HW #9

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Problem 1:
Starting with the expression for μ in class,

$$\mu = mc^2 - kT\ln\left(\frac{gn_Q}{n}\right) \tag{1}$$

$$MB = \frac{g}{h^3}\frac{1}{e^{(E-\mu)/kT}\pm 1}\ \text{, the exponent is much greater than } \pm 1 \text{ so we can drop it} \tag{2}$$

$$= \frac{g}{h^3}\frac{1}{e^{(E-\mu)/kT}}\ , E = \frac{p^2}{2m} + mc^2 \tag{3}$$

$$= \frac{g}{h^3}e^{(\mu-E)/kT} \tag{4}$$

$$= \frac{g}{h^3}e^{(mc^2 - kT\ln\left(\frac{gn_Q}{n}\right) - \frac{p^2}{2m} - mc^2)/kT} \tag{5}$$

$$= \frac{g}{h^3}e^{(-kT\ln\left(\frac{gn_Q}{n}\right) - \frac{p^2}{2m})/kT} \tag{6}$$

$$= \frac{g}{h^3}e^{(-\ln\left(\frac{gn_Q}{n}\right)}e^{-\frac{p^2}{2mkT}} \tag{7}$$

$$= \frac{g}{h^3}e^{(\ln\left(\frac{gn_Q}{n}\right)^{-1}}e^{-\frac{p^2}{2mkT}} \tag{8}$$

$$= \frac{g}{h^3}\left(\frac{gn_Q}{n}\right)^{-1}e^{-\frac{p^2}{2mkT}} \tag{9}$$

$$= \frac{g}{h^3}\left(\frac{n}{gn_Q}\right)e^{-\frac{p^2}{2mkT}} \tag{10}$$

$$= \frac{n}{n_Q h^3}e^{-\frac{p^2}{2mkT}} \tag{11}$$

$$= \frac{n}{h^3}\left(\frac{h^2}{2\pi m_e kT}\right)^{3/2}e^{-\frac{p^2}{2mkT}} \tag{12}$$

$$\boxed{= n\left(\frac{1}{2\pi m_e kT}\right)^{3/2}e^{-\frac{p^2}{2mkT}}} \tag{13}$$

Problem 2a:

$$\frac{\frac{dN_2}{dm_2} \propto m_2^{-\alpha}}{\frac{dN_1}{dm_1} \propto m_1^{-\alpha}} \tag{14}$$

$$\frac{dN_2}{dN_1} \propto \frac{m_2}{m_1}\left(\frac{m_2}{m_1}\right)^{-\alpha} \tag{15}$$

$$\frac{dN_2}{dN_1} \propto \left(\frac{m_2}{m_1}\right)^{-\alpha+1} \tag{16}$$

$$\frac{N_2}{N_1} \propto \left(\frac{150}{0.5}\right)^{-\alpha+1} \text{ , going to conjure black magic and setting } \frac{dN_1}{dN_2} = \frac{N_1}{N_2} \tag{17}$$

$$\boxed{N_2 \propto 2208 N_1} \tag{18}$$

There's about 2200 small $0.5M_\odot$ stars for every $150M_\odot$ star.

Problem 2b:
Let's find N_{tot}

$$\frac{N_1}{N_2} = \beta \tag{19}$$

$$\frac{dN_1}{dm_1} = m_1^{-\alpha} \tag{20}$$

$$N_1 = \beta N_2 \text{ , insert black magic and say } dN_1 = \beta dN_2 \tag{21}$$

$$\beta\frac{dN_2}{dm_1} = m_1^{-\alpha} \tag{22}$$

$$\beta dN_2 = m_1^{-\alpha} dm_1 \tag{23}$$

$$\int dN_2 = \frac{1}{\beta}\int m_1^{-\alpha} dm_1 \tag{24}$$

$$N_2 = \frac{1}{\beta}\int_{m_1}^{m} m_1^{-\alpha} dm_1 \tag{25}$$

$$N_2 = \frac{1}{\beta}\frac{1}{1-\alpha} m_1^{1-\alpha}\Big|_{m_1}^{m} \tag{26}$$

$$N_2 = \frac{1}{\beta}\frac{1}{1-\alpha}\left(m^{1-\alpha} - m_1^{1-\alpha}\right) \tag{27}$$

Use same argument for N_1 and we get:

$$N_1 = \frac{\beta}{1-\alpha}\left(m_2^{1-\alpha} - m^{1-\alpha}\right) \tag{28}$$

We know m_1, m_2, β, and α, so we just need to plot N as a function of m.

$$N_{tot} = N_1 + N_2 \tag{29}$$

$$N_{tot} = \frac{\beta}{1-\alpha}\left(m_2^{1-\alpha} - m^{1-\alpha}\right) + \frac{1}{\beta}\frac{1}{1-\alpha}\left(m^{1-\alpha} - m_1^{1-\alpha}\right) \tag{30}$$

We want to know the total number of stars between $0.5M_\odot$ and $50M_\odot$, so we're going to integrate this.

$$M_{tot} = \int_{0.5M_\odot}^{150M_\odot} \left(\frac{\beta}{1-\alpha}\left(m_2^{1-\alpha} - m^{1-\alpha}\right) + \frac{1}{\beta}\frac{1}{1-\alpha}\left(m^{1-\alpha} - m_1^{1-\alpha}\right)\right) dm \tag{31}$$

$$\boxed{M_{tot} \approx 4847.29M_\odot} \tag{32}$$

Now we're going to use V.T and $\rho = \frac{3M}{4\pi R^3}$.

$$\frac{3}{2}kT = \frac{GM\mu m_p}{2R} \tag{33}$$

$$R = \frac{GM\mu m_p}{3kT} \tag{34}$$

$$\rho = \frac{3M}{4\pi R^3} \tag{35}$$

$$\rho = \frac{3M}{4\pi}\left(\frac{GM\mu m_p}{3kT}\right)^{-3} \tag{36}$$

$$\rho = \frac{3M}{4\pi}\left(\frac{3kT}{GM\mu m_p}\right)^{3} \tag{37}$$

$$\boxed{\rho = 6.04 \times 10^{-25}\ \text{gm cm}^{-3}} \tag{38}$$

$$n = 6 \times 10^{-1}\ \text{cm}^{-3} \tag{39}$$

Problem 3:

$$\mu = mc^2 - kT\ln\left(\frac{gn_Q}{n}\right) \tag{40}$$

$$\mu(H_2) = 2\mu(H) \tag{41}$$

$$\mu(H) = m_H c^2 - kT\ln\left(\frac{n_{Q,H}}{n_H}\right) \tag{42}$$

$$\mu(H_2) = 2m_H c^2 - \chi - kT\ln\left(\frac{n_{Q,H_2}}{n_{H2}}\right) \tag{43}$$

$$2\left(m_H c^2 - kT\ln\left(\frac{n_{Q,H}}{n_H}\right)\right) = 2m_H c^2 - \chi - kT\ln\left(\frac{n_{Q,H2}}{n_{H2}}\right) \tag{44}$$

$$-2kT\ln\left(\frac{n_{Q,H}}{n_H}\right) = -\chi - kT\ln\left(\frac{n_{Q,H2}}{n_{H2}}\right) \tag{45}$$

$$\ln\left(\frac{n_{Q,H}}{n_H}\right)^2 = \frac{\chi}{kT} + \ln\left(\frac{n_{Q,H2}}{n_{H2}}\right) \tag{46}$$

$$\left(\frac{n_{Q,H}}{n_H}\right)^2 = e^{\chi/kT}\left(\frac{n_{Q,H2}}{n_{H2}}\right) \tag{47}$$

$$\frac{n_H n_H}{n_{H2}} = e^{-\chi/kT}\frac{n_{Q,H}n_{Q,H}}{n_{Q,H2}} \tag{48}$$

$$\frac{1}{2}n_H = e^{-\chi/kT}2^{-3/2}n_{Q,H} \tag{49}$$

$$P = \frac{3}{2}n_H kT = 100\text{ Pa} = 1000\text{ cm s}^{-2} \tag{50}$$

$$n_H = \frac{2000}{3kT} \tag{51}$$

$$\frac{1000}{3kT} = e^{-\chi/kT}2^{-3/2}n_{Q,H} \tag{52}$$

Solving for T, I get $\boxed{T \approx 2291K}$.

Problem 4a:

The Saha Eq gives us:

$$\frac{n_p n_e}{n_H} = \frac{g_e g_p}{g_H} e^{-\chi/kT} n_{Q,e} \tag{53}$$

$$\frac{n_p^2}{n_H} = \frac{2 \cdot 1}{2} e^{-\chi/kT} n_{Q,e} \tag{54}$$

$$n = n_p + n_H \text{ , we want } \frac{n_p}{n} \text{ but the Saha Eq. gives us } \frac{n_p}{n_H} \tag{55}$$

$$\frac{n_p^2}{n - n_p} = e^{-\chi/kT} n_{Q,e} \tag{56}$$

$$\frac{\frac{n_p^2}{n}}{1 - \frac{n_p}{n}} = e^{-\chi/kT} n_{Q,e} \tag{57}$$

$$\frac{\frac{n_p^2}{n^2}}{1 - \frac{n_p}{n}} = \frac{1}{n} e^{-\chi/kT} n_{Q,e} \text{ , } \frac{n_p}{n} = F \tag{58}$$

$$\frac{F^2}{1 - F} = \frac{1}{n} e^{-\chi/kT} n_{Q,e} \tag{59}$$

$$F^2 = (1 - F) \frac{1}{n} e^{-\chi/kT} n_{Q,e} \tag{60}$$

$$0 = F^2 + \frac{F}{n} e^{-\chi/kT} n_{Q,e} - \frac{1}{n} e^{-\chi/kT} n_{Q,e} \text{ , solve for } F \tag{61}$$

$$\boxed{F = \frac{-\frac{1}{n} e^{-\chi/kT} n_{Q,e} \pm \sqrt{(\frac{1}{n} e^{-\chi/kT} n_{Q,e})^2 + 4 \frac{1}{n} e^{-\chi/kT} n_{Q,e}}}{2}} \tag{62}$$

Plot at back

Problem 4b:

In class.

$$\frac{n_{n=2}}{n_{n=1}} = 4e^{-10.2\text{eV}/kT} \tag{63}$$

$$\frac{n_{n=2}}{n_H} = 4e^{-10.2\text{eV}/kT} \tag{64}$$

$$\frac{n_{n=2}}{n - n_p} = 4e^{-10.2\text{eV}/kT} \tag{65}$$

$$\frac{\frac{n_{n=2}}{n}}{1 - \frac{n_p}{n}} = 4e^{-10.2\text{eV}/kT} \tag{66}$$

$$\boxed{\frac{n_{n=2}}{n} = \left(1 - \frac{n_p}{n}\right) 4e^{-10.2\text{eV}/kT}} \tag{67}$$

We just plotted $\frac{n_p}{n}$ so we're going to reuse it and multiply by the exponent factor.

Plot at back.

Problem 4c:

$$\Delta E = h\nu \tag{68}$$

$$\Delta E = \frac{hc}{\lambda} \tag{69}$$

$$\Delta E = -13.6\text{eV}\left(\frac{1}{9} - \frac{1}{4}\right) \text{ , from } n = 2 \text{ to } n = 3 \tag{70}$$

$$-13.6\text{eV}\left(\frac{1}{9} - \frac{1}{4}\right) = \frac{hc}{\lambda} \tag{71}$$

$$\boxed{\lambda_\alpha = 6.51 \times 10^{-5} \text{ cm}} \tag{72}$$

$$-13.6\text{eV}\left(\frac{1}{16} - \frac{1}{4}\right) = \frac{hc}{\lambda} \text{ , from } n = 2 \text{ to } n = 4 \tag{73}$$

$$\boxed{\lambda_\beta = 4.8 \times 10^{-5} \text{ cm}} \tag{74}$$

The temperature of A stars is about 1×10^4 K, right about where there is a maximum in the fractional number of electrons in the Balmer Line. M stars are too cold and O stars are too hot.

Problem 4d:

$$-13.6\text{eV}\left(\frac{1}{4} - \frac{1}{1}\right) = \frac{hc}{\lambda} \text{ , from } n = 1 \text{ to } n = 2 \tag{75}$$

$$\boxed{\lambda_\alpha = 1.2 \times 10^{-5} \text{ cm}} \tag{76}$$

The fraction of electrons in the ground state can be interpreted as:

$$n = n_p + n_H \tag{77}$$

$$1 = \frac{n_p}{n} - \frac{n_H}{n} \tag{78}$$

$$\boxed{\frac{n_H}{n} = \frac{n_p}{n} - 1} \tag{79}$$

Since we have a plot for $\frac{n_p}{n}$ and since it goes from 0 to 1, we just flip it upside down to find the plot for $\frac{n_H}{n}$. At the typical M star temperature (3000 K), we see that most of the H is in the ground state and see no prominent Ly_α lines. 3000 K is too cold and there isn't enough thermal energy to bump up an electron to the second energy state. Even though kT can be much less than the ionization E, we can kind of use the argument that:

$$kT \text{ ? } - 13.6\text{eV}\left(\frac{1}{4} - \frac{1}{1}\right) \tag{80}$$

$$4.14 \times 10^{-13} \text{ ergs} < 1.22 \times 10^{-11} \text{ ergs ,} \tag{81}$$

The thermal energy is less than ΔE so we don't see much (if at all) bumped up.