terms in the simplified way[1]
(1)	10	7
(2)	The amplitude vanishes	The amplitude vanishes
(3)	18	6
(4)[2]	4 520	48
(5)	27	27
(6)[3]	64 000	64

[1] The Lagrangian terms only involved from the vertices.

[2] The running time: $T_{ST} \approx 40 \times T_{SM}$.

[3] The running time: $T_{SM} \approx 7$ min while T_{ST} is more than two hours.

The Scalar-Scalar-Graviton Vertex Correction:

(1) (2) (3) (4) (5) (6)

Diagram	The number of Lagrangian terms in the standard way[1]	The number of Lagrangian terms in the simplified way[1]
(1)	The amplitude vanishes	The amplitude vanishes
(2)	240	8
(3)	18	6
(4)	18	6
(5)	360	36
(6)	27	27

[1] The Lagrangian terms only involved from the vertices.

The Scalar-Scalar-Graviton-Graviton Vertex Correction:

(1) (2) (3) (4) (5) (6)

Diagram	The number of Lagrangian terms in the standard way[1]	The number of Lagrangian terms in the simplified way[1]
(1)	The amplitude vanishes	The amplitude vanishes
(2)	678	24
(3)	400	28
(4)	30	21
(5)	30	21
(6)	36	4

[1] The Lagrangian terms only involved from the vertices.

The Scalar-Scalar-Graviton-Graviton Vertex Correction:

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(7) (8) (9) (10) (11) (12)

Diagram	The number of Lagrangian terms in the standard way[1]	The number of Lagrangian terms in the simplified way[1]
(7)	1 017	108
(8)[2]	9 600	32
(9)	720	24
(10)	720	24
(11)	54	18
(12)	54	18

[1] The Lagrangian terms only involved from the vertices.

[2] The running time: $T_{\mathrm{SM}} < 2$ min while T_{ST} is more than one hour.

The Scalar-Scalar-Graviton-Graviton Vertex Correction:

(13) (14) (15) (16)

Diagram	The number of Lagrangian terms in the standard way[1]	The number of Lagrangian terms in the simplified way[1]
(13)	54	18
(14)	14 400	144
(15)	1 080	108
(16)	81	81

[1] The Lagrangian terms only involved from the vertices.

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Conclusions

- Simplification:

Standard Rules $\xrightarrow{\text{Reduced to}}$ Simplified Rules

Especially V_{hhh} from 40 to 4, V_{hhhh} from 113 to 12.

Conclusions

- Simplification:

Standard Rules $\xrightarrow{\text{Reduced to}}$ Simplified Rules

- Verification:

Standard Amplitude $\xleftrightarrow{\text{Equal to}}$ Simplified Amplitude

For Scalar-Graviton and Graviton-Graviton scattering at tree level.

Conclusions

- Simplification:

Standard Rules $\xrightarrow{\text{Reduced to}}$ Simplified Rules

- Verification:

Standard Amplitude $\xleftrightarrow{\text{Equal to}}$ Simplified Amplitude

- Utility:

Standard Calculation $\xrightarrow{\text{Reduced to}}$ Simplified Calculation

For some one-loop diagrams for scalar-graviton scattering.

Future Work

- More complicated diagrams:

(2 560 000) $\xRightarrow{\text{Reduced to}}$ **(256)**

The quadruple graviton vertex at one-loop level in graviton-graviton scattering.

Future Work

- More complicated diagrams:

(2 560 000) $\xrightarrow{\text{Reduced to}}$ **(256)**

- Higher order vertices:

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Future Work

- More complicated diagrams:

(2 560 000) $\xRightarrow{\text{Reduced to}}$ (256)

- Higher order vertices:

- More freedoms:


Freedom???
Freedom!!!
Freedom

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