P8160_Project_2

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We assume,

$$Y_i = N(t_i, \boldsymbol{\theta}) + \epsilon_i,$$

where $N(t_i, \boldsymbol{\theta})$ is the Richard growth function and $\epsilon_i' s$ are i.i.d $N(0, \sigma^2)$.

We aim to minimize the sum of squared errors:

$$\begin{aligned} h(\boldsymbol{\theta}) &= \sum_i (Y_i - N(t_i, \boldsymbol{\theta}))^2 \\ &= \sum_{i=1}^{n} \left[Y_i - a \left\{ 1 + d \exp\left\{ -k(t_i - t_0) \right\} \right\}^{-1/d} \right]^2 \end{aligned}$$

.

By chain rule, calculating the gradient of $h(\boldsymbol{\theta})$ is reduced to calculation of the gradient of $N(t_i, \boldsymbol{\theta})$.

The gradient is:

$$\begin{aligned} \nabla h(\boldsymbol{\theta}) &= \sum_i 2[-Y_i + N(t_i, \boldsymbol{\theta})] \cdot \nabla N(t_i, \boldsymbol{\theta}) \\ &= \sum_i 2[-Y_i + N(t_i, \boldsymbol{\theta})] \cdot \left(\frac{\partial N(t_i, \boldsymbol{\theta})}{\partial a}, \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial d}, \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial k}, \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial t_0} \right) \end{aligned}$$

where:

$$\begin{aligned} \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial a} &= (1 + de^{-k(t-t_0)})^{-1/d} \\ \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial k} &= -\frac{a(t_0 - t)e^{-k(t-t_0)}}{(1 + de^{-k(t-t_0)})^{1+1/d}} \\ \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial d} &= -\frac{a(e^{-k(t-t_0)}d - \ln(1 + e^{-k(t-t_0)}d)(1 + e^{-k(t-t_0)}d))}{d^2(1 + de^{-k(t-t_0)})^{1+1/d}} \\ \frac{\partial N(t_i, \boldsymbol{\theta})}{\partial t_0} &= -\frac{ake^{-k(t-t_0)}}{(1 + de^{-k(t-t_0)})^{1+1/d}} \end{aligned}$$