Shortcuts / Notations

① P.D → Partial differentiation.

P.D.E → Partial differential equation

wr to → with respect to.

② $\frac{\partial z}{\partial x} = p$ $\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial y^2} = t$ $\frac{\partial^2 z}{\partial x \partial y} = s$

③ $f'(x)$ → 1st order differentiation

$f''(x)$ → 2nd order differentiation

MODULE - III

CIE - I

PART - A

① $lx + my + nz = \phi(x^2 + y^2 + z^2)$ — ①

Partially differentiate ① w r to 'x'.

$$l + n \cdot \frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2) \cdot \left(2x + 2z \frac{\partial z}{\partial x}\right).$$

$$l + n\,p = \phi'(x^2+y^2+z^2)(2x + 2zp) \quad \text{—} ②$$

Partially differentiate ① w r to 'y'.

$$m + n \cdot q = \phi'(x^2+y^2+z^2)(2y + 2zq) \quad \text{—} ③$$

② / ③ $\Rightarrow$ $$\frac{l+np}{m+nq} = \frac{x+zp}{y+zq}$$

$\Rightarrow (l+np)(y+zq) = (m+nq)(x+zp)$

$\Rightarrow (l+np)y + z(lq - mp) = (m+nq) \cdot x$

is our required partial differential equation.

(2) $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ — (1)

Partially differentiate (1) w r to x

$$y + yp + z + xp = f'\left(\frac{z}{x+y}\right)\left(\frac{(x+y)p - z}{(x+y)^2}\right)$$

$$(y+z) + p(y+x) = f'\left(\frac{z}{x+y}\right)\left(\frac{(x+y)p - z}{(x+y)^2}\right)$$

$$\Rightarrow \frac{(x+y)^2}{(x+y)p - z}\left(p(x+y) + (y+z)\right) = f'\left(\frac{z}{x+y}\right) \quad \text{— (2)}$$

Partially differentiate (1) w r to y.

$$x + yq + z + xq = f'\left(\frac{z}{x+y}\right)\left(\frac{(x+y)q - z}{(x+y)^2}\right)$$

$$(x+z) + q(x+y) = f'\left(\frac{z}{x+y}\right)\left(\frac{(x+y)q - z}{(x+y)^2}\right)$$

$$\Rightarrow \frac{(x+y)^2}{(x+y)q - z}\left(q(x+y) + (x+z)\right) = f'\left(\frac{z}{x+y}\right) \quad \text{— (3)}$$

~~(2) & (3)~~ Compare (2) & (3)

$$\frac{\cancel{(x+y)^2}}{(x+y)p-z}\left[p(x+y)+(y+z)\right]=\frac{\cancel{(x+y)^2}}{(x+y)q-z}\left[(x+z)+q(x+y)\right]$$

$$\Rightarrow \left[p(x+y)+(y+z)\right]\left[(x+y).q-z\right]=\left[(x+z)+q(x+y)\right]\left[(x+y)p-z\right]$$

$$\Rightarrow z(x+y)(q-p)+(x+y)\left[(y+z)q-(z+x)p\right]=z(x-y)$$

$$\Rightarrow (p-q)z(x+y)+(x+y)\left[p(z+x)-q(y+z)\right]=z(y-x)$$

$$\Rightarrow (x+y)\left[p(x+2z)-q(y+2z)\right]=z(y-x)$$

is our required partial differential equation.

(3) $f(x^2-y^2, x^2-z^2)=0$

let; $x^2-y^2=u$ and $x^2-z^2=v$.

$f(u,v)=0$ — (1)

Partially differentiate (1) w r to x.

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]=0$$

i.e; $\frac{\partial f}{\partial u}(2x-0)+\frac{\partial f}{\partial v}\left(2x-2z.\frac{\partial z}{\partial x}\right)=0$

$x.\frac{\partial f}{\partial u}+(x-pz)\frac{\partial f}{\partial v}=0$ — (2)

Differentiating (1) partially wr to y, we get,

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=0$$

$$\frac{\partial f}{\partial u}\left[\underline{-2y}+0\right]+\frac{\partial f}{\partial v}\left[0-2z\cdot\frac{\partial z}{\partial y}\right]=0$$

$$y\frac{\partial f}{\partial u}+qz\frac{\partial f}{\partial v}=0 \quad -(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between (2) and (3), we have,

$$\begin{vmatrix} x & x-pz \\ y & qz \end{vmatrix}=0$$

$$xqz-(xy-ypz)=0$$

$$xqz-xy+ypz=0$$

$$z(xq+yp)=xy$$

$$xq+yp=\frac{xy}{z}$$ is our required

Partial differential equation.

(4) $z = f(x+ct) + g(x-ct)$. — (1)

P. D (1) w r to 'x' and 't'.

$$\frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct) \quad \text{— (2)}$$

$$\frac{\partial z}{\partial t} = c\, f'(x+ct) - c\, g'(x-ct) \quad \text{— (3)}$$

Take 'c' common.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad \text{— (4)}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(f''(x+ct) - g''(x-ct)\right) \quad \text{— (5)}$$

(4) in (5)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

is our required partial differential equation.

(5) $z = f(x - iy) + g(x + iy)$ — (1)

P.D (1) w r to "x"

$$\frac{\partial z}{\partial x} = f'(x+iy) \cdot 1 + g'(x+iy) \cdot 1.$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+iy) \cdot 1 + g''(x+iy) \cdot 1 \quad \text{— (2)}$$

P.D (1) w r to "y"

$$\frac{\partial z}{\partial y} = f'(x+iy) \cdot i + g'(x+iy) \cdot i.$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+iy) \cdot i^2 + g''(x+iy) \cdot i^2 \quad \text{— (3)}$$

Substitute (2) in (3)

from (3) $\quad \dfrac{\partial^2 z}{\partial y^2} = i^2 \left(\dfrac{\partial^2 z}{\partial x^2} \right) \qquad i^2 = -1$

$$\frac{\partial^2 z}{\partial y^2} = - \frac{\partial^2 z}{\partial x^2}$$

$\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial^2 z}{\partial x^2} = 0$ is our required PDE

$t + r = 0$

PART-B CIE-I

① ~~f(xy~~ $f(x^2+y^2+z^2,\ z^2-2xy)=0$

let; $x^2+y^2+z^2=u$; $z^2-2xy=v$

$f(u,v)=0$ — ①

P.D ① w r to x.

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]=0$$

$$\frac{\partial f}{\partial u}\left[2x+2z\cdot p\right]+\frac{\partial f}{\partial v}\left[-2y+2z\cdot p\right]=0 \quad —②$$

P.D ① w r to y.

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=0$$

$$\frac{\partial f}{\partial u}\left[2y+2z\cdot q\right]+\frac{\partial f}{\partial v}\left[-2x+2z\cdot q\right]=0 \quad —③$$

Eliminating $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$ between ② & ③,

we get

$$\begin{vmatrix} x+pz & pz-y \\ y+qz & qz-x \end{vmatrix}=0$$

$\Rightarrow (x+pz)(qz-x)-(pz-y)(y+qz)=0$

$xz(q-p) + yz(q-p) + (y^2 - x^2) = 0$

$z(q-p)(x+y) + (y^2 - x^2) = 0$ is our

required P.D.E.

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — (1)

P.D (1) w r to "x"

$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0$

$\frac{x}{a^2} + \frac{z}{c^2} \cdot p = 0$ — (2)

P.D (1) w r to "y"

$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0$

$\frac{y}{b^2} + \frac{z}{c^2} \cdot q = 0$ — (3)

differentiate (2) w r to x, we get

$\frac{1}{a^2} + \frac{p}{c^2} \underbrace{\frac{\partial z}{\partial x}}_{p} + \frac{z}{c^2} \cdot \frac{\partial p}{\partial x} = 0$

$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z}{c^2} \cdot r = 0$ — (4)

(4) $\times x$: $\Rightarrow \dfrac{x}{a^2} + \dfrac{p^2}{c^2}\cdot x + \dfrac{z}{c^2} r \cdot x = 0$ — (5)

(5) - (2) $\Rightarrow \dfrac{1}{c^2}(xp^2 + xzr - zp) = 0$

$\Rightarrow xp^2 + xzr = zp$ is our

required P.D.E.

(3) $z = ax^3 + by^3$ — (1)

P.D (1) w r to x.

$\dfrac{\partial z}{\partial x} = 3ax^2 + 0 \Rightarrow p = 3ax^2$ — (2)

$p/3x^2 = a$

P.D (1) w r to y

$\dfrac{\partial z}{\partial y} = 0 + 3by^2 \Rightarrow q = 3by^2$ — (3)

$q/3y^2 = b$

Substitute (2) & (3) in (1)

$z = \dfrac{p}{3\cancel{x^2}}\cdot x^{\cancel{3}} + \dfrac{q}{3\cancel{y^2}}\cdot y^{\cancel{3}}$

$z = \dfrac{px + qy}{3} \Rightarrow 3z = px + qy$ is our

required P.D.E.

(4) $(x-h)^2 + (y-k)^2 + z^2 = a^2$ — (1)

P.D. (1) w.r.to x, we get,

$\Rightarrow 2(x-h) + 2z \cdot \frac{\partial z}{\partial x} = 0$

$2(x-h) + 2zp = 0$

$\cancel{2}(x-h) = -zp$ — (2)

P.D. (1) w.r.to y, we get,

$\Rightarrow 2(y-k) + 2z \cdot \frac{\partial z}{\partial y} = 0$

$2(y-k) + 2zq = 0$

$(y-k) = -zq$ — (3)

Substitute (2) & (3) in (1)

$z^2p^2 + z^2q^2 + z^2 = a^2$ is our

required P.D.E.

(5) $z = f(x) + e^y g(x)$ — (1)

P.D (1) w r to x.

$$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x) \quad \text{— (2)}$$

P.D (1) w r to y.

$$\frac{\partial z}{\partial y} = 0 + e^y g(x) \quad \text{— (3)}$$

$$q = e^y g(x) \quad \text{— (4)}$$

P.D (3) w r to y.

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x).$$

$$t = e^y g(x) \quad \text{— (5)}$$

Compare (4) & (5).

$$q = t$$

$\Rightarrow q - t = 0$ (or) $t - q = 0$ is our

required P.D.E.

(6) Centre lies on z-axis with radius 'r'

Sphere $\rightarrow$ $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

As per the question

$\Rightarrow$ $(x-0)^2+(y-0)^2+(z-c)^2=r^2$

$\Rightarrow$ $x^2+y^2+(z-c)^2=r^2$ — (1)

P. D. (1) w r to x

$\Rightarrow$ $2x+2(z-c)\cdot\frac{\partial z}{\partial x}=0$

$\Rightarrow$ $x+(z-c)p=0$

$\Rightarrow$ $z-c=-\frac{x}{p}$ — (2)

P. D (1) w r to y

$\Rightarrow$ $2y+2(z-c)\cdot\frac{\partial z}{\partial y}=0$

$\Rightarrow$ $y+(z-c)\cdot q=0$

$\Rightarrow$ $z-c=-\frac{y}{q}$ — (3)

compare (2) & (3)

$\Rightarrow$ $\cancel{-}\frac{x}{p}=\cancel{-}\frac{y}{q}$

$yp-xq=0$
(or)

$\Rightarrow$ $xq=yp$ $\Rightarrow$ $xq-yp=0$ is our required PDE.

(7) $z = xy + f(x^2+y^2)$ — (1)

P. D (1) w r to x

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2) \cdot 2x$$

$$p - y = 2x \cdot f'(x^2+y^2) \quad — (2)$$

P. D (1) w r to y.

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2) \cdot 2y$$

$$q - x = 2y \cdot f'(x^2+y^2) \quad — (3)$$

(2) / (3) $\Rightarrow$ $$\frac{p-y}{q-x} = \frac{x}{y}$$

$$py - y^2 = qx - x^2$$

$$x^2 - y^2 = qx - py$$ is our

required P. D. E

(8) Repeated — Part B 1Q

(9) $\log(az-1) = x + ay + b$ — (1)

P.D (1) w r to x.

$$\frac{1}{az-1} \cdot a \cdot \frac{\partial z}{\partial x} = 1$$

$$\frac{1}{az-1} \cdot ap = 1. \quad \Rightarrow ap = az-1 \quad \text{— (2)}$$

P.D (1) w r to y.

$$\frac{1}{az-1} \cdot a \cdot \frac{\partial z}{\partial y} = a$$

$$\Rightarrow aq = a(az-1) \quad \text{— (3)}$$

(3)/(2) $\frac{q}{p} = \frac{a}{1}$

$$ap = q. \quad \text{— (4)}$$

(4) in (2) $\Rightarrow q = \frac{q}{p}z - 1.$

$$pq = qz - p$$ is our

required P.D.E.

(10) $xyz = f(x^2+y^2+z^2)$ — (1)

P.D (1) w r to x, we get

$$yz + xyp = (2x+2zp)\, f'(x^2+y^2+z^2) \quad \text{— (2)}$$

P.D (1) w r to y, we get

$$xz + xyq = (2y+2zq)\, f'(x^2+y^2+z^2) \quad \text{— (3)}$$

from (2) & (3)

$$\frac{yz+xyp}{xz+xyq} = \frac{2x+2zp}{2y+2zq}$$

By simplification, we get

$z(x^2-y^2) + z^2(px-qy) - xy(py-qx) = 0$ is

our required PDE.

verified !

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MODULE - III

CIE - II

PART - A

IOBS → Integrating on both sides.

(7) $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ → Given

Form: $Pp + Qq = R$

where; $P = \sqrt{x}$; $Q = \sqrt{y}$; $z = \sqrt{z}$

Langrange's Auxillary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

I.O.B.S

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$2\sqrt{x} = 2\sqrt{y} + c_1$$

$$c_1 = \sqrt{x} - \sqrt{y}.$$

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

IOBS

$$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{y} = 2\sqrt{z} + c_2$$

$$c_2 = \sqrt{y} - \sqrt{z}$$

∴ The general solution is $f(c_1, c_2) = 0$

i.e $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$ ✓

(6) Given; $(z^2 - 2yz - y^2)p + (xy + xz)q = (xy - zx)$

form: $Pp + Qq = R$

where; $P = z^2 - 2yz - y^2$

$Q = xy + xz = x(y+z)$

$R = xy - zx = x(y-z)$.

Langrange's subsidiary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

Let; $\frac{dy}{\cancel{x}(y+z)} = \frac{dz}{\cancel{x}(y-z)}$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$(y-z)dy = (y+z)dz$

$\int y\,dy - \int d(yz) - \int z\,dz = 0$.

$$\boxed{\frac{y^2}{2} - yz - \frac{z^2}{2} = c_1}$$

Consider; x, y, z as multipliers

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$\Rightarrow$ Each Fraction $= \dfrac{x\,dx + y\,dy + z\,dz}{x(z^2 - 2yz - y^2) + y(x(y+z)) + z(x(y-z))}$

As per method of multipliers, Here denominator $= 0$. Therefore, numerator $= 0$

$\therefore\ x\,dx + y\,dy + z\,dz = 0$

I O B S

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{x^2 + y^2 + z^2 = c_2}$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e

$$f(y^2 - 2yz - z^2,\ x^2 + y^2 + z^2) = 0.$$

(8) Given; $\frac{p}{x^2} + \frac{q}{y^2} = z$

$$\Rightarrow \left(\frac{1}{x^2}\right)\cdot p + \left(\frac{1}{y^2}\right)\cdot q = z$$

form: $Pp + Qq = R$

where, $P = 1/x^2$; $Q = 1/y^2$; $R = z$

Langrange's subsidiary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx \times x^2}{1} = \frac{dy \times y^2}{1} = \frac{dz}{z}$$

$$x^2\,dx = y^2\,dy = \frac{1}{z}\,dz.$$

$x^2\,dx = y^2\,dy$

IOBS

$\frac{x^3}{3} = \frac{y^3}{3} + c_1$

$x^3 - y^3 = c_1$

$y^2\,dy = \frac{1}{z}\,dz$

IOBS

$\frac{y^3}{3} = \log z + c_2.$

$\frac{y^3}{3} - \log z = c_2$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e,

$$f\left(x^3 - y^3, \frac{y^3}{3} - \log z\right) = 0.$$

(9) Given; $xp + yq = 1$

form : $Pp + Qq = R$

where; $P = x$; $Q = y$; $R = 1$

Langrange's subsidiary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{1}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

IOBS

$$\log x = \log y + \log c_1$$

$$\frac{x}{y} = c_1$$

$$\frac{dy}{y} = \frac{dz}{1}$$

IOBS

$$\log y = z + c_2$$

$$\log y - z = c_2$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e,

$$f\left(\frac{x}{y}, \log y - z\right) = 0$$

(10) Given; $y^2 z p + x^2 z q = x y^2$

form: $Pp + Qq = R$

where; $P = y^2 z$; $Q = x^2 z$; $R = x y^2$

Langrange's subsidiary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{x y^2}$$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$x^2\, dx = y^2\, dy$$

I O B S

$$\frac{x^3}{3} = \frac{y^3}{3} + c_1$$

$$x^3 - y^3 = c_1$$

$$\frac{dx}{\cancel{y^2} z} = \frac{dz}{x \cancel{y^2}}$$

$$x\, dx = z\, dz$$

I O B S

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = c_2$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e

$$f(x^3 - y^3,\ x^2 - z^2) = 0$$

PART-B

(11) Given; $px^2 + qy^2 = z(x+y)$

(16) form: $Pp + Qq = R$

$P = x^2$; $Q = y^2$; $R = z(x+y)$

Langrange's subsidiary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$$

(i) $\dfrac{dx}{x^2} = \dfrac{dy}{y^2}$

I O B S

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\boxed{\frac{1}{x} - \frac{1}{y} = c_1}$$

(ii) choose $1, -1, 0$ as multipliers.

~~d~~ Then,

Each Fraction $= \dfrac{1\,dx - 1.dy + 0}{1.x^2 - 1.y^2 + 0}$

$$\Rightarrow \frac{dx - dy}{x^2 - y^2} = \frac{dx - dy}{\cancel{(x+y)(x-y)}\ (x+y)(x-y)}$$

Now;

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)} = \frac{dx-dy}{(x+y)(x-y)}$$

$$\frac{dz}{z(\cancel{x+y})} = \frac{d(x-y)}{(\cancel{x+y})(x-y)}$$

$$\frac{1}{z}\,dz = \frac{1}{x-y}\,d(x-y)$$

I.O.B.S

$$\log z = \log(x-y) + \log c_2$$

$$\boxed{c_2 = \frac{z}{x-y}}$$

∴ The general solution is given by

$$f(c_1, c_2) = 0 \quad \text{i.e} \quad f\left(\frac{1}{x} - \frac{1}{y}, \frac{z}{x-y}\right) = 0$$

(12) $p - x^2 = y^2 + q$ → Given

$p - q = y^2 + x^2$

$1 \cdot p - 1 \cdot q = y^2 + x^2$

form: $Pp + Qq = R$

where; $P = 1$; $Q = -1$; $R = y^2 + x^2$.

Langrange's subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{y^2 + x^2}.$$

$x^2, -y^2, 1$

(i) $\frac{dx}{1} = \frac{-dy}{1}$

I.O.B.S we get; $x = -y + c_1$

$\boxed{x + y = c_1}$

(ii) choose $-x^2, y^2, 1$ as multiplies.

∴ Each Fraction $= \dfrac{-x^2 dx + y^2 dy + 1 \cdot dz}{-x^2 \cdot 1 - 1 \cdot y^2 + 1 \cdot (y^2 + x^2)}$

we observe that, denominator $= 0$, i.e

$-x^2 - y^2 + y^2 + x^2 = 0$

As per method of multipliers, if denominator $= 0$, then numerator $= 0$ i.e,

$$-x^2 dx + y^2 dy + 1 \cdot dz = 0$$

I.O.B.S

$$\boxed{-\frac{x^3}{3} + \frac{y^3}{3} + z = c_2}$$

$$\boxed{-x^3 + y^3 + 3z = c_2}$$

~~(or)~~

$$\cancel{x^3 - y^3 - 3z = c_2}$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e

$$f(x+y, -x^3 + y^3 + 3z) = 0$$

(or)

The general solution is $f(c_1, c_2) = 0$ i.e

$$f\left(x+y, -\frac{x^3}{3} + \frac{y^3}{3} + z\right) = 0$$

(13) Given: $y^2 p + x^2 q = x^2 y^2 z^2$

form: $Pp + Qq = R$

where; $P = y^2$; $Q = x^2$, $R = x^2 y^2 z^2$

Langrange's subsidiary equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$$

$$\frac{dx}{\cancel{y^2}} = \frac{dz}{x^2 \cancel{y^2} z^2}$$

$$x^2 dx = \frac{1}{z^2} dz$$

I OBS

$$\frac{x^3}{3} = -\frac{1}{z} + c_1$$

$$\boxed{\frac{x^3}{3} + \frac{1}{z} = c_1}$$

$$\frac{dy}{\cancel{x^2}} = \frac{dz}{x^2 y^2 z^2}$$

$$y^2 dy = \frac{1}{z^2} dz$$

I OBS.

$$\frac{y^3}{3} = -\frac{1}{z} + c_2$$

$$\boxed{\frac{y^3}{3} + \frac{1}{z} = c_2}$$

$\therefore$ The General Solution is $f(c_1, c_2) = 0$

i.e $f\left(\frac{x^3}{3} + \frac{1}{z}, \frac{y^3}{3} + \frac{1}{z}\right) = 0$

(14) Given; $p \tan x + q \tan y = \tan z$.

form: $Pp + Qq = R$

$P = \tan x$; $Q = \tan y$; $R = \tan z$

Langrange's subsidiary equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$\cot x \, dx = \cot y \, dy$

IOBS

$\log|\sin x| = \log|\sin y| + \log c_1$

$$\boxed{c_1 = \frac{\sin x}{\sin y}}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$\cot y \, dy = \cot z \, dz$

IOBS

$\log|\sin y| = \log|\sin z| + \log c_2$

$$\boxed{c_2 = \frac{\sin y}{\sin z}}$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$

i.e; $$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

(15) Given ; $(x-a)p + (y-b)q + (c-z) = 0$

$\Rightarrow (x-a)p + (y-b)q = (z-c)$

$\Rightarrow Pp + Qq = R$

where ; $P = x-a$; $Q = y-b$; $R = z-c$,

Langrange's subsidiary equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

$$\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{dz}{z-c},$$

$$\frac{dx}{x-a} = \frac{dy}{y-b}.$$

I.O.B.S

$$\int \frac{dx}{x-a} = \int \frac{dy}{y-b}$$

$$\log(x-a) = \log(y-b) + \log c_1$$

$$\log \frac{(x-a)}{(y-b)} = \log c_1$$

$$c_1 = \frac{x-a}{y-b}$$

$$\frac{dy}{y-b} = \frac{dz}{z-c}$$

I.O.B.S.

$$\int \frac{dy}{y-b} = \int \frac{dz}{z-c}$$

$$\log(y-b) = \log(z-c) + \log c_2$$

$$\log \frac{(y-b)}{(z-c)} = \log c_2$$

$$c_2 = \frac{y-b}{z-c}.$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$

i.e $f\left(\frac{x-a}{y-b}, \frac{y-b}{z-c}\right) = 0.$

(18)

(17) Given; $pz - qz = z^2 + (x+y)^2$.

form : $Pp + Qq = R$

$P = z;\ Q = -z\ ;\ R = z^2 + (x+y)^2$

Langrange's subsidiary equation.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{z} = -\frac{dy}{z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\frac{dx}{\cancel{z}} = -\frac{dy}{\cancel{z}}$$

I OBS

$x = -y + c_1$

$\boxed{x + y = c_1}$

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\frac{dx}{1} = \frac{1}{z^2 + c_1^2}\, z\, dz$$

$\left(\int \frac{f'(x)}{f(x)} = \log f(x)\right)$

I.O.B.S

$x = \log(z^2 + c_1^2) + c_2$

$\boxed{c_2 = x - \log(z^2 + (x+y)^2)}$

$\therefore$ The general solution is $f(c_1, c_2) = 0$

i.e $f\left(x+y,\ x - \log(z^2 + (x+y)^2)\right) = 0$

(18) Given; $\frac{y^2 z}{x} p + xzq = y^2$

$$\frac{y^2 z p + x^2 z q}{x} = y^2$$

$$y^2 z p + x^2 z q = xy^2$$

form: $Pp + Qq = R$

$P = y^2 z$; $Q = x^2 z$; $R = xy^2$

Langrange's subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

(i) $\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$ $\rightarrow$ solving, we get (Refer Part-A Q10)

$$\boxed{c_1 = x^3 - y^3}$$

(ii) $\frac{dx}{y^2 z} = \frac{dz}{xy^2}$ $\rightarrow$ solving, we get

$$\boxed{c_2 = x^2 - z^2}$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e

$f(x^3 - y^3, x^2 - z^2) = 0$.

(19) Given; $x(y^2-z^2)p + y(z^2+x^2)q = z(x^2+y^2)$

form, $Pp + Qq = R$

where; $P = x(y^2-z^2)$

$Q = -y(z^2+x^2)$

$R = z(x^2+y^2)$

Langrange's subsidiary equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

By method of multipliers

(i) choose x, y, z as multipliers

$\therefore$ Each Fraction $= \dfrac{x\,dx + y\,dy + z\,dz}{x(x(y^2-z^2)) + y(-y(z^2+x^2)) + z(z(x^2+y^2))}$

Here; denominator $= 0$. So, as per method of multipliers, Numerator should be equal to zero i.e

$x\,dx + y\,dy + z\,dz = 0$

I. O. B S

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$\boxed{x^2 + y^2 + z^2 = c_1}$$

(ii) Choose $1/x, -1/y, -1/z$ as multipliers

$\therefore$ Each fraction $= \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{\frac{1}{x}\cdot x(y^2 - z^2) - \frac{1}{y}(-y(z^2+x^2)) - \frac{1}{z}\cdot(z(x^2+y^2))}$

Here, denominator $= 0$. so, as per method of multipliers, Numerator should be equal to zero i.e,

$$\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$$

I. O. B. S

$$\log x - \log y - \log z = \log c_2$$

$$\log\left(\frac{x}{yz}\right) = \log c_2$$

$$\boxed{c_2 = x/yz}$$

$\therefore$ The general solution is $f(c_1, c_2) = 0$

i.e, $f(x^2 + y^2 + z^2,\ x/yz) = \underline{0}$.

(20) Given; $(x-y)p + (y-x-z)q = z$

form: $Pp + Qq = R$

$P = x-y$; $Q = y-x-z$, $R = z$

Langrange's Subsidiary equation;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$$

(i) Choose 1, 1, 1 as multipliers,

$\therefore$ Each Fraction $= \dfrac{1.dx + 1.dy + 1.dz}{1(x-y) + 1(y-x-z) + z}$

Here, denominator = 0. So, as per method of multipliers, numerator should be equal to zero i.e; $1.dx + 1.dy + 1.dz = 0$

I.O.B.S

$$\boxed{x + y + z = c_1}$$

~~(ii) Choose 1, 1, 0 as multipliers~~

$\Rightarrow$ ~~Each~~

(ii) Choose 1, 1, 0 as multipliers

$$\Rightarrow \text{Each Fraction} = \frac{dx + dy}{x-y+y-x-z} = \frac{-d(x+y)}{z}$$

$$\therefore \frac{dx}{x-y} = \frac{dy}{y-x-z} = \underbrace{\frac{dz}{z} = \frac{-d(x+y)}{z}}$$

Consider; $\dfrac{dx - dy + dz}{x-y-y+x+z+z}$

$$\frac{dx - dy + dz}{2(x-y+z)} = \frac{d(x-y+z)}{2(x-y+z)}$$

Now; $\dfrac{d(x-y+z)}{2(x-y+z)} = \dfrac{dz}{z}$

$$\frac{d(x-y+z)}{(x-y+z)} = 2 \cdot \frac{dz}{z}$$

$$\ln(x-y+z) = \ln(z^2) + \ln c_2$$

$$\boxed{\frac{x-y+z}{z^2} = c_2}$$

Verified!

$\therefore$ The general solution is $f(c_1, c_2) = 0$ i.e

$$f\left(x+y+z, \frac{x-y+z}{z^2}\right) = 0$$

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