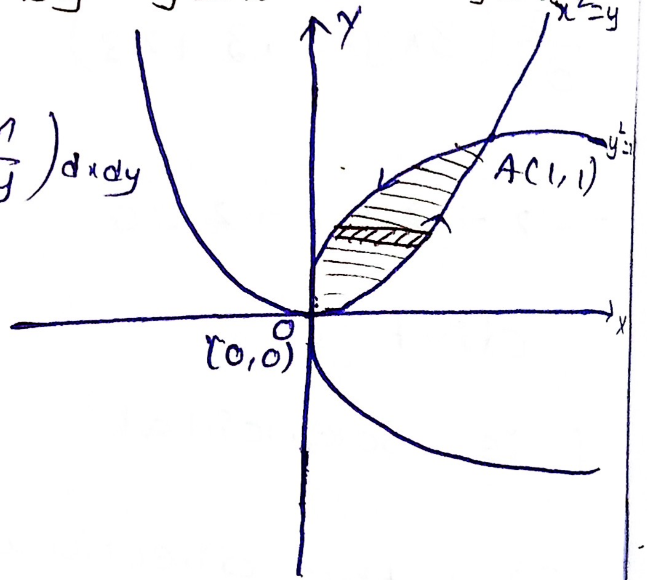
Module

Vector Integration.

Part-A

Green's theorem

3) Using green's theorem in the plane evaluate $\int_c (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where c is the region bounded by $y = x^2$ and $y^2 = x$



$$\Rightarrow \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 2xy - x^2$

$\frac{\partial M}{\partial y} = 2x$

$N = x^2 + y^2$

$\frac{\partial N}{\partial x} = 2x$

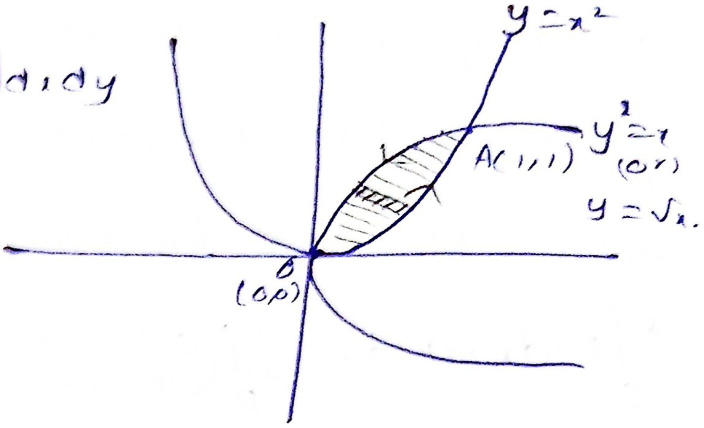
$$RHS = \int_0^1 \int_{y^2}^{\sqrt{y}} (2x - 2x)\,dy\,dx$$

$R.H.S = 0$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy = 0.$$

4) Applying Green's theorem evaluate $\int_c (xy + y^2)\,dx + x^2\,dy$ where c is the region bounded by $y = \sqrt{x}$ and $y = x^2$

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$$M = xy + y^2$$

$$\frac{\partial M}{\partial y} = x + 2y$$

$$N = x^2$$

$$\frac{\partial N}{\partial x} = 2x$$

$$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2x - x - 2y)\, dx\, dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (x - 2y)\, dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} - 2xy\right)_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\left[\frac{1}{2}(\sqrt{y})^2 - 2\sqrt{y}\cdot y\right] - \left[\frac{y^4}{2} - 2y^3\right]\right] dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{3/2} - \frac{y^4}{2} + 2y^3\right) dy$$

$$= \left[\frac{y^2}{4} - 2\frac{y^{5/2}}{5/2} - \frac{1}{2}\frac{y^5}{5} + 2\frac{y^4}{4}\right]_0^1$$

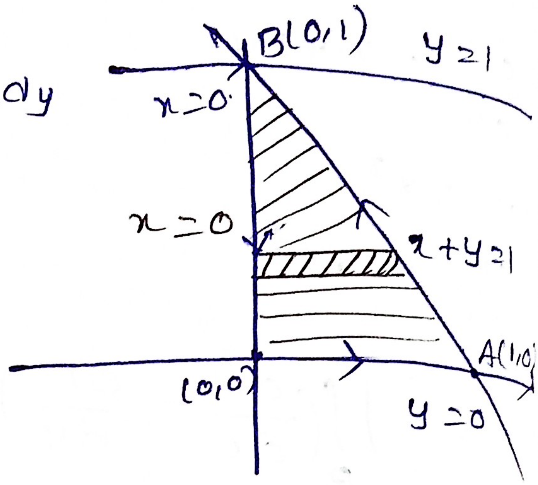
$$= \left[\frac{y^2}{4} - \frac{4}{5}y^{5/2} - \frac{1}{10}y^5 + \frac{1}{2}y^4\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} - \frac{1}{10} + \frac{1}{2}$$

$$= \frac{5-16-2+10}{20} = \frac{15-18}{20} = \frac{-3}{20}$$

5) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $x=0$, $y=0$ and $x+y=1$.

$\Rightarrow \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$M = 3x^2 - 8y^2$

$\frac{\partial M}{\partial y} = -16y$

$N = 4y - 6xy$

$= \frac{\partial N}{\partial x} = -6y$

$$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,.\,dx$$

$$= \int_0^1 10\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5\int_0^1 [y^2]_0^{1-x}\,dx$$

$$= 5\int_0^1 (1-x)^2\,dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3}\left[-(1-\cancel{x}) + (1-0)\right]$$

$$= \frac{5}{3}\left[-\cancel{1} + \cancel{1} + 1 - 0\right] = 5/3$$

$$L.H.S = \int_C M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$$

OA $\Rightarrow y = 0$ x varies from 0 to 1
$dy = 0$

$$\int_{OA} M\,dx + N\,dy = \int_0^1 3x^2\,dx = [x^3]_0^1 = 1$$

Along AB $y = 1-x$
$dy = -dx$
x varies from 1 to 0

$$\int_{AB} M\,dx + N\,dy = \int_1^0 3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)\,dx$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = 8/3$$

Along BO $x = 0$
$dx = 0$
y varies from 1 to 0

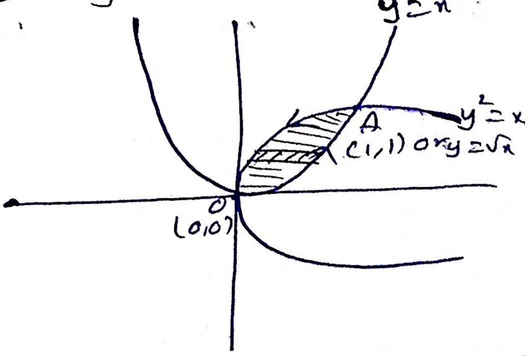
$$\int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\int_C M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = 5/3.$$

R.H.S = L.H.S

$\therefore$ Green's theorem verified.

8) Evaluate Green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the closed curve of the region by $y = \sqrt{x}$, $y = x^2$



$$\Rightarrow \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3x^2 - 8y^2$ $\quad \frac{\partial M}{\partial y} = -16y$

$N = 4y - 6xy$ $\quad \frac{\partial N}{\partial x} = -6y$

$$R.H.S = \int_0^1\int_{y^2}^{\sqrt{y}} (-6y+16y)\,dx\,dy$$

$$= \int_0^1\int_{y^2}^{\sqrt{y}} [10y]\,dx\,dy$$

$$= \int_0^1 [10xy]_{y^2}^{\sqrt{y}}\,dy$$

$$= 10\int_0^1 (y\cdot(\sqrt{y}-y^2))\,dy$$

$$= 10\int_0^1 (y^{3/2}-y^3)\,dy$$

$$= 10\left(\frac{y^{5/2}}{5/2}-\frac{y^4}{4}\right)_0^1$$

$$= 10\left(\frac{2}{5}-\frac{1}{4}\right) = 3/2$$

Q) If $\bar{F} = (x^2+y^2)i - 2xyj$ evaluate $\int \bar{F}\cdot d\bar{r}$ where curve c is the rectangle in x-y plane bound.. by $y=0$, $y=b$, $x=0$, $x=a$.

$\Rightarrow$ $\bar{F} = (x^2+y^2)i - 2xy\,j$

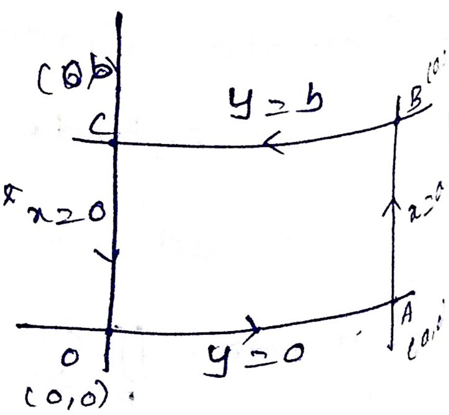
$\bar{r} = xi + yj$

$d\bar{r} = dx\,i + dy\,j$ $\Rightarrow$ $\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$

$\int_c \bar{F}\cdot d\bar{r} = \int_c (x^2+y^2)dx - 2xy\,dy$

$\int_c \bar{F}\cdot d\bar{r} =$

$\int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$



Along OA $y=0$

$dy = 0$.

x varies from 0 to a.

$\int_0^a x^2\,dx = a^3/3$

Along AB $x = a$, $dx = 0$ y varies from 0 to b

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b (-2ay)\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

Along BC $y = b$, $dy = 0$ x varies from a to 0

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO $x = 0$, $dx = 0$ y varies from b to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_b^0 (0)\,dx + (0) = 0$$

$$\oint_C \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2$$

$$= -2ab^2$$

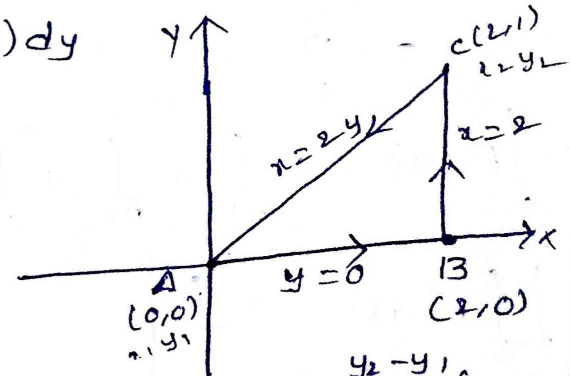
10) If $\bar{f} = (2x+y^2)\,i + (3y-4x)\,j$ evaluate $\oint_C \bar{F}\cdot d\bar{r}$ around a triangle ABC in the xy-plane with $A(0,0)$ $B(2,0)$, $C(2,1)$ in the counter clock wise direction and oppsite direction.

$\Rightarrow \bar{F} = (2x+y^2)\,i + (3y-4x)\,j$

$\int \bar{F}\cdot d\bar{r} = \int (2x+y^2)\,i + (3y-4x)\,dy$

$\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$



Along AB $y = 0$, $dy = 0$

x varies from 0 to 2

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$x = 2y$

$= \int_0^2 (2x)\,dx + (-4x)0$

$= \int_0^2 2x\,dx = [x^2]_0^2 = 4$

Along BC $\quad x = 2$, $dx = 0$ $\quad$ y varies from 0 to 1

$= \int_0^1 (4+y^2)(0) + (3y-8)\,dy$

$= \int_0^1 (3y-8)\,dy = \left(\frac{3y^2}{2} - 8y\right)_0^1 = \frac{3}{2} - 8 = \frac{-13}{2}$

Along CA $\quad x = 2y$, $dx = 2dy$ $\quad$ y varies from 1 to 0

$\Rightarrow \int_1^0 (2(2y)+y^2)2dy + (3y-8y)dy$

$\Rightarrow \int_1^0 (4y+y^2)2dy - 5y\,dy$

$= \left[\frac{8y^2}{2} + \frac{2y^3}{3} - \frac{5y^2}{2}\right]_1^0$

$= 0 - \left[4 + \frac{2}{3} - \frac{5}{2}\right]$

$= 0 - \left[4 - \frac{11}{6}\right]$

$= 0 - \left[\frac{24-11}{6}\right]$

$= \frac{-13}{6}$

$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} + \int_{BC} + \int_{CA} = 4 - \frac{13}{2} - \frac{13}{6} = \frac{-28}{6}$

$\boxed{\int_C \bar{F}\cdot d\bar{r} = \frac{-14}{3}}$

Part - B

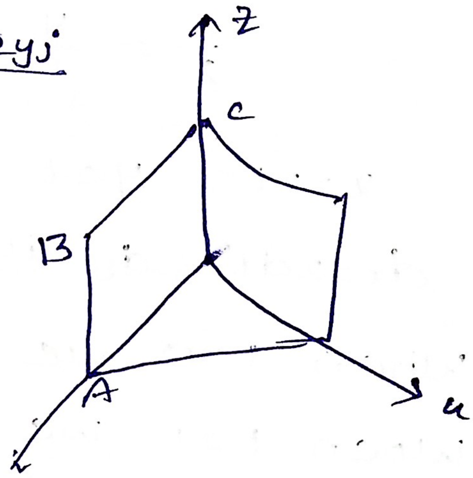
1) Evaluate $\iint_S \bar{F}\cdot ds$ if $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between $z=0$ and $z=2$

$\Rightarrow$ $x^2+y^2=9$

$\nabla\phi = 2xi + 2yj$

$\hat{n} = \dfrac{2xi+2yj}{\sqrt{4(x^2+y^2)}} = \dfrac{2xi+2yj}{6}$

$\hat{n} = \dfrac{xi+yj}{3}$



$\iint_S \bar{F}\cdot\hat{n}\,ds = \iint_R \bar{F}\cdot\hat{n}\,\dfrac{dx\,dz}{\hat{n}\cdot j}$

$I = \frac{1}{3}\iint_R (xyz + 2y^3)\,\dfrac{dx\,dz}{y/3}$

$= \iint_R (xz + 2y^2)\,dx\,dz$

$= \int_0^2\int_0^3 (xz + 2(9-x^2))\,dx\,dz$

$= \int_0^2 \left(\frac{9}{2}z + 18\times 3 - 2\times 9\right)dz$

$= \left[\frac{9}{2}\cdot\frac{z^2}{2} + 36z\right]_0^2$

$= 81$

2) Find the work done in moving a particle in the force field $\bar{F} = 3x^2i + (2xz-y)j + zk$ along the straight line from $(0,0,0)$ to $(2,1,3)$

$\Rightarrow$ $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

$d\bar{r} = dxi + dyj + dzk$

$\bar{F}\cdot d\bar{r} = 3x^2 dx + (2x3 - y)\, dy + 3\, d3$

eq^n of the straight line $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{3 - 3_1}{3_2 - 3_1}$

The eq^n of the line joining two points $(0,0,0)$ to $(2,1,3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{3-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{3}{3} = t$$

$x = 2t$, $y = t$; $3 = 3t$

$dx = 2dt$; $dy = dt$; $d3 = 3dt$

when $t = 0$ we get $(0,0,0)$

when $t = 1$ we get $(2,1,3)$

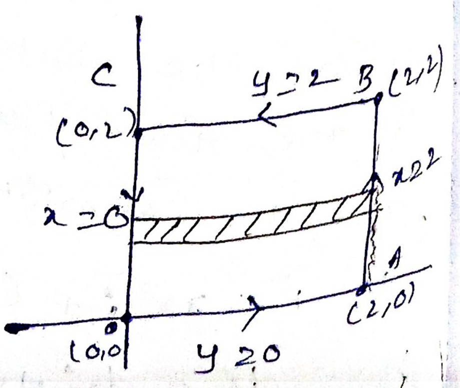
$$\int_C \bar{F}\cdot d\bar{r} = \int_0 3(4t^2)2dt + [2(2t)(33) - t)dt + 33(3dt)$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt = \int_0^1 (36t^2 + 8t)\, dt$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = 16 \text{ units}$$

6) verify Green's theorem in the plane for $\int_C (x^2 - xy)\, dx + (y^2 - 2xy)\, dy$ where C is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$

$\Rightarrow$ $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$



$M = x^2 - xy$; $N = y^2 - 2xy$

$\frac{\partial M}{\partial y} = -x$ $\qquad \frac{\partial N}{\partial x} = -2y$

$$R.H.S = \int_0^2 \int_0^2 (-2y + x)\, dx \cdot dy$$

$$= \int_0^2 \left(-2xy + \frac{x^2}{2}\right)_0^2 dy$$

$$= \int_0^2 (-4y + 2)\, dy$$

$$= \left(-\frac{4y^2}{2} + 2y\right)_0^2 = -2 \times 4 + 4$$

$$= -8 + 4 = -4$$

for L.H.S $= \oint_C M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Along OA $y = 0$, $dy = 0$ x varies from $0 \to 2$

$$= \int_0^2 x^2\, dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

dt)

Along AB $x = 2$, $dx = 0$ y varies from $0 \to 2$

$$= \int_0^2 (4 - 2y)(0) + (y^2 - 4y)\, dy$$

$$= \int_0^2 (y^2 - 4y)\, dy = \left[\frac{y^3}{3} - \frac{4y^2}{2}\right]_0^2$$

$$= \frac{8}{3} - 2 \times 4 = -\frac{16}{3}$$

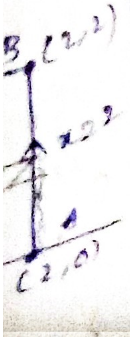
IY

2

Along BC $y = 2$, $dy = 0$ x varies from $2 \to 0$

$$= \int_2^0 (x^2 - 2x)\, dx + 0$$

$$= \left(\frac{x^3}{3} - x^2\right)_2^0 = -\left(\frac{8}{3} - 4\right) = -\left(-\frac{4}{3}\right) = 4/3$$



Along CO $x = 0$, $dx = 0$ y varies from $2 \to 0$

$$\int_2^0 y^2\, dy = \left(\frac{y^3}{3}\right)_2^0 = -8/3$$

$$\oint_C M\,dx + N\,dy = \frac{8}{3} - \frac{16}{3} + \frac{4}{3} - \frac{8}{3}$$

$$= -\frac{12}{3} = -4$$

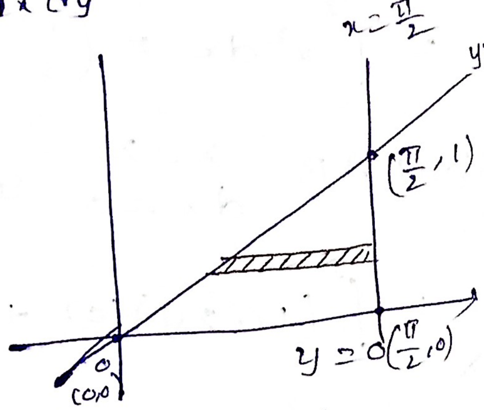
$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$-4 = -4$$

$\therefore$ Green's theorem verified.

7) Applying Green's theorem the plane evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where c is the plane triangle enclosed by $y = 0$, $y = \frac{2x}{\pi}$ and $x = \frac{\pi}{2}$

$\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$M = y - \sin x$ $\qquad$ $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = -\sin x.$

Limits

$x \rightarrow \frac{y\pi}{2}$ to $\pi/2$

$y \rightarrow 0$ to 1

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\pi/2} (-\sin x - 1)\,dx\,dy$$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\pi/2}\,dy$$

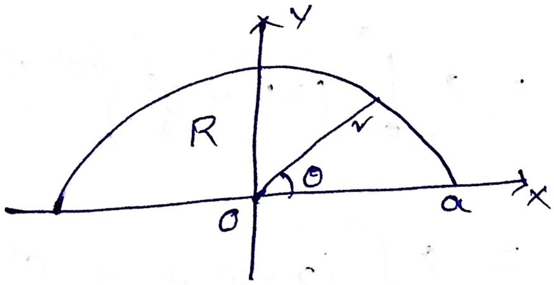
$$= \int_0^1 \left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\frac{y\pi}{2} - \frac{y\pi}{2}\right) dy$$

$$= \int_0^1 \left(0 - \frac{\pi}{2}\right) - \cos\frac{y\pi}{2} + y\frac{\pi}{2}\,dy$$

$$= \left(-\frac{\pi}{2}y - \frac{\sin y\pi/2}{\pi/2} + \frac{\pi}{2}\cdot\frac{y^2}{2}\right)^1$$

$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2} + \frac{\pi}{4} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

8) Apply Green's theorem in the plane for $\int_c (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where c is a boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.



$\Rightarrow M = 2x^2-y^2 \quad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$

By green's theorem

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

$$\int_c (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x+2y)\,dx\,dy$$

$$= 2\iint_R (x+y)\,dx\,dy$$

$x = r\cos\theta; \quad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

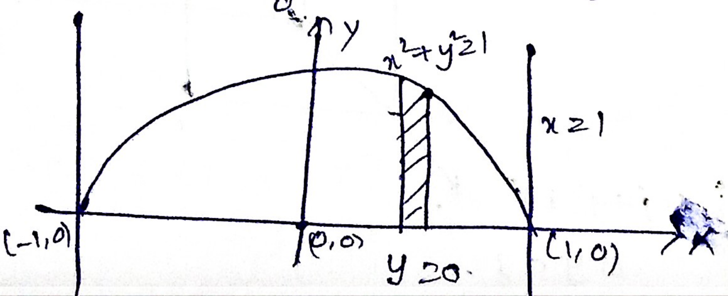
r varies from 0 to a and θ varies from 0 to π

$$\oint_c (2x^2-y^2)\,dx + (x^2+y^2)\,dy = 2\int_0^a\int_0^\pi (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_0^a r^2\,dr \int_0^\pi (\cos\theta+\sin\theta)\,d\theta$$

$$= 2\int_0^a r^2\,dr\,(\sin\theta - \cos\theta)_0^\pi$$

$$= 2\int_0^a r^2\,dr = \frac{4a^3}{3}$$



N' represented as :

$M = 2x^2 - y^2$ and $N = x^2 + y^2$

limits

$x \to -a$ to a

$y \to 0$ to $\sqrt{a^2 - x^2}$

$\frac{\partial M}{\partial y} = -2y$ $\quad \frac{\partial N}{\partial x} = 2x$

$$= \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (2x + 2y)\, dy\, dx$$

$$= 2 \int_{-a}^{a} \left(xy + \frac{y^2}{2} \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 2 \int_{-a}^{a} \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right) dx$$

The function is odd function

So, the value is zero

$\therefore$ we get $\quad 2\int_{-a}^{a} \frac{a^2-x^2}{2}\, dx = \left[\left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} \right]$

$$= \left[a^3 - \frac{a^3}{3} \right] - \left[-a^3 + \frac{a^3}{3} \right]$$

$$= + \frac{4a^3}{3}$$

$\therefore$ Hence proved

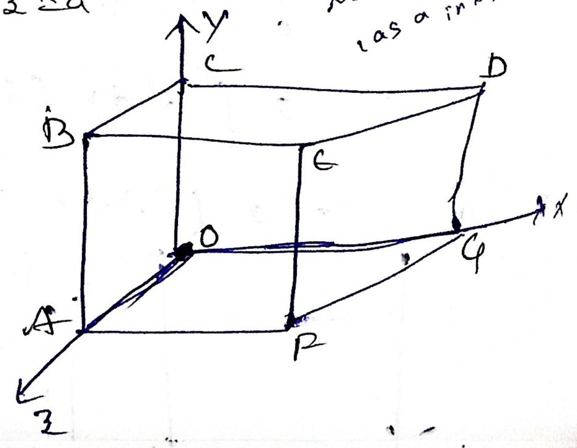
ii) If $\bar{F} = 4xz\,\bar{i} - y^2\bar{j} + yz\,k$ evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$.

where S is the surface of the cube $x = 0$,

$x = a$, $y = 0$, $y = a$, $z = 0$ $z = a$ 12

replace 1 as a in Ans

$\Rightarrow \bar{F} = 4xz\bar{i} - y^2\bar{j} + yzk$



On the face of DEFG

$n = +i \quad x = 1$

$\iint_{DEFG} \bar{F}\cdot\bar{n}\, ds = \int_0^1 \int_0^1 (4z\,i - y j + yz\,k)\cdot i$

$dy\, dz$

$\int_C$

$\int$

$=$

$$= \int_0^1\int_0^1 dy\, dz\, (4z) = 2$$

for face ABCO $n = -i$, $x = 0$

$$\iint_{ABCD} \bar{F}.n ds = \int_0^1\int_0^1 (-y^2 j + yzk)(-i) dy dz = 0$$

for face ABEF $n = j$, $y = 1$

$$\iint_{ABEF} F.n ds = \int_0^1\int_0^1 (4xz i - j + zk).j\, dx dz.$$

$$= \int_0^1\int_0^1 -dx dz = -1$$

face DGOC, $n = -j$, $y = 0$

$$\iint_{DGOC} \bar{F}.n ds = \int_0^1\int_0^1 (4xz) i (-j) dx dz = 0$$

face BCDE : $n = k$; $z = 1$ then

$$\int_0^1\int_0^1 (4xi - y^2 j + yk).k\, dx dy = \int_0^1\int_0^1 y\, dx dy = \frac{1}{2}$$

face AFGO $n = -k$ $z = 0$ then

$$\int_0^1\int_0^1 (-y^2 j).(-k) dx dy = 0$$

$$\iint_S F.n ds = 2 + 0 + (-1) + 0 + \frac{1}{2} + 0 = 3/2$$

12) If $F = (5xy - 6x^2)i + (2y - 4x)j$ evaluate $\int_C \bar{F}.d\bar{r}$. along the curve c in $y = x^3$ plane from $(1,1)$ to $(2,8)$

$\Rightarrow$ $F = (5xy - 6x^2)\bar{i} + (2y - 4x)j$ $\quad d\bar{r} = dx i + dy j$

given c is $y = x^3$ $\quad \bar{F}.d\bar{r} = (5xy - 6x^2)dx$
$dy = 3x^2 dx$ $\quad + (2y - 4x)dy$

Along c_1 x varies from $1 \to 2$

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 (5x^4-6x^2)\,dx + (2x^3-4x)\,3x^2\,dx$$

$$= \int_1^2 5x^4\,dx - 6x^2\,dx + 6x^5 - 12x^3\,dx$$

$$= \left(5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right)_1^2$$

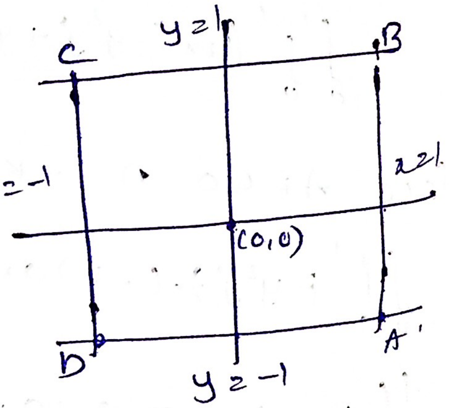
$$= \left(x^5 - 2x^3 + x^6 - 3x^4\right)_1^2$$

$$= (32-16+64-48) - (1-2+1-3)$$

$$= 32+3 = 35 \text{ units}$$

13) Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by lines $x=\pm1$, $y=\pm1$

$\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$



$M = x^2+xy$ $\qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = x$ $\qquad \frac{\partial N}{\partial x} = 2x$

Limits

x varies -1 to 1

y varies -1 to 1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_{-1}^{1}\int_{-1}^{1} (2x-x)\,dy\,dx = \int_{-1}^{1} x\,dx\,dy$$

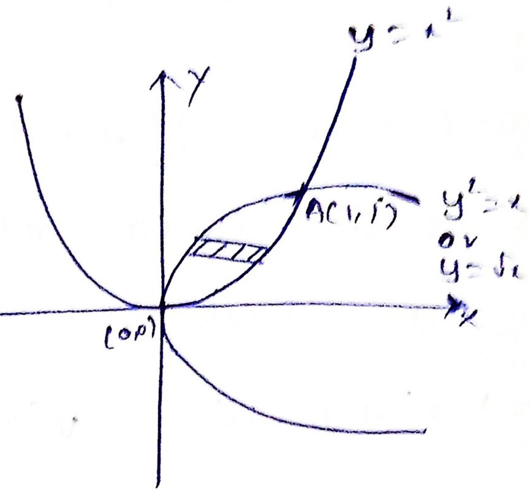
$$= \int_{-1}^{1} \left(\frac{x^2}{2}\right)_{-1}^{1} dy = 2\int_{-1}^{1} [1-1]\,dy$$

$\frac{1}{2} - \frac{(-1)^2}{2}$ $= 0.$

17) verify Green's theorem in the plane for

$\int_C (3x^2-8y^2)dx + (4y-6xy)dy$ where c is a region

bounded by $y=\sqrt{x}$ and $y=x^2$



$\Rightarrow \int_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = 3x^2-8y^2$ $\quad N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y$ $\quad \frac{\partial N}{\partial x} = -6y$

$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

x varies from $y^2 \to \sqrt{y}$

y varies from $0 \to 1$

$= \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y+16y)\,dx\,dy$

$= \int_0^1 \int_{y^2}^{\sqrt{y}} (10y)\,dx\,dy$

$= \int_0^1 (10xy)_{y^2}^{\sqrt{y}}\,dy$

$= 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1 = 10\left(\frac{2}{5}-\frac{1}{4}\right) = \frac{3}{2}$

$\int Mdx+Ndy = \int_{OA} + \int_{AO}$

Along OA $y=x^2 \Rightarrow dy = 2x\,dx$ x varies from 0 to 1

$\int_0^1 (3x^2-8x^4)dx + (4x^2-6x^3)(2x)\,dx = -1$

Along AO, $x=y^2 \Rightarrow dx = 2y\,dy$ y varies from 1 to 0

$\int_1^0 (3y^4-8y^2)2y\,dy + (4y-6y^3)\,dy = 5/2$

$\int M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

L.H.S = R.H.S

Hence Green's theorem is verified.

19) Evaluate $\iint_S \bar{A}\cdot\hat{n}\,ds$ where $\bar{A} = 3\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and s is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

$\Rightarrow x^2+y^2=16 \qquad \nabla(x^2+y^2) = 2x\hat{i}+2y\hat{j}$

$\hat{n}$ = unit vector normal to surface S at any point

$$(x,y,3) = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$x^2+y^2=16$, therefore $\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4(x^2+y^2)}} = \frac{2x\hat{i}+2y\hat{j}}{8}$

$$= \frac{x}{4}\hat{i} + \frac{y}{4}\hat{j}$$

$$\iint_S \bar{A}\cdot\hat{n}\,ds = \iint_R \bar{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} \qquad \hat{n}\cdot\hat{j} = y/4$$

$$= \iint_R \left(\frac{x3}{4} + \frac{xy}{4}\right)\frac{dx\,dz}{y/4}$$

$$= \iint_R \left(\frac{x3}{y} + x\right) dx\,dz$$

$$= \int_0^5\int_0^4 \left(\frac{x3}{\sqrt{16-x^2}} + x\right) dx\,dz$$

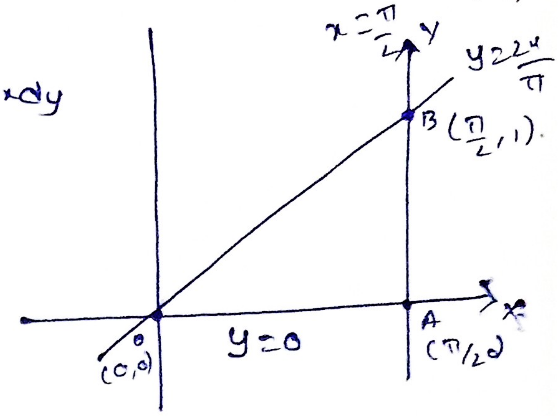
$$= \int_0^5\int_0^4 \left(\frac{-\frac{3}{2}(-2x)}{\sqrt{16-x^2}} + x\right) dx\,dz$$

$$= \int_0^5 \left[-\frac{3}{2}\cdot\frac{\sqrt{16-x^2}}{1/2} + \frac{x^2}{2}\right]_0^4 dz$$

$$= \int_0^5 (4z+8)\,dz = \left[\frac{4z^2}{2} + 8z\right]_0^5$$

20. Evaluate by Green's theorem $\int(y - \sin x)\,dx + \cos x\,dy$ where 'c' is the triangle enclosed by the lines $y = 0$; $x = \pi/2$ $\pi y = 2x$ (or) $y = 2x/\pi$

$$\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$M = y - \sin x$ $\qquad$ $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = -\sin x$

$x \to \frac{y\pi}{2}$ to $\frac{\pi}{2}$

$y \to 0$ to 1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\pi/2} (-\sin x - 1)\,dx\,dy$$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\,dy$$

$$= \int_0^1 \left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\frac{y\pi}{2} - \frac{y\pi}{2}\right) dy$$

$$= \int_0^1 \left(0 - \frac{\pi}{2}\right) - \cos\frac{y\pi}{2} + y\frac{\pi}{2}$$

$$= \left(-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\pi/2} + \frac{\pi}{2}\cdot\frac{y^2}{2}\right)_0^1$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4} = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$