Unit - 5

* Recurrence Relations:-

Def → A Recurrence Relation for the Sequence $a_0, a_1, a_2, \ldots\ldots, a_n$ is an equation that relates a_n to its predecessors such as $a_0, a_1, \ldots\ldots a_{n-1}$.

→ So, Recurrence Relation gives a relation regarding how a_n relates to Predecessors.
↳ previous elements/items.

How get an item from its previous item is called RR.

Ex:- Fibonacci Sequence:- → the first two no's are 0 & 1, next no is by adding previous 2 no's.

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2,\ F_0 = 0,\ F_1 = 1$$

$F_2 = F_1 + F_0$	$F_3 = F_2 + F_1$	$F_4 = F_3 + F_2$	$F_5 = F_4 + F_3$
$F_2 = 1 + 0$	$F_3 = 1 + 1$	$= 2 + 1$	$= 3 + 2$
$F_2 = 1$	$F_3 = 2$	$F_4 = 3$	$= 5$

∴ Sequence is 0, 1, 1, 2, 3, 5

(or)

Def:- A Recurrence Relation is an equation that recursively defines a Sequence based on a rule that gives the next term in the Sequence as a function of the previous term(s) when one/more initial terms are given.

Ex:- ① Find out the Sequence generated by Recurrence Relations below.

ⓐ $T_n = 2 \cdot T_{n-1}$ with $T_1 = 4$

Sol:- Given, Recurrence Relation is

$T_n = 2 \cdot T_{n-1}$

Initial condition is $T_1 = 4$

$T_2 = 2 \cdot T_1 = 2 \cdot 4 = 8 \ (n=2)$

$T_3 = 2 \cdot T_2 = 2 \cdot 8 = 16 \ (n=3)$

$T_4 = 2 \cdot T_3 = 2 \cdot 16 = 32 \ (n=4)$

$T_5 = 2 \cdot T_4 = 2 \cdot 32 = 64 \ (n=5)$

n:	1	2	3	4	5
T_n:	4	8	16	32	64

∴ Recurrence sequence generated by RR is 4, 8, 16, 32, 64 ......

ⓑ $T_n = 3 \cdot T_{n-1} - 4$ with $T_1 = 3$.

Given, Recurrence Relation is

$T_n = 3 \cdot T_{n-1} - 4$

initial condition is $T_1 = 3$

$T_2 = 3 \cdot T_1 - 4 = 3 \cdot 3 - 4 = 9 - 4 = 5 \ (n=2)$

$T_3 = 3 \cdot T_2 - 4 = 3 \cdot 5 - 4 = 15 - 4 = 11 \ (n=3)$

$T_4 = 3 \cdot T_3 - 4 = 3 \cdot 11 - 4 = 33 - 4 = 29 \ (n=4)$

$T_5 = 3 \cdot T_4 - 4 = 3 \cdot 29 - 4 = 83 \ (n=5)$

n:	1	2	3	4	5
T_n:	3	5	11	29	83

∴ Recurrence Sequence generated by RR is
3, 5, 11, 29, 83 ......

Q) Find out the Recurrence Relation for the following Sequence.

ⓐ 2, 6, 18, 54, 162, ... ⓑ 20, 17, 14, 11, 8, ... ③ 1, 3, 6, 10, 15, 21, ...

ⓐ

Given Recursive Sequence

2, 6, 18, 54, 162, ...

T_1, T_2, T_3, T_4, T_5 is the Recursive Sequence.

$T_1 = 2, T_2 = 6, T_3 = 18, T_4 = 54, T_5 = 162$

∴ initial Condition is $T_1 = 2$

$T_2 = 6 = 3 \cdot T_1 = 3 \cdot 2 = 6$

$T_3 = 18 = 3 \cdot T_2 = 3 \cdot 6 = 18$

$T_4 = 54 = 3 \cdot T_3 = 3 \cdot 18 = 54$

$T_5 = 162 = 3 \cdot T_4 = 3 \cdot 54 = 162$

⋮

$\boxed{T_n = 3 \cdot T_{n-1}}$ with $T_1 = 2$

ⓑ Given Recursive Sequence

20, 17, 14, 11, 8, ...

T_1, T_2, T_3, T_4, T_5 is the Recursive Sequence,

$T_1 = 20, T_2 = 17, T_3 = 14, T_4 = 11, T_5 = 8$

∴ initial Condition, $T_1 = 20$

$T_2 = 17 = T_1 - 3 = 20 - 3 = 17$

$T_3 = 14 = T_2 - 3 = 17 - 3 = 14$

$T_4 = 11 = T_3 - 3 = 14 - 3 = 11$

$T_5 = 8 = T_4 - 3 = 11 - 3 = 8$

⋮

$\boxed{T_n = T_{n-1} - 3}$ with $T_1 = 20$

③ Given Recursive Sequence

1, 3, 6, 10, 15, 21, ...

$T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15,$
$T_6 = 21$

∴ initial Condition, $T_1 = 1$

$T_2 = T_1 + 2 = 1 + 2 = 3$

$T_3 = T_2 + 3 = 3 + 3 = 6$

$T_4 = T_3 + 4 = 6 + 4 = 10$

$T_5 = T_4 + 5 = 10 + 5 = 15$

$T_6 = T_5 + 6 = 15 + 6 = 21$

⋮

$\boxed{T_n = T_{n-1} + n}$ with $T_1 = 1$

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* First Order Linear / Homogeneous Recurrence Relation :-

→ Linear Recurrence Relation of first order with Constant Coefficient is

$$\boxed{a_n = c \cdot a_{n-1} + f(n)} \quad , \text{ for } n \geq 1 \rightarrow ①$$

where c is Constant, $f(n)$ is a Known function.

→ Such a relation is Called "a linear recurrence Relation of First order with Constant Coefficient."

→ In eq ①, if $f(n) = 0$, then the relation is called "Homogeneous Recurrence Relation". otherwise, it is called as "Non-homogeneous"/"n-homogeneous Recurrence Relation".

→ eq ① Can be Solved by Substituting n by $n+1$ in eq ①

$$\boxed{a_{n+1} = c \cdot a_n + f(n+1)} \rightarrow ②$$

→ Substitute $n = 0, 1, 2, 3, ...$ in eq ②

put $n = 0 \Rightarrow a_1 = c \cdot a_0 + f(1)$

put $n = 1 \Rightarrow a_2 = c \cdot a_1 + f(2) \Rightarrow a_2 = c \cdot [c \cdot a_0 + f(1)] + f(2) \Rightarrow a_2 = c^2 a_0 + c \cdot f(1) + f(2)$.

put $n = 2 \Rightarrow a_3 = c \cdot a_2 + f(3) \Rightarrow c[c^2 a_0 + c f(1) + f(2)] + f(3) \Rightarrow c^3 a_0 + c^2 f(1) + c f(2) + f(3)$.

⋮

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \dots\dots + c \cdot f(n-1) + f(n)$$

$$a_n = c^n \cdot a_0 + \underbrace{\sum_{k=1}^{n-1} c^{n-k} \cdot f(k)}_{f(n)^{th} \text{ term.}} \quad \text{for } n \geq 1 \longrightarrow (3)$$

this is the general solution of the recurrence relation (2) which is equivalent to eq(1)

In eq (3), if $f(n) = 0$, the recurrence relation is homogeneous,

eq (3) becomes.

$$\boxed{a_n = c^n a_0} \quad \text{for } n \geq 1 \longrightarrow (4)$$

this is the solution for first order linear (or) homogeneous recurrence relation.

Ex (1) Solve the recurrence Relation $a_{n+1} = 4 \cdot a_n$ for $n \geq 0$, given that $a_0 = 3$

Sol:- Give Recurrence Relation is

$$a_{n+1} = 4 \cdot a_n \longrightarrow (1)$$

here $f(n) = 0$ ∴ it is

∴ The given Recurrence Relation is first order linear / homogeneous.

the general solution for first order linear / homogeneous recurrence relation is

$$a_n = c^n \cdot \underbrace{a_0}_{\text{initial condition.}} \longrightarrow (2)$$

if $a_n = c^n$ then $a_{n+1} = c^{n+1}$ (∵ n is replaced with n+1).

$$\Rightarrow 4 \cdot a_n = c^n \cdot c \quad [\because a^{x+y} = a^x \cdot a^y]$$

(given problem)

$$\Rightarrow 4 \cdot \cancel{c^n} = \cancel{c^n} \cdot c \quad [\because a_n = c^n]$$

$$\Rightarrow \underline{c = 4}$$

'c' value is substituted in eq (2)

$$a_n = c^n \cdot a_0$$

$$a_n = 4^n \cdot a_0 \quad [\because a_0 = 3]$$

$$\boxed{a_n = 4^n \cdot 3}$$

Solution of the recurrence Relation.

(2) Solve the recurrence relation $a_n = 7 \cdot a_{n-1}$ for $n \geq 1$, $a_2 = 98$.

Sol:- Given R.R is $a_n = 7 \cdot a_{n-1} \longrightarrow (1)$

∴ the given R.R is first order linear R.R because $f(n) = 0$.

the general solution for first order linear. R.R is

$$a_n = c^n \cdot a_0 \longrightarrow (2)$$

~~If $a_n = c^n$ then a_n~~

In eq ①, Substituting n+1 in place of n.

$$a_{n+1} = 7\cdot a_{\cancel{n+1-1}} \Rightarrow a_{n+1} = 7\cdot a_n \to ③$$

$n \geq 1$
$n+1 \geq 1$
$n+1-1 \geq 0$, $n \geq 0$

if $a_n = c^n$ then $a_{n+1} = c^{n+1}$

$$\Rightarrow 7\cdot a_n = c^n\cdot c^1$$
$$\Rightarrow 7\cdot \cancel{c^n} = \cancel{c^n}\cdot c$$
$$\Rightarrow \underline{c = 7}.$$

Substituting 'c' value in eq ②

$$a_n = c^n\cdot a_0$$
$$a_n = 7^n\cdot a_0 \to ④$$

$\therefore a_2 = 98$

Substituting $n=2$ in eq ④

$$\Rightarrow a_2 = 7^2\cdot a_0$$
$$\Rightarrow 98 = 7^2\cdot a_0$$
$$\Rightarrow 98/49 = a_0$$
$$\Rightarrow a_0 = 2$$

Substitute a_0 in eq ④

$$\boxed{a_n = 7^n\cdot 2}$$

$\therefore a_2 = 98$
~~Substituting n=2 in eq ④~~

Absent:
23-532, 23-543, 545, 54A, 54U, 551, 555, 55N, 55R, 24-519

* <u>Second Order Homogenous</u> (or) <u>Linear Recursion Relation</u>

Let us Consider the recurrence relation $c^n\cdot a_n + c^{n-1}\cdot a_{n-1} + c^{n-2}\cdot a_{n-2} = 0$, for $n \geq 2 \to ①$

where c^n, c^{n-1} and c^{n-2} are real Constants. A relation of equation ① type is called "Second-order linear (or) Homogeneous Recurrence relation" with Constant Coefficients.

$\Rightarrow$ Substitute $a_n = c\cdot k^n$ in eq ①, we get $c^n\cdot c\cdot k^n + c^{n-1}\cdot c\cdot k^{n-1} + c^{n-2}\cdot c\cdot k^{n-2} = 0 \to ②$

if c is a Constant, then above eq ② becomes.

$$\Rightarrow c^n\cdot k^n + c^{n-1}\cdot k^{n-1} + c^{n-2}\cdot k^{n-2} = 0 \to ③$$

Substitute $n=2$ in eq ③

$$c^2 k^2 + c^{2-1} k^{2-1} + c^{2-2}\cdot k^{2-2} = 0$$
$$c^2 k^2 + c^1 k + c^0 = 0 \to ④$$

∴ General term is $c^n . k^2 + c^{n-1} . k + c^{n-2} = 0 \longrightarrow (5)$

The eq (5) is called the "auxilliary equation" (or) "the characteristic equation" for the relation (1)

Now the following three cases of characteristic equation are

Case 1:- the two roots k_1 and k_2 of eq (5) are real & distinct, then we take

$$a_n = A.(k_1)^n + B(k_2)^n$$

where A and B are constants. it is the general solution of eq (1)

Case 2:- the two roots k_1 and k_2 of eq (5) are real & equal, with k as a com

then, $\quad a_n = (A+Bn)k^n$

where A & B are constants. it is the general solution of eq (1)

Case 3:- The two roots of k_1 and k_2 of eq (5) are complex, then k_1 & k_2 a

Complex conjugates of each other so that if

$k_1 = p+iq$, then $k_2 = p-iq$ and we take

$$a_n = r^n (A \cos n\theta + B \sin n\theta).$$

where A & B are ~~constants~~ arbitary complex constants

$$r = |k_1| + |k_2| = \sqrt{p^2+q^2}$$

$$\theta = \tan^{-1}(q/p)$$

it is the general solution of eq (1)

Eg:-1. Solve the recurrence relation $a_n + a_{n-1} - 6.a_{n-2} = 0$ for $n \geq 2$. given that $a_0 = -1$ and $a_1 = 8$

Sol:- Given Recurrence Relation: $a_n + a_{n-1} - 6.a_{n-2} = 0 \longrightarrow (1)$

Generally, Second order linear relation is of the form.

$$c^n.a_n + c^{n-1} a_{n-1} + c^{n-2}.a_{n-2} = 0 \longrightarrow (2)$$

Comparing eq (1) & eq (2) then, we get

$$c^n = 1, c^{n-1} = 1 \text{ and } c^{n-2} = 6$$

The characteristic eq of eq (2) is

$$c^n.k^2 + c^{n-1}.k + c^{n-2} = 0 \longrightarrow (3)$$

Substitute c^n, c^{n-1} and c^{n-2} values in eq ③ for getting the characteristic equation of eq ① is

$$1\cdot k^2 + 1\cdot k - 6 = 0$$

$$k^2 + k - 6 = 0 \longrightarrow ④$$

It is the characteristic equation of given problem. Now, we have to findout the roots of characteristic eq ④

$$k^2 + k - 6 = 0$$

$$k^2 + 3k - 2k - 6 = 0$$

$$k(k+3) - 2(k+3) = 0$$

$$(k+3)(k-2) = 0$$

$$k = -3,\ k = 2$$

$\therefore$ roots are real and distinct.

<u>Case 1</u>:-

$$a_n = A\cdot(k_1)^n + B(k_2)^n$$

$$a_n = A\cdot(-3)^n + B\cdot(2)^n \longrightarrow ⑤$$

the initial conditions are $a_0 = -1$ & $a_1 = 8$

Now take $a_0 = -1$ i.e., $n = 0$

Substitute $n=0$ in eq ⑤

$$a_0 = A\cdot(-3)^0 + B\cdot(2)^0$$

$$-1 = A\cdot 1 + B\cdot 1$$

$$-1 = A + B \longrightarrow ⑥$$

Now take $a_1 = 8$ ie, $n = 1$

$$a_1 = A\cdot(-3)^1 + B(2)^1$$

$$8 = -3A + 2B \longrightarrow ⑦$$

Solve eq ⑥ & ⑦

$$A + B = -1 \ast 2$$

$$-3A + 2B = 8$$

$$2A + 2B = -2$$

$$-3A + 2B = 8$$

$$5A = -10$$

$$A = -10/5 = -2$$

A value is substituted in eq ⑥

$$A + B = -1$$

$$-2 + B = -1$$

$$B = -1 + 2$$

$$B = 1$$

A & B values substituted eq ⑤

$$\boxed{\therefore a_n = (-2)(-3)^n + 1\cdot(2)^n} \longrightarrow ⑧$$

Eg ②:- Solve the recurrence relation $a_n - 6\cdot a_{n-1} + 9\cdot a_{n-2} = 0$ for $n \geq 2$
g.T. $a_0 = 5$ & $a_1 = 12$

Sol:- Given R.R is $a_n - 6\cdot a_{n-1} + 9\cdot a_{n-2} = 0$ for $n \geq 2 \longrightarrow$ ①

Generally. Second order linear R.R is

$$c^n\cdot k_n + c^{n-1}\cdot k_{n-1} + c^{n-2}\cdot k_{n-2} = 0 \rightarrow ② \text{ for } n \geq 2$$

then the characteristic equation is

$$c^n\cdot k^2 + c^{n-1}\cdot k + c^{n-2} = 0 \longrightarrow ③$$

Compare eq ① & ②

$$c^n = 1,\ c^{n-1} = -6,\ c^{n-2} = 9$$

~~Now, the characteristic eqn~~
Now, the Constants Coefficient values Substituted in eq ③

$$k^2 - 6\cdot k + 9 = 0$$

$k^2 - 6k + 9 = 0$ is the characteristic eqn

Roots:- $k^2 - 6k + 9 = 0$
$k^2 - 3k - 3k + 9 = 0$
$k(k-3) - 3(k-3) = 0$
$(k-3)(k-3) = 0 \quad \therefore k = 3, 3$

$\therefore$ roots, $k_1 = 3$ & $k_2 = 3$, Hence roots are Real & equal.

Case 2:- $a_n = (A + Bn)k^n$

A & B are Constants $k = k_1 = k_2 = 3$

Substitute k value in eq ④

$$a_n = (A + Bn)3^n \longrightarrow ⑤$$

initial Conditions, $a_0 = 5$, $a_1 = 12$

first Consider $a_0 = 5$, where $n = 0$

Substitute $n = 0$ in eq ⑤
$a_0 = (A + B(0))3^0$
$5 = (A + 0)\cdot 1$
$\boxed{A = 5}$

$a_1 = 12$, $n = 1$ in eq ⑤
$a_1 = (A + B(1))3^1$
$12 = (A + B)3$
$\boxed{3A + 3B = 12} \rightarrow 6$

Substitute 'A' value in eq ⑥
$12 = 3(5) + 3(B)$
$\boxed{B = -1}$

Substitute A & B values in eq ⑤

$a_n = (5 + (-1)n)\cdot 3^n$

$$\boxed{a_n = (5 - n)\cdot 3^n} \rightarrow ⑦$$

Eg:- 3 Solve the R.R. $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$

G.T $a_0 = 1$, $a_1 = 2$

roots, $k = 1 \pm i$, $k_1 = 1+i$, $k_2 = 1-i$

Case 3:- $p = 1$, $q = 1$, $\theta = \pi/4$

$$\boxed{a_n = (\sqrt{2})^n (A \cos^n \pi/4 + B \sin^n \pi/4)} \rightarrow ⑥$$

$a_0 = 1$, $a_1 = 2$

$A = 1$, $B = 1$

$$\therefore a_n = (\sqrt{2})^n (\cos n\pi/4 + B \sin n.\pi/4)$$

* Solving third order Homogenous linear Recurrence Relation :-

A third-order linear homogenous recurrence relation with constant coefficients has the following form

$$\boxed{c^n \cdot a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} \cdot a_{n-3} = 0, \quad \text{for } n \geq 3} \rightarrow (1)$$

where $c^n, c^{n-1}, c^{n-2}, c^{n-3}, \ldots\ldots, c^{n-k}$ are real constants with $c^n \neq 0$.

The characteristic eqn of eq(1) is $\boxed{c^n \cdot k^3 + c^{n-1} \cdot k^2 + c^{n-2} \cdot k^1 + c^{n-3} = 0} \rightarrow (2)$

Now the following three cases of characteristic eqn and their soln are

$$r = |k_2| = |k_3| = \sqrt{p^2+q^2} \quad ; \quad \theta = \tan^{-1}(q/p).$$

Case-1 :- the three roots k_1, k_2 & k_3 of eq(2) are real and distinct

$$\boxed{a_n = A \cdot k_1^n + B \cdot k_2^n + C \cdot k_3^n}$$

where, A, B & C are real constants

Case 2 :- the three roots k_1, k_2 & k_3 of eq(2) are real & equal.

$$\boxed{a_n = (A + Bn + Cn^2) \cdot k^n}$$

where $k = k_1 = k_2 = k_3$

Case 3 :- the three roots k_1, k_2 & k_3 are real & imaginary

$$a_n = A \cdot k_1^n + r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

where A, c_1, c_2 are constants

Example :- Solve the following R.R

$$a_n = 6 \cdot a_{n-1} - 12 \cdot a_{n-2} + 8 \cdot a_{n-3}$$

given, $a_0 = 1, a_1 = 4$ & $a_2 = 28$.

Sol :- Given R.R is

$$a_n = 6 \cdot a_{n-1} - 12 \cdot a_{n-2} + 8 \cdot a_{n-3}$$

$$a_n - 6 \cdot a_{n-1} + 12 \cdot a_{n-2} - 8 \cdot a_{n-3} = 0 \rightarrow (1)$$

Generally equation is

$$c^n \cdot a_n + c^{n-1} \cdot a_{n-1} + c^{n-2} \cdot a_{n-2} + c^{n-3} \cdot a_{n-3} = 0 \rightarrow (2)$$

the characteristic eqn of eq(2) is

$$c^n \cdot k^3 + c^{n-1} \cdot k^2 + c^{n-2} \cdot k^1 + c^{n-3} = 0 \rightarrow (3)$$

Compare eq(1) & eq(2)

$$c^n = 1, c^{n-1} = -6, c^{n-2} = 12, c^{n-3} = -8$$

Now substituting the above values in eq(3)

$$k^3 - 6k^2 + 12k - 8 = 0$$

$$k = 2, k = 2, \quad k - 2 = 0$$
$$k = 2$$

$$k = k_1 = k_2 = k_3 = 2$$

$\therefore$ roots are real & equal.

Case 2 :- $a_n = (A + Bn + Cn^2) \cdot k^n \rightarrow (4)$

Substitute k value in eq(4)

$$\therefore a_n = (A + Bn + Cn^2)(2)^n \rightarrow (5)$$

Initial conditions, $a_0 = 1, a_1 = 4$ & $a_2 = 28$

Take $a_0 = 1$ i.e., $n = 0$, in eq(5)

$$a_0 = (A + B(0) + C(0)^2) \cdot (2)^0$$

$$1 = (A + 0 + 0)(1)$$

$$\boxed{1 = A}$$

Take $a_1 = 4$, $n = 1$ in eq(5)

$$a_1 = (A + B(1) + C(1)^2)(2)^1$$

$$4 = (A + B + C)(2)$$

$$A + B + C = 4/2 \Rightarrow A + B + C = 2 \rightarrow (6)$$

Take $a_2 = 28$, $n = 2$ in eq(5)

$$a_2 = (A + B(2) + C(2)^2)(2)^2$$

$$28 = (A + 2B + 4C) 4$$

$$\frac{28}{4} = A + 2B + 4C$$

$$A + 2B + 4C = 7 \rightarrow (7)$$

Solve (6) & (7) eqn.

$$A + B + C = 2 \times 2$$
$$A + 2B + 4C = 7$$

$$2A + 2B + 2C = 4$$
$$A + 2B + 4C = 7$$

$$A - 2C = -3$$
$$1 - 2C = -3$$
$$-2C = -3 - 1$$
$$\cancel{-}2C = \cancel{-}4 \; 2$$
$$\boxed{C = 2}$$

Substitute A, C values in eq(6)

$$A + B + C = 2$$
$$1 + B + 2 = 2$$
$$B + 3 = 2$$
$$B = 2 - 3$$
$$\boxed{B = -1}$$

$\therefore$ The characteristic eqn is

$$a_n = (1 - 1(n) + 2n^2)(2)^n$$

~~N Abstract~~ ~~Objectives of the Project~~

~~2) Pre requisite & Requirements~~

~~3) Technologies~~

~~4) Milestones of Project~~

~~5)~~

6)

* **Solving Non-homogenous Recurrence Relation:-**

Non-homogenous Recurrence Relation has the following form.

$$\boxed{C_n \cdot a_n + C_{n-1} \cdot a_{n-1} + C_{n-2} \cdot a_{n-2} + \dots + C_{n-k} \cdot a_{n-k} = f(n)}$$

for $n \geq k \rightarrow (1)$

where, $C_n, C_{n-1}, C_{n-2}, \dots, C_{n-k}$ are real constants with $C_n \neq 0$ & $f(n)$ is a given valued function of n

A general solution of eq (1) is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

where, $a_n^{(h)}$ is the general solu^n of the homogenous part of the eq (1) with $f(n) = 0$,

$a_n^{(P)}$ is any particular solu^n of the eq (1) & $a_n^{(P)}$ can be calculated by using the following cases.

Case 1:- Suppose $f(n)$ is a polynomial of degree 'q' and 1 is not a root of characteris eq^n of homog^n part of the eq (1)

In this case, $a_n^{(P)}$ is

$$a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \dots + A_q \cdot n^q$$

where $A_0, A_1, A_2, \dots, A_q$ are constants, that are evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the eq (1)

Case 2:- Suppose $f(n)$ is a polynomial of degree 'q' and '1' is a root of multiplicity m of the characteristic eq^n of the homogenous part of the eq (1), In this case.

$$a_n^{(P)} = n^m \{A_0 + A_1 n + A_2 n^2 + \dots + A_q n^q\}$$

Case 3:- Suppose $f(n) = \alpha b^n$ where α is constant & b is not a root of charst. eq^n of homog^n part of eq (1), then

$$a_n^{(P)} = A_0 b^n$$

Case 4:- Suppose $f(n) = \alpha \cdot b^n$, where α is constant & b is a root of multiplicity M of characteristic eq^n of homog^n part of eq (1) then $a_n^{(P)} = A_0 n^m b^m$

Summary

$f(n)$	$a_n^{(P)}$
Constant	A_0
n	$A_0 + A_1 n$
n^2	$A_0 + A_1 n + A_2 n^2$
r^n	$A_0 \cdot r^n$

Ex:- Solve the following R.R $a_n + 4 \cdot a_{n-1} + 4 \cdot a_{n-2} = 8$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 2$

Sol:- Given, R.R

$$a_n + 4 \cdot a_{n-1} + 4 \cdot a_{n-2} = 8 \rightarrow (1)$$

Non-homog^n R.R General solu^n is

$$a_n = a_n^{(h)} + a_n^{(P)} \rightarrow (2)$$

To obtain $a_n^{(h)}$, put $f(n) = 0$ in eq (1)

$$a_n + 4 \cdot a_{n-1} + 4 \cdot a_{n-2} = 0 \rightarrow (3)$$

$\because$ the characteristic eq^n of eq (3) is

$$k^2 + 4k + 4 = 0$$

$\because a = 1, b = 4, c = 4$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(4)}}{2 \times 1} = -2$$

$\therefore k = k_1 = k_2 = -2$.

Here the roots of characteristic eq are real & equal

$\therefore$ General solution is $a_n^{(h)}$ is

$$a_n^{(h)} = (A + Bn) k^n$$

$$a_n^{(h)} = (A + Bn)(-2)^n$$

$a_n^{(P)}$ value will be calculated.

$$a_n^{(P)} = A_0 \rightarrow (5)$$

eq (4) & (5) are substituted in eq (2)

$$\because a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = (A + Bn)(-2)^n + A_0 \rightarrow (6)$$

1) ~~22951A0573~~
2) ~~22951A05A0~~
3) ~~22951A0590~~
4) ~~22951A0597~~

For obtaining A_0 value, eq ⑤ is substituted in eq ①

$$a_n + 4\cdot a_{n-1} + 4\cdot a_{n-2} = 8$$
$$A_0 + 4\cdot A_0 + 4\cdot A_0 = 8$$
$$9A_0 = 8$$
$$\boxed{A_0 = 8/9}$$

Now A_0 value substituted in eq ⑥

5) $\boxed{\because a_n = (A+Bn)(-2)^n + 8/9} \rightarrow$ ⑦
6) $a_0 = 1$ i.e., $n=0$
7) in eq ⑦
8) $a_0 = (A+B(0))(-2)^0 + 8/9$
9) $1 = A + 8/9$
10) $A = 1/9$
11)
12)

$a_1 = 2$ i.e., $n=1$

in eq ⑦

$$a_1 = (A+B(1))(-2)^1 + 8/9$$
$$2 = ((A+B)-2) + 8/9$$
$$2 = -2A - 2B + 8/9$$
$$-2A - 2B = 2 - 8/9 = 10/9$$
$$-2A - 2B = 10/9$$
$$-2(1/9) - 2B = 10/9$$
$$\boxed{B = -2/3}$$

∴ A & B values in eq ⑦

$$\therefore a_n = \left(1/9 + (-2/3)n\right)(-2)^n + 8/9$$

∴ this is the required sol^n.

Ex 2:- R.R is $a_n = 3\cdot a_{n-1} + 2n$

$a_1 = 3$.

Ex 3:- R.R is $a_{n+1} - 2\cdot a_n = 2^n$, $n \geq 0$,

$a_0 = 1$

Ex 4:- R.R is $a_{n+2} - 4\cdot a_{n+1} + 3\cdot a_n = -200$ for $n \geq 0$

$a_0 = 3000$, $a_1 = 3300$

* Generating functions :-

Consider a sequence of real numbers $a_0, a_1, \ldots, a_n$ let us denote the sequence by a_r, where $r = 0,1,2,3\ldots$

$\because a_r = a_0, a_1, a_2, a_3 \ldots$

Suppose there exists a function $f(x)$ whose expansion in a series of powers of x is shown below.

$$\boxed{f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots\ldots}$$

$f(x)$ can also be written as

$$f(x) = \sum_{r=0}^{n} a_r \cdot x^r$$

Here $f(x)$ is called as a generating function for the sequence of $a_0, a_1, \ldots$

ex 1:- Find out the sequence generated by the generating function $(1+x)^n$

Sol:- Given generating function: $(1+x)^n$

we already know that,

eq of ...

$$(x+y)^n = \sum_{r=0}^{n} (x)^{n-r} \cdot y^r \cdot {}^nC_r$$

$$(1+x)^n = {}^nC_0 \cdot (1)^n \cdot x^0 + {}^nC_1 \cdot (1)^{n-1} \cdot x^1 + {}^nC_2 (1)^{n-2} \cdot x^2 + {}^nC_3 (1)^{n-3} \cdot x^3 + \ldots$$

$$= \frac{n!}{0!(n-0)!}(1)\cdot(1) + \frac{n!}{1!(n-1)!}(1)(x) + \frac{n!}{2!(n-2)!}(1)\cdot x^2 + \ldots\ldots$$

$$= \frac{\cancel{n!}}{1 \cdot \cancel{n!}} 1 + \frac{n\cancel{(n-1)!}}{1 \times \cancel{(n-1)!}} \cdot x + \frac{n(n-1)\cancel{(n-2)!}}{2! \times \cancel{(n-2)!}} x^2 + \frac{n(n-1)(n-2)\cancel{(n-3)!}}{3!\,\cancel{(n-3)!}} \cdot x^3 + \ldots$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots$$

$$\therefore = \sum_{r=0}^{n} {}^nC_r \cdot x^r$$

$$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \cdot x^r$$

The sequence generated by the generating function $(1+x)^n$ is ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \ldots\ldots$

(or)

$C(n,0), C(n,1), C(n,2), C(n,3) \ldots\ldots$

(i) $(1+x)^1 = (1+x)$

(ii) $(1-x)^1 = (1-x)$

(iii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$