$\boxed{\text{Phillips } 1.4, 1.5, 4.1, 4.2}$

1 Post-Fusion

$$\lambda \sim \frac{h}{p} \gg r_n \approx 10^{-13} \text{ cm} \tag{1}$$

$$r_c \text{ , classical distance distance, of closest approach} \tag{2}$$

Fusion parameters will be QM, but won't have stuff to do with λ. σ for nuclear reactions $\approx$ nuclear physics part (strong/weak interaction) $\cdot$ tunnelling through coulomb barrier. We're going to focus on the tunnelling which sets the physics for the central temps of stars.

Let's consider SE:

$$\left(\frac{\hbar^2}{2m_r}\nabla^2 + V(r)\right)\Psi = E\Psi \text{ , } m_r = \text{reduced mass} \tag{3}$$

$$-\frac{\hbar^2}{2m_r}\frac{d^2}{dr^2}\Psi = E\Psi \text{ , } \Psi = e^{ikr} \text{ , } E = \frac{\hbar^2 k^2}{2m_r} \text{ , } k = \frac{\sqrt{E2m_r}}{\hbar} \tag{4}$$

Good ol review. $P = |\Psi|^2 = \text{constant}$. Now we have

$$\frac{\hbar^2}{2m_r}\frac{d^2}{dr^2}\Psi = (V-E)\Psi \text{ , and} (V-E) > 0 \tag{5}$$

$$\Psi \propto e^{-kr} \text{ , } \frac{\hbar^2 k^2}{2m_r} = (V-E) \tag{6}$$

QM'ally, particle can't be somewhere where it's potential is less than the energy. Now, the probability of tunnelling is $|\Psi|^2 \sim e^{-2kl}$. Tunnelling is generic feature of wave theory not just QM. Sound waves tunnel, waves in the atmosphere tunnel.... WUT.
Now lets imagine a particle with energy $E = \frac{1}{2}m_r v^2$,$v = |\bar{v_1} - \bar{v_2}|$. Now...

$$\left(-\frac{\hbar^2}{2m_r}\nabla^2 + \frac{Z_1 Z_2 e^2}{r}\right)\Psi = E\Psi \tag{7}$$

$$E = V \tag{8}$$

$$= \frac{Z_1 Z_2 e^2}{r_c} \tag{9}$$

$$r_c = \frac{Z_1 Z_2 e^2}{E} \tag{10}$$

There's some finite prob that they can tunnel trough the potential one they reach r_c. We want to compute the probability! One small difference is that with the atom, particles can have angular momentum and we have to use spherical harmonics.

$$\Psi = \frac{f(r)}{r} Y_{l,m}(\theta, \phi) \tag{11}$$

OMG.

$$\left(-\frac{\hbar^2}{2m_r} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2m_r r^2} + \frac{e^2 Z_1 Z_2}{r} \right) \Psi = E\Psi \tag{12}$$

Particles with high angular momentum don't fuse because any small difference in path and they'll fly off. Only possible fusion happens when $l = 0$, so we can cross out the $\frac{l(l+1)\hbar^2}{2m_r r^2}$ term. Now...

$$\left(-\frac{\hbar^2}{2m_r} \frac{d^2}{dr^2} + \frac{e^2 Z_1 Z_2}{r} \right) f = Ef \tag{13}$$

What's the probability of reaching r_n if they start at the classical turning point (r_c)? $P = |f(r_n)|^2$.

Now,

$$\frac{d^2 f(r)}{dr^2} + g(r)f(r) = 0 \ , g(r) = \frac{2m_r}{\hbar^2} \left(E - \frac{e^2 Z_1 Z_2}{r} \right) \tag{14}$$

We're interested in situations where the E is less than the potential, so $g(r) < 0$. This pops up in lot of places, apparently. If $g(r)$ is a constant, we can solve it.

$$f \sim e^{\pm i \sqrt{g} r} \tag{15}$$

This solution isn't vaid if $g(r)$ isn't a constant. For the case of interest, g is *almost* constant. It's slowly varying, in reality. It's a function of position for which there is an analytic solution to the above equation. The analytic equation is known as the WKB solution.

It's plausible that the solution is of the form $f \sim e^{i\phi(r)}$ if we think$g$ doesn't change much over time. If $g =$ constant, $\phi(r) = \sqrt{g} r$.

$$f' = i\phi'(r) e^{i\phi(r)} = i\phi'(r) f \tag{16}$$

$$f'' = i\phi'' f + i\phi' f' = i\phi'' f - (\phi')^2 f \tag{17}$$

$$\frac{d^2 f(r)}{dr^2} + g(r) f(r) = 0 \tag{18}$$

$$i\phi'' - (\phi')^2 + g = 0 \tag{19}$$

Assume ϕ'' is small, and by small we mean $\phi'' \ll g$.

$$(\phi')^2 = g(r) \tag{20}$$
$$\phi' = \sqrt{g(r)} \tag{21}$$
$$\phi(r) = \int^r \sqrt{g(x)}dx \tag{22}$$
$$f \sim e^{i\phi(r)} = e^{\pm i \int^r \sqrt{g}dx} \tag{23}$$

We can check whether our assumption that $\phi'' \ll g$, $\phi'' = \frac{1}{2}g^{-1/2}g'$, and WKB solution is valid if $\frac{g'}{\sqrt{g}} \ll g$. This is what we mean by a "slowly varying" potential. Lets think about this physically.

$$g' \sim \frac{g}{L} \text{ , } L = \text{length over which potential varies} \tag{24}$$
$$\frac{1}{L\sqrt{g}} \ll 1 \tag{25}$$
$$\frac{1}{\sqrt{g}} \ll L \tag{26}$$
$$\phi = \int \sqrt{g}dx \tag{27}$$
$$\phi = \int \frac{dx}{\lambda} \text{ , where } \lambda \text{ is the wavelength to our solution on order } \frac{1}{\sqrt{g}} \tag{28}$$

Our WKB solution is okay if $\frac{1}{\sqrt{g}} \ll L$, $\lambda \ll L$. In our case, λ is the deBroglie wavelength.

$$g = \frac{2m_r}{\hbar^2}\left(E - \frac{e^2 Z_1 Z)_2}{r}\right) \tag{29}$$

1.1 Tunnelling

$$f(r_n) = e^{i\int_{r_n}^{r_c} \sqrt{g}dr} = e^{-\int_{r_n}^{r_c} \sqrt{|g|}dr} \tag{30}$$
$$P = e^{-I} \text{ , } I = 2\int_{r_n}^{r_c} \sqrt{|g|}dr \tag{31}$$
$$I = \frac{2\sqrt{2m_r E}}{\hbar}\int_{r_n}^{r_c}\left(\frac{e^2 Z_1 Z_2}{r} - E\right)^{1/2} dr \tag{32}$$
$$= \frac{2\sqrt{2m_r E}}{\hbar}\int_{r_n}^{r_c}\left(\frac{r_c}{r} - 1\right)^{1/2} dr \tag{33}$$
$$x = \frac{r}{r_c} \tag{34}$$
$$I = r_c \int_{r_n/r_c}^{x=r/r_r c}\left(\frac{1}{x} - 1\right)^{1/2} dx \tag{35}$$
$$\int_0^1 \left(\frac{1}{x} - 1\right)^{1/2} dx = \frac{\pi}{2} \tag{36}$$

Tunnelling is independent of where nuclear reaction becomes important. Tunnelling dominates at classical point. Once you get trough the turning point, it doesn't matter how far you have to go.

$$I = \pi\sqrt{\frac{2m_r e^4 Z_1^2 Z_2^2}{\hbar^2 E}} = \left(\frac{E_g}{E}\right)^{1/2} \quad , E_g = \frac{2\pi^2 m_r e^4 Z_1^2 Z_2^2}{\hbar^2} \tag{37}$$

$$E \sim E_g \ , I \sim 1 \ , \text{Prob of tunnelling } \sim 1 \tag{38}$$

$$E \ll E_g \ , I \gg 1, P \ll 1 \tag{39}$$

$$E_g = 1 \text{ MeV} \frac{M_r}{m_p} Z_1^2 Z_2^2 \tag{40}$$

$$P \approx e^{-(E_g/E)^{1/2}} \tag{41}$$

If E is too low, no significant tunnelling and no significant fusion.

At center of sun....

$$T_{center} \sim 10^7 \text{K} \sim 1\text{KeV} \tag{42}$$

$$\frac{3}{2} kT = 2\text{keV} \tag{43}$$

$$\ll E_g \sim 500\text{keV} \tag{44}$$

$$P \sim 10^{-7} \text{ ,damn, that's low.} \tag{45}$$

Let's imagine particles with 10 times the thermal energy. $E = 10E_{th} = 20\text{keV}$, then $P \sim 10^{-2}$. This tells us that clearly particles whih are more energetic that average are **MUCH** more likely to tunnel and thus undergo fusion. So when do we treat things QM'ically? If deBroglie wavelength is large. Recap, $\lambda = \frac{h}{p} \sim \frac{h}{\sqrt{2Em_r}}$. As $E \uparrow$, $\lambda \downarrow$. Higher E has a smaller classical turning point. Then, $r_c \downarrow$ as $E \uparrow$. Yes, in absolute cm that the r_c goes down, but relatively, it's easier to tunnel at higher E.

$$I = (E_g/E)^{1/2} \tag{46}$$

$$= \frac{\pi\sqrt{2m_r E} r_c}{\hbar} \sim \frac{r_c}{\lambda} \tag{47}$$

At high Z, $E_g \uparrow \rightarrow T \uparrow$. H is easiest to fuse at earliest stages.