MODULE IV (SOLUTIONS)

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PART A

PROBLEM SOLVING AND CRITICAL THINKING QUESTIONS

Problem 1: Show that, $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Here, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

put, $x=\frac{1}{1+y} \Rightarrow dx=-\frac{1}{(1+y)^2}$

then, $1-x=\frac{y}{1+y}$; then, $\beta(m,n)=\int_\infty^0 \frac{y^{n-1}}{(1+y)^{m+n}}(-1)dy=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$

Similarly, $\beta(m,n)=\int_0^1 x^{n-1}(1-x)^{m-1}dx$

$$\Rightarrow \beta(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

Therefore, $2\beta(m,n)=\int_0^\infty \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy=\underbrace{\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy}_{I_1}+\underbrace{\int_1^\infty \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy}_{I_2}$

Put, $y=\frac{1}{z}\Rightarrow dy=-\frac{1}{z^2}\ in\ I_2$

$$\therefore I_2=-\int_1^0 \frac{\frac{1}{z^{m-1}}+\frac{1}{z^{n-1}}}{\left(1+\frac{1}{z}\right)^{m+n}}\frac{dz}{z^2}=\int_0^1 \frac{\frac{z^{n-1}+z^{m-1}}{z^{m-1}z^{n-1}}}{\left(\frac{z+1}{z}\right)^{m+n}}\frac{dz}{z^2}=\int_0^1 \frac{\frac{z^{n-1}+z^{m-1}}{z^{m+n-2}}}{\frac{\left(z+1\right)^{m+n}}{z^{m+n-2}}}dz$$

$$=\int_0^1 \frac{z^{n-1}+z^{m-1}}{\left(z+1\right)^{m+n}}dz=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy$$

Then, $2\beta\left(m,n\right)=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy+\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy=2\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy$

Therefore, $\beta\left(m,n\right)=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Problem 2 and 3 (i): Find the value of $\int_0^1 \frac{1}{\sqrt{(1-x^n)}}dx$

put, $x^n = \sin^2\theta \Rightarrow nx^{n-1}dx = 2\sin\theta\cos\theta d\theta$

$$dx = \frac{2\sin\theta\cos\theta d\theta}{n\sin^2\theta(\sin^{2/n}\theta)^{-1}} = \frac{2\sin\theta\cos\theta d\theta}{n\sin^2\theta(\sin\theta)^{-2/n}} = \frac{2}{n}(\sin\theta)^{2/n-1}\cos\theta d\theta$$

When, x = 0 then $\theta = 0$; and x = 1 then $\theta = \pi/2$

$$Then, \frac{2}{n}\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\theta}}(\sin\theta)^{2/n-1}\cos\theta d\theta = \frac{2}{n}\int_0^{\pi/2}(\sin\theta)^{2/n-1}d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2}\sin^{2/n-1}\theta\cos^0\theta d\theta = \frac{2}{n}*\frac{1}{2}*\beta\left(\frac{2/n-1+1}{2},\frac{0+1}{2}\right)$$

$$= \frac{1}{n}\beta\left(\frac{1}{n},\frac{1}{2}\right) = \frac{1}{n}\frac{\Gamma(\frac{1}{n})*\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n}+\frac{1}{2})} = \frac{\sqrt{\pi}}{n}\frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{2+n}{2n})}$$

Problem 3 (ii): Find the value of $\int_a^b (a-x)^m (x-b)^n dx$, $b > a$.

put, $x-a=(b-a)t \Rightarrow dx=(b-a)dt$

When, x = a then t = 0; and x = b then t = 1

$Then, a-x=-(b-a)t$ and $x - b = a+(b-a)t-b=(b-a)(t-1)$

$= - (b-a)(1-t)$

$$Then, \int_0^1 \{-(b-a)t\}^m \{-(b-a)(1-t)\}^n (b-a)dt$$

$$= (-1)^{m+n}(b-a)^{m+n+1}\int_0^1 t^m (1-t)^n dt = (-1)^{m+n}(b-a)^{m+n+1}\beta(m+1, n+1)$$

$$= (-1)^{m+n}(b-a)^{m+n+1}\frac{\Gamma(m+1)*\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$= (-1)^{m+n}(b-a)^{m+n+1}\frac{m\Gamma(m)*n\Gamma(n)}{(m+n+1)(m+n)\Gamma(m+n)}$$

Problem 4: Find the value of $\int_0^\infty \frac{1}{1+x^4}\,dx$ using the Beta-Gamma function.

Here, $1-t = \frac{1}{1+x^4} \Rightarrow x^4 = \frac{t}{1-t} \Rightarrow x^3 = \left(\frac{t}{1-t}\right)^{3/4}$

Now, $4x^3 dx = \frac{1}{(1-t)^2}dt \Rightarrow dx = \frac{dt}{4x^3(1-t)^2} = \left(\frac{1-t}{t}\right)^{3/4} \frac{dt}{(1-t)^2}$

then the given integral is in the form, $= \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/4}\,dt = \frac{1}{4}\beta\left(\frac{1}{4},\frac{3}{4}\right)$

$$= \frac{1}{4} * \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4}+\frac{3}{4})} = \frac{1}{4} * \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{\Gamma(1)} = \frac{\pi}{4\sin\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4}$$

Problem 5: Show that, $\beta(m,n)=\dfrac{\Gamma m * \Gamma n}{\Gamma(m+n)}$

Here, $\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$

put, x $= zy$; then, $dx = zdy$

Then, $\Gamma n=\int_0^\infty e^{-zy}(zy)^{n-1}zdy = z^n\int_0^\infty e^{-zy}y^{n-1}dy$

$$=z^n\int_0^\infty e^{-zx}x^{n-1}dx \qquad (1)$$

$$\therefore \frac{\Gamma n}{z^n}=\int_0^\infty e^{-zx}x^{n-1}dx$$

Multiplying both sides by $e^{-z}z^{m-1}$ in equation (1),

then integrating with respect to z from o to ∞

$$\Gamma n\int_0^\infty e^{-z}z^{m-1}dz=\int_0^\infty e^{-z}z^{m-1}\left[z^n\int_0^\infty e^{-zx}x^{n-1}dx\right]dz$$

$$\Rightarrow \Gamma n * \Gamma m = \int_0^\infty \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1} dx dz$$

Since, the integration limit is same, so the changing the order of integration

$$\Gamma n * \Gamma m = \int_0^\infty x^{m-1} \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz \right] dx = \int_0^\infty x^{n-1} \left[\frac{\Gamma n * \Gamma m}{(1+x)^{m+n}} \right] dx$$

$$\Rightarrow \frac{\Gamma n * \Gamma m}{\Gamma n+m} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

Problem 6: Show that, $4\int_0^\infty \frac{x^2}{1+x^4}dx = \sqrt{2}\pi$

put, $x^2 = \tan\theta$; then, $2x dx = \sec^2\theta d\theta \Rightarrow dx = \frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}$

when, x = 0 then $\theta = 0$, and when x = ∞ then $\theta = \pi/2$

$$\text{Then,} 4\int_0^{\pi/2} \frac{\tan\theta}{1+\tan^2\theta}\frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}} = 4\int_0^{\pi/2} \frac{\sqrt{\tan\theta}\sqrt{\tan\theta}}{\sec^2\theta}\frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}$$

$$= 4\int_0^{\pi/2} \sqrt{\tan\theta}\frac{d\theta}{2} = 4*\frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta d\theta$$

$$= 2*\frac{1}{2}\beta\left(\frac{{}^1\!/_2+1}{2}, \frac{-{}^1\!/_2+1}{2}\right) = \beta\left(\frac{3}{4},\frac{1}{4}\right) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$= \frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{\pi}{\sin(\pi/4)} = \sqrt{2}\pi \quad \left(\because \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}\right)$$

Problem 7: Show that, $\Gamma(n)\Gamma(1-n) = \pi / \sin n\pi$

(a). Here, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put, $x = \dfrac{y}{1+y}$; then, $\beta(m,n) = \int_0^\infty \dfrac{y^{n-1}}{(1+y)^{m+n}}\,dy = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

letting, $m = 1 - n; 0 < n < 1$

$$\int_0^\infty \frac{y^{n-1}}{(1+y)^{1-n+n}}\,dy = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1-n+n)}$$

$$\Rightarrow \int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy = \Gamma(1-n)\Gamma(n);\ \text{as, } \Gamma(1) = 1$$

$$\Rightarrow \frac{\pi}{\sin n\pi} = \Gamma(1-n)\Gamma(n)$$

$$\textit{Therefore}, \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi};\left(\text{Since}, \int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy = \frac{\pi}{\sin n\pi}\right)$$

Problem 8: Show that, $\beta(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m2^{4m-1}} \cdot \frac{1}{\beta(m,m)}$

$$\beta(m+\tfrac{1}{2}, m+\tfrac{1}{2}) = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)} = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m+1)}$$

$$= \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{2m\Gamma(2m)} = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{2m * \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma\left(m+\frac{1}{2}\right)\Gamma(m)};$$

$$\left[\because \Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma\left(m+\frac{1}{2}\right)\Gamma(m)\right]$$

$$= \frac{\Gamma\left(m+\frac{1}{2}\right)}{2m * \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma(m)} = \frac{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)}{2m * \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma(m)\Gamma(m)} = \frac{\dfrac{\Gamma(2m)}{\left(\dfrac{2^{2m-1}}{\sqrt{\pi}}\right)}}{2m * \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma(m)\Gamma(m)}$$

$$=\frac{1}{2}\left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)\left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)\frac{1}{\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}}$$

$$=\frac{\pi}{2^{4m-1}}\frac{1}{\beta(m,m)}$$

Problem 9: Show that, $\int_0^1 x^m (\log x)^n dx = \dfrac{(-1)^n n!}{(m+1)^{n+1}}$

Here, $\int_0^1 x^m (\log x)^n dx$

put, $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

When, x = 0, then t = ∞ and when x = 1, then t = 0

then, $\int_0^1 x^m (\log x)^n dx = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}) dt = (-1)^n \int_0^\infty t^n (e)^{-(m+1)t} dt$

$$= (-1)^n \int_0^\infty \left(\frac{u}{m+1}\right)^n e^{-u} \frac{du}{m+1} dt; \text{ where, } (m+1)t = u$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty (u)^{n+1-1} e^{-u} dt$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) = \frac{(-1)^n}{(m+1)^{n+1}} n!; \text{ as } \Gamma(n+1) = n!$$

Problem 10: Show that, $\int_0^1 y^{q-1}(\log\frac{1}{y})^{p-1}dx = \frac{\Gamma(p)}{(q)^p}$

Here, the required integral is, $\int_0^1 y^{q-1}(\log\frac{1}{y})^{p-1}dy$

put, $\log {}^1\!/_y = x \Rightarrow \frac{1}{y} = e^x \Rightarrow y = e^{-x} \Rightarrow dy = -e^{-x}dx$

When, y = 0, then t = ∞ and when x = 1, then t = 0

$$\text{then, } \int_0^1 y^{q-1}(\log\frac{1}{y})^{p-1}dy = -\int_\infty^0 (e^{-x})^{q-1}(x)^{p-1}(e^{-x})dx = \int_0^\infty (e^{-x})^{q-1+1}(x)^{p-1}dx$$

$$= \int_0^\infty (e^{-x})^q (x)^{p-1}dx = \int_0^\infty e^{-qx}x^{p-1}dx$$

put, $qx = u \Rightarrow qdx = du \Rightarrow dx = \frac{du}{q}$

Again, x = 0, then u = 0 and when x = ∞, then t = ∞

$$= \int_0^\infty e^{-u}\left(\frac{u}{q}\right)^{p-1}\frac{du}{q} = \int_0^\infty e^{-u}\frac{u^{p-1}}{q^{p-1}}\frac{du}{q} = \frac{1}{q^p}\int_0^\infty e^{-u}u^{p-1}du = \frac{\Gamma(p)}{q^p}$$

PART B

LONG ANSWER QUESTIONS

Problem 1: $\boldsymbol{\dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m,n)}{m+n}}$.

Solution: We know $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Therefore,

$$\beta(m+1,n) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+1+n)} \Rightarrow \Gamma(m+n+1) = \frac{\Gamma(m+1)\Gamma(n)}{\beta(m+1,n)}$$

$$\beta(m,n+1) = \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \Rightarrow \Gamma(m+n+1) = \frac{\Gamma(m)\Gamma(n+1)}{\beta(m,n+1)}$$

From both relations, we get

$$\frac{\Gamma(m+1)\Gamma(n)}{\beta(m+1,n)} = \frac{\Gamma(m)\Gamma(n+1)}{\beta(m,n+1)}$$

Since, $\Gamma(x+1) = x\Gamma(x)$, then we get: $\quad \dfrac{m\Gamma(m)\Gamma(n)}{\beta(m+1,n)} = \dfrac{n\Gamma(m)\Gamma(n)}{\beta(m,n+1)}$

Simplifying, we get

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n}.$$

Solution cont.:

Similarly, we get

$$\beta(m+1,n)=\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+1+n)}=\frac{\Gamma(m+1)\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$\Rightarrow \Gamma(m+n)=\frac{\Gamma(m+1)\Gamma(n)}{(m+n)\beta(m+1,n)}$$

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\Rightarrow \Gamma(m+n)=\frac{\Gamma(m)\Gamma(n)}{\beta(m,n)}$$

From both relations, we get: $\frac{\Gamma(m+1)\Gamma(n)}{(m+n)\beta(m+1,n)}=\frac{\Gamma(m)\Gamma(n)}{\beta(m,n)}$

Since, $\Gamma(x+1)=x\Gamma(x)$, then we get: $\frac{m\Gamma(m)\Gamma(n)}{(m+n)\beta(m+1,n)}=\frac{\Gamma(m)\Gamma(n)}{\beta(m,n)}$

Simplifying, we get

$$\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n)}{(m+n)}.$$

Problem 2: $\boldsymbol{\beta(m+1,n)+\beta(m,n+1)=\beta(m,n)}$

Solution: We know $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Therefore,

$$\beta(m+1,n)=\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+1+n)}=\frac{\mathrm{m}\Gamma(m)\Gamma(n)}{(\mathrm{m+n})\Gamma(m+n)}$$, Since, $\Gamma(x+1)=x\Gamma(x)$

$$\beta(m,n+1)=\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}=\frac{\Gamma(m)\mathrm{n}\Gamma(n)}{(\mathrm{m+n})\Gamma(m+n)}$$

From both relations, we get

$$\boldsymbol{\beta(m+1,n)+\beta(m,n+1)}=\frac{\mathrm{m}\Gamma(m)\Gamma(n)}{(\mathrm{m+n})\Gamma(m+n)}+\frac{\Gamma(m)\mathrm{n}\Gamma(n)}{(\mathrm{m+n})\Gamma(m+n)}$$

$$=\frac{(\mathrm{m+n})\,\Gamma(m)\Gamma(n)}{(\mathrm{m+n})\Gamma(m+n)}=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$=\boldsymbol{\beta(m,n)}.$$

Problem 3: Show that, $\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\,d\theta \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta = \pi$

We know that, $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

Then, $\int_0^{\pi/2} \sin^{-1/2}\theta\cos^0\theta\,d\theta \int_0^{\pi/2} \sin^{-1/2}\theta\cos^0\theta\,d\theta = \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2},\frac{0+1}{2}\right) * \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{0+1}{2}\right)$

$$= \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)\beta\left(\frac{3}{4},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{\pi}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}; \quad \left[\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\pi}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{\pi}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi$$

Problem 4: Solve the integral $\int_0^a x^4 \sqrt{a^2 - x^2}dx$ using Beta-Gamma functions.

Solution: $I = \int_0^a x^4 \sqrt{a^2 - x^2}dx$

Let

$$x^2 = a^2 y \Rightarrow 2xdx = a^2 dy \Rightarrow dx = \frac{a^2 dy}{2x} = \frac{a^2 dy}{2a\sqrt{y}} = \frac{a dy}{2\sqrt{y}}.$$

When $x = 0, y = 0$ and $x = a, y = 1.$

Then,

$$I = \int_0^1 a^4 y^2 \sqrt{a^2 - a^2 y}\frac{a dy}{2\sqrt{y}} = \frac{a^6}{2}\int_0^1 y^{\frac{3}{2}} \sqrt{1 - y} dy = \frac{a^6}{2}\int_0^1 y^{\frac{3}{2}} (1 - y)^{\frac{1}{2}} dy$$

Since $\int_0^1 y^m (1 - y)^n \, dy = \beta(m, n)$, then

$$I = \frac{a^6}{2}\int_0^1 y^{\frac{3}{2}} (1 - y)^{\frac{1}{2}} dy = \frac{a^6}{2}\beta\left(\frac{3}{2}, \frac{1}{2}\right).$$

Solution continued

Again, $\beta(m,n) = \frac{\Gamma(m)\Gamma(\mathrm{n})}{\Gamma(m+n)}$ and $\Gamma(m+1) = m\Gamma(m)$, so we get

$$\beta\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\pi}{2}$$

(since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$)

Thus,

$$I = \frac{a^6}{2} \times \frac{\pi}{2} = \frac{a^6\pi}{4}.$$

Problem 5: Solve the integral $\int_0^1 (x\log x)^4\,dx$ using Beta-Gamma functions.

$$Put, x = e^{-t} \Rightarrow dx = -e^{-t}dt$$

when, x = 1, then t = 0; and when x = 0, then t = ∞

$$\text{Then,} \int_0^{\infty} \left(-te^{-t}\right)^4 (-e^{-t})dt = \int_0^{\infty} e^{-5t}t^4 dt \quad \text{..........(1)}$$

$$Put, 5t = u \Rightarrow 5dt = du$$

when, t = 0, then u = 0; and when t = ∞, then u = ∞

$$then, (1) \Rightarrow \int_0^{\infty} e^{-u}\frac{u^4}{5^4}\frac{du}{5} = \frac{1}{5^5}\int_0^{\infty} e^{-u}u^4 du = \frac{1}{5^5}\int_0^{\infty} e^{-u}u^{5-1}du$$

$$= \frac{\Gamma(5)}{5^5} = \frac{4*3*2*1}{5*5*5*5*5} = \frac{24}{3125}$$

Problem 6: Solve, $\int_0^\infty x^{-3/2}(1-e^{-x})dx$

Here, $\int_0^\infty x^{3/2}(1-e^{-x})dx = \underbrace{\int_0^\infty x^{-3/2}dx}_{I_1} + \underbrace{\int_0^\infty x^{-3/2}e^{-x}dx}_{I_2}$

$I_1 = \int_0^\infty x^{-3/2}dx = 0$

For, $I_{2,}$ put, $x = t^2 \Rightarrow dx = 2tdt$

When, x = 0, then t = 0 and when x = ∞, then t = ∞

then, $\int_0^\infty x^{3/2}(1-e^{-x})dx = \int_0^\infty (e^{-t^2})(t^2)^{3/2}2tdt = 2\int_0^\infty e^{-t^2}t^4dt$

Again, put, $\mathrm{u} = t^2 \Rightarrow du = 2tdt \Rightarrow dt = \dfrac{du}{2t} = \dfrac{du}{2\sqrt{u}}$

When, x = 0, then t = 0 and when x = ∞, then t = ∞

$$=2\int_0^\infty e^{-u}u^2\frac{du}{2\sqrt{u}} = \int_0^\infty e^{-u}u^{\frac{3}{2}}du$$

$$=\int_0^\infty e^{-u}u^{\frac{5}{2}-1}\frac{du}{q}=\Gamma(\frac{5}{2})=\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})=\frac{3\sqrt{\pi}}{4}=I_2$$

$\therefore$ The required solution is, $\dfrac{3\sqrt{\pi}}{4}$

Problem: 7. Solve the integral $\int_0^\infty \sqrt{x}e^{-\frac{x}{3}}dx$ using Gamma function

Solution: Assume $I = \int_0^\infty \sqrt{x}e^{-\frac{x}{3}}dx$

Let $\frac{x}{3} = y \Rightarrow dx = 3dy.$

$$I = \int_0^\infty \sqrt{x}e^{-\frac{x}{3}}dx = \int_0^\infty \sqrt{3y}e^{-y}3dy = 3\sqrt{3}\int_0^\infty \sqrt{y}e^{-y}dy = 3\sqrt{3}\int_0^\infty y^{\frac{1}{2}}e^{-y}dy$$

Since $\int_0^\infty y^{n-1}e^{-y}dy = \Gamma(n)$, then we get

$$I = 3\sqrt{3}\int_0^\infty y^{\frac{1}{2}}e^{-y}dy = 3\sqrt{3}\int_0^\infty y^{\frac{3}{2}-1}e^{-y}dy = 3\sqrt{3}\Gamma\left(\frac{3}{2}\right)$$

Now, as $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$I = 3\sqrt{3}\Gamma\left(\frac{3}{2}\right) = 3\sqrt{3}\Gamma\left(\frac{1}{2}+1\right) = 3\sqrt{3}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2}\sqrt{\pi}.$$

Problem 8: Show that, $\Gamma(n)=\int_0^1\left(\log\frac{1}{x}\right)^{n-1}dx$

put, $\log\frac{1}{x}=y\Rightarrow\frac{1}{x}=e^{y}\Rightarrow x=e^{-y}\Rightarrow dx=-e^{-y}dy$

When, x = 0, then y = ∞ and when x = 1, then y = 0

then, $\int_0^1\left(\log\frac{1}{x}\right)^{n-1}dx=\int_\infty^0 y^{n-1}\left(-e^{-y}\right)dy=\int_0^\infty e^{-y}y^{n-1}dy=\Gamma(n)$

Problem 9: Show that $\beta(n,n) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(\frac{n+1}{2}\right)}$

Solution: Since

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Therefore,

$$\beta(n,n) = \frac{\Gamma(n)\Gamma(n)}{\Gamma(n+n)} = \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}.$$

We also know that

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{n+1}{2}\right)\Gamma(n).$$

Then,

$$\beta(n,n) = \frac{\Gamma(n)\Gamma(n)}{\Gamma(n+n)} = \frac{\Gamma(n)\Gamma(n)}{\frac{2^{2n-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{n+1}{2}\right)\Gamma(n)} = \frac{\Gamma(n)}{\frac{2^{2n-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{n+1}{2}\right)}.$$

Problem: 10. Solve the integral $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$ using Beta-Gamma functions.

Solution: Assume

$$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$$

We know

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Thus,

$$\int_0^\infty \frac{x^8}{(1+x)^{24}}dx = \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}dx = \beta(9,15).$$

Similarly,

$$\int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx = \beta(15,9).$$

Since $\beta(m,n) = \beta(n,m)$, then

$$I = \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \frac{x^{14}}{(1+x)^{24}}dx = \beta(9,15) - \beta(15,9) = 0.$$

Problem 11: Show that $\int_0^\infty x^m e^{-ax^n}dx = \frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{1+m}{n}\right)$ where m and n are positive constants.

Solution: Put, $ax^n = z \Rightarrow anx^{n-1}dx = dz \Rightarrow dx = \dfrac{dz}{an\left(\dfrac{z}{a}\right)^{\frac{n-1}{n}}}$

Again, when the $x = 0$, then z = 0 and when x = ∞ then z = ∞

$$\therefore \int_0^\infty x^m e^{-ax^n} dx = \int_0^\infty \left(\frac{z}{a}\right)^{\frac{m}{n}} e^{-z} \frac{dz}{an\left(\dfrac{z}{a}\right)^{\frac{n-1}{n}}} = \frac{a^{\frac{n-1}{n}}}{na^{\frac{m}{n}+1}} \int_0^\infty z^{\frac{m}{n}-\frac{n-1}{n}} e^{-z} dz$$

$$= \frac{a^{\frac{n-1}{n}-\frac{m}{n}-1}}{n} \int_0^\infty z^{\frac{m-n+1}{n}} e^{-z} dz = \frac{a^{-\frac{m+1}{n}}}{n} \int_0^\infty z^{\frac{m+1}{n}-1} e^{-z} dz$$

$$= \frac{1}{na^{\frac{m+1}{n}}} \Gamma\left(\frac{m+1}{n}\right)$$

Problem: 12. Prove that $\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$ where n is positive constant.

Solution: Assume

$$I=\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{n-1}{n}\right)$$

Also, we can write

$$I=\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(\frac{n-2}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{1}{n}\right)$$

Then,

$$I^2=\left[\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{n-1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{n-2}{n}\right)\right]\cdot\ldots\cdot\left[\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(\frac{1}{n}\right)\right]$$

$$=\left[\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(1-\frac{2}{n}\right)\right]\cdot\ldots\cdot\left[\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(1-\frac{n-1}{n}\right)\right]$$

From reflection formula, we know

$$[\Gamma(x)\cdot\Gamma(1-x)]=\frac{x}{\sin(\pi x)}$$

Solution continued.

Then, we get

$$I^2 = \left[\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(1-\frac{2}{n}\right)\right]\cdot\ldots\cdot\left[\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(1-\frac{n-1}{n}\right)\right]$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{n}\right)}\frac{\pi}{\sin\left(\frac{2\pi}{n}\right)}\cdots\frac{\pi}{\sin\left(\frac{(n-1)\pi}{n}\right)}.$$

We know that $\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$.

Then,

$$I^2 = \frac{\pi}{\sin\left(\frac{\pi}{n}\right)}\frac{\pi}{\sin\left(\frac{2\pi}{n}\right)}\cdots\frac{\pi}{\sin\left(\frac{(n-1)\pi}{n}\right)} = \frac{\pi^{n-1}}{\frac{n}{2^{n-1}}} = \frac{(2\pi)^{n-1}}{n}.$$

Thus,

$$I = \frac{(2\pi)^{\frac{(n-1)}{2}}}{n^{\frac{1}{2}}}.$$

Problem 13: Solve, $\int_0^{\pi/2} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta$

We know that, $\int_0^{\pi/2} \sin^m\theta \cos^n\theta d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$; and $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Then, $\int_0^{\pi/2} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta = \int_0^{\pi/2} \sqrt{\tan\theta} d\theta + \int_0^{\pi/2} \sqrt{\sec\theta} d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta d\theta + \int_0^{\pi/2} \sin^0\theta \cos^{-1/2}\theta d\theta = \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) + \frac{1}{2}\beta\left(\frac{0+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} + \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} + \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$=\frac{1}{2}\frac{\dfrac{\pi}{\sin\dfrac{\pi}{4}}}{1}+\frac{1}{2}\frac{\sqrt{\pi}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}{\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}=\frac{1}{2}\sqrt{2}\pi+\frac{1}{2}\frac{\sqrt{\pi}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}{\Gamma\left(1-\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}$$

$$=\frac{\pi}{\sqrt{2}}+\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}{\dfrac{\pi}{\sin\dfrac{\pi}{4}}}\quad\left[\because\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

$$=\frac{\pi}{\sqrt{2}}+\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}{\sqrt{2}\pi}==\frac{\pi}{\sqrt{2}}+\frac{1}{2\sqrt{2}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Problem: 14. Solve the integral $\int_0^\infty 3^{-4x^2}\,dx$.

Solution: Assume $I = \int_0^\infty 3^{-4x^2}\,dx$.

Let $3^{-4x^2} = e^{-y} \Rightarrow -4x^2 \ln 3 = -y \Rightarrow x = \sqrt{\frac{y}{4\ln 3}}$

Then,

$$8x \ln 3\, dx = dy \Rightarrow dx = \frac{dy}{8x\ln 3} = \frac{dy}{8\ln 3\sqrt{\frac{y}{4\ln 3}}} \Rightarrow dx = \frac{y^{-\frac{1}{2}}dy}{4\sqrt{\ln 3}}.$$

Therefore,

$$I = \int_0^\infty 3^{-4x^2}dx = \int_0^\infty e^{-y}\frac{y^{-\frac{1}{2}}}{4\sqrt{\ln 3}}dy = \frac{1}{4\sqrt{\ln 3}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}dy$$

Solution continued.

Since $\Gamma(n) = \int_0^\infty y^{n-1}e^{-y}dy$ and

$$I = \frac{1}{4\sqrt{\ln 3}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}dy = \frac{1}{4\sqrt{\ln 3}}\int_0^\infty y^{\frac{1}{2}-1}e^{-y}dy.$$

Then,

$$I = \frac{1}{4\sqrt{\ln 3}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}.$$

Problem, 15: Solve the integral $\int_0^1 \frac{1}{\sqrt{(-\log x)}} dx$ using Gamma function

Solution:

put, $-\log x = y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y} dy$

When, x = 0, then y = ∞ and when x = 1, then y = 0

then, $\int_0^1 \frac{1}{\sqrt{-\log x}} dx = \int_\infty^0 \frac{1}{y^{1/2}} \left(-e^{-y}\right) dy = \int_0^\infty e^{-y} y^{1/2-1} dy$

$$= \Gamma(1/2) = \sqrt{\pi}$$

Problem: 16. Solve the integral $\int_0^\infty e^{-u^{-1/m}}du$ *in terms of Gamma function*

Solution:

$$\text{put, } u^{1/m} = z \Rightarrow \frac{1}{m}u^{\frac{1}{m}-1}du = dz \Rightarrow du = \frac{mdz}{u^{\frac{1}{m}-1}} = u^{-\frac{1}{m}+1}mdz$$

$$\Rightarrow du = u^{-\frac{1}{m}+1}mdz = \left(z^{-m}\right)^{\frac{-m+1}{m}}mdz = z^{1-m}mdz$$

When, u = 0, then z = 0 and when u = ∞, then z = ∞

$$\text{then, } \int_0^\infty e^{-u^{1/m}}du = \int_0^\infty e^{-z}z^{1-m}mdz = m\int_0^\infty e^{-z}z^{1-m}dz = m\Gamma(m)$$

Problem: 17. Solve the integral $\int_0^2 (8-x^3)^{\frac{1}{3}}\,dx$ using Beta-Gamma functions

Solution: Assume $I=\int_0^2 (8-x^3)^{\frac{1}{3}}\,dx$.

Let $x=2u \Rightarrow dx=2du$.

When $\mathrm{x}=0, \mathrm{u}=0$ and $\mathrm{x}=2, \mathrm{u}=1$.

Then,

$$I=\int_0^2 (8-x^3)^{\frac{1}{3}}\,dx=\int_0^1 (8-8u^3)^{\frac{1}{3}}\,2du=4\int_0^1 (1-u^3)^{\frac{1}{3}}\,du.$$

Now, let $u^3=t \Rightarrow u=t^{\frac{1}{3}}$.

Then, $3u^2du=dt \Rightarrow du=\frac{dt}{3u^2}=\frac{dt}{3t^{\frac{2}{3}}}$.

Then, we get

$$\mathrm{I}=4\int_0^1 (1-\mathrm{u}^3)^{\frac{1}{3}}\,\mathrm{du}=4\int_0^1 (1-\mathrm{t})^{\frac{1}{3}}\frac{\mathrm{dt}}{3\mathrm{t}^{\frac{2}{3}}}=\frac{4}{3}\int_0^1 \mathrm{t}^{-\frac{2}{3}}(1-\mathrm{t})^{\frac{1}{3}}\,\mathrm{dt}=\frac{4}{3}\int_0^1 \mathrm{t}^{\frac{1}{3}-1}(1-\mathrm{t})^{\frac{2}{3}-1}\,\mathrm{dt}$$

Solution continued.

Since, $\int_0^1 \mathrm{x}^{\mathrm{m}-1}(1-\mathrm{x})^{\mathrm{n}-1}\,\mathrm{dx} = \beta(\mathrm{m},\mathrm{n})$. Then, we get

$$I = \frac{4}{3}\int_0^1 \mathrm{t}^{\frac{1}{3}-1}(1-\mathrm{t})^{\frac{2}{3}-1}\,\mathrm{dt} = \frac{4}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right).$$

Since $\beta(\mathrm{m},\mathrm{n}) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m}+\mathrm{n})}$ and $\Gamma(1) = 1$, then

$$I = \frac{4}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right) = \frac{4}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{4}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{4}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{4}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

Since, $\Gamma(\mathrm{n})\Gamma(1-\mathrm{n}) = \frac{\pi}{\sin \pi n}$, then,

$$I = \frac{4}{3}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{8\pi}{3\sqrt{3}}.$$

Problem: 18. Solve the integral $\int_0^1 (1-x^3)^{\frac{1}{3}}\,dx$ using Beta-Gamma functions

Solution: Assume $I = \int_0^1 (1-x^3)^{\frac{1}{3}}\,dx$.

Now, let $x^3 = t \Rightarrow x = t^{\frac{1}{3}}$.

Then, $3x^2 dx = dt \Rightarrow du = \frac{dt}{3x^2} = \frac{dt}{3t^{\frac{2}{3}}}$.

Then, we get

$$I = \int_0^1 (1-x^3)^{\frac{1}{3}}\,dx = \int_0^1 (1-t)^{\frac{1}{3}} \frac{dt}{3t^{\frac{2}{3}}} = \frac{1}{3}\int_0^1 t^{-\frac{2}{3}}(1-t)^{\frac{1}{3}}\,dt = \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}\,dt$$

Solution continued.

Since, $\int_0^1 \mathrm{x}^{\mathrm{m}-1}(1-\mathrm{x})^{\mathrm{n}-1}\,\mathrm{dx} = \beta(\mathrm{m},\mathrm{n})$. Then, we get

$$I = \frac{1}{3}\int_0^1 \mathrm{t}^{\frac{1}{3}-1}(1-\mathrm{t})^{\frac{2}{3}-1}\,\mathrm{dt} = \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right).$$

Since $\beta(\mathrm{m},\mathrm{n}) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m}+\mathrm{n})}$ and $\Gamma(1) = 1$, then

$$I = \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

Since, $\Gamma(\mathrm{n})\Gamma(1-\mathrm{n}) = \frac{\pi}{\sin \pi n}$, then,

$$I = \frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}.$$

Problem: 19. State and prove the symmetry property of the Beta function.

Symmetry Property of the Beta Function

The **Beta function** is defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \text{for } x, y > 0.$$

The **symmetry property** states that:

$$B(x, y) = B(y, x).$$

Proof of the Symmetry Property

1. **Start with the definition of $B(x, y)$:**

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

2. **Substitute $u = 1 - t$:** Let $t = 1 - u$, so $dt = -du$. When $t = 0$, $u = 1$; and when $t = 1$, $u = 0$.

   Substitute into the integral:

$$B(x, y) = \int_1^0 (1-u)^{x-1} u^{y-1} (-du).$$

   Simplify by reversing the limits:

Simplify by reversing the limits:

$$B(x,y) = \int_0^1 u^{y-1}(1-u)^{x-1}\,du.$$

3. **Compare this with the definition of $B(y,x)$:** From the definition:

$$B(y,x) = \int_0^1 u^{y-1}(1-u)^{x-1}\,du.$$

Clearly:

$$B(x,y) = B(y,x).$$

Thus, the Beta function is symmetric, as required.

Problem: 20. State and prove any two other forms of Beta function

Show that, $\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

This form is also called the Beta function expressed as an improper integral

Put, $x=\frac{y}{1+y} \Rightarrow \mathrm{dx} = \frac{1}{(1+y)^2}dy$

Again, when $x = 0$, then $y=0$; and when $x=1$, then $y = \infty$

Now, $\beta(m,n)=\int_0^1 x^{n-1}(1-x)^{m-1}dx=\int_0^\infty \left(\frac{y}{1+y}\right)^{n-1}(1-\frac{y}{1+y})^{m-1}\frac{dy}{(1+y)^2}$

$$=\int_0^\infty \frac{y^{n-1}}{(1+y)^{n-1}}\frac{1}{(1+y)^{m-1}}\frac{dy}{(1+y)^2}=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dy$$

Again, $\beta(m,n)=\beta(n,m)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dy$

Another form: $$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2}\beta(\frac{m+1}{2}, \frac{n+1}{2})$$

This is called the beta function in terms of trigonometric functions.

Here, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put, $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta d\theta$

When, x = 0, then $\theta = 0$ and when x = 1, then $\theta = \pi/2$

then, $\beta(m,n) = \int_0^{\pi/2} (\sin\theta)^{2n-2}(1-\sin^2\theta)^{m-1} 2\sin\theta\cos\theta d\theta$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2n-1}(\cos\theta)^{2m-1} d\theta$$

Now, $\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^n(\cos\theta)^m d\theta$

$$\Rightarrow \int_0^{\pi/2} (\sin\theta)^n(\cos\theta)^m d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right) * \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

SOME IMPORTANT CONCEPTS AND PROBLEMS

Problem1: Prove that, $\Gamma(1/2) = \sqrt{\pi}$

By alternate form of the Gamma function,

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} x^{2.\frac{1}{2}-1} dx = 2\int_0^\infty e^{-x^2} dx$$

$$\text{So, } \Gamma(1/2)\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx * 2\int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

This double integral in the first quadrant is evaluated by the changing to the polar coordinates, $x = r\cos\theta, y = r\sin\theta$, and $J = r$.

$$or,\ dxdy = rdrd\theta$$

Solution continues:

$$\left[\Gamma(1/2)\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta = 4\int_{0}^{\pi/2}\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_{r=0}^{\infty} d\theta$$

$$= -2\int_{0}^{\pi/2}(e^{-\infty} - e^{0})\,d\theta$$

$$= -2\int_{0}^{\pi/2}(0-1)\,d\theta$$

$$= 2\int_{0}^{\pi/2} d\theta = 2*\theta\Big|_{0}^{\pi/2} = 2*\frac{\pi}{2}$$

Hence, $\Gamma(1/2) = \sqrt{\pi}$

An allternate form of Gamma function is $\overline{|n} = \int_0^\infty e^{-x^2} x^{2n-1} dx; n > 0$

Solution:

Put, $x = t^2 \Rightarrow dx = 2tdt$, in the general form, $\overline{|n} = \int_0^\infty e^{-x} x^{n-1} dx$

Again, when the $x = 0$, then $t = 0$ and when $z = \infty$ then $t = \infty$

$$\overline{|n} = \int_0^\infty e^{-t^2} t^{2n-2} 2tdt = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$$

Now, the changing the variable to x

$$\overline{|n} = 2\int_0^\infty e^{-x^2} x^{2n-1} dt$$

Problem 2: Find the values of $I = \int_0^\infty e^{-x} x^7 dx$

We know that, $\overline{|n+1} = n\overline{|n}$

The given integral is in the form of a gamma function.

$$\therefore \int_0^\infty e^{-x} x^7 dx = \int_0^\infty e^{-x} x^{8-1} dx = \overline{|8} = \overline{|7+1} = 7\overline{|7} = 7*6\overline{|6} = ...7*6*4...2*1 = 7!$$

Problem 3: Find the values of $I = \int_0^\infty e^{-ax} x^{n-1} dx$

Put, $ax = z \Rightarrow adx = dz$

When, $x = 0$, then $z = 0$ and when $x = \infty$ then $z = \infty$

$$\therefore \int_0^\infty e^{-ax} x^{n-1} dx = \int_0^\infty e^{-z} \left(\frac{z}{a}\right)^{n-1} \frac{dx}{a} = \frac{1}{a^n} \int_0^\infty e^{-z} z^{n-1} dx = \frac{\overline{|n}}{a^n}$$

Problem 4: Find the values of $I = \int_0^\infty e^{-4x} x^{10} dx$

We know that, $\Gamma(n+1) = n\Gamma(n)$

From the precceding formula, $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$

$$I = \int_0^\infty e^{-4x} x^{10} dx = \int_0^\infty e^{-4x} x^{11-1} dx = \frac{\Gamma(11)}{4^{11}} = \frac{10!}{4^{11}}$$

Problem 5: Find the values of $I = \int_0^\infty e^{-x} x^{3/2} dx$

$$I = \int_0^\infty e^{-x} x^{3/2} dx = \int_0^\infty e^{-x} x^{(5/2-1)} dx = \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

BETA FUNCTION

The Beta function is defined by the integral:

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \text{ , m, n} > 0 \qquad (1)$$

Relation to Gamma Function:

$$\beta(m,n) = \frac{\Gamma(m) * \Gamma(n)}{\Gamma(m+n)}$$

Symmetry Property

$$\beta(m,n) = \beta(n,m)$$

Problem 6:

Find the values of $\displaystyle\int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx$

Solution: $$\int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx = \int_0^\infty \frac{x^{8-1}}{(1+x)^{4+8}}\,dx = \beta(4,8) = \frac{\Gamma(4)\Gamma(8)}{\Gamma(12)} = \frac{3!*7!}{11!}$$

Problem 7:

Find the values of $\displaystyle\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^9}\,dx$

Solution: $$\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^9}\,dx = \int_0^\infty \frac{x^2}{(1+x)^9}\,dx + \int_0^\infty \frac{x^6}{(1+x)^9}\,dx$$

$$= \int_0^\infty \frac{x^{3-1}}{(1+x)^{6+3}}\,dx + \int_0^\infty \frac{x^{7-1}}{(1+x)^{2+7}}\,dx$$

$$= \beta(6,3) + \beta(2,7) = \frac{\Gamma(6)\Gamma(3)}{\Gamma(9)} + \frac{\Gamma(2)\Gamma(7)}{\Gamma(9)}$$

$$= \frac{5!*2!}{8!} + \frac{1*6!}{8!} = \frac{1}{42}$$

Problem 8: Find the values of $I = \int_0^1 x^7 (1-x)^{10}\, dx$

We know that, $\beta(m,n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$

The given integral is in the form of a beta function.

$$\therefore \int_0^1 x^7 (1-x)^{10}\, dx = \int_0^1 x^{8-1} (1-x)^{11-1}\, dx = \beta(8,11) = \frac{\overline{|8}\,\overline{|11}}{\overline{|19}} = \frac{7!*10!}{18!}$$

Problem 9: Find the values of $I = \int_0^1 x^{7/2} (1-x)^{5/2}\, dx$

We know that, $\beta(m,n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$

$$\therefore I = \int_0^1 x^{7/2} (1-x)^{5/2}\, dx, = \int_0^1 x^{(9/2)-1} (1-x)^{(7/2)-1}\, dx = \beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{\overline{\left|\frac{9}{2}\right.}\,\overline{\left|\frac{7}{2}\right.}}{\overline{|8}}$$

$$= \frac{\overline{\left|\frac{9}{2}\right.}\,\overline{\left|\frac{7}{2}\right.}}{\overline{|8}} = \frac{\left(\frac{7}{2}*\frac{5}{2}*\frac{3}{2}*\frac{1}{2}\overline{\left|\frac{1}{2}\right.}\right)\left(\frac{5}{2}*\frac{3}{2}*\frac{1}{2}\overline{\left|\frac{1}{2}\right.}\right)}{7*6*5*4*3*2*1} = \frac{5\pi}{512}$$

Problem 10: Find the values of $I = \int_0^{\pi/2} \sin^6\theta \cos^3\theta d\theta$

We know that, $\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

Then, $\int_0^{\pi/2} \sin^6\theta \cos^3 d\theta = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{4}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \Gamma\left(\frac{1}{2}\right)}{\frac{9}{2} * \frac{7}{2} * \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \Gamma\left(\frac{1}{2}\right)} = \frac{2}{63}$

Problem 11: Find the values of $I = \int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta d\theta$

Problem 12: Find the values of $I = \int_0^{a} x^2 (a^2 - x^2)^{3/2} dx$

Problem 13: Show that, $\left|\Gamma(n)\right|^2 = \pi / \{n \sinh(n\pi)\}$

Here, $\left|\Gamma(in)\right|^2 = \Gamma(in)\Gamma(-in)$

Here, $\Gamma(in+1) = in\Gamma(in)$, and $\Gamma(-in+1) = -in\Gamma(-in)$

Now, $\Gamma(in) = \dfrac{\Gamma(in+1)}{in}$; then, $\Gamma(-in) = \dfrac{\Gamma(1-in)}{-in}$

$$\text{Therefore, } \left|\Gamma(in)\right|^2 = \frac{\Gamma(in+1)\Gamma(1-in)}{(+in)*(-in)} = \frac{in\Gamma(in)\Gamma(1-in)}{-i^2n^2}$$

$$= \frac{\Gamma(in)\Gamma(1-in)}{-in} = \frac{1}{-in}\left(\frac{\pi}{\sin(n\pi i)}\right)$$

Again, $\sin(n\pi i) = -i\sinh(n\pi)$

$$So, \left|\Gamma(in)\right|^2 = \frac{\pi}{-i^2 n \sinh(n\pi)} = \frac{\pi}{n\sinh(n\pi)}$$

Note

- $\int_0^{\pi/2} \sin^m \theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{1}{2}\right)$
- $\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{n+1}{2}\right)$

Thank You