MODULE - I

PROBABILITY THEORY

PART-A

(1) The possible no. of Exhaustive cases $= {}^9C_3$

(i). Let 'A' be the event of 3 students belong to different classes.

The favourable no. of outcomes for the event 'A' is ${}^2C_1 \times {}^3C_1 \times {}^4C_1$

Therefore, the required probability is.

$$P(A) = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{\cancel{2} \times \cancel{3} \times \cancel{4}2}{\cancel{84}\,\cancel{28}\,\cancel{14}\,7} = \frac{2}{7}.$$

(ii) Let 'B' be the event that two belong to the same class and third to the different class.

The favourable no. of outcomes for the event 'B' is $({}^2C_2 \times {}^7C_1) + ({}^3C_1 \times {}^6C_1) + ({}^4C_2 \times {}^5C_1)$.

Therefore, the required probability is

$$P(B) = \frac{({}^2C_2 \times {}^7C_1) + ({}^3C_1 \times {}^6C_1) + ({}^4C_2 \times {}^5C_1)}{{}^9C_3} = \frac{55}{84}.$$

(iii) Let 'c' be the event that the three belong to the same class.

The favourable no. of outcomes for the event 'c' is $[0 + {}^3c_3 \times {}^4c_0 + {}^4c_3 \times {}^4c_0]$

Therefore, the required probability is,

$$P(C) = \frac{(0 + {}^3c_3 \times {}^4c_0 + {}^4c_3 \times {}^4c_0)}{{}^9c_3} = \frac{5}{84}.$$

---

(2) Let 'A' be the event that the drawn 3 tickets are in Arithematical progression.

Out of $(2n+1)$ tickets, three can be drawn in ${}^{(2n+1)}c_3$ ways. The exhaustive no. of outcomes $= {}^{(2n+1)}c_3$

$$= \frac{(2n+1)\,2n\,(2n-1)}{3 \times 2 \times 1}$$

$$= \frac{n(4n^2-1)}{3}$$

To enumerate the favourable no. of outcomes we are to consider the cases with common difference $d = 1, 2, 3, \ldots n$:

If $d = 1$.

$$\left.\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ \ldots & \ldots & \ldots \\ (2n-1) & 2n & (2n+1) \end{array}\right\} \text{i.e } (2n-1) \text{ cases in all}$$

If $d = 2$

$$\left.\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \\ \ldots & \ldots & \ldots \\ (2n-3) & (2n-1) & (2n+1) \end{array}\right\} \text{i.e } (2n-3) \text{ cases in all.}$$

If $d = n-1$.

$$\left.\begin{array}{ccc} 1 & n & 2n-1 \\ 2 & n+1 & 2n \\ 3 & n+2 & 2n+1 \end{array}\right\} \text{i.e } 3 \text{ cases in all}$$

If $d = n$ $\quad 1 \quad n+1 \quad 2n+1$ only 1 case.

$\therefore$ The favourable no. of outcomes is equal to $= 1 + 3 + \ldots + (2n-3) + (2n-1)$ a.p. (Progression)

$$\Rightarrow \frac{n}{2}\left[\cancel{1} + 2n - \cancel{1}\right] = n^2$$

Therefore, the required probability

$$P(A) = \frac{\text{Favourable no. of outcomes}}{\text{Exhaustive no. of outcomes}}$$

$$P(A) = \frac{n^2}{\left(\frac{n(4n^2-1)}{3}\right)} = \frac{3n^{\cancel{2}}}{\cancel{n}(4n^2-1)}$$

$$P(A) = \frac{3n}{4n^2-1}$$

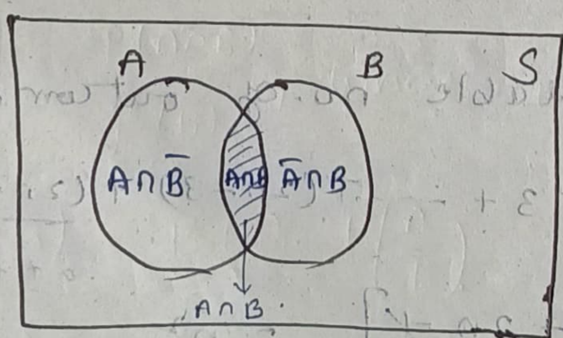
---

③ Law of addition of Probability for two events

Statement: Let 'A' and 'B' be any two events in the sample space 'S' and are not the mutually exclusive, then

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B).}$$

Proof: Consider the following venn diagram



From the venn diagram,

we can write; $(A \cup B) = A \cup (\bar{A} \cap B)$.

Apply probability on both sides

$$P(A \cup B) = P(A \cup \underbrace{(\bar{A} \cap B)}_{B})$$

We know that; A & B are two mutually exclusive ~~efets~~ events; so; $P(A \cup B) = P(A) + P(B)$

$$P(A \cup B) = P(A) + P(\bar{A} \cap B) \quad \text{--- (1)}$$

consider; $P(\bar{A} \cap B) = ?$

From venn diagram

$$B = (\bar{A} \cap B) \cup (A \cap B)$$

Apply probability on both sides.

$$P(B) = P(\underbrace{(\bar{A} \cap B)}_{A} \cup \underbrace{(A \cap B)}_{B})$$

$$P(B) = P(\bar{A} \cap B) + P(A \cap B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

from (1)

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Hence, Proved.

④ **Conditional Probability:** If the occurence of event 'A' effects the occurence of event 'B', then the two events are said to be dependent events.

→ In this case, we have to calculate the probability of 'A' on the assumption that event 'B' already occured and vice-versa, such a probability is called conditional probability.

Multiplication theorem on probability:

Statement: If $A_1, A_2, \ldots, A_n$ are 'n' events in the sample space 's', then

$$P(A_1 \cap A_2 \cap \ldots A_{n-1} \cap A_n) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right) \ldots P\left(\frac{A_n}{A_1 \cap A_2 \ldots A_{n-1}}\right)$$

Proof:

Strategy: By Mathematical Induction

We know that, By conditional probability,

$$\boxed{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}; \text{ where } P(B) \neq 0.}$$

$$P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right) \quad \text{—} ①$$

Now on taking $B = A_1$ and $A = A_2$

$$P(A_1 \cap A_2) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right) \quad \text{—— (2)}$$

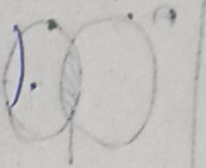
This is true for $n = 2$ events.

Now on taking $A = A_3$ and $B = A_1 \cap A_2$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2) \cdot P\left(\frac{A_3}{A_1 \cap A_2}\right).$$

from (2)

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right) \cdot P\left(\frac{A_3}{A_1 \cap A_2}\right).$$



This is true for $n = 3$ events.

Therefore, By the principle of mathematical Induction, The theorem is true for all the integral values of 'n'

To culminate;

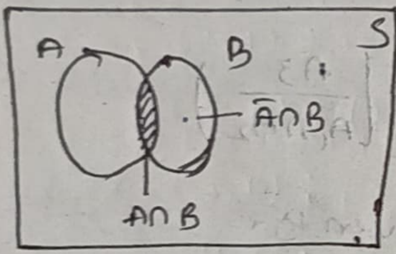
$$P(A_1 \cap A_2 \cap A_3 \cdots A_n) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right) \cdot P\left(\frac{A_3}{A_1 \cap A_2}\right) \cdots P\left(\frac{A_n}{A_1 \cap A_2 \cap A \cdots \cap A_{n-1}}\right)$$

Hence the proof.

(5) Given, that A and B are two independent events. Since, A and B are two independent events, $P(A \cap B) = P(A) \cdot P(B)$ — (1)

(i) To prove $\bar{A}$ and B are independent events.

Consider the following venn diagram,



From the venn diagram,

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B).$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B).$$

$$P(\bar{A} \cap B) = P(B) - P(A) \cdot P(B)$$

$$P(\bar{A} \cap B) = P(B)(1 - P(A))$$

$$P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$$

Therefore; $\bar{A}$ and B are also the independent events.

(ii) To prove: A and $\bar{B}$ are independent events.

From the (i) venn diagram

$$A = (A \cap B) \cup (A \cap \bar{B})$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$P(A \cap \bar{B}) = P(A) - P(A)P(B)$$

$$P(A \cap \bar{B}) = P(A)(1 - P(B))$$

$$P(A \cap \bar{B}) = P(A)\,P(\bar{B})$$

Therefore, A and $\bar{B}$ are independent events.

(iii) To prove $\bar{A}$ and $\bar{B}$ are independent events

consider; $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$P(\bar{A} \cap \bar{B}) = 1 - (P(A) + P(B) - P(A \cap B))$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A).P(B)$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B)(1 - P(A))$$

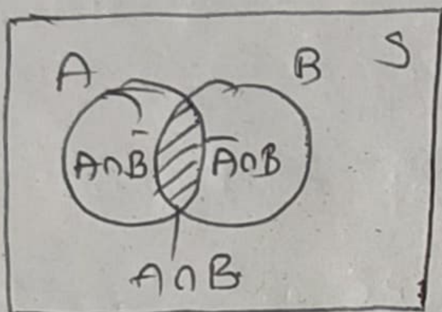
$$P(\bar{A} \cap \bar{B}) = (1 - P(A))(1 - P(B))$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}).P(\bar{B})$$

Therefore, $\bar{A}$ and $\bar{B}$ are also independent events.

(8) To prove: $P(A\cap B) \leq P(A) \leq P(A\cup B) \leq P(A) + P(B)$

Consider the venn diagram,



From the venn diagram,

$$A = (A\cap \bar{B}) \cup (A\cap B)$$

$$P(A) = P\left[(A\cap \bar{B}) \cup (A\cap B)\right)$$

$$P(A) = P(A\cap \bar{B}) + P(A\cap B)$$

since $\quad P(A\cap \bar{B}) \geq 0$

$$P(A) \geq P(A\cap B)$$

$$P(A\cap B) \leq P(A) \quad \text{— (1)}$$

From addition theorem;

$$P(A\cup B) = P(A) + \underbrace{P(B) - P(A\cap B)}_{0} \quad \text{— (2)}$$

also we have; $\quad P(B) \geq P(A\cap B)$

$$P(B) - P(A\cap B) \geq 0.$$

$\therefore P(A\cup B) \geq P(A)$. (from addition (2) theorem)

$$P(A) \leq P(A \cup B) \quad - (2)$$

Also from addition theorem;

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) + \underbrace{P(A \cap B)}_{\supset} = P(A) + P(B)$$

$$P(A \cup B) \leq P(A) + P(B) \quad - (3)$$

From (1), (2) and (3),

we draw the following conclusions,

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

Hence proved

(16)

(7) Let 'A' and 'B' be the events that the sum is greater than '8' and neither '7' nor '11' respectively:

If two dice are thrown, the Exhaustive no. of outcomes = 36

$$S = \left\{ \begin{matrix} (1,1); (1,2); (1,3); (1,4); (1,5); (1,6) \\ (2,1); (2,2); (2,3); (2,4); (2,5); (2,6) \\ (3,1); (3,2); (3,3); (3,4); (3,5); (3,6) \\ (4,1); (4,2); (4,3); (4,4); (4,5); (4,6) \\ (5,1); (5,2); (5,3); (5,4); (5,5); (5,6) \\ (6,1); (6,2); (6,3); (6,4); (6,5); (6,6) \end{matrix} \right\}$$

(i) The favourable no. of outcomes for event 'A' is 10

$$\left\{ \begin{matrix} (3,6); (4,5); (4,6); (5,4); (5,5); (5,6); \\ (6,3); (6,4); (6,5); (6,6). \end{matrix} \right\}$$

$$P(A) = \frac{\text{Favourable No. of outcomes}}{\text{Total No. of outcomes}}$$

$$P(A) = \frac{10}{36}$$ ✓

(ii) The possible cases for the sum '7'

is $\{(1,6)\ (2,5)\ (3,4)\ (4,3)\ (5,2)\ (6,1)\}$

$\hookrightarrow$ 6 cases

The possible cases for the sum '11' is

2 cases only i.e. $\{(5,6);(6,5)\}$

$P(\underbrace{\text{getting } 7}_{\text{(let C)}}) = P(C) = 6/36.$

$P(\underbrace{\text{getting } 11}_{\text{(let D)}}) = P(D) = 2/36$

$P(\text{neither } 7 \text{ nor } 11) = P(\bar{C} \cap \bar{D}) = P(\overline{C \cup D})$

$P(B) = 1 - P(C \cup D)$

mutually exclusive even

$P(B) = 1 - (P(C) + P(D))$

$P(B) = 1 - \left(\frac{6}{36} + \frac{2}{36}\right)$

$P(B) = 1 - \frac{8}{36}$

$P(B) = \frac{28}{36}$ ✓

(8) The probability of getting '6' in a throw of a die $= 1/6$.

The probability of not getting a '6' in a throw of a die $= 1 - 1/6 = 5/6$.

By multiplication theorem on probability, the probability that in four throws of a die no '6' is obtained $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$\boxed{x}$ $\boxed{x}$ $\boxed{x}$ $\boxed{x}$

$\frac{5}{6}$ $\frac{5}{6}$ $\frac{5}{6}$ $\frac{5}{6}$

$= \left(\frac{5}{6}\right)^4$

Hence the probability of obtaining '6' atleast once in four throws of a die is

$\Rightarrow 1 - \left(\frac{5}{6}\right)^4 = 1 - 0.48$

$= 0.52$ — (1) (52% chance)

Now, if a trail consist of throwing of two dice at a time, the probability of getting double '6' $= \frac{1}{36}$ ((6, 6) – only 1 chance)

Thus, the probability of not getting double '6' is

$\Rightarrow 1 - \frac{1}{36} = \frac{35}{36}$.

Now, In 24 throws, with two dice each, the probability that no double of '6' is obtained is $= \frac{35}{36} \times \frac{35}{36} \times \cdots \times \frac{35}{36}$ (24 times)

$\Rightarrow \left(\frac{35}{36}\right)^{24} = 0.508$

Thus the probability a double of '6' atleast once in 24 throws with 2 dice is

$\Rightarrow 1 - \left(\frac{35}{36}\right)^{24} = 1 - 0.508 = 0.492$ — (2)

Since, from (1) is greater than (2), the result follows.

(9) Given; $P(A) = \frac{1}{2}$; $P(B) = \frac{3}{4}$; $P(C) = \frac{1}{4}$

The problem will be solved if anyone solves the problem that means we need to calculate $P(A \cup B \cup C)$.

Since A & B are independent events;

$$P(A \cap B) = P(A) \cdot P(B)$$

we know that;

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(C) \cdot P(A) + P(A) \cdot P(B) \cdot P(C)$$

$$P(A \cup B \cup C) = \frac{1}{2} + \underbrace{\frac{3}{4} + \frac{1}{4}}_{1} - \frac{1}{2} \cdot \frac{3}{4} - \frac{3}{4} \cdot \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4}$$

$$P(A \cup B \cup C) = \cancel{\frac{1}{2}} + 1 - \cancel{\frac{3}{8}} - \frac{3}{16} - \cancel{\frac{1}{8}} + \frac{3}{32}$$

$$P(A \cup B \cup C) = 1 - \underbrace{\frac{6}{32} + \frac{3}{32}}_{3/1} = 1 - \frac{3}{32}$$

$$\boxed{P(A \cup B \cup C) = \frac{29}{32}}$$

(10) Let E_1, E_2, E_3 denotes the events that boxes are selected at random (1, 2, 3)

then $P(E_1) = 1/3$

$P(E_2) = 1/3$

$P(E_3) = 1/3$

Let 'A' denotes the event of getting a red ball then

$$P(A/E_1) = \frac{3C_1}{8C_1} = \frac{3}{8}$$

$$P(A/E_2) = \frac{\cancel{4}C_1}{6C_1} = 1/6$$

$$P(A/E_3) = \frac{4C_1}{9C_1} = \frac{4}{9}$$

The required probability that it comes from box 2 is given by,

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \cdot P\left(\frac{A}{E_2}\right)}{\sum_{i=1}^{3} P(E_i) \cdot P\left(\frac{A}{E_i}\right)}$$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + P(E_3) \cdot P\left(\frac{A}{E_3}\right)}$$

$$P\left(\frac{E_2}{A}\right) = \frac{\left(\frac{1}{3}\times\frac{1}{6}\right)}{\left(\frac{1}{3}\times\frac{3}{8}\right)+\left(\frac{1}{3}\times\frac{1}{6}\right)+\left(\frac{1}{3}\times\frac{4}{9}\right)} = \frac{12}{71}$$

(Take $1/3$ common for simple calculation)

(8) T

PART - B

① **For 'A':**

One share → 1 prize + 2 blanks

Favourable No. of cases $= {}^{1}C_1 = 1$

Total No. of cases $= {}^{3}C_1 = 3$

$$P(A) = \frac{\text{Favourable No. of cases}}{\text{Total No. of cases}} = \frac{1}{3}$$

↓ sucess

For 'B':

Three shares → 3 prizes + 6 blanks

Favourable No. of cases $= {}^{6}C_3 = 20$ (blanks)

Total No. of cases $= {}^{9}C_3 = 84.$

$$P(B) = \frac{84-20}{84} = \frac{64}{84} = \frac{16}{21}$$

↓ sucess

P(A) sucess : P(B) sucess $= \frac{1}{3} : \frac{16}{21}$

$= 7 : 16.$

(2) let 'A' be the event that the later box contains 2 red and 6 black balls

Total No. of balls = 15

No. of ways of choosing 8 balls from 15 is $^{15}C_8$ → Total no. of cases.

No. of ways of choosing 2 red balls out of 5 is $^5C_2$.

No. of ways of choosing 6 black balls out of 10 is $^{10}C_6$.

Favourable no. of cases $= {}^5C_2 \times {}^{10}C_6$.

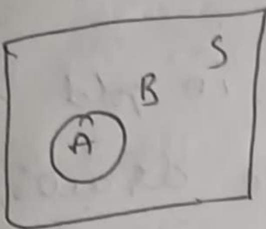
$$P(A) = \frac{\text{Favourable no. of Cases}}{\text{Total No. of Cases}}$$

$$P(A) = \frac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8}$$

$$P(A) = \frac{140}{429}$$

(5) Let A and B be two events. we know that $A \subset B$ then $P(A) \leq P(B)$

Given; $P(A) = 3/4$ and $P(B) = 5/8$.

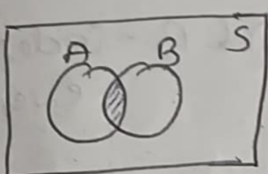
(i) 

Since; $A \subset A \cup B$

$$P(A) \leq P(A \cup B)$$

$$\frac{3}{4} \leq P(A \cup B)$$

$$P(A \cup B) \geq 3/4$$

(ii). 

Since; $A \cap B \subset B$

$$P(A \cap B) \leq P(B)$$

$$P(A \cap B) \leq 5/8 \quad \text{— (1)}$$

From addition theorem;

$$P(A \cup B) \Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$$

$$\Rightarrow \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$$

$$\Rightarrow \frac{6 + 5 - 8}{8} \leq P(A \cap B)$$

$$\Rightarrow \frac{3}{8} \leq P(A \cap B) \quad \text{— (2)}$$

from (1) & (2)

$$\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}.$$

Hence, Proved.

(4) Let 'A' be the event of drawing 4 gold coins in 1st draw. Let 'B' be the event of drawing 4 silver coins in 2nd draw.

In a bag there contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made such that

(i) coins are replaced before second trail,

$P(A \cap B) = P(A) \cdot P(B)$ {A & B - independent events}

$$P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$$

$$P(A \cap B) = \frac{49}{31212}$$

(ii) coins are not replaced before second trail;

$$P(A \cap B) = P(A) \cdot P(B/A)$$

$$P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{14}C_4}$$

$$P(A \cap B) = \frac{35}{7293}$$

2

(5) Given; A, B, C are mutually independent events.

To prove: 'A∪B' and 'C' are independent

i.e $P((A\cup B)\cap C) = P(A\cup B)\cdot P(C)$

Consider; $P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C))$

$= P(A\cap C) + P(B\cap C) - P(A\cap B\cap C)$

$= P(A)\cdot P(C) + P(B)\cdot P(C) - \underbrace{P(A)\,P(B)}_{P(A\cap B)}\,P(C)$

$= (P(A) + P(B) - P(A\cap B))\cdot P(C)$

$= P(A\cup B)\cdot P(C)$

LHS = RHS ✓

Hence proved.

Therefore; 'A∪B' and 'C' are also the mutually independent events.

⑥ BOX → 6 red, 4 white & 5 black balls

Total - 15 balls.

4 balls are drawn at random and the ball drawn. There is at least 1 ball of each colour.

Case - 1 : R R W B   Case - 2 : W W R B   Case - 3 : B B R W

Each possibility, we have the following ways of arranging the balls $= \frac{4!}{2!} = 12$ ways

Case - 1 : $P_1 = \frac{6}{15} \times \frac{5}{14} \times \frac{4}{13} \times \frac{5}{12} \times 12 = \frac{600}{2730}$

Case - 2 : $P_2 = \frac{4}{15} \times \frac{3}{14} \times \frac{6}{13} \times \frac{5}{12} \times 12 = \frac{360}{2730}$

Case - 3 : $P_3 = \frac{5}{15} \times \frac{4}{15} \times \frac{6}{13} \times \frac{4}{12} \times 12 = \frac{480}{2730}$

The required probability $= P_1 + P_2 + P_3$

$= \frac{48}{91}$

(7) let 'A' be an event of first box chosen at random and a ticket is drawn and let 'B' be an event of second box chosen at random and a ticket is drawn. The required event can happen in the following of above any 2 mutually exclusive events,

(i) since the probability of choosing one of the boxes is $1/2$. The required probability is given by

$$P(2 \text{ or } 4) = \underbrace{\left(\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{4}\right)}_{P(A)} + \underbrace{\left(\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{6}\right)}_{P(B)} = \frac{5}{12}$$

(ii) The probability of getting '3' is given by

$$P(\text{getting } 3) = \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\cancel{\frac{1}{2} \times \frac{0}{6}}\right) = \frac{1}{8}$$

(iii) The probability of getting 1 or 9 is given by;

$$P(1 \text{ or } 9) = \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{6}\right) = \frac{5}{24}$$

dice - 36 - 5

(8) Since, 'A' wins the game if he throws '6', so the probability of 'A' wins the game

$P(A) = \frac{5}{36}$ $\{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\}$

$P(\bar{A}) = 1 - P(A) = \frac{31}{36}$

Since, 'B' wins the game if he throws '7', so the probability of 'B' wins the game

$P(B) = \frac{6}{36} = \frac{1}{6}$ $\{(1,6)\ (2,5)\ (3,4)\ (4,3)\ (5,2)\ (6,1)\}$

$P(\bar{B}) = 1 - P(B) = 5/6$

The Probability of 'A' wins the game

$= P(A) + P(\bar{A} \cap \bar{B} \cap A) + P(\bar{A} \cap \bar{B} \cap \bar{A} \cap \bar{B} \cap A) + \cdots$

(Multiplication theorem)

$= \frac{5}{36} + \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{5}{36} + \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{5}{36} + \cdots$

$= \frac{5}{36} \left\{ 1 + \frac{31}{36} \cdot \frac{5}{6} + \left(\frac{31}{36} \cdot \frac{5}{6} \right)^2 + \cdots \right\}$

Geometric Progression.

$a/_{1-r}$

$$= \frac{5}{36}\left\{\frac{1}{1-\frac{155}{216}}\right\}$$

$$= \frac{5}{36}\left\{\frac{216}{216-155}\right\}$$

$$\Rightarrow \frac{5}{\cancel{36}} \times \frac{\overset{6}{\cancel{216}}}{61} = \frac{30}{61}$$

$\therefore$ A begins and his chance of winning $\frac{30}{61}$

Hence proved.

(a) Let $P(A)$; $P(B)$, $P(C)$ be the probabilities of the events that the bolts are manufactured by the machines A, B, C respectively. Then

$$P(A) = \frac{20}{100} = \frac{1}{5};\quad P(B) = \frac{30}{100} = \frac{3}{10};\quad P(C) = \frac{50}{100} = \frac{1}{2}.$$

Let 'D' denote the bolt is defective, then

$$P(D/A) = \frac{6}{100};\quad P(D/B) = \frac{3}{100};\quad P(D/C) = \frac{2}{100}$$

If bolt is defective, then the probability that it is from

(i) machine - A:

$$P(A/D) = \frac{P(D/A) \cdot P(A)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$P(A/D) = \frac{12}{31}.$$

(ii) machine - B:

$$P(B/D) = \frac{P(D/B) \cdot P(B)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$P(B/D) = \frac{9}{31}.$$

(iii) machine - C:

$$P(C/D) = \frac{P(D/C) \cdot P(C)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$P(C/D) = \frac{10}{31}.$$

(10) Let the probabilities of business man going to hotels X, Y, Z be respectively P(X), P(Y) P(Z). Then,

$$P(X) = \frac{20}{100} = \frac{2}{10} ; \quad P(Y) = \frac{50}{100} = \frac{5}{10} ;$$

$$P(Z) = \frac{30}{100} = \frac{3}{10} .$$

Let 'E' be the event that the hotel room has faulty plumbing. Then the probabilities that hotels x, y, z have faulty plumbing are : $P(E/X) = \frac{5}{100} = \frac{1}{20}$; $P(E/Y) = 4/100 = 1/25$

$P(E/Z) = 8/100 = 2/25$.

The probability that the business man's room having faulty plumbing is assigned to hotel Z

$$= P(Z/E) = \frac{P(Z) \cdot P(E/Z)}{P(X) \cdot P\left(\frac{E}{X}\right) + P(Y) \cdot P\left(\frac{E}{Y}\right) + P(Z) \cdot P\left(\frac{E}{Z}\right)}$$

$$= \frac{\frac{3}{10} \times \frac{2}{25}}{\frac{2}{10} \times \frac{1}{20} + \frac{5}{10} \times \frac{1}{25} + \frac{3}{10} \times \frac{2}{25}}$$

$$= \frac{4}{9} .$$

(11) Given; $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$; $P(A \cap B) = \frac{1}{5}$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \frac{5}{6} - \frac{1}{5}$

$\Rightarrow P(A \cup B) = \frac{19}{30}$ ✓

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$P(\bar{A} \cap B) = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15}$

$P(\bar{A} \cap B) = \frac{2}{15}$ ✓

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$P(A \cap \bar{B}) = \frac{1}{2} - \frac{1}{5} = \frac{5-2}{10}$

$P(A \cap \bar{B}) = \frac{3}{10}$ ✓

(iv) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{19}{30}$

$P(\bar{A} \cap \bar{B}) = \frac{11}{30}$ ✓

(12) 6 Men, 4 ladies → Total → 10

Committee members → 5

A Committee of 5 members out of 10

Can be formed in ${}^{10}C_5$ ways = 252.

The favourable cases for one lady,

1 lady + 4 men → $\underset{\text{Men}}{{}^6C_4} \times \underset{\text{ladies}}{{}^4C_1} = 60$

2 ladies + 3 men → ${}^6C_3 \times {}^4C_2 = 120$

3 ladies + 2 men → ${}^6C_2 \times {}^4C_3 = 60$

4 ladies + 1 men → ${}^6C_1 \times {}^4C_4 = 6$

Total — 246 ways

P(atleast one lady) = $\dfrac{\text{fav. no. of cases}}{\text{Total no. of cases}}$

$= \dfrac{246}{252}$

$= \dfrac{41}{42}$

(13) "U N I V E R S I T Y"

No. of Permutations $= \frac{10!}{2!} = \frac{10!}{2}$

No. of words which two I's are never together = Total No. of words. − No. of words in which two I's are together

$= \frac{10!}{2} - 9!$

$= 9! - 4$

Required probability = $\frac{\text{No. of words which two I's are never together}}{\text{No. of permutations}}$

$= \frac{9! - 4}{\left(\frac{10!}{2}\right)}$

$= \frac{4}{5}$

(14) 6 boys, 6 girls — sit in a row

a) 6 girls sit together

$\underbrace{GGGGGG}_{1}\ \underbrace{BBBBBB}_{1}$

Total no. of persons = 6 + 1 = 7

Total no. of arrangements in a row of persons = 7!

The girls interchanges their seat by 6! ways.

Favourable no. of outcomes = 7! × 6!

Total no. of ways = 12! (6 + 6)

∴ Required probability = $\frac{7! \times 6!}{12!} = \frac{1}{132}$

b) The boys & girls sit alternately.

Total no. of ways = 12! (6 + 6)

Case - 1: $G\overset{B}{\uparrow}\ G\overset{B}{\uparrow}\ G\overset{B}{\uparrow}\ G\overset{B}{\uparrow}\ G\overset{B}{\uparrow}\ G\overset{B}{\uparrow}$

No. of fav. cases = 6! × 6!
(B) (G)

Case - 2: $B\overset{G}{\uparrow}\ B\overset{G}{\uparrow}\ B\overset{G}{\uparrow}\ B\overset{G}{\uparrow}\ B\overset{G}{\uparrow}\ B\overset{G}{\uparrow}$

No. of fav. cases = 6! × 6!
(B) (G)

Favourable No. of cases $= (6! \times 6!) + (6! \times 6!)$

$\therefore$ Required probability $= \dfrac{\text{Fav. No. of cases}}{\text{Total No. of cases}}$

$$= \frac{(6! \times 6!) + (6! \times 6!)}{(12!)}$$

$$= \frac{2 \times 6! \times \cancel{6!}}{12 \times 9 \times 8 \times 7 \times 11 \times 10 \times \cancel{6!}}$$

$$= \frac{1}{462}$$

(15) 'A' may win in the first round with the probability $1/2$.

He may win the second round, after all of them failed in first round, with the probability $\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2}$

In the third round with probability $= \left(\frac{1}{2}\right)^6 \cdot \frac{1}{2}$ and so on.

By Addition theorem,

The chance of A's success is

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^6 \cdot \frac{1}{2} + \cdots$$

$$= \frac{1}{2}\Big[\underbrace{1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots}_{G.P}\Big] \qquad \frac{a}{1-r}$$

$$= \frac{1}{2}\left[\frac{1}{1-\left(\frac{1}{2}\right)^3}\right]$$

$$= \frac{4}{7}$$

B' may win the first round after A's failure with probability $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$,

second round probability $= \left(\frac{1}{2}\right)^3 \cdot \frac{1}{4}$ and so on.

$\therefore$ B's chance of success $= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2}\right)^3 + \cdots$

$$= \frac{1}{4}\Big(\underbrace{1 + \left(\tfrac{1}{2}\right)^3 + \cdots}_{G.P}\Big)$$

$$= \frac{1}{4}\left(\frac{1}{1-\left(\frac{1}{2}\right)^3}\right)$$

$$= \frac{2}{7},$$

Similarly, the chance of winning the game for C $= \dfrac{\frac{1}{8}}{1-\left(\frac{1}{2}\right)^3} = \dfrac{1}{7}$,

(16) A Committee of 4 students out of 15 can be formed in $^{15}C_4$ ways.

(10 boys + 5 girls)

Let 'E' be the event of forming a committe with at least 3 girls

1 boy 3 girls + 0 boy 4 girls.

No. of ways of forming the Committee is = The no. of favourable ways to E.

$= \left({}^{10}C_1 \times {}^{5}C_3\right) + \left({}^{10}C_0 \times {}^{5}C_4\right)$

$= 100 + 5$

$= 105$

$P(E) = \dfrac{\text{No. of Favourable ways to 'E'}}{\text{Exhaustive ways to 'E'}}$

$P(E) = \dfrac{105}{{}^{15}C_4} = 0.0769$

(17) Given that 5 men out of 100 and 25 women out of 10000 are color blind.

Now, A color blind person is choosen at random.

The Probability that the chosen person is male $= P(m) = \frac{1}{2}$.

The probability that the chosen person is female $= P(W) = \frac{1}{2}$.

Let B represent a blind person. Then,

$$P(B/M) = 5/100 = 0.05$$

$$P(B/W) = 25/10000 = 1/400 = 0.0025$$

The Probability that the chosen person is male is given by,

$$P(M/B) = \frac{P(B/m) \cdot P(M)}{P(m) \cdot P(B/m) + P(W) \cdot P(B/W)}$$

$$= \frac{0.05 \times 0.5}{(0.05 \times 0.5) + (0.5 \times 0.0025)}$$

$$= 0.95$$

(8) Given; $P(H/M) = \frac{4}{100}$

$P(H/W) = 1/100$

$P(W) = \frac{60}{100} = \frac{3}{5}$ $\quad P(W/H) = ?$

$P(M) = 1 - P(W) \quad (\because P(M) + P(W) = 1)$

$P(M) = 1 - \frac{3}{5}$

$P(M) = \frac{2}{5}$

From Baye's theorem,

$$P(W/H) = \frac{P(W) \cdot P(H/W)}{P(M) \cdot P\left(\frac{H}{M}\right) + P(W) \cdot P\left(\frac{H}{W}\right)}$$

$$P(W/H) = \frac{\frac{3}{5} \cdot \frac{4}{100}}{\frac{2}{5} \cdot \frac{4}{100} + \frac{3}{5} \cdot \frac{1}{100}} = \frac{3}{8+3}$$

$$P(W/H) = \frac{3}{11}$$

(19) Let $P(A)$ be the probability of 'A' hitting target.

$P(B)$ be the probability of 'B' hitting target.

$P(C)$ be the probability of 'C' hitting target.

Given; $P(A) = \frac{3}{5}$; $P(B) = 2/5$; $P(C) = 3/4$

$$P(\bar{A}) = \frac{2}{5} \; ; \; P(\bar{B}) = \frac{3}{5} \; ; \; P(\bar{C}) = 1/4$$

$P(A \cup B \cup C) = ?$

The probability of that none of A, B, C hits the target $= P(\bar{A} \cap \bar{B} \cap \bar{C})$

$$= P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

$$\Rightarrow \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{50}$$

$\therefore$ Required probability $= P(A \cup B \cup C)$

= The probability of atleast one of A, B, C hitting the target.

$$\Rightarrow 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - \frac{3}{50}$$

$$= \frac{47}{50}.$$

(20) Given; $P(A) = \frac{1}{3} \rightarrow P(\bar{A}) = 2/3$

$P(B) = \frac{1}{4} \rightarrow P(\bar{B}) = 3/4$

$P(C) = \frac{1}{5} \rightarrow P(\bar{C}) = 4/5$

(i) Probability that one of them misses target

Required probability = $P(\bar{A} \cap B \cap C) + P(A \cap \bar{B} \cap C) + P(A \cap B \cap \bar{C})$

Here A, B, C are independent events

$= P(\bar{A}) \cdot P(B) \cdot P(C) + P(A) \cdot P(\bar{B}) P(C) + P(A) P(B) P(\bar{C})$

$= \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5}$

$\Rightarrow \frac{2 + 12 + 4}{3 \cdot 4 \cdot 5} = \frac{18}{12 \cdot 5}$

$= \frac{3}{10}$

(ii) probability that only second man misses the target.

Required probability $= P(A \cap \bar{B} \cap C) = P(A) \cdot P(\bar{B}) \cdot P($

$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{5}$

$= \frac{1}{20}$

Here ; A, B, C → independent events.

Verified!

Prepared by:

M. Tharun Charan

— AIML-C