MODULE - III

CIE - I

PART - A

(1)

Poisson distribution can be determined as a limiting case of Binomial distribution under the following assumptions:

(1) p, the probability of the occurrence of the event is very small.

(2) n is very very large, where n is no. of trails i.e, $n \to \infty$.

(3) np is a finite quantity, say $np = \lambda$, then λ is called the parameter of the poisson distribution.

Now, we wish to know that, the limiting form of the Binomial distribution under the above 3 conditions

We have by binomial distribution probability law, the probability of 'x' successes in a series of 'n' independent trails is given by,

$$B(x:n,p) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad ; x = 0, 1, 2, ---, n$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^{x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{x!\cancel{(n-x)!}} \cdot \frac{\lambda^x}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$= \frac{\cancel{n^x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!} \cdot \frac{\lambda^x}{\cancel{n^x}} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

Now, we taking limit as $n \to \infty$, we get,

$$\lim_{n\to\infty} B(x:n,p) = \lim_{n\to\infty} \frac{\lambda^x}{x!} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{\left(1-\frac{\lambda}{n}\right)^{x}} \left(1-\frac{\lambda}{n}\right)^{n}$$

$$= \frac{\lambda^x}{x!} \frac{(1-0)(1-0)\cdots(1-0)}{(1-0)^x} \cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}_{e^{-\lambda}}$$

$$= \frac{\lambda^x}{x!} \cdot e^{-\lambda}$$

$$\therefore \lim_{n\to\infty} B(x:n,p) = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

* The sum of these probabilities is unity as it should be.

(2) Mean of Poisson distribution:

$$Mean = E(X) = \sum_{x=0}^{\infty} x \cdot P(x).$$

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$E(X) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \cdot \lambda \cdot \lambda^{x-1}}{x(x-1)!}$$

$$E(X) = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$E(x) = \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

$$E(X) = \lambda e^{-\lambda} [e^{\lambda}]$$

$$E(X) = \lambda(1)$$

$$E(X) = \lambda$$

Variance of poisson distribution:

$$Variance = E(X^2) - (E(X))^2$$

Consider, $E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot P(x)$

$$E(X^2) = \sum_{x=0}^{\infty} (x(x-1) + x) P(x)$$

$$E(X^2) = \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda} \lambda^{x}}{x!} + \sum_{x=0}^{\infty} x \cdot P(x)$$

$$E(X^2) = e^{-\lambda} \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{\lambda^2 \cdot \lambda^{x-2}}{\cancel{x(x-1)}(x-2)!} + E(X)$$

$$E(x^2) = e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$E(x^2) = e^{-\lambda}\lambda^2\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{4!}+\cdots\right]+\lambda$$

$$E(x^2) = e^{-\lambda}\lambda^2(e^{\lambda})+\lambda$$

$$E(x^2) = \lambda^2+\lambda$$

Now;

$$Var(x) = E(x^2) - (E(x))^2$$

$$Var(x) = \lambda^2+\lambda-(\lambda)^2$$

$$var(x) = \lambda^2+\lambda-\lambda^2$$

$$var(x) = \lambda$$

Finally,

Mean of poisson distribution $= \lambda$

Varience of poisson distribution $= \lambda$

③ Properties of Binomial distribution:

The Binomial Distribution holds under the following conditions:

→ Trails are repeated under identical conditions for a fixed no. of times.

→ The probability of success in each trail remains constant and does not change from trail to trail.

→ There are only two possible outcomes, eg. success or failure for each trail.

→ The trails are independent i.e, the probability of an event in any trail is not affected by the results of any other trail.

Mean of Binomial distribution:

$$Mean = E(X) = \sum_{x=0}^{n} x \cdot P(x)$$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\, x!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n(n-1)!}{((n-1)-(x-1))!\, x(x-1)!} p^{1} \cdot p^{x-1} \cdot q^{((n-1)-(x-1))}$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} p^{x-1} q^{((n-1)-(x-1))}$$

$$= np \; {}^{(n-1)}C_{(x-1)} \; p^{x-1} q^{((n-1)-(x-1))}$$

$$= np \left[{}^{n-1}C_0 \, p^0 q^{n-1} + {}^{n-1}C_1 \, p q^{n-2} + \cdots \right.$$

$$\left. \cdots + {}^{(n-1)}C_{(n-1)} \, p^{n-1} q^0 \right]$$

$$= np \, [q+p]^{n-1}$$

$$= np \, (1)^{n-1}$$

$$= np \, (1)$$

$$= np.$$

$\therefore$ Mean of the Binomial distribution $= np$

④ For a Binomial variable x,

Mean = 4 i.e $np = 4$

Valience = 3 i.e $npq = 3$

we know; $npq = 3$

$4 \cdot q = 3 \quad (\because np = 4)$

$$\boxed{q = \frac{3}{4} = 0.75}$$

But; $p + q = 1$

$p + 0.75 = 1$

$$\boxed{p = 0.25}$$

$n \times 0.25 = 4$

$n = 4/0.25$

$$\boxed{n = 16}$$

i) $P(x = 1) = {}^nc_1 \, p^1 \, q^{n-1}$

$= {}^{16}c_1 \, (0.25)^1 (0.75)^{16-1}$

$= 1.6 \, (0.25)^1 \, (0.75)^{15}$

$= 0.053$

ii) $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - \{P(X = 0)\}$

$P(X = 0) = {}^{16}c_0 \, (0.25)^0 \, (0.75)^{16-0}$

$P(X = 0) = 1 \, (0.25)^0 \, (0.75)^{16}$

$P(X=0) = 0.010$

$P(X \geq 1) = 1 - P(X=0)$

$P(X \geq 1) = 1 - 0.010$

$P(X \geq 1) = 0.99$

iii) $P(0 < X < 3) = \boxed{P(X=1) + P(X=2)}$

$P(X=2) = 16_{C_2} (0.25)^2 (0.75)^{16-2}$

$P(X=2) = 120 (0.25)^2 \boxed{(0.75)^{14}}$

$P(X=2) = 0.133$

$P(0 < X < 3) = P(X=1) + P(X=2)$

$P(0 < X < 3) = 0.053 + 0.133$

$P(0 < X < 3) = 0.186$

(5) To Calculate : The Expected frequencies

Here; n = No. of trails = 6

N = total frequency = Σf_i = 13 + 25 + ---- + 4

= 200

Now; mean $(np) = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

$$6 \times p = \frac{25 + 104 + 174 + 128 + 80 + 24}{200}$$

$$6 \times p = \frac{535}{200}$$

$$6p = 2.675 \Rightarrow \boxed{p = 0.446}$$

we know; $p + q = 1$.

$$q = 1 - p = 0.554$$

$$p/q = \frac{0.446}{0.554} = 0.805$$

The initial frequency $f(0) = N \times q^n$

$$= 200 \times (0.554)^6$$

$$= 5.782$$

Now, we can Tabulate the expected frequencies.

x	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1}\cdot\frac{p}{q}$	$f(x+1) = \left(\frac{n-x}{x+1}\cdot\frac{p}{q}\right)\cdot f(x)$
0	6	4.83	5.782 $\sim$ 6
1	$\frac{5}{2} = 2.5$	2.0125	27.927 $\sim$ 28
2	$\frac{4}{3} = 1.3$	1.0465	56.189 ~~56.45~~ $\sim$ 56
3	$\frac{3}{4} = 0.75$	0.60375	58.801 $\sim$ 59
4	$\frac{2}{5} = 0.4$	0.322	35.48 $\sim$ 35
5	$\frac{1}{6} = 0.16$	0.1288	11.42 $\sim$ 11
6	—	—	1.46 $\sim$ 1

$\therefore$ The successive terms in the expansion give the expected or theoretical frequencies which are:

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4
Expected frequencies	6	28	56	59	35	11	1

CIE - I

PART - B

(1) Given;

Total no. of tape recorders = n = 20.

5 are defective

i.e;

The probability of defective = $p = \frac{5}{20} = \frac{1}{4} = 0.25$

q = The probability of non defective = $1 - p$

$$q = 1 - 0.25$$

$$q = 0.75$$

For 10 randomly choosen tape recorders

The standard deviation = $\sqrt{npq}$

$$S.D = \sqrt{10 \times 0.25 \times 0.75}$$

$$S.D = \sqrt{1.875}$$

$$S.D = 1.369 \simeq 1.37$$

(i) $P(X=0) = {}^{10}C_0 \, (0.25)^0 \, (0.75)^{10-0}$

$= 0.056$

ii) $P(X=1) = {}^{10}C_1 (0.25)^1 (0.75)^{10-1}$

$= 10 \, (0.25)(0.75)^9$

$= 0.187$

iii) $P(X=2) = {}^{10}C_2 (0.25)^2 (0.75)^{10-2}$

$\frac{\overset{5}{\cancel{10}} \times 9}{1 \times \cancel{2}}$

$= 45 \, (0.25)^2 (0.75)^8$

$= 0.281$

iv) $P(0 < X < 4) = P(X=1) + P(X=2) + P(X=3)$

$P(X=3) = {}^{10}C_3 (0.25)^3 (0.75)^7$

$\frac{10 \times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{1 \times \cancel{2} \times \cancel{3}}$

$= 120 \, (0.25)^3 (0.75)^7$

$= 0.250$

Now; $P(0 < X < 4) = 0.187 + 0.281 + 0.250$

$= 0.718$

(2) Given; Mean $(\lambda) = 1.5$

$$P(r) = \frac{e^{-\lambda}\lambda^{r}}{r!}$$

(i) $P(\text{no demand}) = P(0) = \dfrac{e^{-1.5}(1.5)^{0}}{0!}$

$$= 0.2231$$

$\therefore$ No. of days in a year there is no demand of car $= 365 \times 0.2231$

$$\approx 81 \text{ days.}$$

(ii) Some demand is refused if the no. of demands is more than two i.e $r > 2$

$P(\text{demand refused}) = P(r>2) = 1 - (P(0) + P(1) + P(2))$

$$= 1 - \left[e^{-1.5} + \frac{e^{-1.5}(1.5)}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 1 - \left[e^{-1.5}\left(1 + 1.5 + 1.125\right)\right]$$

$$= 1 - 3.625\left(e^{-1.5}\right)$$

$$= 0.1913$$

$\therefore$ No. of days in a year when some demand is refused $= 365 \times 0.1913$

$$= 69.82$$

$$\simeq 70 \text{ days.}$$

(3) Let x be the no. of phone calls / minute coming into a switch board

Given; $\lambda = 2.5$

Now the poisson distribution is,

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-2.5}(2.5)^{x}}{x!}$$

(i) P(4 or fewer calls) $= P(x \leq 4)$

$= P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= e^{-2.5}\left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$

$= e^{-2.5}(1 + 2.5 + 3.125 + 2.6042 + 1.6276)$

$= \underline{0.8912}$

(ii) P(more than 6 calls) $= P(x > 6)$

$= 1 - P(x \leq 6)$

$= 1 - e^{-2.5}\left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \text{---} + \frac{(2.5)^6}{6!}\right]$

$= 1 - e^{-2.5}[1 + 2.5 + 3.125 + 2.6042 + 1.6276 + 0.8138 + 0.3391]$

$= \underline{0.01416}$

④ Given; $\Sigma f_i = N = 1000$

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{365 + 420 + 240 + 112 + 45 + 12 + 7}{1000}$$

$$= 1.201$$

$\therefore$ Mean of poisson distribution $\lambda = 1.201$

The table of therotical frequencies is given by

x	$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	$N \cdot P(x)$
0	$\frac{e^{-1.201}(1.201)^0}{0!} = 0.3$	300
1	0.36	366
2	0.21	210
3	0.08	80
4	0.02	20
5	0.006	6
6	0.001	1
7	0.0002	0.2 ~ 0

Therefore, By the poisson distribution,

The theoretical frequencies are given by,

x	0	1	2	3	4	5	6	7
f	305	365	210.	80	28	9	2	1
Theoritical frequencies	300	366.	210.	80	20	6	1	0

(5) To fit the Binomial distribution of getting number of heads i.e

$p = 1/2$; $q = 1/2$ and $n = 4$.

Also;

$N = \sum f_i = 5 + 22 + 65 + 60 + 8 = 160$

$p/q = (1/2)/(1/2) = 1$

The initial frequency $= N \times q^n$

$= 160 \times \left(\frac{1}{2}\right)^4 = 10$

x	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1} \cdot \frac{p}{q}$	$f(x+1) = \left(\frac{n-x}{x+1} \cdot \frac{p}{q}\right) f(x)$
0	4	4	10
1	$\frac{3}{2} = 1.5$	1.5	40
2	$\frac{2}{3} = 0.6$	0.6	60
3	$\frac{1}{4} = 0.25$	0.25	36
4	—	—	9

∴ The Binomial distribution of getting no. of heads.

x	0	1	2	3	4
f	5	22	65	60	8
Expected frequencies	10	40	60	36	9

⑥ Let the no. of boys in each family = x

p = The probability of each boy = $\frac{1}{2}$

(equal probability for boys and girls)

Number of children, n = 5

The probability distribution is

$$P(r) = {}^{n}C_{r}\, p^{r} q^{n-r} = {}^{5}C_{r} \left(\frac{1}{2}\right)^{r} \left(\frac{1}{2}\right)^{5-r}$$

$$= {}^{5}C_{r} \cdot \frac{1}{2^5} \text{ Per family}$$

(i) $P(3 \text{ boys}) = P(r=3) = {}^{5}C_{3} \cdot \frac{1}{2^5} = \frac{10}{32}$

$= \frac{5}{16}$ Per family

∴ For 800 families, the probability of no. of families having 3 boys $= \frac{5}{16} \times 800$

$= 250$ families

(ii) P(5 girls) = P(no. boys) = P(r = 0)

$= P(0)$

$= \frac{1}{2^5} \cdot {}^{5}C_{0}$

$= \frac{1}{32}$ Per family

$\therefore$ For 800 families, the probability of no. of families having 5 girls $= \frac{1}{32} \times 800$

$= 25$ families

(iii) P(either 2 or 3 boys) $= P(r=2) + P(r=3)$

$= P(2) + P(3)$

$= \frac{1}{2^5} \cdot {}^5C_2 + \frac{1}{2^5} \cdot {}^5C_3$

$\Rightarrow \frac{1}{2^5}(10+10) = \frac{20}{32} = \frac{5}{8}$ Per family.

$\therefore$ Expected no. of families with 2 or 3 boys

$= \frac{5}{8} \cdot (800)$

$= 500$ families

(7) Given; $P(x=1) = \frac{3}{2} \cdot P(x=3)$

$$\frac{\cancel{e^{-\lambda}} \cancel{\lambda}}{1!} = \frac{3}{2} \cdot \frac{\cancel{e^{-\lambda}} \lambda^{\cancel{3}\,2}}{\underbrace{3!}_{3\times 2!}}$$

$$\frac{1}{\cancel{1!}} = \frac{\cancel{3}}{2} \times \frac{\lambda^2}{\cancel{3}\times 2\times \cancel{1}}$$

$$\lambda^2 = 2\times 2$$

$$\boxed{\lambda = 2} \quad (\because \lambda > 0)$$

Hence; $P(X=x) = P(x) = \frac{e^{-2}\, 2^x}{x!}$

$e^{-2} = 0.1353$

(i) $P(x \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$$P(x\geq 1) = 1 - \frac{e^{-2}\, 2^0}{0!} = 1 - e^{-2} = 0.8646$$

(ii) $P(X\leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right]$$

$$\Rightarrow e^{-2}\left[1 + 2 + \frac{4}{2} + \frac{8}{6}\right] = 0.8526$$

(iii) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= e^{-2}\left[\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!}\right]$$

$$\Rightarrow e^{-2}\left[\frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120}\right] = 0.5751$$

(8) Given; Mean $= \lambda = 1.8$.

we have; $P(X = x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$P(X = x) = \frac{e^{-1.8}(1.8)^{x}}{x!}$$

(i) $P(\text{atleast one}) = P(x \geq 1) = 1 - P(X = 0)$

$$= 1 - \frac{e^{-1.8}(1.8)^{0}}{0!}$$

$$= 1 - e^{-1.8}$$

$$= \underline{0.8347}$$

(ii) $P(\text{atmost one}) = P(x \leq 1)$

$$= P(X = 0) + P(X = 1)$$

$$= e^{-1.8}\left[\frac{(1.8)^{0}}{0!} + \frac{(1.8)^{1}}{1!}\right]$$

$$= e^{-1.8}(1.8 + 1)$$

$$= e^{-1.8}(2.8)$$

$$= \underline{0.4628}$$

(9) Here; $n = 5$

$$N = \Sigma f_i = 2 + 14 + 20 + 34 + 22 + 8 = 100$$

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{14 + 40 + 102 + 88 + 40}{100} = 2.84$$

mean of Binomial distribution = np

$$np = 2.84$$

$$5p = 2.84$$

$$\boxed{p = 0.568}$$

$q = 1 - p = 0.432$

$$p/q = \frac{0.568}{0.432} = 1.314$$

The Initial frequency $= N q^n$

$$= 100 \times (0.432)^5$$

$$= 1.50$$

Now, we can fit the Binomial distribution as follows:

x	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1} \cdot \frac{p}{q}$	$f(x+1) = \left\{\frac{n-x}{x+1} \cdot \frac{p}{q}\right\} f(x)$
0	5	6.57	1.50 ~ 2
1	2	2.62	9.85 ~ 10
2	1	1.31	25.8 ~ 26
3	0.5	0.65	33.79 ~ 34
4	0.2	0.26	21.96 ~ 22
5	–	–	5.70 ~ 6

∴ The Binomial distribution with the Expected frequencies is as follows.

x	0	1	2	3	4	5
f	2	14	20	34	22	8
Expected frequencies	2	10	26	34	22	6

(10) Given; $n = 5$

P = Probability of hitting target $= \frac{1}{3} = 0.33$

$q = 1 - P = \frac{2}{3} = 0.66$

$f(x) = P(X = x) = {}^nC_x \, p^x q^{n-x}$

$= {}^5C_x \, p^x q^{5-x}$

(i) P(atmost 3 times) $= P(x \le 3)$

$= P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= {}^5C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{5-0} + {}^5C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4 + {}^5C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3$

$+ {}^5C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2$

$= 1(0.33)^0 (0.66)^5 + 5(0.33)^1 (0.66)^4 + 10(0.33)^2 (0.66)^3$

$+ 10(0.33)^3 (0.66)^2$

$= 0.12 + 0.31 + 0.31 + 0.15$

$= \underline{0.89}$

(ii) P(Atleast 2 times) $= P(x \geq 2)$

$= 1 - (P(X < 2))$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left({}^5C_0 (0.33)^0 (0.66)^5 + {}^5C_1 (0.33)^1 (0.66)^4 \right)$

$= 1 - (0.12 + 0.31)$

$= \underline{0.57}$

Verified!

Prepared by:
M. Sai Charan
AI & ML - C