Module-3

Part-A:

1. from the partial differential Equation by Eliminating arbitary function $lx+my+nz = \phi(x^2+y^2+z^2)$

Sol $lx+my+ny = \phi(x^2+y^2+z^2)$ — ①

diff w.r.to x.

$l+np = f'(x^2+y^2+z^2)(2x+2zp)$ — ②

diff w.r.to y.

$m+nq = f'(x^2+y^2+z^2)(2y+2zq)$ — ③

Divide Eq ② & ③

$$\frac{l+np}{m+nq} = \frac{f'(2x+2zp)}{f'(2y+2zq)}$$

$$\frac{l+np}{m+nq} = \frac{x+zp}{y+zq}$$

$(l+np)(y+zq) = (m+nq)(x+zp)$

$ly+lzq+npy+\cancel{npzq} = mx+mzp+nqx+\cancel{npzq}$

$\Rightarrow npy-mzp+lzq-nqx = ly-mx$

$\Rightarrow (ny-mz)p + (lz-nx)q = -ly+mx$

② from the partial differential Equation by eliminating the Arbitary function $xy+yz+zx = f\left(\frac{z}{x+y}\right)$

$xy+yz+zx = f\left(\frac{z}{x+y}\right)$

$y+yp+xp+z = f'\left[\frac{(x+y)p-z}{(x+y)^2}\right]$ — ①

$x+yq+z+xq = f'\left[\frac{(x+y)q-z}{(x+y)^2}\right]$ — ②

divide Eq ① & ②

$$\frac{y+yp+z+xp}{x+yq+z+xq} = \frac{f'\left(\frac{(x+y)p-z}{(x+y)^2}\right)}{f'\left(\frac{(x+y)q-z}{(x+y)^2}\right)}$$

$$\frac{y+yp+z+xp}{x+yq+z+xq} = \frac{(x+y)p-z}{(x+y)q-z}$$

$$\Rightarrow \frac{(y+yp+z+xp)}{x+yq+z+xq} = \frac{(x+y)p-z}{(x+y)q-z}$$

$$= (y+yp+z+xp)((x+y)q-z)$$

$$= (x+yq+z+xq)((x+y)p-z)$$

$$\rightarrow (xy+2yz+2zx+x^2)p -$$

$$(xy+2yz+2zx+y^2)q$$

$$(\cancel{xy+2yz+2zx+y^2})$$

$$= xz-yx$$

(3) from the partial diff eqn by eliminating Arbitary function $f(x^2+y^2, x^2-z^2)=0$

Sol Given:

$$xy+yz+zx = f\left(\frac{z}{(x+y)}\right)$$

$$y+yp+z+xq = f'\left(\frac{z'}{x+y}\right)$$

$$x^2-y^2 = f(x^2-z^2) \quad -(1)$$

$$2x = f'(x^2-z^2)(2x-2zp) \quad -(2)$$

$$-2y = f'(-x^2-z^2)(-2zq)$$

dividing eq (2) & (3)

$$\frac{2y}{-2y} = \frac{2(x-zp)f'(x^2-z^2)}{-2zq\, f'(x^2-z^2)}$$

$$\frac{x}{y} = \frac{x-zp}{zq}$$

$$xzp = xy-yzp$$

$$\boxed{(yz)p+(2z)q = xy}$$

(4) from the partial diff eqn by eliminating the arbitary function 'c' from $z = f(x+ct)+g(x-ct)$.

Sol $z = f(x+ct)+g(x-ct)$

$$\frac{\partial z}{\partial x} = p = f'(x+ct) + g(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) = 0$$

$$\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial t}$$

(5) di E fu Sol

$\frac{\partial z}{\partial y} = q = cf'(x+ct) - cg'(x-ct)$

$\frac{\partial^2 z}{\partial t^2} = t = c^2(f''(x+ct) + g''(x-ct)$

$\Rightarrow rc^2 = t$

⑤ from the partial differential Eqn by Eliminating the Arbitary function $z = f(x+y) + g(x+iy)$.

Sol $z = f(x+iy) + g(x+iy)$

$\frac{\partial z}{\partial x} = f'(x+iy) + g'(x+iy)$

$\frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x+iy)$ — ①

$\frac{\partial z}{\partial y} = if'(x+iy) + ig(x+iy)$

$\frac{\partial^2 z}{\partial y^2} = f''(x+iy)i^2 + g''(x+iy)i^2$

$\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - g''(x+iy)$ — ②

Eq. ① + ②

$r + t = 0$

part-B.

① Form the PDE by Eliminating orbitary function from $f(x^2+y^2+z^2), z^2-2xy) = 0$

$f(x^2+y^2+z^2, z^2-2xy) = 0$

$u = x^2+y^2+z^2$, $v = z^2-2xy$

$u = f(v)$

$x^2+y^2+z^2 = f(z^2-2xy)$ — ①

$2x + 2zp = f'(2zp - 2y)$ — ②

$2y + 2zq = f'(2zq - 2x)$ — ③

Eq. ② divided by ③

$\frac{2(x+2p)}{2(y+3q)} = \frac{\cancel{f'}2(3p-y)}{f'\cancel{2}(3q-x)}$

$\Rightarrow (x+2p)(2q-x) = (3p-y)(y+3q)$

$\Rightarrow x2q - x^2 + \cancel{z}pq - 2xp$

$= 2py + \cancel{z}pq - y^2 - 2yq$

$\Rightarrow zpy + xzp - zyq - xzq = x^2 + y^2$

$\Rightarrow (zy+zx)p - (zy+zx)q = -x^2+y^2$

$\Rightarrow \boxed{zp - zq = y - x}$

② Form a PDE by Eliminating a, b, c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\Rightarrow \frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0$$

$$c^2x + a^2zp = 0 \quad \text{— ②}$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$$

$$c^2y + b^2zq = 0 \quad \text{— ③}$$

Again d.w. to x,

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0.$$

On sub $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ from ② we get

$$= -\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

$$xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\left(\frac{\partial z}{\partial x}\right) = 0 \quad \text{— ①}$$

Again d.w.r. to y and sub $\frac{c^2}{b^2}$ from ③ in the resultant Equation of

$$\Rightarrow yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\left(\frac{\partial z}{\partial y}\right) = 0 \quad \text{— ③}$$

③ Solve the PDE $(x^2 - yz)p + (y^2 - 3z)q = z^2 - xy$ form the partial deq^n.

Constants $z = ax^3 + by^3$

$Pp + Qq = R$

$P = x^2 - yz$

$Q = y^2 - 3z$

$R = z^2 - xy$ Lts

$$\frac{dx}{P} = \frac{dy}{q^2 - zx\,9} = \frac{dz}{R}$$

$$\frac{dx}{x^2 - y^2} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad -(1)$$

Fraction of Eq^n can be taken as

$$\frac{dx - dy}{(x-y)(x+y+z)}$$

$Z = ax^3 + by^3$

$\frac{\partial z}{\partial x} = p = 3ax^2$ — (1)

$a = p/3x^2$ — (1)

$a = p/3x^2$ — (2)

$a = p/3y^2 b$

$b = q/3y^2$ — (3)

from Eq (2) & (3).

$$z = \frac{p}{3x^3} + \frac{q}{3y^2}$$

$$z = \frac{p}{3}x + \frac{q}{3}y$$

$$\boxed{3z = px + qy}$$

(9) From the PDE by Eliminating arbitary Constants h, k from $(x-h)^2 + (y-k)^2 + z^2 = a^2$

$(x-h)^2 + (y-k)^2 + z^2 = 0$ — (1)

$2(x-h) + 2zp = 0$

$(x-h) = -zp$ — (2)

$2(y-k) = -2zq$

$(y-k) = -zq$ — (3)

(2) & (3) sub in Eq (1)

$(-zp)^2 + (-zq)^2 + z^2 = a^2$

$z^2(p^2 + q^2 + 1) = a^2$

(8) From the PDE by eliminating the arbitary function

$Z = f(x) + e^y g(x)$

$z = f(x) + e^y g(x)$

$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x) = p$

$\frac{\partial^2 z}{\partial x^2} = f''(x) + e^y g''(x)$

$= q$

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$\frac{\partial z}{\partial y} = q = g(x) e^y$

$\frac{\partial^2 z}{\partial y^2} = t = e^y g(x)$

$\boxed{q - t = 0}$

⑥ find the d.eqn of all spheres lies on z-axis with a given radius 'r'.

$(x-0)^2 + (y-0)^2 + (z-c)^2 = r^2$ — ①

$x^2 + y^2 + (z-c)^2 = r^2$ — ②

d.w.r to x

$2x + 2(z-c) p = 0$

$(z-c) = -x/p$ — ③

d.w.r to y

$2y + 2(z-c) q = 0$

$(z-c) = -y/q$ — ④

divide ③ & ④

$\frac{\cancel{z-c}}{\cancel{z-c}} = \frac{\cancel{-}x/p}{\cancel{-}y/q}$

$\frac{y}{q} = \frac{x}{p}$

$\boxed{yp = xq}$

⑦ From the PDE by Eliminating arbitrary function of from

$z = xy + f(x^2+y^2)$

$z = xy + f(x^2+y^2)$

$z - xy = f(x^2+y^2)$ — ①

$p - y = f'(x^2+y^2) 2x$ — ②

$q - x = f'(x^2+y^2) 2y$ — ③

Taking ② ÷ ③

$\frac{p-y}{q-x} = \frac{\cancel{2}x f}{\cancel{2}y f(x)}$

$y(p-y) = x(q-x)$

$\boxed{py - qx = y^2 - x^2}$

⑧ form the PDE by Eliminating the Arbitrary function from

$f(x^2+y^2+z^2, x^2-2xy) = 0$

$f(x^2+y^2+z^2, z^2-2xy)=0$

$u = x^2+y^2+z^2, v = z^2-2xy$

$x^2+y^2+z^2 = f(z^2-2xy)$

$2x+2zp = f'(2zp-2y)$ — ①

$2y+2zq = f'(2zq-2x)$ — ②

Take 1 ÷ 2

$$\frac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)} = \frac{\cancel{2}(zp-y)\cancel{f'}}{\cancel{2}(zq-x)\cancel{f'}}$$

$(x+zp)(zq-x) = (y+zq)(zp-y)$

$zpy + xzp - zyq - xzq = y^2-x^2$

$z(x+y)p - z(y+x)q = (y+x)(y-x)$

$$\boxed{zp - zq = y-x}$$

⑨ form the PDE by Eliminating a & b from $\log(az-1) = x+ay+b$

$\log(az-1) = x+ay+b$

$\frac{1}{az-1}(a \cdot p) = 1$ — ①

$\frac{1}{az-1}(aq) = a$

$aq = a(az-1)$ — ②

Eqn (2) ÷ (1)

$$\frac{aq}{ap} = \frac{a(\cancel{az-1})}{(\cancel{az-1})}$$

$$\frac{q}{p} = a \Rightarrow ap = q \quad \text{--- (3)}$$

Sub eq (3) in eq (1)

$$q = \frac{q}{p} z - 1$$

$$pq = qz - p$$

$$\boxed{p(q+1) = qz}$$

$$\Rightarrow (y+zq)(xyp+yz) = (xyq+xz)(x+zp)$$

$$\Rightarrow xy^2p + y^2z + xyzpq + yz^2q = x^2yq + xyzpq + x^2z + xz^2p$$

$$\Rightarrow \boxed{x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)}$$

(10) form the PDE by Eliminating arbitary function of from

$$xyz = f(x^2+y^2+z^2)$$

$$xyz = f(x^2+y^2+z^2)$$

$$(xyp+yz) = f'(2x+2zp) \quad \text{--- (1)}$$

$$(xyq+xz) = f'(2y+2zq) \quad \rightarrow (2)$$

$1 \div 2$

$$\frac{xyp+yz}{xyq+xz} = \frac{\cancel{2}(x+zp)\cancel{f'}}{\cancel{2}(y+zq)\cancel{f'}}$$

Module - III

part-A

6. Solve the partial Differential Eqn

$(z^2-2yz-y^2)p+(xy+xz)q=xy-yz$.

Auxiliary Eqn:

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\frac{dy}{x(y+z)}=\frac{dx}{z^2-2yx-y^2}=\frac{dz}{xy-zx}$

let, $\frac{dy}{x(y+z)}=\frac{dx}{x(y-z)}$

$ydy-zdy=ydx+zdx$

$d(zy)=zdy+ydx$

$ydy-zdz-d(zy)=0$

$\int ydy-\int zdz-\int d(zy)=0$

$y^2/2-z^2/2-zy=c_1$

$y^2-z^2-2zy=c_1$

$\Rightarrow \frac{xdx+ydy+zdx}{xz^2-2xyz-xy^2+xy^2+xyz+xyz-z^2x}$

$\Rightarrow xdx+ydy+zdz=0$

$\int xdx+\int ydy+\int zdx=c_2$

$x^2/2+y^2/2+z^2/2=c_2$

$x^2+y^2+z^2=c_2$

$\therefore f(y^2-z^2-2zy, x^2+y^2+z^2)=c_2$

7. Solve the partial Differential solution

$p\sqrt{x}+q\sqrt{y}=\sqrt{z}$.

Let, $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}$

$\sqrt{x}=\sqrt{y}+c_1$

$\Rightarrow \sqrt{x}-\sqrt{y}=c_1$

also, $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

$=\int\frac{1}{\sqrt{y}}dy=\int\frac{1}{\sqrt{x}}dx$

$\sqrt{y}=\sqrt{z}+c_2$

$\boxed{\sqrt{y}-\sqrt{z}=c_2}$

$\therefore f(\sqrt{x}-\sqrt{y}, \sqrt{y}-\sqrt{x})=0$

8. Solve the partial Differential Equation $p/x^2+q/y^2=z$

Sol $\frac{dx}{p}=\frac{dy}{q}=\frac{dz}{R}$

$\frac{dx}{1/x^2}=\frac{dy}{1/y^2}=\frac{dy}{1/y^2}$

$\int x^2 dx=\int y^2 dy$

Also, $\frac{dy}{y}=\frac{dz}{1}$

$\int \frac{1}{y}dy=\int \frac{1}{1}dz$

$\log y = z+\log c_2$

$\log y - z = c_2$

$\boxed{\therefore F(x/y, \log y - z)=0}$

9) Solve the partial differential Equation $xp+yq=1$.

Auxiliary Equation

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{1}$

Let $\frac{dx}{x}=\frac{dy}{y}$

$\Rightarrow \int \frac{1}{x}dx=\int \frac{1}{y}dy$

$\log x = \log y + \log c_1$

$\Rightarrow \log x - \log y = \log c_1$

$\Rightarrow \log \frac{x}{y} = \log c_1$

$\Rightarrow \frac{x}{y}=c_1$

$\int x dx + \int y dy + \int z dx = 0$

$(x)\, x^2/2 + y^2/2 + z^2/2 = c_2$

Also $\frac{dy}{y}=\frac{dz}{1}$

$\int \frac{1}{y}dy=\int \frac{1}{1}dz$

$\log y = z+\log c_2$

$\log y - z = c_2$

$\therefore f(x/y, \log y - z)=0.$

10) Solve the partial differential Eq

$y^2p+x^2q=x^2y^2z^2$

Sol $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$

$\frac{dx}{y^2} = \frac{dy}{x^2}$

$x^2dx = y^2dy$

$\Rightarrow x^3 - y^3 = c_1$

$\frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$

$y^2dy = \frac{1}{z^2}dz$

$\frac{y^3}{3} + \frac{1}{2} = c_2$

$\boxed{\therefore f\left(x^3-y^3, \frac{y^3}{3}+\frac{1}{2}\right) = 0}$

part-B.

(11) solve the partial differential Equation $x^2p + y^2q = z(x+y)$

Auxillary eqn: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$

Let, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$= \int \frac{1}{x^2}dx = \int \frac{1}{y^2}dy$

$= -\frac{1}{x} = -\frac{1}{y} + c_1$

$\frac{1}{x} - \frac{1}{y} = \frac{x-y}{xy}\ c_1$

$= \frac{1 \cdot dx - 1 \cdot dy + 0 \cdot dz}{x^2 - y^2} = \frac{dz}{z(x+y)}$

$= \frac{dx - dy}{(x-y)(x+y)} = \frac{dz}{z(x+y)}$

$= \int \frac{dx-dy}{(x-y)(x+y)} = \int \frac{dz}{z(x+y)}$

$= \int \frac{d(x-y)}{(x-y)} = \int \frac{dz}{z}$

$\Rightarrow \log(x-y) = \log z + \log c_2$

$= \frac{x-y}{z} = c_2$

$\boxed{\therefore f\left(\frac{x-y}{xy}, \frac{x-y}{z}\right) = 0}$

(12) solve the partial differ Eqn $p - x^2 = y^2 + q$

Sol Auxillary Eq:

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{y^2+z^2}$$

let, $\frac{dx}{1} = \frac{dy}{-1}$

$$\int \frac{dx}{1} = \int \frac{dy}{-1}$$

$$\int dx + \int dy = 0$$

$$\Rightarrow x+y = c_1$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{y^2+x^2}$$

$$\Rightarrow \frac{-x^2dx + y^2dy + 1dz}{-x^2 - y^2 + 1(y^2+x^2)}$$

$$\Rightarrow \frac{-x^2dx + y^2dy + 1dz}{0}$$

$$\Rightarrow -x^2dx + y^2dy + 1dz = 0$$

$$\int -x^2dx + \int y^2dy + \int 1dz = 0$$

$$\Rightarrow -\frac{x^3}{3} + \frac{y^3}{3} + z = c_2$$

$$\therefore f\left(x+y, -\frac{x^3}{3} + \frac{y^3}{3} + z\right) = 0$$

(13) Solve the partial differential Eqn,

$$y^2zp + x^2zq = xy^2$$

Eqn $y^2zp + x^2zq = xy^2$

Auxillary Eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}}$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c_1$$

$$x^3 - y^3 = c_1$$

also, $\frac{dx}{y^2z} = \frac{dz}{xy^2}$

$$\int x\,dx = \int z\,dz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = c_2$$

$$\boxed{\therefore f(x^3 - y^3, x^2 - z^2) = 0}$$

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14. Solve the partial differential Equation

$p\tan x + q\tan y = \tan z$

Auxiliary eqⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

let $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$$= \int \frac{1}{\tan x} dx = \int \frac{1}{\tan y} dy$$

$$= \log \sin x = \log \sin y + \log c_1$$

$$= \log \sin x - \log \sin y = \log c_1$$

$$= \frac{\sin x}{\sin y} = c_1$$

also $\frac{dy}{\tan y} = \frac{dx}{\tan x}$

$$= \int \frac{1}{\tan y} dy = \int \frac{1}{\tan z} dz$$

$$\log \sin y = \log \sin x + \log c_2$$

$$\log \sin y - \log \sin z = \log c_2$$

$$\boxed{\frac{\sin y}{\sin z} = c_2}$$

$$\therefore F\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin x}\right) = 0$$

(17) Solve the partial Differential eqⁿ $px - qz = z^2 + (x-y)^2$.

Sol Auxiliary eqⁿ.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{-z}$$

$$\Rightarrow \int \frac{dx}{1} = \int \frac{dy}{-1}$$

$$\Rightarrow x = -y + c_1$$

$$x + y = c_1$$

Also, $\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$

$$\Rightarrow \frac{dx}{1} = \frac{z}{z^2 + (x+y)^2} dz$$

$$= \int \frac{dx}{1} = \int \frac{z}{z^2 + c_1^2} dz$$

$$x = \log(z^2 + c_1^2) + c_2$$

$x - \log(z^2 + (x+y)^2) = c_2$

$\therefore f(x+y, x - \log(z^2+(x+y)^2)) = 0$

(14) Solve the partial Differential Eqn

$x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Auxilliary Eqn, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$

Let $\Rightarrow \frac{xdx + ydy + zdx}{0}$

$\Rightarrow xdx + ydy + zdz = 0$

$= \int xdx + \int ydy + \int zdz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$

$\frac{\frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z}}{0}$

$= \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0$

$= \int \frac{1}{x} dx - \int \frac{1}{y} dy - \int \frac{1}{z}$

$= \log x - \log y - \log z$

$\log x - \log(yz) = \log$

$= \frac{x}{yz} = c_2$

$\therefore f\left(\frac{x^2}{2}, \frac{y^2}{2} + \frac{z^2}{2}, \frac{x}{yz}\right)$

(15) Solve the partial D.E of $(x-a)p + (y-b)q + (c-z) = 0$

Sol $\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{dz}{-(c-z)}$

$\int \frac{1}{x-a} dx = \int \frac{1}{y-b} dy$

$\log(x-a) - \log(y-b) = \log c_1$

$\log\left(\frac{x-a}{y-b}\right) = \log c_1$

$\boxed{\frac{x-a}{y-b} = c_1}$

$$\frac{dy}{y-b} = \frac{dz}{-(c-z)}$$

$$= \int \frac{1}{y-b}\,dy = \int \frac{1}{z-c}\,dz$$

$$= \log(y-b) = \log(z-c) + \log c_2$$

$$c_2 = \frac{y-b}{z-c}$$

$$\boxed{\therefore f\left(\frac{x-a}{y-b}, \frac{y-b}{z-c}\right) = 0}$$

(18): Solve the differential Equation

$$\frac{y^2z}{x}p + xzq = y^2$$

Sol: $\dfrac{dx}{\frac{y^2z}{x}} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$

$$\Rightarrow \frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz}$$

$$= \frac{x\,dx}{y^2} = \frac{dy}{x}$$

$$x^2dx = y^2dy$$

$$x^3 - y^3 = c_1$$

$$\Rightarrow \frac{dx}{\frac{y^2z}{x}} = \frac{dz}{y^3}$$

$$x\,dx = z\,dz$$

$$\Rightarrow x^3 - y^3 = c_1$$

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dx}{y^2}$$

$$x\,dx = z\,dz$$

$$x^2 - y^2 = c_2$$

$$\boxed{\therefore f(x^3-y^3, x^2-y^2) = 0}$$

(20) Solve $(x-y)p + (y-x-z)q = z$

Sol: $\dfrac{dx}{x-y} = \dfrac{dy}{y-x-z} = \dfrac{dz}{z}$

$$\frac{dx+dy+dz}{x-y+y-x-z+z} = 0$$

$$\int dx + \int dy + \int dz = 0$$

$$= x + y + z = c_1$$

$$\frac{dx-dy+dz}{2(x-y+z)}$$

$$= \frac{1}{2}\frac{d(x-y+z)}{x-y+z} = \frac{dz}{z}$$

$$\frac{2\,dz}{z} = \frac{d(x-y+z)}{x-y+z}$$

$$\log(x-y+z) = \log z^2 + \log c_2$$

$$c_2 = \frac{x-y+z}{z^2}$$

$$\Rightarrow \boxed{f\left(x+y+z,\ \frac{x-y+z}{z^2}\right) = 0}$$