Maths

Assignment

Module-I

Part-A

1) Construct the analytic function $f(z)$ in terms of z if $f(z)$ is an analytic function of z such $u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Sol: Now, $V = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial v}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} \; ; \; \frac{\partial v}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

let $f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$$f'(z) = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$= i\,\frac{2\cos^2 z - 2}{(1-\cos 2z)^2} \quad (u = z,\ y = 0) = i\int \frac{-2}{1-\cos 2z}\,dz$$

$$= -2i\int \frac{1}{\sin^2 z}\,dz = i\cot z + c'$$

$$\Rightarrow (1+i)\,f(z) = i\cot z + c'$$

$$f(z) = \frac{i}{1+i}\cot z + c'$$

2) Show that if $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$, both u and v satisfy laplace's eqn but u+iv is not a regular analytic function of z

A) $u_x = 2x \qquad u_y = -2y$

$u_{xx} = 2 \qquad u_{yy} = -2$

$2 - 2 = 0$

$$v_x = \frac{x^2+y^2(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = \frac{(x^2+y^2)^{\cancel{2}}(2y) - 2xy(2(\cancel{x^2+y^2})(2x))}{(x^2+y^2)^{\cancel{4}}}$$

$$= \frac{(x^2+y^2)(2y) - 2xy(4x)}{(x^2+y^2)^3} = \frac{2yx^2+2y^3-8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$v_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_{yy} = \frac{(x^2+y^2)^{\cancel{2}}(2y) - (y^2-x^2)\,2(\cancel{x^2+y^2})\,2y}{(x^2+y^2)^{\cancel{4}}} = \frac{(x^2+y^2)(2y) - (y^2-x^2)(4y)}{(x^2+y^2)^3}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{2y^3-6x^2y}{(x^2+y^2)^3} + \frac{6x^2y-2y^3}{(x^2+y^2)^3} = 0$$

CR eq are not Satisfied, means not analytic

3) Construct the analytic function $f(z)$ given $u-v=$

$$\frac{(\cos x+\sin x-e^{(-y)})}{2\cos x-e^{y}-e^{-y}} = \frac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}$$ and $f(z)$ is Subjected

to the Condition $f\left(\frac{\pi}{2}\right)=0$

A) Now

$$U = \frac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}} = \frac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}$$

$$\frac{\partial u}{\partial y} = \frac{(\cos x-\cosh y)(-\sin x+\cos x)-(\cos x+\sin x-e^{-y})(-\sin x)}{2(\cos x-\cosh y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(\cos x-\cosh y)(e^{-y})-(\cos x+\sin x-e^{-y})(-\sinh y)}{2(\cos x-\cosh y)^2}$$

$$f(z) = u+iv = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{(\cos x-\cosh y)(-\sin x+\cos x)-(\cos x+\sin x-e^{-y})(\sin x)}{2(\cos x-\cos xy)^2}$$

$$-i\,\frac{(\cos x-\cosh y)(e^{-y})-(\cos x+\sin x-e^{-y})(-\sinh y)}{2(\cos x-\cosh y)^2}$$

$$f'(z) = \frac{(\cos z-1)(-\sin z+\cos z) - (\cos z+\sin z-1)(-\sin z)}{2(\cos z-1)^2}$$

$$-i\,\frac{(\cos z-1)(1) - (\cos z+\sin z-1)(0)}{2(\cos z-1)^2}$$

$$f'(z) = \frac{(1-\cos z) - i(\cos z-1)}{2(\cos z-1)^2} = \frac{-1(1+i)}{2(\cos z-1)}$$

$$f'(z) = \int \frac{-(1+i)}{2(\cos z-1)}\,dz = -(1+i)\int \frac{1}{2(\cos z-1)}\,dz = -\frac{(1+i)}{2}\cot\frac{z}{2}$$

$$\therefore f(z) = \frac{1}{2}\cot\left(\frac{z}{2}\right) + c'$$

4) Construct the analytic function $f(z)$ whose real part of it is $u = e^x\left[\left((x^2-y^2)\cos y - 2xy\sin y\right)\right]$

A) $u = e^x\left[(x^2-y^2)\cos y - 2xy\sin y\right]$

$= e^x x^2 \cos y - e^x y^2 \cos y - 2xy\sin y\, e^x$

$\frac{\partial u}{\partial x} = (e^x 2x + x^2 e^x)\cos y - e^x y^2\cos y - (2xe^x + e^x)y\sin y$

$\frac{\partial v}{\partial y} = e^x x^2 \sin y - e^x\left(y^2(-\sin y) + \cos y(2y)\right) - 2xe^x(y\cos y + \sin y)$

$= e^x x^2\sin y + e^x y^2\sin y - e^x\cos y(2y) - 2xe^x y\cos y - 2xe^x\sin y$

$f(z) = u+iv = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial u} - i\frac{\partial u}{\partial y}$

$$= \left[e^x 2x\cos y + x^2 e^x \cos y - e^x y^2 \cos y - xe^x y\sin y - e^x y\sin y\right] - i$$

$$- e^x x^2 \sin y + e^x y^2 \sin y - e^x \cos y (2y) - 2xe^x y\cos y - 2xe^x \sin y]$$

$$= \left[e^z 2z\cos(0) + z^2 e^z \cos(0) - e^z(0)\cos 0 - ze^z 0 \sin 0 - e^z 0 \sin 0\right]$$

$$- i\left[-e^z z^2 \sin(0) + e^z (0)^2 \sin 0 - e^z \cos 0 (2(0)) - 2ze^z 0 \cos 0 - 2ze^z \sin 0\right]$$

$$= e^z 2z + z^2 e^z$$

$$f'(z) = 2e^z z + z^2 e^z$$

$$\int f'(z) = \int 2ze^z + z^2 e^z$$

$$f(z) = 2\left[ze^z - e^z + z^2 e^z e^z \frac{z^3}{3}\right]$$

$$= 2\left[ze^z - e^z + \left[z^2 e^z - \int e^z 2z\right]\right]$$

$$= 2\left[ze^z - e^z\right] + z^2 e^z - 2\int ze^z$$

$$= 2\left[ze^z - e^z\right] + z^2 e^z - 2\left(ze^z - e^z\right)$$

$$\boxed{f(z) = z^2 e^z + c}$$

5) State that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0$ where $w = f(z)$ is an analytic function.

2 A) Proof: If $f(z) = u+iv \Rightarrow |f(z)| = \sqrt{u^2+v^2}$

$$\Rightarrow \log|f(z)| = \log\left[\sqrt{u^2+v^2}\right]$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \log\sqrt{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2} \log\left[\sqrt{u^2+v^2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} \log\sqrt{u^2+v^2}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{uu_x+vv_x}{u^2+v^2}\right] = \left\{\frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2) - (uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}\right\}$$

Illy

$$\frac{\partial^2}{\partial y^2} \log\left[\sqrt{u^2+v^2}\right] = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} \log\left(\sqrt{u^2+v^2}\right)\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{uu_y+vv_y}{u^2+v^2}\right] = \left\{\frac{(u^2+v^2)(uu_{yy}+u_y^2+vv_{yy}+v_y^2) - (uu_y+vv_y)(2uu_y+2vv_y)}{(u^2+v^2)^2}\right\}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| =$$

$$\frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2) - (uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$+\frac{(u^2+v^2)(uu_{yy}+u_y^2+vv_{yy}+v_y^2) - (uu_y+vv_y)(2uu_y+2vv_y)}{(u^2+v^2)^2}$$

$= \{2(u^2+$

Since By CR

and $u_{xx}+$

7) Show that t

Continous at

yet derivative

A)

(1) $L_1 = \lim_{y\to}$

$= \lim_{y\to 0}\{$

$= \lim_{y\to 0}\{$

$= \lim_{y\to 0}\{$

$\Rightarrow \boxed{L_1 = 0}$

(3) $L_3 = \lim_{x\to 0}\{$

$= \lim_{x\to 0}\{$

$= \lim_{x\to 0}\{ 2x$

$$= \left(\frac{2(u^2+v^2)(u_x^2+u_y^2) - 2(u^2+v^2)(u_x^2+u_y^2)}{(u^2+v^2)^2} \right) = 0$$

Since By CR eq^n and harmonic eq^n $u_x = v_y$ & $u_y = -v_x$

and $u_{xx} + v_{yy} = 0$, $v_{xx} + u_{yy} = 0$.

7) Show that the function defined by $\begin{cases} \frac{2xy(x+iy)}{x^2+y^2}, & z \neq 0 \\ 0 & z = 0 \end{cases}$ is

Continous at origin, Cr equations also Satisfied at origin, yet derivative doesnot exist at origin.

A)

① $L_1 = \lim_{y\to 0} \left\{ \lim_{x\to 0} f(x,y) \right\}$

$= \lim_{y\to 0} \left\{ \lim_{x\to 0} \frac{2xy(x+iy)}{x^2+y^2} \right\}$

$= \lim_{y\to 0} \left\{ \frac{2(0)\,y(0+iy)}{(0)^2+y^2} \right\}$

$= \lim_{y\to 0} \left\{ \frac{0}{y^2} \right\} = \lim_{y\to 0} \{0\}$

$\Rightarrow \boxed{L_1 = 0}$

② $\lim_{x\to 0} \left\{ \lim_{y\to 0} f(x,y) \right\}$

$= \lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{2xy(x+iy)}{x^2+y^2} \right\}$

$= \lim_{x\to 0} \left\{ \frac{2x(0)(x+i(0))}{x^2+(0)^2} \right\}$

$= \lim_{x\to 0} \left\{ \frac{0}{x^2} \right\} = \lim_{x\to 0} \{0\}$

$\Rightarrow \boxed{L_2 = 0}$

③ $L_3 = \lim_{x\to 0} \left\{ \lim_{y\to mx} f(x,y) \right\}$

$= \lim_{x\to 0} \left\{ \lim_{y\to mx} \frac{2xy(x+iy)}{x^2+y^2} \right\}$

$= \lim_{x\to 0} \left\{ \frac{2x\,mx(x+imx)}{x^2+(mx)^2} \right\}$

$= \lim \left\{ \frac{2m\cancel{x^3}(1+im)}{\cancel{x^2}(1+m^2)} \right\}$

$= \lim_{x\to 0} \left\{ \frac{2mx(1+im)}{1+m^2} \right\}$

$= \frac{2m(0)(1+im)}{1+m^2}$

$= \frac{0}{1+m^2} = 0$ $\boxed{L_3 = 0}$

2 (4) $L_4 = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2} f(x,y)\right\}$

$$= \lim_{x\to 0}\left\{\lim_{y\to mx^2}\frac{2xy(x+iy)}{x^2+y^2}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{2x\,mx^2\cdot(x+imx^2)}{x^2+(mx^2)^2}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{2m\cancel{x^4}(1+imx)}{\cancel{x^4}(1+m^2x^2)}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{2mx^2(1+imx)}{1+m^2x^2}\right\}$$

$$= \frac{2m(0)(1+im0)}{1+m^2(0)^2}$$

$$= \frac{0}{1} = 0$$

$\boxed{L_4 = 0}$

(5) $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to my^2} f(x,y)\right\}$

$$= \lim_{y\to 0}\left\{\lim_{x\to my^2}\frac{2xy(x+iy)}{x^2+y^2}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{2my^2y(my^2+iy)}{y^2+(my^2)^2}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{2m\cancel{y^4}(my+i)}{\cancel{y^2}(1+m^2y^2)}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{2my^2(my+i)}{1+m^2y^2}\right\}$$

$$= \frac{2m(0)(0+i)}{1+m^2(0)^2}$$

$$= \frac{0}{1} = 0$$

$\boxed{L_5 = 0}$

$L_1 = L_2 = L_3 = L_4 = L_5 = L = 0$

$\therefore f(x,y)$ is continous at $(0,0)$

8) Show that the function $(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at origin

A) $f(z) = u+iv = \sqrt{|xy|}$

$u = \sqrt{|xy|}$, $v = 0$

$u_x = \frac{\partial u}{\partial x} = \ldots$

$u_y = \frac{\partial u}{\partial y} = \ldots$

$v_x = \frac{\partial v}{\partial x} = \ldots$

$v_y = \frac{\partial v}{\partial y} = \ldots$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\ldots}$

$f'(0) = \lim\limits_{z\to 0} \ldots$

$= \ldots$

i) $x\to 0, y\ldots$

ii) $y\to 0, x\ldots$

iii) $y\to mx, \ldots$

$\therefore$ It is ...

9) utilize the ... $w = \phi + i\varphi$ w... function ϕ

A) $w = \phi + i\ldots$

$\varphi = x^2 - y^2 + \ldots$

$u_x = \frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$u_y = \frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$v_x = \frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$v_y = \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ => CR eqns are satisfied

$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{z-0} = \lim_{z\to 0} \frac{\sqrt{|xy|}}{z}$

$= \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$

i) $x\to 0, y\to 0 = 0$

ii) $y\to 0, x\to 0 = 0$

iii) $y\to mx, x\to 0 = \frac{\sqrt{m}}{1+im} \neq 0$

$\therefore$ It is not analytic

---

9) utilize the complex potential for an electric field $w=\phi+i\varphi$ where $\varphi = x^2y^2 + \frac{x}{x^2+y^2}$ and determine the function ϕ

A) $w = \phi + i\varphi$

$\varphi = x^2-y^2+\frac{x}{x^2+y^2} \Rightarrow \frac{\partial\varphi}{\partial x} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\frac{\partial \varphi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

let, $f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$$= -2y - \frac{2xy}{(x^2+y^2)^2} + i\left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

$$\Rightarrow f'(z) = 0 + i\left(2z - \frac{1}{z^2}\right)$$

$$\Rightarrow \boxed{f(z) = i\left(z^2 - \frac{1}{z}\right)}$$

10) In a two dimensional flow of a fluid, the velocity potential $\phi = x^2 - y^2$. Find the stream function φ.

A) $w = \phi + i\varphi$

$$\phi = x^2 - y^2 \Rightarrow \frac{\partial \phi}{\partial x} = 2x$$

$$\frac{\partial \phi}{\partial y} = -2y$$

let, $f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$$= 2x - i(-2y)$$

$$\Rightarrow f'(z) = 2z - i(-2(0))$$

$$\Rightarrow f'(z) = 2z \Rightarrow f(z) = z^2 + C$$

$$\boxed{\varphi = 2xy + C}$$

Part-B :

1) Find Constants a, b such that the following function is analytic, where $f(z) = 3x - y + 5 + i(ax + by - 3)$

A) Given $f(z) = (3x - y + 5) + i(ax + by - 5)$

$$u = 3x - y + 5 \qquad v = ax + by - 5$$

$$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial v}{\partial x} = a$$

$$\frac{\partial u}{\partial y} = -1 \qquad \frac{\partial v}{\partial y} = b$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \boxed{b = 3}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow +1 = +a \Rightarrow \boxed{a = 1}$$

2) Show that the real part of an analytic function $f(z)$ where $u = \log|z|^2$ is a harmonic function. If so find the analytic function by Milne thomson method.

A) Given $u = \log|z|^2$ let $f(z) = u + iv$

To Show the function is harmonic function

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2) - (2x)(2x)}{x^2+y^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2) - (2y)(2y)}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$\therefore u_{xx} + u_{yy} = 0$

$f'(z) = u_x + i v_x$

$f'(z) = u_x - i u_y \rightarrow$ By CR eqn

$f'(z) = \frac{2x}{x^2+y^2} - i \frac{2y}{x^2+y^2}$

Put $x = z \quad y = 0$

$f'(z) = \frac{2z}{z^2} - i(0) = 2\frac{1}{z}$

$f(z) = 2\int \frac{1}{z} dz = \boxed{2 \log z + C}$ ~~there~~

The required eqn $f(z) = 2\log xe^{i\theta} = 2(\log x + i\theta)$

$f(x) = 2 \log \sqrt{x^2+y^2} + i \tan^{-1}\left(\frac{y}{x}\right)$

3) Use Milne-Thompson's method to find the imaginary Part of an analytic function $f(z)$ whose real part of an analytic function is $e^{2x}(x\cos 2y - y\sin 2y)$

A) $u = xe^{2x}\cos 2y - y\sin 2y\, e^{2x}$

$\Rightarrow \frac{\partial u}{\partial x} = \cos 2y(x2e^{2x} + e^{2x}) - y \sin 2y \, 2e^{2x}$

$\frac{\partial u}{\partial y} = xe^{2x}(-2\sin 2y) - e^{-2x}(y\, 2\cos 2y + \sin 2y)$

let $f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ CR equations

$\Rightarrow f'(z) = \{\cos 2y\,(2xe^{2x} + e^{2x}) - 2y\sin 2y\, e^{2x}\} -$

$i\,\{-2xe^{2x}\sin 2y - e^{2x}(2y\cos 2y + \sin 2y)\}$

(Taking $x = z$ & $y = 0$)

$\Rightarrow f'(z) = 2ze^{2z} + 2e^{2z}$

$\Rightarrow \int f'(z)\,dz = 2\int (ze^{2z} + e^{2z})\,dz$

$\Rightarrow f(z) = 2ze^{2z} - 2(1)e^{2z} + e^{2z} + c$

$\boxed{\Rightarrow f(z) = ze^{2z} + c}$

4) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$

where $w = f(z)$ is an analytic function.

A) If $f(z) = u + iv \Rightarrow |Re.f(z)| = \sqrt{u^2} \Rightarrow |Re\, f(z)|^2 = u^2$

let $\phi = |Re.f(z)|^2$ $\quad \frac{\partial \phi}{\partial x} = 2uu_x$

$\frac{\partial^2 \phi}{\partial x} = \{2\, uu_{xx} + u_x.u_x\} = 2[uu_{xx} + (u_x)^2]$

Illy $\frac{\partial^2 \phi}{\partial y^2} = 2[uu_{yy} + (u_y)^2]$

$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2[u(u_{xx} + u_{yy}) + (u_x)^2 + (u_y)^2]$ - (1)

Since $f(z)$ is Analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow (u_x)^2 = (v_y)^2 \text{ and } (u_y)^2 = (v_x)^2$$

and $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$

$$\text{from (1)} \Rightarrow 2\left[u(0) + (\cancel{v_x})^2 + (u_x)^2\right]$$

$$= 2\left[(u_x)^2 + (v_x)^2\right]$$

$$= 2|f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re.f(z)|^2 = 2|f'(z)|^2$$

5) Find an analytic function $f(z)$ whose real part of an analytic function is $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ by Milne-Thompson method.

A)

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(-2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x).0 - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

let, $f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$$f'(z) = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} - i\frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow f'(z) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} + i.0$$

$$\Rightarrow f'(z) = \frac{-2}{1-\cos 2z} \Rightarrow \int f'(z)\,dz = -2\int \frac{1}{2\sin^2 z}\,dz$$

$$\Rightarrow f(z) = -\int \operatorname{cosec}^2 z\,dz$$
$$= \cot z + c$$

$\therefore \boxed{f(z) = \cot z + c}$

b) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ If $f(z)$ is a regular function of z.

A) If $f(z) = u+iv \Rightarrow |f(z)| = \sqrt{u^2+v^2} \Rightarrow |f(z)|^2 = u^2+v^2$

let $\phi = |f(z)|^2 \quad \frac{\partial \phi}{\partial x} = 2uu_x + 2vv_x$

$$\frac{\partial^2 \phi}{\partial x^2} = 2[u u_{xx} + u_x \cdot u_x + v v_{xx} + v_x \cdot v_x]$$

$$= 2[u u_{xx} + (u_x)^2 + v v_{xx} + (v_x)^2]$$

Illy

$$\frac{\partial^2 \phi}{\partial y^2} = 2[u_{yy} + (u_y)^2 + v v_{yy} + (v_y)^2]$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2[(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + (u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2]$$

$$= 2[u(0) + v(0) + (u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2]$$

$$= 2 \cdot 2[(u_x)^2 + (v_x)^2]$$

$$= 4|f'(z)|^2$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

7) Show that the function defined by $\begin{cases} \frac{x^2 y}{x^4+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not continous at origin.

A) (1) $L_1 = \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x^2 y}{x^4+y^2} \right\}$

$$= \lim_{y \to 0} \left\{ \frac{0}{y^2} \right\}$$

$$= 0$$

$$\boxed{L_1 = 0}$$

② $L_2 = \lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\}$

$= \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x^2y}{x^4+y^2}\right\}$

$= \lim_{x\to 0}\left\{\frac{0}{x^4}\right\}$

$= 0$

$\boxed{L_2 = 0}$

③ $L_3 = \lim_{x\to 0}\left\{\lim_{y\to mx} f(x,y)\right\}$

$= \lim_{x\to 0}\left\{\lim_{y\to mx} \frac{x^2y}{x^4+y^2}\right\}$

$= \lim_{x\to 0}\left\{\frac{x^2\cdot mx}{x^4+m^2x^2}\right\}$

$= \lim_{x\to 0} \frac{m\cancel{x^3}}{\cancel{x^2}(1+m^2)}$

$= \lim_{x\to 0} \frac{mx}{1+m^2}$

$= 0$

$\boxed{L_3 = 0}$

④ $L_4 = \lim_{x\to 0}\left\{\lim_{y\to mx^2} f(x,y)\right\}$

$= \lim_{x\to 0}\left\{\lim_{y\to mx^2} \frac{x^2y}{x^4+y^2}\right\}$

$= \lim_{x\to 0}\left\{\frac{x^2\cdot mx^2}{x^4+m^2x^4}\right\}$

$= \lim_{x\to 0} \frac{m\cancel{x^4}}{\cancel{x^4}(1+m^2)}$

$= \lim_{x\to 0} \frac{m}{1+m^2}$

$\neq 0$

$\boxed{L_4 \neq 0}$

$\therefore$ not Continous.

8) ~~Show that the function defined by~~ $\begin{cases} \frac{x^2y}{x^4+y^2}, & z\neq 0 \\ 0, & z=0 \end{cases}$ is

not Continous at origin.

8) Show that real part $u = x^3 - 3xy^2$ of an analytic function $f(z)$ is harmonic. Hence find the Conjugate harmonic function and the analytic function.

A) $u = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2$ | $u_y = -6xy$

$u_{xx} = 6x$ | $u_{yy} = -6x$

$u_{xx} + u_{yy} = 0$

$6x - 6x = 0$

u is a harmonic function

To find Conjugate of v'

$dv = \left(\frac{\partial v}{\partial x}\right) \cdot dx + \left(\frac{\partial v}{\partial y}\right) \cdot dy$

IOBS

$v = \int 6xy\,dx + \int (3x^2 - 3y^2)\,dy$

$v = 3x^2y - y^3$

$f(z) = u + iv$

$f(z) = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$= z^3 + c$

$f(z) = z^3 + c$

9) Find weather the following function is analytic or not, where $f(z) = \sin x \cosh y + i \cos x \sinh y$.

A) $u = \sin x \cosh y$ $\quad v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\quad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin u \sinh y$ $\quad \frac{\partial v}{\partial y} = \cos x \cosh y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial o}$$

$f(z)$ is analytic

10) Show that an analytic function $f(z)$ with Constant imaginary part is always Constant.

A) let $f(z) = u+iv$ which is analytic

Given $v = c_1$

$\frac{\partial u}{\partial x} = 0$ $\quad \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad \Rightarrow \frac{\partial v}{\partial x} = 0$

i.e., $\frac{\partial v}{\partial x} = 0$

By CR eqn $u_x = v_y$

$\& \ u_y = -v_x$

$$\boxed{f(z) = u+iv = c_2 + ic_1 = C}$$

11) Find the analytic function $f(z)$ whose imaginary Part of the analytic function is

$$v = e^x(x\sin y + y\cos y)$$

A) $v = xe^x \sin y + y\cos y e^x$

$v_x = (xe^x + e^x)\sin y + y\cos y$

$v_y = xe^x \cos y + e^x(y(-\sin y) + \cos y)$

$= xe^x\cos y - ye^x \sin y + e^x\cos y$

let $f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}$

$\Rightarrow f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = [xe^x\cos y - ye^x\sin y + e^x\cos y] +$

$i[xe^x\sin y + e^x\sin y + \cos y]$

$\Rightarrow f'(z) = [ze^z\cos 0 - 0\cdot e^z\sin 0 + e^z\cos 0] +$

$i[ze^z\sin 0 + e^z\sin 0 + 0\cos 0]$

$\because [x = z, y = 0]$

$\Rightarrow f'(z) = [ze^z + e^z]$

$\Rightarrow f'(z) = [ze^z + e^z]$

$\int f'(z) = \int ze^z + e^z$

$f(z) = [ze^z \cancel{-e^z} + \cancel{e^z}] = ze^z + ci$

$$\boxed{f(z) = ze^z + ci}$$

12) Show that the real part of an analytic function $f(z)$ where $u = 2\log(x^2+y^2)$ is harmonic.

A) Given $u = 2\log(x^2+y^2)$

$$u_x = 2\frac{1}{x^2+y^2}(2x) = \frac{4x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)4 - 4x(2x)}{(x^2+y^2)^2} = \frac{4x^2+4y^2-8x^2}{(x^2+y^2)^2} = \frac{4y^2-4x^2}{(x^2+y^2)^2}$$

$$u_y = 2\frac{1}{(x^2+y^2)}2y = \frac{4y}{(x^2+y^2)}$$

$$u_{yy} = \frac{(x^2+y^2)(4) - 4y(2y)}{(x^2+y^2)^2} = \frac{4x^2+4y^2-8y^2}{(x^2+y^2)^2} = \frac{4x^2-4y^2}{(x^2+y^2)}$$

$$u_{xx}+u_{yy} = \frac{4y^2-4x^2}{(x^2+y^2)^2} + \frac{4x^2-4y^2}{(x^2+y^2)^2}$$

$$= \frac{4y^2-4x^2+4x^2-4y^2}{(x^2+y^2)^2} = 0$$

$$= u_{xx}+u_{yy} = 0$$

$\therefore$ U is harmonic.

13) Show that the function $f(z) = \exp(z)$ is continous at all points of z. find $f'(z)$

A) Given $f(z) = \exp(z) = e^z = e^{x+iy} = e^x e^{iy}$

$$f(z) = e^x(\cos y + i\sin y)$$

f(z) is continous, Satisfy all 5 Conditions.

$$L_1 = \lim_{y\to 0}\left\{\lim_{x\to 0} e^x \cos y + i\sin y\right\} = \lim_{y\to 0} e^0(\cos 0 + i\sin 0) = e^0 = 1$$

$$L_2 = \lim_{x\to 0}\left\{\lim_{y\to 0} e^x(\cos 0 + i\sin 0)\right\} = \lim_{x\to 0} e^0(1) = 1$$

$$L_3 = \lim_{x\to 0}\left\{\lim_{y\to mx} e^x(\cos mx + i\sin mx)\right\} = \lim_{x\to 0} e^0 = 1$$

$$L_4 = \lim_{x\to 0}\left\{\lim_{y\to mx^2} e^x(\cos mx^2 + i\sin mx^2)\right\} = \lim_{x\to 0} e^0 = 1$$

$$L_5 = \lim_{y\to 0}\left\{\lim_{x\to my^2} e^{my^2}(\cos y + i\sin y)\right\} = \lim_{y\to 0} e^0(\cos 0) = 1$$

$$\therefore L_1 = L_2 = L_3 = L_4 = L_5 = L = 0$$

All Conditions are staisfied Hence it is Continous

$$f(z) = u + iv, \quad f'(z) = u_x + iv_x$$

$$u = e^x\cos y, \quad v = e^x\sin y$$

$$\therefore f'(z) = e^x\cos y + ie^x\sin y$$

14) list all the values of K such that $f(z) = e^x(\cos ky + i\sin ky)$ is an analytic function.

A) Given: $f(z) = e^x(\cos ky + i\sin ky)$

$u = e^x\cos ky \qquad v = e^x\sin ky$

$\frac{\partial u}{\partial x} = e^x\cos ky \qquad \frac{\partial v}{\partial x} = e^x\sin ky$

$\frac{\partial u}{\partial y} = e^x k(-\sin ky) \qquad \frac{\partial v}{\partial y} = e^x k\cos ky$

CR eq^n are satisfied it is said to be analytic

$$u_x = v_y \; \& \; u_y = -v_x$$

$$e^x \cos ky = e^x k \cos ky \;\Big|\; e^x - k \sin ky = -e^x \sin ky$$

$$\boxed{k=1} \qquad\qquad \boxed{k=1}$$

$$\boxed{\therefore\ k=1}$$

15) Show that an analytic function $f(z)$ with Constant realpart is always Constant.

A) let $f(z) = u+iv$ is analytic

$$u = c$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow -\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$$

Since, $\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$

$$V = \text{Constant} = C_2$$

$$f(z) = u+iv = c_1 + ic_2 = c$$

Illy $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ i.e., $v_{xx} + v_{yy} = 0$

i.e., the imaginary part v satisfies Harmonic (or) laplace eq^n

16) Show that an analytic function $f(z)$ with Constant modulus is always Constant.

A) Modulus is Constant

$\Rightarrow f(z) = u + iv \Rightarrow |f(z)| = \sqrt{u^2+v^2} = k$

$u^2 + v^2 = k$ - (1)

differentiate (1) partially wrt x; $2uu_x + 2vv_x = 0$

$\Rightarrow uu_x + vv_x = 0$ - (2)

differentiate (1) partially wrt to y; $2uu_y + 2vv_y = 0$

$\Rightarrow uu_y + vv_y = 0$ - (3)

(1) satisfies CR eqn. $u_x = v_y$ & $u_y = -v_x$

(2) $\Rightarrow uu_x + v(-u_y) = 0 \Rightarrow uu_x - vu_y = 0$

(3) $\Rightarrow uu_y + v(u_x) = 0 \Rightarrow uu_y + vu_x = 0$

Square and add to above eqns

$$(uu_x - vu_y)^2 + (uu_y + vu_x)^2 = 0$$

$\Rightarrow (u^2+v^2)(u_x^2+u_y^2) = 0$

But $u^2+v^2 = c^2$, $u_x^2 + u_y^2 = 0$ - (4)

Since $f(z) = u+iv \Rightarrow f'(z) = u_x + iv_x$

$= u_x - iu_y$ [by CReqns]

$|f'(z)| = \sqrt{u_x^2+u_y^2}$

$\Rightarrow |f'(z)|^2 = u_x^2 + u_y^2 = 0$

$\therefore f'(z) = 0 \Rightarrow f(z)$ is Constant.

17) Show that u and v are harmonic functions if u and v is conjugate harmonic functions.

A) Since $w = u + iv$ is analytic function in a region R of the z-plane u and v satisfy the CR's eqn namely

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

If u and v are assumed to have continous second order partical derivatives in the region R then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Differentiating b.s of (2) partially w.r.t y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad -(3)$$

Differentiating b.s of (3) partialy w.r.t y

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad -(4)$$

adding (3) & (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0 \text{ i.e, } u_{xx} + u_{yy} = 0$$

i.e., the real part u satisfies laplace eqn

18) If $w = \phi + i\psi$ represents the Complex potential for an electric field and $\psi = 3x^2y - y^3$. find ϕ

A) Given,

$$\psi = 3x^2y - y^3 \quad v = 3x^2y - y^3$$

$$\frac{\partial v}{\partial x} = 6x \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\text{let } f(z) = u + iv = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = (3x^2 - 3y^2) + i\,6xy \quad [\text{put } x = z, y = 0]$$

$$f'(z) = 3z^2 - 3(0)^2 + i\,6(z)(0) = 3z^2$$

$$\int f'(z) = \int 3z^2 \Rightarrow f(z) = \frac{\cancel{3}z^3}{\cancel{3}} = z^3$$

$$z^3 = (x + iy)^3 \Rightarrow \phi = (x + iy)^3$$

19) Find the analytic function $f(z)$ whose imaginary part of the analytic function is $v = e^x(x\cos y - y\sin y)$ by Milne thompson method

A) Given

$$v = e^x(x\cos y - y\sin y) = xe^x\cos y - e^x y\sin y$$

$$\frac{\partial v}{\partial x} = xe^x + e^x\cos y - e^x y\sin y$$

$$= xe^x\cos y + e^x\cos y - ye^x\sin y$$

$$\frac{\partial v}{\partial y} = xe^x(-\sin y) - e^x(y\cos y + \sin y)$$

$$= -xe^x\sin y - ye^x\cos y - e^x\sin y$$

let $f(z) = u+iv = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$f'(z) = \left(-xe^x\sin y - ye^x\cos y - e^x\sin y\right) +$

$i\left(xe^x\cos y + e^x\cos y - ye^x\sin y\right)$

(Put $x=z, y=0$)

$\Rightarrow f'(z) = \left(-ze^z\sin 0 - e^z\cos 0 - e^z\sin 0\right) +$

$i\left(ze^z\cos 0 + e^z\cos 0 - 0e^z\sin 0\right]$

$\Rightarrow f'(z) = i\left(ze^z + e^z\right)$

$\Rightarrow \int f'(z) = i\int\left(ze^z + e^z\right) = i\left(ze^z - \cancel{e^z} + \cancel{e^z}\right)$

$f(z) = ize^z$

$\boxed{f(z) = ize^z}$

20) If $u-v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of z by Milne thompson method.

A) Here

$u+v =$ imaginary

$u-v =$ real part

$F(z) = u+iv \Rightarrow U = (x-y)(x^2+4xy+y^2)$

$\Rightarrow U = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$

$U = x^3 + 3x^2y - 3xy^2 - y^3$

$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$ | $\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$

let $f(z) = u+iv = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$\Rightarrow f'(z) = ((3x^2+6xy)-3y^2) - i(3x^2-6xy-3y^2)$

Put $x=z,\ y=0$

$\Rightarrow f'(z) = 3z^2 - i3z^2 = 3(z^2-iz^2)$

$\int f'(z) = 3\int z^2 - i\int z^2$

$= 3\left[\frac{z^3}{3} - i\frac{z^3}{3}\right]$

$= z^3 - iz^3 + c$

$f(z) = z^3 - iz^3 + c'$

$f(z) + if(z) = z^3 - iz^3 + c'$

$(1+i)(f(z)) = z^3(1-i) + c'$

$f(z) = z^3\frac{1-i}{1+1} + c'$

$= z^3 \cdot \frac{1-i}{1+i} \times \frac{1-i}{1-i} + c$

$= z^3\frac{(1-i)^2}{1-i^2} = z^3\left(\frac{1-2i+i^2}{1+1}\right)$

$= z^3\left(\frac{-\cancel{2}i}{\cancel{2}}\right)$

$= -iz^3 + c$

$\boxed{\therefore f(z) = -iz^3 + c}$