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MLMC: Machine Learning Monte Carlo for Lattice Gauge Theory

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We present a trainable framework for efficiently generating gauge configurations, and discuss ongoing work in this direction. In particular, we consider the problem of sampling configurations from a 4D $SU(3)$ lattice gauge theory, and consider a generalized leapfrog integrator in the molecular dynamics update that can be trained to improve sampling efficiency.

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1. Introduction

We would like to calculate observables O:

$$\langle O\rangle \propto \int [\mathcal{D}x]\; O(x)\, \pi(x) \tag{1}$$

where $\pi(x) \propto e^{-\beta S(x)}$ is our target distribution.
If these were independent, we could approximate the integral as $\langle O\rangle \simeq \frac{1}{N}\sum_{n=1}^{N} O(x_n)$ with variance

$$\sigma_O^2 = \frac{1}{N}\operatorname{Var}\left[O(x)\right] \Longrightarrow \sigma_O \propto \frac{1}{\sqrt{N}}. \tag{2}$$

Instead, nearby configurations are correlated, causing us to incur a factor of τ_{int}^{O} in the variance expression

$$\sigma_O^2 = \frac{\tau_{\text{int}}^{O}}{N}\operatorname{Var}\left[O(x)\right]. \tag{3}$$

1.1 Hamiltonian Monte Carlo (HMC)

The typical approach [8, 9] is to use Hamiltonian Monte Carlo (HMC) algorithm for generating configurations distributed according to our target distribution $\pi(x)$. This can be done by sequentially constructing a chain of states:

$$x_0 \longrightarrow x_1 \longrightarrow x_i \longrightarrow \cdots \longrightarrow x_n \tag{4}$$

such that, as $n \longrightarrow \infty$:

$$\{x_i, x_{i+1}, x_{i+2}, \ldots, x_n\} \sim \pi(x). \tag{5}$$

To do this, we begin by introducing a fictitious momentum[1] $v \sim \mathcal{N}(0,1)$ normally distributed, independent of x. We can write the joint distribution $\pi(x, v)$ as

$$\pi(x, v) = \pi(x)\pi(v) \propto e^{-S(x)} e^{-\frac{1}{2}v^T v} \tag{6}$$

$$= e^{-\left[S(x) + \frac{1}{2}v^T v\right]} \tag{7}$$

We can evolve the Hamiltonian dynamics of the $(\dot{x}, \dot{v}) = (\partial_v H, -\partial_x H)$ system using operators $\Gamma: v \longrightarrow v'$ and $\Lambda: x \longrightarrow x'$. Explicitly, for a single update step of the leapfrog integrator:

$$\tilde{v} := \Gamma(x, v) = v - \frac{\varepsilon}{2}F(x) \tag{8}$$

$$x' := \Lambda(x, \tilde{v}) = x + \varepsilon\tilde{v} \tag{9}$$

$$v' := \Lambda(x', \tilde{v}) = \tilde{v} - \frac{\varepsilon}{2}F(x'), \tag{10}$$

where we've written the force term as $F(x) = \partial_x S(x)$. Typically, we build a trajectory of N_{LF} leapfrog steps

$$(x_0, v_0) \longrightarrow (x_1, v_1) \longrightarrow \cdots \longrightarrow (x', v'), \tag{11}$$

and propose x' as the next state in our chain. This proposal state is then accepted according to the Metropolis-Hastings criteria [25]

$$A(x'|x) = \min\left\{1, \frac{\pi(x')}{\pi(x)}\left|\frac{\partial x'}{\partial x}\right|\right\}. \tag{12}$$

[1] Here $\sim$ means *is distributed according to*.

2. Method

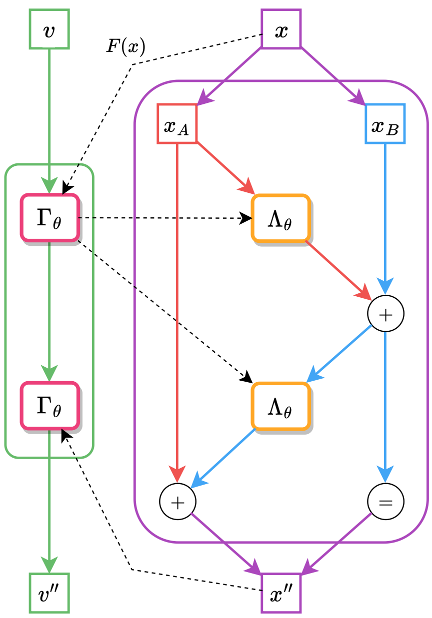
Unfortunately, HMC is known to suffer from long auto-correlations and often struggles with multi-modal target densities. To combat this, we propose building on the approach from [8–10]. We introduce two (invertible) neural networks `xNet` : $(x, v) \to (\alpha_x, \beta_x, \gamma_x)$, `vNet` : $(x, F) \to (\alpha_v, \beta_v, \gamma_v)$. Here, (α, β, γ) are all of the same dimensionality as x and v, and are parameterized by a set of weights θ. These network outputs (α, β, γ) are then used in a generalized MD update (as shown in Fig 2) via:

$$\boxed{\Gamma_\theta^\pm} : (x, v) \to (x, v') \tag{13}$$

$$\boxed{\Lambda_\theta^\pm} : (x, v) \to (x', v). \tag{14}$$

where the superscript $\pm$ on $\Gamma_\theta^\pm$, $\Lambda_\theta^\pm$ correspond to the direction $d \sim \mathcal{U}(-1, +1)$ of the update. To ensure that our proposed update remains reversible, we split the x update into two sub-updates on complementary subsets ($x = x_A \cup x_B$):

Figure 1: Generalized MD update.



$$
\begin{aligned}
v' &= \Gamma_\theta(x, v) && (15)\\
x' &= x_B + \Lambda_\theta(x_A, v') && (16)\\
x'' &= x'_A + \Lambda_\theta(x'_B, v') && (17)\\
v'' &= \Gamma_\theta(x'', v') && (18)
\end{aligned}
$$

2.1 Algorithm

1. `input:` x
   - Re-sample $v \sim \mathcal{N}(0, 1)$
   - Construct initial state $\xi := (x, v)$
2. `forward:` Generate proposal ξ' by passing initial ξ through N_{LF} leapfrog layers:

$$\xi \xrightarrow{\text{LF Layer}} \xi_1 \to \cdots \to \xi_{N_{\mathrm{LF}}} = \xi' := (x'', v'') \tag{19}$$

   - Metropolis-Hastings accept / reject:

$$A(\xi'|\xi) = \min\left\{1, \frac{\pi(\xi')}{\pi(\xi)} \left|\mathcal{J}(\xi', \xi)\right|\right\}, \tag{20}$$

   where $|\mathcal{J}(\xi', \xi)|$ is the determinant of the Jacobian.

3. `backward:` (if training)

- Evaluate the loss function $\mathcal{L}(\xi', \xi)$ and back propagate

4. `return:` x_{i+1}

- Evaluate MH criteria (Eq. 23) and return accepted config:

$$x_{i+1} \leftarrow \begin{cases} x'' & \text{w/ prob.} \quad A(\xi'|\xi) \\ x & \text{w/ prob.} \quad 1 - A(\xi'|\xi) \end{cases} \tag{21}$$

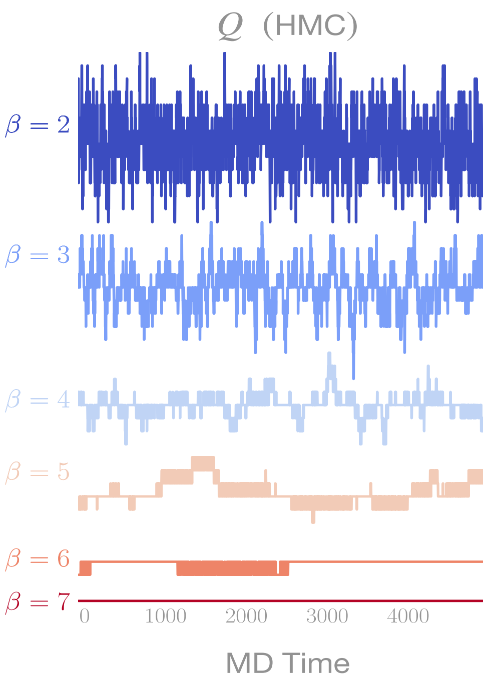
3. Lattice Gauge Theories

3.1 2D $U(1)$ Model

We build upon the approach originally introduced in [17], which was successfully applied to the 2D $U(1)$ lattice gauge model in [8–10]. In particular, we are interested in measuring the (scalar) topological charge $Q \in \mathbb{Z}$ on the lattice. Since different lattice configurations with the same value of Q are related by a gauge transformation, they do not meaningfully contribute to our statistics.

Because of this, we would like to generate configurations from different *topological sectors*[2] to reduce uncertainty in our statistical estimates. By repeating this procedure at increasing spatial resolution[3] ($\beta \propto 1/a$), we are able to extrapolate our estimates to the continuum limit where they can be compared with experimental measurements. Current approaches such as HMC are known to suffer from auto-correlation times which scale exponentially in this limit, significantly limiting their effectiveness. This phenomenon can be seen in Fig 4, where fluctuations in the topological charge between sequential configurations (the *tunneling rate*) $\delta Q = |Q^{i+1} - Q^i|$ decreases as $\beta = 2 \rightarrow 3 \rightarrow \cdots$, and disappear completely (Q = const.) by $\beta = 7$.

Figure 2: $\delta Q \rightarrow 0$ with increasing β for the 2D $U(1)$ model. Image from [9].



(a) $|\mathcal{J}|$ vs LF step.

(b) δQ for trained model (blue) vs HMC (red).

Figure 3: Results from trained 2D $U(1)$ model at $\beta = 4$ showing increasing $|\mathcal{J}|$ towards the middle of the trajectory (left), resulting in improved tunneling rate (δQ) (right).

[2]Characterized by different values of Q.
[3]Here a is the lattice spacing.

3.2 4D $SU(3)$ Model

We would like to generalize this approach to handle 4D $SU(3)$ link variables $U_\mu(n) \in SU(3)$:

$$
\begin{aligned}
U_\mu(n) &= \exp\left[i\omega_\mu^k(n)\lambda^k\right] && (22)\\
&= e^{iW}, \quad W \in \mathfrak{su}(3) && (23)
\end{aligned}
$$

where $\omega_\mu^k(n) \in \mathbb{R}$ and λ^k are the generators of $SU(3)$. We consider the standard Wilson gauge action

$$
\begin{aligned}
S_G &= -\frac{\beta}{6}\sum \mathrm{Tr}\left[U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)\right], \quad \text{where} && (24)\\
U_{\mu\nu}(n) &= U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n). && (25)
\end{aligned}
$$

3.3 Generic MD Updates

As before, we introduce momenta $P_\mu(n) = P_\mu^k(n)\lambda^k$ conjugate to the real fields $\omega_\mu^k(n)$. We can write the Hamiltonian as

$$
H[P,U] = \frac{1}{2}P^2 + S_G[U] \Longrightarrow \boxed{\frac{d\omega^k}{dt} = \frac{\partial H}{\partial P^k}}, \quad \boxed{\frac{dP^k}{dt} = -\frac{\partial H}{\partial \omega^k}}. \tag{26}
$$

To update the gauge field $U_\mu = e^{i\omega_\mu^k\lambda^k}$, write,

$$
\frac{d\omega^k}{dt}\lambda^k = P^k\lambda^k \Longrightarrow \boxed{\frac{dW}{dt} = P}. \tag{27}
$$

Discretizing with step size ε,

$$
\begin{aligned}
W(\varepsilon) &= W(0) + \varepsilon P(0) \Longrightarrow && (28)\\
-i\log U(\varepsilon) &= -i\log U(0) + \varepsilon P(0) && (29)\\
U(\varepsilon) &= e^{i\varepsilon P(0)}U(0) \Longrightarrow && (30)\\
&\boxed{\Lambda : U \longrightarrow U' = e^{i\varepsilon P}U}. && (31)
\end{aligned}
$$

Similarly for the momentum update,

$$
\begin{aligned}
&\boxed{\frac{dP^k}{dt} = -\frac{\partial H}{\partial \omega^k}} = -\frac{\partial H}{\partial W} = -\frac{dS}{dW} \Longrightarrow && (32)\\
&P(\varepsilon) = P(0) - \varepsilon \left.\frac{dS}{dW}\right|_{t=0} = P(0) - \varepsilon F[U] && (33)\\
&\boxed{\Gamma : P \longrightarrow P' = P - \frac{\varepsilon}{2}F[U]} && (34)
\end{aligned}
$$

where $F[U] = \frac{dS}{dW}$ is the force term (see **??**).

3.4 Generalized MD Update

As in Sec.3, we introduce `pNet`: $(U, F) \longrightarrow (\alpha_P, \beta_P, \gamma_P)$ and `uNet`: $(U, P) \longrightarrow (\,\cdot\,, \beta_U, \gamma_U)$[4]. In terms of the generalized update operators,

$$\boxed{\Gamma_\theta^\pm} : (U, P) \xrightarrow{(\alpha_P, \beta_P, \gamma_P)} (U, P') \tag{35}$$

$$\boxed{\Lambda_\theta^\pm} : (U, P) \xrightarrow{(\,\cdot\,, \beta_U, \gamma_U)} (U', P) \tag{36}$$

we can write the complete update:

$$P' = \Gamma_\theta^\pm(U, P) \tag{37}$$

$$U' = U_B + \Lambda_\theta^\pm(U_A, P') \tag{38}$$

$$U'' = U'_A + \Lambda_\theta^\pm(U'_B, P') \tag{39}$$

$$P'' = \Gamma_\theta^\pm(U'', P') \tag{40}$$

3.5 Momentum Update

In this case, our `pNet` : $(U, F) = (e^{iW}, F) \longrightarrow (\alpha_P, \beta_P, \gamma_P)$. We can write the generalized momentum update as $P^\pm := \Gamma_\theta^\pm(U, P)$, where[5]:

1. `forward`, $(+)$:
$$P^+ := \Gamma_\theta^+(U, P) = P \cdot e^{\frac{\varepsilon}{2}\alpha_P} - \frac{\varepsilon}{2}\left[F \cdot e^{\varepsilon \beta_P} + \gamma_P\right] \tag{41}$$

2. `backward`, $(-)$:
$$P^- := \Gamma_\theta^-(U, P) = e^{-\frac{\varepsilon}{2}\alpha_P} \cdot \left\{P + \frac{\varepsilon}{2}\left[F \cdot e^{\varepsilon \beta_P} + \gamma_P\right]\right\}. \tag{42}$$

By introducing the above modifications, we incur a factor of $\log\left|\frac{\partial P^\pm}{\partial P}\right| = \pm\frac{\varepsilon}{2}\sum \alpha_P$ in the Metropolis Hastings accept / reject $A(U'|U)$, and the sum is taken over the full trajectory.

3.6 Link Update

Similarly to the momentum update, the outputs from our `uNet` : $(U, P) \longrightarrow (\,\cdot\,, \beta_U, \gamma_U)$ are used in the generalized link update $U^\pm := \Lambda^\pm(U, P) = e^{i\varepsilon \tilde{P}^\pm} U$ (where $\tilde{P}^\pm \in \mathfrak{su}(3)$). Explicitly:

1. `forward`, $(+)$:
$$U^+ := \Lambda_\theta^+(U, P) = e^{i\varepsilon \tilde{P}^+} U, \quad \text{with} \quad \tilde{P}^+ = \left[P \odot e^{\varepsilon \beta_U} + \gamma_U\right] \tag{43}$$

2. `backward`, $(-)$:
$$U^- := \Lambda_\theta^-(U, P) = e^{i\varepsilon \tilde{P}^-} U, \quad \text{with} \quad \tilde{P}^- = e^{-\varepsilon \beta_U} \cdot [P - \gamma_U] \tag{44}$$

[4]Note that we have omitted the U scaling term (α_U) term in this update since $U \in SU(3)$

[5]Note that $\left(\Gamma^+\right)^{-1} = \Gamma^-$, i.e. $\Gamma^+\left[\Gamma^-(U, F)\right] = \Gamma^-\left[\Gamma^+(U, F)\right] = (U, F)$, and similarly for $\Lambda^\pm$

3.7 Training

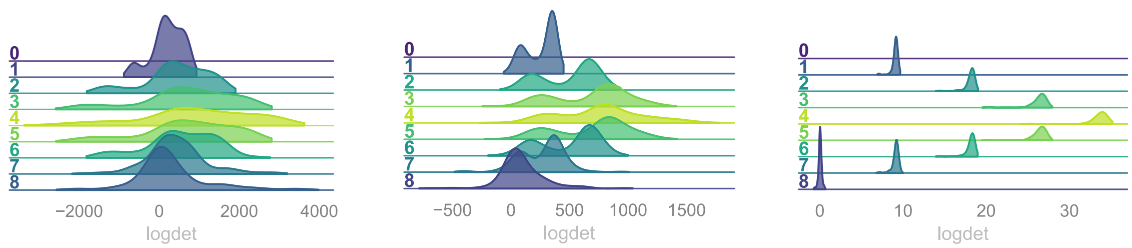
We construct a loss function using the expected squared charge difference

$$\mathcal{L}_\theta(U, U') = \mathbb{E}\left[A(U'|U) \cdot \delta_Q^2(U, U')\right], \tag{45}$$

where $\delta_Q^2(U, U') = |Q' - Q|^2$ is the squared topological charge (see **??**) difference between the initial and proposal configurations.

4. Results

For the trained 2D $U(1)$ model (Fig **??**), we see in Fig **??** that $|\mathcal{J}|$ increases towards the middle of the trajectory, allowing for the sampler to overcome the large energy barriers between different topological sectors. This results in a greater *tunneling rate* (δQ) when compared to generic HMC. Identical behavior is observed after a short training run for the 4D $SU(3)$ model, as shown in Fig 5.



(a) 100 train steps **(b)** 500 train steps **(c)** 1000 train steps

Figure 4: Evolution of $|\mathcal{J}|$ during the first 1000 training iterations for the 4D $SU(3)$ model.

5. Conclusion

In this work we've introduced a generalized MD update for generating 4D SU(3) gauge configurations that can be trained to improve sampling efficiency. Note that this is a relatively simple proof of concept demonstrating how to construct such a sampler. In a future work we plan to further investigate (and quantify) the cost / benefit when compared to alternative approaches such as traditional HMC and purely generative (OT / KL-Divergence [2–4, 15] based approaches.

6. Acknowledgements

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A. Appendix

A.1 Force Term

We can write the force term as

$$F = \frac{dS}{dQ} = -\frac{1}{\lambda^2}\sum_k \lambda^k \,\mathrm{Tr}\left[i\left(UA - A^\dagger U^\dagger\right)\lambda^k\right] \tag{46}$$

where A is the sum over staples

$$\begin{aligned} A = &\sum_{\mu\neq\nu} U_\mu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x) &(47)\\ &+\sum_{\mu\neq\nu} U_{-\nu}(x+\hat{\mu})\, U_\mu^\dagger(x-\hat{\nu})\, U_{-\nu}^\dagger(x). &(48)\end{aligned}$$

Since, $i\left(UA - A^\dagger U^\dagger\right) \in \mathfrak{su}(3)$, we can write it in terms of the generators λ^k as

$$\begin{aligned} \sum_k \lambda^k\,\mathrm{Tr}\left[\lambda^k \sum_j c_j\,\lambda^j\right] &= \sum_k\sum_j c_j\,\lambda^j\,\mathrm{Tr}\left[\lambda^k\lambda^j\right] &(49)\\ &= \frac{1}{2}\sum_k\sum_j c_j\, t^k\,\delta_{jk} &(50)\\ &= \frac{1}{2}\sum_k c_k\, t^k &(51)\end{aligned}$$

consequently, we can simplify the force term as

$$F[U] = \frac{dS}{dW} = -\frac{1}{2g^2}\, i\left(UA - A^\dagger U^\dagger\right). \tag{52}$$

A.2 Topological Charge Q

In lattice field theory, the topological charge Q is defined as the 4D integral over spacetime of the topological charge density q. In the continuum,

$$Q = \int d^4 x q(x), \text{ where} \tag{53}$$

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\lambda} \mathrm{Tr}\left\{F_{\mu\nu} F_{\rho\lambda}\right\} \tag{54}$$

On the lattice, we choose a discretization[6] $q_L(x)$ such that $Q = a^4 \sum_x q_L(x)$. The most obvious discretization of q_L uses the 1×1 plaquette $P_{\mu\nu}(x)$, and can be written as

$$q_L^{\text{plaq}}(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\lambda} \mathrm{Tr}\left\{P_{\mu\nu}(x) P_{\rho\lambda}(x)\right\} \tag{55}$$

this has the advantage of being computationally inexpensive, but leads to lattice artifacts of order $O(a^2)$.

[6]We are free to choose a specific discretization as long as it gives the right continuum limit