Multiple body orbits: An investigation

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Abstract

In this paper, we explored complex multiple-body orbits using different integration methods such as the Euler, velocity Verlet and Runge-Kutta methods. Most complex orbits are unsolvable analytically, hence why these methods are particularly useful. The beginning of this paper starts with some simple models, where we focused on the accuracy of the methods before progressing on to more complex orbits, such as a multiple body choreography and extensions of the simple Lagrange problem, using the above integration methods to propagate them. It was concluded that the above methods were very accurate for the simple orbits, producing accurate graphs and conserving energy and angular momentum. The integration methods were also able to simulate a figure-of-eight choreography as well as the simple Lagrange problem. When extending the Lagrange problem to multiple bodies, the orbits quickly became unstable when reaching 10 or more bodies, requiring much more computational time to produce any accurate trajectory.

1 Introduction

Gravitational motion is the main idea behind the movement of astronomical objects and various other astrophysical processes. Surprisingly, however, when it comes to simulating three or more bodies, there doesn't exist a single definite solution, or sometimes any solution at all. This means that we can't use a single analytical equation to predict the movements of a three-body system except at the initial conditions[1], hence we resort to using computational physics to approximate our complex orbits. In this paper, we begin by using a few well-known integration methods to propagate some simple orbits to test the stability of these methods as well as their ability to conserve properties of the system like energy and angular momentum. After verifying their efficacy, we then delved into more complex orbits, using the integration methods mentioned previously to propagate them. The force due to gravity between two bodies is given by[2]:

$$\mathbf{F_1} = \frac{Gm_1m_2}{|\mathbf{r}_{12}^3|}\mathbf{r}_{12} \quad \text{where} \quad \mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{1}$$

Once we have more than two bodies, however, we realise that our equations become much more difficult, as we're left with multiple coupled differential equations which are very challenging to solve analytically. Hence, we resort to solving these equations numerically through various integration techniques. Specifically, this paper explores the Euler, velocity Verlet and the fourth-order Runge-Kutta method. We investigated the following orbits initially: 1) Two bodies of distinct masses, one being significantly heavier, using the Euler method 2) Two bodies of similar masses orbiting their centre of mass using the velocity Verlet method 3) Three bodies of different masses with the velocity Verlet method 4) 3 bodies similar to previously using the Runge-Kutta method. After verifying the accuracy of the integrating methods, we then used them to investigate: 1) A figure-of-eight choreography of 3 equal masses 2) A unit circle orbit consisting of 3 equal masses evenly spaced on a unit circle (simple Lagrange problem) 3) Multiple bodies of equal mass on a unit circle orbiting its circumference.

2 Procedure

We will outline the three integration methods used in the investigation, as well as the starting conditions derived from each orbit.

2.1 Integration Methods

2.1.1 The Euler method

An easy way of solving differential equations is using the Euler method:

$$y(t + \Delta t) = y(t) + \Delta t \frac{dy}{dt} \tag{2}$$

which we can derive from the forward difference formula. Using Euler's method allows us to find the value of a function y at different intervals of Δt, although it is particularly unstable so is not used for more complicated integrations. Using this formula, we can derive the basic equations of motion:

$$F = m \frac{d^2 x}{dt^2} \tag{3}$$

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = \frac{F}{m} \tag{4}$$

where we can substitute our gravitational force equation (Eqn. 1) for F. The Euler method demands Δt to be a small timestep to have any stability or accuracy at all but it still may result in an unstable period over a long period. With both the Euler and velocity Verlet method, the positions and velocity of each body at different timesteps are stored in pre-created position and velocity arrays. We then update these position and velocity arrays using equations (2) and (4) using a loop to increment steps in time. If we were to use R_i and V_i as position and velocity arrays respectively, the following equations hold:

$$R_{i+1} = R_i + V_n dt \tag{5}$$

$$V_{i+1} = V_n + \frac{F}{m} dt \tag{6}$$

2.1.2 The velocity Verlet method

Contrary to the Euler method, the velocity Verlet method is a far more accurate integrator, which is given by the equations:

$$x(t + \Delta t) = x(t) + \Delta t v(t) + \Delta t^2 \frac{F(t)}{2m} \tag{7}$$

$$v(t + \Delta t) = v(t) + \Delta t \frac{F(t) + F(t + \Delta t)}{2m} \tag{8}$$

One key aspect of the Verlet method is its ability to conserve energy and time-reversibility. It belongs to a group called Symplectic Integrators, a family of equations that can solve systems that can be written using Hamiltonian equations, allowing it to conserve energy. From the dt^2 term, it's obvious that the Verlet method has a second-order global error. As the Verlet method conserves energy, we chose it to propagate our complex orbits later on. We can write the Verlet method as:

$$R_{n+1} = R_n + v_n dt + \frac{F_n}{2m} dt^2 \tag{9}$$

$$V_{n+1} = V_n + \frac{F_n + F_{n+1}}{2m} dt \tag{10}$$

where F_n gives the force at step n. When investigating systems of bodies with more than three bodies, we used the Verlet method for convenience. Since with n bodies, there are at least $n(n-1)$ forces to consider, which can become very quickly tedious when the number of bodies becomes large. We can consider all the forces on a body of mass m_1 using the summation notation:

$$\vec{F_1} = \sum_{i \neq 1}^{N_{bodies}} \frac{G m_1 m_i}{\mid \vec{r}_{1i} \mid^3} \vec{r}_{1i} \tag{11}$$

and hence all forces on the system:

$$\vec{F} = \sum_{j}^{N_{bodies}} \sum_{i \neq j}^{N_{bodies}} \frac{G m_i m_j}{\mid \vec{r}_{ij} \mid^3} \vec{r}_{ij} \tag{12}$$

2.1.3 Fourth-order Runge-Kutta method

The fourth-order Runge-Kutta method (RK4) is an integrator which utilises the midpoint method of finding derivatives, which results in it being second-order in Δx. It's given by the equations:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{13}$$

$$k_1 = \Delta x f(x_n, y_n) \tag{14}$$

$$k_2 = \Delta x f(x_n + \Delta x/2, y_n + k_1/2) \tag{15}$$

$$k_3 = \Delta x f(x_n + \Delta x/2, y_n + k_2/2) \tag{16}$$

$$k_4 = \Delta x f(x_n + \Delta x, y_n + k_3) \tag{17}$$

where the gradients k_n define midpoint evaluations of the derivative along a step size. k_1 gives the Euler method derivative, finding the gradient during the beginning of the interval, k_2 and k_3 are derivatives located at the interval's midpoint, and k_4 is the interval's gradient at the end. It's this depth that allows for a very high accuracy, with an error proportional to Δx^4. Nonetheless, a drawback of the Runge-Kutta method is the lack of energy and angular momentum conversation over a long period. Due to each step also requiring four intermediate steps, it also is computationally more expensive, hence simulations take longer. However, this does mean in return that the Runge-Kutta method can provide greater accuracy compared to the Verlet method. To solve the gravitational equations of motion, we decoupled equation 4, as the Runge-Kutta requires the position and velocity arrays to be combined into a single array. An entry of this array, Y_n would contain a 2x2 matrix containing the velocity and position vectors at a specified step n.

2.2 Prelusive Investigation: Simulating simple orbits

We used the Python programming language to code and model all our investigations. In our prelusive investigation, we used the following common assumptions:

- Orbits are coplanar and exist only in 2 dimensions
- Orbits last between 1,000 - 15,000 timesteps with a timestep of 10^{-2}
- A simple function finding the force between two objects (Eqn 1) was used for all orbits
- G, the gravitational constant is equal to 1
- Positions and shape of the orbits are plotted against time
- Kinetic energy ($\frac{1}{2}mv^2$), potential energy ($\frac{-GMm}{r}$), and angular momentum ($L = mvr$) were properties calculated and plotted against time to see if they were conserved

Note: All simulations are documented in the PHAS0030 Week 4 Assignment.
We investigated these simple orbits to verify that the Verlet method was the best method for propagating our choreography and simple Lagrange problem.

2.2.1 Two bodies with different masses

In our first scenario, we investigated a system with a large mass fixed at the origin and another mass orbiting it, similar to a star-planet system. We used the parameters:

- $m_1 = 1$, $m_2 = 0.001$, $dt = 0.01$
- m_1 is significantly heavier than m_2 such that m_1 if fixed to the origin
- The distance between m_1 and m_2 is originally 1

For our m_2 body, we assume that it is undergoing circular motion (hence its acceleration, a_2 is equal to v_2^2/r_{12}), which is caused by the gravitational force between m_1 and m_2. Hence we can solve for its tangential anticlockwise velocity, v_2:

$$F_{12} = m_2 a_2 \tag{18}$$

$$\frac{Gm_1m_2}{r_{12}^2} = \frac{m_2v_2^2}{r_{12}} \tag{19}$$

$$|v_2| = \sqrt{\frac{Gm_1}{r_{12}}} \tag{20}$$

since $r_{12} = m_1 = m_2 = G = 1$. We can therefore write the initial positions and velocities as:

$$\vec{r_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad and \quad \vec{r_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{21}$$

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad and \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{22}$$

We used Euler's method to propagate this orbit for 15,000 steps, using equation 4 as our decoupled differential equations, and only plotted m_2 as m_1 was assumed to be fixed at the origin.

2.2.2 Two bodies with a similar mass

We used the Verlet integrator for this scenario of two similar masses. We defined $m_1 = 3$ and and $m_2 = 1$, $x_1 = 0.25$ and $x_2 =$ -0.75. To calculate the initial positions of the bodies, we considered their centre of mass, due to both masses being similar and not being able to fix one of them. The centre of mass of two bodies is given by:

$$\vec{r}_{CoM} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} \tag{23}$$

which when we substitute in our values, the numerator becomes zero. This very conveniently makes our centre of mass at the origin. To find our initial velocities such that the bodies orbit around their centre of the mass (or the origin), we start with the generic equations:

$$\frac{m_1 v_1^2}{r_1} = \frac{G m_1 m_2}{r_{12}^2} \tag{24}$$

where r_1 is the distance from the origin and r_{12} is the distance between the bodies. Since we chose our positions such that $r_{12} = 1$, we can can rearrange for v_1:

$$|v_1| = \sqrt{G m_2 r_1} \tag{25}$$

Hence, when we substitute our values for both bodies, we get $|v_1| = 0.5$ and $|v_2| = 1.5$. We also know that our velocities will be perpendicular to their positions, hence when we choose our bodies to rotate anti-clockwise, we get our initial positions and velocities:

$$\vec{r_1} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix} \quad and \quad \vec{r_2} = \begin{pmatrix} -0.75 \\ 0 \end{pmatrix} \tag{26}$$

$$\vec{v_1} = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \quad and \quad \vec{v_2} = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \tag{27}$$

We propagated this orbit for 15000 steps.

2.2.3 Three bodies

Our three-body system consists of a star, planet and moon. The star will be significantly heavier than the planet and moon, whilst the moon will be about 100 times lighter than the planet. This means the centre of mass can be assumed to be where the star will be located. The planet and moon will also be very far from the star compared to how close they are to each other.

$$m_1 = 1, \quad m_2 = 3 * 10^{-6}, \quad m_3 = 3.6 * 10^{-8} \tag{28}$$

For our initial conditions, we will set all planets to be on the x-axis with the star at the origin, the planet on the right and the moon even further. The distance between the moon and planet, r_{23} will be 400 times smaller than between the star and planet, r_{12}.

$$\vec{r_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{r_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{r_3} = \begin{pmatrix} 1.025 \\ 0 \end{pmatrix} \tag{29}$$

For our initial velocities, we will set our star to be stationary since the gravity of the moon and planet make virtually no difference. The planet's velocity will be determined by equation 20 where m_1 will be the star and we assume the moon has a negligible difference to the net force on the planet. Finally, we calculate the moon's velocity with respect to the planet and then add the velocity of the planet to it. This in turn takes into account how the moon is orbiting an object which itself is moving.

$$v_2 = \sqrt{\frac{Gm_1}{r_{12}}} \tag{30}$$

$$v_3 = v_2 + \sqrt{\frac{Gm_2}{r_{23}}} = \sqrt{\frac{Gm_1}{r_{12}}} + \sqrt{\frac{Gm_2}{r_{23}}} \tag{31}$$

Hence our initial velocities when tangentially anticlockwise are:

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} 0 \\ \sqrt{\frac{Gm_1}{r_{12}}} \end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix} 0 \\ \sqrt{\frac{Gm_1}{r_{12}}} + \sqrt{\frac{Gm_2}{r_{23}}} \end{pmatrix}, \tag{32}$$

Due to the moon and planet being relatively close to each other, it's difficult to distinguish them apart when you consider their orbit around the star. Hence we plotted some short sections of the orbit at different times to show their paths more clearly. These short sections lasted for 20 timesteps, while the whole orbit was propagated for 15,000 timesteps.

2.2.4 Using a different integrator: The Runge-Kutta Method

In this section, we will try using a more accurate integrator for our orbits, specifically the Runge-Kutta method. We decided to adjust the Runge-Kutta function since we found it easier to do it in an alternative way. It still uses the same formula but we changed the right-hand side function for the Runge-Kutta solver to one which calculated the orbital velocity at each point. The Runge-Kutta function would take this in and project the position using the above equations.
We also tested out the accuracy of the Runge-Kutta method timesteps between 0.001s and 1s, with a constant total time of 15s. Furthermore, we pushed both the RK4 and Verlet methods to compute 1,500,000 timesteps for our three-body orbit to determine their long-term stability in extreme scenarios and visually compare the difference between their accuracy.

2.3 Further investigations: N-body Choreographs

To explore further, we considered systems of more than 3 bodies which no longer have a fixed significantly heavy mass at the centre of the orbit. This means for each body, we were required to consider every other body's interaction with it. This meant for an N-body system, we typically considered at least N(N-1) interactions. Hence why it's known that these three body problems are typically much harder to solve[3].

2.3.1 Three body figure-of-eight choreography

To start, we first investigated a figure-of-eight choreography consisting of 3 bodies of equal mass. To achieve this, we use the initial positions:

$$\vec{r_1} = \begin{pmatrix} 0.97000436 \\ -0.24308753 \end{pmatrix}, \quad \vec{r_2} = -\vec{r_1} = \begin{pmatrix} -0.97000436 \\ 0.24308753 \end{pmatrix}, \quad \vec{r_3} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

and the initial velocities:

$$\vec{v_1} = \vec{v_2} = \frac{-\vec{v_3}}{2} = \begin{pmatrix} 0.46620369 \\ 0.43236573 \end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix} -0.93240737 \\ 0.86473146 \end{pmatrix} \tag{34}$$

which result in the 3 bodies following a figure-of-eight pattern under their own gravity[4]. These conditions were discovered in the paper: A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses by Alain Chenciner and Richard Montgomery. To propagate the system, we used the Verlet method, with 15,000 steps. The suggested conditions were given to 8 significant figures, so we also investigated how stable our orbit was when we used less accurate values. We tested the orbits with initial conditions between 2 to 7 significant figures with 15,000 timesteps, still using the Verlet method.

2.3.2 Simple Lagrange problem

After establishing our figure-of-eight choreography, we explored the Lagrange problem of three bodies with an equal mass being located on the vertices of an equilateral triangle. This problem can also be thought of as three bodies being equally spaced out on a unit circle, with the vertices of the equilateral triangle touching the circle. We first solved for the initial positions and velocities and then later on extended this to N-bodies equally spaced on a unit circle. We ensured all the bodies followed each other in a unit circle, orbiting around their centre of mass under their gravity, through the correct initial positions and velocities, and then used the velocity Verlet method to propagate the system.
To make our problem simpler, we always kept one body at the bottom of the circle, specifically (0,-1). This body acted as our 'anchor' and reference point. We then considered θ as the angle between our anchor point and another body through the centre of the circle in an anticlockwise direction. For example, the point (0,1) will be located at $\pi/2$ radians (on the unit circle). This coordinate is the same as (sin(0), cos (π)). We then considered the relationship between the trigonometric equations and the unit circle and pointed out that sin(θ) gave our x-coordinate and cos(π - θ) gave our y-coordinate.

$$\vec{r}_\theta = \begin{pmatrix} sin(\theta) \\ cos(\pi - \theta) \end{pmatrix} \tag{35}$$

For the simple Lagrange problem, we used trigonometric equations to find the initial positions. Our first coordinate will always be (sin(0), cos (π)) as mentioned above. Since we have three bodies that are equally distant from each other, the angles between them will be $2\pi/3$. So the other two bodies will start where $\theta = 2\pi/3$ and $\theta = 4\pi/3$, therefore their initial coordinates will be at (sin($2\pi/3$), cos ($\pi/3$)) and (sin($4\pi/3$), cos($-\pi/3$)). We plotted these points as well as traced out a unit circle to confirm they lie on it.

$$\vec{r}_1 = \begin{pmatrix} sin(0) \\ cos(\pi) \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} sin(2\pi/3) \\ cos(\pi/3) \end{pmatrix}, \quad \vec{r}_3 = \begin{pmatrix} sin(4\pi/3) \\ cos(-\pi/3) \end{pmatrix} \tag{36}$$

To find the initial velocities of the three bodies in a simple way, we considered the body undergoing circular motion due to the force from both of the other bodies. Using equation 1, we have:

$$\frac{m_1 v_1^2}{r} = \frac{G m_1 m_2}{|r_{12}^2|}\hat{r}_{12} + \frac{G m_1 m_3}{|r_{13}^2|}\hat{r}_{13} \tag{37}$$

Focusing on the body at the bottom of the circle, we wanted it to orbit a circle of radius one so that r would equal 1. The right-hand side (RHS) should only consider components of the force which are parallel to the line connecting the bottom body to the centre (upwards or north) as these are the only components which contribute to the centripetal force. Hence, by using simple geometry and trigonometry, our unit vectors are:

$$\hat{r}_{12} = sin(\pi/3), \quad \hat{r}_{13} = sin(2\pi/3) \tag{38}$$

Using the cosine rule, we can also calculate $|r_{12}^2|$ and $|r_{13}^2|$:

$$|r_{12}^2| = r^2 + r^2 - 2cos(2\pi/3) = 2 - 2cos(2\pi/3), \qquad |r_{13}^2| = 2 - 2cos(4\pi/3). \tag{39}$$

Since G and all our masses are equal to 1, we end up with:

$$|v| = \sqrt{\frac{sin(\pi/3)}{2 - 2cos(2\pi/3)} + \frac{sin(2\pi/3)}{2 - 2cos(4\pi/3)}}. \tag{40}$$

Due to the polar symmetry of the problem, this means all bodies will have the same magnitude of velocity as above. The direction of our velocities will be tangent in an anticlockwise direction. In our model, this means a rotation of $\pi/2$ radians anticlockwise of the position vector, which is a transformation of the coordinates from (x_0, y_0) to $(-y_0, x_0)$:

$$\vec{v}_1 = |v| \begin{pmatrix} -cos(\pi) \\ sin(0) \end{pmatrix}, \quad \vec{v}_2 = |v| \begin{pmatrix} -cos(\pi/3) \\ sin(2\pi/3) \end{pmatrix}, \quad \vec{v}_3 = |v| \begin{pmatrix} -cos(-\pi/3) \\ sin(4\pi/3) \end{pmatrix} \tag{41}$$

We propagated this system for only 1000 steps due to the lack of stability of the orbits (which we will explore further later).

2.3.3 Extension of the simple Lagrange problem: 5 and 8 bodies in a unit circle

After successfully solving our simple Lagrange problem, we investigated a similar problem with 5 and 8 bodies evenly spaced on a unit circle. For 5 bodies, since they're evenly spread out, they have an angle of $2\pi/5$ between them. Hence, using the same as for the simple Lagrange problem, we were able to trivially find their positions:

$$\vec{r}_1 = \begin{pmatrix} sin(0) \\ cos(\pi) \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} sin(2\pi/5) \\ cos(3\pi/5) \end{pmatrix}, \quad \vec{r}_3 = \begin{pmatrix} sin(4\pi/5) \\ cos(\pi/5) \end{pmatrix} \tag{42}$$

$$\vec{r}_4 = \begin{pmatrix} sin(6\pi/5) \\ cos(-\pi/5) \end{pmatrix}, \quad \vec{r}_5 = \begin{pmatrix} sin(8\pi/5) \\ cos(-3\pi/5) \end{pmatrix} \tag{43}$$

Force on one body due to all other bodies is given by:

$$\frac{m_1 v_1^2}{r} = \frac{Gm_1m_2}{|r_{12}^2|}\hat{r_{12}} + \frac{Gm_1m_3}{|r_{13}^2|}\hat{r_{13}} + \frac{Gm_1m_3}{|r_{14}^2|}\hat{r_{14}} + \frac{Gm_1m_5}{|r_{15}^2|}\hat{r_{15}} \tag{44}$$

Using geometry, the distances between the bodies are given by:

$$|r_{12}^2| = 2 - 2cos(2\pi/5), \quad |r_{13}^2| = 2 - 2cos(4\pi/5), \quad |r_{14}^2| = 2 - 2cos(6\pi/5), \quad |r_{15}^2| = 2 - 2cos(8\pi/5) \tag{45}$$

and using trigonometry, the unit vectors are:

$$\hat{r_{12}} = sin(\pi/5), \quad \hat{r_{13}} = sin(2\pi/5), \quad \hat{r_{14}} = sin(3\pi/5), \quad \hat{r_{15}} = sin(4\pi/5). \tag{46}$$

Once again, their velocities will be tangentially anticlockwise, hence we can write the unit vectors for the initial velocities using the same coordinate transformation idea as in the simple Lagrange problem:

$$\hat{v}_1 = \begin{pmatrix} -cos(\pi) \\ sin(0) \end{pmatrix}, \quad \hat{v}_2 = \begin{pmatrix} -cos(3\pi/5) \\ sin(2\pi/5) \end{pmatrix}, \quad \hat{v}_3 = \begin{pmatrix} -cos(\pi/5) \\ sin(4\pi/5) \end{pmatrix} \tag{47}$$

$$\hat{v}_4 = \begin{pmatrix} -cos(-\pi/5) \\ sin(6\pi/5) \end{pmatrix}, \quad \hat{v}_5 = \begin{pmatrix} -cos(-3\pi/5) \\ sin(8\pi/5) \end{pmatrix} \tag{48}$$

Due to G and all masses being equal to one, the magnitude of velocity for all bodies to stay in orbit is therefore:

$$|v| = \sqrt{\frac{sin(\pi/5)}{2 - 2cos(2\pi/5)} + \frac{sin(2\pi/5)}{2 - 2cos(4\pi/5)} + \frac{sin(3\pi/5)}{2 - 2cos(6\pi/5)} + \frac{sin(4\pi/5)}{2 - 2cos(8\pi/5)}}.$$

Thus, our final velocities will be:

$$\vec{v}_1 = |v| \begin{pmatrix} -cos(\pi) \\ sin(0) \end{pmatrix}, \quad \vec{v}_2 = |v| \begin{pmatrix} -cos(3\pi/5) \\ sin(2\pi/5) \end{pmatrix}, \quad \vec{v}_3 = |v| \begin{pmatrix} -cos(\pi/5) \\ sin(4\pi/5) \end{pmatrix} \tag{49}$$

$$\vec{v}_4 = |v| \begin{pmatrix} -cos(-\pi/5) \\ sin(6\pi/5) \end{pmatrix}, \quad \vec{v}_5 = |v| \begin{pmatrix} -cos(-3\pi/5) \\ sin(8\pi/5) \end{pmatrix} \tag{50}$$

Looking at our initial conditions for three and five-body simulations, we noticed it was quite tedious to manually calculate the positions each time. Fortunately, we realised that there was a pattern for determining the initial positions and velocities of any N-body unit circle orbit. When looking at our initial positions, we recognised that they took the form:

$$\vec{r_i} = (sin((i-1) * 2\pi/n), cos(\pi - (i-1) * 2\pi/n) \tag{51}$$

where n was the number of bodies and i is an integer running from 0 to $n-1$. We can also figure out a pattern for initial velocity. When looking at the formula for the magnitude of our velocity for 3 bodies and 5 bodies, we notice that it takes the form:

$$|v_n| = \sqrt{\sum_{i=1}^{n-1} \frac{sin(i * \pi/n)}{2 - 2cos(i * 2\pi/n)}} \tag{52}$$

where n was once again the number of bodies. Reminding ourselves of how we can use the position transformation to find our velocity vectors, we were able to determine a systematic way of finding our initial velocities and positions for any scenario with n bodies:

$$\vec{v_j} = |v_n|(-cos(\pi - (j-1) * 2\pi/n), sin((j-1) * 2\pi/n)) \tag{53}$$

This was particularly useful for our code since we were able to create a generalised function that would be able to propagate any system of n bodies in a unit circle. Using equations 51 and 52, we determined our initial positions and velocities for 8 bodies and propagated both the 5 and 8-body systems for 1000 steps.

2.3.4 Pushing the limits of the Verlet method

For the final part of our investigation, we examined the stability of our Verlet model when dealing with 10, 20 or 50 bodies. Additionally, We further analysed if smaller timesteps of 0.025s or 0.001s made any difference, whilst keeping the total time of the simulation, 10s, constant.

3 Results and Analysis

3.1 Two bodies with different masses

We observed that m_2 moved around m_1 in a roughly circular orbit as expected. From our knowledge of trigonometric functions, we can also infer that the positions horizontally and vertically were sinusoidal.

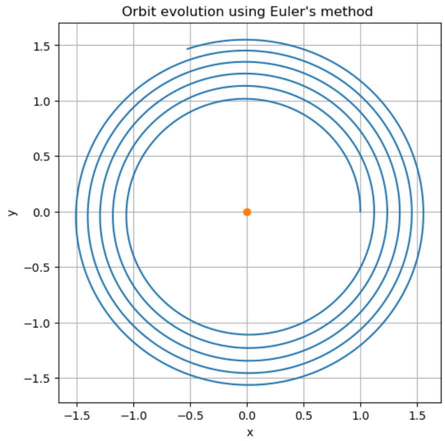

Figure 1: Propagation of star-planet system using Euler's method

We can see however that the orbit in Figure 1 is rather imperfect, which highlights the lack of stability of the Euler method. Furthermore, we predicted that the Euler method would not conserve angular momentum and Energy very well. This is something we can evidently see in Figure 2, as m_2 lost 37.05% of its initial energy after propagating 5000 steps. We also see in Figure 3 that the angular momentum of m_2 is increased 27.02% after 5000 steps which, along with the path in Figure 1, indicates that the error of the orbit would have increased over time.

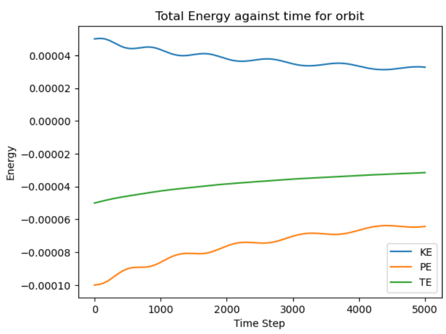

Figure 2: Energy of m_2 against time

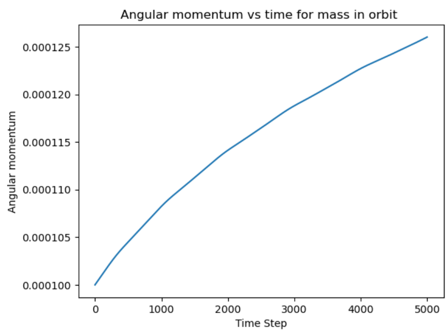

Figure 3: Angular momentum of m_2 against time

3.2 Two bodies with a similar mass

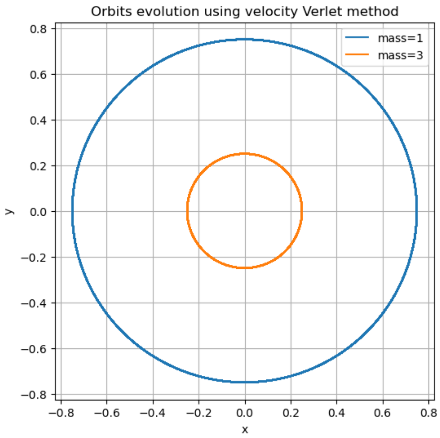

Figure 4: Propagation of two similar massed bodies using the Verlet method

When the masses of our two bodies were the same, they orbited around their centre of mass which we we conveniently put at the origin. Once again, due to their circular motion, we can infer that both their horizontal and vertical displacement over time was sinusoidal. We notice also that despite having three times as many identically sized timesteps as our Euler method, the orbit was far more stable than for the Euler method. This accuracy quickly breaks down however when we increase our timesteps while keeping total time the same, as evident in figures 5 and 6. When dt is equal to 1s, we no longer have any resemblance of an orbit.

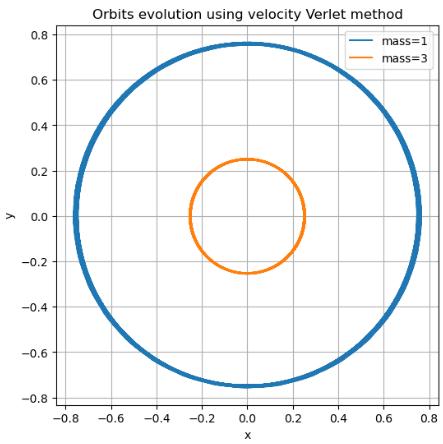

Figure 5: Verlet orbit when $dt = 0.1$

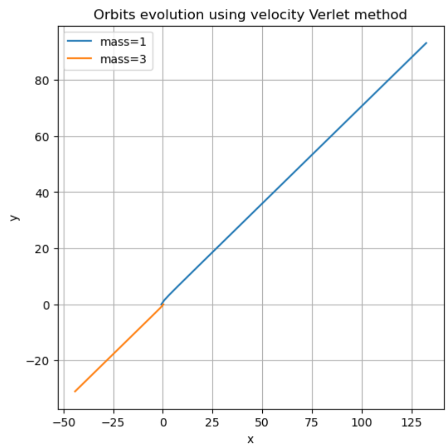

Figure 6: Verlet orbit when $dt = 1.0$

We can find the total energy and angular momentum of the system by adding the individual energies and angular momentum of both bodies. When we do this, we see these quantities are conserved extremely well. Our system lost 0.0094% of its initial energy and gained 4.99 x 10^{-7}% more angular momentum after 15,00 timesteps. The Verlet method took longer to compute than the Euler method as expected due to Verlet having a complex equation and there now being two bodies to consider each step. Where the Euler method took 0.15s for 5,000 steps, the Verlet method took 1.25s for 15,000 steps.

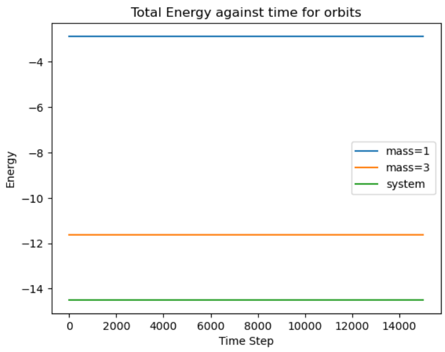

Figure 7: Energy of system for two bodies

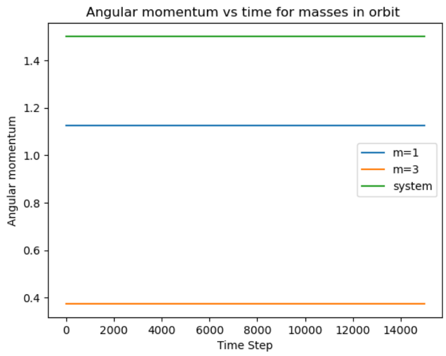

Figure 8: Angular momentum of system for two bodies

3.3 Three bodies

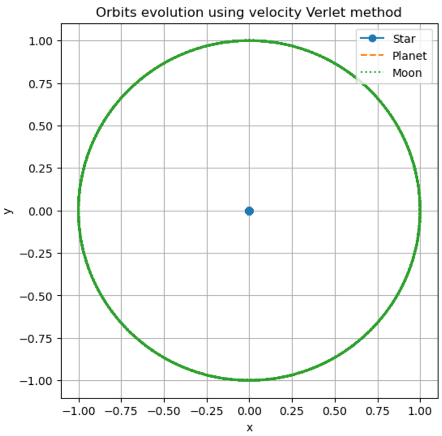

Figure 9: Three body orbit using Verlet method

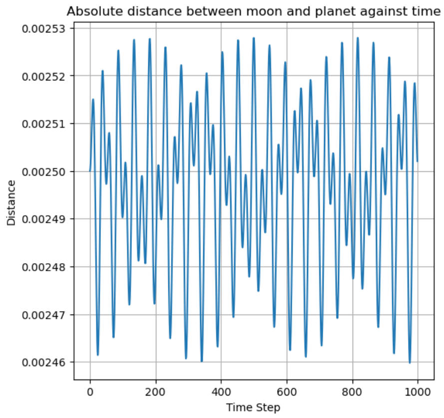

Figure 10: Distance for 1000 timesteps

Our three-body system was similar to that of a star-planet-moon system, where the star was fixed at the origin. When looking at the orbits, the paths of the planet and moon are nearly indistinguishable, hence we plotted some short paths in Figure 11, lasting 20 timesteps, of just the planet and star. From these graphs, we can see how the moon orbits around the planet with the paths crossing each other roughly once every 25 steps, while the planet is also moving through space. Figure 10 also confirms that the orbit around the planet. When we analyse further, we notice that it also shows a wave that resembles a superposition of two waves: a carrier wave and a modulating wave. The carrier wave is of a higher frequency which is related to the frequency of how often the moon orbits the planet, while the modulating wave is of a lower frequency which relates to how often the planet orbits the star.

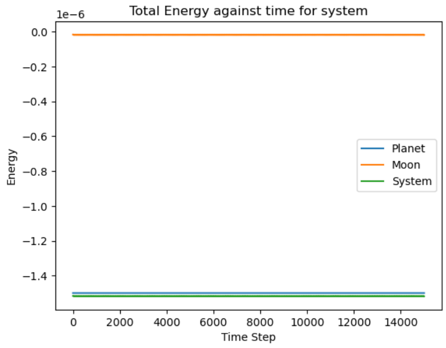

Figure 12: Energy of system with three bodies

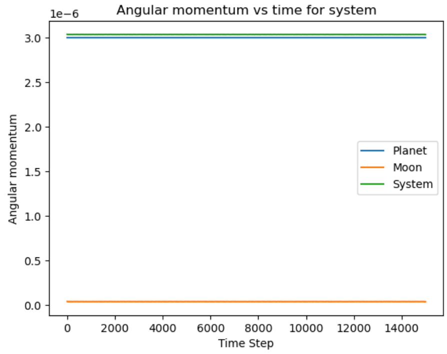

Figure 13: Angular momentum of system with three bodies

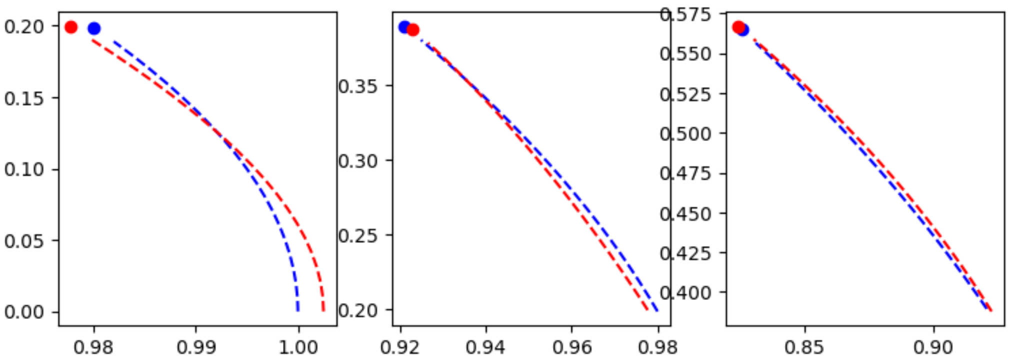

Figure 11: Orbits of planet and moon, between timesteps of 0-20, 20-40 and 40-60

Our system gained 0.14% more energy and lost 0.070% of its angular momentum after 15,000 timesteps. While this is still a good level of conservation, it's far worse than what our 2 body system was able to conserve. Our previous system lost 0.0094% of its initial energy, about 15 times smaller in magnitude. This is likely due to the larger complexity of a three-system, resulting in our Verlet method being able to less accurately conserve all conservable quantities. The increase in energy is likely due to the interacting forces from the bodies causing each other to inaccurately accelerate ever so slightly. Our system was also losing angular momentum, indicating that after a long enough period, the bodies would crash into each other.

3.4 Using a different integrator: The Runge-Kutta method

We used the RK4 method to propagate the same three-body orbit as in the previous section, hence we won't display it for 15,000 steps in this paper. The orbits were almost identical except the RK4 method took a longer time to execute due to more terms being computed in each step, though this indeed resulted in a more visibly smooth and accurate orbit when using the same number of timesteps as the previous section (15,000). We noted before that the RK4 method is not energy-conserving, which is something we observed in our energy plots for the planet. Looking at Figure 14, we see that the RK4 method results in the planet losing energy over time whilst the Verlet method in Figure 15 conserves energy very well. Angular momentum was almost identical to Figure 13, losing 0.041% of its initial angular momentum which was similar to the same three-body system with the Verlet method.

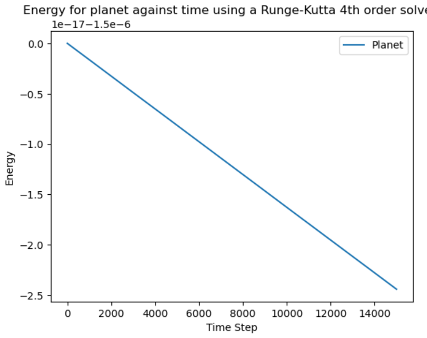

Figure 14: Energy of system with RK4 method

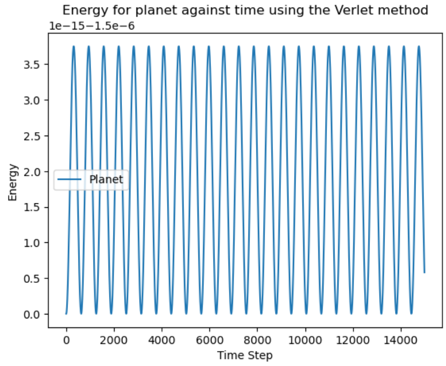

Figure 15: Energy of system with Verlet method

We also investigated how timesteps between 0.001s and 1s affected the accuracy of the moon's orbit with a constant time of 15 seconds. This is shorter than our previous orbit which was 150s but we chose a shorter time so that the orbits for each timestep could be distinguished better. The results in Figure 16 were as expected, with a smaller timestep resulting in a more accurate orbit. Any timestep equal to or below 0.01s allowed for a reasonably accurate orbit.

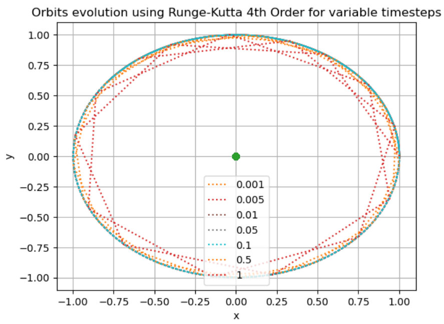

Figure 16: Orbit of the moon with variable timesteps using RK4

When pushing the limits for the RK4 and the Verlet method by propagating 1,500,000 timesteps for our three-body orbit, we surprisingly observed that both methods were able to keep an almost perfect orbit. The RK4 method however provides that extra detail of accuracy when you closely observe both the orbits in Figures 17 and 18, as its graph is thinner than the Verlets. Ideally, it would have had no thickness at all except for the width of our illustrated line but this is extremely impressive regardless and hints at how powerful and accurate a higher-order Runge-Kutta function would be. However, this would of course require a great amount of computational time, as both methods took 2-3 minutes to propagate 1,500,000 steps.

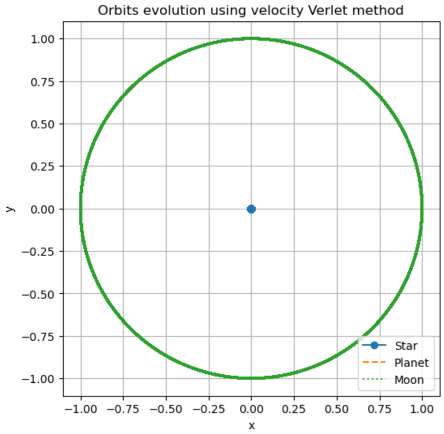

Figure 17: Propagation with 1,500,000 steps using Verlet

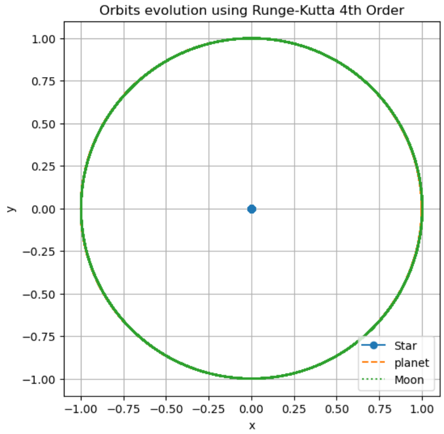

Figure 18: Propagtaion with 1,500,000 steps using RK4

3.5 Three-body Choreography

For our choreography, we used the initial conditions outlined in section 2.3.1 and were able to successfully create our figure-of-eight orbit. The orbit was fairly stable with 15,000 timesteps.

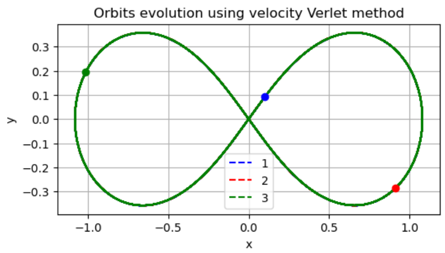

Figure 19: Energy of system with three bodies

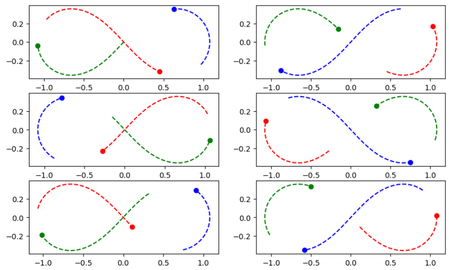

Figure 20: Angular momentum of system with three bodies

Figure 19 shows the bodies creating the choreography, however, they were challenging to distinguish hence, in Figure 20, we plotted short sections of the orbits lasting 150 timesteps to illustrate how the bodies travelled with respect to each other. We can see whenever any of the equal masses are heading towards the centre, they would accelerate and travel a greater distance before slowing down and turning at the edges. The orbit has some similarities to a system undergoing simple harmonic motion, as the net force on a body is opposite to its displacement resulting in it pointing towards the centre when the body is not in the middle of the system. The velocity is also at its maximum when displacement is 0 and minimum when displacement is maximum, analogous to the $\pi/2$ phase difference observed in simple harmonic motion. However, Figure 20 also shows that the net force of an object is not necessarily 0 when displacement is also zero, which means we can't declare it definitely as a simple harmonic system.

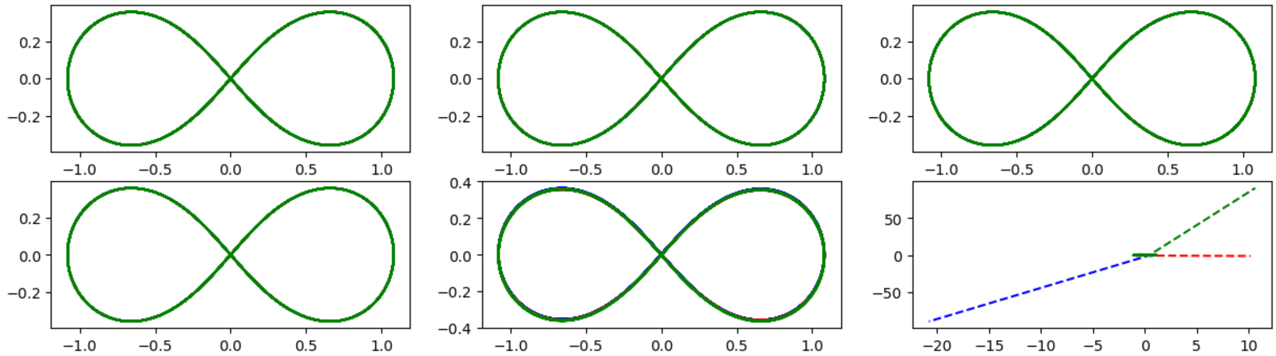

Figure 21: Figure-of-eight choreography with initial conditions between 2-7 significant figures

When testing the stability of our choreography with less accurate initial conditions, we can see in Figure 21 that the orbits are all perfectly stable until we reach our last plot with only 2 significant figures. These orbits were only propagated for 15,000 steps so it is likely however that we would see more of the plots losing stability if we had a longer simulation time.

3.6 Simple Lagrange problem

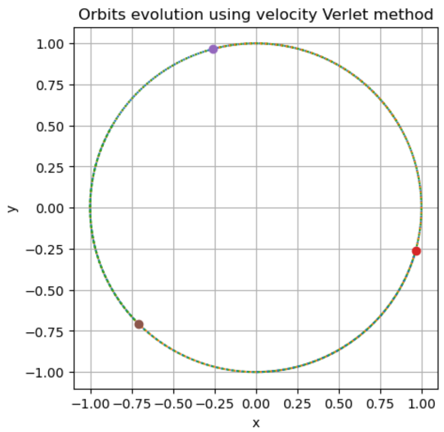

Figure 22: Simple Lagrange Problem with 1000 timesteps

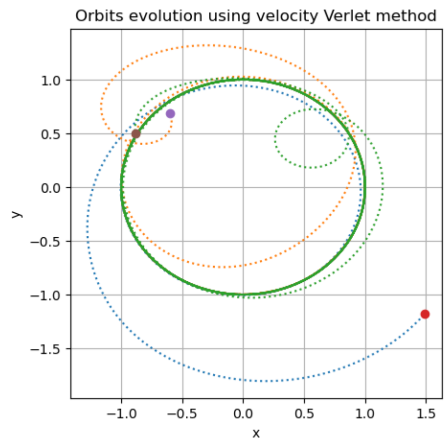

Figure 23: Simple Lagrange Problem with 7000 timesteps

Our simple Lagrange problem was able to be propagated successfully with 1000 timesteps as shown in Figure 22, however, it was quite unstable and would start to deviate after 7000 timesteps as seen in Figure 23. This suggests that the simple Lagrange problem is much more unstable than our previous plots which were able to work perfectly well, even when testing with 1,500,000 timesteps. This is likely due to each body relying on two other bodies to provide the required centripetal force. Both of these bodies also have positional errors, unlike our star-planet-moon system which had a fixed point, hence any error from one body is exaggerated to the rest. The paths of the orbits of the paths are clearer however when we plot short sections lasting 150 timesteps. From these paths, we observe that the planets follow each other and stay equidistant at all points from our perspective. If we were to plot these sections further along time, however, we would see that the planet's paths would start to deviate, from the small inaccuracies compounding.

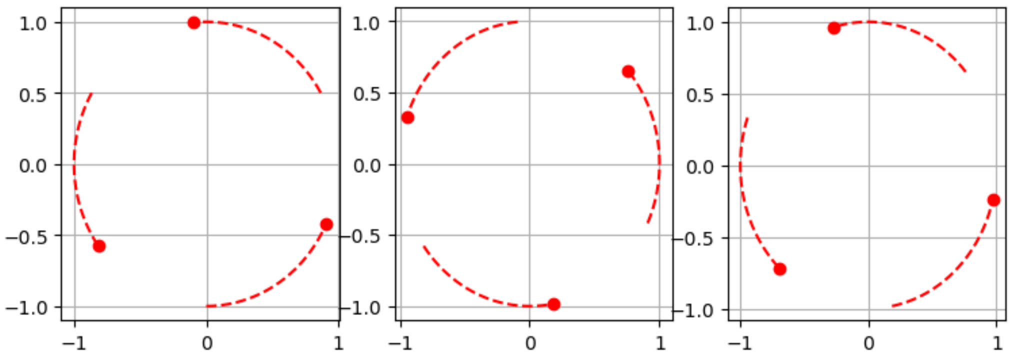

Figure 24: Sections of orbit between 0-150, 150-300, 300-450 timesteps

3.7 3 and 5 bodies in a unit circle

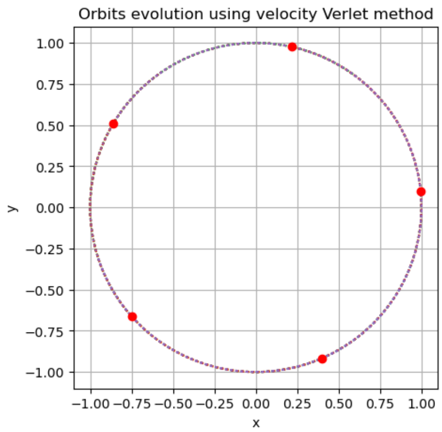

Figure 25: 5 Bodies in orbit

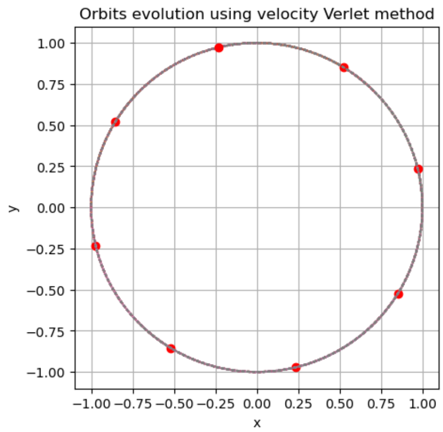

Figure 26: 8 bodies in orbit

Our plots for 5 and 8 bodies were similar to our simple Lagrange problem. We see they can sustain a circular orbit for 1000 timesteps, although anything more results in the orbits deviating. Hence these orbits are possible in the physical world although naturally finding such orbits is almost impossible as we discovered how susceptible they are to any sort of error. The vast majority of real-life orbits are not perfectly circular either, so for the planets to rely on each other's gravity is extremely unreliable. Instead, we have elliptical orbits dominating our galaxies and investigating their eccentricity would have been a better experiment.

3.8 Pushing the limited of the Verlet method

The orbits in Figures 27-29 were all unstable and resulted in chaos. Figure 27 had a timestep of 0.01 for 1,000, the same as we used for 3, 5 and 8 bodies but it seems 10 more bodies result in these parameters being ineffacious. This further supports the idea that each body depends on every other body to stay in orbit, so a greater number of bodies results in more positional errors affecting the force it feels, leading to the system going into chaos sooner. Making the timestep smaller whilst keeping the time constant stabilised the orbits as the planets are closer to their unit circle orbit. However, it's clear that a system with many bodies would require an extremely small timestep, which along with the exponentially increasing number of forces required to be computed, would result in an exceptionally long computation time. The 50-body orbit for example took about 30 seconds to propagate 1,000 steps, despite producing a very chaotic and inaccurate orbit. It might have been more useful instead to consider a centre-of-mass (CoM) approach for this type of problem due to its symmetry and perhaps this would have significantly reduced the computational time to propagate such a system. It is something to definitely note and consider next time when simulating another comparable orbit.

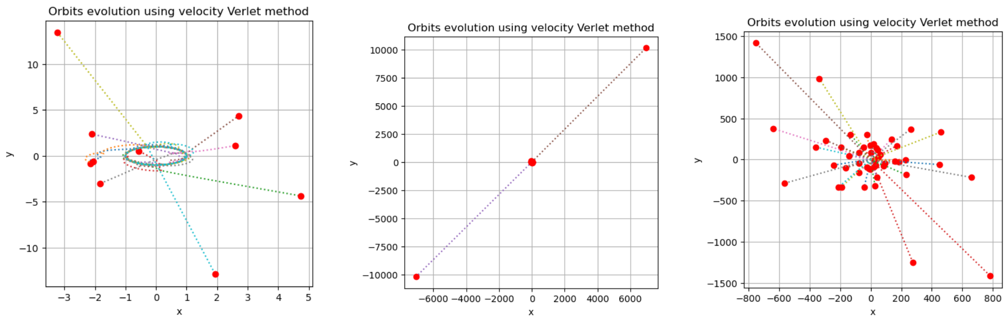

Figure 27: 10, 20 and 50 bodies with timestep = 0.01

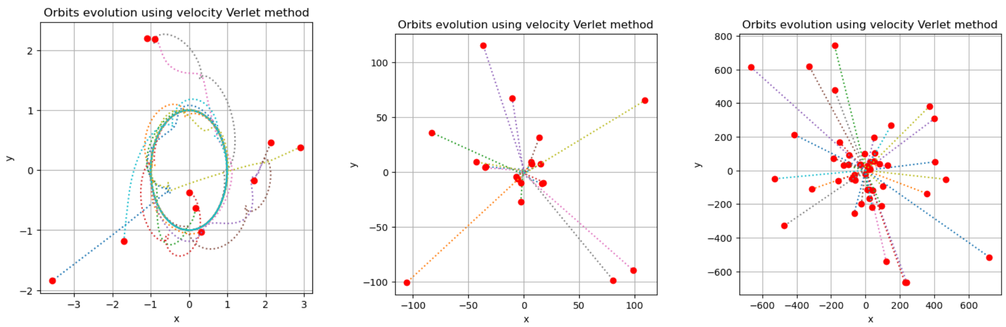

Figure 28: 10, 20 and 50 bodies with timestep = 0.0025

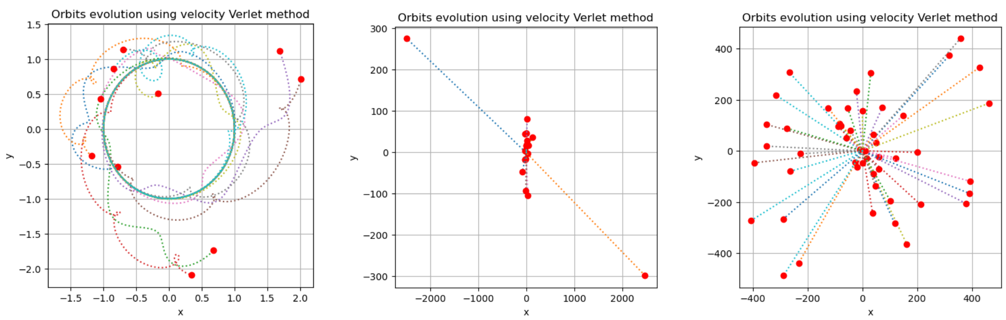

Figure 29: 10, 20 and 50 bodies with timestep = 0.001

4 Conclusion

The Euler, Verlet and RK4 integrators all delivered fairly correct approximations of our simulated orbits. The Verlet method was able to conserve energy very well whilst the Euler and RK4 methods struggled, although the RK4 method was far more precise than the Euler method. The Verlet and RK4 methods were able to conserve angular momentum very well while the Euler method was once again ineffective. The Verlet and RK4 methods, however, became much less effective at conserving these properties once an extra body was introduced, due to the greater amount of errors generated from this addition. When determining the optimal integrator for propagating our orbits, all methods conserved energy and angular momentum, so had some merit. Despite this, the Euler method however was largely inaccurate in these conservations as well as produced sizably unstable orbits, hence could not be considered the optimal integrator. Furthermore, the RK4 method wasn't energy-conserving or time-reversible, regardless of whether it was more accurate position-wise than the Verlet method. It also involved many more calculations per step which resulted in a longer computation time. Hence it was apparent that the Verlet method would be the most optimal integrator for our extended investigation due to its combination of accuracy, efficiency and ability to conserve system properties.

When propagating our three-body choreography, we found it to be remarkably stable even truncating the initial conditions from 8 to 3 significant figures and propagating it for 15,000 steps. However, our N-bodies in unit circle orbits were far more unstable, even using a smaller timestep of 1,000 or decreasing our timestep. As we added more bodies, we realised the orbits becoming even more unstable due to the larger number of positional errors which compounded as each body was responsible for providing the centripetal force of every other body.

To conclude, this concise exploration of orbital mechanics was found to be rigorous yet rewarding. Should someone have more time and a much more powerful computer, they might investigate the rotation of an asteroid belt. It would also be interesting to simulate a four-body orbit such as a blackhole star-planet-moon system or a three-body system similar to section 2.2.3 but with a binary star system. Additionally, it's clear that these methods could have clear applications in statistical and fluid mechanics simulations or, in fact, any type of system with movable bodies.

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