INSTITUTE OF AERONAUTICAL ENGINEERING

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LECTURE NOTES:

COMPLEX ANALYSIS AND SPECIAL FUNCTIONS(AHSB05))

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Chapter 1

COMPLEX FUNCTIONS AND DIFFERENTIATION

Course Outcomes

After successful completion of this module, students should be able to:

CO 1	**Identify** the fundamental concepts of analyticity and differentiability for finding complex conjugates , conformal mapping of complex transformations.	Understand

COMPLEX FUNCTIONS AND DIFFERENTIATION:

1.1 Introduction:

In this part of the course we will study some basic complex analysis. This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematics and physics.We will extend the notions of derivatives and integrals, familiar from calculus,to the case of complex functions of a complex variable. In so doing we will come across analytic functions, which form the centerpiece of this part of the course. In fact, to a large extent complex analysis is the study of analytic functions.
After a brief review of complex numbers as points in the complex plane, we will first discuss analyticity and give plenty of examples of analytic functions. We will then discuss complex integration, culminating with the generalised Cauchy Integral Formula, and some of its applications. We then go on to discuss the power series representations of analytic functions and the residue calculus, which will allow us to compute many real integrals and infinite sums very easily via complex integration.
Historically,Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is helpful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including the branches of hydrodynamics, thermodynamics, and particularly quantum mechanics.
For a complex number z = x + iy, the number Re z = x is called the real part of z and the number Im z = y is said to be the its imaginary part. If x = 0, z is said to be a purely imaginary number.
Definition :
$\sqrt{x^2+y^2}$ is said to be the absolute value or the modulus of the complex number z.
Functions of a Complex Variable : Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function f : D → C is a map that assigns to each complex argument z = x + iy in D a unique complex number w = u + iv. We write w = f(z).
The set D is called the domain of the function f and the set f(D) is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w the image of z under f.
If $z = x+iy \in D$, we shall write f(z) = u(x, y) + iv(x, y) or f(z) = u(z) + iv(z). The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.
Example 1.
The function f : C → C, defined by $f(z) = z^3$, can be written as f(z) = u(x, y) + iv(x, y), with u, $v : R^2 \to R$ given by $u(x,y) = x^3-3xy^2$, $v(x,y) = 3x^2y-y^3$.

Example 2.

For the function f : C → C, defined by f(z) = e z , we have $u(x,y) = excosy, v(x,y) = exsiny$, for any $(x,y) \in R^2$

Limits of Functions : It is defined as force acting per unit area. Let $D \subseteq C, a \in D$ and f : D → C. A number $l \in C$ is called a limit of the function f at the point a if for any $V \in V(l)$, there exists $U \in V(a)$ such that, for any $z \in U \cap D$, it follows that $f(z) \in V$.

We shall use the notation $\lim_{z \to z_0} =$ f(z).

Remark : If a complex function f : D → C possesses a limit l at a given point a, then this limit is unique.

1.2 Differentiability of complex function:

Let w = f(z) be a given function defined for all z in a neighbourhood ofz_0.If $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$exists,the function f(z) is said to be derivable atz_0 and the limit is denoted by . if exists is called the derivative of f(z) at z_0.

Exercise : 1

$f(z) = \left|z^2\right| isafunctionwhichiscontinuousatallzbutnotderivableatanyz \neq 0$

Solution :

$Let f(z) = \left|z^2\right| = z\bar{z}$

$Then f(z) = z_0\bar{z}_0$

$Wez\bar{z} = z_0\bar{z}_0 havetoprovethat \underset{z \to z_0}{lt} z = z_0 \; and \underset{z \to z_0}{lt} \bar{z} = \bar{z}_0$

$Thus \underset{z \to z_0}{lt} z\bar{z} = z_0\bar{z}_0$

$\underset{z \to z_0}{lt} f(z) = f(z_0)$

$The functioniscontinuousatallz$

$f(z_0+\Delta z) = (z_0+\Delta z)(\bar{z}+\Delta\bar{z}) = z_0\bar{z}_0 + z_0\Delta\bar{z} + \Delta z\bar{z}_0 + \Delta z\Delta\bar{z}$

$\frac{now f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{z_0\Delta\bar{z}+\Delta z\bar{z}_0+\Delta z\Delta\bar{z}}{\Delta z}$

$Considerthelimitas\Delta z \to 0$

$Case1: \; let\Delta z \to 0alongx - axisthen\Delta x = \Delta x, \Delta y = 0 \Rightarrow \Delta z = \Delta x$

$\underset{z \to z_0}{lt} \frac{f(z_0+\Delta z)-f(z)}{\Delta z} = \underset{\Delta x \to 0}{lt} \frac{z_0\Delta x+\Delta x\bar{z}_0+\Delta x\Delta x}{\Delta x} = z_0 + \bar{z}_0 - - - - - - - - - - (1)$

$Case2: Let \; \Delta z \to 0alongy - axisthen\Delta x = 0, \Delta y = \Delta y \Rightarrow \Delta z = i\Delta y$

$\underset{\Delta z \to 0}{lt} \frac{f(z_0+\Delta z)-f(z)}{\Delta z} = \underset{\Delta y \to 0}{lt} \frac{z_0(-i\Delta y)+i\Delta y\bar{z}_0+(i\Delta y)(-i\Delta y)}{i\Delta y} = -z_0 + \bar{z}_0 - - - - - - - - (2)$

$Thus, from (1) \, and (2) \, for \, f(z_0) \, toexists$

$i.ez_0 = -z_0 \Rightarrow 2z_0 = 0 \Rightarrow z_0 \neq 0$

$f(z) \, doesnotexiststhough f(z) = \left|z^2\right| iscontinuousatallz.$

1.3 Polar form of Cauchy-Riemann equation:

Theorem
$If f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)\ and f(z)\ isderivableatz_0 = r_0e^{i\theta_0}\ then$
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$
Proof : $Let\ z = re^{i\theta} Then f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$
$Differentiating\ \ itwithrespecttorpartially,$
$\frac{\partial}{\partial r} f(z) = f'(z)\frac{\partial z}{\partial r} = f'(z)e^{i\theta}$

$f(z) = \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r) ------(1)$
$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).rie^{i\theta}$
$f'(z) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta) --- (2)$
$From\,(1)\,and\,(2)\,wehave$
$\frac{1}{e^{i\theta}}(u_r + iv_r) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$
$u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$
$Equatingrealandimaginaryparts\ ,weget$
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} and \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

1.3.1 Analytic function:

A complex function is said to be analytic on a region R if it is complex differentiable at every point in R . The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with ”analytic function”. Many mathematicians prefer the term ”holomorphic function” (or ”holomorphic map”) to ”analytic function” .
Singularities: A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts.

Eg. f(z)=$\frac{1}{z}$is analytic every where except at z=0

At z=0 $f'(z)$ does not exist.
So z=0 is an isolated singular point.

1.3.2 Entire function:

A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a meromorphic function.

1.4 Cartisian Form of Cauchy–Riemann equations:

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables u(x,y) and v(x,y) are the two equations:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
2. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable z = x + iy, f(x + iy) = u(x,y) + iv(x,y)

1.4.1 Relation with harmonic functions:

Analytic functions are intimately related to harmonic functions. We say that a real-valued function h(x, y) on the plane is harmonic if it obeys Laplace's equation:

$\frac{\partial^2 h}{\partial^2 x} + \frac{\partial^2 h}{\partial^2 y} = 0$

In fact, as we now show, the real and imaginary parts of an analytic function are harmonic. Let f = u + i v be analytic in some open set of the complex plane.

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x} \qquad (usingCauchy--Riemann) \\
&= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} \\
&= 0
\end{aligned}
$$

A similar calculation shows that v is also har monic. This result is important in applications because it shows that one can obtain solutions of a second order partial differential equation by solving a system of first order partial differential equations. It is particularly important in this case because we will be able to obtain solutions of the Cauchy–Riemann equations without really solving these equations.
Given a harmonic function u we say that another harmonic function v is its harmonic conjugate if the complex-valued function f = u+i v is analytic.

1.4.2 Conjugate harmonic function:

If two harmonic functions u and v satisfy the Cauchy-Reimann equations in a domain D and they are real and imaginary parts of an analytic function f in D then v is said to be a conjugate harmonic function of u in D.If f(z)=u+iv is an analytic function and if u and v satisfy Laplace's equation ,then u and v are called conjugate harmonic functions.

Polar form of cauchys Riemann equations:
The Cauchy-Riemann equations can be written in other coordinate systems. For instance, it is not difficult to see that in the system of coordinates given by the polar representation z = r e these equations take the following form:

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$
$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

1.4.3 Problem:

Show that the function $f : C \to C$, defined by f(z) = $\bar{z}$ Solution: Indeed, since $u(x,y) = x, v(x,y) = -y$, it follows that $\frac{\partial u}{\partial x}$ while $\frac{\partial u}{\partial x}$. So, this function, despite the fact that it is continuous everywhere on C, it is R differentiable on C, is nowhere C-derivable. Problem: Show that the function $f(z) = e^z$ satisfies the Cauchy-Riemann equations. Solution:
$e^z = e^x(cosy + isiny)$
$And\ \frac{\partial u}{\partial x} \quad = e^x cosy = \frac{\partial v}{\partial y};\ \ \frac{\partial u}{\partial y} \ = e^x siny = -\frac{\partial v}{\partial x};$
Moreover, $e^z\ iscomplexderivableand\ itfollowsimmediatelythatitscomplexderivativeise^z.$

1.4.4 Holomorphic functions:

Holomorphic functions are complex functions, defined on an open subset of the complex plane, that are differentiable. In the context of complex analysis, the derivative of f at z_0 is defined to be

Construction of analytic function whose real or imaginary part is known: Suppose $f(z) = u + iv$is an analytic function ,whose real part u is known .We can find v, the imaginary part and also the function f(z).

Problem:
Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f'(z)| = 0$where f(z) is an analytic function.

Solution : $Taking x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2}=\frac{-i}{2}(z-\bar{z})$
$we have \frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$
$and \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$
$\therefore \frac{\partial^2}{\partial z\partial \bar{z}}= \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right).\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$
$hence \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\log|f'(z)|=4 \quad \frac{\partial^2}{\partial z\partial \bar{z}}\left(\frac{1}{2}\log|f'(z)|^2\right)$
$2\frac{\partial^2}{\partial z\partial \bar{z}}[(\log f'(z)f'(\bar{z}))](|z|^2=z\bar{z})$
$2\frac{\partial^2}{\partial z\partial \bar{z}}[(\log f'(z)+f'(\bar{z}))]$
$2\left[\frac{\partial}{\partial \bar{z}}\frac{f''(z)}{f'(z)}+\frac{\partial}{\partial z}\frac{f''(z)}{f'(z_{-})}\right]$
$=2(0+0)=0$
$Since f(z) is analytic, f(z) is analytic, is also analytic and \frac{\partial f'(z)}{\partial \bar{z}}=0, \frac{\partial f'(\bar{z})}{\partial z}=0$
Problem:

$$Show that \ f(z)=\begin{cases}\frac{xy(x+iy)}{x^2+y^4} & ,z\neq 0\\ 0 & ,z=0\end{cases} \qquad is not analytic \ at z=0 although C-R equations satisified at origin.$$

Solution : $\frac{f(z)-f(0)}{z-0}=\frac{f(z)-0}{z}=\frac{f(z)}{z}$
$\frac{xy^2(x+iy)}{(x^2+y^4).z}=\frac{xy^2(z)}{(x^2+y^4).z}=\frac{xy^2}{(x^2+y^4)}$
$Clearly \ \lim_{\substack{x\to 0\\ y\to 0}}\frac{xy^2}{(x^2+y^4)}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{xy^2}{(x^2+y^4)} \qquad =0$
$Along \ path y=mx$
$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x(m^2.x^2)}{x^2+m^4.x^4}=\lim_{x\to 0}\frac{m^2.x^2}{1+m^4.x^2}=0$
$Along \ path x=my^2$
$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{y^2(m.y^2)}{y^4+m^2.y^4}=\lim_{y\to 0}\frac{m}{1+m^2}\neq 0 \lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{y^2(m.y^2)}{y^4+m^2.y^4}=\lim_{y\to 0}\frac{m}{1+m^2}\neq 0$

Limit value depends on m i.e on the path of approach and its different for the different paths Followed and therefore limit does not exists.
Hence f(z) is not differentiable at z=0.Thus f(z) is not analytic at z=0
To prove that C-R conditions are satisified at origin
$f(z)=u+iv=\frac{xy^2(x+iy)}{(x^2+y^4)}$
$Then u(x,y)=\frac{x^2y^2}{(x^2+y^4)} and and \quad v(x,y)=\frac{xy^3}{(x^2+y^4)} for \ z\neq 0$
$Also u(0,0)=0 \ and v(0,0)=0 \quad [f(z)=0 at z=0]$
$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0}{x}=0$
$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{x\to 0}\frac{0}{y}=0$
$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0}{x}=0$
$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{x\to 0}\frac{0}{y}=0$
$Thus C-R equations are satisified are satisified at the origin$
$Hence f(z) is not analytic at z=0 even C-R equations are satisified at origin.$

1.4.5 Problem:

Find the regular function whose imaginary part is
$\log(x^2+y^2)+x-2y$

Solution : $Given v = \log(x^2+y^2)+x-2y$
$\therefore \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2}+1---(1) and \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2}-2---(2)$
$f(z) \ = \ \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}\,(Using C-R equation)$
$= \frac{2y}{x^2+y^2}-2+\left(\frac{2x}{x^2+y^2}+1\right)(using\,(1),(2))$
$By Milne Thomson method, f(z)\, is expressed in terms of z by replacing x z and y by 0.$
$f'(z) = -2+i\left(\frac{2z}{z^2}+1\right) = -2+i\left(\frac{2}{z}+1\right)$
$On\ integrating,\ f(z) = \int\left[-2+i\left(\frac{2}{z}+1\right)\right]dz+c$
$= -2z+i(2\log z+z)+c = 2i\log z-(2-i)z+c$

Problem: Show that the function$u = 4xy-3x+2$ is harmonic .construct the corresponding analytic function f(z)=u+iv in terms of z.

solution:
$Given u = 4xy-3x+2\,(1)$
$Differentiating\,(1)\, partially w.r.t.x, \frac{\partial u}{\partial x} = 4y-3$
$Again differentiating\ \frac{\partial^2 u}{\partial x^2} = 0$
$Again differentiating\,(1)\ partially w.r.t.y, \frac{\partial u}{\partial x} = 4x$
$Again differentiating\ \frac{\partial^2 u}{\partial y^2} = 0$
$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$
$Hence u is Harmonic.$
$Now f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \Rightarrow f'(z) = 4y-3-i.4x$
$Using Milne Thomson method$
$f'(z) = -3-i4z\,(Putting x = z and y = 0)$
$Integrating, f(z) = -3z-i2z^2+c$

1.4.6 Bilinear Transformation-Mobius Transformations:

The effect of temperature and pressure on a liquid can be described in terms of kinetic-molecular theory. An increase in the temperature Another important class of elementary mappings was studied by August Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations. They arise naturally in mapping problems involving the function arc tan(z). In this section, we show how they are used to map a disk one-to-one and onto a half-plane. An important property is that these transformations are conformal in the entire complex plane except at one point. There exists a unique bilinear transformation that maps three distinct points onto three distinct points , respectively. An implicit formula for the mapping is given by the equation
$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$.

1.4.7 Example 1.

Construct the bilinear transformation w = S(z) that maps thepoints $z_1 = -i, z_2 = 1, z_3 = i$onto the points $w_1 = -1, w_2 = 0, w_3 = 1$ respectively.

Solution:

$\frac{(z-(-i))(1-i)}{(z-i)(1-(-i))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$

$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{(w+1)(0-1)}{(w-1)(0+1)}$ Expanding this equation, collecting terms involving w and zw on

$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{(w+1)}{(-w+1)}$

the left and then simplify.

$\rightarrow (z+i)(1-i)(w+1) = (z-i)(1+i)(-w+1)$

$= (1+i)zw + (1-i)w + (1+i)z + (1-i)$

$(-1+i)zw + (-1-i)w + (1-i)z + (1+i)$

$zw + izw - w - iw + z - iz + 1 + i$

$2zw + 2w = -2iz + 2i$

$w(1+z) = i(1-z)$

$Therefore the desired bilinear transformation is$

$w = s(z) = \frac{i(1-z)}{1+z}$

1.4.8 Example 2.

Construct the bilinear transformation w = S(z) that maps thepoints $z_1 = -2, z_2 = -1-i, z_3 = 0$onto the points $w_1 = -1, w_2 = 0, w_3 = 1$ respectively.

Solution:

$\frac{(z-(-2))(-1-i)}{(z-i)(1-(-i))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$

$\frac{(z+2)(-1-i)}{(z)(-1-i+2)} = \frac{(w+1)(0-1)}{(w-1)(0+1)}$

$\frac{(z+2)(-1-i)}{(z)1-i} = \frac{(w+1)}{(-w+1)}$

$\rightarrow (z+2)(1-w) = iz(w+1)$

$z - iz + 2 = zw + izw + 2w$

$(1-i)z + 2 = w(z + iz + 2)$

$(1-i)z + 2 = w((1+i)z + 2)$

$which can be solved for w in terms of z, giving the desired solution$

$w = s(z) = \frac{(1-i)z+2}{(1+i)z+2}$

1.4.8.1 Fixed Point:

A fixed point of a mapping $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$ Example: Find the fixed points of $w = s(z) = \frac{4z+3}{2z-1}$

Solution:
$s(z) = \frac{4z+3}{2z-1}$
$z = \frac{4z+3}{2z-1}$
$z(2z-1) = 4z+3$
$2z^2 - z - 4z - 3 = 0$
$2z^2 - 5z - 3 = 0$
$(2z+1)(z-3) = 0$
$(z+\frac{1}{2})(z-3) = 0$
$Therefore, the fixed points of \ s(z) = \frac{4z+3}{2z-1} are$
$z = -\frac{1}{2}, 3$

1.4.8.2 EXCERCISE PROBLEMS:

1) Show that the real part of an analytic function f (z) where $u = e^{-2xy} \sin\sin\left(x^2 - y^2\right)$
2) Prove that the real part of analytic function f (z) where $u = log|z|^2$
3) Obtain the regular function f (z) whose imaginary part of an analytic function is $\frac{x-y}{x^2+y^2}$
4) Find an analytic function f (z) whose real part of an analytic function is $u = \frac{sin2x}{\cos h2y - \cos 2x}$ by Milne-Thompson method.
5) Find an analytic function $f(z) = u + iv$ if the real part of an analytic function is $u = a(1 + cos\theta)$ using Cauchy-Riemann equations in polar form
6) State and Prove the necessary condition for f (z) to be an analytic function in Cartesian form.
7) If u and v are conjugate harmonic functions then show that uv is also a harmonic function.
8) Find an analytic function whose real part is $u = \frac{\sin 2x}{\cos h2y - \cos 2x}$
9) Find the orthogonal trajectories of the family of curves $x^3y - xy^3 = C$= constant
10) If f(z) is an analytic function of z and i $u - v = (x-y)(x^2 + 4xy + y^2)$)if find f(z) in terms of z.

Chapter 2

COMPLEX INTEGRATION

Course Outcomes

After successful completion of this module, students should be able to:

CO 2	**Apply** integral theorems of complex analysis and its consequences consequences for the analytic function with derivatives of all orders in simple connected region.	Apply

COMPLEX INTEGRATION:

2.1 Introduction:

Integration of complex functions plays a significant role in various areas of science and engineering. In this chapter, we will deal with the notion of integral of a complex function along a curve in the complex plane. We start with the definition of integration of a complex-valued function of a real variable and extend this idea to the integration of a complex-valued function of a complex variable. Using integration, we will prove an important result on analytic functions. This chapter also includes the Cauchy–Goursat theorem, Cauchy's integral formula, some related theorems, maximum modulus principle and their applications.

2.1.1 Definition:

In mathematics, a line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, W = F • s) have natural continuous analogs in terms of line integrals $(W = \int_C \mathbf{F} \cdot d\mathbf{s})$. . The line integral finds the work done on an object moving through an atomic or gravitational field. In complex analysis, the line integral is defined in terms of multiplication and addition of complex numbers. Let us consider F(t)= u(t)+i v(t) , $a \leq t \leq b$. Where u and v are real valued continuous functions of t in [a,b]. we define $\int\limits_a^b F(t)\, dt = \int\limits_a^b u(t)\, dt + i\int\limits_a^b v(t)\, dt$ Thus $\int\limits_a^b F(t)\, dt$, is a complex number such that real part of $\int\limits_a^b F(t)\, dt$ is $\int\limits_a^b u(t)\, dt$ and imaginary part of is $\int\limits_a^b u(t)\, dt$ is $\int\limits_a^b v(t)\, dt$

2.1.1.1 Problem:

Evaluate $\int_0^{1+i} (x^2 - iy)dz$ along the paths 1)y=x 2)y=x2
Solution:1) along the line y=x, dy= dx so that dz = dx+idx=(1+i) dx

$\int_0^{1+i}(x^2-iy)dz = \int\limits_0^1 (x^2-ix)(1+i)dx,$

$Since y = x$

$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$

$= (1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$

$= \frac{5}{6} - \frac{1}{6}i$

$2) along the parabola y = x^2, dy = 2xdx so that dz = dx + 2ixdx$

$dz = (1+2ix)\, dx and x varies from 0 to 1$

$\therefore \int_0^{1+i}(x^2-iy)dz = \int\limits_0^1 (x^2-ix^2)(1+2ix)dx$

$= (1-i)\int_0^1 x^2(1+2ix)dx$

$= (1-i)\left(\frac{1}{3}+\frac{1}{2}i\right)$

$= \frac{(1-i)(2+3i)}{6}$

$= \frac{5}{6} + \frac{1}{6}i$

Problem

Evaluate $\int_{z=0}^{z=1+i}(x^2+2xy+i(y^2-x))dz$ along $y = x^2$

Solution:

$f(z) = x^2+2xy+i(y^2-x))dz$

$Z = x+iy, dz = dx+idy$

$\therefore the curve y = x^2, dy = 2xdx$

$\therefore dz = dx+2xidx = (1+2ix)dx$

$f(z) = x^2+2x\left(x^2\right)+i\left(x^4-x\right)$

$= x^2+2x^3 \ +i\left(x^4-x\right)$

$f(z)\ dz = \left(x^2\ +2x^3\right)+i\left(x^4-x\right)(1+2ix))dx$

$= x^2+2x^3+i\left(x^4-x\right)+2ix^3+4ix^4-2x^5+2x^2$

$\therefore \int\limits_c f(z)dz = \int_{z=0}^{1+i} x^2+2xy+i(y^2-x)dz$

$\int_0^1(-2x^5+3x^2+2x^3+i(5x^4-x+2x^3))dx$

$\left[-\frac{x^6}{3}+x^3+\frac{x^4}{2}+i(\frac{5x^5}{5}-\frac{x^2}{2}+\frac{x^4}{2}\right]_0^1$

$\left[\left(\frac{-1}{3}+1+\frac{1}{2}\right)+\left(\frac{5}{5}-\frac{1}{2}+\frac{1}{2}\right)\right]-0$

$\frac{7}{6}+\frac{5}{5}i = \frac{7}{6}+i$

$\int\limits_c f(z)dz = \frac{7}{6}+i$

2.1.2 Cauchy-Goursat Theorem:

Let f(z) be analytic in a simply connected domain D. If C is a simple closed contour that lies inD, then

$\int\limits_c f(z)dz = 0$

$Letusrecallthat\ \ e^z,\ \cos z, z^n\ \ (wherenisapositiveinteger)\ areallentirefunctionsandhavecontinuousderivatives.\ T$

$(a) \int\limits_c e^z dz = 0$

$(b) \int\limits_c \cos z dz = 0$

and

$(c) \int\limits_c z^n dz = 0$

2.1.3 Cauchy Theorem:

STATEMENT : let F(z)=u(x,y)+iv(x,y) be analytic on and within a simple closed contour (or curve) ‘c’ and let f ‘(z) be continuous there,then$\int f(z)\,dz = 0$

Proof: f (z)=u(x,y)+iv(x,y)

And dz=dx+idy

f(z).dz = (u(x,y)+iv(x,y))dx+idy

f(z).dz = u(x,y)dx+i u(x,y)dy+iv(x,y)dx+i2 v(x,y)dy

f(z).dz= u(x,y)dx- v(x,y)dy+i(u(x,y)dy+ v(x,y)dx

Integrate both sides, we get

$\int f(z)\,dz = \int (\mathrm{udx} - \mathrm{vdy}) + i\,(\mathrm{udy} + \mathrm{vdx})$

$Bygreenstheorem, wehave$

$\int Mdx + Ndy = \iint \frac{\partial \mathrm{N}}{\partial \mathrm{x}} - \frac{\partial \mathrm{M}}{\partial \mathrm{Y}} \mathrm{dxdy}$

$\int\int f(z)\,dz = \iint \left(-\frac{\partial \mathrm{v}}{\partial \mathrm{x}} - \frac{\partial \mathrm{u}}{\partial \mathrm{Y}}\right) \mathrm{dxdy} + i\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}} - \frac{\partial \mathrm{v}}{\partial \mathrm{Y}}\right) \mathrm{dxdy}$

$UsingGreen'sTheoreminplane, assumingthatRistheregionboundedbyC.itisgiventhatf(z) = u(x,y) + iv(x,y)isan$

$Itisgiventhatf\,(z) = u\,(x,y) + iv\,(x,y)\,isanalyticonandwithinc.\ Hence, Usingthiswehave$

$\int\limits_c f(z)dz = 0$

$Hence\ \ the\,theorem.$

2.1.4 Cauchy’s integral formula:

$f(z_0) = \frac{1}{2\pi i} \oint\limits_r \frac{f(z)dz}{z - z_0}$

Where the integral is a contour integral along the contour r enclosing the point z_0.

Problem:Evaluate using cauchy’s integral formula $\int\limits_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle$|z| = 3$

Solution:

$\int\limits_c \frac{e^{2z}}{(z-1)(z-2)} dz ------(1)$

$Boththepointsz = 1, z = 2lineinside\,|z| = 3$

$Resolving\;\; intopartial fractions$

$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$A = -1, B = 1$

$From\,(1)$

$\int\limits_c \frac{e^{2z}}{(z-1)(z-2)} dz = \int\limits_c \frac{-e^{2z}}{(z-1)} dz + \int\limits_c \frac{e^{2z}}{(z-2)} dz$

$(bycauchysintegral formula)$

$= -2\pi i f\,(1) + 2\pi i f\,(2)$

$= -2\pi i e^{2.1} + 2\pi i e^{2.2}$

$= -2ie^2 + 2e^4 = 2\pi i(e^4 - e^2)$

Problem:

Using Cauchy's integral formula to evaluate $\int\limits_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)z-2)} dz$,

where c is the circle $|z| = 3$

Solution:

$\int\limits_c \frac{f(z)}{(z-1)z-2)} dz = (\int\limits_c \frac{1}{(z-2)} dz + \int\limits_c \frac{1}{(z-1)} dz) f\,(z)\,dz$

$\int\limits_c \frac{f(z)}{(z-2)} dz + \int\limits_c \frac{f(z)}{(z-1)} dz$

$= 2if\,(2) - 2\prod if\,(1)$

$= 2i\,(sin4 + cos4) - (sin + cos))$

$= 2i\,(1 - (-1)) = 4i$

$\int\limits_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)z-2)} dz = 4\pi i$

Problem: :

Evaluate $\int\limits_c \frac{(z-1)}{(z+1)^2(z-2)} dz$whrere c is $|Z - i| = 2$

Solution:the singularities of$\frac{(z-1)}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2)=0$

$Z=-1 and z=2$

$Z=-1 lies inside the circle since |-1-i|-2<0$

$Z=2 lies outside the circle since I2-iI-2>0$

$The given line integral can be written as$

$\int\limits_c \frac{(z-1)}{(z+1)^2(z-2)}dz=\int\limits_c \frac{\frac{(z-1)}{(z-2)}}{(z+1)^2} \quad ---(1)$

$The derivative of analytic function is given by$

$\int\limits_c \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2\pi i f^n(a)}{n!} ----(2)$

$From (1) and (2) f(z)=\frac{(z-1)}{(z-2)}, a=-1, n=1$

$f^1(z)=\frac{1(z-2)-1(z-1)}{(z-2)^2}=\frac{1}{(z-2)^2}$

$f^1(-1)=\frac{1}{-9}$

$Substituting in (2), we get$

$\int\limits_c \frac{(z-1)}{(z+1)^2(z-2)}dz=\frac{2\pi i}{1}(-\frac{1}{9})$

$\frac{-2}{9}\pi i$

Problem:

Evaluate $\int\limits_c \frac{e^{2z}}{(z+1)^4}dz$ where c:$|z-1|=1$

Solution:

The singular points of $\int\limits_c \frac{e^{2z}}{(z+1)^4}dz$ are givenby

$(z+1)^4=0\rightarrow z=-1$

The singular point z=-1 lies insidethecircle

Applying cauchy's integral formula for derivatives

$\int\limits_c \frac{f(z)}{(z-a)^{n+1}}dz=\int\limits_c \frac{2\pi i f^n(-1)}{n!}dz$

$F(z)=e^{2z}, n=3, a=-1$

$f(z)=2e^{2z}$

$f^1(z)=4e^{2z}$

$f^{11}(z)=8e^{2z}$

$f^{111(}z)=16e^{2z}$

$f^{111}(-1)=16e^{-2}$

$Substituting in (1)$

$\int\limits_c \frac{e^{2z}}{(z+1)^4}dz=\int\limits_c \frac{2\pi i f^{111}(-1)}{n!}$

$=\frac{2\pi i 16e^{-2}}{2!}$

$=16\pi i e^{-2}$

Problem:

Evaluate$\oint_c \frac{3z^2+z}{z^2-1}dz$ where cisthe circle $|z-1|=1$

Solution:

$Given f(z) = 3z^2 + z$

$Z = a = +1 or - 1$

$The circle\, |z-1| = 1\; has centre at z = 1 and radius 1 and includes the point z = 1,$

$f(z) = 3z^2 + z\; is an analytic function$

$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$\oint \frac{3z^2+z}{z^2-1} = \frac{1}{2}\left[\int\limits_c \frac{3z^2+z}{z-1}dz\right] - \frac{1}{2}\left[\int\limits_c \frac{3z^2+z}{z+1}dz\right]$ From equation(1)

$Since z = 1 lies inside c,$

$we have by cauchys integral formula$

$\oint \frac{3z^2+z}{z^2-1} = 2\pi i f(i)$

$= 2\pi i * 4$

$By Cauchys integral theorem, since z = -1 lies outside c, we have$

$\oint_c \frac{3z^2+z}{z-1}dz = 0$

we have $\oint_c \frac{3z^2+z}{z-1}dz = \frac{1}{2}(8\pi i) - 0$
$= 4\pi i$

EXCERCISE PROBLEMS:

$(1) Evaluate\ \int \frac{dz}{z-z_0} where c := |z-z_0| = r$

$(2) Evaluate \int_{(1,1)}^{(2,2)} (x+y)dx + (y-x)dy\;\; along the parabola\;\; y^2 = x$

$(3) Evaluate \int\limits_c \frac{z^2+4}{z^2-1} dz where C:\; |z| = 2 using Cauchys Integral formula$

$(4) Evaluate \int\limits_c \frac{e^{2z}}{(z-1)(z-2)} dz where C: |z| = 4 using Cauchys Integral formula$

$(5) Evaluate \int\limits_c \frac{z^3-z}{(z-2)^3} dz where C: |z| = 3 using Cauchys Integral formula$

$(6) Expand f(z) = \int\limits_c \frac{e^{2z}}{(z-1)^3} at a point\; z = 1$

$(7) Expand f(z) = \int\limits_c \frac{1}{z^2-4z+3} for\;\; 1 < |z| < 3$

Chapter 3

POWER SERIES EXPANSION OF COMPLEX FUNCTION

Course Outcomes

After successful completion of this module, students should be able to:

CO 3	**Extend** the Taylor and Laurent series for expressing the function in terms of complex power series.	Understand
CO 4	**Apply** Residue theorem for computing definite integrals by using the singularities and poles of real and complex analytic functions over closed curves.	Apply

POWER SERIES EXPANSION OF COMPLEX FUNCTION:

3.1 Introduction:

Power series, in mathematics, an infinite series that can be thought of as a polynomial with an infinite number of terms, such as $1+x+x^2+x^3+..............$ Usually, a given power series will converge (that is, approach a finite sum) for all values of x within a certain interval around zero—in particular, whenever the absolute value of x is less than some positive number r, known as the radius of convergence. Outside of this interval the series diverges (is.infinite), while the series may converge or diverge when x = ± r. The radius of convergence can often be determined by a version of the ratio test for power series: given a general power series $a_0+a_1x+a_2x^2+................$,in which the coefficients are known, the radius of convergence is equal to the limit of the ratio of successive coefficients.
Power series are useful in mathematical analysis, where they arise as Taylor series of infinitely differentiable functions. In fact, Borel's theorem implies that every power series is the Taylor series of some smooth function..

3.2 Power series:

A series expansion is a representation of a particular function as a sum of powers in one of its variables, or by a sum of powers of another (usually elementary) function f(z). A power series in a variable is an infinite sum of the form $\sum a_i z^i$ A series of the form $\sum a_n z^n$ is called as power series.

that is
$$1+x+x^2+x^3+..............$$
$$a_0+a_1x+a_2x^2+................$$

$$\sum a_n z^n = a_1 z + a_2 z^2 + ...+a_n z^n + ...$$

3.3 Taylor's series:

Taylor's series Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series.
The Taylor series is an infinite series, whereas a Taylor polynomial is a polynomial of degree n and has a finite number of terms. The form of a Taylor polynomial of degree n for a function
f (z) at x = a is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + ...$$

$|z-a| < r$

3.4 Maclaurin series:

Maclaurin series:
A Maclaurin series is a Taylor series expansion of a function about x=0,

$$f(z) = f(0) + f'(0)(z) + f''(0)\frac{(z)^2}{2!} + ...$$

This series is called as maclurins series expansion of f(z).

Problems
1. Determine the first four terms of the power series for

$$\sin 2x$$

using Maclaurin's series.

Solution:
Let

$$\begin{aligned}
&f(x) = sin2x && f(0) = sin0 = 0\\
&f\prime(x) = 2cos2x && f\prime(0) = 2cos0 = 2\\
&f\prime\prime(x) = --4sin2x && f\prime\prime(0) = --4sin0 = 0\\
&f\prime\prime\prime(x) = --8cos2x && f\prime\prime\prime(0) = --8cos0 = --8\\
&f^{iv}(x) = 16sin2x && f^{iv}(0) = 16sin0 = 0\\
&f^{v}(x) = 32cos2x(0) && f^{v}(0) = 32cos0 = 32\\
&f^{vi}(x) = --64sin2x && f^{vi}(0) = --64sin0 = 0\\
&f^{vii}(x) = --128cos2x && f^{vii}(0) = --128cos0 = --128\\
&(x) = sin2x = 0 + 2x + 0x^2 + (-8) + 0.x^4 + 32 && = 2x - +
\end{aligned}$$

Problem : Find the Taylor series about z = -1 for f (x) = 1/z. Express your answer in sigma notation.
Solution:
let

$$f(z) = z^{-1} \qquad f(-1) = -1$$
$$f'' = 2z^{-3} \qquad f''(-1) = -2$$
$$f''' = -6z^{-4} \qquad f'''(-1) = -6$$
$$f'''' = 24z^{-5} \qquad f''''(-1) = -24$$
$$f(z) = -1 - 1(z+1) - \frac{2}{2!}(z+1)^2 - \frac{6}{3!}(z+1)^3 - \frac{24}{4!}(z+1)^4 - \dots\dots\dots = \sum_{n=0}^{\infty} -1(z+1)^n$$

Problem :

Find the Maclaurin series for

$$f(z) = ze^z$$

Express your answer in sigma notation.

Solution:

Let

$$f(z) = ze^z \qquad f(0) = 0$$
$$f' = e^z + ze^z \qquad f'(0) = 1+0 = 1$$
$$f'' = e^z + e^z + ze^z \qquad f''(0) = 1+1+0 = 2$$
$$f''' = e^z + e^z + e^z + ze^z \qquad f'''(0) = 1+1+1+0 = 3$$
$$f'''' = e^z + e^z + e^z + e^z + ze^z \qquad f''''(0) = 1+1+1+1+0 = 4$$
$$f(z) = 0 + 1z + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \frac{4}{4!}z^4 + \dots\dots\dots\dots$$
$$z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \dots\dots\dots\dots$$

3.5 Laurents series:

Laurent series:

In mathematics, the Laurent series of a complex function f(z) is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied

The Laurent series for a complex function f(z) about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

where the

$$a_n and a$$

are constants.

Laurent polynomials:
A Laurent polynomial is a Laurent series in which only finitely many coefficients are non-zero. Laurent polynomials differ from ordinary polynomials in that they may have terms of negative degree.
Principal part:
The principal part of a Laurent series is the series of terms with negative degree, that is

$$f(z) = \sum_{K=-\infty}^{-1} a_K (z-a)^K$$

If the principal part of f is a finite sum, then f has a pole at c of order equal to (negative) the degree of the highest term; on the other hand, if f has an essential singularity at c, the principal part is an infinite sum (meaning it has infinitely many non-zero terms). Two Laurent series with only finitely many negative terms can be multiplied: algebraically, the sums are all finite; geometrically, these have poles at c, and inner radius of convergence 0, so they both converge on an overlapping annulus.
Thus when defining formal Laurent series, one requires Laurent series with only finitely many negative terms.
Similarly, the sum of two convergent Laurent series need not converge, though it is always defined formally, but the sum of two bounded below Laurent series (or any Laurent series on a punctured disk) has a non-empty annulus of convergence.

Zero's of an analytic function:
A zero of an analytic function f(z) is a value of z such that f(z)=0 .Particularly a point a is called a zero of an analytic function f (z) if f(a) = 0.
Example:

$$f(z) = \frac{(z+1)^2}{(z^2+1)^2}$$

Now,

$$(z+1)^2 = 0$$

Z = -1, z = -1 are zero's of an analytic function.
Zero's of mth order:

If an analytic function f(z) can be expressed in the form

$$f(z) = (z-a)^m \Phi(z)$$

where

$$\Phi(z)$$

is analytic function and

$$\Phi(a) \neq 0$$

then z=a is called zero of mth order of the function f(z).
• A simple zero is a zero of order 1.
Example:1.

$$f(z) = (z-1)^3$$

$$\Rightarrow (z-1)^3 = 0$$

z=1 is a zero of order 3 of the function f(z).
2.

$$f(z) = \frac{1}{1-z}$$

that is

$$z = \infty$$

is a simple zero of f(z).
3.

$$f(z) = \sin z$$

that is

$$z = n\pi \quad \forall n = 0, 1, 2, 3, ...$$

are simple zero's of f(z).

3.6 Problems:

Problem: Find the first four terms of the Taylor's series expansion of the complex function

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

About z =2.Find the region of convergence. Solution:
The singularities of the function

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

are z = 3 and z = 4 Draw a circle with centre at z=2 and radius 1 .Then the distance of singularities from the centre are 1 and 2. Hence within the circle

$$|z-2| = 1$$

the given function is analytic .Hence ,it can be extended in Taylor's series within the circle

$$|z-2| = 1$$

. Hence

$$|z-2| = 1$$

is the circle of convergence.
Now,

$$f(z) = \frac{5}{z-4} - \frac{4}{z-3}$$

, f(2)= 3/2

$$f''(z) = -\frac{8}{(z-3)^3} + \frac{10}{(z-4)^3}$$

,

$$f^{''}(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4}$$

,

$$f^{'''}(2) = \frac{177}{8}$$

Taylor's series expansion for f(z) at z=2 is

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!}\left(\frac{27}{4}\right) + \frac{(z-2)^3}{3!}\left(\frac{177}{8}\right)$$

Singular point of an analytic function: A point at which an analytic function f(z) is not analytic, i.e. at which f '(z) fails to exist, is called a singular point or singularity of the function.
There are different types of singular points:
Isolated and non-isolated singular points: A singular point z0 is called an isolated singular point of an analytic function f(z) if there exists a deleted ∈-spherical neighborhood of z0 that contains no singularity. If no such neighborhood can be found, z0 is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. In fig 1a where z1, z2 and z3 are isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ∈-spherical neighborhood of it contains singular points. See Fig. 1b where z0 is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential

singularities and branch points.

3.7 Types of singularities:

1. Pole:
An isolated singular point z0 such that f(z) can be represented by an expression that is of the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}$$

Where n is a positive integer.. The integer n is called the order of the pole. If n = 1, z0 is called a simple pole.
Example: 1.The function

$$f(z) = \frac{5z+1}{(z-2)^3 (z+3)(z-2)}$$

has a pole of order 3 at z = 2 and simple poles at z = -3 and z = 2.
A point z is a pole for f if f blows up at z (f goes to infinity as you approach z). An example of a pole is z=0 for f(z) = 1/z. Simple pole: A pole of order 1 is called a simple pole whilst a pole of order 2 is called a double pole.
2. Removable singular point: An isolated singular point z0 such that f can be defined, or redefined, at z0 in such a way as to be analytic at z0. A singular point z0 is removable if $\lim_{z \to z_0} f(z)$Exist.
Example: 1.The singular point z = 0 is a removable singularity of f(z) = (sin z)/z since $\lim_{z \to z_0} \frac{\sin z}{z} = 1$

3. Essential singular point: A singular point that is not a pole or removable singularity is called an essential singular point.
Example: 1. $f(z) = e^{1/z-3}$ has an essential singularity at z = 3. Singular points at infinity: The type of singularity of f(z) at $z = \infty$is the same as that of f(1/w) at w = 0.

Example: The function $f(z) = z^2$ has a pole of order 2 at $z = \infty$, since f(1/w) has a pole of order 2 at w = 0.

3.8 Residues:

Residues:

The constant a-1 in the Laurent series
$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ of about a point z0 is called the residue of f(z). If is analytic at z0, its residue is zero, but the converse is not always true (for example, of about a point z0 is called the residue of f(z). If is analytic at z0, its residue is zero, but the converse is not always true
for example,

$$\frac{1}{z^2}$$

has residue of 0 at z=0 but is not analytic at z=0 . The residue of a function f at a point z0 may be denoted $\operatorname*{Res}_{Z\to Z_0} f(z)$

Residues at Poles:
(i) If f(z) has a simple pole at z0 , then $Res[f,z_0] = \lim_{Z\to Z_0}(z-z_0)f(z)$ (ii) If f(z) has a pole of order 2 at z0 , then $Res[f,z_0] = \lim_{Z\to Z_0}\frac{d}{dz}(z-z_0)^2 f(z)$ (iii) If f(z) has a pole of order 3 at z0 , then $Res[f,z_0] = \frac{1}{2!}\lim_{Z\to Z_0}\frac{d^2}{dz^2}((z-z_0)^3 f(z))$ (v) If f(z) has a pole of order k at z0 , then $Res[f,z_0] = \frac{1}{(k-1)!}\lim_{Z\to Z_0}\frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z))$

3.9 Cauchy's Residue Theorem:

Cauchy's Residue Theorem:
If the contour C encloses multiple poles, then the theorem gives the general result $\int_c f(z)dz = 2\pi i \sum_{a\in A} \operatorname*{Res}_{z=a_i} f(z)$
Where A is the set of poles contained inside the contour. This amazing theorem therefore says that the value of a contour integral for any contour in the complex plane depends only on the properties of a few very special points inside the contour.

Residue at infinity:
The residue at infinity is given by:
$Res[f(z)]_{Z=\infty} = -\frac{1}{2\pi i}\int_C f(z)dz$ Where f is an analytic function except at finite number of singular points and C is a closed countour so all singular points lie inside it.

3.10 Problems on Residues:

Problem:
Determine the poles of the function $f(z) = \frac{z+2}{(z+1)^2(z-2)}$ and the residue at each pole.

Solution: The poles of f(z) are given by (z+1)2(z-2)=0

Here z=2 is a simple pole and z= -1 is a pole of order 2 .

Residue at z=2 is

$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z+2}{(z+1)^2(z-2)} = \frac{4}{9}$

Residue at z=-1 is $\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z) = \lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{z+2}{(z+1)^2(z-2)}$

$\lim_{z\to -1}\frac{d}{dz}\frac{(z+2)}{(z-2)} = \lim_{z\to -1}\frac{-4}{(z-2)^2} = \frac{-4}{9}$

Problem: Find the residue of the function $f(z) = \frac{1-e^{2z}}{z^4}$ at the poles

Solution:

Let $f(z) = \frac{1-e^{2z}}{z^4}$

z =0 is a pole of order 4

Residue of f(z) at z=0 is

$\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(z-0)^4\frac{(1-e^{2z})}{z^4}$

$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$

$=\frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$

$=\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$

$=\frac{1}{3!}\lim_{z\to 0}(-8e^{2z})$

$=\frac{-8}{3!} = \frac{-4}{3}$

Problem: Find the residue of the function

$f(z) = z^3\cos\left(\frac{1}{z}\right)$ at $z = \infty$ Solution:

Let $f(z) = z^3\cos\left(\frac{1}{z}\right)$

$g(t) = f\left(\frac{1}{t}\right) = \frac{1}{t^3}\cos t$

$= \frac{1}{t^3}\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - ....\right]$

$= \left[\frac{1}{t^3} - \frac{1}{2t} + \frac{t}{24} - ....\right]$

There fore = - coefficient ot t in the eapansion of g(t) about t=0

= -1/24.

3.11 Exercise:

1) Evaluate

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}$$

where

$$C: \quad |z| \ = 1$$

2) Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{a}$$

3) Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{(\mathrm{x}+1)^3} = \frac{3\pi}{8}$$

4) Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

5) Evaluate

$$\int (y^2+z^2)dx + (z^2+x^2)dy + (x^2+y^2)dz$$

from (0,0,0) to (1,1,1) , where C is the curve

$$x = t, x = t^2, x = t^3$$

6)) Obtain the Taylor series expansion of

$$f(z) = \frac{1}{z}$$

about the point z = 1

7) Obtain Laurent's series expansion of

$$f(z) = \frac{z^2-4}{z^2+5z+4}$$

valid in 1¡ z ¡ 2

8)Find the residue of the function $f(z) = \frac{z^2-2z}{(z^2+1)(z+1)^2}$ at each pole

Chapter 4

SPECIAL FUNCTIONS-I

Course Outcomes

After successful completion of this module, students should be able to:

CO 5	**Determine** the characteristics of special functions generalization on elementary factorial function for the proper and improper integrals.	Apply

SPECIAL FUNCTIONS-I:

4.1 Introduction:

Functions play a vital role in Mathematics. It is defined as a special association between the set of input and output values in which each input value correlates one single output value. We know that there are two types of Euler integral functions. One is a beta function, and another one is a gamma function. The domain, range or codomain of functions depends on its type. In this page, we are going to discuss the definition, formulas, properties, and examples of beta functions.
Beta functions are a special type of function, which is also known as Euler integral of the first kind. It is usually expressed as B(x, y) where x and y are real numbers greater than 0. It is also a symmetric function, such as B(x, y) = B(y, x). In Mathematics, there is a term known as special functions. Some functions exist as solutions of integrals or differential equations. Beta Function Definition The beta function is a unique function where it is classified as the first kind of Euler's integral. The beta function is defined in the domains of real numbers. The notation to represent the beta function is

$$\beta$$

. The beta function is meant by B(p, q), where the parameters p and q should be real numbers.

The beta function in Mathematics explains the association between the set of inputs and the outputs. Each input value the beta function is strongly associated with one output value. The beta function plays a major role in many mathematical operations.
Formulation of differential equation from the given physical situation, called modeling.
Solutions of this differential equations evaluating the arbitrary constants from the given conditions and Physical interpretation of the solution.

4.2 Beta functions:

DEFINITION:
IMPROPER INTEGRAL:
The integral for which

$$\int_a^b f(x)\ dx$$

i) Either the interval of integration is not finite i. e

$$a = -\infty \; or \; b = \infty$$

or both
ii) The function f(x) is unbounded at one or more point in [a, b] is called das improper integral.
NOTE: Integral of (i) and (ii) are called the improper integrals of first and second kinds respectively.
Examples:
1. $\int\limits_{0}^{\infty} \frac{1}{1+x^4}\, dx$ and $\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ are improper integrals of the first kind.
2. $\int\limits_{0}^{1} \frac{1}{1-x^2}\, dx$ is an improper integral of the second kind

DEFINITION:

BETA FUNCTION:

The definite integral $\int\limits_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the Beta function and is denoted by . $\beta\left(m,n\right)=\int\limits_0^1 x^{m-1}(1-x)^{n-1}\,dx$ The integral converges for m¿0,n¿0. NOTE:

Beta function is also called as Eulerian integral of first kind

4.3 Properties of Beta functions:

i) SYMMETRY PROPERTY OF BETA FUNCTION

$\beta\left(m,n\right)=\beta\left(n,m\right)$

Proof:

By definition, we have

$\beta\left(m,n\right)=\int\limits_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put 1-x=y so that dx=-dy

$\therefore \beta\left(m,n\right)=\int\limits_1^0 (1-y)^{m-1}y^{n-1}\,\left(-dy\right)$

$=\int\limits_0^1 y^{n-1}(1-y)^{m-1}\,dy$

$=\int\limits_0^1 x^{n-1}(1-x)^{m-1}\,dx$

$\beta\left(m,n\right)$

$\left[\int\limits_a^b f\left(t\right)\,dt=\int\limits_a^b f\left(x\right)\,dx\right]$

Hence $\beta\left(m,n\right)=\beta\left(n,m\right)$

ii) Prove that

$\beta\left(m,n\right)=\int\limits_1^0 \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$

Proof:

By definition, we have

$\beta\left(m,n\right)=\int\limits_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x=\sin^2\theta$ so that $dx=\sin 2\theta\,d\theta$

$\beta\left(m,n\right)=\int\limits_0^{\pi/2}\left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1}\,\sin 2\theta\,d\theta$

$= 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$
Hence proved

iii) Show that

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

Proof:
By definition, we have

$$\beta(m+1,n) + \beta(m,n+1)$$

=

$$\int_0^1 x^m (1-x)^{n-1} \, dx + \int_0^1 x^{m-1} (1-x)^n \, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} [x + (1-x)] \, dx$$

!

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

=

$$\beta(m,n)$$

Hence

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

Iv) If m and n are positive integers, then

$$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Proof:

We have

$$\beta\,(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Integrating by parts

$$\left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 \frac{(1-x)^n}{n(-1)}(m-1)\,x^{m-2}dx$$

=

$$\frac{m-1}{n}{}_0^1 - \int_0^1 x^{m-2}(1-x)^n dx = \frac{m-1}{n}\beta\,(m-1,n+1)$$

. . . .(1) Now we have to find

$$\beta\,(m-1,n+1)$$

To obtain this put m=m-1 and n=n+1 in (1). Then, we have

$$\beta\,(m-1,n+1) = \frac{m-2}{n+1}\beta\,(m-2,n+2)$$

Putting this value of

$$\beta\,(m-1,n+1)$$

in (1) we have

$$\beta\,(m,n) = \frac{m-1}{n}.\frac{m-2}{n+1}\beta\,(m-2,n+2)$$

. . . . (2)

Changing m to m-2 and n to n-2 from (1) we have

$$\beta\,(m-2,n+2)$$

=

$$\frac{m-3}{n+2}.\frac{m-2}{n+1}\beta\,(m-3,n+3)$$

From (2)

$$\beta\,(m,n) = \frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\beta\,(m-3,n+3)$$

Proceeding like this we get

$$\beta\left(m,n\right)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}...........\frac{\left[m-\left(m-1\right)\right]}{\left[n+\left(m-2\right)\right]}\beta\left(m-\left(m-1\right),n+\left(m-1\right)\right)$$

$$=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}$$

....

$$\frac{1}{\left(n+m-2\right)}\beta\left(1,n+m-1\right)$$

...........(3)

$$\beta\left(m,n\right)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}...........\frac{1}{\left(n+m-2\right)}..\frac{1}{\left(n+m-1\right)}$$
$$=\frac{\left(m-1\right)!}{\left(n+m-1\right)\left(n+m-2\right)..........................\left(n+2\right)\left(n+1\right)n}$$

Multiplying the numerator and denominator by

$$\left(n-1\right)!,$$

we have

$$\beta\left(m,n\right)=\frac{\left(m-1\right)!\left(n-1\right)!}{\left(m+n-1\right)!}$$

4.4 Standard forms of Beta functions:

FORM I:

To show

$$\beta\left(m,n\right)=$$

$$\int_{0}^{\infty}\frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx$$

=

$$\int_{0}^{\infty}\frac{x^{n-1}}{\left(1+x\right)^{m+n}}dx$$

Proof:

We have

$$\beta\left(m,n\right)=\int_{0}^{1}x^{m-1}(1-x)^{n-1}\,dx$$

.............(1)
put

$$x=\frac{1}{1+y}$$

so that

$$dx=\frac{dy}{(1+y)^{2}}$$

From (1)
We have

$$\beta\left(m,n\right)=$$

$$\int_{\infty}^{0}\left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}.-\frac{dy}{(1+y)^{2}}$$

$$=\int_{0}^{\infty}\frac{y^{n-1}dy}{(1+y)^{m+1}(1+y)^{n-1}}$$

$$=\int_{0}^{\infty}\frac{y^{n-1}dy}{(1+y)^{m+n}}$$

$$\therefore\beta\left(m,n\right)$$

=

$$\int_{0}^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Hence proved

FORM II:

To show that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Proof: From form we have

$$\beta(m,n) =$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

=

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Now putting

$$x = \frac{1}{y} and\ dx = -\frac{1}{y^2} dy$$

in the second integral, we get

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

=

$$\int_\infty^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \frac{-1}{y^2} dy$$

$$\int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}} \cdot \frac{-1}{y^{m+1}} dy$$

$$\int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Hence

$$\beta(m,n) =$$

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

FORM III:

$$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx$$
$$a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx = a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(\frac{ax}{b}+1\right)^{m+n}}dx$$
$$\frac{ax}{b} = t then \ \frac{a\,dx}{b} = dt$$
$$\frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{\frac{b^{m-1}}{a^{m-1}}t^{m-1}}{(t+1)^{m+n}} \frac{b}{a}\ dt$$
$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$$
$$\beta(m,n)$$

Hence Proved

FORM IV:

To show

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\ dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

Put

$$x = \frac{(1+a)t}{t+a}$$

then

$$dx = (1+a)\left[\frac{(t+a)1-t(1+0)}{(t+a)^2}\right]$$
$$= \frac{a(1+a)}{(t+a)^2}$$
$$dx = \frac{a(1+a)}{(t+a)^2}dt$$

Also when x=0, t=0 and x=1,t=1. Now (1) become

$$\beta(m,n) = \int_0^1 \frac{(1+a)^{m-1}t^{n-1}}{(t+a)^{m-1}}\left(1-\frac{(1+a)t^1}{(t+a)^1}\right)^{n-1}\frac{a(1+a)}{(t+a)^2}dt$$

$$\beta(m,n) = \int_0^1 \frac{(1+a)^{m-1} t^{n-1}}{(t+a)^{m-1}} \left(\frac{a-at}{t+a}\right)^{n-1} a\,dt$$

Also we have

$$\beta(m,n) = \frac{\gamma(m)\,\gamma(n)}{\gamma(m+n)}$$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{\gamma(m)\,\gamma(n)}{\gamma(m+n)}$$

Taking m + n=1 so that m=n-1, we get

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{\gamma(1-n)\,\gamma(n)}{\gamma(1)}$$

Or

$$\therefore \int_0^\infty \frac{t^{\left[(2m+1)/2n\right]-1}\, t^{\frac{1}{2n}-1}}{(1+t)} dt = \frac{\pi}{2n} \cos ecs\pi$$

$$\therefore \int_0^\infty \frac{t^{s-1}}{(1+t)} dt = \frac{\pi}{2n \sin s\pi}$$

$$\therefore \int_0^\infty \frac{x^{s-1}}{(1+x)} dt = \frac{\pi}{2n \sin s\pi}$$

.............(2) From (1) and (2) we have

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\, dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

Hence Proved

4.5 Problems Beta functions:

1. Show that

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Solution:

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin^{m-1}\theta \cos^{n-1}\theta \,(\,\sin\theta \,\cos\theta\,)\, d\theta$$

$$= \int_0^{\pi/2} \left(\sin^2\theta\right)^{\frac{m-1}{2}} \left(\cos^2\theta\right)^{\frac{n-1}{2}} (\,\sin\theta \,\cos\theta\,)\, d\theta$$

Put

$$\sin^2\theta = x$$

so that

$$(\,\sin\theta \,\cos\theta\,)\, d\theta = \frac{dx}{2}$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} \, dx$$

$$= \int_0^1 x^{\left(\frac{m+1}{2}\right)-1} (1-x)^{\left(\frac{n-1}{2}\right)-1} \, dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Hence proved

2. Express the following integrals in terms of Beta function:

i)

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

ii)

$$\int_0^4 \frac{x}{\sqrt{9-x^2}}dx$$

Answer:

$$\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

Solution: Put

$$x^2 = y$$

so that

$$dx = \frac{1}{2}y^{-1/2}dy$$

When x=0, y=0 when x=1, y=1.

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \int_0^1 \frac{y^{1/2}}{\sqrt{1-y}}\frac{1}{2}y^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1}dy$$

$$= \frac{1}{2}\beta\left(1,\frac{1}{2}\right)$$

Exercise Problems:

1. Prove that

$$\int_0^a (a-x)^{m-1}x^{n-1}dx = a^{m+n-1}\beta\,(m,\,n)$$

Hint: put x=ay 2. Show that

$$\int_0^a x^{m-1}(1-x^n)^p dx = \frac{1}{n}\beta\left(\frac{m+1}{n},\ p+1\right)$$

Hint: put

$$x^n = y$$

3. Show that

$$\int_0^a (1+x)^{m-1}(1-x)^{n-1} dx = 2^{m+n-1}\beta\,(m,\ n)$$

Hint: put

$$x = \frac{1+y}{2}$$

4. Show that i)

$$\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}} dx = 0$$

ii)

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = 2\beta\,(m,\ n)$$

5. Prove that

$$\frac{\beta\,(p,\ q+1)}{q} = \frac{\beta\,(p+1,\ q)}{p} = \frac{\beta\,(p,\ q)}{p+q}$$

where p¿0, q¿0. 6. Show that

$$\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1}\beta\,(m+1, n+1)$$

4.6 Gamma functions:

Definition:
The definite integral

$$\int_0^\infty e^{-x}x^{n-1} dx$$

is called the Gamma function and is denoted by

$$\Gamma(n) = \int\limits_0^\infty e^{-x} x^{n-1} dx$$

And read as "gamma n".

NOTE:
1. The integral converges for n¿0.
2. Gamma function is also called Eulerian integral of the second kind.
3. The integral Gamma function does not converges if $n \leq 0$.

4.7 Properties of functions:

I. To show that

$$\Gamma(1) = 1$$

Proof: By definition of Gamma function, we have

$$\Gamma(n) = \int\limits_0^\infty e^{-x} x^{n-1} dx$$
$$\therefore \Gamma(n) = \int\limits_0^\infty e^{-x} x^0 dx = \int\limits_0^\infty e^{-x} dx = (e^{-x})_0^\infty = 1$$

II. To show that $\Gamma(n) = (n-1)\Gamma(n-1)$where n ¿ 1.
Proof: By definition of Gamma function, we have
$\Gamma(n) = \int\limits_0^\infty e^{-x} x^{n-1} dx = \left[x^{n-1} \frac{e^{-x}}{(-1)}\right]_0^\infty - \int\limits_0^\infty (n-1) x^{n-2}. \left(\frac{e^{-x}}{-1}\right) dx$ Integrate by parts
$= (n-1) \int\limits_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1)$

Note:
1. $\Gamma(n+1) = (n)\Gamma(n)$
2. If n is a positive fraction, then we write $\Gamma(n) = (n-1)(n-2)(n-3)(n-4)................\Gamma(n-r)$
Where $(n-r) > 0$ 3. If n is a non-negative integer, then $\Gamma(n+1) = (n)!$

4.8 Relation between Beta and Gamma functions:

1. $\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Where m¿0, n¿0

Proof: : By definition of Gamma function, we have $\Gamma(m) = \int\limits_0^\infty e^{-x}x^{m-1}dx$...........(1) Put x = yt so that dx =y dt then (1) gives

$\Gamma(m) = \int\limits_0^\infty e^{-yt}yt^{m-1}t^{m-1}ydt = \int\limits_0^\infty e^{-yt}y^m t^{m-1}dt = \int\limits_0^\infty e^{-yx}y^m x^{m-1}dx$..........(2) Or

$\frac{\Gamma(m)}{y^m} = \int\limits_0^\infty e^{-yx}x^{m-1}dx$..................(3)

Multiplying both sides of (3)

$\Gamma(m)\int\limits_0^\infty e^{-y}y^{n-1}dy = \int\limits_0^\infty \left\{\int\limits_0^\infty e^{-y(x+1)}y^{m+n-1}x^{m-1}dx\right\}dy$..................(4)

$\Gamma(m)\Gamma(n) = \int\limits_0^\infty \left\{\int\limits_0^\infty e^{-y(x+1)}y^{m+n-1}dy\right\}x^{m-1}dx$

by interchanging the order of integration

$\Gamma(m)\Gamma(n) = \int\limits_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx\,\Gamma(m)\Gamma(n) = dx\int\limits_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx = \Gamma(m+n)\beta(m,\ n) \therefore \beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Hence proved

2. To prove that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ Proof:

By Form I of Beta function

$\beta(m,n) = \int\limits_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$ Also we have

$\therefore \beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}\int\limits_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Taking m + n=1 so that m=1-n, we get $\int\limits_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\gamma(1-n)\gamma(n)}{\gamma(1)}$

$\gamma(1-n)\gamma(n) = \int\limits_0^\infty \frac{x^{n-1}}{(1+x)}dx$ We have

$\int\limits_0^\infty \frac{x^{2m}}{(1+x^{2n})}dx = \frac{\pi}{2n}\cos ec\frac{(2m+1)\pi}{2n}$ where m¿0, n¿0 and n¿m

Put $x^{2m} = t$ and $\frac{(2m+1)}{2n} = s$we have

$\int\limits_0^\infty \frac{t^{\left(2m/2n\right)}t^{1/2n}}{(2n)(1+t)t}dt = \frac{\pi}{2n}\cos ec\, s\pi$ or

$\int\limits_0^\infty \frac{t^{\left(2m+1/2n\right)^{-1}}}{(1+t)}dt = \pi\cos ec\, s\pi$ or

$\int\limits_0^\infty \frac{t^{s-1}}{(1+t)}dt = \frac{\pi}{\sin\, n\pi}\ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ Hence proved

3. To show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: we know that

$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ Taking $m = n = \frac{1}{2}$, we have

$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{\gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\left[\Gamma\left(\frac{1}{2}\right)\right]^2\ \gamma(1)=1$..............(1)

$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int\limits_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx=\int\limits_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$ $x=\sin^2\theta$so that $dx=2\sin\theta\cos\theta d\theta$ Also when

$x=0,\theta=0$when $x=1,\theta={}^1\!/_2\ \therefore\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int\limits_0^{\pi/2}dx=\int\limits_0^1\frac{1}{\sin\theta}\frac{1}{\cos\theta}2\sin\theta\cos\theta\ d\theta=2\int\limits_0^{\pi/2}d\theta=\pi$........(2)

From (1) and (2) we have

$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

4. To show that $\int\limits_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

Proof: we have $\Gamma(n)=\int\limits_0^\infty e^{-x}x^{n-1}dx$

Taking $n={}^1\!/_2$we have

$\Gamma\left(\frac{1}{2}\right)=\int\limits_0^\infty e^{-x}x^{-1/2}dx$ Put $x=t^2$ so that d x=2t d t

Also when x=0, t=0: when $x\to\infty,t\to\infty$

$\therefore\Gamma\left(\frac{1}{2}\right)=\int\limits_0^\infty e^{-t^2}\left(t^2\right)^{-1/2}2t\,dt=2\int\limits_0^\infty e^{-t^2}dt$ or$2\int\limits_0^\infty e^{-x^2}dx=\sqrt{\pi}\ \therefore\int\limits_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

4.9 Problems on Gamma functions:

1. Compute

i) $\Gamma\left(\frac{11}{2}\right)$

ii) $\Gamma\left(-\frac{1}{2}\right)$

iii) $\Gamma\left(-\frac{7}{2}\right)$

Solutions: i) We have

$\Gamma(n+1)=(n)\Gamma(n)$ Taking $n=\frac{7}{2}$

$$\begin{aligned}
&\Gamma\left(\tfrac{11}{2}\right)=\tfrac{9}{2}\Gamma\left(\tfrac{9}{2}\right)\\
&=\tfrac{9}{2}\tfrac{7}{2}\Gamma\left(\tfrac{7}{2}\right)\\
&=\tfrac{9}{2}\tfrac{7}{2}\tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right)\\
&=\tfrac{9}{2}\tfrac{7}{2}\tfrac{5}{2}\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right)\\
&=\tfrac{9}{2}\tfrac{7}{2}\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\\
&=\tfrac{9}{2}\tfrac{7}{2}\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi}
\end{aligned}$$

3. Evaluate i)$\int\limits_0^1 x^5(1-x)^3dx$ ii) $\int\limits_0^1 x^4(1-x)^2dx$ Answer: 1/105

iii)$\int\limits_0^1 x(1-x)^{1/3}dx$ answer$\frac{16\sqrt{\pi}}{9\sqrt{3}}$

iv) $\int\limits_0^1 x^{5/2}\left(1-x^2\right)^{3/2}dx$Answer: $\frac{8}{65}\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$

Solution: i)
$\int_0^1 x^5(1-x)^3dx = \int_0^1 x^{6-1}(1-x)^{4-1}dx\ \beta\,(6,4) = \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}\ \frac{5!3!}{9!} = \frac{1}{504}$

5. Evaluate
i) $\int_0^\infty x^6 e^{-2x}dx$
ii) $\int_0^\infty x^{3/2} e^{-4x}dx$
iii) $\int_0^\infty x^2 e^{-x^2}dx$
iv) $\int_0^\infty \sqrt{x}e^{-x^2}dx$

Solution: Put
$2x = y$ so that $dx = \frac{1}{2}dy$
$\int_0^\infty x^6 e^{-2x}dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\frac{1}{2}dy$
$\frac{1}{2}\int_0^\infty y^6 e^{-y}\frac{1}{2}dy$
$\frac{1}{2}\int_0^\infty y^{7-1}e^{-y}\frac{1}{2}dy = \frac{1}{2^7}6!$

6. Evaluate $\int_0^{\pi/2} \sin^5\theta\cos^{7/2}\theta\ d\theta$ Solution: i)

we have $\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta = \frac{1}{2}\beta\,(m,n)$
Put 2 m-1=5 and 2n-1=1/2 so that m=3, n=9/4
Therefore $\int_0^{\pi/2} \sin^5\theta\cos^{7/2}\theta\ d\theta = \frac{1}{2}\beta\,(3,\ 9/4)$
$\frac{1}{2}\frac{\Gamma(3)}{\Gamma\left(3+\frac{9}{4}\right)} = \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(21/4\right)} = \frac{64}{1989}$

EXERCISE
1. $\int_0^{\pi/2} \sin^7\theta\ d\theta$
2. $\int_0^{\pi/2} \cos^{11}\theta\ d\theta$
3. $\int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta$

Chapter 5

SPECIAL FUNCTIONS-II

Course Outcomes

After successful completion of this module, students should be able to:

CO 6	**Apply** the role of Bessel functions in the process of obtaining the series solutions for second order differential equation	Analyze

SPECIAL FUNCTIONS-II:

5.1 Introduction:

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. A goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.
Bessel's equation :

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0$$

is called Bessel's equation.
Solution of Bessel's Equation: Because x=0 is a regular singular point of Bessel's equation we know that there exists at least one solution of the form $y = \sum\limits_{n=o}^{\infty} c_n x^{n+r}$

5.2 Solution of Bessel Function of the First Kind:

Solution of Bessel Function of the First Kind:
Using the coefficients

$$C_{2n}$$

just obtained and r=v, a series solution is $y = \sum\limits_{n=0}^{\infty} c_{2n} x^{2n+v}$
solution is usually denoted by $J_v(x)$: where
$J_v(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$
The functions

$$J_v(x) \ and \ J_{-v}(x)$$

are called Bessel functions of the first kind of order v and –v, respectively.
Depending on the value of v, solution may contain negative powers of x and hence converge on

$$(0,\infty)$$

The function

$$J_n\left(x\right)$$

is called the Bessel function of the first kind of order n and is denoted by Jn(x).
Thus $J_n(x) = \sum\limits_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1)\cdot r!}$

5.3 Properties of Bessel's function:

1. Show that $J_{-n}(x) = (-1)^n J_n(x)$where n is a positive integer.

Proof: By definition of Bessel's function, we have
$J_n(x) = \sum\limits_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1)\cdot r!}$.............(1)
Hence, $J_{-n}(x) = \sum\limits_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{)(-n+r+1)\cdot r!}$...........................(2)
But gamma function is defined only for a positive real number. Thus we write (2) in the following from
$J_{-n}(x) = \sum\limits_{r=n}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{)(-n+r+1)\cdot r!}$.............(3)
Let r – n = s or r = s + n. Then (3) becomes

$J_{-n}(x) = \sum\limits_{s=0}^{\infty}(-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{-n+2s+2n} \cdot \frac{1}{)(s+1)\cdot(s+n)!}$
We know that

$$\gamma(s+1) = s!\ and\ (s+n)! = \gamma(s+n+1)$$

$= \sum\limits_{s=0}^{\infty}(-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{)(s+n+1)\cdot s!}$

$= (-1)^n \sum\limits_{s=0}^{\infty}(-1)^{s} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{)(s+n+1)\cdot s!}$

Comparing the above summation with (1), we note that the RHS is $J_n\left(x\right)$
Thus, $J_{-n}(x) = (-1)^n J_n(x)$Since, $(-1)^n J_n(x) = J_{-n}(x)$we have $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$

2. Show that $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$where n is a positive integer
Proof : By definition
$J_n(x) = \sum\limits_{r=0}^{\infty}(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1)\cdot r!}$
$J_n(-x) = \sum\limits_{r=0}^{\infty}(-1)^r \cdot \left(-\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1)\cdot r!}$

$= \sum_{r=0}^{\infty} (-1)^r \cdot (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1) \cdot r!}$
$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1) \cdot r!}$
Thus, $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$
Since, $(-1)^n J_n(x) = J_{-n}(x)$we have $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$

5.4 Recurrence relations of Bessel's function:

Recurrence Relations are relations between Bessel's functions of different order.
Recurrence Relations 1:
$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ From definition,

$x^n J_n(x) = x^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1) \cdot r!}$
$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2(n+r)} \cdot \frac{1}{)(n+r+1) \cdot r!}$

$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)x^{2(n+r)-1}}{2^{n+2r})(n+r+1) \cdot r!}$
$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+r)x^{n+2r-1}}{2^{n+2r-1}(n+r))(n+r) \cdot r!}$
$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(x/2)^{(n-1)+2r}}{)(n-1+r+1) \cdot r!} \quad = x^n J_{n-1}(x)$.............(1)
Thus,
$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Recurrence Relations 2:

$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$
From definition,

$x^{-n} J_n(x) = x^{-n} \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1) \cdot r!}$
$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{)(n+r+1) \cdot r!}$
$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2rx^{2r-1}}{2^{n+2r})(n+r+1) \cdot r!}$
$= -x^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{x^{n+1+2(r-1)}}{2^{n+1+2(r-1)})(n+r+1) \cdot (r-1)!}$
Let k = r – 1
$= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{n+1+2k}}{2^{n+1+2k})(n+1+k+1) \cdot k!} \quad = -x^{-n} J_{n+1}(x)$

Thus, $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$..................(2)

Recurrence Relations 3:

$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$ We know that

$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ Applying product rule on LHS, we get $x^n J_n^{/}(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)$

Dividing byx^nwe get

$J_n^{/}(x) + (n/x)J_n(x) = J_{n-1}(x)$................(3)

Also differentiating LHS of $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$we get

$x^{-n}J_n^{/}(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$

Dividing by x^{-n}

we get $-J_n^{/}(x) + (n/x)J_n(x) = J_{n+1}(x)$............(4)

Adding (3) and (4), we obtain

$2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$ That is $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Recurrence Relations 5:

$J_n^{/}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ This recurrence relation is another way of writing the Recurrence relation 2.

Recurrence Relations 6:

${}^{/}_{n}(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

This recurrence relation is another way of writing the Recurrence relation 1.

Recurrence Relations 7:

$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

This recurrence relation is another way of writing the Recurrence relation 3.

5.5 Generating function for Bessel's function:

To Prove that

$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

or

If n is an integer then $J_n(x)$ is the coefficient of t^n in the expansion of $e^{\frac{x}{2}(t-1/t)}$

Proof:

We have

$e^{\frac{x}{2}(t-1/t)} = e^{xt/2} \times e^{-x/2t} = \left[1 + \frac{(xt/2)}{1!} + \frac{(xt/2)^2}{2!} + \frac{(xt/2)^3}{3!} + \cdots\cdots\right] \bullet \left[1 + \frac{(-xt/2)}{1!} + \frac{(-xt/2)^2}{2!} + \frac{(-xt/2)^3}{3!} + \cdots\cdots\right]$

(using the expansion of exponential function)

$= \left[1 + \frac{xt}{2\cdot 1!} + \frac{x^2t^2}{2^2 2!} + \cdots + \frac{x^n t^n}{2^n n!} + \frac{x^{n+1}t^{n+1}}{2^{n+1}(n+1)!} + \cdots\cdots\right] \bullet$

$\left[1 - \frac{x}{2t\cdot 1!} + \frac{x^2}{2^2 t^2 2!} - \cdots + \frac{(-1)^n x^n}{2^n t^n n!} + \frac{(-1)^{n+1}x^{n+1}}{2^{n+1}t^{n+1}(n+1)!} + \cdots\cdots\right]$

If we collect the coefficient of t^nin the product, they are

$= \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!1!} + \frac{x^{n+4}}{2^{n+4}(n+2)!2!} - \cdots\cdots = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4} - \cdots\cdots$

$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!} = J_n(x)$ Similarly, if we collect the coefficients of

$$t^{-n}$$

in the product, we get $= J_{-n}(x)$ Thus,

$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

5.6 Orthogonality of Bessel Functions:

If

$$\alpha \; and \; \beta$$

are the two distinct roots of

$$J_n(x) = 0$$

then

$\int_0^{\pi} x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0, & if\ \alpha \neq \beta \\ \frac{1}{2}\left[J_n^{/}(\alpha)\right]^2 = \frac{1}{2}[J_{n+1}(\alpha)]^2, & if\ \alpha = \beta \end{cases}$

Proof:

We know that the solution of the equation

$$x^2 u^{//} + x u^{/} + (a^2x^2 - -n^2)u = 0$$

$$x^2 v^{//} + x v^{/} + (b^2x^2 - -n^2)v = 0$$

are

$$u = J_n(ax)$$

and

$$v = J_n(bx)$$

respectively.
Multiplying (1) by

$$v/x$$

and (2) by

$$u/x$$

and subtracting, we get

$$\left(u^{/}v - -uv^{/}\right) + (b^2 - -a^2)xuv = 0$$

or
$\frac{d}{dx}\left\{x\left(u^{/}v - uv^{/}\right)\right\} = \left(\beta^2 - \alpha^2\right)xuv$ Now integrating both sides from 0 to 1, we get
$\left(\beta^2 - \alpha^2\right)\int\limits_0^1 xuvdx = \left[x\left(u^{/}v - uv^{/}\right)\right]_0^1 = \left(u^{/}v - uv^{/}\right)_{x=1}$......................(3)
Since $[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)] \cdot \frac{d(\alpha x)}{dx} = \alpha J_n^{/}(\alpha x)$
Similarly

$$v = J_n(bx)$$

gives $v^{/} = \frac{d}{dx}[J_n(\beta x)] = \beta J_n^{/}(\beta x)$
Substituting these values in (3), we get
$\int\limits_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n^{/}(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n^{/}(\beta)}{\beta^2 - \alpha^2}$...............(4)
if

$$\alpha$$

and

$$\beta$$

are the two distinct roots of

$$\mathbf{J_n(x) = 0}$$

then

$$J_n(\alpha) = 0,$$

and

$$J_n(\beta) = 0,$$

and hence (4) reduces to $\int\limits_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = 0$
This is known as Orthogonality relation of Bessel functions.

5.7 Trigonometrical expansions using Bessel functions:

Problem 1

a) $J_n(x) = \frac{1}{\pi}\int\limits_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$n being an integer

b) $J_0(x) = \frac{1}{\pi}\int\limits_0^{\pi} \cos(x\cos\theta)d\theta$

c)$J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Solution :

We know that

$e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum\limits_{n=1}^{\infty} [t^n + (-1)^n t^{-n}] J_n(x)$

$= J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \cdots\cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \cdots\cdots$ since$J_{-n}(x) = (-1)^n J_n(x)$we have

$e^{\frac{x}{2}(t-1/t)} = J_0(x) + J_1(x)(t - 1/t) + J_2(x)\left(t^2 + 1/t^2\right) + J_3(x)\left(t^3 - 1/t^3\right) + \cdots\cdots$...............(1)

Let

$$t = cos\theta + isin\theta$$

so that

$$t^p = \cos p\theta + i\sin p\theta$$

and

$$1/_{t^p} = \cos p\theta - i\sin p\theta$$

From this we get,

$$t^p + 1/t^p = 2cosp\theta \; andt^p - -1/t^p = 2isinp\theta$$

Using these results in (1), we get

$e^{\frac{x}{2}(2i\sin\theta)} = e^{ix\sin\theta} = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots] + 2i[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots]$

Since $e^{ixsin}\theta = cos(xsin\theta) + isin(xsin\theta)$.........(2)

equating real and imaginary parts in (2) we get, $\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots]$.........(3)

$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots]$...........(4)

These series are known as Jacobi Series.

Now multiplying both sides of (3) by $\cos n\theta$ and $\sin n\theta$ both sides of (4) by $sinn\theta$

and integrating each of the resulting expression between 0 and π,

we obtain $\frac{1}{\pi}\int\limits_0^{\pi} \cos(x\sin\theta)\cos n\theta d\theta = \begin{cases} J_n(x), & \text{nisevenorzero} \\ 0, & \text{nisodd} \end{cases}$ and

$\int\limits_0^\pi \sin(x\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & \text{niseven} \\ J_n(x), & \text{nisodd} \end{cases}$ Here we used the standard result

$\int\limits_0^\pi \cos p\theta \cos q\theta d\theta = \int\limits_0^\pi \sin p\theta \sin q\theta d\theta = \begin{cases} \frac{\pi}{2}, & if\ p = q \\ 0, & if\ p \neq q \end{cases}$

From the above two expression,

in general, if n is a positive integer,

we get $J_n(x) = \frac{1}{\pi}\int\limits_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta =$

$\frac{1}{\pi}\int\limits_0^\pi \cos(n\theta - x\sin\theta) d\theta$

5.8 Problems on Bessel functions:

Prove that

$(a)\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad (b)\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ By definition,

$J_n(x) = \sum\limits_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{)(n+r+1)\cdot r!}$

Putting n = ½, we get $J_{1/2}(x) = \sum\limits_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{1/2+2r} \cdot \frac{1}{)(r+3/2)\cdot r!}$

$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)2!} - \cdots\cdots\right]$

Using the results

$$\gamma(1/2) = \sqrt{\pi}\, and\ \gamma(n) = (n--1)\,\gamma(n--1),$$

we get

$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, \Gamma(7/2) = \frac{15\sqrt{\pi}}{8}$ and so on Using these values in (1), we get

$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x}{4}^2 \frac{4}{3\sqrt{\pi}} + \frac{x}{16}^4 \frac{8}{15\sqrt{\pi}.2} - \cdots\cdots\right]$

$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x}\left[x - \frac{x}{6}^3 + \frac{x}{120}^5 - \cdots\cdots\right] = \sqrt{\frac{2}{x\pi}}\left[x - \frac{x}{3!}^3 + \frac{x}{5!}^5 - \cdots\cdots\right]$

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

Putting n = - 1/2, we get

$J_{-1/2}(x) = \sum\limits_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-1/2+2r} \cdot \frac{1}{)(r+1/2)\cdot r!}$

$J_{-1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(1/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(3/2)1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(5/2)2!} - \cdots\cdots\right]$ ......(2)

$$\gamma(1/2) = \sqrt{\pi}\, and\ \gamma(n) = (n--1)\,\gamma(n--1),$$

$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x}{4}^2\frac{2}{\sqrt{\pi}} + \frac{x}{16}^4\frac{4}{3\sqrt{\pi}.2} - \cdots\cdots\right]$
$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x}{4}^2\frac{2}{\sqrt{\pi}} + \frac{x}{16}^4\frac{4}{3\sqrt{\pi}.2} - \cdots\cdots\right]$
$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

2. Prove the following results:
$(a) \quad J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$ $(b) \quad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x + \frac{3}{x}\sin x\right]$

Solution:

We prove this result using the recurrence relation
$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$........................(1)
Putting n = 3/2 in (1), we get $J_{1/2}(x) + J_{5/2}(x) = \frac{3}{x}J_{3/2}(x)$
$\therefore \quad J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$
$i.e., \quad J_{5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\sin x$
$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$
$= \sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)}{x^2}\sin x - \frac{3}{x}\cos x\right]$ Also putting n = - 3/2 in (1), we get
$J_{-5/2}(x) + J_{-1/2}(x) = -\frac{3}{x}J_{-3/2}(x)$
$\therefore \quad J_{-5/2}(x) = -\frac{3}{x}J_{-3/2}(x) - J_{-1/2}(x)$

$$= \left(\frac{-3}{x}\right)\left(-\sqrt{\frac{2}{\pi x}}\right)\left[\frac{x\sin x + \cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$i.e., \quad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x + 3\cos x - x^2\cos x}{x^2}\right]$
$= \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$

5.9 Exercise Problems on Bessel functions:

1. Show that $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = \frac{2}{x}\left[nJ_n^2(x) - (n+1)J_{n+1}^2(x)\right]$
2. Prove that $J_0^{//}(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right]$
3. Show that a) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$
b)$\int xJ_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

BIBLIOGRAPHY

1.Erwin Kreyszig, “Advanced Engineering Mathematics”, John Wiley and Sons Publishers, 10th Edition,2010

2.B. S. Grewal, “Higher Engineering Mathematics”, Khanna Publishers, 43rd Edition,2015.

3.T.K.V Iyengar, B.Krishna Gandhi, “Engineering Mathematics - III”, S. Chand and Co., 12th Edition, 2015.

4. Churchill, R.V. and Brown, J.W, “Complex Variables and Applications”, Tata Mc Graw-Hill, 8th Edition, 2012.

5. Axler, Sheldon , ”Harmonic Functions from a Complex Analysis Viewpoint,”(1986)

6. L. V. Ahlfors, Complex Analysis. An Introduction to the Theory of Analytic Functions of One Complex Variable, 3rd edition, McGraw–Hill, New York, 1979; firstedition, 1953.

7. M. J. Ablowitz and A. S. Fokas, Complex Variables: Introduction and Applications, 2nd edition, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2003; first edition, 1997.

8. Ablowitz, M. J., Fokas, A. S. . Complex variables: introduction and applications (2nd ed). Cambridge University Press.(2003)

9. Needham, T. . Visual Complex Analysis. Oxford University Press, Oxford(1997)

10. Brown, J. W., Churchill, R. V. Complex Variables and Applications. 8th Edition. New York: McGraw-Hill Higher Education.(2009)

11. Davis, Philip J. (1972), ”6. Gamma function and related functions”, in Abramowitz, Milton; Stegun, Irene A.

12. Askey, R. A.; Roy, R. (2010), ”Beta function”, in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W.)

13. Paris, R. B. (2010), ”Incomplete beta functions”, in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W.

14. Ramanujan’s, R. Askey, extension of the gamma and beta functions, Amer. Math. Monthly, 87(1980)

15.Beals, Richard; Wong, Roderick. Special Functions: A Graduate Text. Cambridge University Press. (2010)