PROGRESS REPORT-3

GRAPH THEORY

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Title of the project: Local Antimagic Labelling on Broom Graph

Keywords: Antimagic, Labelling, Chromatic Index, Broom Graph

Summary of the progress report

Hypothesis :

Broom graph $B_{n,d}$ with $n \geq 5$ and $d \geq 3$ have $\chi_{la}(B_{n,d}) = n - d + 2$.

Definitions :

1. Local Antimagic Labelling

Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$. A bijection $f\colon E \rightarrow \{1, 2,.., m\}$ is called a local antimagic labeling if for any two adjacent vertex $w(u) \neq w(u)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. Thus, any local antimagic labeling induces a proper vertex coloring of G where the vertex v is assigned the color $w(v)$. [Hartsfield and Ringel]

2. **Local Antimagic Chromatic Index**

For a vertex $x \in V$, define weight $w(x)$ as a sum from the label of the edges that is adjacent with the vertex x .

Local antimagic chromatic index ($\chi_{la}(G)$) is defined as the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G .

[Alison Marr]

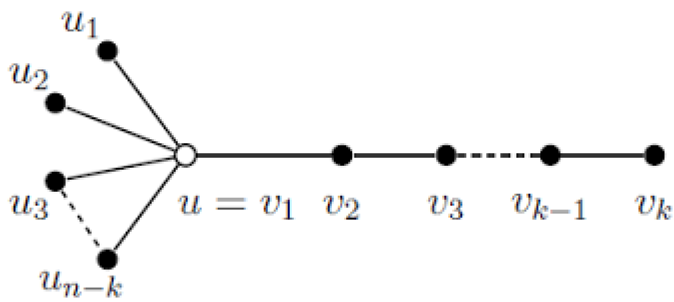
3. **Broom Graph**

A Broom Graph $B_{n,d}$ is a graph of n vertices, which have a path P with d vertices and $(n - d)$ pendant, all of these being adjacent to the end of path P.

[M.J. Morgan, S. Mukwembi, H.C.]

Figure of Broom Graph $B_{n,d}$

General form of broom graph $B_{n,d}$:(d=k)



Broom graph $B_{n,d}$ consist of path P with vertex k together with $(n - d)$.pendant vertices adjacent to vertices at the end of P

4. Proofing Broom graph $B_{n,d}$ with $n \geq 5$ and $d \geq 3$ have $\chi_{la}(B_{n,d}) = n - d + 2$.
   a. Finding formula of f as labeling local antimagic and weight
      i. From all this proofing below, we claim that we can have:

$$f(v_1 u_j) = n - j$$

$$f(v_i, v_{i+1}) = \begin{cases} \dfrac{d-i+1}{2}, & d \text{ genap dan } i \text{ ganjil atau } d \text{ ganjil dan } i \text{ genap} \\ \dfrac{d+i}{2}, & d \text{ genap dan } i \text{ genap atau } d \text{ ganjil dan } i \text{ ganjil} \end{cases}$$

f adalah pelabelan *local antimagic* dan *weight* dari simpul adalah:

$$w(v_1) = \begin{cases} \dfrac{n^2 - d^2 - n + 2d}{2}, & d \text{ genap} \\ \dfrac{n^2 - d^2 - n + 2d + 1}{2}, & d \text{ ganjil} \end{cases}$$

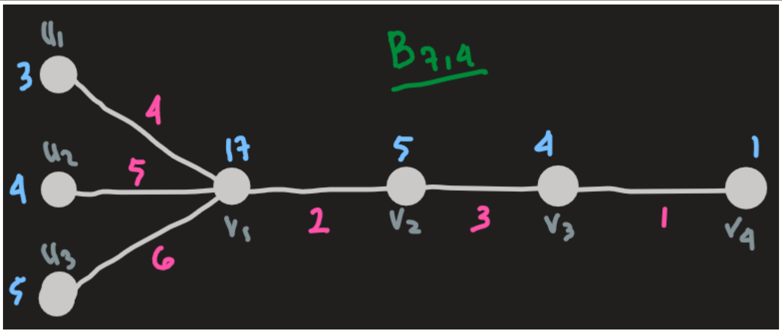
$$w(v_i) = \begin{cases} d, & d \text{ genap dan } i \text{ ganjil atau } d \text{ ganjil dan } i \text{ genap} \\ d+1, & d \text{ genap dan } i \text{ genap atau } d \text{ ganjil dan } i \text{ ganjil} \end{cases}$$

$$w(v_d) = 1$$

$$w(u_j) = n - j$$

❖

❖ **Proving by giving example**

- 

  - $d\ =\ 4\ (even)$ and $n\ =\ 7(odd)$ and $f(v_1u_j)$ and $f(v_iv_{i+1})$, with $i: 1, 2, 3\ =\ 4\ -\ 1\ =\ d\ -\ 1,\ j: 1, 2, 3\ =\ n\ -\ d$
  - $f(v_1u_1)\ =\ 6\ =\ 7\ -\ 1\ =\ n\ -\ j$

    $f(v_1u_2)\ =\ 5\ =\ 7\ -\ 2\ =\ n\ -\ j$

    $f(v_1u_3)\ =\ 4\ =\ 7\ -\ 3\ =\ n\ -\ j$

    $f(v_1v_2)\ =\ 2\ \ =\frac{4-1+1}{2}\ =\frac{d-i+1}{2}$

    $f(v_2v_3)\ =\ 3\ \ =\frac{4+2}{2}\ =\frac{d+i}{2}$

    $f(v_3v_4)\ =\ 1\ =\frac{4-3+1}{2}\ =\frac{d-i+1}{2}$

  - $w(v_1)\ =\ 17\ =\frac{7^2-4^2-7+2(4)}{2}\ =\frac{n^2-d^2-n+2d}{2}$

    $w(v_2)\ =\ 5\ =\ 4\ +\ 1\ =\ d\ +\ 1$

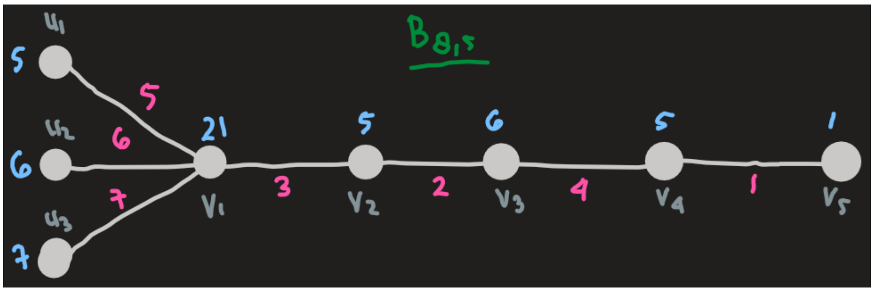
    $w(v_3)\ =\ 4\ =\ d$

    $w(v_3)\ =\ 1$

    $w(u_1)\ =\ 6\ =\ 7\ -\ 1\ =\ n\ -\ j$

    $w(u_2)\ =\ 5\ =\ 7\ -\ 2\ =\ n\ -\ j$

    $w(u_3)\ =\ 4\ =\ 7\ -\ 3\ =\ n\ -\ j$

- 

  - $d\ =\ 5\ (odd)$ and $n\ =\ 8(even)$ and $f(v_1u_j)$ and $f(v_iv_{i+1})$, with $i: 1, 2, 3, 4\ =\ d\ -\ 1, j: 1, 2, 3\ =\ n\ -\ d$
  - $f(v_1u_1)\ =\ 7\ =\ 8\ -\ 1\ =\ n\ -\ j$

    $f(v_1u_2)\ =\ 6\ =\ 8\ -\ 2\ =\ n\ -\ j$

    $f(v_1u_3)\ =\ 5\ =\ 8\ -\ 3\ =\ n\ -\ j$

    $f(v_1v_2)\ =\ 3\ \ =\frac{5+1}{2}\ =\frac{5+i}{2}$

$f(v_2v_3) = 2 = \frac{5-2+1}{2} = \frac{d-i+1}{2}$

$f(v_3v_4) = 4 = \frac{5+3}{2} = \frac{5+i}{2}$

$f(v_4v_5) = 1 = \frac{5-4+1}{2} = \frac{d-i+1}{2}$

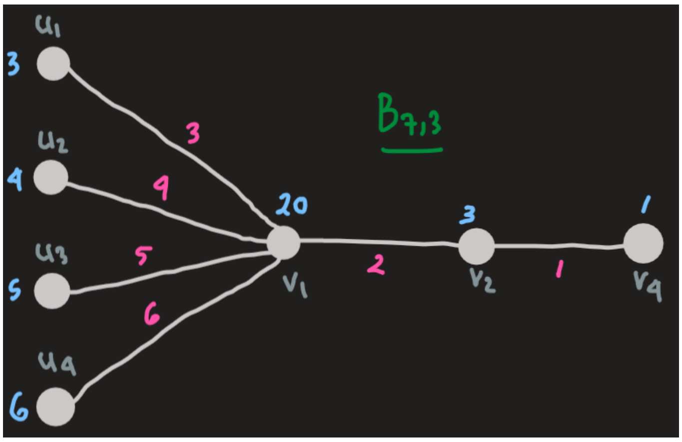
- $w(v_1) = 21 = \frac{8^2-5^2-8+2(5)+1}{2} = \frac{n^2-d^2-n+2d+1}{2}$

  $w(v_2) = 5 = d$

  $w(v_3) = 6 = 5 + 1 = d + 1$

  $w(v_4) = 5 = d$

  $w(v_5) = 1$

- 

  - $d = 3\ (odd)$ and $n = 7(odd)$ and $f(v_1u_j)$ and $f(v_iv_{i+1})$, with i: $1, 2 = d - 1$, j: $1, 2, 3, 4 = n - d$
  - $f(v_1u_1) = 6 = 7 - 1 = n - j$

    $f(v_1u_2) = 5 = 7 - 2 = n - j$

    $f(v_1u_3) = 4 = 7 - 3 = n - j$

    $f(v_1u_4) = 3 = 7 - 4 = n - j$

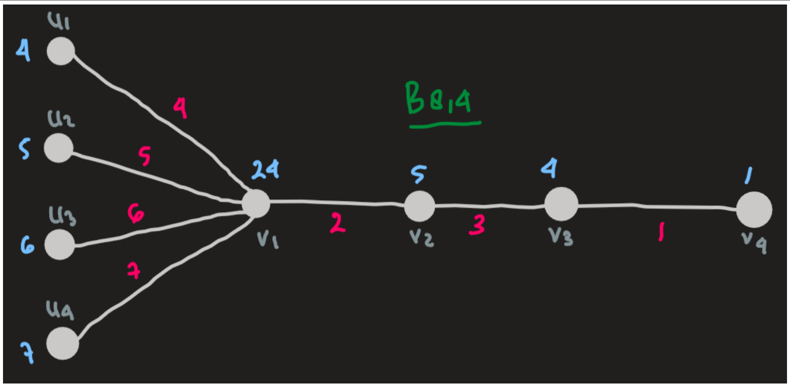
    $f(v_1v_2) = 2 = \frac{3+1}{2} = \frac{5+i}{2}$

    $f(v_2v_3) = 1 = \frac{3-2+1}{2} = \frac{d-i+1}{2}$

  - $w(v_1) = 20 = \frac{7^2-3^2-7+2(3)+1}{2} = \frac{n^2-d^2-n+2d+1}{2}$

    $w(v_2) = 3 = d$

    $w(v_3) = 1$



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  - $d = 4\ (even)$ and $n = 8(even)$ and $f(v_1u_j)$ and $f(v_iv_{i+1})$, with $i: 1, 2, 3 = d - 1,\ j: 1, 2, 3, 4 = n - d$
  - $f(v_1u_1) = 7 = 8 - 1 = n - j$

    $f(v_1u_2) = 6 = 8 - 2 = n - j$

    $f(v_1u_3) = 5 = 8 - 3 = n - j$

    $f(v_1u_4) = 4 = 8 - 4 = n - j$

    $f(v_1v_2) = 2 = \frac{4-1+1}{2} = \frac{d-i+1}{2}$

    $f(v_2v_3) = 3 = \frac{4+2}{2} = \frac{d+i}{2}$

    $f(v_3v_4) = 1 = \frac{4-3+1}{2} = \frac{d-i+1}{2}$
  - $w(v_1) = 24 = \frac{8^2-4^2-8+2(4)}{2} = \frac{n^2-d^2-n+2d}{2}$

    $w(v_2) = 5 = 4 + 1 = d + 1$

    $w(v_3) = 4 = d$

    $w(v_4) = 1$

    $w(u_1) = 7 = 8 - 1 = n - j$

    $w(u_2) = 6 = 8 - 2 = n - j$

    $w(u_3) = 5 = 8 - 3 = n - j$

    $w(u_4) = 4 = 8 - 4 = n - j$

❖ **Proving by Induction**

❖ $f(v_1u_j)$

➢ Assume its true for and $n = 5\ and\ j = 1$, so that we have:

$f(v_1u_1) = 4 = 5 - 1 = n - j$

➢ Assume its true for and $n = k\ and\ j = m$, so that we have:

$f(v_1u_m) = k - m$

➢ Induction for $n = k + 1\ and\ j = m + 1$, so that we have:

$f(v_1 u_{m+1}) = (k + 1) - (m + 1) = k - m + 1 - 1 = k - m$
$= f(v_1 u_m)$

➢ Proven.

❖ $f(v_i v_{i+1})$

$$f(v_i, v_{i+1}) = \begin{cases} \frac{d-i+1}{2}, & d\ genap\ dan\ i\ ganjil\ atau\ d\ ganjil\ dan\ i\ genap \\ \frac{d+i}{2}, & d\ genap\ dan\ i\ genap\ atau\ d\ ganjil\ dan\ i\ ganjil \end{cases}$$

➢ d even and i odd
- Assume its true for $d = 6$ and $i = 3$, so that we have:

  $f(v_3 v_{3+1}) = 2 = \frac{6-3+1}{2} = \frac{d-i+1}{2}$
- Assume its true for $d = k\ and\ i = m$, so that we have:

  $f(v_m v_{m+1}) = \frac{k-m+1}{2}$
- Because d even and i odd, we need to proof that this formula also true for $d = 2k\ \ and\ i = 2m + 1$:

  $f(v_{2m+1} v_{2m+2}) = \frac{2k-(2m+1)+1}{2}$

  with: $g = 2k\ \ and\ h = 2m + 1$

  we can get: $f(v_{2m+1} v_{2m+2}) = \frac{2k-(2m+1)+1}{2} = \frac{g-h+1}{2}$
- Proven

➢ d odd and i even
- Assume its true for $d = 5$ and $i = 4$, so that we have:

  $f(v_4 v_{4+1}) = 1 = \frac{5-4+1}{2} = \frac{d-i+1}{2}$
- Assume its true for $d = k\ and\ i = m$, so that we have:

  $f(v_m v_{m+1}) = \frac{k-m+1}{2}$
- Because d odd and i even, we need to proof that this formula also true for $d = 2k + 1\ \ and\ i = 2m$:

  $f(v_{2m} v_{2m+1}) = \frac{(2k+1)-2m+1}{2}$

  with: $g = 2k + 1\ \ and\ h = 2m$

  we can get: $f(v_{2m} v_{2m+1}) = \frac{(2k+1)-2m+1}{2} = \frac{g-h+1}{2}$
- Proven

➢ d even and i even
- Assume its true for $d = 6$ and $i = 4$, so that we have:

  $f(v_4 v_{4+1}) = 5 = \frac{6+4}{2} = \frac{d+i}{2}$
- Assume its true for $d = k\ and\ i = m$, so that we have:

$f(v_m v_{m+1}) = \frac{k+m}{2}$

- Because both of d and i are even, we need to proof that this formula also true for $d = 2k\ and\ i = 2m$:

  $f(v_{2m} v_{2m}) = \frac{(2k)+(2m)}{2}$

  with: $g = 2k\ and\ h = 2m$

  we can get: $f(v_{2m} v_{2m}) = \frac{(2k)+(2m)}{2} = \frac{g+h}{2}$

- Proven

➢ d odd and i odd

- Assume its true for $d = 5$ and $i = 3$, so that we have:

  $f(v_3 v_{3+1}) = 4 = \frac{5+3}{2} = \frac{d+i}{2}$

- Assume its true for $d = k\ and\ i = m$, so that we have:

  $f(v_m v_{m+1}) = \frac{k+m}{2}$

- Because both of d and i are odd, we need to proof that this formula also true for $d = 2k + 1\ and\ i = 2m + 1$:

  $f(v_{2m+1} v_{2m+2}) = \frac{(2k+1)+(2m+1)}{2} = \frac{g+h}{2}$

  with: $g = 2k + 1\ and\ h = 2m + 1$

  we can get: $f(v_{2m+1} v_{2m+2}) = \frac{(2k+1)+(2m+1)}{2} = \frac{g+h}{2}$

- Proven

❖ $w(v_1)$

$$w(v_1) = \begin{cases} \dfrac{n^2 - d^2 - n + 2d}{2}, & d\ genap \\ \dfrac{n^2 - d^2 - n + 2d + 1}{2}, & d\ ganjil \end{cases}$$

➢ d even and n even

- Assume its true for $n = 6\ and\ d = 4$, so that we have:

$w(v_1) = 11 = \frac{6^2-4^2-6+8}{2} = \frac{n^2-d^2-n+2d}{2}$

- Assume its true for $d = k$ and $n = m$

  $\sum_{j=1}^{m-k} f(v_1 u_j) + f(v_1 v_2)$

  $= \sum_{j=1}^{m-k} m - j + (\frac{k-1+1}{2})$

  $= (m - 1) + (m - 2) + ... + (m - (m - k)) + (\frac{k}{2})$

  $= m(m - k) - (1 + 2 + ... + (m - k)) + (\frac{k}{2})$

$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k}{2})$

$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k}{2})$

$= m^2 - mk - \frac{m^2+k^2-2k-2mk+m}{2}$

$= \frac{2m^2-2mk-(m^2+k^2-2k-2mk+m)}{2} = \frac{m^2-k^2-m+2k}{2}$

- Because both d and n are even, we need to proof that this formula also true for $d = 2k$ and $n = 2m$

$$\sum_{j=1}^{2m-2k} 2m - j + (\frac{2k-1+1}{2})$$

$= (2m - 1) + (2m - 2) +...+ (2m - (2m - 2k)) + (\frac{2k}{2})$

$= 2m(2m - 2k) - (1 + 2 +...+ (2m - 2k)) + (\frac{2k}{2})$

$= 2m(2m - 2k) - \frac{(2m-2k)(2m-2k+1)}{2} + (\frac{2k}{2})$

$= 2m(2m - 2k) - \frac{(2m-2k)(2m-2k+1)}{2} + (\frac{2k}{2})$

$= 4m^2 - 4mk - \frac{4m^2-8km+4k^2+(2m-2k)}{2} + (\frac{2k}{2})$

$= \frac{8m^2-8mk-(4m^2-8km+4k^2+(2m-2k))+2k}{2} = \frac{4m^2-4k^2-2m+4k}{2}$

with: $g = 2m,\ h = 2k$ we can get:

$\frac{4m^2-4k^2-2m+4k}{2} = \frac{g^2-h^2-g+2h}{2}$

- Proven

➢ d even and n odd

- Assume its true for $n = 5\ and\ d = 4$, so that we have:

$w(v_1) = 6 = \frac{5^2-4^2-5+8}{2} = \frac{n^2-d^2-n+2d}{2}$

- Assume its true for $d = k\ and\ n = m$, so that we have:

$$\sum_{j=1}^{m-k} f(v_1u_j) + f(v_1v_2)$$

$= \sum_{j=1}^{m-k} m - j + (\frac{k-1+1}{2})$

$= (m - 1) + (m - 2) +...+ (m - (m - k)) + (\frac{k}{2})$

$= m(m - k) - (1 + 2 +...+ (m - k)) + (\frac{k}{2})$

$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k+1}{2})$

$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k}{2})$

$$= m^2 - mk - \frac{m^2+k^2-2k-2mk+m}{2}$$

$$= \frac{2m^2-2mk-(m^2+k^2-2k-2mk+m)}{2} = \frac{m^2-k^2-m+2k}{2}$$

- Because d even and n odd, we need to proof that this formula also true for $d = 2k\ and\, n = 2m + 1$:

$$\sum_{j=1}^{2m+1-2k} 2m + 1 - j + (\frac{2k-1+1}{2})$$

$$= (2m + 1 - 1) + (2m + 1 - 2) +...$$

$$+ (2m + 1 - (2m + 1 - 2k)) + (\frac{2k}{2})$$

$$= (2m + 1)(2m + 1 - 2k) - (1 + 2 +... +$$

$$+ (2m + 1 - 2k)) + (\frac{2k}{2})$$

$$= (2m + 1)(2m + 1 - 2k) - \frac{(2m+1-2k)(2m+1-2k+1)}{2} + (\frac{2k}{2})$$

$$= 4m^2 + 4m + 1 - 4km - 2k - \frac{4m^2+4k^2+6m-8mk-6k+2}{2}$$

$$+ (\frac{2k}{2})$$

$$= \frac{(8m^2+8m+2-8km-4k)-(4m^2+4k^2+6m-8mk-6k+2)+2k}{2}$$

$$= \frac{4m^2-4k^2-2m+4k}{2}$$

with: $g = 2m,\ h = 2k$ we can get:

$$\frac{4m^2-4k^2-2m+4k}{2} = \frac{g^2-h^2-g+2h}{2}$$

- Proven

➢ d odd and n odd

- Assume its true for $n = 5\ and\, d = 3$, so that we have:

$$w(v_1) = 9 = \frac{5^2-3^2-5+6+1}{2} = \frac{n^2-d^2-n+2d+1}{2}$$

- Assume its true for $d = k\ and\, d = m$, so that we have:

$$\sum_{j=1}^{m-k} f(v_1u_j) + f(v_1v_2)$$

$$= \sum_{j=1}^{m-k} m - j + (\frac{k+1}{2})$$

$$= (m - 1) + (m - 2) +... + (m - (m - k)) + (\frac{k+1}{2})$$

$$= m(m - k) - (1 + 2 +... + (m - k)) + (\frac{k+1}{2})$$

$$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k+1}{2})$$

$$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k+1}{2})$$

$$= m^2 - mk - \frac{m^2+k^2-2k-2mk+m-1}{2}$$

$$= \frac{2m^2-2mk-(m^2+k^2-2k-2mk+m-1)}{2} = \frac{m^2-k^2-m+2k+1}{2}$$

- Because both d and n are odd, we need to proof that this formula also true for $d = 2k + 1$ *and* $d = 2m + 1$:

$$\sum_{j=1}^{2m+1-(2k+1)} 2m + 1 - j + (\frac{2k+1+1}{2})$$

$$\sum_{j=1}^{2m-2k} 2m + 1 - j + (\frac{2k+2}{2})$$

$$= (2m + 1 - 1) + (2m + 1 - 2) + ...$$
$$+ (2m + 1 - (2m + 1 - (2m - 2k)))$$
$$+ (\frac{2k+2}{2})$$

$$= (2m + 1)(2m + 1 - (2k + 1)) - (1 + 2 + ... +$$
$$+ (2m - 2k)) + (\frac{2k+2}{2})$$

$$= (2m + 1)(2m - 2k) - \frac{(2m-2k)(2m-2k+1)}{2} + (\frac{2k+1+1}{2})$$

$$= 4m^2 + 2m - 4km - 2k - \frac{4m^2+4k^2+2m-8mk-2k}{2}$$
$$+ (\frac{2k+1+1}{2})$$

$$= \frac{(8m^2+4m-8km-4k)-(4m^2+4k^2+2m-8mk-2k)+2k+2}{2}$$

$$= \frac{4m^2-4k^2+2m-2k+(2k+2)}{2} = \frac{4m^2-4k^2+2m+2}{2}$$

with:

$g = 2m + 1,\ h = 2k + 1$ we can get:

$$\frac{4m^2-4k^2+2m+2}{2}$$

$$= \frac{4m^2-4k^2+4m-2m+2}{2}$$

$$= \frac{4m^2-4k^2+4m-2m-4k+4k+1+1-1-1+2}{2}$$

$$= \frac{4m^2+4m+1-(4k^2+4k+1)-(2m+1)+2(2k+1)+1}{2}$$

$$= \frac{g^2-h^2-g+2h+1}{2}$$

- Proven.

➢ d odd and n even

- Assume its true for $n = 4$ *and* $d = 3$, so that we have:

$$w(v_1) = 5 = \frac{4^2-3^2-4+6+1}{2} = \frac{n^2-d^2-n+2d+1}{2}$$

- Assume its true for $d = k$ *and* $d = m$, so that we have:

$$\sum_{j=1}^{m-k} f(v_1 u_j) + f(v_1 v_2)$$

$$= \sum_{j=1}^{m-k} m - j + (\frac{k+1}{2})$$

$$= (m - 1) + (m - 2) +...+ (m - (m - k)) + (\frac{k+1}{2})$$

$$= m(m - k) - (1 + 2 +...+ (m - k)) + (\frac{k+1}{2})$$

$$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k+1}{2})$$

$$= m(m - k) - \frac{(m-k)(m-k+1)}{2} + (\frac{k+1}{2})$$

$$= m^2 - mk - \frac{m^2+k^2-2k-2mk+m-1}{2}$$

$$= \frac{2m^2-2mk-(m^2+k^2-2k-2mk+m-1)}{2} = \frac{m^2-k^2-m+2k+1}{2}$$

- Because d odd and n even, we need to proof that this formula also true for $d = 2k + 1$ $and\, n = 2m$:

$$\sum_{j=1}^{2m-(2k+1)} 2m - j + (\frac{2k+1+1}{2})$$

$$\sum_{j=1}^{2m-2k-1} 2m - j + (\frac{2k+2}{2})$$

$$= (2m - 1) + (2m + 1 - 2) +...$$

$$+ (2m - (2m + 1 - (2m - 2k - 1)))$$

$$+ (\frac{2k+2}{2})$$

$$= (2m)(2m - 2k - 1) - (1 + 2 +...+$$

$$+ (2m - 2k - 1)) + (\frac{2k+2}{2})$$

$$= (2m)(2m - 2k - 1) - \frac{(2m-2k-1)(2m-2k)}{2} + (\frac{2k+2}{2})$$

$$= 4m^2 - 4km - 2m - \frac{4m^2+4k^2-2m-8mk+2k}{2}$$

$$+ (\frac{2k+2}{2})$$

$$= \frac{(8m^2-8km-4m)-(4m^2+4k^2-2m-8mk-2k)+2k+2}{2}$$

$$= \frac{4m^2-4k^2-2m-2k+(2k+2)}{2} = \frac{4m^2-4k^2-2m+2}{2}$$

with:

$g = 2m$, $h = 2k + 1$ we can get:

$$\frac{4m^2-4k^2-2m+2}{2}$$

$$= \frac{4m^2-4k^2-2m+2-1+1}{2}$$

$$= \frac{4m^2-4k^2-2m-4k+4k+1-1+2}{2}$$

$$= \frac{4m^2-(4k^2+4k+1)-(2m+1)+2(2k+1)+1}{2}$$

$$= \frac{g^2-h^2-g+2h+1}{2}$$

■ Proven.

❖ $w(v_i)$

$$w(v_i) = \begin{cases} d, & d\ genap\ dan\ i\ ganjil\ atau\ d\ ganjil\ dan\ i\ genap \\ d+1, & d\ genap\ dan\ i\ genap\ atau\ d\ ganjil\ dan\ i\ ganjil \end{cases}$$

➢ d even and i odd

- Assume its true for $n = 6$ and $d = 3$, so that we have:

  $$w(v_1) = 11 = \frac{6^2-3^2-6+6}{2} = \frac{n^2-d^2-n+2d}{2}$$

- Assume its true for $d = k\ and\ i = m$, so that we have:

  $$w(v_m) = k$$

- Because d odd and i even, we need to proof that this formula also true for $d = 2k\ and\ n = 2m$:

  $$w(v_{2m+1}) = 2k$$

  with:

  $g = 2m,\ h = 2k + 1$ we can get: $w(v_{2m+1}) = 2k = g$

- Proven

➢ d odd and i even

- Assume its true for $d = 5$ and $i = 2$, so that we have:

  $$w(v_5) = 5$$

- Assume its true for $d = k\ and\ i = m$, so that we have:

  $$w(v_m) = k$$

- Because d odd and i even, we need to proof that this formula also true for $d = 2k + 1\ and\ i = 2m$:

  $$w(v_{2m}) = 2k + 1$$

  with:

  $g = 2k + 1$ we can get: $w(v_{2m}) = 2k + 1 = g$

- Proven

➢ d even and i even

- Assume its true for $d = 6$ and $i = 2$, so that we have:

  $$w(v_2) = 7 = 6 + 1 = d + 1$$

- Assume its true for $d = k\ and\ i = m$, so that we have:

$w(v_m) = k + 1$

- Because both d and i are even, we need to proof that this formula also true for $d = 2k$ $and\ i = 2m$:

  $w(v_{2m}) = 2k + 1$

  with:

  $g = 2k$ we can get: $w(v_{2m})$ $=2k + 1 = g + 1$

- Proven

➢ d odd and i odd

- Assume its true for $d = 5$ and $i = 3$, so that we have:

  $w(v_3) = 6 = 5 + 1 = d + 1$

- Assume its true for $d = k\ and\ i = m$, so that we have:

  $w(v_m) = k + 1$

- Because both d and i are even, we need to proof that this formula also true for $d = 2k + 1\ and\ i = 2m + 1$:

  $w(v_{2m+1}) = (2k + 1) + 1$

  with:

  $g = 2k + 1$ we can get:

  $w(v_{2m+1})$ $=(2k + 1) + 1 = g + 1$

- Proven

❖ $w(v_d) = 1$

➢ Assume its true for and $i = 1$, so that we have:

$w(v_i) = 1$

➢ Assume its true for and $i = k$, so that we have:

$w(u_k) = 1$

➢ Induction for and $i = k + 1$, so that we have:

$w(u_{k+1}) = 1 = w(u_k)$

➢ Proven.

❖ $w(u_j) = n - j$

➢ Assume its true for and $n = 5\ and\ j = 1$, so that we have:

$w(u_j) = 5 = 5 - 1 = n - j$

➢ Assume its true for and $n = k\ and\ j = m$, so that we have:

$w(u_j) = k - m$

➢ Induction for and $k + 1\ and\ m + 1$, so that we have:

$w(u_{j+1}) = (k + 1) - (m + 1) = k - m + 1 - 1 = k - m$

$= w(u_j)$

➢ Proven.

b. Finding upper and lower bound of $\chi_{la}(B_{n,d})$

Upper bound

For $j = n - d$ **we will get:**

$w\left(u_{n-d}\right) = (n - d) + d - (n - d) = d$

For $j = n - d - 1$ **we will get:**

$w\left(u_{n-d-1}\right) = (n - d) + d - (n - d - 1) = d + 1$

So, $w\left(v_i\right)$ **will produce same color with** $w\left(u_{n-d}\right)$ **and** $w\left(u_{n-d-1}\right)$

$w\left(u_j\right)$ **will produce** $n - d$ **different color.**

$w\left(v_1\right)$ **will produce** 1 **different color.**

$w\left(v_d\right)$ **will produce 1 different color.**

We will have $(n - d) + 1 + 1 = n - d + 2$ **different color, so** $\chi_{la}(B_{n,d}) \leq n - d + 2$

Lower bound

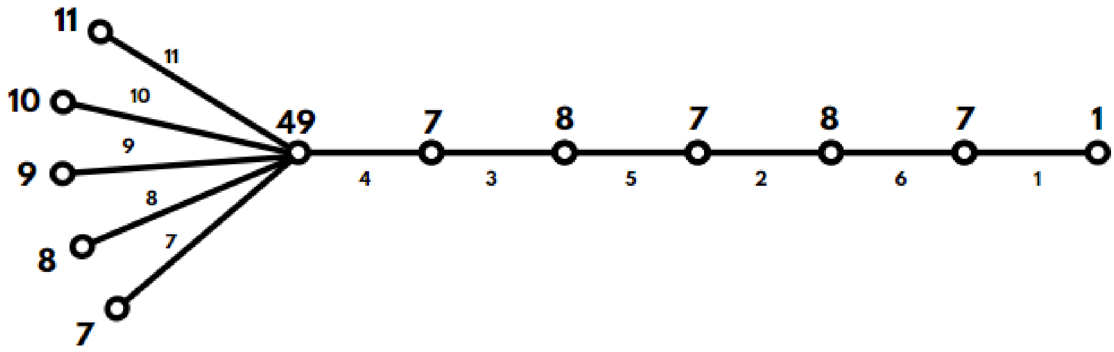
We can use the theorem "For any tree T **with** l **leaves,** $\chi_{la}(G) \geq l + 1$**".**

We know that graph $B_{n,d}$ **has** $n - d + 1$ **leaves, so** $l = n - d + 1$**.**

According to the theorem, we get $\chi_{la}(B_{n,d}) \geq n - d + 2$**.**

By finding upper and lower bound its proving Broom graph $B_{n,d}$ **with** $n \geq 5$ **and** $d \geq 3$ **have** $\chi_{la}(B_{n,d}) = n - d + 2$

Example :



Local antimagic labeling on $B_{12,7}$ **with** $\chi_{la}(B_{12,7}) = 7$

- Edge labelling

$f(v_1u_1) = 11 = 12 - 1 = n - j$

$f(v_1u_2) = 10 = 12 - 2 = n - j$

$f(v_1u_3) = 9 = 12 - 3 = n - j$

$f(v_1u_4) = 8 = 12 - 4 = n - j$

$f(v_1u_5) = 7 = 12 - 5 = n - j$

$f(v_1v_2) = \frac{d+i}{2} = \frac{7+1}{2} = 4$

$f(v_2v_3) = \frac{d-i+1}{2} = \frac{7-2+1}{2} = 3$

$f(v_3v_4) = \frac{d+i}{2} = \frac{7+3}{2} = 5$

$f(v_4v_5) = \frac{d-i+1}{2} = \frac{7-4+1}{2} = 2$

$f(v_5v_6) = \frac{d+i}{2} = \frac{7+5}{2} = 6$

$f(v_6v_7) = \frac{d-i+1}{2} = \frac{7-6+1}{2} = 1$

- Vertex Labelling

$w(v_1) = \frac{n^2-d^2-n+2d+1}{2} = \frac{12^2-7^2-12+2(7)+1}{2} = 49$

$$w(v_i) = \begin{cases} d = 7 & i = 3,5 \\ d+1 = 8 & i = 2,4,6 \end{cases}$$

$w(v_7) = 1$

$w(u_1) = n - j = 12 - 1 = 11$

$w(u_2) = n - j = 12 - 2 = 10$

$w(u_3) = n - j = 12 - 3 = 9$

$w(u_4) = n - j = 12 - 4 = 8$

$w(u_5) = n - j = 12 - 5 = 7$

- Lower Bound

  There are 7 color needed which is 1,7,8,9,10,11, and 49. So $\chi_{la}(B_{12,7}) \leq 7$

- Upper Bound

  There are 6 leaf node which makes the upper bound $\chi_{la}(B_{12,7}) \geq 7$

- Conclusion

  By $\chi_{la}(B_{12,7}) \leq 7$ and $\chi_{la}(B_{12,7}) \geq 7$, or in other way $7 \leq \chi_{la}(B_{12,7}) \leq 7$

  we get $\chi_{la}(B_{12,7}) = 7$

From all explanation and example above, we can conclude that broom graph $B_{n,d}$ with $n \geq 5$ and $d \geq 3$ will have $\chi_{la}(B_{n,d}) = n - d + 2$.

NOTES:

We will do labeling the edges of the graph (the broom graph) $B_{n,d}$ with using bijection function $f: E \rightarrow \{1, 2, n - 1\}$.

We will do labeling the edges of the graph (the broom graph) $B_{n,d}$ with using bijection function $f: E \rightarrow \{1, 2, n - 1\}$.

f defined as local antimagic labeling, and we get:

$$f(v_i u_j) = n - j$$

$$f(v_i v_{i+1}) = \begin{cases} \dfrac{d+i}{2}, & for\ d\ even\ and\ i\ odd, or\ d\ odd\ and\ i\ even \\ \dfrac{d-i+1}{2}, & for\ d\ even\ and\ i\ even, or\ d\ odd\ and\ i\ odd \end{cases}$$

w **defined as weight of the vertices, and we get:**

$$w(v_1) = \begin{cases} \dfrac{n^2 - d^2 - n + 2d + 1}{2}, & for\ d\ odd \\ \dfrac{n^2 - d^2 - n + 2d}{2}, & for\ d\ even \end{cases}$$

$$w(v_i) = \begin{cases} d, & for\ d\ even\ and\ i\ odd, or\ d\ odd\ and\ i\ even \\ \dfrac{d+1}{2}, & for\ d\ even\ and\ i\ even, or\ d\ odd\ and\ i\ odd \end{cases}$$

$$w(v_d) = 1$$

$$w(u_j) = n - j$$

$V = \{v_1\} \cup \{v_i|$

$v_1\, v_2\, v_3\, v_d$

$u_1\, u_2\, u_3\, u_{n-d}$

Broom Graph $B_{n,d}$ **with** $n \geq 5$ **and** $d \geq 3$ **where the vertices set of is**

$$V = \{v_1\} \cup \{v_i|2 \leq i \leq d - 1\} \cup \{u_j|1 \leq j \leq n - d\}$$

and the set of edges is

$$E = \{v_i v_{i+1}|2 \leq i \leq d - 1\} \cup \{v_1 u_j|1 \leq j \leq n - d\}$$