1) Solve the double integral $\int_{-1}^{2}\int_{2}^{x+2} dy\,dx$

$$\int_{-1}^{2}\int_{2}^{x+2} dy\,dx = \int_{x=-1}^{2}\int_{y=2}^{x+2} (1)\,dy\,dx = \int_{x=-1}^{2} [x]_{2}^{x+2}\,dx$$

$$\Rightarrow \int_{x=-1}^{2} (x+2-2)\,dx = \int_{x=-1}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{2} = \left[\frac{4}{2} - \frac{1}{2}\right] = \frac{3}{2}$$

2) Evaluate where R is the region bounded by the plane

$\iiint_R (x+y+z)\,dx\,dy\,dz$. Where $x=y=z=0$, $x=2$, $y=x$, $z=x+y$

$$\iiint_R (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{x}\int_{z=0}^{x+y} (x+y+z)\,dz\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{x} \left[xz+yz+\frac{z^2}{2}\right]_0^{x+y} dy\,dz$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{x} \left[x(x+y)+y(x+y)+\frac{(x+y)^2}{2}\right] dy\,dx$$

$$\Rightarrow \cancel{\int_{x=0}^{2}\int_{y=0}^{x} \left\{\frac{(x+y)}{2}\left[2x+2y+x^2+y^2+2xy\right]\right\} dy\,dx}$$

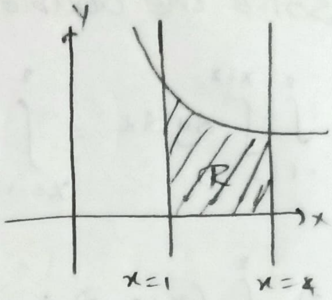
$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{x} \left[x^2+xy+xy+y^2+\frac{x^2+y^2+2xy}{2}\right] dy\,dx$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{x} \left[\frac{3x^2}{2}+\frac{3y^2}{2}+3xy\right] dy\,dx$$

$$\Rightarrow \int_{x=0}^{2} \left[\frac{3x^2y}{2}+\frac{\cancel{3}y^3}{\cancel{3}\times 2}+\frac{3xy^2}{2}\right]_0^{x} dx \Rightarrow \int_{x=0}^{2} \left(\frac{3x^3}{2}+\frac{x^3}{2}+\frac{3x^3}{2}\right) dx$$

$$\Rightarrow \int_{0}^{2} \frac{7x^3}{2}\,dx = \left(\frac{7x^4}{2\times 4}\right)_0^{2} = \frac{7\times 16}{8} = 14$$

3] Evaluate $\iint x^2\,dx\,dy$ over the region bounded by hyperbola $xy=4$, $y=0$, $x=1$, $x=4$



The limits of x are $x=1$ to 4 and

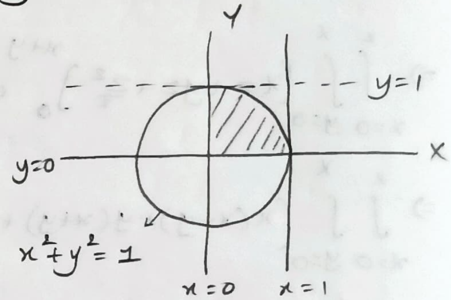
the limits of y are $y=0$ to $y=\frac{4}{x}$

$$\therefore \iint_R x^2\,dx\,dy = \int_{x=1}^{4}\int_{y=0}^{4/x} x^2\,dy\,dx$$

$$\Rightarrow \int_{x=1}^{4}\int_{y=0}^{4/x} x^2\,dy\,dx = \int_{x=1}^{4}\left[x^2 y\right]_0^{4/x} dx = \int_{x=1}^{4}\left[x^2 \times \frac{4}{x}\right] dx = \int_{x=1}^{4} 4x\,dx$$

$$\Rightarrow \left[\frac{4x^2}{2}\right]_1^4 = \left[2x^2\right]_1^4 = 32-2 = 30$$

4] Find change the order of integration and hence evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$



The limits of x are 0 to 1

the limits of y are 0 to $\sqrt{1-x^2}$

After changing the limits

the limits of x are 0 to $\sqrt{1-y^2}$

the limits of y are 0 to 1

$$\therefore \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy$$

$$\Rightarrow \int_{y=0}^{1}\left[y^2 x\right]_0^{\sqrt{1-y^2}} dy = \int_{y=0}^{1} y^2\sqrt{1-y^2}\,dy$$

Substitute $y=\sin\theta$

$$\Rightarrow \frac{dy}{d\theta} = \cos\theta \Rightarrow dy = \cos\theta\,d\theta$$

The limits changes to $\Rightarrow$ If $y=0 \Rightarrow \theta=0$

$y=1 \Rightarrow \theta=\frac{\pi}{2}$

$$\Rightarrow \int_{y=0}^{1} y^2\sqrt{1-y^2}\,dy = \int_{\theta=0}^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi/2} \sin^2\theta\cos\theta\cos\theta\,d\theta = \int_{\theta=0}^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$\Rightarrow \frac{1}{4}\int_{\theta=0}^{\pi/2} (2\sin\theta\cos\theta)^2\,d\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} (\sin 2\theta)^2\,d\theta$$

$$\Rightarrow \frac{1}{4}\int_{\theta=0}^{\pi/2} \left[\frac{1-\cos 4\theta}{2}\right] d\theta = \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$\Rightarrow \frac{1}{8}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{16}$$

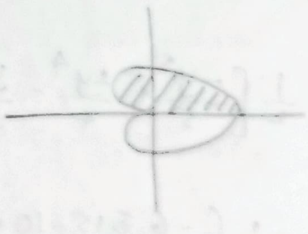
5) Calculate the double integral $\int_0^2\int_0^x \exp^{(x+y)}\,dy\,dx$

$$\int_0^2\int_0^x e^{x+y}\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{x} e^x.e^y\,dy\,dx = \int_{x=0}^{2} e^x\left[e^y\right]_0^x dx$$

$$\Rightarrow \int_{x=0}^{2} e^x[e^x - 1]\,dx = \int_{x=0}^{2} [e^{2x} - e^x]\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^2$$

$$\Rightarrow \left[\frac{e^4}{2} - e^4 - \left[\frac{1}{2} - 1\right]\right] = \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{e^4 - 2e^2 + 1}{2} = \frac{(e^2-1)^2}{2}$$

6) Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ and above the initial line



$$\iint_R r\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi}\left[\frac{r^2\sin\theta}{2}\right]_0^{a(1+\cos\theta)} d\theta = \int_0^{\pi} \frac{a^2(1+\cos\theta)^2\sin\theta}{2}\,d\theta$$

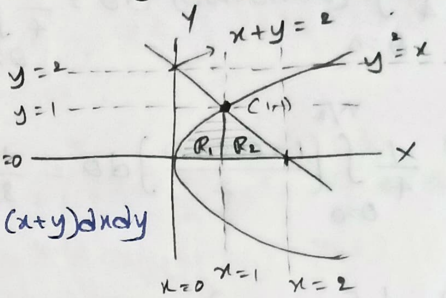
let $t = 1+\cos\theta$, if $\theta = 0 \Rightarrow t = 2$

$dt = -\sin\theta\,d\theta$, $\theta = \pi \Rightarrow t = 0$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi} \frac{(1+\cos\theta)^2\sin\theta\,d\theta}{2} = \frac{a^2}{2}\int_0^{2} -t^2\sin\theta\,d\theta$$

$$\Rightarrow \frac{-a^2}{2}\int_0^{2} t^2\,dt = \frac{-a^2}{2}\left[\frac{t^3}{3}\right]_2^0 = \frac{-a^2}{2}\left[0-\frac{8}{3}\right] = \frac{8a^2}{6} = \frac{4a^2}{3}$$

7) Evaluate $\iint_R (x+y)\,dx\,dy$ over the region bounded by $y=0$, $x+y=2$ and $y^2=x$



$$\iint_R (x+y)\,dx\,dy = \iint_{R_1}(x+y)\,dx\,dy + \iint_{R_2}(x+y)\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}}(x+y)\,dx\,dy + \int_{x=1}^{2}\int_{y=0}^{2-x}(x+y)\,dx\,dy$$

$x=y^2$, $x+y=2$

$y^2+y-2=0$

$y^2+2y-y-2=0$

$y(y+2)-1(y+2)=0$

$y=1$, $y=-2$

$y=1 \Rightarrow x=1$

[or]

$$\iint_R (x+y)\,dx\,dy = \int_{y=0}^{1}\int_{x=y^2}^{2-y}(x+y)\,dx\,dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\frac{x^2}{2}+yx\right]_{y^2}^{2-y} dy = \int_{y=0}^{1}\left[\frac{4+y^2-4y}{2}+2y-y^2-\frac{y^4}{2}-y^3\right]dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\frac{4+y^2-4y+4y-2y^2-y^4-2y^3}{2}\right]dy = \frac{1}{2}\int_{y=0}^{1}\left[-y^4-2y^3-y^2+4\right]dy$$

$$\Rightarrow \frac{1}{2}\left[\frac{-y^5}{5}-\frac{2y^4}{4}-\frac{y^3}{3}+4y\right]_0^1 = \frac{1}{2}\left[\frac{-1}{5}-\frac{1}{2}-\frac{1}{3}+4\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{-6-15-10+120}{30}\right] = \frac{89}{60}$$

[or]

$$\iint_R (x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}}(x+y)\,dx\,dy + \int_{x=1}^{2}\int_{y=0}^{2-x}(x+y)\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{1} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{x}} dx + \int_{x=1}^{2} \left[xy + \frac{y^2}{2}\right]_0^{2-x} dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x\sqrt{x} + \frac{x}{2}\right]dx + \int_{x=1}^{2}\left[2x - x^2 + \frac{4+x^2-4x}{2}\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x^{3/2}+\frac{x}{2}\right]dx + \int_{x=1}^{2}\left[\frac{-x^2+4}{2}\right]dx$$

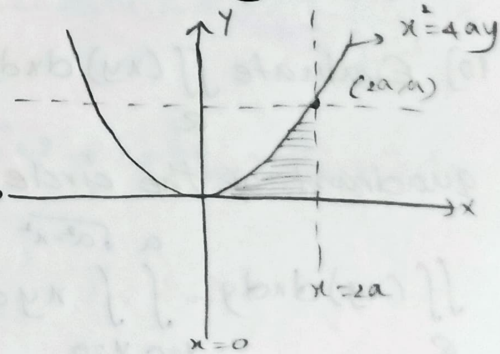
$$\Rightarrow \left[\frac{x^{5/2}}{5/2}+\frac{x^2}{4}\right]_0^1 + \frac{1}{2}\left[\frac{-x^3}{3}+4x\right]_1^2$$

$$\Rightarrow \frac{2}{5}+\frac{1}{4}+\frac{1}{2}\left[\frac{-8}{3}+8-\left(\frac{-1}{3}+4\right)\right] = \frac{2}{5}+\frac{1}{4}+\frac{1}{2}\left[\frac{16}{3}-\frac{11}{3}\right]$$

$$\Rightarrow \frac{8+5}{20}+\frac{1}{2}\left[\frac{5}{3}\right] = \frac{13}{20}+\frac{5}{6} = \frac{39+50}{60} = \frac{89}{60}$$

8] Evaluate $\iint_R (xy)\,dx\,dy$ over the driangular bounded by X-axis, ordinate $x=2a$ and the curve $x^2=4ay$

$$\iint_R (xy)\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} (xy)\,dy\,dx$$



$x=2a,\ x^2=4ay$

$4a^2=4ay \Rightarrow y=a$

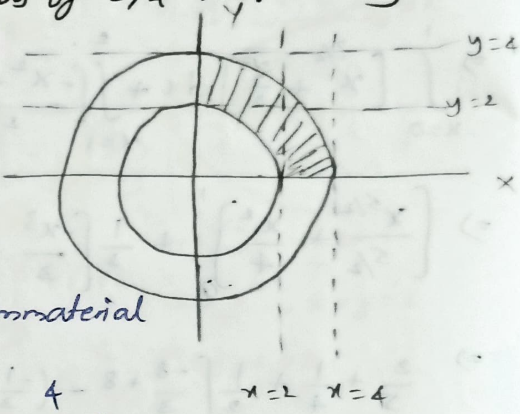
$$\Rightarrow \int_{x=0}^{2a}\left[\frac{xy^2}{2}\right]_0^{\frac{x^2}{4a}} dx = \int_{x=0}^{2a} \frac{x}{2}\left[\frac{x^2}{4a}\right]^2 dx$$

$$\Rightarrow \int_{x=0}^{2a} \frac{x}{2}\times\frac{x^4}{16a^2}\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$\Rightarrow \frac{1}{32a^2}\times\frac{16\times 4a^6}{6} = \frac{a^4}{3}$$

9) Determine the area of $\iint r^3 dr d\theta$ over the area bounded in the first quadrant between the circles $r=2$ and $r=4$

9) Evaluate $\iint_R r^3 dr d\theta$ where R is the region in the first quadrant bounded by two concentric circles with centre at origin and radius of 2, 4 respectively



$$\iint_R r^3 dr d\theta = \int_{\theta=0}^{\pi/2}\int_{r=2}^{4} r^3 dr d\theta$$

Both the limits are Constants

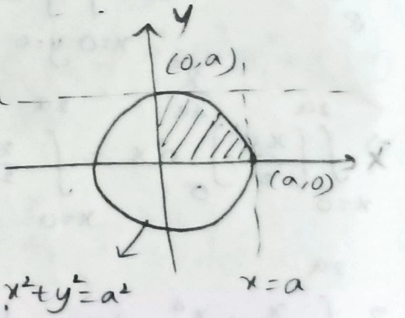
then the order of Integration is Immaterial

$$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=2}^{4} (1) r^3 dr d\theta = \int_{\theta=0}^{\pi/2} (1) d\theta \int_{r=2}^{4} r^3 dr$$

$$\Rightarrow \cancel{\left[\frac{\pi}{2}/0\right]\left[\frac{r^4}{4}\right]} = [\theta]_0^{\pi/2}\left[\frac{r^4}{4}\right]_2^4$$

$$\Rightarrow \left(\frac{\pi}{2}-0\right)\left[64-4\right] = 30\pi$$

10) Evaluate $\iint_R (xy)\, dx dy$, Where R is the positive quadrant of the circle $x^2+y^2=a^2$



$$\iint_R (xy) dx dy = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$\Rightarrow \int_{x=0}^{a}\left[\frac{xy^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \int_{x=0}^{a}\frac{x(a^2-x^2)}{2} dx$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{a}(a^2x-x^3)dx = \frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$\Rightarrow \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{a^4}{4}\times\frac{1}{2} = \frac{a^4}{8}$$

Module-V Part-B

1) Evaluate the triple Integral $\int_0^1\int_0^{1-z}\int_0^{1-y-z} xyz\,dx\,dy\,dz$

$$\int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{1-y-z} xyz\,dx\,dy\,dz = \int_{z=0}^{1}\int_{y=0}^{1-z}\left(\frac{x^2yz}{2}\right)_0^{1-y-z} dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_{z=0}^{1}\int_{y=0}^{1-z} yz[1-y-z]^2\,dy\,dz = \frac{1}{2}\int_{z=0}^{1}\int_{y=0}^{1-z}[(1-z)-y]^2\,dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_{z=0}^{1}\int_{y=0}^{1-z} yz\,[(1-z)^2-2y(1-z)+y^2]\,dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_{z=0}^{1}\int_{y=0}^{1-z} z[(1-z)^2y-2(1-z)y^2+y^3]\,dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_{z=0}^{1}\int_{y=0}^{1-z} z\left[\frac{(1-z)^2(1-z)^2}{2}-\frac{2}{3}(1-z)(1-z)^3+\frac{(1-z)^4}{4}\right]dz$$

$$\Rightarrow \frac{1}{2}\int_{z=0}^{1} z(1-z)^4\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]dz = \frac{1}{2}\int_{z=0}^{1} z[1-4z+6z^2+4z^3+z^4]\frac{1}{12}\,dz$$

$$\Rightarrow \frac{1}{24}\int_{z=0}^{1}(z-4z^2+6z^3\mp 4z^4+z^5)\,dz$$

$$\Rightarrow \frac{1}{24}\left[\frac{z^2}{2}-\frac{4z^3}{3}+\frac{6z^4}{4}\mp\frac{4z^5}{5}+\frac{z^6}{6}\right]_0^1$$

$$\Rightarrow \frac{1}{24}\left[\frac{1}{2}-\frac{4}{3}+\frac{3}{2}\mp\frac{4}{5}+\frac{1}{6}\right]$$

$$\Rightarrow \frac{1}{24}\left[\frac{15-40+45\mp 24+5}{30}\right]$$

$$\Rightarrow \frac{1}{24}\left[\frac{1}{30}\right] = \frac{1}{720}$$

2) Solve the double integral $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$\Rightarrow \frac{1}{3}\int_0^{\pi} \cos\theta\left[a^3(1+\cos\theta)^3\right] d\theta = \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta$$

let

$$I = \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta \rightarrow (1)$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \frac{a^3}{3}\int_0^{\pi}\cos(\pi-\theta)\,(1+\cos(\pi-\theta))^3 d\theta$$

$$I = \frac{a^3}{3}\int_0^{\pi}(-\cos\theta)(1-\cos\theta)^3 d\theta \rightarrow (2)$$

$$(1)+(2) \Rightarrow 2I = \frac{a^3}{3}\left[\int_0^{\pi}\cos\left[(1+\cos\theta)^3 - (1-\cos\theta)^3\right] d\theta\right]$$

$$(a+b)^3 - (a-b)^3 = 2b^3 + 6a^2b$$

$$2I = \frac{a^3}{3}\int_0^{\pi}\cos\theta(2\cos^3\theta + 6\cos\theta)\,d\theta$$

$$\Rightarrow \frac{a^3}{3}\int_0^{\pi}\cos\theta\left[2\cos\theta\,(\cos^2\theta+3)\right] d\theta = \frac{2a^3}{3}\int_0^{\pi}\cos^2\theta(\cos^2\theta+3)\,d\theta$$

$$\Rightarrow \frac{2a^3}{3}\int_0^{\pi}(3\cos^2\theta + \cos^4\theta)\,d\theta = \frac{2a^3}{3}\int_0^{\pi}\left[3\left(\frac{1+\cos2\theta}{2}\right) + \left(\frac{1+\cos2\theta}{2}\right)^2\right] d\theta$$

$$\Rightarrow \frac{2a^3}{3}\int_0^{\pi}\left[\frac{3}{2}(1+\cos2\theta) + \frac{1}{4}(1+\cos^2 2\theta + 2\cos2\theta)\right] d\theta$$

$$\Rightarrow \frac{2a^3}{3}\int_0^{\pi}\left[\frac{3}{2}(1+\cos2\theta) + \frac{1}{4}\left(1+\frac{1+\cos4\theta}{2} + 2\cos2\theta\right)\right] d\theta$$

$$\Rightarrow \frac{2a^3}{3}\left[\frac{3}{2}\left(\theta + \frac{\sin2\theta}{2}\right) + \frac{1}{4}\left(\theta + \frac{\theta}{2} + \frac{\sin4\theta}{8} + \frac{2\sin2\theta}{2}\right)\right]_0^{\pi}$$

$$\Rightarrow \frac{2a^3}{3}\left[\frac{3}{2}\pi + \frac{1}{4}\left(\pi + \frac{\pi}{2}\right)\right] = \frac{2a^3}{3}\left[\frac{3\pi}{2} + \frac{3\pi}{8}\right] = \frac{5\pi a^3}{4}$$

$$2I = \frac{5\pi a^3}{4} \Rightarrow I = \frac{5\pi a^3}{8}$$

3) Calculate the double integral $\int_0^1\int_x^{\sqrt{x}} [x^2+y^2]\,dx\,dy$

$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} [x^2+y^2]\,dy\,dx = \int_{x=0}^{1}\left(x^2y+\frac{y^3}{3}\right)_x^{\sqrt{x}}$$

$$\Rightarrow \int_{x=0}^{1}\left[x^{5/2}+\frac{x^{3/2}}{2}-x^3-\frac{x^3}{3}\right]dx = \int_0^1\left[x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right]dx$$

$$\Rightarrow \left[\frac{2x^{7/2}}{7}+\frac{1}{3}\frac{2x^{5/2}}{5}-\frac{4x^4}{3\times4}\right]_0^1 = \left[\frac{2}{7}+\frac{1}{15}-\frac{1}{3}\right] = \frac{3}{35}$$

4) Solve the double integral $\int_0^5\int_0^{x^2} x[x^2+y^2]\,dx\,dy$

$$\int_0^5\int_0^{x^2}[x^3+xy^2]\,dx\,dy = \int_{x=0}^{5}\int_{y=0}^{x^2}[x^3+xy^2]\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{5} \cancel{\frac{x^4}{4}}+\left[x^3y+\frac{xy^3}{3}\right]_0^{x^2}dx = \int_{x=0}^{5}\left[x^5+\frac{x^7}{3}\right]dx$$

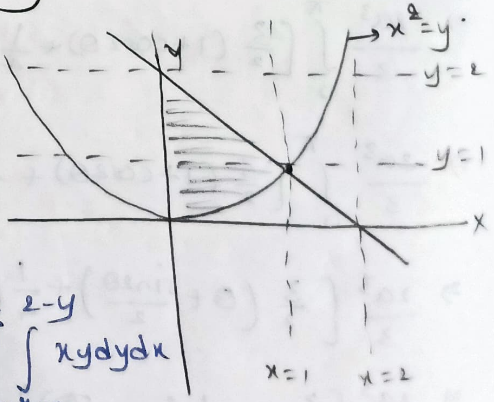
$$\Rightarrow \left[\frac{x^6}{6}+\frac{x^8}{24}\right]_0^5 = \frac{5^6}{6}+\frac{5^8}{24} = \frac{15625}{24}$$

5) Evaluate the double integral $\int_0^1\int_0^{\pi/2} r\sin\theta\,dr\,d\theta$

$$\int_0^1\int_0^{\pi/2} r\sin\theta\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^1\left[-\cos\theta\right]_0^{\pi/2}$$

$$\Rightarrow \left[\frac{1}{2}\right]\left[+1\right] = \frac{1}{2}$$

6) By changing the order of integration evaluate the double integral $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$



By changing the order of integration

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dy\,dx + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dy\,dx$$

$$\Rightarrow \int_{y=0}^{1}\left[\frac{x^2y}{2}\right]_0^{\sqrt{y}}dy + \int_{y=1}^{2}\left[\frac{x^2y}{2}\right]_0^{2-y}dy$$

$$\Rightarrow \int_{y=0}^{1} \frac{y^2}{2}\, dy + \int_{y=1}^{2} \frac{(2-y)^2 y}{2}\, dy$$

$$\Rightarrow \frac{1}{2}\int_{y=0}^{1} y^2 dy + \int_{y=1}^{2} (4y+y^3-4y^2)\, dy$$

$$\Rightarrow \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right]_1^2$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{3}\right) + \left[8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right]$$

$$\Rightarrow \frac{1}{6} + \left[10 + \frac{16-128-3}{12}\right] = \frac{1}{6}+\frac{1}{2}\left[\frac{5}{12}\right]$$

$$\Rightarrow \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$$

7) Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dy\,dx$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dy\,dx = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} xy\,dy\,dx$$

$$\Rightarrow \int_{y=0}^{a} \left(\frac{x^2 y}{2}\right)_0^{\sqrt{a^2-y^2}} dy = \int_{y=0}^{a} \frac{(a^2-y^2)y}{2}\, dy = \int_{y=0}^{a} \left[\frac{a^2y-y^3}{2}\right] dy$$

$$\Rightarrow \frac{1}{2}\left[\frac{a^2y^2}{2}-\frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{a^4}{8}$$

8) Solve the triple integral $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$

$$\int_{x=0}^{\log^2 y = x}\int_{y=0}^{x+\log y}\int_{z=0} e^{(x+y+z)}\, dx\,dy\,dz = \int_{x=0}^{\log 2}\int_{y=0}^{x} (e^{x+y})(e^{z})_0^{x+\log y}\, dy\,dx$$

$$\Rightarrow \int_{x=0}^{\log 2}\int_{y=0}^{x} [e^{x+y}][e^{x+\log y}-1]\, dy\,dx = \int_{x=0}^{\log 2}\int_{y=0}^{x} (ye^{2x}e^{y}-e^{x}e^{y})\, dy\,dx$$

$$\Rightarrow \int_{x=0}^{\log 2} [e^{2x}(e^{y}(y-1))]_0^{x}\, dx - \int_{x=0}^{\log 2} [e^{x}e^{y}]_0^{x}\, dx$$

$$\Rightarrow \int_{x=0}^{\log 2} e^{3x}(x-1)+e^{2x}\, dx - \int_{x=0}^{\log 2} (e^{2x}-e^{x})\, dx$$

$$\Rightarrow \int_{x=0}^{\log 2} (x e^{3x} - e^{3x} + e^{x})\,dx = \int_{x=0}^{\log 2} (x-1) e^{3x} + e^{x}\, dx$$

$$\Rightarrow \left[(x-1)\frac{e^{3x}}{3} - \left(\frac{e^{3x}}{9}\right) + e^{x} \right]_0^{\log 2}$$

$$\Rightarrow \left[(\log 2 - 1)\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2} - \left(\frac{-1}{3} - \frac{1}{9} + 1\right)\right]$$

$$\Rightarrow (\log 2 - 1)\frac{8}{3} - \frac{8}{9} + 2 - \left(\frac{-1-3+9}{9}\right)$$

$$\Rightarrow \frac{8}{3}(\log 2 - 1) - \frac{8}{9} + 2 - \frac{5}{9} = \frac{5}{9} + \frac{8}{3}(\log 2 - 1)$$

$$\Rightarrow \frac{-19}{9} + \frac{8}{3}\log 2$$

9) Evaluate the triple integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-y^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}(1)\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{\pi}{2}\,dy\,dx = \int_{x=0}^{1} \left[\frac{\pi}{2} y\right]_0^{\sqrt{1-x^2}} = \frac{\pi}{2}\int_{x=0}^{1} \sqrt{1-x^2}\,dx$$

put $x = \sin\theta$, $\Rightarrow$ If $x=0 \Rightarrow \theta = 0$

$dx = \cos\theta\, d\theta$, $x = 1 \Rightarrow \theta = \pi/2$

$$\frac{\pi}{2}\int_{\theta=0}^{\pi/2} \cos\theta(\cos\theta\, d\theta) = \frac{\pi}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$\frac{\pi}{2}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{\pi}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

10) Find the value of $\iint xy\,dx\,dy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{1-x^2}} xy\,dx\,dy = \int_{x=0}^{a}\left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$\Rightarrow \int_{x=0}^{a} \frac{x}{2}\frac{b^2}{a^2}(a^2-x^2)\,dx = \int_{x=0}^{a}\frac{b^2}{2a^2}(a^2x-x^3)\,dx$$

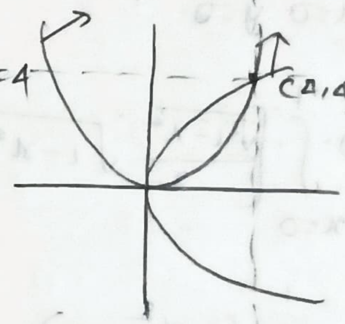
$$\Rightarrow \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{b^2}{2a^2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{a^4b^2}{8a^2} = \frac{a^2b^2}{8}$$

11) Change the order of integration and evaluate $\int_0^4\int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$

$$\int_0^4\int_{\frac{x^2}{4}}^{2\sqrt{x}} (1)\,dx\,dy = \int_{x=0}^{4}[y]_{\frac{x^2}{4}}^{2\sqrt{x}}\,dx = \int_{x=0}^{4}\left[2\sqrt{x}-\frac{x^2}{4}\right]dx$$

$x^2=4y$ $\quad y^2=4x$

$$\Rightarrow \left[\frac{4x^{3/2}}{3}-\frac{x^3}{12}\right]_0^4 = \frac{4\times4\times2}{3}-\frac{\cancel{64}^{18}}{\cancel{12}_{3}} = \frac{14}{3}$$



By changing the order of Integration

$x^2=4y,\ y^2=4x$

$$\int_{y=0}^{4}\int_{x=\frac{y^2}{4}}^{x=2\sqrt{y}} (1)\,dx\,dy = \int_{y=0}^{4}(x)_{\frac{y^2}{4}}^{2\sqrt{y}}\,dy$$

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

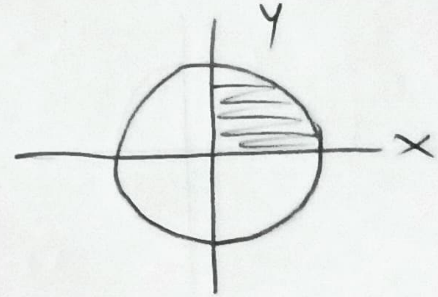
$$x^4 = 64x$$

$$\Rightarrow \int_{y=0}^{4}\left[2\sqrt{y}-\frac{y^2}{4}\right]dy = \left[\frac{4x^{3/2}}{3}-\frac{y^3}{12}\right]_0^4$$

$x=0,\ x=4$

$$\Rightarrow \frac{4\times4\times2}{3}-\frac{64}{12} = \frac{14}{3}$$

12) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, where R is the poridive quadrant of the circle $x^2+y^2=1$



$$\iint_R (x^2+y^2)\,dx\,dy$$

put $x = r\cos\theta$; $y = r\sin\theta$; θ raires from $\theta = 0$ to $\frac{\pi}{2}$

$\Rightarrow dx\,dy = r\,dr\,d\theta$ $\qquad$ r raires from $r = 0$ to 1

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\, r\,dr\,d\theta$$

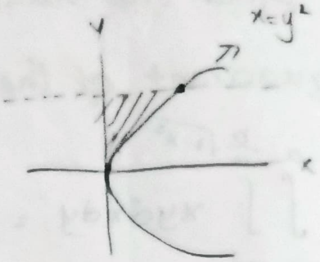
$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(1) = r^2$$

$$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\,dr\,d\theta = \int_0^{\pi/2} 1\,d\theta \int_0^1 r^3\,dr$$

$$\Rightarrow (\theta)_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1$$

$$\Rightarrow \left(\frac{\pi}{2}\right)\left(\frac{1}{4}\right) = \frac{\pi}{8}$$

13) Change the order of interaction and hence evaluate $\int_0^2\int_0^{y^2} x^2\,dx\,dy$



$y^2 \quad \sqrt{x} \quad 2$

$$\int_{x=0}\int_{y=0} x^2\,dx\,dy = \int_{y=0}^{2}\left[\frac{x^3}{3}\right]_0^{y} dy$$

$$\Rightarrow \int_{y=0}^{2}\left[\frac{y^6}{3}\right]dy = \left[\frac{y^7}{21}\right]_0^2 = \frac{128}{21}$$

14) Evaluate the double integral $\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dx\,dy$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

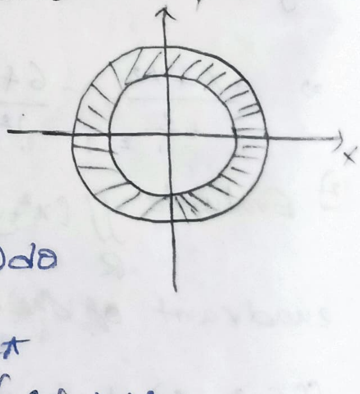
$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sqrt{(\sqrt{1-x^2})^2-y^2}\,dy\,dx = \int_{x=0}\left[\frac{y}{2}\sqrt{1-x^2-y^2} + \frac{1-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}} dx$$

$$\Rightarrow \int_{x=0}^{1}\left[\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-1+x^2} + \frac{1-x^2}{2}\sin^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[\frac{1-x^2}{2}\cdot\frac{\pi}{2}\right]dx = \frac{\pi}{2}\int_{x=0}^{1}\left(\frac{1}{2}-\frac{x^2}{2}\right)dx = \frac{\pi}{2}\left[\frac{1}{2}x-\frac{x^3}{6}\right]_0^1$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{1}{2}-\frac{1}{6}\right] = \frac{\pi}{2}\left[\frac{3-1}{6}\right] = \frac{\pi}{2}\left[\frac{2}{6}\right] = \frac{\pi}{3}$$

15) Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$



$$\iint_R r^3\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{4}\int_0^{2\pi}(256\sin^4\theta-16\sin^4\theta)\,d\theta$$

$$\Rightarrow \frac{1}{4}\int_0^{2\pi} 240\sin^4\theta\,d\theta = 60\int_0^{2\pi}\sin^4\theta\,d\theta = 60\int_0^{2\pi}(\sin^2\theta)^2\,d\theta$$

$$\Rightarrow 60\int_0^{2\pi}\left(\frac{1-\cos2\theta}{2}\right)^2 d\theta = 15\int_0^{2\pi}(1+\cos^2 2\theta-2\cos2\theta)\,d\theta$$

$$\Rightarrow 15\int_0^{2\pi}\left(1+\left(\frac{1+\cos 4\theta}{2}\right)-2\cos 2\theta\right)d\theta$$

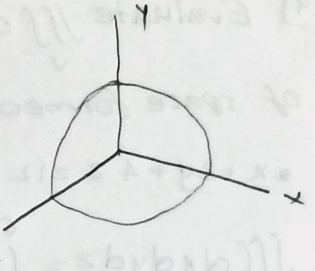
$$\Rightarrow 15\int_0^{2\pi}\left(\frac{3}{2}+\frac{\cos 4\theta}{2}-2\cos 2\theta\right)d\theta$$

$$\Rightarrow 15\left[\frac{3}{2}\theta+\frac{\sin 4\theta}{8}-\frac{2\sin 2\theta}{2}\right]_0^{2\pi}$$

$$\Rightarrow 15\left[\frac{3}{2}(2\pi)\right]=45\pi$$

16) Evaluate $\iiint_V dx\,dy\,dz$ where V is the positive ~~quadrant~~ octant of the volume of the sphere $x^2+y^2+z^2=a^2$

$$\iiint_V dx\,dy\,dz=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$



$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}[z]_0^{\sqrt{a^2-x^2-y^2}}dy\,dx=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx$$

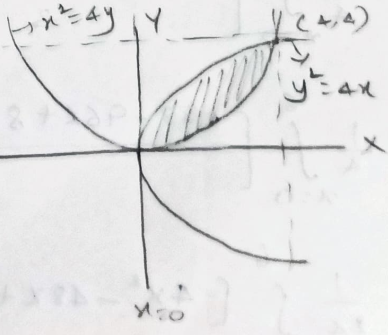
$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dx\,dy=\int_{x=0}\left[\frac{y}{2}\sqrt{a^2-x^2-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}}dx$$

$$\Rightarrow \int_{x=0}^{a}\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right)dx=\frac{\pi}{4}\int_{x=0}^{a}(a^2-x^2)\,dx$$

$$\Rightarrow \frac{\pi}{4}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{\pi}{4}\left[a^3-\frac{a^3}{3}\right]=\frac{\pi}{4}\left[\frac{2a^3}{3}\right]=\frac{a^3\pi}{6}$$

17) Evaluate $\iint_R (y)\,dx\,dy$ over the area between by the curves $y^2=4x$ and $x^2=4y$

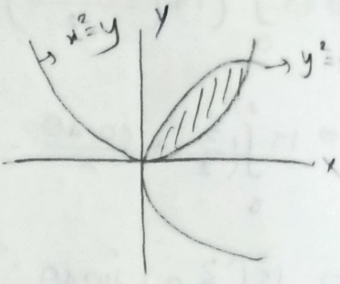


$$\iint_R y\,dx\,dy=\int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{\sqrt{4x}} y\,dx\,dy$$

$$\int_{x=0}^{4}\left[\frac{y^2}{2}\right]_{\frac{x^2}{4}}^{2\sqrt{x}}dx=\int_{x=0}\left[\frac{4x}{2}-\left(\frac{x^2}{4}\right)^2\frac{1}{2}\right]dx$$

$$\Rightarrow \int_{x=0}^{4}\left(\frac{4x}{2}-\frac{x^4}{32}\right)dx=\left[\frac{4x^2}{4}-\frac{x^5}{160}\right]_0^4\Rightarrow 16-\frac{64\times 16}{160}=\frac{48}{5}$$

18) Evaluate $\iint_R dx\,dy$ over the area between by the curve $y = x^2$ and $x = y^2$

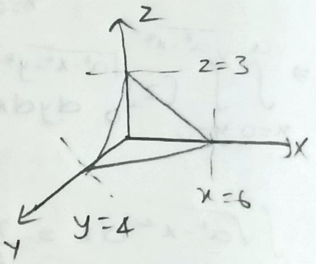


$$\iint_R dx\,dy = \int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{\sqrt{4x}} dx\,dy$$

$$\Rightarrow \int_{x=0}^{4} [y]_{\frac{x^2}{4}}^{\sqrt{4x}}\, dx = \int_{x=0}^{4} \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$\Rightarrow \left[\frac{4x^{3/2}}{3} - \frac{x^3}{12}\right]_0^4 = \left[\frac{4\times 4\times 2}{3} - \frac{4\times 4\times \cancel{4}}{\cancel{12}\,3}\right] = \frac{16}{3}$$

19) Evaluate $\iiint_V dx\,dy\,dz$ where V is the finite region of space formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$



$$\iiint_V dx\,dy\,dz = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}}\int_{z=0}^{\frac{12-2x-3y}{4}} dx\,dy\,dz$$

$$\Rightarrow \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} [z]_0^{\frac{12-2x-3y}{4}}\, dy\,dx = \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12-2x-3y)\,dy\,dx \qquad \frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1$$

$$\Rightarrow \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12-2x-3y)\,dy\,dx = \frac{1}{4}\int_{x=0}^{6} \left[12y - 2xy - \frac{3y^2}{2}\right]_0^{\frac{12-2x}{3}} dx$$

$$\Rightarrow \frac{1}{4}\int_{x=0}^{6} \left[\frac{12(12-2x)}{3} - \frac{2x(12-2x)}{3} - \frac{\cancel{3}(12-2x)^2}{\cancel{9}\,3\times 2}\right] dx$$

$$\Rightarrow \frac{1}{4}\int_{x=0}^{6} \left[\frac{144-24x-24x+4x^2}{3} - \frac{144+4x^2-48x}{6}\right] dx$$

$$\Rightarrow \frac{1}{4}\int_{x=0}^{6} \left[\frac{288-96x+8x^2-144-4x^2+48x}{6}\right] dx$$

$$\Rightarrow \frac{1}{24}\int_{x=0}^{6} [4x^2-48x+144]\,dx = \frac{1}{6}\int_{x=0}^{6} [x^2-12x+38]\,dx$$

$$\Rightarrow \frac{1}{6}\int_{x=0} \left[\frac{x^3}{3} - \frac{12x^2}{2} + 38x\right]_0^6$$

$$\Rightarrow \frac{1}{6}\left[\frac{216}{3} - \frac{12\times 36}{2} + 288\right]$$

$$\Rightarrow \frac{1}{6}\left[72 - 216 + 288\right] = \frac{144}{6} = 24$$

20) Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{1}{\sqrt{(\sqrt{1-x^2})^2 - y^2}}\, dy\,dx = \int_{x=0}^{1} \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}} dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_{x=0}^{1} \left[\sin^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)\right] dx = \int_{x=0}^{1} \frac{\pi}{2}\, dx = \frac{\pi}{2}$$