

MODULE-II

RANDOM VARIABLES.

PART-A

① Given ; $f(x) = \begin{cases} 3x^2 & ; 0 < x < 1 \\ 0 & ; \text{otherwise.} \end{cases}$

To find ; $a = ?$ If $P(a \leq x \leq 1) = \frac{19}{81}$

Given ; $P(a \leq x \leq 1) = \frac{19}{81}$

$$\int_a^1 f(x) dx = \frac{19}{81}$$

$$\int_a^1 3x^2 dx = \frac{19}{81}$$

$$3 \left[\frac{x^3}{3} \right]_a^1 = \frac{19}{81}$$

$$3 \left[\frac{1-a^3}{3} \right] = \frac{19}{81} \Rightarrow 1 - a^3 = \frac{19}{81}$$

$$1 - \frac{19}{81} = a^3$$

$$\frac{62}{81} = a^3 \Rightarrow$$

$$\boxed{a = 0.914}$$

✓

② Given:

$$f(x) = \begin{cases} \frac{1}{q} x e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Total Production = 12 million KW-hour

To find: $P(x > 12) = ?$ - Power cut/ shortage.

Now; we need to find

$$P(0 \leq x \leq 12) = \int_0^{12} f(x) dx = \int_0^{12} \frac{1}{q} x e^{-x/3} dx$$

$$\begin{aligned} \int_0^{12} \frac{1}{q} x e^{-x/3} dx &= \frac{1}{q} \left\{ x \cdot \frac{e^{-x/3}}{-1/3} - 1 \cdot \frac{e^{-x/3}}{1/3} \right\} \Big|_0^{12} \\ &= \frac{1}{q} \left\{ -36 e^{-4} - q e^{-4} - (0 - q) \right\} \end{aligned}$$

$$\Rightarrow \frac{1}{q} \{ q - 45 e^{-4} \} = 1 - 5 e^{-4}$$

$$\text{Now;} P(x > 12) = 1 - P(0 \leq x \leq 12)$$

$$= 1 - (1 - 5 e^{-4})$$

$$= 5 e^{-4}$$

$$= 0.0915781$$

D.I.P.C. =

③ If head occurs first time there will be only one toss. On the other hand, if first one is tail, second occurs. If head occurs there will be only two tosses. Suppose second one is also tail third occurs. If head occurs there will be three tosses and so on. Therefore,

$$P(1) = P(H) = \frac{1}{2}$$

$$P(2) = P(TH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(3) = P(TTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(4) = P(TTTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(5) = P(TTTTH) + P(TTTTT) = \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$$

The probability distribution function of x is,

x	1	2	3	4	5
$P(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

$$\text{Hence; } E(x) = \sum P_i x_i$$

$$= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{16}\right)$$

$$\Rightarrow \frac{8+8+6+4+5}{16} = \frac{31}{16} = 1.9375$$

(4) Let x denote the twice the number appearing on the face when a die is thrown. Then x is a discrete random variable whose probability distribution is given by,

(i)

$X = x_i$	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Above table represents the probability distribution of x .

(ii) Mean (μ) = $E(x) = \sum_i P_i x_i$

$$\mu = 2\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) + 8\left(\frac{1}{6}\right) + 10\left(\frac{1}{6}\right) + 12\left(\frac{1}{6}\right)$$

$$\mu = \frac{1}{6}(2+4+6+8+10+12) = \frac{42}{6} = 7$$

(iii) Variance (σ^2) = ? $\sigma^2 = E(x^2) - (E(x))^2$

$$E(x^2) = \sum_i x_i^2 P_i$$

$$E(x^2) = 4\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) + 64\left(\frac{1}{6}\right) + 100\left(\frac{1}{6}\right) + 144\left(\frac{1}{6}\right)$$

$$E(x^2) = \frac{1}{6}(4+16+36+64+100+144) = \frac{364}{6} = 60.67$$

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$\sigma^2 = 60.67 - (7)^2 = 60.67 - 49$$

$$\sigma^2 = 11.67$$

$$⑤ \text{ Given: } f(x) = k e^{-|x|}, -\infty < x < \infty \quad (x \in \mathbb{R})$$

We know that: $\int_{-\infty}^{\infty} f(x) dx = 1.$

$$\int_{-\infty}^{\infty} k e^{-|x|} dx = 1$$

$$k \int_{-\infty}^{\infty} e^{-|x|} dx = 1.$$

Since $e^{-|x|}$ is an even function, so it can be written as $k \cdot 2 \int_0^{\infty} e^{-x} dx = 1.$

In $0 \leq x \leq \infty, |x| = x.$

$$2k \left(-e^{-x} \right)_0^{\infty} = 1$$

$$2k(-0 - (-1)) = 1$$

$$2k = 1 \quad (\Rightarrow) \quad k = \frac{1}{2}$$

$$\text{Hence: } f(x) = k e^{-|x|} = \frac{1}{2} e^{-|x|}$$

$$\text{Mean (M)} = E(x) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|} dx \\ = 0$$

Since, integrand is odd

$$\begin{aligned}
 \text{Variance } (\sigma^2) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\
 &= \int_{-\infty}^{\infty} (x - 0)^2 f(x) dx \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx \\
 &= \frac{1}{2} \cdot 2 \int_0^{\infty} x^2 e^{-x} dx, \text{ since integrand is even.} \\
 &= \int_0^{\infty} x^2 e^{-x} dx \\
 &= \left(x^2 \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{-1} + 2 \frac{e^{-x}}{-1} \right)_0^{\infty}
 \end{aligned}$$

$$\Rightarrow (0 - (-2)) = 2 \quad \checkmark$$

$$\boxed{P(0 < x < 4)} = \frac{1}{2} \int_0^4 e^{-|x|} dx = \frac{1}{2} \int_0^4 e^{-x} dx$$

$\because 0 < x < 4 \rightarrow |x| = x$

$$= \frac{1}{2} (e^{-x})^4 \Big|_0^4$$

$$= \frac{1}{2} (e^{-4} - 1)$$

$$= \frac{1}{2} (1 - e^{-4})$$

$$= 0.4908 \quad \checkmark$$

⑥ Given; $f(x) = Ax^r$; $0 < r < 1$

To finds Value of 'A' = ?

We know that; $\int_0^1 f(x) dx = 1$

$$\int_0^1 Ax^r dx = 1$$

$\therefore A$ is constant.

$$A \int_0^1 x^r dx = 1$$

$$A \left[\frac{x^{r+1}}{r+1} \right]_0^1 = 1$$

$$A \left(\frac{1}{r+1} - 0 \right) = 1$$

$$\frac{A}{r+1} = 1$$

$$\boxed{A=3}$$

\therefore The value of 'A' = 3.

⑦ Given: $f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

$$\begin{aligned}
 \text{(i) } E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \underbrace{\int_{-\infty}^0 x(0) dx}_{\cancel{= 0}} + \int_0^{\infty} x e^{-x} dx \\
 &= \left[-e^{-(x+1)} \right]_0^{\infty} \\
 &= (-1) \lim_{x \rightarrow \infty} \frac{x+1}{e^x} + (0+1) \\
 &= 0+1 \\
 &= 1 \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 f(x) dx \\
 &= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x}) \right]_0^{\infty} \\
 &= \left[-e^{-x} (x^2 + 2x + 2) \right]_0^{\infty} \\
 &= (-1) \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 2}{e^{-x}} + 2 \\
 &\Rightarrow 0+2 = 2 \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) } \text{Var}(X) &= E(X^2) - (E(X))^2 = 2 - (1)^2 \\
 &= 2-1 \\
 &= 1 \checkmark
 \end{aligned}$$

⑧ Given; $E(X) = 10$; $V(X) = 1$

To find, $E(2X(X+10)) = ?$

We know that; $V(X) = 1$

$$E(X^2) - (E(X))^2 = 1$$

$$E(X^2) = 1 + (10)^2 = 101$$

$$\text{Now}; E(2X(X+10)) = E(2X^2 + 20X)$$

$$= E(2X^2) + E(20X)$$

$$= 2E(X^2) + 20E(X)$$

$$= 2(101) + 20(10)$$

$$= 202 + 200$$

$$= 402$$

$$\therefore E(2X(X+10)) = 402$$

⑨ Given, probability distribution

x	1	2	3	4	5	6	7	8
$P(x=x_i)$	$2K$	$4K$	$6K$	$8K$	$10K$	$12K$	$14K$	$4K$

(i) From the properties of probability;

$$\sum_i P(x_i) = 1$$

$$\Rightarrow 2\underline{K} + 4\underline{K} + 6\underline{K} + 8\underline{K} + 10\underline{K} + 12\underline{K} + 14\underline{K} + 4\underline{K} = 1$$

$$\Rightarrow 60K = 1 \Rightarrow K = \frac{1}{60}$$

The updated probability distribution is,

x	1	2	3	4	5	6	7	8
$P(x=x_i)$	$\frac{2}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{8}{60}$	$\frac{10}{60}$	$\frac{12}{60}$	$\frac{14}{60}$	$\frac{4}{60}$

$$(ii) P(x < 3) = P(x=1) + P(x=2)$$

$$\Rightarrow \frac{2}{60} + \frac{4}{60} = \frac{6}{60} = \frac{1}{10} = 0.1$$

$$(iii) P(x > 5) = P(x=6) + P(x=7) + P(x=8)$$

$$\Rightarrow \frac{12}{60} + \frac{14}{60} + \frac{4}{60} = \frac{30}{60} = \frac{1}{2} = 0.5$$

⑩ Given; $f(x) = \begin{cases} cx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

(i) Since, the total probability is unity,

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^2 cx(2-x) dx = 1$$

$$\Rightarrow C \int_0^2 (2x - x^2) dx = 1$$

$$\Rightarrow C \left[2x - \frac{x^3}{3} \right]_0^2 = 1$$

$$\Rightarrow C \left(4 - \frac{8}{3} \right) = 1 \Rightarrow 4C/3 = 1$$

$$C = \frac{3}{4}$$

$$\therefore f(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$(ii) \text{ Mean of } X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = \int_0^2 x \cdot \frac{3x}{4}(2-x) dx = \frac{3}{4} \int_0^2 (2x^2 - x^3) dx$$

$$\mu = \frac{3}{4} \left[2 \cdot \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{3}{4} \left[\frac{2 \cdot 2^3}{3} - \frac{2^4}{4} \right]$$

$$\mu = \frac{3}{4} \cdot 2^4 \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$\mu = 1 \quad \checkmark$$

$$(iii) E(X^2) = \int_0^2 x^2 \cdot \frac{3x}{4}(2-x) dx = \frac{3}{4} \int_0^2 (2x^3 - x^4) dx$$

$$E(X^2) = \frac{3}{4} \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^2$$

$$E(X^2) = \frac{3}{4} \left[2 \cdot \frac{2^4}{4} - \frac{2^5}{5} \right]$$

$$E(X^2) = \frac{3}{4} \cdot \frac{2^8}{8} \left[\frac{1}{4} - \frac{1}{5} \right]$$

$$E(X^2) = \frac{6}{5} \quad \checkmark$$

$$\text{Variance} = E(X^2) - (E(X))^2 = \frac{6}{5} - (1)^2 = \frac{6}{5} - 1$$

$$= \frac{1}{5} \quad \checkmark$$

(14) PART - B

① Let x denotes the maximum of the 2 numbers that appear when a pair of fair dice is thrown at once.

The sample space is given by,

$$S = \left\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \right\}$$

If the random variable x assigns the maximum of its number in S , then the sample space

$$S = \left\{ \begin{array}{ccccccc} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 5 & 5 & 5 & 5 & 5 \\ 6 & 5 & 4 & 4 & 4 & 4 \\ 6 & 5 & 4 & 3 & 3 & 3 \\ 6 & 5 & 4 & 3 & 2 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{array} \right\}$$

The maximum number could be

1, 2, 3, 4, 5, 6

for maximum 1, the favourable cases are
 $P(X=1) = 1/36$ ($\because (6,6)$ is possible).

For maximum 2, the favourable cases are

$$P(X=2) = 3/36 \quad (\because (1,1), (2,1), (1,2))$$

similarly

$$P(X=3) = 5/36 \quad (\because (1,2), (2,2), (3,1), (2,3), (1,3))$$

$$P(X=4) = 7/36$$

$$P(X=5) = 9/36$$

$$P(X=6) = 11/36$$

\therefore The probability distribution is

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$(iii) E(X) = \sum P_p x_i$$

$$= 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36}$$

$$= \frac{1+6+15+28+45+66}{36} = \frac{161}{36} \approx 4.4722$$

$$\therefore M = 4.4722$$

$$(iii) \text{ Variance} = (E(X^2)) - (E(X))^2$$

$$= 1 \cdot \frac{1}{36} + 4 \cdot \frac{3}{36} + 9 \cdot \frac{5}{36} + 16 \cdot \frac{7}{36} + 25 \cdot \frac{9}{36} + 36 \cdot \frac{11}{36} - (4.47)^2$$

$$= \frac{1+12+45+112+225+396}{36} - (4.47)^2$$

$$= 21.97 - 19.98$$

$$= 1.99$$

$\therefore \sigma^2 = 1.99$

	1	2	3	4	5	6	7	X
\bar{x}	1	2	3	4	5	6	7	1.99
$\bar{x} - x$	0	-1	-2	-3	-4	-5	-6	-0.01

② X denotes the number of heads in a single toss of 4 fair coins.

The required probability distribution is

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$\text{i)} P(X < 2) = P(X=0) + P(X=1)$$

$$\Rightarrow \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$\text{ii)} P(1 < X \leq 3) = P(X=2) + P(X=3)$$

$$\Rightarrow \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$$

⑤ The given probability distribution

X	-1	0	1	2	3
$P(X=x)$	0.3	0.1	0.1	0.3	0.2

(i) Expectation = Mean = $\sum_i f_i x_i$

$$\mu = -1(0.3) + 0(0.1) + 1(0.1) + 2(0.3) + 3(0.2)$$

$$\mu = -0.3 + 0 + 0.6 + 0.6$$

$$\boxed{\mu = 1}$$

(ii) $E(X^2) = (-1)^2(0.3) + 0^2(0.1) + 1^2(0.1) + 2^2(0.3) + 3^2(0.2)$

$$+ 3^2(0.2)$$

$$E(X^2) = 0.3 + 0.1 + \underbrace{4.2}_{\frac{12}{3}} + 1.8$$

$$E(X^2) = 3.4$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = 3.4 - (1)^2 = 2.4$$

(iii) $V(X) = 2.4 = \sigma^2$

$$\text{Standard deviation} = \sqrt{\sigma^2}$$

$$= \sqrt{2.4}$$

$$\sigma = 1.54$$

④ Given: $f(x) = \frac{1}{n}$ for $x = 1, 2, 3, \dots, n$

The probability distribution is

x	1	2	3	...	n
$P(X=x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$...	$\frac{1}{n}$

(i) Mean: $\sum p_i x_i$

$$\mu = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \dots + n \cdot \frac{1}{n}$$

$$\mu = (1+2+\dots+n) \cdot \frac{1}{n} = \frac{n(n+1)}{2} \cdot \frac{1}{n}$$

$$\boxed{\mu = E(X) = \frac{n+1}{2}} \quad \checkmark$$

(ii) $E(X^2) = 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{1}{n} + 3^2 \cdot \frac{1}{n} + \dots + n^2 \cdot \frac{1}{n}$

$$E(X^2) = (1^2 + 2^2 + \dots + n^2) \cdot \frac{1}{n}$$

$$E(X^2) = \frac{n(n+1)(2n+1)}{6} \cdot \frac{1}{n}$$

$$E(X^2) = \frac{(n+1)(2n+1)}{6}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$V(X) = \frac{n+1}{12} (4n+2 - 3n-3)$$

$$V(X) = \frac{(n+1)(n-1)}{12}$$

$$\boxed{\text{Variance} = V(X) = \frac{n^2-1}{12}}$$

⑤ Given, probability distribution function

x	8	12	16	20	24
P(x)	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

$$(i) E(X) = ? = \sum p_i x_i$$

$$E(X) = 8\left(\frac{1}{8}\right) + 12\left(\frac{1}{6}\right) + 16\left(\frac{3}{8}\right) + 20\left(\frac{1}{4}\right) + 24\left(\frac{1}{12}\right)$$

$$E(X) = 1 + 2 + 6 + 5 + 2$$

$$E(X) = 16$$

$$\text{Mean} = \mu = 16$$

$$(ii). E(X^2) = ? = \sum_{i=1}^4 x_i^2 p(x)$$

$$E(X^2) = 8^2 \left(\frac{1}{8}\right) + (12)^2 \cdot \frac{1}{8} + (16)^2 \cdot \frac{3}{8} + (20)^2 \cdot \frac{1}{4}$$
$$= 24^2 \left(\frac{1}{12}\right)$$

$$E(X^2) = 8 + 24 + 96 + 100 + 48$$

$$E(X^2) = 276$$

$$\text{Variance } (V(X)) = E(X^2) - (E(X))^2$$

$$V(X) = 276 - (16)^2$$

$$V(X) = 276 - 256 = 20 \checkmark$$

$$(iii) N(X) = r = 20$$

$$\text{Standard deviation} = \sqrt{r}$$

$$= \sqrt{20}$$

$$= 4.47 \text{ (or) } 2\sqrt{5}$$

$$\textcircled{6} \text{ Given if } f(x) = \begin{cases} Ae^{-x/5}; & x \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

(i) In order that $f(x)$ should be a probability density function, $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} A e^{-x/5} dx = 1$$

$$\Rightarrow A \left[\frac{e^{-x/5}}{-1/5} \right]_0^{\infty} = 1$$

$$\Rightarrow -5A [e^{-\infty} - e^0] = 1$$

$$\Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$$

$$(ii) P(X \geq 20) = \int_{20}^{\infty} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_{20}^{\infty}$$

$$= \frac{1}{5} \left[e^{-\infty} - e^{-4} \right] = \frac{1}{5} e^{-4}$$

$$\Rightarrow -1 \left[-\frac{1}{e^4} \right] = \frac{1}{e^4} = 0.0183$$

④ Let X denote the sum of the two nos. that appear when a pair of fair dice is tossed.

(i) The sum X of the two nos. which turn up must be an integer between 2 and 12.

$$X=2 \rightarrow (1,1) \rightarrow P(X=2) = 1/36$$

$$X=3 \rightarrow (2,1) (1,2) \rightarrow P(X=3) = 2/36$$

Similarly till $X=12$.

Now the Probability distribution in this case is given by the following table.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$(i) \text{ Mean} = \mu = E(x) = \sum_i P(x=x_i) \cdot x_i$$

$$\mu = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \dots + 12\left(\frac{1}{36}\right)$$

$$\mu = \frac{1}{36} [2+6+12+20+30+42+40+36+30+22+$$

(2)

$$\mu = \frac{252}{36} = 7 \checkmark$$

$$(ii) \text{ Variance} = V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = 2^2\left(\frac{1}{36}\right) + 3^2\left(\frac{2}{36}\right) + \dots + 12^2\left(\frac{1}{36}\right)$$

$$E(x^2) = \frac{1}{36} [4+18+48+100+180+294+320+324$$

$$+ 300+242+144] = \frac{1974}{36} = 54.83$$

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = 54.83 - 49$$

$$V(x) = 5.83 \checkmark$$

$$\textcircled{8} \text{ Given; } f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) Clearly; $f(x) \geq 0, \forall x \text{ in } (1, 2)$, and

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 \cdot dx + \int_0^{\infty} e^{-x} dx = 0 + [e^{-x}]_0^{\infty} = -(0 - 1) = 1.$$

Hence the function $f(x)$ is a density function. ✓

$$\text{(ii) Required probability} = P(1 \leq x \leq 2) = \int_1^2 f(x) dx$$

$$\Rightarrow \int_1^2 e^{-x} dx = -[e^{-x}]_1^2 = -(e^{-2} - e^{-1}) = e^{-1} - e^{-2}$$

$$= 0.368 - 0.135 = 0.233$$

(iii) Cumulative Probability function

$$F(x) = \int_{-\infty}^x f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^x e^{-x} dx$$

$$\Rightarrow 0 + [e^{-x}]_0^x (-1) = -(e^{-x} - 1) = 1 - e^{-x}$$

$$F(2) = 1 - 0.135$$

$$F(2) = 0.865$$

④ Given; $f(x) = \begin{cases} Kx^3 & ; 0 \leq x \leq 3 \\ 0 & ; \text{otherwise} \end{cases}$

we have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_{-\infty}^0 0 \cdot dx + \int_0^3 Kx^3 dx = 1$$

$$\Rightarrow 0 + K \left[\frac{x^4}{4} \right]_0^3 = 1 \Rightarrow \left(\frac{81}{4} - 0 \right) K = 1$$

$$K = \frac{4}{81}$$

\therefore The value of K is $4/81$

$$P\left(\frac{1}{2} < x < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x) dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{4}{81} \cdot x^3 dx$$

$$= \frac{4}{81} \left[\frac{x^4}{4} \right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\Rightarrow \frac{4}{81} \left[\frac{3^4}{2^4} - \frac{1^4}{2^4} \right] = \frac{4}{81} \times \frac{80}{16}$$

$$= \frac{5}{81} = 0.0617$$

⑩ Given, the probability distribution table,

x	0	1	2	3	4	5	6
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$

(i) We know that $\sum_i P(X=x_i) = 1$

$$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 = 7K^2 + 18 = 1$$

$$\Rightarrow 10K^2 + 18 - 1 = 0$$

$$\cancel{K = \frac{-4 + \sqrt{19}}{2} ; \quad \cancel{K = \frac{-4 - \sqrt{19}}{2}}}$$

$$K = \cancel{0.78} ; \quad \cancel{K = -2.78}$$

$$\Rightarrow 10K^2 + 10K - 1 = 0$$

$$\Rightarrow (K+1)(10K-1) = 0$$

$$\Rightarrow K = -1 \quad (\text{or}) \quad \frac{1}{10} \quad \checkmark$$

$$K = \gamma_{10}$$

$$(i) P(X < 6)$$

$$P(X < 6) = P(5) + P(4) + P(3) + P(2) + P(1) + P(0)$$

$$P(X < 6) = K^2 + 3K + 2K + \frac{1}{2}K + K + 0$$

$$P(X < 6) = K^2 + 8K$$

$$P(X < 6) = \frac{1}{100} + \frac{80}{100}$$

$$P(X < 6) = \frac{81}{100}$$

$$(ii) P(X \geq 6)$$

$$P(X \geq 6) = P(6) + P(7)$$

$$P(X \geq 6) = 2K^2 + 7K + K$$

$$P(X \geq 6) = 9K + K$$

$$P(X \geq 6) = \frac{9}{100} + \frac{10}{100}$$

$$P(X \geq 6) = \frac{19}{100}$$

⑪ Let x denote the minimum of the two numbers that appear when a pair of fair dice is thrown once.

(i) When 2 dice are thrown, the sample space is given by.

$$S = \left\{ \begin{array}{ccccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

If random variable x assigns the minimum of its number ins, then the sample space

$$S = \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \right\}$$

The minimum number could be

$$1, 2, 3, 4, 5, 6$$

For '1', the fav. cases are

(1, 1) —— (1, 6); (2, 1) (3, 1) (4, 1) (5, 1) (6, 1)

$$P(X=1) = \frac{11}{36}$$

Similarly for

$$P(X=2) = \frac{9}{36}; P(X=3) = \frac{7}{36}; P(X=4) = \frac{5}{36}$$

$$P(X=5) = \frac{3}{36}; P(X=6) = \frac{1}{36}$$

∴ The probability distribution is

x	1	2	3	4	5	6
$P(X=x_i)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) Expectation = Mean = $\sum_i p_i x_i$

$$E(X) = 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{1}{36}\right)$$

$$E(X) = \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6) = \frac{91}{36}$$

$$\mu = E(X) = 2.5278$$

$$(iii) \text{ Variance} = E(X^2) - (E(X))^2$$

$$E(X^2) = 1\left(\frac{11}{36}\right) + 4\left(\frac{9}{36}\right) + 9\left(\frac{7}{36}\right) + 16\left(\frac{5}{36}\right) + 25\left(\frac{3}{36}\right) + 36\left(\frac{1}{36}\right)$$
$$= \frac{1}{36}(11 + 36 + 63 + 80 + 75 + 36)$$
$$= \frac{301}{36} = 8.3611$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = 8.3611 - (2.5278)^2$$

$$V(X) = 8.3611 - 6.3898$$

$$V(X) = 1.9713$$

≈

(12) Given; Probability distribution table

x	-3	-2	-1	0	1	2	3
$P(x)$	K	0.1	K	0.2	$2K$	0.4	$2K$

(i) We know that; $\sum P(x_i) = 1$

$$\Rightarrow K + 0.1 + K + 0.2 + 2K + 0.4 + 2K = 1$$

$$\Rightarrow 6K = 0.3 \Rightarrow K = 0.05$$

(ii) Mean = $\sum x_i P(x_i) = E(x)$

$$\mu = -3(K) + (-2)0.1 + (-1)K + 0(0.2) + 1(2K) + 2(0.4) + 3(2K)$$

$$\mu = -3K - 0.2 - K + 0 + 2K + 0.8 + 6K$$

$$\mu = 4K + 0.6 = 4(0.05) + 0.6 = 0.8$$

(iii) $E(x^2) = (-3)^2 K + (-2)^2 0.1 + (-1)^2 K + 0(0.2) + 1(2K) + 2^2 (0.4) + 3^2 (2K)$

$$E(x^2) = 9K + 0.4 + K + 0 + 2K + 1.6 + 18K$$

$$E(x^2) = \underbrace{30K + 2}$$

$$E(x^2) = 10.5 + 2 = 3.5$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = 3.5 - (0.8)^2$$

$$V(X) = 3.5 - 0.64 = 2.86 \text{ (or), } \frac{143}{50}$$

⑬ Given: $f(x) = \begin{cases} Kx e^{-\lambda x}; & x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

(i) since the total probability is unity, we

have, $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \underbrace{\int_{-\infty}^0 f(x) dx}_{0} + \int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow 0 + \int_0^{\infty} Kx e^{-\lambda x} dx = 1$$

$$\Rightarrow K \int_0^{\infty} x e^{-\lambda x} dx = 1$$

$$\Rightarrow K \left\{ x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 1 \left(\frac{e^{-\lambda x}}{\lambda} \right) \right\} \Big|_0^\infty = 1$$

$$K \left(0 - 0 \right) - \left(0 - \frac{1}{\lambda^2} \right) = 1$$

$$\boxed{K = \lambda^2}$$

$$(ii), \mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = \int_0^{\infty} x \cdot \cancel{\lambda^n} x e^{-\lambda x} dx = \lambda^n \int_0^{\infty} x^n e^{-\lambda x} dx$$

$$\mu = \lambda^n \left\{ x^n \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right\}$$

$$\mu = \lambda^n \left[(0 - 0 + 0) - (0 - 0 - \frac{2}{\lambda^3}) \right]$$

$$\mu = \frac{2}{\lambda} \checkmark = E(X)$$

$$(iii) E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \lambda^n x e^{-\lambda x} dx$$

$$E(x^2) = \lambda^n \int_0^{\infty} x^3 e^{-\lambda x} dx$$

$$E(x^2) = \lambda^n \left\{ x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right\}_0^{\infty} = \lambda^n \left[(0 - 0 + 0 - 0) - (0 - 0 + 0 - \frac{6}{\lambda^4}) \right]$$

$$E(x^2) = 6/\lambda^2$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$V(x) = \frac{2}{\lambda^2} \checkmark$$

(14) Given: $f(x) = \begin{cases} K(1-x^3) & ; \text{ for } 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(i) we know that ; $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_0^{\infty} K(1-x^3) dx = 1.$$

$$\Rightarrow \int_0^1 K(1-x^3) dx + \int_1^{\infty} \underbrace{K(1-x^3) dx}_{=0} = 1$$

$$\Rightarrow K \left[x - \frac{x^3}{3} \right]_0^1 = 1.$$

$$\Rightarrow K \left(1 - \frac{1}{3} \right) = 1 \Rightarrow \boxed{K = \frac{3}{2}} \quad \checkmark$$

(ii) $P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x) dx$

$$= \int_{0.1}^{0.2} K(1-x^3) dx$$

$$= \frac{3}{2} \int_{0.1}^{0.2} (1-x^3) dx$$

$$= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_{0.1}^{0.2}$$

$$= \frac{3}{2} \left[0.21 - \frac{0.007}{3} \right]$$

$$= 0.2965 \quad \checkmark$$

$$(i) P(X > 0.5) = \int_{0.5}^{\infty} f(x) dx$$

$$= \int_{0.5}^1 f(x) dx + \int_1^{\infty} f(x) dx$$

$$\Rightarrow \frac{3}{2} \int_{0.5}^1 (1-x^3) dx = \frac{3}{2} \left(x - \frac{x^3}{3} \right) \Big|_{0.5}^1$$

$$= \frac{3}{2} \left[\left(1 - \frac{1}{3} \right) - \left(0.5 - \frac{0.125}{3} \right) \right]$$

$$= \frac{3}{2} \left[\frac{2}{3} - 0.4583 \right]$$

$$= 0.3125 \quad \checkmark$$

(15) Given; Probability function table

X	4	5	6	8
$P(X)$	0.1	0.3	0.4	0.2

$$(i) E(X) = \sum_i x_i P(x_i) = 4(0.1) + 5(0.3) + 6(0.4) + 8(0.2)$$

$$E(X) = 0.4 + 1.5 + 2.4 + 1.6$$

$$E(X) = 5.9 \checkmark$$

$$(ii) E(X^2) = \sum_i x_i^2 P(x_i) = 16(0.1) + 25(0.3) + 36(0.4) + 64(0.2)$$

$$E(X^2) = 1.6 + 7.5 + 14.4 + 12.8$$

$$E(X^2) = 36.3$$

$$V(X) = E(X^2) - (E(X))^2 = 36.3 - (5.9)^2$$

$$V(X) = 36.3 - 34.81$$

$$V(X) = \sigma^2 = 1.49$$

$$(iii) \sigma = \sqrt{1.49} \rightarrow \text{standard deviation.}$$

$$\sigma = 1.22$$

⑯ Given; $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^1 x \cdot x dx + \int_1^2 x(2-x) dx + \int_2^{\infty} 0 \cdot dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx + 0$$

$$= \left[\frac{x^3}{3} \right]_0^1 + \left[2x - \frac{x^3}{3} \right]_1^2$$

$$\Rightarrow \frac{1}{3} + \left(4 - \frac{8}{3} \right) - \left(1 - \frac{1}{3} \right) = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^3 dx + \int_1^2 (x^3 - 2x^2) dx$$

$$= \left[\frac{x^4}{4} \right]_0^1 + \left[\frac{x^4}{4} - \frac{2x^3}{3} \right]_1^2$$

$$\Rightarrow \frac{1}{4} + \left(4 - \frac{16}{3} \right) - \left(\frac{1}{4} - \frac{2}{3} \right) = \frac{-2}{3}$$

$$E(25X^2 + 30X - 5) = 25 E(X^2) + 30 E(X) - 5$$

$$= 25 \left(-\frac{2}{3} \right) + 30(1) - 5$$

$$= \frac{25}{3} \checkmark$$

$$(x) f(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) The density function $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 1/2 e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$(ii) \text{ Mean} = \mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \cdot \frac{1}{2} e^{-2x} dx$$

$$\mu = \frac{1}{2} \left[x \left(\frac{e^{-2x}}{-2} \right) - 1 \left(\frac{e^{-2x}}{-2} \right) \right]_0^{\infty}$$

$$\mu = -\frac{1}{8} \left[e^{-2x} (2x+1) \right]_0^{\infty} = -\frac{1}{8} (0-1) = \frac{1}{8}$$

$$(iii) \text{ Variance} = E(X^2) - \mu^2$$

$$E(X^2) = \int_0^{\infty} x^2 \cdot \frac{1}{2} e^{-2x} dx - \left(\frac{1}{8}\right)^2$$

$$E(X^2) = \frac{1}{2} \left[x^2 \left(\frac{e^{-2x}}{-2} \right) - 2x \left(\frac{e^{-2x}}{-2} \right) + 2 \left(\frac{e^{-2x}}{-8} \right) \right]_0^{\infty} - \frac{1}{64}$$

$$E(X^2) = \frac{1}{8} \left[e^{-2x} (2x^2 + 2x + 1) \right]_0^{\infty} - \frac{1}{64}$$

$$E(X^2) = \frac{1}{8} [0 - (0+0+1)] - \frac{1}{64} = -\frac{1}{8} - \frac{1}{64} = -\frac{9}{64}$$

(18) Two coins are tossed simultaneously,
 'x' denotes the no. of heads

$$S = \{ TT, TH, HT, HH \}.$$

The probability distribution function is

x	0	1	1	2
$P(x=x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$(i) E(X) = \sum_i p_i x_i$$

$$E(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 2\left(\frac{1}{4}\right).$$

$$E(X) = 0 + \frac{2}{4} + \frac{2}{4}$$

$$E(X) = 1 = \mu.$$

$$(ii) E(X^2) = \sum_i p_i x_i^2$$

$$E(X^2) = \frac{1}{4}(0)^2 + \frac{2}{4}(1)^2 + \frac{1}{4}(2)^2$$

$$E(X^2) = 0 + \frac{2}{4} + \frac{4}{4}$$

$$E(X^2) = \frac{6}{4} = \frac{3}{2}$$

$$(iii) E(X^3) = \sum_i p_i x_i^3$$

$$E(X^3) = (0)^3 \cdot \frac{1}{4} + (1)^3 \cdot \frac{2}{4} + (2)^3 \cdot \frac{1}{4}$$

$$E(X^3) = 0 + \frac{2}{4} + \frac{8}{4}$$

$$E(X^3) = \frac{10}{4} = \frac{5}{2} \quad \checkmark$$

$$(iv) V(X) = E(X^2) - (E(X))^2$$

$$V(X) = \frac{3}{2} - (1)^2$$

$$V(X) = \frac{3}{2} - 1$$

$$V(X) = \frac{1}{2} \quad \checkmark$$

(19) Given: $f(x) = \begin{cases} 0, & x < 2 \\ \frac{(2x+3)}{18}, & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$

(i) For all points x in $-\infty \leq x \leq \infty$, $f(x) \geq 0$

and $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^2 0 \cdot dx + \int_2^4 \frac{2x+3}{18} dx + \int_4^{\infty} 0 \cdot dx$

$$\Rightarrow \frac{1}{18} \int_2^4 (2x+3) dx = \frac{1}{18} \left[\frac{(2x+3)^2}{4} \right]_2^4$$

$$= \frac{1}{72} (121 - 49) = 1$$

Hence, $f(x)$ is a probability density function. ✓

$$\begin{aligned} \text{(ii)} P(2 \leq x \leq 3) &= \int_2^3 f(x) dx \\ &= \frac{1}{18} \int_2^3 (2x+3) dx \\ &= \frac{1}{18} \left[x^2 + 3x \right]_2^3 \\ &\Rightarrow \frac{1}{18} (18 - 10) = \frac{8}{18} = \frac{4}{9} \end{aligned}$$

$$② f(x) = \frac{K}{x^n + 1}; -\infty < x < \infty$$

$$\text{since, } \int_{-\infty}^{\infty} f(x) dx = 1 \rightarrow \int_{-\infty}^{\infty} \frac{K}{x^n + 1} dx = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{1}{x^n + 1} dx = 1$$

$$\Rightarrow K (\tan^{-1} x) \Big|_{-\infty}^{\infty} = 1$$

$$\Rightarrow K (\tan^{-1} \infty - \tan^{-1} (-\infty)) = 1 \Rightarrow K \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 1$$

$$K \pi = 1$$

$$K = \frac{1}{\pi}$$

By the definition, the distribution function is given by,

$$F(x) = \int_{-\infty}^x f(x) dx = \int_{-\infty}^x \frac{K}{1+x^n} dx$$

$$F(x) = \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1+x^n} dx$$

$$F(x) = \frac{1}{\pi} (\tan^{-1} x) \Big|_{-\infty}^x$$

$$F(n) = \frac{1}{\pi} \left[\tan^{-1} n - \tan^{-1}(-n) \right]$$

$$F(n) = \frac{1}{\pi} \left[\tan^{-1} n + \frac{\pi}{2} \right] \quad \checkmark$$

~~Naivef! See~~

H - Repeated from B (Part 1 and 2)

Prepared by

M. Shw Chary

★DMR-C