MODULE - I

Prepo

PART - A

① Given, $x^2y\,dx - (x^3+y^3)\,dy = 0$ — ①

It is of the form. $M\,dx + N\,dy = 0$

$M = x^2y$; $N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2$; $\frac{\partial N}{\partial x} = -3x^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ Not exact differential equation.

① is Homogenous differential equation

$Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$

$\therefore$ I.F $= \frac{1}{Mx+Ny} = \frac{-1}{y^4}$

① × I.F $\Rightarrow -\frac{x^2y}{y^4}dx + \frac{x^3+y^3}{y^4}dy = 0$

It is of the form $M_1\,dx + N_1\,dy = 0$

$M_1 = -\frac{x^2}{y^3}$; $N_1 = \frac{x^3+y^3}{y^4}$

y^{-3}

y^{-3+1}

$-3.y^{-3-1}$

$-3/y^4$

$\frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4}$; $\frac{\partial N_1}{\partial y} = \frac{3x^2}{y^4}$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \rightarrow \text{exact d.E}$$

The General solution is

$$\underset{(y-\text{const})}{\int M_1\,dx} + \underset{(\cancel{y}-x)-\text{ terms independent of } x}{\int N_1\,dy} = c$$

$$\int -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{x^3}{3y^3} + \log|y| = \underline{c} \rightarrow \text{General solution}$$

② Given, $2xy\,dy - (x^2+y^2+1)\,dx = 0$ —①

It is of the form $M\,dx + N\,dy = 0$.

$M = -(x^2+y^2+1)$ $\quad N = 2xy$

$\frac{\partial M}{\partial y} = -2y$ $\quad \frac{\partial N}{\partial x} = 2y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow \underline{\text{Non Exact d.E}}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y - 2y = -4y$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy} \times -4y = -\frac{2}{x} = f(x)$$

$$I \cdot F = e^{\int f(x)\,dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}}$$

$$= 1/x^2$$

$$① \times I \cdot F \Rightarrow \frac{2y}{x} dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = 0$$

It is of the form $M_1 dx + N_1 dy = 0$

$$M_1 = -1 - \frac{y^2}{x^2} - \frac{1}{x^2} \qquad N_1 = 2y/x$$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \rightarrow \text{Exact d.E}$$

General solution is,

$$\underset{(y-\text{const})}{\int M_1 dx} + \underset{(x-x)}{\int N_1 dy} = c$$

$$\int \left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right) dx + \int 0\, dy = c$$

$-x + \frac{y^2}{x} + \frac{1}{x} = c$ is our required solution.

(3) Given $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

It is of the form $\frac{dy}{dx} + Py = Q$ (we need to convert)

$$\Rightarrow (1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{4x^2}{1+x^2}$$

form: $\frac{dy}{dx} + Py = Q$

$$\int \frac{f'(x)}{f(x)}dx = \log(f(x))$$

$$P = \frac{2x}{1+x^2} \; ; \; Q = \frac{4x^2}{1+x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)}$$

$$= 1+x^2$$

General solution is given by,

$$y \times I.F = \int Q \times I.F \, dx + c$$

$$y \times (1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)\,dx + c$$

$y(1+x^2) = \frac{4x^3}{3} + c$ is our required solution,

(4) $\frac{dy}{dx} + 2y = e^x + x$

Form: $\frac{dy}{dx} + Py = Q \rightarrow P = 2;\ Q = e^x + x.$

$I.F = e^{\int P dx} = e^{\int 2 dx} = e^{2x}.$

solution is given by, $y(I.F) = \int Q(IF)\,dx + c$

$\Rightarrow y(e^{2x}) = \int (e^x + x)e^{2x}\,dx + c.$

$\Rightarrow y(e^{2x}) = \int (e^{3x} + x e^{2x})\,dx + c$

$\int uv\,dx = uv_1 - u'v_2$

$\Rightarrow y e^{2x} = \frac{e^{3x}}{3} + x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$ is our

required solution.

(5) Given parabola is $y^2 = 4a(x+a).$ — (1)

$\Rightarrow y^2 = 4ax + 4a^2$

Diff. w.r.to 'x'

$2y.\frac{dy}{dx} = 4a.1 + 0$

$2yy_1 = 4a.$

$a = \frac{yy_1}{2}.$ — (2)

Substitute (2) in (1)

$$y^2 = 4\,\frac{yy_1}{2}\,x + 4\cdot\frac{y^2y_1^2}{4}$$

(or) $$y^2 = 2xyy_1 + y^2y_1^2 \quad — (3)$$

Above equation is a d.E

Replace y_1 with $-1/y_1$ in (3)

$$y^2 = 2xy\left(\frac{-1}{y_1}\right) + y^2\left(\frac{-1}{y_1}\right)^2$$

$$y^2 = \frac{-2xy}{y_1} + \frac{y^2}{y_1^2}$$

$$y^2y_1^2 = -2xyy_1 + y^2$$

$$y^2 = 2xyy_1 + y^2y_1^2 - (4)$$

(3) & (4) are same, hence the given system is self orthogonal.

(6) Let 'T' is the temperature of coffee at time 't',

According to NLC, (Newton's law of Cooling)

$$\frac{dT}{dt} = -K\,(T - T_0),\quad K > 0$$

Given $T_0 = 24°c$

separating variables & Integrating,

$$\int \frac{dT}{T-24} = -K \int dt$$

$$\log(T-24) = -Kt + \log c$$

$$T-24 = c e^{-Kt} \quad — ①$$

when $t=0$, $T = 92°c$

from ① substitute t & T, we get $\boxed{c=68}$

$$\hookrightarrow T-24 = 68 e^{-Kt} \quad — ②$$

When $t=1$; $T = 80°c$

from ②; $e^{K} = \frac{68}{56} \Rightarrow K = \log\left(\frac{68}{56}\right)$

When; $T = 65°c$, $t = ?$

from ②; $65 - 24 = 68 e^{-\log\left(\frac{68}{56}\right)t}$

$$41 = 68 \cdot \left(\frac{56}{68}\right) \cdot t$$

$$t = \frac{41}{56} \text{ mins}$$

(7) Given; $x^2 + y^2 = a^2$ — (1)

Diff eq (1) w.r.to 'x'

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2x + 2yy' = 0$$

$$x + yy' = 0 \Rightarrow y' = \frac{-x}{y}$$

Replacing y' with $-\frac{1}{y'}$, we get d.E Corresponding to orthogonal trajectories as.

$$\cancel{x + y\left(\frac{-1}{y}\right)} \quad +\frac{1}{y'} = \frac{x}{y}$$

$$y' = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating on Both sides

$$\int \frac{1}{y}\, dy = \int \frac{1}{x}\, dx$$

$$\log|y| = \log|x| + c$$ is

our required orthogonal trajectory.

(8) Given ; $(x^4 e^x - 2mxy^2)dx + (2myx^2)dy = 0$ — (1)

It is of the form $M\,dx + N\,dy = 0$

$M = x^4 e^x - 2mxy^2$ | $N = 2myx^2$

$\frac{\partial M}{\partial y} = -4mxy$ | $\frac{\partial N}{\partial x} = 4mxy$

$\partial M/\partial y \neq \partial N/\partial x \rightarrow$ Non Exact d.E

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4mxy - 4mxy = -8mxy.$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\cancel{2}\cancel{m}x^{\cancel{2}}\cancel{y}}(-\cancel{8}\cancel{m}\cancel{x}\cancel{y}) = -4/x$$

$$f(x) = -4/x$$

$$I.F = e^{\int f(x)\,dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

① $\times$ I.F

$$\Rightarrow \left(e^x - \frac{2my^2}{x^3}\right)dx + \frac{2my}{x^2}\,dy = 0$$

form: $M\,dx + N\,dy = 0$

$$M = e^x - \frac{2my^2}{x^3} \qquad N = \frac{2my}{x^2}$$

$$\frac{\partial M}{\partial y} = -\frac{4my}{x^3} \qquad \frac{\partial N}{\partial x} = -\frac{4my}{x^3}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact d.E} \checkmark$$

The Solution is, $\int M\,dx + \int N\,dy = 0$
(y- const) (x - X)

$$\Rightarrow \int \left(e^x - \frac{2my^2}{x^3}\right) dx + \int 0\,dy = c$$

$$\Rightarrow e^x - 2my^2\left(-\frac{1}{2x^2}\right) = c$$

$$\Rightarrow e^x + \frac{my^2}{x^2} = c$$ is our required solution,

(a) Given; $x^2+y^2+2gx=0$ — (1)

diff (1) w.r.to 'x'

$$2x+2yy'+2g=0 \qquad y'=dy/dx.$$

$$x+yy'+g=0$$

$$g=-x-yy' \quad — (2)$$

(2) substitute in (1) $\Rightarrow x^2+y^2+2x(-x-yy')=0$

$$\Rightarrow y^2-x^2-2xyy'=0 \quad — (3)$$

Replace y' with $-1/y'$ (or) $dy/dx \to -dx/dy$

$$\Rightarrow y^2-x^2+2xy\frac{dx}{dy}=0$$

$$\Rightarrow 2xy\frac{dx}{dy}=x^2-y^2$$

$$\Rightarrow 2xy\frac{dx}{dy}=\frac{x^2}{y}-y$$

Consider / Let ; $x^2=u$.

$$-2x\frac{dx}{dy}=\frac{du}{dx}.$$

$$\frac{du}{dx}=-\frac{u}{y}=-y.$$

$$I.F=e^{\int -1/y\,dy}=e^{-\log y}=1/y$$

General solution is, $\frac{1}{y}\cdot u = \int -y \cdot \frac{1}{y}\, dy + c$

$$\Rightarrow \frac{1}{y}\cdot x^2 = -y + c$$

(10) By Newton's law of cooling, we have,

$$\frac{d\theta}{dt} = -K(\theta - \theta_0) \quad \text{--- (1)}$$

$\theta_0 = 20^\circ c$ (surrounding temp.)

$d\theta = -K(\theta - 20)\, dt$

$$\frac{d\theta}{\theta - 20} = -K\, dt$$

I. O. B. S (Integrating)

$$\log(\theta - 20) = -Kt + \log c$$

$$\frac{\theta - 20}{c} = e^{-Kt} \Rightarrow \theta - 20 = ce^{-Kt} \quad \text{--- (2)}$$

When; $\theta = 100$, $100 - 20 = c$

$$c = 80$$

from (2) $\theta - 20 = 80e^{-Kt}$ --- (3)

when $t = 10$, $\theta = 75^\circ c$

$$75 - 20 = 80e^{-10K}$$

$$\frac{55}{80} = e^{-10K}$$

$$\frac{11}{16} = e^{-10K} \quad — (4)$$

when, $t = 30$; $\theta - 20 = 80e^{-30K}$

$$\theta - 20 = 80\left(e^{-10K}\right)^3$$

$$\theta - 20 = 80\left(\frac{11}{16}\right)^3 \text{ from } (4)$$

$$\theta = 20 + 80\left(\frac{1331}{4096}\right)$$

when; $\theta = 25^\circ \Rightarrow 25 - 20 = 80e^{-Kt}$

from (3) $\Rightarrow 5/80 = e^{-Kt} = \left(\frac{11}{16}\right)^{t/10}$

$$\log\left(\frac{5}{80}\right) = \frac{t}{10}\log\left(\frac{11}{16}\right)$$

$$t = \frac{10(\log 5 - \log 80)}{\log 11 - \log 6}$$

PART-B

① $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

form: $M\,dx + N\,dy = 0$

$M = 1+e^{x/y}$ | $N = e^{x/y}(1-x/y)$

$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$ | $\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\frac{1}{y}$

$\frac{\partial N}{\partial x} = e^{x/y}\left(-x/y^2\right)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ Exact d.E

General Solution is $\underset{y-\text{const}}{\int M\,dx} + \underset{x-x}{\int N\,dy} = c$

$\Rightarrow \int\left(1+e^{x/y}\right)dx + \int 0\,dy = c$

$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$

$\Rightarrow x + ye^{x/y} = c$ is our required soln

(2) $(xe^{xy} + 2y)\frac{dy}{dx} + ye^{xy} = 0$

form : $M\,dx + N\,dy = 0$

$M = ye^{xy}$ | $N = xe^{xy} + 2y$

$\frac{\partial M}{\partial y} = y.e^{xy}.x + e^{xy}(1)$ | $\frac{\partial N}{\partial x} = xe^{xy}.y + e^{xy}1$

$\frac{\partial M}{\partial y} = e^{xy}(xy+1)$ | $\frac{\partial N}{\partial x} = e^{xy}(xy+1)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Exact d.E

General solution is $\underset{y-\text{const}}{\int M\,dx} + \underset{x-x}{\int N\,dy} = c$

$\Rightarrow \int ye^{xy}\,dx + \int 2y\,dy = c$

$y\frac{e^{xy}}{y} + y^2 = c$

$e^{xy} + y^2 = c$ is our required sol^n

(3) $x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x.$

form:

$M\,dx + N\,dy = 0$

$M = 3x^2 \tan y - \cos x$ $\quad N = x^3 \sec^2 y$

$\frac{\partial M}{\partial y} = 3x^2 \sec^2 y$ $\quad \frac{\partial N}{\partial x} = 3x^2 \sec^2 y.$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Exact D.E

General solution is $\int_{y\text{-con}} M\,dx + \int_{x\text{-}\cancel{x}} N\,dy = c$

$\int (3x^2 \tan y - \cos x)\,dx + \int 0\,dy = c$

$\tan y \int \underbrace{3x^2}_{x^3} dx - \int \cos x\,dx = c$

$x^3 \tan y - \sin x = c$ is our required solution.

(4) $(x^2-y^2)\,dx = 2xy\,dy$

$\rightarrow (x^2-y^2)\,dx - 2xy\,dy = 0$

form: $M\,dx + N\,dy = 0$

$M = x^2 - y^2$ | $N = -2xy$

$\frac{\partial M}{\partial y} = -2y$ | $\frac{\partial N}{\partial x} = -2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ "Exact differential equation"

The solution is given by

$$\int_{y\text{-const}} M\,dx + \int_{x-x} N\,dy = c$$

$$\int (x^2-y^2)\,dx + \int 0\,dy = c$$

$$\frac{x^3}{3} - xy^2 = c$$

$x^3 - 3xy^2 = c$ is our required solution.

✓

(5) Let $\theta \rightarrow$ temperature of object at time t

$\theta_1 \rightarrow$ initial temperature of object

$\theta_0 \rightarrow$ temperature of surrounding air.

Given; $\theta_1 = 120°F$; $\theta_0 = 70°F$

By Newton's law of cooling

$$\theta = \theta_0 + (\theta_1 - \theta_0)\, e^{-kt}$$

$$\theta = 70 + (120 - 70)\, e^{-kt}$$

$$\theta = 70 + 50e^{-kt}$$

use the condition $\theta(30) = 95°F$ to determine k

$$95 = 70 + 50\, e^{-30k}$$

$$e^{-30k} = \frac{25}{50} = \frac{1}{2} \Rightarrow -30k = \log(1/2)$$

$$k = \frac{1}{30}\log 2 = 0.023$$

Thus the required solution which gives the temperature of the body at any time t is

$$\theta(t) = 70 + 50e^{-0.023t}$$

$t = 30$ mins.

$$\theta(30) = 70 + 50\, e^{\frac{-0.023(30)}{-0.69}}$$

$$\simeq 95.08°F.$$

⑥ Given, $x^r - y^r = a^r$; $a \to$ parameter. — ①

↓

family of rectangular hyperbola's

diff ① w.r.to x.

$2x - 2y \cdot y' = 0 \Rightarrow y' = x/y$ — ②

Replace dy/dx with $-dx/dy$

$$-\frac{dx}{dy} = \frac{x}{y} \quad (\text{or}) \quad \frac{dy}{dx} = -y/x \quad \text{— ③}$$

$$\frac{dy}{y} + \frac{dx}{x} = c$$

I.O.B.S

$$\log(y) + \log(x) = \log c^r$$

$$\log xy = \log c^r$$

$$xy = \underline{c^r} \quad \text{— ④}$$

④ (This) is the family of orthogonal trajectories of ①

$\therefore$ ① & ④ are mutually orthogonal trajectories

Hence proved.

(7) $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

$$\frac{dy}{dx} - \frac{y}{x(x-1)} = x(x-1)$$

form: $\frac{dy}{dx} + Py = Q$

$$P = \frac{-1}{x(x-1)} \;;\; Q = x(x-1)$$

$$I.F = e^{\int P\,dx} = e^{-\int \frac{1}{x(x-1)}dx} = e^{\int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx}$$

$$= e^{\log x - \log(x-1)}$$

$$= \frac{x}{x-1}$$

General solution is $y(I.F) = \int Q(I.F)\,dx + c$

$$y \cdot \frac{x}{x-1} = \int x(x-1)\frac{x}{x-1}\,dx + c$$

$$\frac{xy}{x-1} = \int x^2\,dx$$

$$\frac{xy}{x-1} = \frac{x^3}{3} + c$$ is our

required solution.

(8) $e^x \cdot \frac{dy}{dx} = 2xy^2 + ye^x$

$$\frac{dy}{dx} = \frac{2xy^2}{e^x} + \frac{y\cancel{e^x}}{\cancel{e^x}}$$

$$\frac{dy}{dx} + (-1)\cdot y = e^{-x}\, 2xy^2$$

Bernoulli's equation.

Dividing with y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = e^{-x}(2x)$$

let; $1/y = u \Rightarrow -1/y^2 \frac{dy}{dx} = \frac{du}{dx}$

$$-\frac{du}{dx} - u = 2xe^{-x}$$

$$\underset{P}{\frac{du}{dx} + u} = \underset{Q}{2xe^{-x}}$$

$$I.F = e^{\int 1 \cdot dx} = e^x$$

General $sol^n \Rightarrow u(I.F) = \int 2x\cancel{e^{-x}} \cdot \cancel{e^x}\, dx + c$

$$\frac{1}{y} e^x = x^2 + c$$ is the

general solution.

(9) By Newton's law of cooling, we have

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$\theta_0 \to$ temp of surrounding medium

$$\frac{d\theta}{dt} = -K(\theta - 25) \Rightarrow \frac{d\theta}{\theta - 25} = -K\,dt$$

I O B S

$$\log(\theta - 25) = -Kt + c \quad -(1)$$

$t = 0, \theta = 140°c \Rightarrow \log(140 - 25) = -K(0) + c$

$$c = \log(115).$$

from (1) $\log(\theta - 25) = -Kt + \log(115)$

$$Kt = \log(115) - \log(\theta - 25) \quad -(2)$$

$t = 20, \theta = 80°c \Rightarrow 20K = \log(115) - \log(80 - 25)$

$$20K = \log(115) - \log(55) \quad -(3)$$

$(2)/(3) \Rightarrow \dfrac{t}{20} = \dfrac{\log(115) - \log(\theta - 25)}{\log(115) - \log(55)}$

when $\theta = 35°c$

$$\Rightarrow \frac{t}{20} = \frac{\log(115) - \log(10)}{\log(115 - \log(55)}$$

$$\frac{t}{20} = \frac{\log(11.5)}{\log\left(\frac{23}{11}\right)} \Rightarrow \frac{t}{20} = 3.31$$

$$t = 66.2 \text{ minutes.}$$

$\therefore$ The temperature will be $35°C$ after 66.2 minutes.

(10) $x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$

$$\frac{dy}{dx} + \frac{(2x^2-1)}{(x(1-x^2))}y = \frac{x^3}{x(1-x^2)}.$$

form: $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x^2-1}{x(1-x^2)} = \frac{2x^2-1}{x(1-x)(1+x)} = \frac{-1}{x} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$$

$$\int P\,dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= -\log\left[x\cdot(1-x)^{1/2}(1+x)^{1/2}\right]$$

$$= \log\frac{1}{x\sqrt{1-x^2}}$$

$$I.F = e^{\int P\,dx} = e^{\log\frac{1}{x\sqrt{1-x^2}}} = \frac{1}{x\sqrt{1-x^2}}$$

$Sol^n \to y(I.F) = \int Q\,(I.F)\,dx + c.$

$$y\,\frac{1}{x\sqrt{1-x^2}} = \int \frac{x^2}{1-x^2} \times \frac{1}{x\sqrt{1-x^2}}\,dx + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-3/2}(-2x)\,dx + c$$

$$= (1-x^2)^{-1/2} + c$$

$$y = x + cx\sqrt{1-x^2}.$$

(11) $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \quad -(1)$$

Bernoulli's equation with 'x'

Divide with x^2, (1)

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}\cdot y = y^3 \quad -(2)$$

let; $1/x = u$

$$\frac{du}{dy} = \frac{-1}{x^2}\frac{dx}{dy}$$

Substituting (2) in (1),

$$-\frac{du}{dy} - uy = y^3$$

$$\frac{du}{dy} + uy = -y^3$$

$$I.F = e^{\int y\, dx} = e^{y^2/2}$$

General solution is $\rightarrow u \cdot e^{y^2/2} = \int -y^3 \cdot e^{y^2/2} dy$

$$u e^{y^2/2} = -2\left(\frac{y^2}{2} - 1\right) e^{y^2/2} + c$$

Substitute; $u = 1/x$; we get,

$$\frac{1}{x} e^{y^2/2} = (2 - y^2) e^{y^2/2} + c$$

$$\Rightarrow \cancel{x(2-y^2) + c e^{y^2}}$$

$$\Rightarrow x(2-y^2) + cx e^{-y^2/2} = 1$$ is our required solution

(12) 2. $\frac{dy}{dx} - y \sec x = y^3 \tan x$

Divide y^3 on both sides.

$$\frac{2}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\sec x = \tan x$$

let $1/y^2 = u \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$$-\frac{du}{dx} - u\sec x = \tan x$$

$$\frac{du}{dx} + u\sec x = -\tan x$$

$$I.F = e^{\int \sec x\,dx} = e^{\log|\sec x+\tan x|}$$

$$= \sec x + \tan x.$$

General solution $\Rightarrow u(\sec x+\tan x) = \int \tan x(\sec x+\tan x)dx$
$+c$

$$\Rightarrow u(\sec x+\tan x) = -\int \tan x\sec x\,dx + \int \underbrace{\tan^2 x}_{\sec^2 x-1}\,dx + c$$

$$\Rightarrow u(\sec x+\tan x) = -\sec x - \tan x + x + c$$

$\therefore$ The required solution is

$$\frac{1}{y^2}(\sec x+\tan x) = x - (\sec x+\tan x) + c$$

(13) $(1-x^2)\frac{dy}{dx} + xy = y^3\sin^{-1}x.$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)y = \left(\frac{\sin^{-1}x}{1-x^2}\right)y^3$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)\cdot\frac{1}{y^2} = \frac{\sin^{-1}x}{1-x^2} \quad -① $$

Put $1/y^2 = v$ $\Rightarrow$ $-\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$.

from ①

$$-\frac{1}{2}\frac{dv}{dx} + \frac{x}{1-x^2}v = \frac{\sin^{-1}x}{1-x^2}.$$

$$\frac{dv}{dx} - \frac{2x}{1-x^2}v = \frac{-2\sin^{-1}x}{1-x^2} \quad —②$$

$$I.F = e^{\int P dx} = e^{-\int \frac{2x}{1-x^2}dx} = e^{\log(1-x^2)}$$

$$= (1-x^2)$$

Gen sol^n is $\rightarrow$ $v(I.F) = \int Q(IF)\,dx + c$

$$\Rightarrow v(1-x^2) = -2\int \frac{\sin^{-1}x}{\cancel{1-x^2}}\cancel{(1-x^2)}\,dx + c$$

$$\Rightarrow v(1-x^2) = -2\int \sin^{-1}x.\,dx + c$$

$$\Rightarrow \frac{1}{y^2}(1-x^2) = -2\left(x\sin^{-1}x + \sqrt{1-x^2}\right) + c$$ is our required solution.

(14) Question Error ! (Ref: JNTU TB)

Q) $\Rightarrow$ solve $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

Given;

$$(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0 \quad \text{--- (1)}$$

form: $M\,dx + N\,dy = 0$

$M = xy^2 - x^2$ | $N = 3x^2y^2 + x^2y - 2x^3 + y^2$

$\frac{\partial M}{\partial y} = 2xy$ | $\frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow \text{Non Exact d.E}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{M}\left(6xy^2 + \cancel{2xy} - 6x^2 - \cancel{2xy}\right)$$

$$= \frac{1}{\cancel{xy^2 - x^2}} \times 6\,(\cancel{xy^2 - x^2})$$

$$= 6 = g(y)$$

$$\therefore I.F = e^{\int g(y)\,dy} = e^{\int 6\,dy} = e^{6y}$$

(1) $\times$ I.F

$$\Rightarrow e^{6y}(xy^2 - x^2)\,dx + e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$$

form: $M_1\,dx + N_1\,dy = 0$

$$M_1 = \underset{u}{e^{6y}}\,\underset{v}{(xy^2 - x^2)}$$

$$\frac{\partial M_1}{\partial y} = 6e^{6y}(xy^2 - x^2) + e^{6y}(2xy)$$

$$\frac{\partial m_1}{\partial y} = e^{6y}(6xy^2 - 6x^2 + 2xy)$$

$$N_1 = e^{6y}(3x^2y^2 + x^2y - 2x^3 + \cancel{y^2})^0$$

solving:

$$\frac{\partial N_1}{\partial x} = e^{6y}(6xy^2 - 6x^2 + 2xy)$$

$$\therefore \frac{\partial m_1}{\partial y} = \frac{\partial N_1}{\partial x} \rightarrow \text{Exact d.E}$$

The solution is $\underset{y\text{-const}}{\int M\,dx} + \underset{x-x}{\int N\,dy} = c$

$$\Rightarrow \int e^{6y}(xy^2 - x^2)\,dx + \int y^2\,dy = c$$

$$\Rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) + \frac{y^3}{3} = c$$ is our required solution.

Note:

Q) If $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3)\,dy = 0$

$\downarrow$

Sol$^n \rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) = c$

Also If $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

$\downarrow$

Sol$^n \rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) + \frac{y^3}{3} = c$

(15) $x^2 \frac{dy}{dx} = e^y - x$

$$\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$$

$$e^{-y} \frac{dy}{dx} = \frac{1}{x^2} - e^{-y} \cdot \frac{1}{x} \quad - (1)$$

Put $e^{-y} = z$

$$e^{-y} \frac{dy}{dx} = -\frac{dz}{dx}$$

from (1) $-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$

$$\frac{dz}{dx} - \frac{z}{x} = \frac{-1}{x^2}$$

$$I.F = e^{\int -1/x \, dx} = e^{-\log x} = 1/x$$

$Sol^n \rightarrow$ $z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x} \, dx + c$

$$\frac{e^{-y}}{x} = \int -\frac{1}{x^3} \, dx + c$$

$$\frac{e^{-y}}{x} = \frac{1}{2x^2} + c$$ is our

required solution.

(16) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx = 0$

form: $M\,dx + N\,dy = 0$

$M = y\cos x + \sin y + y$ | $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ | $\frac{\partial N}{\partial x} = \cos x + \cos y + 1.$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Exact d.E

sol^n $\rightarrow$ $\int\limits_{y-\text{const.}} M\,dx + \int\limits_{x-x} N\,dy = c$

$\int (y\cos x + \sin y + y)\,dx + \int 0 \cdot dy = c$

$\Rightarrow y\sin x + x\sin y + xy = c$ is our required solution.

(17) $x^2 + y^2 + 2gx + c = 0$ —— (1)

diff. w.r.to x.

$2x + 2y\frac{dy}{dx} + 2g = 0$

$2g = -\left(2x + 2y\frac{dy}{dx}\right)$ —— (2)

(2) in (1)

$$x^2+y^2-\left(2x+2y\frac{dy}{dx}\right)x+c=0$$

$$y^2-x^2-2xy\frac{dy}{dx}+c=0 \quad \text{— (3)}$$

Replace dy/dx with $-dx/dy$

$$y^2-x^2+2xy\frac{dx}{dy}+c=0$$

$$2x\frac{dx}{dy}-\frac{x^2}{y}=\frac{-(c+y^2)}{y} \quad \text{— (4)}$$

Put $x^2=u$

$$2x\frac{dx}{dy}=\frac{du}{dy} \quad \text{— (5)}$$

(5) in (4) $$\frac{du}{dy}-\frac{u}{y}=-\left(\frac{c+y^2}{y}\right) \quad \text{— (6)}$$

$$I.F=e^{\int -\frac{1}{y}dy}=e^{-\log y}=1/y$$

Gen sol^n of (6)

$$u\cdot\frac{1}{y}=-\int\left(\frac{c+y^2}{y}\right)\frac{1}{y}\,dy+K$$

$$\frac{u}{y}=-\int\left(1+\frac{c}{y^2}\right)dy+K$$

$\Rightarrow u = -y^2 + C + Ky$

$\Rightarrow x^2 + y^2 - Ky - C = 0$

~~or~~ Above equation is family of orthogonal trajectories of family of circles $x^2 + y^2 + 2gx + C = 0$

(18) $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

$$\frac{dy}{dx} - \left(\frac{y}{x+1}\right) = e^{3x}(x+1)$$

form: $\frac{dy}{dx} + Py = Q$

$P = -\frac{1}{x+1}$; $Q = e^{3x}(x+1)$

$I.F = e^{\int P dx} = e^{-\int \frac{1}{x+1} dx} = e^{-\log(x+1)} = 1/x+1$

General sol^n is $\rightarrow y \times I.F = \int Q\,(I.F)\,dx + C$

$$y \cdot \frac{1}{x+1} = \int e^{3x}(x+1) \cdot \frac{1}{x+1}\, dx + C$$

$$y/x+1 = e^{3x}/3 + C$$

$$y = \left(\frac{e^{3x}}{3} + C\right)(x+1)$$ is our

required solution.

(19) $(x^2 - ay)dx = (ax - y^2)dy$

$\Rightarrow (x^2 - ay)\,dx + (y^2 - ax)\,dy = 0$

form: $M\,dx + N\,dy = 0$

$M = x^2 - ay$ | $N = y^2 - ax$

$\frac{\partial M}{\partial y} = -a$ | $\frac{\partial N}{\partial x} = -a$

$\partial M/\partial y = \partial N/\partial x \rightarrow$ Exact d.E

The required general solution is

$\int M\,dx + \int N\,dy = c$
(y-const) (x-x)

$\Rightarrow \int (x^2 - ay)\,dx + \int y^2\,dy = c$

$\Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = c$ is our required Solution.

(20) $y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0$ — (1)

form: $M\,dx + N\,dy = 0$

$M = y(x^2y^2+2)$ | $N = x(2-2x^2y^2)$

$\frac{\partial M}{\partial y} = 3x^2y^2+2$ | $\frac{\partial N}{\partial x} = 2-6x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ Non exact d.E

$Mx - Ny = 3x^3y^3 \neq 0$

$I.F = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3} \neq 0$

(1) $\times$ I.F

$\Rightarrow \frac{x^2y^2+2}{3x^3y^2}dx + \frac{2-2x^2y^2}{3x^2y^3}dy = 0$ — (2)

form $M_1 dx + N_1 dy = 0$

$M_1 = \frac{x^2y^2+2}{3x^3y^2}$ | $N_1 = \frac{2-2x^2y^2}{3x^2y^3}$

$\frac{\partial M_1}{\partial y} = \frac{\frac{2x^2y^2}{3x^3 \cdot 2y}}{}$ | $\frac{\partial N_1}{\partial x} = \frac{-4x}{}$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \rightarrow$ exact d.E

General solution is given by

$$\int_{(y\text{-const})} M_1\, dx + \int_{(x-X.)} N_1\, dy = 0$$

$$\frac{1}{3}\int \frac{1}{x}\, dx + \frac{2}{3y^2}\int \frac{1}{x^3}\, dx - \frac{2}{3}\int \frac{1}{y}\, dy = 0$$

$$\frac{1}{3}\log|x| + \frac{-1}{3x^2y^3} - \frac{2}{3}\log|y| = c$$

is our required solution.

Verified!

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