MODULE - II

RANDOM VARIABLES.

PART - A

① Given ; $f(x) = \begin{cases} 3x^2 ; & 0 < x < 1 \\ 0 ; & \text{otherwise.} \end{cases}$

To find ; $a = ?$ If $P(a \le x \le 1) = \frac{19}{81}$

Given; $P(a \le x \le 1) = \frac{19}{81}$

$$\int_a^1 f(x)\, dx = \frac{19}{81}$$

$$\int_a^1 3x^2\, dx = \frac{19}{81}$$

$$3 \cdot \left[\frac{x^3}{3}\right]_a^1 = \frac{19}{81}$$

$$\cancel{3}\left[\frac{1-a^3}{\cancel{3}}\right] = \frac{19}{81} \Rightarrow 1 - a^3 = \frac{19}{81}$$

$$1 - \frac{19}{81} = a^3$$

$$\frac{62}{81} = a^3 \Rightarrow \boxed{a = 0.914}$$

(2) Given;

$$f(x) = \begin{cases} \frac{1}{9} x e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Total Production = 12 million KW-hours.

To find : $P(x > 12) = ?$ - Power cut/ shortage.

Now; we need to find

$$P(0 \leq x \leq 12) = \int_0^{12} f(x)\, dx = \int_0^{12} \frac{1}{9} x e^{-x/3}\, dx$$

$$= \frac{1}{9} \left\{ x \cdot \frac{e^{-x/3}}{-1/3} - 1 \cdot \frac{e^{-x/3}}{1/9} \right\}_0^{12}$$

$$= \frac{1}{9} \left\{ -36 e^{-4} - 9 e^{-4} - (0 - 9) \right\}$$

$$\Rightarrow \frac{1}{9} \left\{ 9 - 45 e^{-4} \right\} = 1 - 5e^{-4}$$

Now; $P(x > 12) = 1 - P(0 \leq x \leq 12)$

$$= 1 - (1 - 5e^{-4})$$

$$= 5e^{-4}$$

$$= 0.0915781$$

(3) If head occurs first time there will be only one toss. On the other hand, if first one is tail, second occurs. If head occurs there will be only two tosses. Suppose second one is also tail third occurs. If head occurs there will be three tosses and so on. Therefore

$P(1) = P(H) = 1/2$

$P(2) = P(TH) = 1/2 \cdot 1/2 = 1/4$

$P(3) = P(TTH) = 1/2 \cdot 1/2 \cdot 1/2 = 1/8$

$P(4) = P(TTTH) = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/16$

$P(5) = P(TTTTH) + P(TTTTT) = 1/32 + 1/32 = 1/16$

The probability distribution function of x is,

x	1	2	3	4	5
$P(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

Hence; $E(x) = \sum_i p_i x_i$

$$= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{16}\right)$$

$$\Rightarrow \frac{8+8+6+4+5}{16} = \frac{31}{16} = 1.9375$$

(4) Let x denote the twice the number appearing on the face when a die is thrown. Then x is a discrete random variable whose probability distribution is given by,

(i)

$X = x_i$	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Above table represents the probability distribution of x.

(ii) Mean $(\mu) = E(X) = \sum_i P_i x_i$

$$\mu = 2\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) + 8\left(\frac{1}{6}\right) + 10\left(\frac{1}{6}\right) + 12\left(\frac{1}{6}\right)$$

$$\mu = \frac{1}{6}(2+4+6+8+10+12) = \frac{42}{6} = 7$$

(iii) Varience $(\sigma^2) = ?$ $\sigma^2 = E(X^2) - (E(X))^2$

$$E(X^2) = \sum_i x_i^2 P_i$$

$$E(X^2) = 4\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) + 64\left(\frac{1}{6}\right) + 100\left(\frac{1}{6}\right) + 144\left(\frac{1}{6}\right)$$

$$E(X^2) = \frac{1}{6}(4+16+36+64+100+144) = \frac{364}{6} = 60.67$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$\sigma^2 = 60.67 - (7)^2 = 60.67 - 49$$

$$\sigma^2 = 11.67$$

(5) Given; $f(x) = Ke^{|-x|}$, $-\infty < x < \infty$ $(x \in R)$

we know that; $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

$$\int_{-\infty}^{\infty} K e^{|-x|} dx = 1$$

$$K \int_{-\infty}^{\infty} e^{|-x|} dx = 1.$$

Since $e^{|-x|}$ is an even function, so it can be written as $K \cdot 2 \int_{0}^{\infty} e^{-x} dx = 1.$

In $0 \leq x \leq \infty$, $|x| = x$.

$$2K \left(-e^{-x}\right)_0^{\infty} = 1$$

$$2K\left(-0-(-1)\right) = 1$$

$$2K = 1 \Rightarrow \boxed{K = \frac{1}{2}}$$

Hence; $f(x) = Ke^{|-x|} = \frac{1}{2} e^{|-x|}$

Mean $(\mu) = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \frac{1}{2} \int_{-\infty}^{\infty} x e^{|-x|} dx$

$= 0$

Since, integrand is odd

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-0)^2 f(x)\,dx.$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$$

$$= \frac{1}{2}\cdot 2\int_{0}^{\infty} x^2 e^{-|x|}\,dx, \text{ since integrand is even.}$$

$$= \int_{0}^{\infty} x^2 e^{-x}\,dx$$

$$= \left(x^2 \frac{e^{-x}}{-1} - 2x\cdot\frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$\Rightarrow (0-(-2)) = \underline{\underline{2}}.$$

$$P(0<X<4) = \frac{1}{2}\int_0^4 e^{-|x|}\,dx = \frac{1}{2}\int_0^4 e^{-x}\,dx$$

$\because 0<x<4 \rightarrow |x| = x$

$$= -\frac{1}{2}\left(e^{-x}\right)_0^4$$

$$= -\frac{1}{2}\left(e^{-4}-1\right)$$

$$= \frac{1}{2}\left(1-e^{-4}\right)$$

0.98).

$$= \underline{\underline{0.4908}}$$

(6) Given; $f(x) = Ax^2 ; 0 < x < 1$

To find; Value of 'A' = ?

we know that; $\int_0^1 f(x)\,dx = 1$

$$\int_0^1 Ax^2\,dx = 1$$

$\because$ A is constant.

$$A\int_0^1 x^2\,dx = 1$$

$$A\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$A\left(\frac{1}{3} - 0\right) = 1$$

$$\frac{A}{3} = 1$$

$$\boxed{A = 3}$$

$\therefore$ The value of 'A' = 3.

⑦ Given: $f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \underbrace{\int_{-\infty}^{0} x(0)\,dx}_{0} + \int_{0}^{\infty} x e^{-x}\,dx$

$\cancel{= \int_{-\infty}^{0} 0 \cdot f(x)}$

$= \left[-(x+1)e^{-x}\right]_0^{\infty}$

$= (-1) \lim_{x \to \infty} \frac{x+1}{e^{x}} + (0+1)$

$= 0 + 1$

$= 1$ ✓

(ii) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} x^2 \underbrace{f(x)}_{e^{-x}}\,dx$

$= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})\right]_0^{\infty}$

$= \left[-e^{-x}(x^2 + 2x + 2)\right]_0^{\infty}$

$= (-1) \lim_{x \to \infty} \frac{x^2 + 2x + 2}{e^{x}} + 2$

$\Rightarrow 0 + 2 = 2$ ✓

(iii) $Var(X) = E(X^2) - (E(X))^2 = 2 - (1)^2$

$= 2 - 1$

$= 1$ ✓

(8) Given: $E(X) = 10$; $V(X) = 1$.

To find, $E(2X(X+10)) = ?$

We know that; $V(X) = 1$

$$E(X^2) - (E(X))^2 = 1$$

$$E(X^2) = 1 + (10)^2 = 101$$

Now; $E(2X(X+10)) = E(2X^2 + 20X)$

$$= E(2X^2) + E(20X)$$

$$= 2E(X^2) + 20E(X)$$

$$= 2(101) + 20(10)$$

$$= 202 + 200$$

$$= 402$$

$$\boxed{\therefore E(2X(X+10)) = 402}$$

(a) Given, probability distribution

X	1	2	3	4	5	6	7	8
$P(X=x)$	2K	4K	6K	8K	10K	12K	14K	4K

(i) From the properties of probability;

$$\sum_i P(x_i) = 1$$

$\Rightarrow 2K + 4K + 6K + 8K + 10K + 12K + 14K + 4K = 1.$

$\Rightarrow 60K = 1 \quad \Rightarrow \boxed{K = \frac{1}{60}}$

The updated probability distribution is,

X	1	2	3	4	5	6	7	8
$P(X=x)$	$\frac{2}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{8}{60}$	$\frac{10}{60}$	$\frac{12}{60}$	$\frac{14}{60}$	$\frac{4}{60}$

(ii) $P(X<3) = P(X=1) + P(X=2)$

$\Rightarrow \frac{2}{60} + \frac{4}{60} = \frac{6}{60} = \frac{1}{10} = 0.1$

(iii) $P(X>5) = P(X=6) + P(X=7) + P(X=8)$

$\Rightarrow \frac{12}{60} + \frac{14}{60} + \frac{4}{60} = \frac{30}{60} = \frac{1}{2}$

$= 0.5$

(10) Given;

$$f(x) = \begin{cases} cx(2-x), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Since, the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_0^2 cx(2-x)\, dx = 1$$

$$\Rightarrow c \int_0^2 (2x - x^2)\, dx = 1$$

$$\Rightarrow c \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} - \frac{x^3}{3} \right]_0^2 = 1$$

$$\Rightarrow c\left(4 - \frac{8}{3}\right) = 1 \Rightarrow \frac{4c}{3} = 1$$

$$\boxed{c = \frac{3}{4}}$$ ✓

$$\therefore f(x) = \begin{cases} \frac{3}{4} x(2-x), & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Mean of $X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$$\mu = \int_0^2 x \cdot \frac{3x}{4}(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$\mu = \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{2.2^3}{3} - \frac{2^4}{4}\right]$$

$$\mu = \frac{3}{4} \cdot 2^4 \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\mu = 1$$

(iii) $E(X^2) = \int_0^2 x^2 \frac{3x}{4}(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$

$$E(X^2) = \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$E(X^2) = \frac{3}{4}\left[2 \cdot \frac{2^4}{4} - \frac{2^5}{5}\right]$$

$$E(X^2) = \frac{3}{4} \cdot 2^5 \left[\frac{1}{4} - \frac{1}{5}\right] \frac{1}{20}$$

$$E(X^2) = \frac{6}{5}$$

Variance $= E(X^2) - (E(X))^2 = \frac{6}{5} - (1)^2 = \frac{6}{5} - 1$

$$= \frac{1}{5}$$

PART - B

1. Let x denote the minimum of the two numbers that appear when a pair of fair dice is thrown once.

(i) when 2 dice are thrown, the sample space is given by.

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

If random variable x assigns the minimum of its number in s, then the sample space

$$S = \left\{ \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \right\}$$

The minimum number could be

1, 2, 3, 4, 5, 6

For '1', the favo cases are

$(1,1)\ \_\_\_\_\ \underbrace{(1,6)}_{6};\ \underbrace{(2,1)(3,1)(4,1)(5,1)(6,1)}_{5}$

$P(X=1) = \frac{11}{36}$

Similarly for

$P(X=2) = \frac{9}{36}$; $P(X=3) = \frac{7}{36}$; $P(X=4) = \frac{5}{36}$

$P(X=5) = \frac{3}{36}$; $P(X=6) = \frac{1}{36}$

∴ The probability distribution is

X	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) Expectation = Mean = $\sum_i p_i x_i$

$E(X) = 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right) + 5\left(\frac{3}{36}\right) +$
$6\left(\frac{1}{36}\right)$

$E(X) = \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6) = \frac{91}{36}$

$\mu = E(X) \doteq 2.5278$

(iii) Varience = $E(x^2) - (E(x))^2$

$$E(x^2) = 1\left(\frac{11}{36}\right) + 4\left(\frac{9}{36}\right) + 9\left(\frac{7}{36}\right) + 16\left(\frac{5}{36}\right) + 25\left(\frac{3}{36}\right) +$$

$$36\left(\frac{1}{36}\right)$$

$$= \frac{1}{36}(11 + 36 + 63 + 80 + 75 + 36)$$

$$= \frac{301}{36} = 8.3611$$

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = 8.3611 - (2.5278)^2$$

$$V(x) = 8.3611 - 6.3898$$

$$V(x) = 1.9713$$

(2) X denotes the number of heads in a single toss of 4 fair coins.

The required probability distribution is

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{4}{16}$

(i) $P(X<2) = P(X=0) + P(X=1)$

$\Rightarrow \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$ ✓

(ii) $P(1<X\leq 3) = P(X=2) + P(X=3)$

$\Rightarrow \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$ ✓

(5) The given probability distribution

x	-1	0	1	2	3
$P(X=x)$	0.3	0.1	0.1	0.3	0.2

(i) Expectation = Mean $= \sum_i f_i x_i$

$\mu = -1(0.3) + 0(0.1) + 1(0.1) + 2(0.3) + 3(0.2)$

$\mu = -0.3 + 0.1 + 0.6 + 0.6$

$\boxed{\mu = 1}$ ✓

(ii) $E(x^2) = (-1)^2(0.3) + 0^2(0.1) + 1^2(0.1) + 2^2(0.3)$
$+ 3^2(0.2)$

6x

$E(x^2) = 0.3 + 0.1 + \underbrace{\cancel{1}.2 + 1.8}_{3}$

1.2
1.8
———
3.0

$E(x^2) = 3.4$ ✓

$V(X) = E(x^2) - (E(X))^2$

$V(X) = 3.4 - (1)^2 = 2.4$ ✓

(iii) $V(X) = 2.4 = \sigma^2$

Standard deviation $= +\sqrt{\sigma}$

$= +\sqrt{2.4}$

$\sigma = 1.54$ ✓

(4) Given; $f(x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$

The probability distribution is

x	1	2	3	----	n
$P(X=x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	----	$\frac{1}{n}$

(i) Mean = $\sum_i p_i x_i$

$$\mu = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n}$$

$$\mu = (1 + 2 + \ldots + n) \cdot \frac{1}{n} = \frac{n(n+1)}{2} \cdot \frac{1}{n}$$

$$\boxed{\mu = E(X) = \frac{n+1}{2}}$$ ✓

(ii) $$E(X^2) = 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{1}{n} + 3^2 \cdot \frac{1}{n} + \ldots + n^2 \cdot \frac{1}{n}$$

$$E(X^2) = (1^2 + 2^2 + \ldots + 3^2) \cdot \frac{1}{n}$$

$$E(X^2) = \frac{\cancel{n}(n+1)(2n+1)}{6} \cdot \frac{1}{\cancel{n}}$$

$$E(X^2) = \frac{(n+1)(2n+1)}{6}$$ ✓

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$V(X) = \frac{n+1}{12}(4n+2 - 3n - 3)$$

$$V(X) = \frac{(n+1)(n-1)}{12}$$

$$\boxed{\text{variance} = V(X) = \frac{n^2-1}{12}}$$

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(5) Given, probability distribution function

x	8	12	16	20	24
P(x)	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

(i) $E(X) = ? = \sum_i p_i x_i$

$$E(X) = \cancel{8}\left(\frac{1}{\cancel{8}}\right) + \overset{2}{\cancel{12}}\left(\frac{1}{\cancel{6}}\right) + \overset{2}{\cancel{16}}\left(\frac{3}{\cancel{8}}\right) + \overset{5}{\cancel{20}}\left(\frac{1}{\cancel{4}}\right) + \overset{2}{\cancel{24}}\left(\frac{1}{\cancel{12}}\right)$$

$$E(X) = 1 + 2 + 6 + 5 + 2$$

$$E(X) = 16$$

$$\text{Mean} = \mu = 16$$

(ii). $E(x^2) = ? = \sum_i x_i^2 P(x)$

$E(x^2) = 8^2\left(\frac{1}{8}\right) + (12)^2 \cdot \frac{1}{6} + (16)^2 \cdot \frac{3}{8} + (20)^2 \cdot \frac{1}{4} +$

$24^2\left(\frac{1}{12}\right)$

$16 \cdot \frac{16 \cdot 3}{8}$ $\quad 20 \cdot \frac{20 \cdot 1}{4}$ $\quad 24 \cdot \frac{24 \cdot 1}{12}$

$E(x^2) = 8 + 24 + 96 + 100 + 48$

$E(x^2) = 276$

Varience $(V(X)) = E(x^2) - (E(X))^2$

$V(X) = 276 - (16)^2$

$V(X) = 276 - 256 = 20$ ✓

(iii) $V(X) = \sigma = 20$

Standard deviation $= +\sqrt{\sigma}$

$= +\sqrt{20}$

$= 4.47$ (or) $2\sqrt{5}$

✓

(6) Given; $f(x) = \begin{cases} Ae^{-x/5} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

(i) In order that $f(x)$ should be a probability density function, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} A e^{-\frac{x}{5}}\,dx = 1$$

$$\Rightarrow A\left[\frac{e^{-x/5}}{-1/5}\right]_{0}^{\infty} = 1$$

$$\Rightarrow -5A\left[e^{-\infty} - e^{0}\right] = 1$$

$$\Rightarrow 5A = 1 \Rightarrow \boxed{A = 1/5}$$

(ii) $P(X > 20) = \int_{20}^{\infty} \frac{1}{5} e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{20}^{\infty}$

$20/5$

$$= \frac{1}{\cancel{5}} \times -\frac{\cancel{5}}{1}\left[\underset{0}{e^{-\infty}} - e^{-4}\right]$$

$$\Rightarrow -1\left[-\frac{1}{e^4}\right] = \frac{1}{e^4} = 0.0183$$

(7) let x denote the sum of the two no's that appear when a pair of fair dice is tossed.

(i) The sum x of the two no's which turn up must be an integer between 2 and 12.

$x = 2 \rightarrow (1,1) \rightarrow P(X=2) = 1/36$

$x = 3 \rightarrow (2,1)(1,2) \rightarrow P(X=3) = 2/36$

⋮

Similarly till $x = 12$.

Now the probability distribution in this case is given by the following table.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(ii) Mean $= \mu = E(X) = \sum_i P(X = x_i) \cdot x_i$

$$\mu = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \cdots + 12\left(\frac{1}{36}\right)$$

$$\mu = \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

$$\mu = \frac{252}{36} = 7$$

(iii) Varience $= V(X) = E(X^2) - (E(X))^2$

$$E(X^2) = 2^2\left(\frac{1}{36}\right) + 3^2\left(\frac{2}{36}\right) + \cdots + 12^2\left(\frac{1}{36}\right)$$

$$E(X^2) = \frac{1}{36}[4 + 18 + 48 + 100 + 180 + 294 + 320 + 324$$

$$+ 300 + 242 + 144] = \frac{1974}{36} = 54.83$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = 54.83 - 49$$

$$V(X) = 5.83$$

(8) Given; $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$

(i) Clearly; $f(x) \geq 0, \forall x$ in $(1,2)$, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} e^{-x}\,dx = 0 + (e^{-x})_0^{\infty}(-1)$$

$$= -(0-1)$$

$$= 1 ✓$$

Hence the function $f(x)$ is a density function. ✓

(ii) Required probability $= P(1 \leq x \leq 2) = \int_1^2 f(x)\,dx$

$$\Rightarrow \int_1^2 e^{-x}\,dx = -(e^{-x})_1^2 = -(e^{-2} - e^{-1})$$

$$= e^{-1} - e^{-2}$$

$$= 0.368 - 0.135$$

$$= 0.233 ✓$$

(iii) Cumulative Probability function

$$F(2) = \int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^2 e^{-x}\,dx$$

$$\Rightarrow 0 + (e^{-x})_0^2(-1) = -(e^{-2} - 1) = 1 - e^{-2}$$

$$F(2) = 1 - 0.135$$

$$F(2) = 0.865 ✓$$

(a) Given; $f(x) = \begin{cases} Kx^3 ; & 0 \le x \le 3 \\ 0 ; & \text{otherwise} \end{cases}$

we have $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

$\Rightarrow \int_{-\infty}^{0} 0 . dx + \int_{0}^{3} Kx^3 \, dx = 1$

$\Rightarrow 0 + K \left[\frac{x^4}{4}\right]_0^3 = 1 \Rightarrow \left(\frac{81}{4} - 0\right) K = 1$

$\boxed{K = \frac{4}{81}}$ ✓

∴ The value of K is 4/81.

$P\left(\frac{1}{2} < x < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\, dx = \int_{1/2}^{3/2} \frac{4}{81} . x^3 \, dx$

$= \frac{4}{81} \left[\frac{x^4}{4}\right]_{1/2}^{3/2}$

$\Rightarrow \frac{\cancel{4}}{81} \cdot \frac{1}{\cancel{4}} \left[\frac{3^4}{2^4} - \frac{1^4}{2^4}\right] = \frac{\cancel{4}}{81} \times \frac{\cancel{80}\,5}{\cancel{16}\,\cancel{4}}$

$= \frac{5}{81} = 0.0617$ ✓

(10) Given, the Probability distribution table,

x	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

(i) we know that $\sum_i P(X=x_i) = 1$

$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0$

$\cancel{K = \frac{-4+\sqrt{19}}{3}}$; $\cancel{\frac{-4-\sqrt{19}}{3}}$

$\cancel{K = 0.119}$, $\cancel{-2.78}$

$\Rightarrow 10K^2 + 10K - K - 1 = 0$

$\Rightarrow (K+1)(10K-1) = 0$

$\Rightarrow K = -1$ (or) $\frac{1}{10}$

$\boxed{K = 1/10}$

(ii) $P(X<6)$

$$P(X<6) = P(5) + P(4) + P(3) + P(2) + P(1) + P(0)$$

$$P(X<6) = K^2 + 3K + 2K + 2K + K + 0$$

$$P(X<6) = K^2 + 8K$$

$$P(X<6) = \frac{1}{100} + \frac{80}{100}$$

$$P(X<6) = \boxed{\frac{81}{100}}$$

(iii) $P(X \geq 6)$

$$P(X \geq 6) = P(6) + P(7)$$

$$P(X \geq 6) = 2K^2 + 7K^2 + K$$

$$P(X \geq 6) = 9K^2 + K$$

$$P(X \geq 6) = \frac{9}{100} + \frac{10}{100}$$

$$P(X \geq 6) = \frac{19}{100}$$

(11) Question Modified (Repeated from 1 part A)

To prove: $V(aX+b) = a^2 V(X)$

$V(X) \rightarrow$ Variance

$a, b \rightarrow$ constants

Consider ; let $Y = aX + b$ — (1)

$$E(Y) = E(aX+b).$$

$$E(Y) = a\,E(X) + b \quad — (2)$$

(1) - (2) $\quad Y - E(Y) = a[X - E(X)]$

Squaring and taking Exception on both sides

$$E[\{Y - E(Y)\}^2] = a^2[E\{X - E(X)\}^2]$$

$$V(Y) = a^2 V(X)$$

$$V(aX+b) = a^2 V(X).$$

Hence proved

Results:

(1) If $b = 0$ then $V(aX) = a^2 V(X)$

(2) If $a = 0$ then $V(b) = 0$.

(3) If $a = 1$ then $V(X+b) = V(X)$.

(12) Given; Probability distribution table.

x	-3	-2	-1	0	1	2	3
P(x)	K	0.1	K	0.2	2K	0.4	2K

(i) We know that; $\sum_i P(x_i) = 1$

$\Rightarrow K + 0.1 + K + 0.2 + 2K + 0.4 + 2K = 1$

$\Rightarrow 6K = 0.3 \Rightarrow \boxed{K = 0.05}$

(ii) Mean $= \sum_i x_i P(x_i) = E(X)$

$\mu = -3(K) + (-2)0.1 + (-1)K + 0(0.2) + 1(2K) + 2(0.4) + 3(2K)$

$\mu = -3K - 0.2 - K + 0 + 2K + 0.8 + 6K$

$\mu = 4K + 0.6 = 4(0.05) + 0.6 = 0.8$

(iii) $E(X^2) = (-3)^2 K + (-2)^2 0.1 + (-1)^2 K + 0(0.2) + 1(2K) + 2^2(0.4) + 3^2(2K)$

$E(X^2) = 9K + 0.4 + K + 0 + 2K + 1.6 + 18K$

$E(X^2) = 30K + 2$

$E(X^2) = 1.5 + 2 = 3.5$

$V(X) = E(X^2) - (E(X))^2$

$V(X) = 3.5 - (0.8)^2$

$V(X) = 3.5 - 0.64 = 2.86$ (or) $\frac{143}{50}$

---

(13) Given: $f(x) = \begin{cases} kx e^{-\lambda x} ; x \geq 0, \lambda > 0 \\ \cancel{\lambda > 0} \\ 0, \text{otherwise} \end{cases}$

(i) since the total probability is unity, we have, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_{0}^{\infty} f(x)\,dx = 1$

$\Rightarrow 0 + \int_{0}^{\infty} kx e^{-\lambda x}\,dx = 1$

$\Rightarrow k \int_{0}^{\infty} x e^{-\lambda x}\,dx = 1$

$\Rightarrow k \left\{ x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right\}_{0}^{\infty} = 1.$

$k \left\{ (0 - 0) - \left(0 - \frac{1}{\lambda^2}\right)\right\} = 1$

$\boxed{k = \lambda^2}$

(ii). $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$$\mu = \int_0^\infty x \cdot \lambda^2 x e^{-\lambda x}\, dx = \lambda^2 \int_0^\infty x^2 e^{-\lambda x}\, dx$$

$$\mu = \lambda^2 \left\{ x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right\}_0^\infty$$

$$\mu = \lambda^2 \left[(0 \mp 0 + 0) - \left(0 - 0 - \frac{2}{\lambda^3}\right) \right]$$

$$\mu = \frac{2}{\lambda} \checkmark = E(X)$$

(iii) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^\infty x^2 \lambda^2 x e^{-\lambda x}\, dx$

$$E(X^2) = \lambda^2 \int_0^\infty x^3 e^{-\lambda x}\, dx$$

$$E(X^2) = \lambda^2 \left\{ x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right\}_0^\infty = \lambda^2 \left[(0-0+0-0) - \left(0-0+0-\frac{6}{\lambda^4}\right) \right]$$

$$E(X^2) = 6/\lambda^2$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$V(X) = \frac{2}{\lambda^2} \checkmark$$

(14) Given; $f(x) = \begin{cases} K(1-x^2) \text{ ; for } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$

(i)
we know that ; $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{0}^{\infty} K(1-x^2)\, dx = 1.$$

$$\Rightarrow \int_{0}^{1} K(1-x^2)\, dx + \int_{1}^{\infty} \underbrace{K(1-x^2)\, dx}_{0} = 1$$

$$\Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1.$$

$$\Rightarrow K\left(1 - \frac{1}{3}\right) = 1 \quad \Rightarrow \boxed{K = 3/2}$$

(ii) $P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x)\, dx$

$$= \int_{0.1}^{0.2} K(1-x^2)\, dx$$

$$= \frac{3}{2}\int_{0.1}^{0.2} (1-x^2)\, dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right]$$

$$= 0.2965$$

(ii.) $$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \underbrace{\int_{0.5}^{1} f(x)\,dx}_{} + \underbrace{\int_{1}^{\infty} f(x)\,dx}_{0}$$

$$\Rightarrow \frac{3}{2}\int_{0.5}^{1}(1-x^2)\,dx = \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1-\frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= \frac{3}{2}\left[\frac{2}{3} - 0.4583\right]$$

$$= 0.3125$$

(15) Given; Probability function table

X	4	5	6	8
P(X)	0.1	0.3	0.4	0.2

(i) $E(X) = \sum_i x_i P(x_i) = 4(0.1) + 5(0.3) + 6(0.4) + 8(0.2)$

$E(X) = 0.4 + 1.5 + 2.4 + 1.6$

$E(X) = 5.9$ ✓

(ii) $E(X^2) = \sum_i x_i^2 P(x_i) = 16(0.1) + 25(0.3) + 36(0.4) + 64(0.2)$

$E(X^2) = 1.6 + 7.5 + 14.4 + 12.8$

$E(X^2) = 36.3$

$V(X) = E(X^2) - (E(X))^2 = 36.3 - (5.9)^2$

$V(X) = 36.3 - 34.81$

$V(X) = \sigma^2 = 1.49$ ✓

(iii) $\sigma = \sqrt{1.49}$ → standard deviation.

$\sigma = 1.22$ ✓

(16) Given; $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx + \int_2^{\infty} 0 \cdot dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx + 0$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right]_1^2$$

$$\Rightarrow \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^3\,dx + \int_1^2 (x^3 - 2x^2)\,dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left[\frac{x^4}{4} - 2\frac{x^3}{3}\right]_1^2$$

$$\Rightarrow \frac{1}{4} + \left(4 - \frac{16}{3}\right) - \left(\frac{1}{4} - \frac{2}{3}\right) = \frac{-2}{3}$$

$$E(25X^2 + 30X - 5) = 25\,E(X^2) + 30\,E(X) - 5$$

$$= 25\left(-\frac{2}{3}\right) + 30(1) - 5$$

$$= \frac{25}{3}$$

(17) $F(x) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(i) The density function $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 1/2\, e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(ii) Mean $= \mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x \cdot \frac{1}{2} e^{-2x}\, dx$

$$\mu = \frac{1}{2}\left[x\left(\frac{e^{-2x}}{-2}\right) - 1\left(\frac{e^{-2x}}{4}\right)\right]_0^{\infty}$$

$$\mu = -\frac{1}{8}\left[e^{-2x}(2x+1)\right]_0^{\infty} = -1/8\,(0-1) = 1/8$$

(iii) Varience $= E(x^2) - \mu^2$

$$\sigma^2 = \int_0^{\infty} x^2 \cdot \frac{1}{2} e^{-2x}\, dx - \left(\frac{1}{8}\right)^2$$

$$\sigma^2 = \frac{1}{2}\left[x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)\right]_0^{\infty} - \frac{1}{64}$$

$$\sigma^2 = \frac{1}{8}\left[e^{-2x}(2x^2+2x+1)\right]_0^{\infty} - \frac{1}{64}$$

$$\sigma^2 = \frac{1}{8}\left[0-(0+0+1)\right] - \frac{1}{64} = -\frac{1}{8} - \frac{1}{64} = \frac{-9}{64}$$

(18) Two coins are tossed simultaneously, 'x' denotes the no. of heads

$S = \{TT, TH, HT, HH\}$.

The probability distribution function is

X	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

(i) $E(X) = \sum_i p_i x_i$

$E(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 2\left(\frac{1}{4}\right)$.

$E(X) = 0 + \frac{2}{4} + \frac{2}{4}$

$E(X) = 1 = \mu$.

(ii) $E(X^2) = \sum_i p_i x_i^2$

$E(X^2) = \frac{1}{4}(0)^2 + \frac{2}{4}(1)^2 + \frac{1}{4}(2)^2$

$E(X^2) = 0 + \frac{2}{4} + \frac{4}{4}$

$E(X^2) = \frac{6}{4} = \frac{3}{2}$

(iii) $E(X^3) = \sum_i p_i x_i^3$

$E(X^3) = (0)^3 \cdot \frac{1}{4} + (1)^3 \frac{2}{4} + (2)^3 \cdot \frac{1}{4}$

$E(X^3) = 0 + \frac{2}{4} + \frac{8}{4}$

$E(X^3) = \frac{10}{4} = \frac{5}{2}$

(iv) $V(X) = E(X^2) - (E(X))^2$

$V(X) = \frac{3}{2} - (1)^2$

$V(X) = \frac{3}{2} - 1$

$1.5 - 1 = 0.5$

$V(X) = \frac{1}{2}$

(19) Given: $f(x) = \begin{cases} 0, & x < 2 \\ \frac{(2x+3)}{18}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$

(i) For all points x in $-\infty \le x \le \infty$, $f(x) \ge 0$

and $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} 0\cdot dx + \int_{2}^{4} \frac{2x+3}{18}\,dx + \int_{4}^{\infty} 0\cdot dx$

$\Rightarrow \frac{1}{18}\int_{2}^{4}(2x+3)\,dx = \frac{1}{18}\left[\frac{(2x+3)^2}{4}\right]_2^4$

$= \frac{1}{72}(121-49) = 1$ ✓

Hence, $f(x)$ is a probability density function. ✓

(ii) $P(2 \le x \le 3) = \int_{2}^{3} f(x)\,dx$

$= \frac{1}{18}\int_{2}^{3}(2x+3)\,dx$

$= \frac{1}{18}(x^2+3x)_2^3$

$\Rightarrow \frac{1}{18}(18-10) = \frac{8}{18} = \frac{4}{9}$ ✓

(20) $f(x) = \frac{K}{x^2+1}$; $-\infty < x < \infty$

since, w.k.T $\rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow K \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = 1$

$\Rightarrow K \left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$

$\Rightarrow K\left(\tan^{-1}\infty - \tan^{-1}(-\infty)\right) = 1 \Rightarrow K\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$

$K\pi = 1$

$\boxed{K = 1/\pi}$

By the defination, the distribution function is given by,

$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{K}{1+x^2}\,dx$

$F(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+x^2}\,dx$

$F(x) = \frac{1}{\pi} \left(\tan^{-1} x\right)_{-\infty}^{x}$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1}x - \tan^{-1}(-\infty)\right]$$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right]$$

Verified

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