# Module-3 Partial differential equations

Sem-II

PART-A

① $lx+my+nz=\phi(x^2+y^2+z^2)$

<u>Diff w.r.t x</u>

$l+0+np=\phi'(x^2+y^2+z^2)$
$(2x+2zp) \longrightarrow ①$

<u>Diff. w.r.t y</u> $\quad \frac{\partial z}{\partial x}=p \quad \frac{\partial z}{\partial y}=q$

$0+m+n.q=\phi'(x^2+y^2+z^2)$
$(2y+2zq) \longrightarrow ②$

Divide ① by ②

$$\frac{l+np}{m+nq}=\frac{2x+2zp}{2y+2zq}$$

$$\frac{l+np}{m+nq}=\frac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)}$$

$(l+np)(y+zq)=(x+zp)(m+nq)$

$ly+zlq+npy+\cancel{npzq}$
$=xm+xnq+\cancel{zpnq}+zpm$

$\cancel{zlq+npy+xnq}$ $\cancel{=zpnq}$

$zlq+npy-xnq-zpm=\cancel{ly+xm}\ \ ly-xm$

$p(\cancel{\quad} \cancel{\theta} ny - \cancel{\theta} zm) + q(lz \cancel{\theta} nn)$

$$\cancel{= ly \, \theta \, mx}$$

$$= ly - mx$$

$\cancel{\text{③ } x+y+}$

③ sol $xy + yz + 2x = f\left(\frac{z}{((x+y))}\right)$

Diff w.r.t $x$

$\cancel{y+z+2x}$ $y + z + 2x' = f'\frac{z}{((x+y))} * \frac{z}{((x+y))} * \frac{z}{\left(-(x+y)^2\right)} *$

$$* (1+y)'$$

Diff W.r.t toy

$x + y + 2y' = f'\frac{z}{((x+y)} * \frac{z}{(x+y)} * \frac{z}{(-(x+y)^2)} * (1+x')$

eliminate these two equations

$y + z + 2x' = x + \cancel{\theta} z + 2y'$

Simplifying this equation we get

$x' - y' = \frac{y}{x+y} - \frac{x}{(x+y)}$

rearranging terms

$x' + \frac{x}{(x+y)} = y' + \frac{y}{x+y}$

∴ The partial differential of given equation

$xy + yz + 2x = f\frac{\cancel{\theta} z}{((x+y))}$

③ $f(x^2-y^2, x^2-z^2)$

$u = x^2-y^2$ & $V = x^2-z^2$ ~~So~~, Then we have

$f(u, v) = 0$

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dy} + \frac{\partial f}{\partial z}\frac{dz}{dx} = 0$$

$$\frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial V}\frac{dv}{dx} = 0$$

find $\frac{du}{dx}$ & $\frac{dv}{dx}$

$$\frac{du}{dx} = 2x \ \& \ \frac{dv}{dx} = 2x$$

Partial derivatives of f in terms of u and v:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\frac{1}{-2y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{1}{-2z}$$

Substituting these derivatives into the equation

$$\frac{\partial f}{\partial x}\frac{1}{-2y}(2x) + \frac{\partial f}{\partial x}\frac{1}{-2z}(2x) = 0$$

$$x\frac{\partial f}{\partial x}\left(\frac{1}{-y} + \frac{1}{-z}\right) = 0$$

∴ Partial differential equation by given equation

$f(x^2-y^2, x^2-z^2) = 0.$

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4) $z = f(x+ct) + g(x-ct)$

Given $z = f(x+ct) + g(x-ct)$

diff w.r.t $x$

$$\frac{\partial z}{\partial x} = f'(x+ct)(1) + g'(x-ct)(-1)$$

diff w.r.t 't'

$$\frac{\partial z}{\partial t} = f'(x+ct)c + g'(x-ct)(-c)$$

~~To eliminate arbitrary constants~~

Divide ② × ①

$$\frac{1}{c}\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)(-1) + f'(x+ct) + g'(x-ct)(-1)$$

$$\frac{1}{c}\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} = 2f'(x+ct) + 2g'(x-ct)(-1)$$

$$\frac{1}{c}\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} = 2f'(x+ct) - 2g'(x-ct)$$

5) $z = f(x+iy) + g(x+iy)$

Diff. Partially w.r.t $x$

$$\frac{\partial z}{\partial x} = f'(x+iy)\cdot 1 + g'(x+iy)\cdot 1$$

$$p = f'(x+iy) + g'(x+iy) \rightarrow ①$$

Diff. Partially w.r.t $y$

$$\frac{\partial z}{\partial x} = f'(x+iy)(i) + g'(x+iy)(i)$$

$$q = f'(x+iy)i + g'(x+iy)\cdot(i) \rightarrow ②$$

$\frac{\partial P}{\partial x} = \frac{\partial^2 x}{\partial x^2} = r = f''(x+iy) + g''(x+iy) \rightarrow (6)$

$t = \frac{\partial q}{\partial y} = f''(x+iy)i^2 + g''(x+iy)i^2$

$t = -(f''(x+iy) + g''(x+iy))$

$t = -r$

$r+t=0$

$p = \frac{\partial z}{\partial x}$ $t = \frac{\partial^2 z}{\partial y^2}$

$q = \frac{\partial z}{\partial y}$ $s = \frac{\partial^2 z}{\partial x \partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$z \rightarrow$ dep. var

$x, y \rightarrow$ ind p-variable

Part B

(11) $px^2 + qy^2 = z(x+y)$

Sol $\frac{px^2}{x} + \frac{qy^2}{y} = z(x+y)$

$px + qy = z(x+y)$

$\frac{px+qy}{x+y} = z.$

Let's substitute $u$ for $\frac{x}{x+y}$

$z(u) = \frac{pu + q(1-u)}{u + (1-y)}$

$x = u(x+y)$

$y = (1-y)(x+y)$

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solving these eqn's from x and y

$x = \frac{u}{u+(1-u)}$

$y = \frac{1-u}{u+(1-u)}$

$z(u) = pu + q(1-u)$

expressing x & y terms of u

$x = \frac{u}{u+(1-u)}$

$y = \frac{1-u}{u+(1-u)}$

(2) $p - x^2 = y^2 + q$

Sol $p - x^2 - y^2 - q = 0$

$p - (x^2 + y^2) = q$

$x^2 + y^2$ is the equa

(3) $p - x^2 = y^2 + q$

$p^2 - x^2 - y^2 - q = 0$

$p - (x^2 + y^2) = q$

$x^2 + y^2$ is the eqn for a circle centered at the origin

when radius $r = \sqrt{x^2 + y^2}$

$x^2 + y^2 = r^2$

(12) $p - x^2 = y^2 + q$

Sol $p - \frac{x^2}{y^2} = q$

Add $\frac{x^2}{y^2}$ to both sides

$p = q + \frac{x^2}{y^2}$

Subtract $q$ from both sides

$p - q = \frac{x^2}{y^2}$

Isolate $x^2$, we multiply both sides $\frac{1}{y^2}$

$y^2 (p-q) = x^2$

$x = \pm \sqrt{y^2(p-q)}$

(13) $y^2 z p + x^2 z q = x y^2$

Let's rewrite this equation

$y^2 \frac{\partial z}{\partial p} + x^2 \frac{\partial z}{\partial q} = x y^2$

$\frac{1}{z} \frac{\partial z}{\partial p} + \frac{x}{y} \frac{\partial z}{\partial q} = \frac{1}{y}$

$\int \frac{1}{z} \frac{\partial z}{\partial p} dp + \int \frac{x}{y} \frac{\partial z}{\partial q} dq = \int \frac{1}{y} dq$

$\ln |z| + \frac{x}{y} z = \frac{q}{y} + c_1 (p)$

$z e^{\frac{x}{y}} = c_2 e^{\frac{x}{y}}$

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$z = ce^{\frac{x}{y}} - \frac{x}{y}$

$\therefore c$ is arbitrary constant

$p\tan x + q\tan y = \tan z$

Sol $Pp + Qq = R$

$P = \tan x$

$Q = \tan y$

$R = \tan z$

A.E $= \frac{dx}{P} = \frac{dy}{dQ} = \frac{dz}{R}$

$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

$\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$\int \cot x\,dx = \int \cot y\,dy$

$\log \sin x = \log \sin y + \log c_1$

$\log \sin x - \log \sin y = \log c_1$

$\log\left[\frac{\sin x}{\sin y}\right] = \log c_1$

$\boxed{\frac{\sin x}{\sin y} = c_1}$ —— (1)

$\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$\int \cot y\,dy = \int \cot z\,dz$

$\log \sin y = \log \sin z + \log c_2$

$\log \sin y - \log \sin z = \log c_2$

$\log \left(\frac{\sin y}{\sin z}\right) = \log c_2$

$\frac{\sin y}{\sin z} = c_2 \rightarrow (2)$

$\phi(c_1, c_2) = 0$

$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

$\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right)$

or

$\frac{\sin y}{\sin z} = \phi\left(\frac{\sin x}{\sin y}\right)$

(15) $(x-a)p + (y-b)q + (c-z)z$

Sol This is Lagrange's linear equation the arbitary equation is

$\frac{dx}{(x-a)} = \frac{dy}{(y-b)} = \frac{dz}{(c-z)}$

$\frac{dx}{x-a} = \frac{dy}{y-b}$

Integrating we get

$$\int \frac{dx}{x-a} = \int \frac{dy}{y-b}$$

$$\log(x-a) = \log(y-b) = c$$

$$\frac{a-x}{y-b} = c$$

$$\frac{dy}{y-b} = \frac{dz}{c-z}$$

Integrating we get

$$\cancel{\int \frac{dy}{b-y} = \int \frac{dz}{c-z}}$$

$$\int \frac{dy}{y-b} = \int \frac{dz}{c-z}$$

$$\log(y-b) = \log(c-z) = c_2$$

$$\frac{y-b}{c-z} = c_2$$

The solution is

$$\left(\frac{x-a}{y-b}, \frac{y-b}{c-z}\right) = 0$$

(16) $px^2 + qy^2 = z(x+y)$

Sol The auxiliary equation are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$$

From the $1^{st}$ & $2^{nd}$ fractions

$$\frac{dx}{x^2} + \frac{dy}{y^2} = 0 \text{ w.r.t}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{c} \quad \cancel{\text{or } -\frac{1}{x} - \frac{1}{y} = -\frac{1}{c}} \quad \cancel{\text{or } \frac{1}{x} \frac{1}{y} = \frac{1}{c}}$$

$$\frac{dx+dy}{(x^2+y^2)} = \frac{dx+dy}{(x+y)(x+y)} = \frac{dz}{z(x+y)}$$

$$\frac{dx+dy}{x+y} = \frac{dz}{z}$$

Integration

$$\ln(x+y) = \ln z + c_2$$

$$\text{or } \frac{x+y}{z} = c_2$$

General Solution is

$$\phi F\left(\frac{1}{x} + \frac{1}{y} + \frac{x+y}{c+z}\right) = 0$$

⑰ $pz - qz = z^2 + (x+y)^2$

Sol $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

By ~~a~~grans

⑰ $pz - qz = z^2 + (x+y)^2$

Sol ~~$a = (z^2 + (x+y$~~

Sol $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

By using this weget

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

Taking ~~fird~~ first two eqn's

~~Is~~ $\frac{dx}{z} = \frac{dy}{-z}$

$dx = -dy$

Integration on bothsides

$x + y = c$

taking last two equations

$$\frac{dy}{-z} = \frac{dz}{-z^2 + c}$$

$$dy = \frac{-z\,dz}{z^2 + c}$$

Integrate on both Sides

$y = -\frac{1}{2} \ln(z^2 + c) + \ln a$

$y = \ln \frac{a}{\sqrt{z^2 + c}}$

$e^y = \frac{a}{\sqrt{z^2 + c}}$

$a = (z^2 + c)^{\frac{1}{2}} e^y$

$a = (z^2 + (x+y)^2)^{\frac{1}{2}} e^y$

(18) $\frac{y^2 z}{x} p + xyq = y^2$

Sol $Pp + Qq = R$

$P = \frac{y^2 z}{x}$ ; $Q = xz$ ; $R = y^2$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$\int x^2\,dx = \int y^2\,dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3}{3} - \frac{y^3}{3} = c$$

$$x^3 - y^3 = 3c$$

$$\boxed{x^3 - y^3 = c_1} \rightarrow (1)$$

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}$$

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + A$$

$$\boxed{\frac{x^2}{2} - \frac{z^2}{2} = A}$$

$$x^2 - z^2 = 2A$$

$$\boxed{x^2 - z^2 = c_2} \rightarrow (2)$$

$$\phi(c_1, c_2) = 0$$

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

or

$$x^3 - y^3 = \phi(x^2 - z^2)$$

or

$$x^2 - z^2 = \phi(x^3 - y^3)$$

(19) $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Sol Let's rewrite the eqn by ~~a~~ rearranging terms

$xy^2p - xz^2p - y z^2 q - x^2yq = zx^2 + zy^2$

we can rearrange terms to solve for one variable say $z$, in terms of the others

$z(x^2+y^2+ py p - yq) = xy^2 p - x^2 y q$

$z(x^2+y^2+(p-q)y) = xy^2 p - x^2 y q$

$$z = \frac{xy^2 p - x^2 y q}{x^2+y^2+(p-q)y}$$

(20) $(x-y)p + (y-x-z)q = z$

Sol $xp - yp + yq - xq - zq = z$

Rearranging terms

$xp - yp + yq - xq = z + zq$

$xp - xq = yp - yq + z + zq$

$x(p-q) = y(p-q) + z(1+q)$

Isolating $y$

$yp - yp + yq = xp - xq + zq + z$

$$y(p+q) = x(p-q) + z(1+q)$$

$$y = \frac{x(p-q) + z(1+q)}{p+q}$$

We can also isolate z

$$z = \frac{\cancel{xp}\, xp - yp + yq - xq}{q+1}$$

$$z = \frac{(p-q)(y-x) + yq}{q+1}$$

The solutions ~~of~~ for x, y, and z in term p, q

$$x = \frac{y(pq) + z(1+q)}{p-q}$$

~~$y = x$~~

$$y = \frac{x(p-q) + z(1+q)}{p+q}$$

$$z = \frac{(p-q)(y-x) + yq}{q+1}$$

∴ These are the general solutions of given equation.