# MODULE - 2

## FORMULAE'S

**line integral:**

$\int_C \bar{F} \cdot d\bar{r}$ $\quad$ $\bar{F} \rightarrow$ vector point function

$\Downarrow$ $\quad$ $d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\int_C f_1\,dx + f_2\,dy + f_3\,dz$ $\quad$ $\bar{F} = f_1\,\hat{i} + f_2\,\hat{j} + f_3\,\bar{k}$

**work done**

$\int_A^B \bar{F} \cdot d\bar{r}$ $\rightarrow$ same as line integral

work done by force is independent of path

$\Downarrow$

curl $\bar{F} = 0$

**Surface integral:**

$\int_S \bar{F} \cdot \hat{n}\, ds = \iint_S \bar{F} \cdot \hat{n}\, ds$

$\hat{n} \rightarrow$ outward drawn unit normal to the surface

Volume integral:

$$\int_V \bar{f} \cdot dv = \iiint_V \bar{f} \, dx \, dy \, dz$$

Gauss divergence theorem:

$$\int_S \bar{f} \cdot \hat{n} \, ds = \int_V \operatorname{div} \bar{f} \, d\bar{v}$$

$$\iint_S \bar{f} \cdot \hat{n} \, ds = \iiint_V (\nabla \cdot \bar{f}) \, dx \, dy \, dz$$

→ double & triple I' relationship

Green's theorem: xy plane.

$$\int_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy$$

→ line & double relationship

Stokes theorem:

$$\int_C \bar{F} \cdot d\bar{r} = \int_S (\operatorname{curl} \bar{F} \cdot \hat{n}) \, ds$$

→ line & $\underbrace{\text{surface}}_{\text{double}}$ relationship

# MODULE - $\overline{V}$

## PART - A

① Gauss divergence theorem.

Given; $\bar{f} = x^2 \bar{i} + y^2 \bar{j} + z^2 \bar{k}$

cuboid $\quad x=0, x=a;\ y=0;\ y=b,\ z=0,\ z=c$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \nabla\bar{F}\, dv$$

LHS: $\quad$ 6 regions

(i) $R_1$ is in $yz$ plane; $x=a$; $ds = dy\,dz$

$y: 0 \to b;\quad z: 0 \to c$

$\bar{n} = i$

$|\bar{n}\cdot i| = |i\cdot i| = 1$

$\bar{f}\cdot\bar{n} = (x^2\bar{i} + y^2\bar{j} + z^2\bar{k})\cdot i = x^2$

$$\iint_S \bar{f}\cdot\bar{n}\, ds = \int_{z=0}^{c}\int_{y=0}^{b} x^2\, dy\, dz$$

$$= \int_0^c x^2 [y]_0^b\, dz$$

$$\Rightarrow \int_0^c x^2 b\, dz = x^2 b [z]_0^c$$

$$= x^2 bc$$

$$= a^2 bc \quad (\because x=a)$$

(ii) $R_2$ is in $yz$ plane; $ds = dy\,dz$; $\bar{n} = -\bar{i}$,

$x = 0$

$|\bar{n}\cdot\bar{i}| = |-\bar{i}\cdot\bar{i}| = -1$

$\bar{f}\cdot\bar{n} = -x^2$

$$\iint_{R_2} \bar{f}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} -x^2\,dy\,dz$$

$$= 0.$$

(iii) $R_3$ is in $xy$ plane; $ds = dx\,dy$; $\bar{n} = \bar{k}$;

~~$z = 0$~~ $z = c$

$|\bar{n}\cdot\bar{k}| = |\bar{k}\cdot\bar{k}| = 1$

$\bar{f}\cdot\bar{n} = z^2$

$$\iint_{R_3} \bar{f}\cdot\bar{n}\,ds = \int_0^b\int_0^a z^2\,dx\,dy = c^2ab$$

(iv) $R_4$ is in $xy$ plane; $ds = dx\,dy$; $\bar{n} = -\bar{k}$

$z = 0$

$|\bar{n}\cdot\bar{k}| = |-\bar{k}\cdot\bar{k}| = 1$

$\bar{f}\cdot\bar{n} = -z^2$

$$\iint_{R_4} \bar{f}\cdot\bar{n}\,ds = \int_0^b\int_0^a -z^2\,dx\,dy = 0 \quad (\because z = 0)$$

(v) $R_5$ is in $x$-$z$ plane; $ds = dx\,dz$; $\bar{n} = \bar{j}$

$$y = b$$

$$|\bar{n}\cdot\bar{j}| = |\bar{j}\cdot\bar{j}| = 1$$

$$\bar{f}\cdot\bar{n} = y^2$$

$$\iint_{R_5} \bar{f}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} y^2\,dx\,dz = b^2 ac$$

(vi) $R_6$ is in $x$-$z$ plane; $ds = dx\,dz$; $\bar{n} = -\bar{j}$

$$y = 0$$

$$|\bar{n}\cdot\bar{j}| = |-\bar{j}\cdot\bar{j}| = -1$$

$$\bar{f}\cdot\bar{n} = -y^2$$

$$\iint_{R_6} \bar{f}\cdot\bar{n}\,ds = \int_0^c\int_0^a -y^2\,dx\,dz = 0 \quad (\because y = 0)$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,ds + \iint_{R_2} \bar{f}\cdot\bar{n}\,ds + \iint_{R_3} \bar{f}\cdot\bar{n}\,ds$$

$$+ \iint_{R_4} \bar{f}\cdot\bar{n}\,ds + \iint_{R_5} \bar{f}\cdot\bar{n}\,ds + \iint_{R_6} \bar{f}\cdot\bar{n}\,ds.$$

$$= a^2bc + 0 + abc^2 + 0 + ab^2c + 0$$

$$= abc(a+b+c).$$

RHS:

$$\nabla \bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2\bar{i} + y^2\bar{j} + z^2\bar{k}\right)$$

$$\nabla \bar{f} = 2x + 2y + 2z$$

$$\iiint \nabla \bar{f}\, dv = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_{0}^{c}\int_{0}^{b}\left[\frac{x^2}{2} + xy + xz\right]_{x=0}^{a} dy\, dz$$

$$= 2\int_{0}^{c}\int_{0}^{b}\left(\frac{a^2}{2} + ay + az\right) dy\, dz$$

$$= 2\int_{0}^{c}\left[\frac{a^2}{2}y + \frac{ay^2}{2} + ayz\right]_{y=0}^{b} dz$$

$$= 2\int_{0}^{c}\left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_{z=0}^{c}$$

$$= \cancel{2}\left(\frac{a^2bc + ab^2c + abc^2}{\cancel{2}}\right)$$

$= a^2bc + ab^2c + abc^2$

$= abc(a+b+c)$

LHS = RHS

Hence; verified!

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② Gauss divergence theorem

$\iint_S \bar{F}\cdot\bar{n}\, ds = ?$ $\bar{F} = y\bar{i} + x\bar{j} + z^2\bar{k}$

cylinder $\rightarrow$ $x^2 + y^2 = a^2$ $z = 0;\ z = b$

$\nabla \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(y\bar{i} + x\bar{j} + z^2\bar{k}\right)$

$\nabla \bar{F} = y + x + z^2$

$x^2 + y^2 = a^2$

$z : 0 \rightarrow b$ $y = 0$

$x : -a \rightarrow +a$ $x^2 = a^2$

$y : -\sqrt{a^2 - x^2} \rightarrow +\sqrt{a^2 - x^2}$ $x = \pm a$

$x^2 + y^2 = a^2$

Now;

$y^2 = a^2 - x^2$

$$\iiint \nabla\cdot\bar{F}\cdot dv = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{b} (y + x + z^2)\, dx\, dy\, dz$$ $y = \pm\sqrt{a^2 - x^2}$

x y z

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{b} (x + y + z^2)\, dx\, dy\, dz$$ ($\because$ Integrand is even function)

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{b}(x\,dz+y\,dz+z^2\,dz)\,dx\,dy$$

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[xz+yz+\frac{z^3}{3}\right]_{z=0}^{b}dx\,dy$$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\left(xb+yb+\frac{b^3}{3}\right)dx\,dy$$

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\left(xb\,dy+yb\,dy+\frac{b^3}{3}dy\right)dx$$

$$\int_0^a\left[xyb+\frac{y^2}{2}b+\frac{yb^3}{3}\right]_{y=0}^{\sqrt{a^2-x^2}}dx$$

$$\int_0^a\left(xb\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}b+\frac{b^3}{3}\left(\sqrt{a^2-x^2}\right)\right)dx$$

Put $x = a\sin\theta$ $\qquad x=a \qquad x=0$

$dx = a\cos\theta\,d\theta$ $\qquad \theta=\pi/2 \qquad \theta=0$

$\theta: 0 \to \pi/2$

$$\Rightarrow \int_0^{\pi/2} \left( ab\sin\theta \underbrace{\sqrt{a^2 - a^2\sin^2\theta}}_{a^2\cos^2\theta} + \frac{a^2 - a^2\sin^2\theta}{2} \cdot b + \frac{b^3}{3}\sqrt{a^2 - a^2\sin^2\theta} \right) d\,a\cos\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \left( ab\sin\theta \cdot (a\cos\theta) + \frac{a^2\cos^2\theta}{2} \cdot b + \frac{b^3}{3}(a\cos\theta) \right) a\cos\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \left( a^2 b\sin\theta\cos\theta + \frac{a^2 b\cos^2\theta}{2} + \frac{ab^3\cos\theta}{3} \right) a\cos\theta \, d\theta$$

$$\Rightarrow \underbrace{\int_0^{\pi/2} a^3 b\sin\theta\cos^2\theta \, d\theta}_{(a)} + \underbrace{\int_0^{\pi/2} \frac{a^3 b\cos^3\theta}{2} d\theta}_{(b)} + \underbrace{\int_0^{\pi/2} \frac{a^2 b^3\cos^2\theta}{3} d\theta}_{(c)}$$

(1) ~~$I_{m,n} = \int \sin^m x \cos^n x \, dx$~~

$I_{m,n}$

$a^3 b\sin\theta \underbrace{\cos^2\theta}$

$\sin\theta(1 - \sin^2\theta)$

$\sin\theta \, d\theta - \sin^3\theta \, d\theta$

(1) $\int_0^{\pi/2} a^3 b \sin\theta (1 - \sin^2\theta)\, d\theta$

$\int_0^{\pi/2} (a^3 b \sin\theta - a^3 b \sin^3\theta)\, d\theta$

$\int_0^{\pi/2} a^3 b \sin\theta \, d\theta - \int_0^{\pi/2} a^3 b \sin^3\theta \, d\theta$

$a^3 b \int_0^{\pi/2} \sin\theta \, d\theta - a^3 b \int_0^{\pi/2} \sin^3\theta \, d\theta$ — (8)

$$\int_0^{\pi/2} \sin\theta \, d\theta = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & n \to \text{even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & n \to \text{odd} \end{cases}$$

$\int_0^{\pi/2} \sin\theta \, d\theta = 1$ $(n = \text{odd} = 1)$

$\int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{3-1}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2} = \frac{\pi}{6}$

from (8)

$a^3 b(1) - a^3 b(\pi/6) = a^3 b - \frac{a^3 b \pi}{6}$

$= \frac{6a^3 b - a^3 b \pi}{6}$

(b) $\int_0^{\pi/2} \frac{a^3 b \cos^3\theta}{2} \, d\theta = \frac{a^3 b}{2} \int_0^{\pi/2} \cos^3\theta \, d\theta$

$n = 3$

$\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{3-1}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{6}$

$\Rightarrow \frac{a^3 b}{2} \cdot \frac{\pi}{6} = \frac{a^3 b \pi}{12}$

(c) $\int_0^{\pi/2} \frac{a^2 b^3}{3} \cos^2\theta \, d\theta = \frac{a^2 b^3}{3} \int_0^{\pi/2} \cos^2\theta \, d\theta$

$n = 2$ $\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{1}{2} \cdot \frac{\pi}{2} = \pi/4$

$\Rightarrow \frac{a^2 b^3}{3} \cdot \frac{\pi}{4} = \frac{a^2 b^3 \pi}{12}$

Now;

$= \frac{6a^3 b - a^3 b \pi}{6} + \frac{a^3 b \pi}{12} + \frac{a^2 b^3 \pi}{12}$

$= \frac{12a^3 b - 2a^3 b\pi + a^3 b \pi + a^2 b^3 \pi}{12}$

$= \frac{12a^3 b - a^3 b\pi + a^2 b^3 \pi}{12}$ (or) $a^3 b - \frac{a^3 b}{12}\pi + \frac{a^2 b^3}{12}\pi$

(3) Green's theorem

$$\int_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$$

2 parabolas $y^2 = x$ and $y = x^2$

+

intersects at $(0,0)$ & $(1,1)$

$\frac{\partial M}{\partial y} = 2x$ $\quad (\because M = 2xy - x^2)$

$\frac{\partial N}{\partial x} = 2x$ $\quad (\because N = x^2 + y^2)$.

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$

By green's theorem,

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (0)\,dx\,dy$$

$$= 0$$

$$\therefore \int_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy = 0$$

(4) Green's theorem

$$\int_C (xy + y^2)\,dx + (x^2)\,dy.$$

$y = x$ and $y = x^2$

intersect at $(0,0)$ & $(1,1)$

$M = xy + y^2$ | $N = x^2$

$\frac{\partial M}{\partial y} = 2y + x$ | $\frac{\partial N}{\partial x} = 2x$

$$\int_C (xy + y^2)\,dx + x^2\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1}\int_{x^2}^{x} (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$\Rightarrow \int_0^1 \left(xy - y^2\right)_{x^2}^{x} dx = \int_0^1 \left((x^2 - x^2) - (x^3 - x^4)\right) dx.$$

$$\Rightarrow \int_0^1 (x^4 - x^3)\,dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1$$

$$= 1/5 - 1/4$$

$$= -1/20$$

(5) Green's theorem

$$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

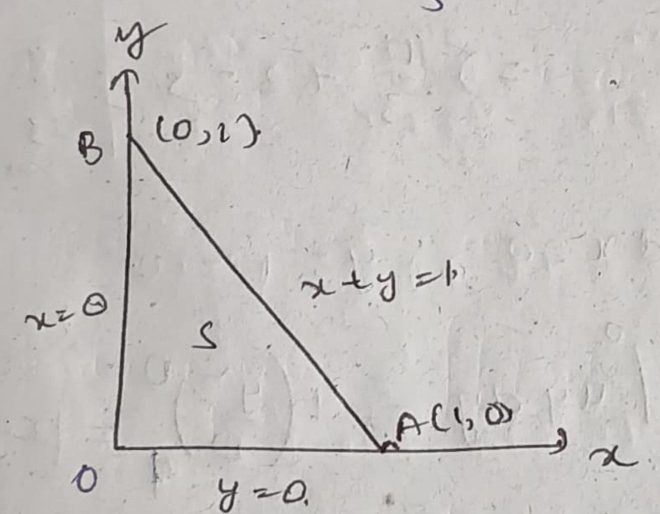
$x=0$; $y=0$ & $x+y=1$

$M = 3x^2-8y^2$ | $N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y$ | $\frac{\partial N}{\partial x} = -6y$.

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

LHS




$$\int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BO} M\,dx + N\,dy. \quad \text{— (1)}$$

$OA \rightarrow y=0 \rightarrow dy=0$

$$\int_{OA} M\,dx + N\,dy = \int_0^1 3x^2\,dx = 3\left(\frac{x^3}{3}\right)_0^1 = 1$$

$AB \longrightarrow \quad x+y=1 \qquad x=1-y$

$dy=-dx \qquad y: 0\to 1.$

$$\int_{AB} M\,dx+N\,dy = \int_0^1 (3(y-1)^2-8y^2)(-dy)+(4y+6y(y-1))\,dy$$

$$= \int_0^1 (-5y^2-6y+3)(-dy)+(6y^2-2y)\,dy.$$

$$\Rightarrow \int_0^1 (11y^2+4y-3)\,dy = \left(11\frac{y^3}{3}+4\frac{y^2}{2}-3y\right)_0^1$$

$$\Rightarrow \frac{11}{3}+2-3 = \frac{8}{3}.$$

$BO \to x=0 \;;\; dx=0 \qquad y: 1\to 0.$

$$\int_{BO} M\,dx+N\,dy = \int_1^0 4y\,dy = 4\left(\frac{y^2}{2}\right)_1^0 = \left[2y^2\right]_1^0$$

$$=-2$$

from (1)

$$\int_C M\,dx+N\,dy = 1+\frac{8}{3}-2 = \frac{5}{3} \quad -(2)$$

Now ; $\underline{RHS}$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (-6y + 16y)\, dx\,dy$$

$$= 10 \int_0^1 \int_0^{1-x} y\, dy\, dx.$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx.$$

$$= \frac{10}{2} \int_0^1 \frac{(1-x)^2}{2}\, dx.$$

$$= 5 \int_0^1 (1-x)^2\, dx.$$

$$= 5 \left(\frac{(1-x)^3}{-3}\right)_0^1$$

$$\Rightarrow -\frac{5}{3}\left((1-1)^3 - (1-0)^3\right) = 5/3$$

— (3)

from (2) & (3)

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Hence verified

Stokes's theorem

(6) $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$

cube $\rightarrow$ $x=0, y=0; z=0$ & $x=2, y=2; z=1$

above xy plane $\rightarrow \bar{n} = k$

By stokes theorem,

$$\int \text{curl } \bar{F} \cdot \bar{n} \, ds = \int \bar{F} \cdot d\bar{r}$$

LHS

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= y\bar{i} - (1-z)\bar{j} - \bar{k}$$

$$\text{curl } \bar{F} \cdot \bar{n} = (y\bar{i} - (1-z)\bar{j} - \bar{k}) \cdot \bar{k}$$

$$= -1$$

$$\therefore \int \text{curl } \bar{F} \cdot \bar{n} \, ds = \int_0^2 \int_0^2 -1 \, dx \, dy \qquad (z=0; dz=0)$$

$$= -4. \quad \text{— (1)}$$

RHS

$$\int \bar{F}\cdot d\bar{r} = \int \left((y-z+2)\bar{i} + (yz+4)\bar{j} - (xz)\bar{k}\right).$$
$$(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

let; $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$= \int (y-z+2)\,dx + (yz+4)\,dy - (xz)\,dz.$$

S is the surface of the cube above the xy-plane $\to z=0;\ dz=0$

$$\therefore \int \bar{F}\cdot dr = \int (y+2)\,dx + \int 4\,dy.$$

Along $\overline{OA}$; $y=0;\ z=0;\ dy=0;\ dz=0;\ x: 0\to 2$

$$\int_0^2 2\,dx = 2[x]_0^2 = 4$$

Along $\overline{BC}$; $y=2;\ z=0;\ dy=0;\ dz=0;\ x: 2\to 0$

$$\int_2^0 4\,dx = 4[x]_2^0 = -8.$$

Along $\overline{AB}$; $x=2;\ z=0;\ dx=0;\ dz=0;\ y: 0\to 2$

$$\int_0^2 4\,dy = [4y]_0^2 = 8.$$

Along $\overline{CO}$; $x=0;\ z=0;\ dx=0;\ dz=0;\ y: 2\to 0$

$$\int_2^0 4\,dy = -8.$$

Above the surface, when $z=2$

Along $O'A'$, $\int_0^2 \bar{F}\cdot d\bar{r} = 0$

Along $A'B'$; $x=2$, $z=2$; $dx=0$, $dz=0$; $y: 0 \to 2$

$$\int_0^2 \bar{F}\cdot d\bar{r} = \int_0^2 (2y+4)\,dy = 2\cdot \frac{y^2}{2}\Big]_0^2 + 4y\Big]_0^2 = 4+8$$

$$= 12$$

Along $B'C'$; $y=2$; $z=2$; $dy=0$; $dz=0$; $x: 2 \to 0$

$$\int_2^0 \bar{F}\cdot d\bar{r} = 0$$

Along $C'O'$; $x=0$; $z=2$; $\cancel{dy}\ dx = 0$, $dz=0$; $y: 2 \to 0$

$$\int_2^0 (2y+4)\,dy = 2\cdot\frac{y^2}{2}\Big]_2^0 + 4y\Big]_2^0 = -12$$

$$\int_C \bar{F}\cdot d\bar{r} = OA + AB + BC + CO + O'A' + A'B' + B'C' + C'O'$$

$$= 4 - \cancel{8} + \cancel{8} - 8 + 0 + \cancel{12} + 0 - \cancel{12}$$

$$= -4 \quad \text{--- (2)}$$

from (1) & (2), we have

$$\int \bar{F}\cdot d\bar{r} = \int \text{curl}\,\bar{F}\cdot \bar{n}\,ds = -4.$$

Hence verified

(7) $\bar{F} = (x^3 - yz)\bar{i} - 2xy\bar{j} + 2z\bar{k}$

$x = y = z = 0$ to $x = y = z = a$

Gauss divergence theorem

$$\int_S \bar{F}\cdot n\, ds = \int_V \nabla\cdot\bar{F}\, dv$$

RHS

$\int \cancel{\nabla\cdot F\, dv}$

$$\nabla F = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left((x^3 - yz)\bar{i} - 2xy\bar{j} + 2z\bar{k}\right)$$

$$\nabla F = \partial/\partial x\,(x^3 - yz) + \partial/\partial y\,(-2xy) + \partial/\partial z\,(2z)$$

$$\nabla F = 3x^2 - 2x + 2$$

$$\int_V \nabla\cdot\bar{F}\, dv = \int_0^a\int_0^a\int_0^a (3x^2 - 2x + 2)\, dx\, dy\, dz$$

$$= \int_0^a\int_0^a \left[\frac{3x^3}{3} - \frac{2x^2}{2} + 2x\right]_0^a dy\, dz$$

$$= \int_0^a\int_0^a (a^3 - a^2 + 2a)\, dy\, dz$$

$$\Rightarrow \int_0^a \left[a^3y - a^2y + 2ay\right]_0^a dz = \int_0^a (a^3\cdot a - a^2\cdot a + 2a\cdot a)\, dz$$

$$= \int_0^a (a^4 - a^3 + 2a^2)\, dz$$

$$= \left[a^4 z - a^3 z + 2a^2 \cdot z\right]_0^a$$

$$= a^4 \cdot a - a^3 \cdot a + 2a^2 \cdot a$$

$$= a^5 - a^4 + 2a^3 \quad - (1)$$

LHS

(i) $R_1$ is in $yz$ plane, $x = a$ ; $ds = dy\,dz$

$y : 0 \to a$ ; $z : 0 \to a$

$\bar{n} = \bar{i}$

$|\bar{n} \cdot \bar{i}| = |\bar{i} \cdot \bar{i}| = 1$

$\bar{f} \cdot \bar{n} = x^3 - yz \quad (x = a)$

$= a^3 - yz$

$$\iint_{S_1} \bar{F} \cdot \bar{n}\, ds = \int_0^a \int_0^a a^3 - yz\, dy\, dz$$

$$= \int_0^a \left(a^3 y - \frac{y^2 z}{2}\right)_0^a dz$$

$$= \int_0^a \left(a^4 - \frac{a^2}{2} z\right) dz = a$$

$$= \left[a^4 z - \frac{a^2}{2} \cdot \frac{z^2}{2}\right]_0^a = a^5 - \frac{a^4}{4}$$

(ii) $R_2$ is in $yz$ plane; $x=0$; $ds = dy\,dz$

$y: 0 \to a$; $z: 0 \to a$.

$\bar{n} = -\bar{i}$ ; $|\bar{n}\cdot\bar{i}| = |-\bar{i}\cdot\bar{i}| = 1$.

$\bar{F}\cdot\bar{n} = x^3 - yz \cdot -i = + yz \quad (\because x=0)$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (+yz)\,dy\,dz$$

$$= \int_0^a \left(+\frac{y^2}{2}z\right)_0^a dz$$

$$\Rightarrow \int_0^a + \frac{a^2}{2} z\,dz = +\frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^a$$

$$\Rightarrow +\frac{a^2}{2}\cdot\frac{a^2}{2} = +\frac{a^4}{4}$$

(iii) $R_3$ is in $xz$ plane; $y=a$; $ds = dx\,dz$

$x: 0 \to a$; $z: 0 \to a$

$\bar{n} = \bar{j}$ ; $|\bar{n}\cdot\bar{j}| = |\bar{j}\cdot\bar{j}| = 1$

$\bar{F}\cdot\bar{n} = -2xy \quad = -2xa \quad (\because y=a)$

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a -2x\cancel{y}a\,dx\,dz = -a\int_0^a \left[2\cdot\frac{x^2}{2}\right]_0^a dz$$

$$\Rightarrow -a\int_0^a a^2\,dz = -a^4$$

(iv) $R_4$ is in $xz$ plane; $ds = dx\,dz$; $y = 0$

$x: 0 \to a$; $z: 0 \to a$

$\bar{n} = -\bar{j}$; $|\bar{n}\cdot\bar{j}| = |-\bar{j}\cdot\bar{j}| = -1$

$\bar{f}\cdot\bar{n} = -2xy = 0$ $(\because y = 0)$

$$\iint_{S_4} \bar{f}\cdot\bar{n}\,ds = \int_0^a\int_0^a 0\,dx\,dz = \underline{0}$$

(v) $R_5$ is in $xy$ plane; $ds = dx\,dy$; $z = a$

$x: 0 \to a$; $y: 0 \to a$

$\bar{n} = \bar{k}$; $|\bar{n}\cdot\bar{k}| = |\bar{k}\cdot\bar{k}| = \underline{1}$

$\bar{f}\cdot\bar{n} = 2z = 2a$ $(\because z = a)$

$$\iint_{S_5} \bar{f}\cdot\bar{n}\,ds = \int_0^a\int_0^a 2a\,dx\,dy = 2a\,a^2 = 2a^3$$

(vi) $R_6$ is in $xy$ plane; $ds = dx\,dy$; $z = 0$

$x: 0 \to a$; $y: 0 \to a$

$\bar{n} = -\bar{k}$; $|\bar{n}\cdot\bar{k}| = |-\bar{k}\cdot\bar{k}| = -1$

$\bar{f}\cdot\bar{n} = 2z = 0$

$$\iint_{S_6} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a 0\,dx\,dy = \underline{0}$$

Now,

$$\int_S \bar{F}\cdot\hat{n}\,ds = \iint_{S_1} \bar{F}\cdot\hat{n}\,ds + \iint_{S_2} \bar{F}\cdot\hat{n}\,ds + \iint_{S_3} \bar{F}\cdot\hat{n}\,ds + \iint_{S_4} \bar{F}\cdot\hat{n}\,ds$$

$$+ \iint_{S_5} \bar{F}\cdot\hat{n}\,ds + \iint_{S_6} \bar{F}\cdot\hat{n}\,ds$$

$$= a^5 - \cancel{\frac{a^4}{4}} + \cancel{\frac{a^4}{4}} - a^4 + 0 + 2a^3$$

$$= a^5 - a^4 + 2a^3 \quad \text{--- (2)}$$

from (1) & (2), we have,

$$\int_S \bar{F}\cdot\hat{n}\,ds = \int_V \nabla\cdot\bar{F}\,dv = a^5 - a^4 + 2a^3$$

Hence proved.

(8) $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$y = \sqrt{x}$ and $y = x^2$

Ans. $M = 3x^2 - 8y^2$ | $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ | $\frac{\partial N}{\partial x} = -6y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 16y - 6y = 10y$

Now; $\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

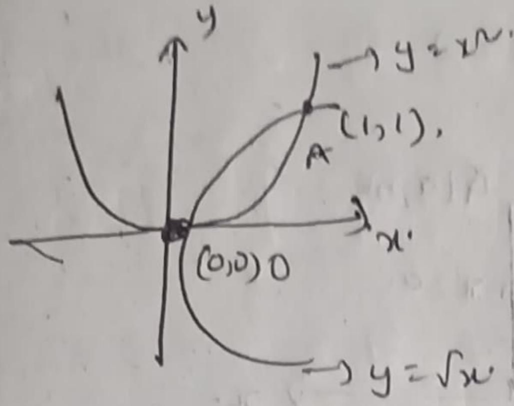
$\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx = 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$

$= \frac{10}{2}\int_0^1 (x - x^4)\,dx$

$\Rightarrow 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right)$

$= \frac{3}{2}$ — (1)

LHS




$$\int_C M\,dx + N\,dy = \int_{(OA)} I_1 + \int_{(AO)} I_2$$

$y = x^2 \qquad x = y^2$

$$I_1 = \int_0^1 \left(\{3x^2 - 8(x^2)^2\}\,dx + \{4x^2 - 6x(x^2)\}\,2x\,dx\right)$$

$$\Rightarrow \int_0^1 (3x^3 + 8x^3 - 20x^4)\,dx = -1$$

$$I_2 = \int_1^0 (3x^2 - 8x)\,dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_1^0 (3x^2 - 11x + 2)\,dx = 5/2$$

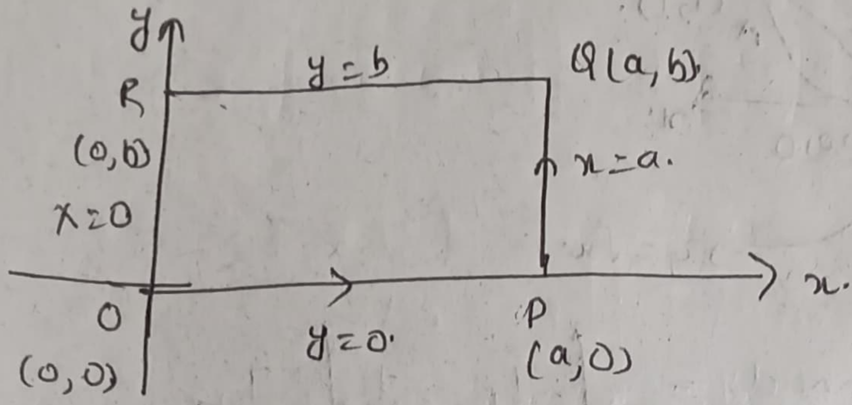
$$\int_C M\,dx + N\,dy = -1 + 5/2 = 3/2 \quad \text{— (2)}$$

from (1) & (2), we have

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Hence verified

(9) Since xy plane → $z=0$




$\bar{F} = (x^2+y^2)\,\bar{i} - 2xy\,\bar{j}$ ; $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left((x^2+y^2)\bar{i} - 2xy\,\bar{j}\right)\cdot(dx\,\bar{i}+dy\,\bar{j})$$

$$= \int_C (x^2+y^2)\,dx + (-2xy)\,dy$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r} \quad -(1)$$

(i) OP : $y=0$; $dy=0$; $x: 0 \to a$.

$$\int_{OP} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx = a^3/3$$

(ii) PQ : $x=a$ ; $dx=0$ ; $y: 0 \to b$

$$\int_{PQ} \bar{F}\cdot d\bar{r} = \int_0^b (a^2+y^2)\cdot 0 + (-2ay)\,dy = -2a\left[\frac{y^2}{2}\right]_0^b$$

$$= -\frac{\cancel{2}ab^2}{\cancel{2}}$$

$$= -ab^2$$

(iii) QR : $y = b$; $dy = 0$; $x : a \to 0$.

$$\int_{QR} \bar{F} \cdot d\bar{r} = \int_a^0 (x^2 + b^2)\, dx = -\frac{a^3}{3} - ab^2.$$

(iv) RO : $x = 0$; $dx = 0$; $y : b \to 0$

$$\int_{RO} F \cdot d\bar{r} = \int_b^0 0\, dy = \underline{0}.$$

from (1)

$$\int_C \bar{F} \cdot d\bar{r} = \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 + 0$$

$$= \underline{-2ab^2}.$$

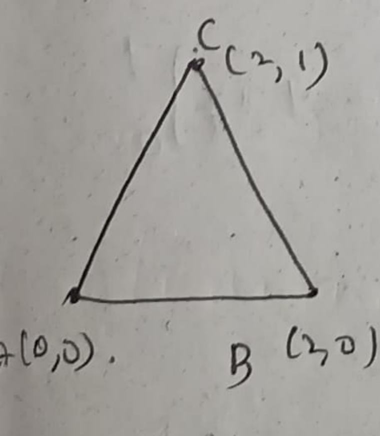
---

(10) $\bar{F} = (2x + y^2)\bar{i} + (3y - 4x)\bar{j}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\bar{F} \cdot d\bar{r} = (2x + y^2)\,dx + (3y - 4x)\,dy$.

xy plane : $\longrightarrow$ $z = 0$




$$\int_C \bar{F} \cdot d\bar{r} = \int_{AB} \bar{F} \cdot dr + \int_{BC} \bar{F} \cdot dr + \int_{CA} \bar{F} \cdot d\bar{r}$$

(i) AB ; $x : 0 \to 2$ ; $y = 0$ ; $dy = 0$.

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^2 2x\,dx = 2 \cdot \left[\frac{x^2}{2}\right]_0^2 = 4$$

(ii) BC ; $x = 2$ ; $dx = 0$ ; $y : 0 \to 1$

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_0^1 (3y - 4(2))\,dy = \int_0^1 (3y - 8)\,dy$$

$$= \left[3\frac{y^2}{2} - 8y\right]_0^1$$

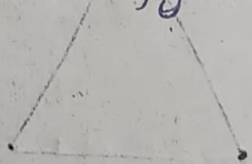
$$\Rightarrow \frac{3}{2} - 8 = \frac{3 - 16}{2} = \frac{-13}{2}$$

Insufficient data / Question mistake

(iii) CA ; $x : 2 \to 0$ ; $y : 1 \to 0$

$$\int_{CA} \bar{F} \cdot d\bar{r} = \int_2^0 (2x + y^2)\,dx + \int_1^0 (3y - 4x)\,dy$$

$$= \left[2\frac{x^2}{2} + y^2 x\right]_2^0 + \left[3\frac{y^2}{2} - 4xy\right]_0^1$$



$$= 4 + 2y^2 + \frac{3}{2} - 4x$$

$$= \frac{11}{2} - 4x + 2y^2$$

$=$

PART-B

(1) $\bar{F} = yz\,\bar{i} + 2y^2\,\bar{j} + xz^2\,\bar{k}$ $\qquad x^2+y^2=9$

$\qquad\qquad z=0 \;\&\; z=2$

$\iint_S \bar{F}\cdot d\bar{s} = ?$

$\nabla\phi = 2(x\bar{i}+y\bar{j})$ ; $|\nabla\phi| = 2\sqrt{x^2+y^2} = 2(3)$
$= 6.$

$\hat{n} = \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \dfrac{x\bar{i}+y\bar{j}}{3}$

let R be the projection of S on yz plane

$\int \bar{F}\cdot\bar{n}\,ds = \int \bar{F}\cdot ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \iint_R \dfrac{(xyz + 2y^3)\,dy\,dz}{(x/3)}$

yz plane $x=0$ ; $y=3$ $\quad z \to 0 \to 2$

$\iint \bar{F}\cdot\bar{n}\,ds = 3\int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \dfrac{2y^3}{x}\right)dy\,dz$ $\quad x \to \sqrt{9-y^2}$

$= 3\int_0^3 \left(2y + \dfrac{4y^3}{\sqrt{9-y^2}}\right)dy$

$= \left[6y\right]_0^3 + 12\int_0^3 \dfrac{y^3}{\sqrt{9-y^2}}\,dy$

Put $y = 3\sin\theta$

we get; $\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy = 18$

$\Rightarrow 6(3) + 12(18) = \underline{234}$.

---

② $P_1(0,0,0)$; $P_2(2,1,3)$ ; $\bar{F} = 3x^2\bar{i} + (2zx-y)\bar{j} + z\bar{k}$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$; $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

work done $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} 3x^2dx + (2zx-y)dy + zdz$

$$= \left[x^3 + 2xyz - \frac{y^2}{2} + \frac{z^2}{2}\right]_{(0,0,0)}^{(2,1,3)}$$

$$= \left(8 + 12 - \frac{1}{2} + \frac{9}{2}\right) - 0$$

$$= \underline{\underline{24}}.$$

---

③ $\bar{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$

$x^2+y^2=4$ xy plane $\rightarrow z=0$ & $dz=0$

let; $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

Circulation $= \int_C \bar{F}\cdot d\bar{r}$

$\bar{F}\cdot d\bar{r} = (2x-y+2z)dx + (x+y-z)dy + (3x-2y-5z)dz$

Put $z=0$ ($\because$ xy plane) $dz=0$

$\bar{F}\cdot d\bar{r} = (2x-y)\,dx + (x+y)\,dy$

Take $x = 2\cos\theta$ ; $y = 2\sin\theta$

$dx = -2\sin\theta\,d\theta$ $dy = 2\cos\theta\,d\theta$

$\theta : 0 \to 2\pi$

Circulation $= \int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx + (x+y)\,dy$

$$= \int_{\theta=0}^{2\pi} (4\cos\theta - 2\sin\theta)(-2\sin\theta\,d\theta) + (2\cos\theta + 2\sin\theta)2\cos\theta\,d\theta$$

$$= \int_0^{2\pi} (-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (4 - 4\sin\theta\cos\theta)\,d\theta$$

$$\Rightarrow \int_0^{2\pi} (4 - 2\sin 2\theta)\,d\theta = \left(4\theta + \cancel{2}\frac{\cos 2\theta}{\cancel{2}}\right)_0^{2\pi}$$

$$= (8\pi + \cos 4\pi) - (0 + \cos 0)$$

$$= \underline{8\pi}$$

---

(4) Refer part-A 1st Question

Substitute $a = b = c = 1$

$\therefore$ LHS = RHS = 1

(5) $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ $\quad x^2 + y^2 = 4$

$z = 0 ; z = 3.$

$\nabla \bar{F} = 4 - 4y + 2z$

By Gauss divergence theorem.

$$\iint_S \bar{F} \cdot ds = \iiint_V \nabla \cdot \bar{F} \, dv$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - \cancel{4}y + 2z) \, dx \, dy \, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[ (4 - 4y) z + z^2 \right]_0^3 dy \, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \underbrace{(12(1-y) + 9)}_{21 - 12y} \, dx \, dy$$

$$= \int_{-2}^{2} \left[ 21 y - \overset{6}{\cancel{12}} \cdot \frac{y^2}{\cancel{2}} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left( \left[ 21 \, y \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} - \left( 6 \underbrace{(\cancel{(4-x^2)} - \cancel{(4-x^2)})}_{0} \right) \right) dx$$

$$= \int_{-2}^{2} 21 \times 2 \; y \Big]_0^{\sqrt{4-x^2}} dx$$

$$= 42 \int_{-2}^{2} \left(\sqrt{4-x^2}\right) dx$$

$$= 42 \times 2 \int_0^2 \sqrt{4-x^2}\, dx$$

$a^2 - x^2$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

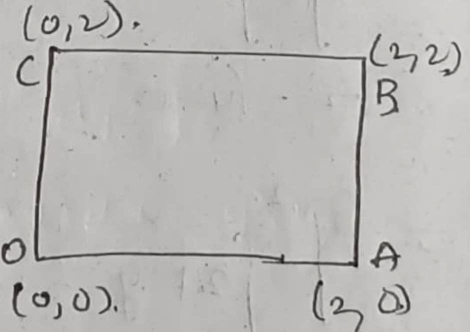
$$\Rightarrow 84\left[0 + 2\cdot\frac{\pi}{2} - 0\right] = 84\pi$$

---

Question mistake

(6) $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$.

$(0,0); (2,0); (2,2); (0,2)$




$M = x^2 - xy^3$ | $N = y^2 - 2xy$

$\frac{\partial M}{\partial y} = -3xy^2$ | $\frac{\partial N}{\partial x} = -2y$

RHS

$$\int_0^2\int_0^2 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\int_0^2\int_0^2 (-2y + 3xy^2)\,dx\,dy$$

$$= \int_0^2 \left[-2xy + \frac{3x^2}{2} y^2\right]_0^2 dy.$$

$$= \int_0^2 (-4y + 6y^2)\, dy = \left(-2y^2 + 2y^3\right)_0^2$$

$$\Rightarrow -8 + 16 = 8. \quad -(1)$$

LHS

(i) OA : $y = 0$

$$\int_0^2 x^2 dx = \left(x^3/3\right)_0^2 = 8/3.$$

(ii) AB : $x = 2$ ; $dx = 0$

$$\int_0^2 (y^2 - 4y)\, dy. = \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \frac{8}{3} - 8$$

$$= -\frac{16}{3}$$

(iii) BC : $y = 2$; $dy = 0$

$$\int_0^2 (x^2 - 8x)\, dx = \left(\frac{x^3}{3} - 4x^2\right)_2^0 = -\left(\frac{8}{3} - 16\right)$$

$$= \frac{40}{3}$$

(iv) CO ; $x = 0$ ; $dx = 0.$

$$\int_2^0 y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = -8/3$$

$\int_C M\,dx + N\,dy = ① + ② + ③ + ④$

$= 8$

$\therefore \int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$

$\Rightarrow \frac{24}{3} = 8$ — ②

from ① & ②;

$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Hence Green's theorem is verified

---

⑦ $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ $\quad y = 2x^2$ $\quad xy$ plane $\rightarrow z = 0$

$dy = 4x\,dx$ $\quad (0,0)$ to $(1,2)$.

$\bar{F} \cdot d\bar{r} = (3xy\,\bar{i} - y^2\,\bar{j}) \cdot (dx\,\bar{i} + dy\,\bar{j})$

$= \cancel{(3xy)} - y^2 = (3xy)\,dx - y^2\,dy$

$\int_C \bar{F} \cdot d\bar{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - 4x^4(4x)\,dx$

$\Rightarrow \int_0^1 (6x^3 - 16x^5)\,dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$

$= -\frac{7}{6}$

(8) $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$

$M = 2x^2 - y^2 \qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x \qquad x^2+y^2 = a^2$

$y^2 = a^2 - x^2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x+y) \qquad y = \pm\sqrt{a^2-x^2}$

$y : 0 \to \sqrt{a^2-x^2}$ ($\because$ upper half of circle).

Put $y = 0$

$x : -a \to a.$

$x^2 = a^2$

$x = \pm a.$

$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} 2(x+y)\,dx\,dy$ $dy\,dx$

$= 2\int_{-a}^{a} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx.$

$\int f(x)^n \cdot f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}$

$= 2\int_{-a}^{a} \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right) dx$

$0$

$\Rightarrow \cancel{2}\left\{ \frac{-(a^2-x^2)^{3/2}}{3/2} + a^2x - \frac{x^3}{3} \right\}_{-a}^{a} = 2\left\{a^3 - \frac{a^3}{3}\right\}$

$= \frac{4a^3}{3}$

⑨ $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

$x^2 + y^2 + z^2 = 1 \rightarrow$ sphere

upper half surface & projection of xy-plane
$\downarrow$
$z=0$

$x^2+y^2=1;\ z=0$

$x = \cos\theta;\ y = \sin\theta,\ \theta: 0 \rightarrow 2\pi$

$dx = -\sin\theta d\theta;\ dy = \cos\theta d\theta.$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)dx + (-yz^2)dy + (-y^2z)dz$$

$z=0$ & $dz=0$

$$= \int_C (2x-y)dx$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta$$

$$\Rightarrow \int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$= \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + \tfrac{1}{2}\cos 2\theta\right]_0^{2\pi}.$$

$$= \tfrac{1}{2}(2\pi - 0) + 0 + \tfrac{1}{2}(\cos 4\pi - \cos 0)$$

$= \pi$ — ①

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1)$$

$$= \bar{k}$$

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_S \bar{k} \cdot \bar{n}\, ds = \iint dx\, dy$$

let R is the projection on xy plane and

$\bar{k} \cdot \bar{n}\, ds = dx\, dy$

$$\iint_R dx\, dy = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = 4 \int_0^1 \sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4\left[\frac{1}{2}\sin^{-1}(1)\right]$$

$$= 2 \cdot \frac{\pi}{2}$$

$$= \pi \quad \text{—(2)}$$

from (1) & (2)

$$\int_C \bar{F} \cdot d\bar{r} = \int (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

Stoke's theorem is verified.

(10) $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ $\quad x^2+y^2 \le 1,\ z=0$

$x = \cos\theta\ ;\ y = \sin\theta$

$\theta : 0 \to 2\pi$

$dx = -\sin\theta\, d\theta\ ;\ dy = \cos\theta\, d\theta$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [-\sin^3\theta(-\sin\theta) + \cos^3\theta \cos\theta]\, d\theta$$

$$\Rightarrow \int_0^{2\pi} \underbrace{(\cos^4\theta + \sin^4\theta)}_{1-2\sin^2\theta\cos^2\theta}\, d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}(2\sin\theta\cos\theta)^2 d\theta$$

$$\Rightarrow [\theta]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \sin^2 2\theta\, d\theta = 2\pi - \frac{1}{4}\int_0^{2\pi}(1-\cos 4\theta)\, d\theta$$

$$\Rightarrow 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{3\pi}{2} \quad -(1)$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$$

$$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = 3\int_S (x^2+y^2)\, \bar{k}\cdot\bar{n}\, ds$$

$(\bar{k}\cdot\bar{n})\, ds = dx\, dy$ $\quad R \to$ region on $xy$ plane

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3 \iint_R (x^2+y^2)\, dx\, dy$$

Put $x = r\cos\phi;\ y = r\sin\phi$ $\quad dx\,dy = r\,dr\,d\phi$

$r : 0 \to 1$

$\phi : 0 \to 2\pi$

$$= 3 \int_{\phi=0}^{2\pi} \int_{r=0}^{1} r \cdot r^2\, dr\, d\phi$$

$$\Rightarrow 3 \int_0^{2\pi} \int_0^1 r^3\, dr\, d\phi = \frac{3\pi}{2} \quad \text{--- (2)}$$

from (1) & (2)

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

Hence, stoke's theorem is verified.

---

(11) $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$ $\quad$ cube:

$x = 0;\ x = a;\ y = 0;\ y = a;\ z = 0;\ z = a$

(i) $R_1$ is $yz$ plane; $x = a$; $dx = 0$; $ds = dy\,dz$

$\bar{n} = \bar{i}$ $\quad ; y : 0 \to a;\ z : 0 \to a$

$\bar{F} \cdot \bar{n} = 4xz = 4az \quad (\because x = a)$

$$\iint_{R_1} \bar{F} \cdot \bar{n}\, ds = \int_0^a \int_0^a 4az\, dz\, dy = 4a \int_0^a \left(\frac{z^2}{2}\right)_0^a dy = 2a^4$$

(ii) $R_2$; $x=0$; $ds = dy\,dz$; $\bar{n} = -\vec{i}$

$\bar{F} \cdot \bar{n} = -4xz = 0$ $(\because x = 0$

$$\iint_{R_2} \bar{F} \cdot \bar{n}\, ds = 0$$

(iii). $R_3$; $z = a$; $ds = dx\,dy$; $\bar{n} = \vec{k}$

$\bar{F} \cdot \bar{n} = yz = ay$ $(\because \cancel{z} = a)$

$$\iint_{R_3} \bar{F} \cdot \bar{n}\, ds = \int_0^a \int_0^a ay\, dy\, dx = a^4/2$$

(iv). $R_4$; $z = 0$; $ds = dx\,dy$; $\bar{n} = -\vec{k}$

$\bar{F} \cdot \bar{n} = -yz = 0$; $(\because z = 0)$

$$\iint_{R_4} \bar{F} \cdot \bar{n}\, ds = 0$$

(v). $R_5$; $y = a$; $ds = dx\,dz$; $\bar{n} = \vec{j}$

$\bar{F} \cdot \bar{n} = -y^2 = -a^2$

$$\iint_{R_5} \bar{F} \cdot \bar{n}\, ds = \int_0^a \int_0^a (-a^2)\, dz\, dx = -a^4$$

(vi) $R_6$; $y = 0$; $ds = dx\,dz$; $\bar{n} = -\vec{j}$

$\bar{F} \cdot \bar{n} = y^2 = 0$ $(\because y = 0$

$$\iint_{R_6} \bar{F} \cdot \bar{n}\, ds = 0$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\, ds = \iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} + \iint_{R_5} + \iint_{R_6}$$

$$\Rightarrow 2a^4 + 0 + \frac{a^4}{2} + 0 - a^4 + 0 = \underline{\underline{\frac{3a^4}{2}}}$$

---

(12) $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$  $y = x^3$

$(1,1)$ to $(2,8)$

$\int_C \bar{F}\cdot d\bar{r} = ?$

$dy = 3x^2\, dx.$

Put $y = x^3$ in given $\bar{F}$

$$\bar{F}\cdot d\bar{r} = (5x^4 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx.$$

$$= (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx,$$

$$\int_{y=x^3} \bar{F}\cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \left[6\frac{x^6}{6} + 5\frac{x^5}{5} - 12\frac{x^4}{4} - 6\frac{x^3}{3}\right]_1^2$$

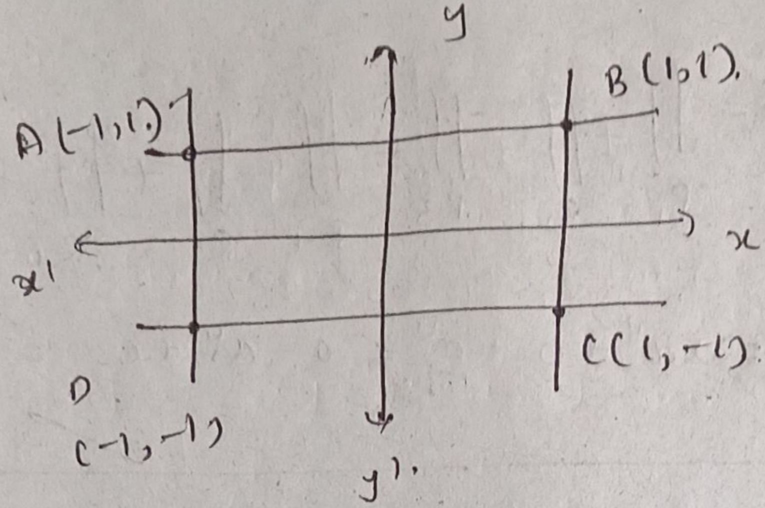
$$= (x^6 + x^5 - 3x^4 - 2x^3)_1^2$$

$$= 16(4 + 2 - 3 - 1) - (1 + 1 - 3 - 2)$$

$$= 32 + 3$$

$$= \underline{35}$$

(13)




$\oint \vec{F}\cdot d\vec{r} = ?$ $\quad \vec{F} = (x^2+xy)\,\hat{i} + (x^2+y^2)\,\hat{j}$

$x = \pm 1 \;;\; y = \pm 1$

(i) AB

$x : -1 \to 1 \;;\; y = 1 \;;\; dy = 0$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_C (x^2+xy)\,dx + \underbrace{(x^2+y^2)\,dy}_{0}$$

$$\Rightarrow \int_{-1}^{1} (x^2+x)\,dx = 2/3$$

(ii) BC ; $x = 1 \;;\; dx = 0 \;;\; y : 1 \to -1$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_C \underbrace{(x^2+xy)\,dx}_{0} + (x^2+y^2)\,dy$$

$$\Rightarrow \int_{1}^{-1} (1+y^2)\,dy = -8/3$$

(iii) CD ; $x : 1 \to -1 \;;\; y = -1 \;;\; dy = 0$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_C (x^2+xy)\,dx + \underbrace{(x^2+y^2)\,dy}_{0}$$

$$\Rightarrow \int^{-1} (x^2-x)\,dx = -2/3$$

(iv) DA ; $x=-1$; $dx=0$; $y: -1 \to 1$

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_C \underbrace{(x^2+xy)\,dx}_{0} + (x^2+y^2)\,dy$$

$$\Rightarrow \int_{-1}^{1} (1+y^2)\,dy = 8/3$$

$$\int_L \bar{F}\cdot d\bar{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = \cancel{2/3} - \cancel{8/3} - \cancel{2/3} + \cancel{8/3}$$

$$= 0$$

---

(14) $\int_C (e^x dx + 2y\,dy - dz)$ $\quad x^2+y^2=9$, $z=2$

Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \int (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix} = \bar{i}(0) - \bar{j}(0) + \bar{k}(0)$$

$$= 0$$

Since curl $\bar{F} = 0$

$$\int(\nabla\times\bar{F})\cdot\bar{n}\,ds = 0$$

$$\therefore \int_C (e^x dx + 2y\,dy - dz) = 0.$$

(15) $\bar{F} = x^2\bar{i} + xy\bar{j}$   $x=0;\ x=a;\ y=0;\ y=a$

$x: 0 \to a;\ y: 0 \to a$   $\bar{F}\cdot d\bar{r} = x^2 dx + xy\,dy$

Stoke's theorem

LHS

(i) OA ; $y=0;\ dy=0;\ x: 0 \to a$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx = a^3/3$$

(ii) AB; $x=a;\ dx=0;\ y: 0 \to a$.

$$\int_{AB} \bar{F}\cdot d\bar{r} = a\int_{y=0}^{a} y\,dy = a^3/2$$

(iii) BC ; $y=a;\ dy=0;\ x: a \to 0$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 x^2\,dx = -a^3/3$$

(iv) CO ; $x=0;\ dx=0;\ y: a \to 0$

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_a^0 0\,dy = 0$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= a^3/3 + a^3/2 - a^3/3 + 0$$

$$= a^3/2 \quad \text{--- (1)}$$

RHS

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix} = ky$$

let R be the projection of S in xy plane,

$z=0$; $dz=0$; $\bar{n}=k$; $ds = dx\,dy$

$\text{curl } \bar{F} \cdot \bar{n} = y$

$$\int_S (\text{curl } \bar{F} \cdot \bar{n})\, ds = \int_{x=0}^{a} \int_{y=0}^{a} y \, dy \, dx$$

$$\Rightarrow \int_0^a \left[\frac{y^2}{2}\right]_0^a dx = \frac{a^2}{2}\left[x\right]_0^a$$

$$= a^3/2 \quad -②$$

from ① & ②

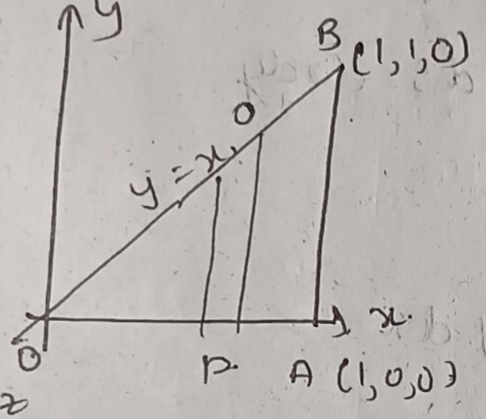
$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int (\text{curl } \bar{F} \cdot \bar{n})\, ds$$

Hence Stoke's theorem is verified

(16) $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$

$(0,0,0)$; $(1,0,0)$; $(1,1,0)$

let; $\bar{F}\cdot d\bar{r} = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$




$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= 2\bar{i} + \bar{k}$$

$\bar{n} = \bar{k}$ ($\because$ xy plane)

$\text{curl}\,\bar{F}\cdot\bar{n} = (2\bar{i}+\bar{k})\cdot\bar{k} = 1$

$ds = dx\,dy$

$$\int_C \bar{F}\cdot d\bar{r} = \underbrace{\iint_S dx\,dy}_{\text{stokes theorem}} = \iint_S dA = A = \text{area of } \Delta OAB$$

$$= \frac{1}{2} \times OA \times OB$$

$$\Rightarrow \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

$\therefore \int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz = \frac{1}{2}$

(17) Gauss divergence theorem

cube

$\bar{F} = (2x - z)\bar{i} + x^2 y \bar{j} - xz^2 \bar{k}$

$x = 0;\ x = 1;\ y = 0;\ y = 1;\ z = 0;\ z = 1.$

<u>RHS</u>

$\nabla \bar{f} = \frac{\partial}{\partial x}(2x - z) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(-xz^2)$

$\nabla \bar{f} = 2 + x^2 - 2xz$

$$\iiint \nabla \bar{f}\, dv = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (x^2 - 2xz + 2)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 \left[\frac{x^3}{3} - \frac{\cancel{2}x^2 z}{\cancel{2}} + 2x\right]_0^1 dy\, dz$$

$$= \int_0^1\int_0^1 (\underbrace{\tfrac{1}{3} + 2}_{7/3} - z)\, dy\, dz$$

$$= \int_0^1 \left(\frac{7}{3}y - yz\right)_0^1 dz$$

$$\Rightarrow \int_0^1 \left(\frac{7}{3} - z\right) dz = \left[\frac{7}{3}(1) - \frac{z^2}{2}\right]_0^1$$

$$= \frac{7}{3} - \frac{1}{2}$$

$$= 11/6 \quad \text{— (1)}$$

LHS

(i) $R_1$ is in $yz$ plane; $x=1$; $dx=0$

$y: 0 \to 1$; $z: 0 \to 1$; $\bar{n} = \bar{i}$

$ds = dy\,dz$ $\qquad |\bar{n}\cdot\bar{i}| = 1$

$\bar{F}\cdot\bar{n} = (2x - z) = 2 - z \quad (\because x = 1)$

$$\iint_{R_1} \bar{F}\,\bar{n}\,ds = \int_0^1\int_0^1 (2-z)\,dy\,dz = \int_0^1 (2-z)\,dz$$

$$= \left| 2z - \frac{z^2}{2} \right|$$

$$\Rightarrow 2 - \tfrac{1}{2} = \tfrac{3}{2}$$

(ii) $R_2$ is in $yz$ plane; $x=0$; $dx=0$

$y: 0 \to 1$; $z: 0 \to 1$; $\bar{n} = -\bar{i}$; $|\bar{n}\cdot\bar{i}| = 1$

$ds = dy\,dz$

$\bar{F}\cdot\bar{n} = (2\underset{0}{x} - z)(-1) = z$

$$\iint_{R_2} \bar{F}\cdot\bar{n}\,ds = \int_0^1\int_0^1 z\,dy\,dz = \int_0^1 z\,dz = \frac{1}{2}$$

(iii) $R_3$ is in $zx$ plane; $ds = dx\,dz$; $\bar{n} = \bar{j}$

$x: 0 \to 1$; $z: 0 \to 1$ $\quad y = 1$ $\quad |\bar{n}\cdot\bar{j}| = 1$

$\bar{F}\cdot\bar{n} = x^2 y = x^2$

$$\iint_{R_3} \bar{F}\cdot\bar{n}\,ds = \int_0^1\int_0^1 x^2\,dx\,dz = \int_0^1 \left(\frac{x^3}{3}\right)_0^1 dz = \frac{1}{3}\int_0^1 dz$$

$$= \frac{1}{3}$$

(iv) $R_4$ is in $zx$ plane; $ds = dx\,dz$; $\bar{n} = -\bar{j}$

$x: 0 \to 1$; $z: 0 \to 1$; $y = 0$ $\quad |\bar{n}\cdot\bar{j}| = -1$

$\bar{f}\cdot\bar{n} = x^2 y = 0$

$\iint_{R_4} \bar{f}\cdot\bar{n}\,ds = \underline{0}$

(v) $R_5$ is in $xy$ plane; $z = 1$; $ds = dx\,dy$

$\bar{n} = \bar{k}$; $|\bar{n}\cdot\bar{k}| = 1$; $y: 0 \to 1$; $x: 0 \to 1$.

$\bar{f}\cdot\bar{n} = -xz^2 = -x(1)^2 = -x$

$$\iint_{R_5} \bar{f}\cdot\bar{n}\,ds = \int_0^1\int_0^1 -x\,dx\,dy = -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy = -\frac{1}{2}\int_0^1 dy$$

$$= \underline{-1/2}$$

(vi) $R_6$ is in $xy$ plane; $z = 0$; $ds = dx\,dy$.

$\bar{n} = -\bar{k}$; $|\bar{n}\cdot\bar{k}| = -1$; $y: 0 \to 1$; $x: 0 \to 1$

$\bar{f}\cdot\bar{n} = xz^2 = \underline{0}$

$\iint_{R_6} \bar{f}\cdot\bar{n}\,ds = \underline{0}$

$$\int_C \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} + \iint_{R_5} + \iint_{R_6}$$

$$= \frac{3}{2} + \cancel{\frac{1}{2}} + \frac{1}{3} + 0 - \cancel{\frac{1}{2}} + 0$$

$$\Rightarrow \frac{3}{2} + \frac{1}{3} = \underline{\frac{11}{6}} \quad - (2)$$

from ① & ②

$$\int_C \bar{F}\cdot\hat{n}\,ds = \int_V \nabla \cdot F\,dv$$

Hence Gauss divergence theorem verified!

(18) $\bar{F} = (3x^2 - 6yz)\,\bar{i} + (2y + 3xz)\,\bar{j} + (1 - 4xyz^2)\,\bar{k}$

$(0,0,0)$ to $(1,1,1)$.

$x = t;\ y = t^2;\ z = t^3$

$\bar{F}\cdot d\bar{r} = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$

$t = 0$ & $t = 1$

| | | |
|---|---|---|
| $x = t$ | $y = t^2$ | $z = t^3$ |
| $dx = dt$ | $dy = 2t\,dt$ | $dz = 3t^2\,dt$ |

Work done $= \int_{t=0}^{1} (3t^2 - 6t^5)\,dt + (2t^2 + 3t^4)\,2t\,dt + (1 - 4t^9)\,3t^2\,dt$

$= \int_0^1 (-12t^{11} + 4t^3 + 6t^2)\,dt$

$= \left[-t^{12} + t^4 + 2t^3\right]_0^1$

$= 2.$

(19) let; $\phi = x^2+y^2-16$ ; $\bar{A} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$

$z=0$ & $z=5$

$x^2+y^2=16$ $\rightarrow$ $x=0$ ; $y=4$.

Normal $(\nabla\phi) = (2x\bar{i}+2y\bar{j})$

$$\bar{n} = \frac{\cancel{2}(x\bar{i}+y\bar{j})}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{4}$$

let R be the projection of S on YZ plane

$$\iint_S \bar{A}\cdot\bar{n}\,ds = \iint_R \bar{A}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$\bar{A}\cdot\bar{n} = \frac{1}{4}(xz+xy)$. $\because i\cdot i = j\cdot j = 1$

$\bar{A}\cdot\bar{i} = x/4$

$$\iint_R \bar{A}\cdot\bar{n}\,ds = \int_{y=0}^{4}\int_{z=0}^{5} \frac{xz+xy}{\cancel{4}}\cdot\frac{\cancel{4}}{x}\,dy\,dz$$

$$= \int_{y=0}^{4}\int_{0}^{5=z} (y+z)\,dz\,dy$$

$$= \int_0^4 \left[yz+\frac{z^2}{2}\right]_0^5 dy$$

$$\Rightarrow \int_0^4 \left(5y+\frac{25}{2}\right) dy$$

$$= \left[ 5\frac{y^3}{2} + \frac{25}{2}y \right]_0^4$$

$$= 40 + 50$$

$$= 90.$$

---

(20) Green's theorem: $y = 0$; $x = \pi/2$; $\pi y = 2x$

$$\int_C (y - \sin x)\,dx + \cos x\,dy$$

$M = y - \sin x$ | $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = -\sin x$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (-1 - \sin x)$$

as per green's theorem

$$\int_C (y - \sin x)\,dx + \cos x\,dy = -\iint_S (1 + \sin x)\,dx\,dy$$

$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (1 + \sin x)\,dx\,dy$$

$$= -\int_0^{\pi/2} (\sin x + 1) \cdot [y]_0^{2x/\pi}$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(\sin x+1)\,dx$$ $\int uv = u\int v - \int u'\int v\,dx$

$$= -\frac{2}{\pi}\left[x(-\cos x+x)\right]_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x+x)\,dx$$

$$= -\frac{2}{\pi}\left[x(-\cos x+x)+\sin x-\frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[-x\cos x+\frac{x^2}{2}+\sin x\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}+1\right]$$

$$= -\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

$$\therefore \int_C (y-\sin x)\,dx+\cos x\,dy = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

Verified!

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