Problem 3a

$$\frac{dM}{dr} = 4\pi r^2 \rho$$
$$\int dM = \int 4\pi r^2 \rho dr$$
$$M = 4\pi a^3 \rho_c \int \theta^n \xi^2 d\xi$$
$$\xi^2 \frac{d\theta}{d\xi} = -\int \xi^2 \theta^n d\xi$$
$$M = -4\pi a^3 \rho_c \xi^2 \frac{d\theta}{d\xi}$$
$$\rho_c = \frac{M}{4\pi a^3 \xi^2 \frac{d\theta}{d\xi}} \text{ , set } a = \frac{R}{\xi_R} \text{ and evaluate } \xi \text{ at } R$$
$$\rho_c = \frac{M}{4\pi R^3} \frac{\xi_R}{\frac{d\theta}{d\xi_R}}$$
$$\rho_c = \frac{3M}{4\pi R^3} \frac{\xi_R}{3\frac{d\theta}{d\xi_R}}$$
$$\rho_c = \frac{3M}{4\pi R^3} a_n \text{ , where } a_n = \frac{\xi_r}{3\frac{d\theta}{d\xi_R}} \text{ and is dimensionless}$$

Problem 3b

$$P = \kappa \rho^\gamma$$
$$P_c = \kappa \rho_c^\gamma$$
$$= \kappa \frac{\rho_c^\gamma \rho_c^2}{\rho_c^2}$$
$$= \kappa \rho_c^{\gamma-2} \rho_c^2$$
$$= \kappa \rho_c^{1/n-1} \rho_c^2$$
$$= \kappa \frac{n+1}{n+1} \frac{4\pi G}{4\pi G} \rho_c^{1/n-1} \rho_c^2$$
$$= \frac{4\pi G \rho_c^2}{n+1} \left(\frac{\kappa(n+1)\rho_c^{1/n-1}}{4\pi G} \right)$$
$$P_c = \frac{4\pi G \rho_c^2}{n+1} a^2 \text{ , where } a^2 = \left(\frac{\kappa(n+1)\rho_c^{1/n-1}}{4\pi G} \right)$$

$$
\begin{aligned}
P_c &= \frac{4\pi G}{n+1} a^2 \left(\frac{3M}{4\pi R^3} a_n \right)^2 \\
&= \frac{9GM^2 a^2 a_n^2}{(n+1)4\pi R^6} \\
&= \frac{9GM^2 \alpha^2 R^2 a_n^2}{(n+1)4\pi R^6} \text{ , where } a = \alpha R \\
&= \frac{GM^2}{R^4} \frac{9\alpha^2 a_n^2}{(n+1)4\pi} \\
P_c &= \frac{GM^2}{R^4} c_n \text{ , where } c_n = \frac{9\alpha^2 a_n^2}{(n+1)4\pi}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\rho} &= \frac{M}{\frac{4\pi}{3} R^3} \\
\bar{\rho}^{4/3} &= \left(\frac{M}{\frac{4\pi}{3} R^3} \right)^{4/3} \\
R^4 &= \left(\frac{M}{\bar{\rho} \frac{4\pi}{3}} \right)^{4/3}
\end{aligned}
$$

$$
\begin{aligned}
P_c &= \frac{GM^2}{R^4} c_n \\
&= GM^2 \left(\frac{M}{\bar{\rho} \frac{4\pi}{3}} \right)^{-4/3} c_n \\
&= GM^{2/3} \bar{\rho}^{4/3} \left(\frac{4\pi}{3} \right)^{4/3} c_n \\
&= GM^{2/3} \bar{\rho}^{4/3} \left(\frac{4\pi}{3} \right)^{4/3} c_n \text{ , } \bar{\rho} = \frac{\rho_c}{a_n} \text{ ,} \\
&= GM^{2/3} \left(\frac{\rho_c}{a_n} \right)^{4/3} \left(\frac{4\pi}{3} \right)^{4/3} c_n \\
&= GM^{2/3} \rho_c^{4/3} d_n \text{ , where } d_n = c_n \left(\frac{4\pi}{3a_n} \right)^{4/3}
\end{aligned}
$$

Problem 3c

Set the two P_c equations equal to each other

$$\begin{aligned}
d_n G M^{2/3} \rho_c^{4/3} &= \left(\frac{GM^2}{R^4}\right) c_n \\
d_n &= \frac{M^{4/3}}{R^4} c_n \rho_c^{-4/3} \\
&= \frac{M^{4/3}}{R^4} c_n \left(\frac{3M}{4\pi R^3} a_n\right)^{-4/3} \\
&= \frac{M^{4/3}}{R^4} c_n \left(\frac{1}{\frac{3M}{4\pi R^3} a_n}\right)^{4/3} \\
&= c_n \left(\frac{4\pi}{3a_n}\right)^{4/3}
\end{aligned}$$

$$\begin{aligned}
d_n(n=3) &= 11.05 \cdot \left(\frac{4\pi}{3 \cdot 54.183}\right)^{4/3} \approx .3639 \\
d_n(n=1.5) &= 0.77 \cdot \left(\frac{4\pi}{3 \cdot 5.99}\right)^{4/3} \approx .477
\end{aligned}$$

Problem 3d

$$\begin{aligned}
P_c &= \frac{\rho_c k T_c}{\mu m_p} \\
&= c_n \frac{GM^2}{R^4} \text{ , from class notes} \\
c_n \frac{GM^2}{R^4} &= \frac{\rho_c k T_c}{\mu m_p} \text{ , } \rho_c = \left(\frac{3M}{4\pi R^3}\right) a_n \\
T_c &= \frac{\mu m_p c_n}{a_n} \frac{GM 4\pi}{3kR}
\end{aligned}$$

Problem 3e

For n=3

$$
\begin{aligned}
T_c &= \frac{\mu m_p c_n}{a_n}\frac{GM4\pi}{3kR} \\
&= \frac{.5\cdot 1.67\times 10^{-24}\cdot 11.05}{54.183}\frac{6.67\times 10^{-8}\cdot 2\times 10^{33}\cdot 4\pi}{6.96\times 10^{10}\cdot 1.38\times 10^{-16}} \\
&= 2.97\times 10^7 K \\
\rho_c &= \frac{3M}{4\pi R^3}a_n \\
&= \frac{3\cdot 2\times 10^{33}}{4\pi(6.96\times 10^{10})^3}54.183 \\
&= 76.7 \text{ gm cm}^{-3} \\
P_c &= \frac{GM^2}{R^4}c_n \\
&= \frac{6.67\times 10^{-8}\cdot(2\times 10^{33})^2}{(6.96\times 10^{10})^4}11.05 \\
&= 1.25\times 10^{17}\text{dyne cm}^{-3}
\end{aligned}
$$

For n=1.5

$$
\begin{aligned}
T_c &= \frac{\mu m_p c_n}{a_n}\frac{GM4\pi}{3kR} \\
&= \frac{.5\cdot 1.67\times 10^{-24}\cdot 0.77}{5.99}\frac{6.67\times 10^{-8}\cdot 2\times 10^{33}\cdot 4\pi}{6.96\times 10^{10}\cdot 1.38\times 10^{-16}} \\
&= 1.87\times 10^7 K \\
\rho_c &= \frac{3M}{4\pi R^3}a_n \\
&= \frac{3\cdot 2\times 10^{33}}{4\pi(6.96\times 10^{10})^3}5.99 \\
&= 8.48 \text{ gm cm}^{-3} \\
P_c &= \frac{GM^2}{R^4}c_n \\
&= \frac{6.67\times 10^{-8}\cdot(2\times 10^{33})^2}{(6.96\times 10^{10})^4}0.77 \\
&= 8.7\times 10^{15}\text{dyne cm}^{-3}
\end{aligned}
$$

I think the $n = 3$ polytrope index better approximates the interior environment of the sun better than $n = 1.5$ because a higher polytropic index corresponds to $\gamma = 4/3$, which is for a relativistic gas. The center of the sun is very dense and hot and energy is carried through radiation.

Problem 4

$$P_c = \frac{GM^2}{R^4} c_n$$

$$P_{ph} = \frac{GM}{R^2 \kappa_{ph}}$$

$$\begin{aligned}
\left(\frac{P_c}{P_{ph}}\right)^{2/5} &= \frac{Tc}{T_{eff}} \\
T_{eff} &= T_c \left(\frac{P_{ph}}{P_c}\right)^{2/5} \\
&= T_c \left(\frac{GM}{R^2 \kappa_{ph}} \frac{R^4}{GM^2 c_n}\right)^{2/5} \\
&= T_c \left(\frac{R^2}{\kappa_{ph} M c_n}\right)^{2/5} \\
&= T_c \left(\frac{R^2}{\kappa_{ph} M c_n} \frac{\rho_{ph}}{\rho_{ph}}\right)^{2/5} \\
&= T_c \left(\frac{l R^2 \rho_{ph}}{M c_n}\right)^{2/5}
\end{aligned}$$

$$\begin{aligned}
\rho_{ph} &= \bar{\rho} \\
&= \frac{M}{\frac{4\pi R^3}{3}}
\end{aligned}$$

$$\begin{aligned}
T_{eff} &= T_c \left(\frac{3l}{R 4\pi c_n}\right)^{2/5} \\
&= 0.626 \cdot T_c \left(\frac{l}{R}\right)^{2/5} \quad , c_n = 0.77
\end{aligned}$$