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Module - 3

PROBABILITY DISTRIBUTION

Introduction: The value of the random variables are corresponding probabilities among arranges such a way that one can suit the mathematical function for probabilities in terms of the value of the random variable.

There are two types of the probability distribution or theoretical distribution:

i) Discreted distribution ii) Continuous distribution

i) Discrete Theoretical distribution:

a) Binomial distribution

b) Poisson distribution

c) Rectangular distribution

d) Negative binomial distribution

e) Geometric distribution.

ii) Continuous Theoretical Distribution:

a) Normal distribution

b) Student's t distribution

c) Student square distribution

d) F - distribution

Discrete Uniform Distribution: A random variable X has an distribution if and only if its probability distribution is given by: $(p(x)) = 1/K$ for $x = x_1, x_2, x_3, x_4, ----, x_n$. The random variable X is then called Discrete Uniform Distribution.

Ex:

(i)

x	0	1
$P(x)$	1/2	1/2

(ii)

x	0	1	2	3
$P(x)$	1/4	1/4	1/4	1/4

Binomial distribution: A random variable x has a binomial distribution if it assumes only non-negative values and its probability density function given by

$$P(x=r) = \begin{cases} nC_r p^r q^{n-r} & \text{where } p+q=1,\ r=0,1,2,3,4,\ldots n \\ 0 & \text{otherwise} \end{cases}$$

Constants of binomial distribution:

i) Mean of the binomial distribution: The Binomial distribution is given by $p(r) = nC_r p^r q^{n-r}$ where $p+q=1$, $r=0,1,2,3,\ldots,n$

mean of x, $\mu = E(x) = \sum_{r=0}^{1} r\,p(r)$

$$= 0 \cdot xq^n + 1 \cdot nC_1 pq^{n-1} + 2 \times nC_2 p^2 q^{n-2} + \ldots + n\ldots$$

$$= npq^{n-1} + 2\frac{n(n-1)}{2!} p^2 q^{n-2} + 3\frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + \ldots + n\ldots$$

$$= np\left\{q^{n-1} + 2\frac{(n-1)}{2!} pq^{n-2} + 3\frac{(n-1)(n-2)p^2q^{n-3}}{3!} + \ldots + p^{n-1}\right\}$$

$$= np\{q+p\}^{n-1}$$

$$\boxed{\therefore \mu = np}.$$

ii) Variance: $\sigma^2 = (\Sigma p_i x_i^2 - \mu^2)$

$$= \Sigma r^2 \cdot P(r) - \mu^2$$

$$= \Sigma (r^2 - r + r) + \Sigma r P(r) - \mu^2.$$

$$= 2 \cdot 1\ {}^nC_2 p^2 q^{n-2} + 3 \cdot 2\ {}^nC_3 p^3 q^{n-3} + 4 \cdot 3 \cdot {}^nC_4 p^4 q^{n-4} + \ldots + n(n-1)\ {}^nC_n p^n + \mu - \mu^2$$

$$= 2\cdot\frac{n(n-1)}{2!}p^2q^{n-2} + 3\cdot 2\,\frac{n(n-1)(n-2)}{3!}p^3q^{n-3} +$$

$$4\cdot 3\,\frac{n(n-1)(n-2)(n-3)}{4!}p^4q^{n-4} + \dots + n(n-1)p^n$$

$$+\mu-\mu^2.$$

$$= n(n-1)p^2\left[q^{n-2} + (n-2)p\cdot q^{n-3} + \frac{(n-2)(n-3)}{2!}p^2q^{n-4}\right.$$

$$\left.+ \dots p^{n-2}\right] + \mu - \mu^2$$

$$= n(n-1)p^2[q+p]^{n-2} + \mu^2 - \mu^2$$

$$= np(1-p)$$

$$= npq.$$

iii) Mode of Binomial Distribution: The mode of binomial distribution is the value of x at which $P(x)$ is maximum then mode $= \begin{cases} \text{integral part of } (n+1)p, \text{ if it is not integer.} \\ (n+1)p \text{ and } (n+1)p-1, \text{ if it is integer.} \end{cases}$

Recurrence Relation:

$$p(r) = nC_r p^r q^{n-r} \rightarrow (1).$$

$$p(r+1) = nC_{r+1}\,p^{r+1}\,q^{n-r-1} \rightarrow (2)$$

Equation (2) divided by eq (1).

$$\frac{p(r+1)}{p(r)} = \frac{nC_{r+1}\,p^{r+1}\,q^{n-r-1}}{nC_r\,p^r q^{n-r}}$$

$$\frac{p(r+1)}{p(r)} = \frac{(n-r)p}{(r+1)q} \Rightarrow p(r+1) = \left[\frac{n-r}{r+1}\right]\frac{p}{q}\cdot p(r).$$

Q. 10 coins are tossed simultaneously. Find the probability of atleast i) 7 heads ii) 6 heads iii) Atleast 1 head.

Sol: Given, $n = 10$.

p = probability of getting head = $\frac{1}{2}$

q = probability of getting tail = $\frac{1}{2}$

By binomial distribution,

$$P(x=r) = {}^nC_r\, p^r q^{n-r}$$

(i) P(atleast 7 heads) $= P(x \geq 7)$ $= P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$$P(x\geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}C_7 \left(\tfrac{1}{2}\right)^7\left(\tfrac{1}{2}\right)^3 + {}^{10}C_8\left(\tfrac{1}{2}\right)^8\left(\tfrac{1}{2}\right)^2 + {}^{10}C_9\left(\tfrac{1}{2}\right)^9\left(\tfrac{1}{2}\right) + {}^{10}C_{10}\left(\tfrac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}\left[120 + 45 + 10 + 1\right]$$

$$= \frac{176}{2^{10}}$$

(ii) P(atleast 6 heads) $= P(x \geq 6)$ $= P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$$P(x\geq 6) = P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}C_6\left(\tfrac{1}{2}\right)^6\left(\tfrac{1}{2}\right)^4 + {}^{10}C_7\left(\tfrac{1}{2}\right)^7\left(\tfrac{1}{2}\right)^3 + {}^{10}C_8\left(\tfrac{1}{2}\right)^8\left(\tfrac{1}{2}\right)^2 + {}^{10}C_9\left(\tfrac{1}{2}\right)^9\left(\tfrac{1}{2}\right) + {}^{10}C_{10}\left(\tfrac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}\left[210 + 120 + 45 + 10 + 1\right]$$

$$= \frac{386}{2^{10}}$$

(iii) $P(x \geq 1) = 1 - P(x < 1)$

$$P(x \geq 1) = P(x=0)$$

$$= {}^{10}C_0 \cdot p^0 \cdot q^{10-0}$$

$$= \frac{10!}{10!\,0!}(1)\left[\frac{1}{2}\right]^{10} \Rightarrow \left(\frac{1}{2}\right)^{10} \Rightarrow \frac{1}{2^{10}}$$

Q. In incidence of occasional disease in an Industry is such that the workers have a 20% of chance of suffering from it. What is the probability that out of 6 workers choosen at random 4 or more will suffer the disease.

Sol: Given, $n = 6$

p = probability of getting diseases = 20% = 0.2

q = probability of not getting disease = 80% = 0.8.

By Binomial distribution,

$$P(x=r) = {}^nC_r\, p^r q^{n-r}$$

(i) P(Four or more) = $P(x \geq 4)$

$P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$

$P(x \geq 4) = {}^6C_4 (0.2)^4 (0.8)^2 + {}^6C_5 (0.2)^5 (0.8)^1 + {}^6C_6 (0.2)^6$

$= 15(0.2)^4 (0.8)^2 + 6(0.2)^5 (0.8) + 1(0.2)^6$

$=$

Q. In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails, that is it.

Sol: Given, $n = 12$

p = probability of getting head = $\frac{1}{2}$

q = probability of getting tail = $1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

By Binomial distribution,

$$P(x=r) = {}^nC_r\, p^r q^{n-r}$$

(i) P(8 head and 4 tails) = P(x = 8)

$P(x=8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4$

$= \frac{1}{2^{12}} (495)$

$= \frac{495}{2^{12}}$

For 256 sets 8 heads & 4 tails.

$= 256 \times \left(\frac{495}{2^{12}}\right)$

$= 30.93 \simeq 31.$

Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys, (ii) 5 Girls, (iii) Either 2 or 3 boys, (iv) Atleast 1 boy. Assume equal probabilities of boys & girls.

Sol:- Given, $n=5$

p = probability of boys = 1/2

q = probability of girls = 1/2

By binomial distribution,

$P(X=r) = {}^nC_r\, p^r q^{n-r}$

(i) P(at least 3) $= P(X \geq 3)$

$= P(X=3) + P(x=4) + P(X=5)$

$= {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + {}^5C_5 \left(\frac{1}{2}\right)^5$

$= \frac{1}{2^5} [10+5+1] = \frac{16}{2^5} = \frac{1}{2}$

$= 800 \times \frac{1}{2}$

$= 400.$

(ii) P(at least 5 girls) $= P(X=5)$

$$= {}^5C_5 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5$$

$$= \frac{1}{2^5}$$

$$= 800 \times \frac{1}{2^5}$$

$$= 25.$$

(iii) P(at least 2 or 3 boys) $= P(X=2) + P(X=3)$

$$= {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2^5}(10+10)$$

$$= \frac{20}{2^5} \times 800$$

$$= 20 \times 25 = 500.$$

(iv) P(at least 1 boy) $= P(X \geq 1)$

$$= 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5$$

$$= 1 - \frac{1}{2^5}$$

$$= 1 - \frac{1}{32}$$

$$= \frac{31}{32}.$$

Poisson Distribution: Poisson distribution as rare distribution of rare events i.e the events whose probability of the occurence is very small but number of trails which could lead to occurence of event are very large.

Conditions:

1) The number of trails 'n' is very large.

2) The probability of trails success 'p' is very small.

3) $np=\lambda$ is finite consonants of Binomial distribution.

$$P(x,\lambda) = P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & \text{if } x=0,1,2,n, \text{ there } \lambda>0 \text{ is parameter} \\ 0 & \text{otherwise.} \end{cases}$$

Examples:

* The number of printing mistakes per page in a large text.

* The number of cars passing a certain point in one minute.

i) Mean of poisson distribution:

$$\text{mean} = E(x) = \sum p_i x_i$$

$$\mu = \sum x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum \frac{x \cdot e^{-\lambda}\lambda^{x}}{x(x-1)!}$$

$$= e^{-\lambda}\sum \frac{\lambda^{x}}{(x-1)!}$$

we put $x-1=t \Rightarrow x=t+1$

$$= e^{-\lambda} \sum \frac{\lambda^{t+1}}{t!}$$

$$= e^{-\lambda} \lambda \cdot \sum \frac{\lambda^{t}}{t!}$$

$$= \underbrace{e^{-\lambda} \cdot \lambda \cdot e^{\lambda}}$$

$$\mu = \lambda.$$

(i) Variance of Poisson distribution:

$$\text{variance } (\sigma^2) = E(x)^2 - (E(x))^2$$

$$= \sum_{x}^{n} x^2 p(x) - \mu^2$$

$$= \sum x^2 \frac{e^{-\lambda} \lambda^x}{x!} - \mu^2$$

$$= e^{-\lambda} \sum \frac{x \lambda^x}{(x-1)!} - \lambda^2$$

$$= e^{-\lambda} \sum \frac{(x-1+1)\lambda^x}{(x-1)!} - \lambda^2$$

$$= e^{-\lambda} \left\{ \sum \frac{(x-1)\lambda^x}{(x-1)(x-2)!} + \sum \frac{\lambda^x}{(x-1)!} \right\} - \lambda^2$$

$$= e^{-\lambda} \left\{ \sum \frac{\lambda^x}{(x-2)!} + \sum \frac{\lambda^x}{(x-1)!} \right\} - \lambda^2$$

$$= e^{-\lambda} \left\{ \sum \frac{\lambda^{y+2}}{y!} + \sum \frac{\lambda^{z+1}}{z!} \right\} - \lambda^2$$

$$= e^{-\lambda} \lambda^2 \sum \frac{\lambda^y}{y!} + e^{-\lambda} \lambda \sum \frac{\lambda^z}{z!} - \lambda^2$$

$$\sigma^2 = \lambda$$

∴ Similarly standard deviation is $\sqrt{\text{variance}}$

$$\therefore \sigma = \sqrt{\text{variance}} = \sqrt{\lambda}.$$

iii) Mode of poisson Distribution :

$$P(x) \geq P(x+1) = \frac{e^{-\lambda}\lambda^{x}}{x!} \geq \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

$$= 1 \geq \frac{\lambda}{(x+1)} \Rightarrow \lambda \leq (x+1)$$

$$= \lambda - 1 \leq x \rightarrow (1)$$

Similarly,

$$P(x) \geq P(x-1) = \frac{e^{-\lambda}\lambda^{x}}{x!} \geq \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$\Rightarrow \frac{\cancel{\lambda^{x}}}{x!} \geq \frac{\cancel{\lambda^{x}}\lambda^{-1}}{(x-1)!}$$

$$= \frac{(x-1)! \geq \lambda^{-1}}{x!} \Rightarrow \frac{1}{x!} = \lambda^{-1}$$

$$\Rightarrow x \leq \lambda \rightarrow (2)$$

From (1) & (2)

$$\Rightarrow \lambda - 1 \leq x \leq \lambda.$$

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Note:

i) If λ is integer and $\lambda - 1$ is integer then a poisson distribution have two maximum values, is bi-model & the modes are $(\lambda-1, \lambda)$.

ii) If λ is not an integer the mode of poisson distribution is integral part of λ.

Recurence relation of poisson Distribution:

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

$$= \frac{\lambda}{x+1} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\boxed{P(x+1) = \frac{\lambda}{(x+1)} \cdot p(x)} \qquad \boxed{P(r+1) = \frac{n-r}{r+1} \frac{p}{q} \cdot p(r)}$$

Q- The probability that an individual suffers a bad reaction of a certain injection is 0.001%, determine the probability that out of 2000 individuals (i) Exactly 3 (ii) More than 2 persons (iii) None (iv) More than 1 individuals suffer a bad reaction.

Sol:- Given, $P = 0.001$

$n = 2000$

Poisson distribution $P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$

Binomial distribution mean $\lambda = np$

$\lambda = 2000(0.001) = 2$

(i) $P(X=3) = \frac{e^{-2}2^3}{3!}$

$= \frac{\cancel{8}^{4} e^{-2}}{3 \times \cancel{2}}$

$= \frac{4e^{-2}}{3}$.

(ii) $P(X>2) = 1 - P(X \leq 2)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$= 1 - \left\{\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!}\right\}$

$= 1 - [e^{-2} + 2e^{-2} + 2e^{-2}] \Rightarrow 1 - 5e^{-2}$.

(iii) $P(X=0) = \frac{e^{-2}}{0!} = e^{-2}$.

(iv) $P(X>1) = 1 - P(X \leq 1)$

$= 1 - \{P(X=0) + P(X=1)\}$

$= 1 - \left\{\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!}\right\}$

$= 1 - [e^{-2} + 2e^{-2}]$

$= 1 - 3e^{-2}$.

Q. A hospital switch board received an average of 4 emergency calls, in a ten minutes interval. What is probability that (i) there are atmost 2 emergency calls in a ten minute interval (ii) there are exactly 3 emergency calls in a ten minutes interval.

Sol: Given, $\lambda = 4$

poisson distribution $(P(X=r)) = \frac{e^{-\lambda}\lambda^r}{r!}$

(i) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$P(X \leq 2) = \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!}$

$= e^{-4} + 4e^{-4} + 8e^{-4}$

$= 13e^{-4}$

(ii) $P(X=3) = \frac{e^{-4}4^3}{3!} = \frac{\cancel{64}^{32}}{3 \cdot \cancel{2}} e^{-4}$

$= \frac{32}{3} e^{-4}$.

Q. Average no. of accidents on any day on a national highway is 1.8 determine the probability that no. of accidents are (i) Atleast 1 (ii) Atmost 1.

Sol:- Given, $\lambda = 1.8$

Poisson distribution $P(X=r) = \frac{e^{-\lambda} \lambda^r}{r!}$

i) $P(\text{atleast } 1) = P(X \geq 1) = 1 - P(X=0)$

$$= \frac{e^{-1.8} \cdot 1.8^0}{0!} = e^{-1.8} = 0.834$$

ii) $P(\text{atmost } 1) = P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{e^{-1.8} \cdot 1.8^1}{1!} = e^{-1.8} \cdot 1.8$$

$= 0.463.$

Q. If a poisson distribution $\frac{3}{2} P(x=1) = P(x=3)$ then find the probability of $P(x \geq 1)$, $P(x \leq 3)$, $P(2 \leq x \leq 5)$

Sol:- Given, $\frac{3}{2} P(X=1) = P(X=3)$

we have, $P(X=r) = \frac{e^{-\lambda} \lambda^r}{r!}$

$$\Rightarrow \frac{3}{2} \cdot \frac{e^{-\lambda} \lambda^1}{1!} = \frac{e^{-\lambda} \lambda^3}{3!}$$

$$\Rightarrow \frac{3}{2} = \frac{\lambda^2}{3 \times 2}$$

$$\Rightarrow \lambda^2 = 3 \times 3$$

$$\Rightarrow \boxed{\lambda = 3}.$$

(i) $P(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0)$

$$= 1 - \frac{e^{-3} 3^0}{0!}$$

$= 1 - e^{-3}$

(ii) $P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$$= \frac{e^{-3} \cdot 3^0}{0!} + \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} \cdot 3^3}{3!}$$

$$= e^{-3} + 3e^{-3} + \frac{9e^{-3}}{2} + \frac{9e^{-3}}{2}$$

$$= 13e^{-3}.$$

Q. If the variance of poisson distribution is 3. Find the probabilities that : (i) $x = 0$ (ii) $x \le 1$ (iii) $0 < x \le 3$ (iv) $1 \le x < 5$

$\because$ mean = variance in PD

Sol:- Given, $\lambda = 3$,

(i) $P(x=0) = \frac{e^{-3} \cdot 3^0}{0!} = e^{-3}$

(ii) $P(x \le 1) = P(x=0) + P(x=1)$

$$= \frac{e^{-3} \cdot 3^0}{0!} + \frac{e^{-3} \cdot 3^1}{1!}$$

$$= e^{-3} + 3e^{-3}$$

$$= 4e^{-3}$$

(iii) $P(0 < x \le 3) = \cancel{P(x=0)} + P(x=1) + P(x=2) + P(x=3)$

$$= \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} 3^3}{3!}$$

$$= 3e^{-3} + \frac{9}{2}e^{-3} + \frac{9}{2}e^{-3}$$

$$= 12e^{-3}$$

(iv) $P(1 \le x < 5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$$= \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} 3^3}{3!} + \frac{e^{-4} 3^4}{4!}$$

$$= 3e^{-3} + \frac{9}{2}e^{-3} + \frac{9}{2}e^{-3} + \frac{27}{8}e^{-3}$$

$$= \frac{12 \cdot 3}{8} e^{-3}.$$

Q. Fit a poisson distribution to the following data and find the expected frequency.

x	0	1	2	3	4
$f(x)$	109	65	22	3	1

Sol: Given, $N = \Sigma f_i$

$= 109 + 65 + 22 + 3 + 1 = 200$

Mean $(\lambda) = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

$\lambda = \dfrac{0(109) + 1(65) + 2(22) + 3(3) + 4(1)}{200}$

$\lambda = 0.61$

Expected frequency $= N\,p(x)$

$\Rightarrow 200 \cdot p(0) = 200 \cdot p(0)$

$\Rightarrow 200 \cdot p(0) = 220 \cdot \dfrac{e^{-0.61} \cdot (0.61)^0}{0!} = 108.67$

$\Rightarrow 200 \cdot p(1) = 220 \cdot \dfrac{e^{-0.61} \cdot (0.61)^1}{1!} = 66.28.$

x	0	1	2	3	4
$f(x)$	109	65	22	3	1
E.f	109	66.	20.21 ≈ 20	4.1 $= 4$	0.63 $= 1$

Q. In thousand sets of trials for an event a small probability frequencies of the no. of success: ~~305, 365, 210~~, find expected frequencies.

x	0	1	2	3	4	5	6	7
$f(x)$	305	365	210	80	28	9	2	1
E.f	309	361	217.0 $= 217$	87	26	6	1	0

Sol: $N = \sum f_i = 1000$

Mean $= \lambda = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+365+420+240+112+112+7+16}{1000}$

$= 1.201$

Expected frequency $= N \cdot P(x)$.

$\Rightarrow 200 \cdot P(0) = 1000 \cdot \frac{e^{-1.201} \cdot (1.201)^0}{0!} = 300.89$

$\Rightarrow 200 \cdot P(1) = 1000 \cdot \frac{e^{-1.201} \cdot (1.201)^1}{1!} = 361.37$

x	0	1	2	3	4	5	6	7
$f(x)$	305	365	210	80	28	9	2	1
E.f	301	361	217	87	26	6	1	0

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Normal Distribution: A continuous random variable X is said to follow normal distribution if its probability function is given by $f(x, \mu, \sigma) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

for $-\infty < x < \infty$, $\sigma > 0$

Here, the x is continuous & it is normal distribution.

- The distribution has two parameters μ and σ is denoted by $N(\mu, \sigma^2)$.

Properties:

Mean of Normal Distribution:

By the definition of normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{mean } (\mu) = E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

We put $\frac{x-\mu}{\sigma} = z \Rightarrow \frac{dx}{\sigma} = dz$

$$dx = \sigma dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-1/2(z)^2} (\sigma dz)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \sigma z \cdot e^{-z^2/2} dz + \mu \int_{-\infty}^{\infty} e^{-z^2/2} dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + 2\mu \int_{0}^{\infty} e^{-z^2/2} dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\mu \sqrt{\pi/2} \right]$$

$$\therefore \boxed{\text{Mean} = \mu}$$

Variance of normal distribution:

By the defination of variance

$$\text{variance } (\sigma^2) = E((x-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $\frac{x-\mu}{\sigma} = z \rightarrow dx = \sigma dz$

$$x = \sigma z + \mu \rightarrow x - \mu = \sigma z$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty} (\sigma z)^2 e^{-z^2/2} (\sigma) \, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} \, dz \Rightarrow \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2} \, dz$$

let, $z^2/2 = t$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} (2t) e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{4a^2}{\sqrt{2} \times \sqrt{2\pi}} \int_0^{0} e^{-t} + t^{1/2} \, dt \Rightarrow \frac{2a^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{3/2 - 1} \, dt$$

$$= \frac{2a^2}{\sqrt{\pi}} \Gamma(3/2) \Rightarrow \frac{2a^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma(1/2)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$\boxed{\text{variance} = \sigma^2}$$

Mode: $\boxed{x = \mu}$

Median: $\boxed{f(x) = 1/2}$.

11/11/24

Properties of normal distribution:

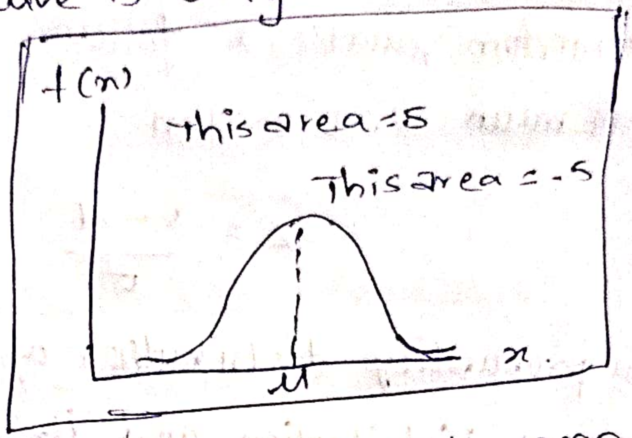
A normal distribution with parameters μ & σ has the following properties -

1. The curve is of Bell-shaped.
2. It is symmetrical about $x = \mu$.
3. It is also called as uni-modal distribution. Since mean, median & mode are equal.

4. The curve is asymptotic to X-axis. That is the curve touches x-axis at only $-\infty$ and $+\infty$.

5. The curve has points of inflexion at $\mu - \sigma$ & $\mu + \sigma$.

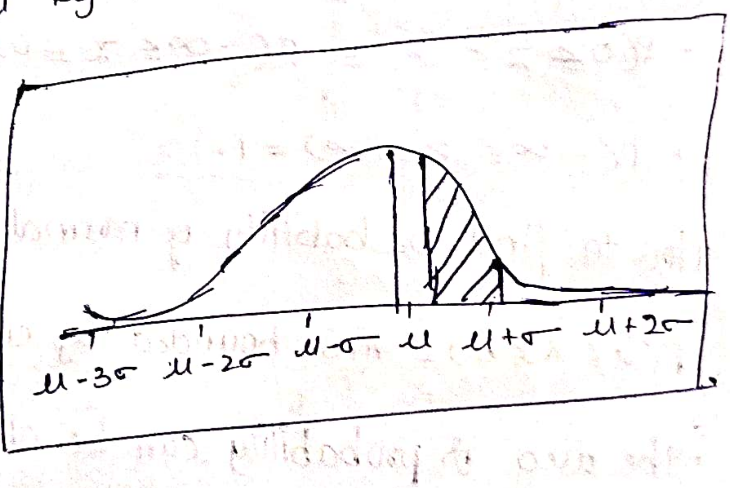
6. Total area under the curve is unity.



7. The probability that the normal variate with mean μ & variance σ^2 lies b/w a and b is given by:

$$P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$P(a \le x \le b)$ = Area bounded by curve b/w coordinate a and b.

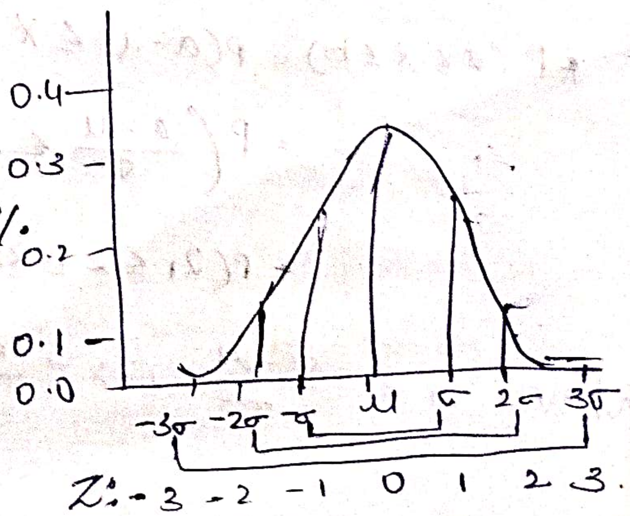


8. Area under the normal curve:

$P[\mu - \sigma < X \le \mu + \sigma] = 68.27\%$

$P[\mu - 2\sigma < X \le \mu + 2\sigma] = 95.44\%$

$P[\mu - 3\sigma < x \le \mu + 3\sigma] = 99.73\%$



Standard normal variate ...

- Normal variate with mean $\mu = 0$ and standard deviation 1 is called standard normal variate, denoted by z.
- If a random variable X follows normal distribution with mean μ S.D σ then

$$z = \frac{x-\mu}{\sigma}$$

Under:	$\leq$
Over:	$\geq$

The probability distribution of z is called standard normal distribution and it's probability density function is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty.$$

- $P(0 \leq x \leq \infty) = P(-\infty \leq x \leq 0) = 0.5$
- $P(0 \leq z \leq \infty) = P(-\infty \leq z \leq 0) = 0.5$
- $P(-\infty \leq z \leq \infty) = 1.$

How to find probability of normal curve:

$P(a \leq x \leq b) =$ Area bounded by curve b/w ordinates a and b

- The area of probability can be obtained using area under the standard normal curve.

$$*P(a \leq X \leq b) = P(a-\mu \leq X-\mu \leq b-\mu)$$

$$= P\left(\frac{a-\mu}{\sigma} \leq \frac{x-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$$

$$= P(z_1 \leq z \leq z_2)$$

where, $z_1 = \frac{a-\mu}{\sigma}$, $z_2 = \frac{x-\mu}{\sigma}$, $z_2 = \frac{b-\mu}{\sigma}$.

case 1: If z_1 and z_2 are positive (or both negative), then

$P(a \le x \le b) = P(z_1 \le z \le z_2)$

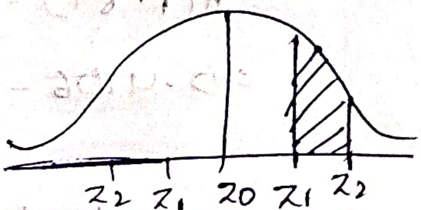
$=$ (Area under curve from 0 to z_2)

$-$ (Area under curve from 0 to z_1)

$= |A(z_2) - A(z_1)|$.

I. $P(x_1 \le x \le x_2) \Rightarrow P(z_1 \le z \le z_2)$

$z_1 = \dfrac{x_1 - \mu}{\sigma}$, $z_2 = \dfrac{x_2 - \mu}{\sigma}$

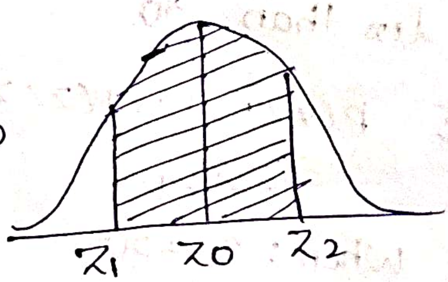


(i) $z_1 > 0, z_2 > 0$ (or) $z_1 < 0, z_2 < 0$

$A(z_1 \text{ to } z_2) = A(0 \text{ to } z_2) - A(0 \text{ to } z_1)$

$= A(z_2) - A(z_1)$

(ii) $z_1 < 0, z_2 > 0$ (or) $z_1 > 0, z_2 < 0$



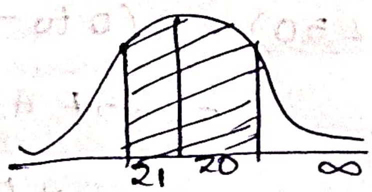
$= A(0 \text{ to } z_1) + A(0 \text{ to } z_2)$

$= A(z_1) + A(z_2)$.

II. $P(x > x_1) \Rightarrow P(z > z_1)$

(i) $z_1 = \dfrac{x_1 - \mu}{\sigma}$

$z_1 < 0$



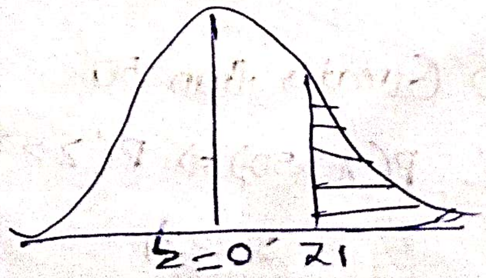
$= A(0 \text{ to } \infty) + A(0 \text{ to } z_1)$

$= 0.5 + A(z_1)$

(ii) $z_1 > 0$

$= A(0 \text{ to } \infty) - A(0 \text{ to } z_1)$

$= 0.5 - A(z_1)$

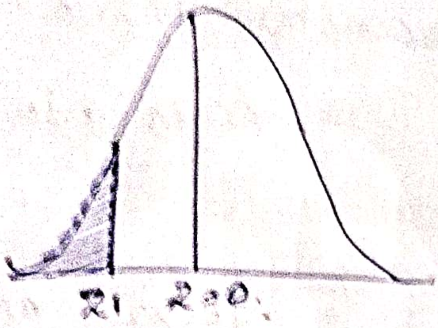


III $P(X < x_1) \Rightarrow P(z < z_1)$

$z_1 = \frac{x_1 - \mu}{\sigma}$

(i) $z_1 < 0$

$= 0.5 - A(z_1)$

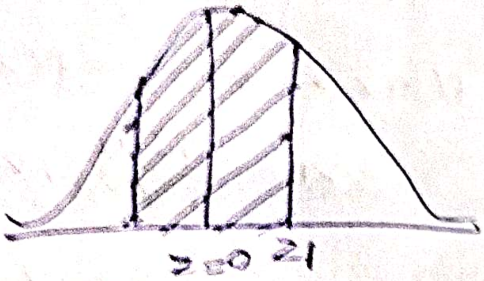


(ii) $z_1 > 0$

$= 0.5 + A(z_1)$

$= A(1.89) - A(2)$

$= 0.4706 - 0.4772.$



Q. x is a normal variate with mean 42 & Standard deviation 4. find the probability that a value taken by X is:

(i) less than 50

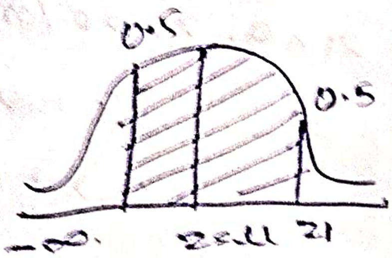
Sol: $P(x < 50) \Rightarrow P(z < z_1)$

$z < z_1$

When: $x_1 = 50.$

$\Rightarrow z_1 = \frac{50 - 42}{4} = \frac{8}{4} = 2$

$z_1 > 0$



$P(x < 50) = A(0 \text{ to } -\infty) + A(z_1 \text{ to } 0)$

$= 0.5 + A(2)$

$= 0.5 + 0.4772$

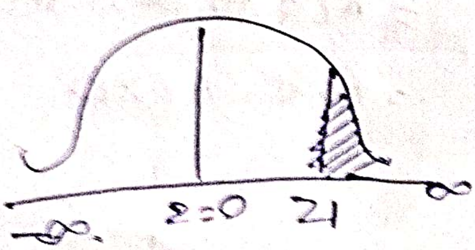
$= 0.9772.$

(ii) Greater than 50

Sol: $P(x > 50) \Rightarrow P(z > z_1)$

$z > z_1$

When: $x_1 = 50$



$\Rightarrow z_1 = \frac{50-42}{4} = \frac{8}{4} = 2$

$z_1 > 0$

$P(x<50) = A(0 \text{ to } \infty) - A(0 \text{ to } z_1)$

$= 0.5 - A(2)$

$= 0.5 - 0.4772$

$= 0.0228.$

(iii) less than 40.

Sol: $P(x<40) \Rightarrow P(z<z_1)$

$z<z_1$

when: $x_1 = 40$

$\Rightarrow z_1 = \frac{40-42}{4} = \frac{-2}{4} = -1/2$

$\therefore z_1 < 0$

$P(x<40) = A(0 \text{ to } -\infty) + A(z_1 \text{ to } 0)$

$= 0.5 + A(-1/2)$

$= 0.5 + A(0.25)$

$= 0.5 + 0.0793$

$= 0.5793.$

(iv) $P(x>40) = A(0 \text{ to } \infty) - A(z_1 \text{ to } 0)$

Sol: $= 0.5 - A(-1/2)$

$= 0.5 - A(0.25)$

$= 0.5 - 0.0793$

$= 0.4207.$

(v) $P(43 \le x \le 46) \Rightarrow P(z_1 \le z \le z_2)$; when $x_1 = 43$

Sol: $z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow \frac{43-42}{4} = \frac{1}{4} = 0.25$

when $x_2 = 46$

$z_1 = \frac{x_2 - \mu}{\sigma} \Rightarrow \frac{46-42}{4} = \frac{4}{4} = 1 > 0$

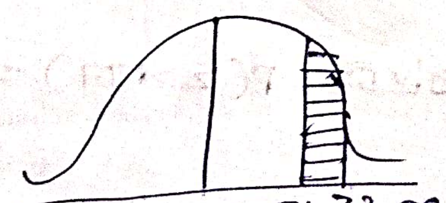
$P(43 \le x \le 46) = A(0 \text{ to } z_2) - A(0 \text{ to } z_1)$

$= A(z_2) - A(z_1)$

$= A(1) - A(0.25)$

$= 0.3413 - 0.0987$

$= 0.2426.$

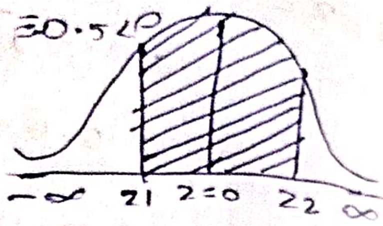


(vi) $P(40 < x < 44)$

Sol: $P(40 < x < 44) \Rightarrow P(z_1 \le z \le z_2)$

when: $x_1 = 40$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{40-42}{4} = \frac{-2}{4} = -0.5 < 0$$



when: $x_2 = 44$

$$z_2 = \frac{44-42}{4} = \frac{2}{4} = 0.5 < 0$$

$P(40 < x < 44) = A(0 \text{ to } z_2) + A(0 \text{ to } z_1)$

$= A(z_2) + A(z_1)$

$= A(0.5) + A(-0.5)$

$= 0.1919 + 0.1919$

$= 0.3838.$

Q. Height of students is normally distribute with mean of 165 cms. And standard deviation 5cms. Find the probability of height of student is:

(i) More than 177 cms.

Sol: Given, mean$(\mu) = 165$

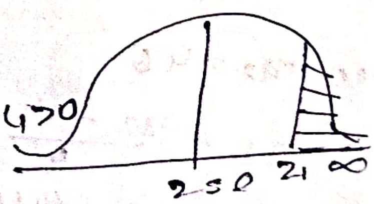
standard deviation $(\sigma) = 5$

Normal distribution $z = \frac{x-\mu}{\sigma}$

$P(x > 177) \Rightarrow P(z > z_1)$

when: $x_1 = 177$

$$z_1 = \frac{x-\mu}{\sigma} = \frac{177-165}{5} = \frac{12}{5} = 2.4 > 0$$



$P(z > 177) = 0.5 - A(z_1)$

$= 0.5 - A(2.4)$

$= 0.5 - 0.4793$

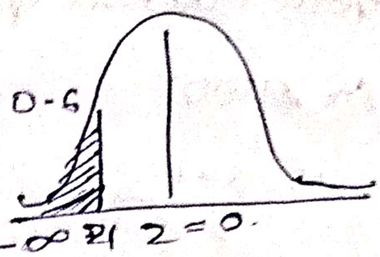
$= 0.0607$

(ii) less than 162 cms.

Sol: $P(x < 162) \Rightarrow P(z < z_1)$

when: $x_1 = 162$

$$z_1 = \frac{x-\mu}{\sigma} = \frac{162-165}{5} = -\frac{3}{5} = -0.6$$



$= 0.5 - A(z_1)$

$= 0.5 - A(-0.6)$

$= 0.5 - 0.2258$

$= 0.2742$

Q. Mean life of electric bulbs manufactured by a firm is 1200 hours. The standard deviation is 200 hours:

(i) In a lot of 10,000 bulbs, how many bulbs are expected to have life 1050 hrs or more?

Sol: Given, mean $(\mu) = 1200$

standard deviation $(\sigma) = 200$

Normal distribution $z = \frac{x-\mu}{\sigma}$

(i) 10,000 bulbs & 1050 hrs. $\Rightarrow P(x \geq 1050) \Rightarrow P(z \geq z_1)$

when $x_1 = \cancel{10000}$ 1050

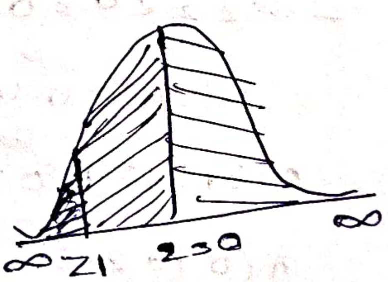
$$z_1 = \frac{1050 - 1200}{200} = \frac{-150}{200} = -3/4 = -0.75 < 0$$

$P(x \geq 1050) = 0.5 + A(z_1)$

$= 0.5 + A(-0.75)$

$= 0.5 + 0.2734$

$= 0.7734$



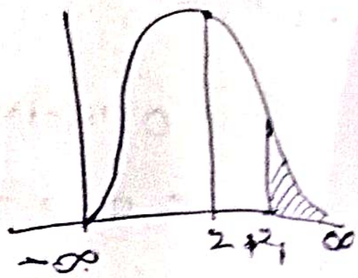
$P(x \geq 1050) = 10000 \times 0.7734$

$= 7734$ bulbs

(ii) What is the percentage of bulbs which are expected to find before 1500 hours of service.

Sol: $P(x < 1500)$

$z_1 = \dfrac{1500-1200}{200} = \dfrac{300}{200} = 1.5 > 0$



$P(x < 1500) = 0.5 + A(1.5)$

$= 0.5 + 0.4332$

$= 0.9332$

The % of bulbs expected to fail before 1500 hours

$= 0.9332 \times 100\%$

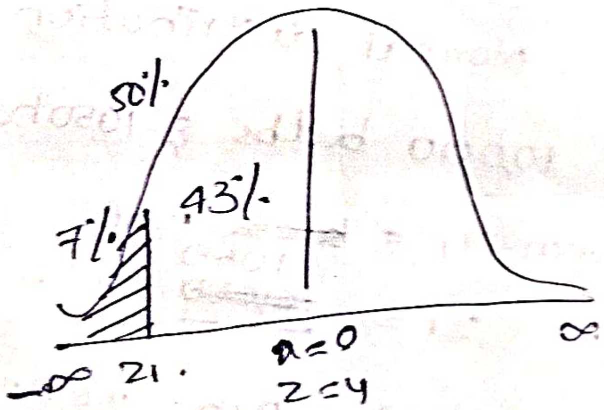
$= 93.32\%$

2. In a normal distribution 7% of items are under 35 and 89% are under 63. Determine mean and variance of the distribution.

Sol: $P(x < 35) = 7\% \Rightarrow P(z < 35)$

(i) $z_1 = \dfrac{x_1 - \mu}{\sigma}$

$z_1 = \dfrac{35 - \mu}{\sigma}$



$-\infty < z < 0 = 0.5$

$A(z_1) \Rightarrow A(0 \text{ to } z_1)$

$A(0 \text{ to } z_1) = A(0 \text{ to } -\infty) - A(z_1 \text{ to } -\infty)$

$A(z_1) = 0.5 - 0.07$ (or) $50\% - 7\%$

$A(z_1) = 0.43$

$z_1 = 1.40 + 0.08$

$z_1 = +1.48$

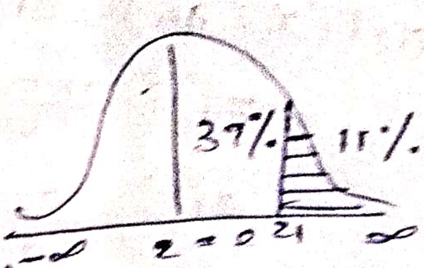
$z_1 =$

Subs z_1 in eq ①

$$-1.48 = \frac{35-\mu}{\sigma} \rightarrow ②$$

(ii) $P(x < 63) = 89\% = 0.89$

$$P(x > 63) = 1 - P(x < 63)$$
$$= 1 - 0.89$$
$$= 0.11$$



$$A(z_2) = A(0 \text{ to } \infty) - A(z_2 \text{ to } \infty)$$
$$= 0.5 - 0.11$$
$$A(z_2) = 0.39$$
$$z_2 = 1.2 + 0.03$$
$$z_2 = 1.23$$

Sub z_2 in eq ②

$$1.23 = \frac{63-\mu}{\sigma} \rightarrow ③.$$

Solving eq ② & eq ③.

$$1.23\sigma = 63 - \cancel{\mu}$$
$$1.48\sigma = 35 - \cancel{\mu}$$
$$2.71\sigma = 28.$$

$$\sigma = \frac{28}{2.71}$$

$$\boxed{\sigma = 10.33} \rightarrow \boxed{\sigma^2 = 106.75}$$

Substitute σ in eq ② or eq ③.

$$\boxed{\mu = 50.29}.$$

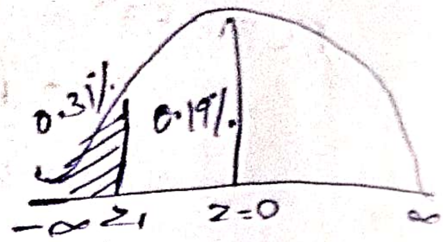
Q. In a normal distribution 31% of items are under 45 and 8% are over 64. Find mean and variance of the distribution.

Sol:-

(i) $P(x < 45) = 31 \Rightarrow P(z < 45)$

$$z_1 = \frac{x_1 - \mu}{\sigma}$$

$$z_1 = \frac{45 - \mu}{\sigma}$$



$-\infty < z < 0 = 0.5$

$$\boxed{A(z_1) = 0.19}$$

$A(0.31)$.

$A(z_1) = A(0 \text{ to } -\infty) - A(z_1 \text{ to } -\infty)$

$A(z_1) = 0.5 - 0.31 = 0.19$

$$0.58 = \frac{45 - \mu}{\sigma} \rightarrow (1).$$

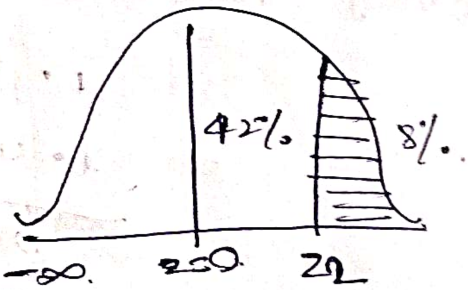
(ii) $P(x > 64) = P(z > 64)$

$P(z > 64) = A(0 \text{ to } \infty) - A(z_2 \text{ to } \infty)$

$A(z_2) = 0.5 - 0.08$

$A(z_2) = 0.42$

$$1.426 = \frac{64 - \mu}{\sigma} \rightarrow (2)$$



Solving eq (1) & eq (2):

$$0.56 = 45 - \mu$$
$$1.426 = 64 - \mu$$

$$-1.96 = -19$$

$$\sigma = \frac{19}{1.96} = 9.94$$

$$\boxed{\sigma^2 = 98.8}$$

from eq (1),

$$\boxed{\mu = 50.02}$$