MODULE - III

CIE - II

PART - A

⑥, ⑦, ⑧ - common

a) **Properties of Normal distribution:**

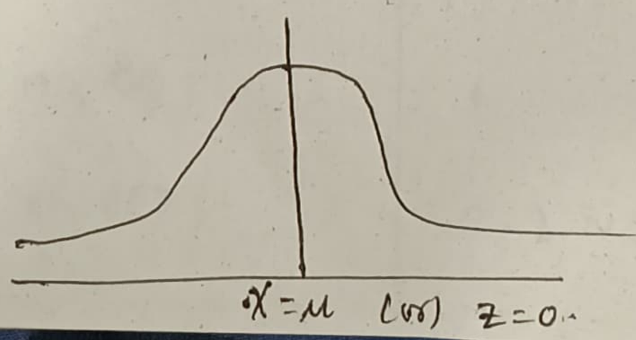
→ We shall use the notation $X \sim N(\mu, \sigma^2)$ to denote the random variable x follows normal distribution with the parameters μ and σ^2.

→ If $X \sim N(\mu, \sigma^2)$, then $z = \frac{X-\mu}{\sigma}$ is called the standard normal variate whose P.D.F is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ with $E(z)=0$ and $Var(z) = 1$.

→ Since $f(x)$ is the P.D.F of normal distribution, so we have,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

→ The normal distribution curve is a Bell-shaped curve and it is symmetrical about the line $x = \mu$ (or) $z = 0$



6(a) Mean of Normal distribution:

Consider the normal distribution with μ and σ^2 as parameters

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Put $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z$

$dx = \sigma\, dz$

When; $x = \infty \to z = \infty$ and $x = -\infty \to z = -\infty$

$$E(X) = \int_{-\infty}^{\infty} (\mu + \sigma z) \cdot \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2} \cdot \cancel{\sigma}\, dz$$

$$E(X) = \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot e^{-z^2/2}\, dz$$

Consider; $f(z) = z e^{-z^2/2}$

$f(-z) = -z e^{-z^2/2} = -f(z)$

$\therefore f(z)$ is odd function

$$\therefore \int_{-a}^{a} f(x)\, dx = \begin{cases} 0 \; ; \text{odd func}^n \\ 2\int_{0}^{a} f(x)\, dx \; ; \text{even func}^n \end{cases}$$

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^\infty e^{-z^2/2} dz + \frac{\sqrt{5}}{\sqrt{2\pi}} (0).$$

$$E(X) = \frac{\sqrt{2}\mu}{\sqrt{\pi}} \int_0^\infty e^{-z^2/2} dz$$

Put $\frac{z^2}{2} = t \Rightarrow z^2 = 2t$

$$2z\,dz = 2\,dt \Rightarrow dz = \frac{dt}{z} = \frac{dt}{\sqrt{2t}}$$

$z = 0 \rightarrow t = 0$ and $z = \infty \rightarrow t = \infty$

$$E(X) = \frac{\sqrt{2}\mu}{\sqrt{\pi}} \int_0^\infty e^{-t} \frac{dt}{\sqrt{2t}}$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-t} \cdot t^{-1/2} dt$$

Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1} dx$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-t} \cdot t^{1/2 - 1} dt}_{\Gamma(\frac{1}{2})}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$E(X) = \frac{\mu}{\sqrt{\pi}} \times \sqrt{\pi}$$

$$E(X) = \mu$$

$\therefore$ Mean of normal distribution $= \mu$.

13) Mode in Normal distribution:

From the normal distribution.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Take 'log' on Both sides

$$\log f(x) = \log \frac{1}{\sigma\sqrt{2\pi}} + \log e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\log f(x) = \log \frac{1}{\sigma\sqrt{2\pi}} + \left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Differentiate with respect to 'x'.

$$\frac{1}{f(x)} f'(x) = 0 - \frac{1}{2\sigma^2} 2(x-\mu).1$$

$$f'(x) = -\frac{(x-\mu)}{\sigma^2} . f(x) \quad \text{--- } 1$$

Again Differentiate with respect to 'x'

$$f''(x) = -\frac{1}{\sigma^2}\left(1.f(x) + (x-\mu).f'(x)\right)$$

$$f''(x) = -\frac{1}{\sigma^2}\left(f(x) - \frac{(x-\mu)^2}{\sigma^2} f(x)\right)$$

$$f''(x) = -\frac{f(x)}{\sigma^2}\left(1 - \frac{(x-\mu)^2}{\sigma^2}\right)$$

To find mode

(1) $f'(x) = 0 \Rightarrow -\frac{(x-\mu)}{\sigma^2} \cdot f(x) = 0$

$$(x-\mu) = 0$$

$$\boxed{x = \mu} \rightarrow \text{Stationary point}$$

Now at $x = \mu$, $f''(x) = -\frac{f(\mu)}{\sigma^2}\left[1 - \frac{(\mu-\mu)^2}{\sigma^2}\right]$

$$f''(x) = -\frac{f(\mu)}{\sigma^2}$$

$$f''(x) < 0$$

By principles of maxima & minima;

$x = \mu$. is the point of maximum. Therefore,

Mode of the Normal distribution $= \mu$.

✓

8b) Median of the Normal distribution:

If "M" is the median of normal distribution then

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}.$$

Consider,

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$$

$$\int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2} \quad \text{—} \quad (1)$$

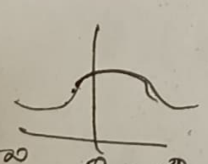
But,

$$\int_{-\infty}^{\mu} f(x)\,dx = \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Put; $\frac{x-\mu}{\sigma} = z$

$\Rightarrow X = \mu + \sigma z \Rightarrow dx = \sigma\,dz$

$$= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\,dz$$



$X = -\infty \rightarrow z = \infty$

$X = \mu \rightarrow z = 0$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz.$$

$$= \frac{1}{2} \quad (\because \text{By symmetry})$$

From (1)

$$\frac{1}{2} + \int_{\mu}^{M} f(x)\, dx = \frac{1}{2}$$

$$\int_{\mu}^{M} f(x)\, dx = 0$$

The above integrals satisfies only when $M = \mu$. Therefore,

Median of Normal distribution $= \mu$

Note :

In Normal distribution

$$\left.\begin{array}{l}\text{Mean} \\ \text{Median} \\ \text{Mode}\end{array}\right\} = \mu$$

Varience $= \sigma^2$

standard deviation $= \sigma$.

Mean deviation $= \frac{4}{5}\sigma$

Let μ be the Mean (at $z=0$) and σ be the S.D of Normal distribution.

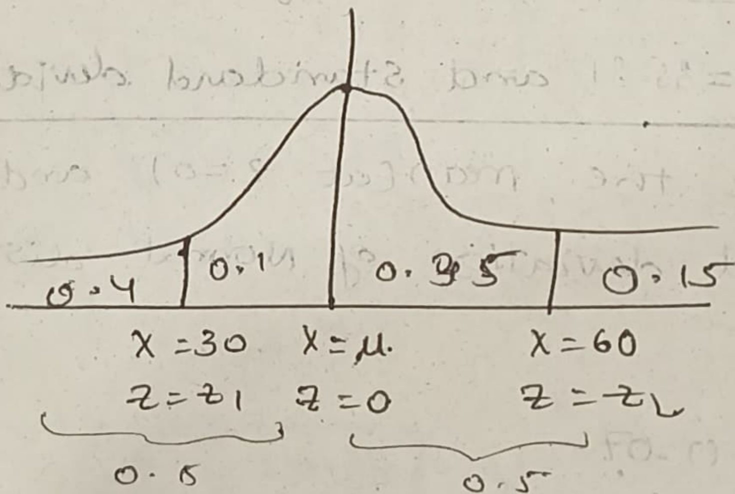
Let the variable 'x' denote the marks in Statistics.

Then given that

$P(X<30) = 0.4$ and $P(X \geq 60) = 0.15$

when; $X = 30 \Rightarrow z = \dfrac{x-\mu}{\sigma} \Rightarrow -z_1 = \dfrac{30-\mu}{\sigma}$ (say)

when; $X = 60 \Rightarrow z = \dfrac{X-\mu}{\sigma} \Rightarrow z_2 = \dfrac{60-\mu}{\sigma}$ (say)



By symmetry,

$\therefore P(0 < z < z_2) = 0.5 - 0.15 = 0.35$

$P(0 < z < z_1) = 0.5 - 0.4 = 0.1$

From normal tables, we get

$z_1 = 0.25$

$z_2 = 1.04$

Hence; $\frac{30-\mu}{\sigma} = -0.25$

$\Rightarrow \frac{\mu-30}{\sigma} = 0.25$ — (3)

also; $\frac{60-\mu}{\sigma} = 1.04$ — (4)

(3) + (4) $\frac{30}{\sigma} = 1.29$

$\boxed{\sigma = 23.26}$

from (3) $\mu = 0.25(23.26) + 30$

$\boxed{\mu = 35.81}$

$\therefore$ Mean $= 35.81$ and standard deviation $= 23.26$

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(10) Let μ be the mean (at $z=0$) and σ be the standard deviation of Normal distribution

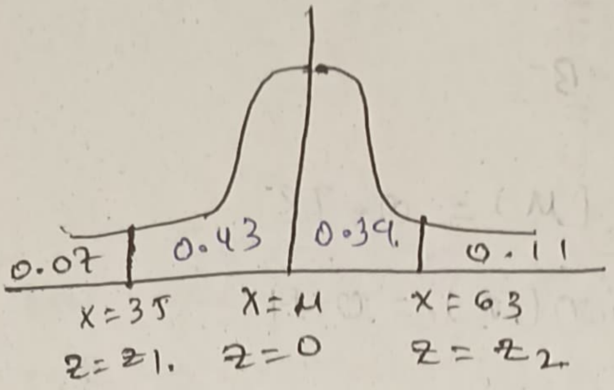
Given that

$P(X<35) = 0.07$

$P(X>63) = 0.11$

when; $X = 35 \Rightarrow z = \frac{X-\mu}{\sigma} \Rightarrow \frac{35-\mu}{\sigma} = -z_1$ (say)

when; $X = 63 \Rightarrow z = \frac{X-\mu}{\sigma} \Rightarrow \frac{63-\mu}{\sigma} = z_2$ (say)



$\therefore P(0 < z < z_1) = 0.5 - 0.07 = 0.43$

$P(0 < z < z_2) = 0.5 - 0.11 = 0.39$

From normal tables, we get

$z_1 = 1.48$ and $z_2 = 1.23$

Hence; $\frac{35-\mu}{\sigma} = -1.48$ | $\frac{63-\mu}{\sigma} = 1.23$ — (4)

$\frac{\mu-35}{\sigma} = 1.48$ — (3)

(3)+(4)

$\frac{28}{\sigma} = 2.71$

$\boxed{\sigma = 10.33}$

from (3)

$\mu = 35 + 1.48(10.33)$

$\boxed{\mu = 50.288}$

Varience $= \sigma^2 = (10.33)^2 = 106.75$

$\therefore$ Mean $= 10.33$ and Varience $= 106.75$

CIE - II

PART - B

(11) Given; mean $(\mu) = 0.78$

Standard deviation $(\sigma) = 0.11$

(i) When $x = 0.9$

$$z = \frac{x-\mu}{\sigma} = \frac{0.9 - 0.78}{0.11}$$

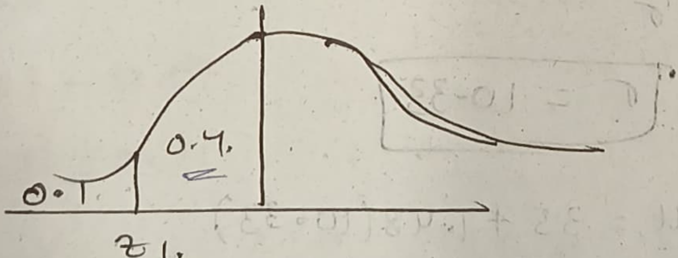
$$z = 1.09 = z_1 \text{ (say)}$$

Hence the number of students with marks more than 90 %.

$$= 0.1379 \times 1000$$

$$= 137.9 \simeq 138$$

(ii) The 0.1 area to the left of z corresponds to the lowest 10% of the students.



$0.4 = 0.5 - 0.1 = 0.5 -$ Area from 0 to z_1

$$P(0 < z < z_1) = 0.4$$

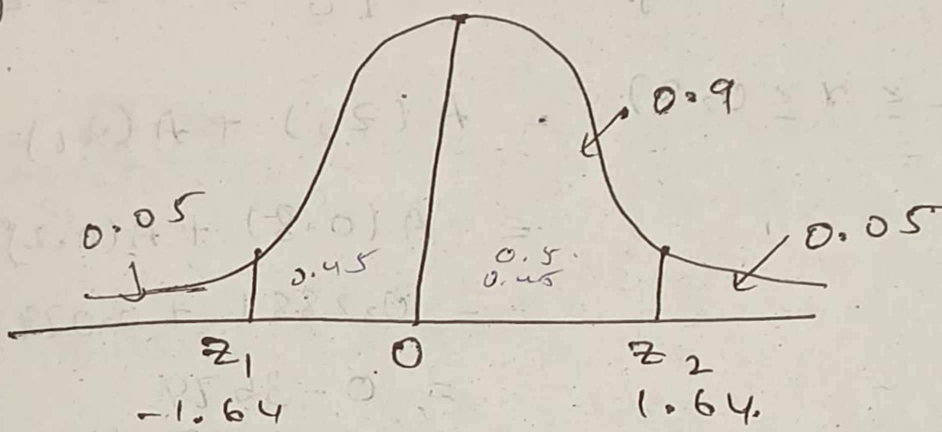
$$z_1 = -1.28$$

Thus: $-1.28 = \frac{x-\mu}{\sigma} \Rightarrow -1.28 = \frac{x-0.78}{0.11}$

$x = 0.6392$

Hence, the highest mark obtained by the lowest 10% of students $= 0.6392 \times 1000$
$\simeq 64\%$

(iii)



Middle 90% correspond to 0.9 area, leaving 0.05 area on both sides. Then the corresponding z's are ± 1.64 (since area $= 0.45$, $z = 1.64$)

$\Rightarrow -1.64 = \frac{x_1 - \mu}{\sigma} \Rightarrow x_1 = -1.64(0.11) + 0.78$

$x_1 = 0.5996$ (or) 59.96%

$\Rightarrow 1.64 = \frac{x_2 - \mu}{\sigma} \Rightarrow x_2 = 1.64(0.11) + 0.78$

$x_2 = 0.9604$ (or) 96.04%

Thus the middle 90% have marks in between 60 to 96.

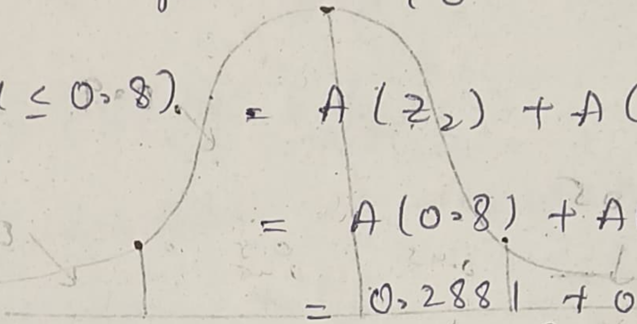
(12) let μ be the mean & σ be the S.D

$\mu = 140$ and $\sigma = 10$

(i) $P(138 \le x \le 148) = ?$

$$x = 138 \Rightarrow z = \frac{X-\mu}{\sigma} = \frac{138-140}{10} = -0.2 = z_1$$

$$x = 148 \Rightarrow z = \frac{X-\mu}{\sigma} = \frac{148-140}{10} = 0.8 = z_2$$



$$P(-0.2 \le x \le 0.8) = A(z_2) + A(z_1)$$

$$= A(0.8) + A(0.2)$$

$$= 0.2881 + 0.0793$$

$$= 0.3674$$

No. of students $= 0.3674 \times 800 \approx 294$

(ii) $P(x > 152) = ?$

$$x = 152 \Rightarrow z = \frac{X-\mu}{\sigma} = \frac{152-140}{10} = 1.2 = z_1$$

$$P(x > z_1) = P(z > z_1)$$

$$= 0.5 - P(z_1)$$

$$= 0.5 - 0.3849$$

$$= 0.1151$$

No of students $= 0.1151 \times 800$

$$\approx 92$$

(8) Given; $\mu = 1$ and $\sigma = 3$

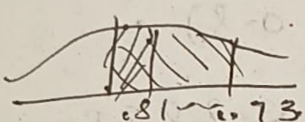
(i) $P(3.43 \le x \le 6.19) = ?$

$x = 3.43 \Rightarrow z = \dfrac{X-\mu}{\sigma} = \dfrac{3.43-1}{3} = 0.81 = z_1$.

$x = 6.19 \Rightarrow z = \dfrac{X-\mu}{\sigma} = \dfrac{6.19-1}{3} = 1.73 = z_2$.

$P(0.81 \le x \le 1.73) = |A(z_2) - A(z_1)|$

$= A(1.73) - A(0.81)$



$= 0.4582 - 0.2910$

$= 0.1672$

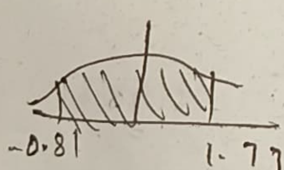
(ii) $P(-1.43 \le x \le 6.19)$

$x = -1.43 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{-1.43-1}{3} = -0.81 = z_1$

$x = 6.19 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{6.19-1}{3} = 1.73 = z_2$

$P(-0.81 \le z \le 1.73) = A(z_2) + A(z_1)$

$= A(1.73) + A(-0.81)$



$= 0.4582 + 0.2910$

$= 0.7492$

(4) Given; $\mu = 30$ and $\sigma = 5$

(i) $P(26 \le X \le 40) = ?$

$x = 26 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{26-30}{5} = -0.8 = z_1$

$x = 40 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{40-30}{5} = 2 = z_2$

$P(-0.8 \le x \le 2) = A(z_2) + A(z_1)$

$= A(2) + A(-0.8)$

$= 0.4772 + 0.2881$

$= \underline{\underline{0.7653}}$

(ii) $P(x \ge 45) = ?$

When $x = 45 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3 = z_1$

$P(x \ge 45) = P(z_1 \ge 3)$

$= 0.5 - A(z_1)$

$= 0.5 - 0.49865$

$= \underline{\underline{0.00135}}$

✓

(15) Given; $\mu = 75$ kg and $\sigma = 7$ kg

(i) $P(60 \leq x \leq 78)$

$x = 60 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{60-75}{7} = -2.14 = z_1$

$x = 78 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{78-75}{7} = 0.42 = z_2$

$P(-2.14 \leq z \leq 0.42) = A(0.42) + A(-2.14)$

$= 0.1628 + 0.4838$

$= 0.6466$

No. of students $= 0.6466 \times 500$

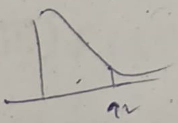
≈ 323 students

(ii) $P(x \geq 92)$

when; $x = 92 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{92-75}{7} = 2.42$

$P(z \geq 2.42) = 0.5 - A(2.42)$

$= 0.5 - 0.4922$



$= 0.0078$

No. of students $= 0.0078 \times 500$

≈ 4 students

(16) Given; $\mu = 34.5$ and $\sigma = 16.5$

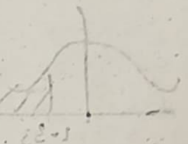
when; $x = 30 \Rightarrow z = \frac{x - \mu}{\sigma} = \frac{30 - 34.5}{16.5} = -0.27 = z_1$

when; $x = 60 \Rightarrow z = \frac{x - \mu}{\sigma} = \frac{60 - 34.5}{16.5} = 1.54 = z_2$

$P(30 \le x \le 60) = P(z_1 \le z \le z_2)$

$= A(z_2) + A(z_1)$

$= A(1.54) + A(-0.27)$



$= 0.4382 + 0.1084$

$= 0.5916$

$\therefore$ The number of students who get marks between 30 and 60

$= 0.5916 \times 1000$

$= 591.6$

Hence, 592 students get marks between 30 and 60.

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(18) Repeated from Part A 10 Q

(7) Let μ be the mean and σ be the standard deviation of distribution. Let the variable x denote the masses of students.

$$\mu = 68 \text{ Kgs} \qquad \sigma = 3 \text{ Kgs}$$

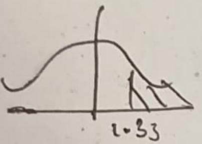
(i) $P(x > 72) = ?$

$$x = 72 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{72-68}{3} = 1.33$$

$$P(z > 1.33) = 0.5 - A(1.33)$$

$$= 0.5 - 0.4082$$

$$= 0.0918$$



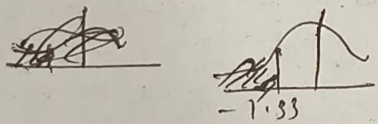
No. of students $= 0.0918 \times 300 \approx 28$

(ii) $P(x \leq 64)$

$$x = 64 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{64-68}{3} = -1.33$$

$$P(x \leq 64) = P(z \leq -1.33) = 0.5 - A(1.33)$$

$$= 0.0918$$



No. of students $= 0.0918 \times 300 \approx 28$

(iii) $P(65 \leq x \leq 71) = ?$

$$x = 65 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{65-68}{3} = -1 = z_1$$

$$x = 71 \Rightarrow z_1 = \frac{x-\mu}{\sigma} = \frac{71-68}{3} = 1 = z_2$$

$$P(z_1 \leq z \leq z_2) = A(z_2) + A(z_1) \checkmark$$

$$= A(-1) + A(1) = 2A(1) = 2(0.3413)$$

$$= 0.6826$$

No of students $= 0.6826 \times 300 \approx 205$

(19) Given; $\mu = 155$ hrs and $\sigma = 19$ hrs

(i) $P(136 < X < 174) = ?$

when $X = 136 \Rightarrow Z = \frac{X-\mu}{\sigma} = \frac{136-155}{19} = -1 = Z_2$

when $X = 174 \Rightarrow Z = \frac{X-\mu}{\sigma} = \frac{174-155}{19} = 1 = Z_1$.

$P(-1 \leq Z \leq 1) = A(Z_2) + A(Z_1)$

$= A(-1) + A(1)$

$\Rightarrow 2A(1) = 2(0.3413)$

$= 0.6826$

(ii) $P(X < 117) = ?$

when; $X = 117 \Rightarrow Z = \frac{X-\mu}{\sigma} = \frac{117-155}{19} = -2$

$P(Z < -2) = 0.5 - A(2)$

$= 0.5 - 0.4772 = 0.0228.$

(iii) $P(X > 195) = ?$

when; $X = 195 \Rightarrow Z = \frac{X-\mu}{\sigma} = \frac{195-155}{19} = 2.10$

$P(Z > 2.1) = 0.5 - A(2.1)$

$= 0.5 + 0.4821$

$= 0.0179.$

(20) Given; $\mu = 35$ and $\sigma = 5$

(i) $P(25 \le X \le 40) = ?$

$x = 25 \Rightarrow z = \frac{x - \mu}{\sigma} = \frac{25-35}{5} = -2 = z_1$

$x = 40 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{40-35}{5} = 1 = z_2$

$P(z_1 \le z \le z_2) = A(z_2) + A(z_1)$

$= A(1) + A(-2)$

$= A(1) + A(2)$

$= 0.3415 + 0.4772$

$= \underline{0.8185}$

No. of students $= 0.8185 \times 1000 = 819$

(ii) $P(x \ge 40) = ?$

$x = 40 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{40-35}{5} = 1$

$P(z \ge 1) = 0.5 - A(1)$

$= 0.5 - 0.3415$

$= \underline{0.1585}$

No. of students $= 0.1585 \times 1000 = 159$

(iii) $P(x \le 20) = ?$

$x = 20 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{20-35}{5} = \underline{-3}$

$P(z \le -3) = 0.5 - A(-3)$

$= 0.5 - A(3)$

$= \cancel{0.5 - 0.4998}\ 499$

$= 0.5 - 0.499$

$= 0.01$

$\therefore$ No. of students $= 1000 \times 0.001 = 1$

(iv). $P(x > 50) = ?$

$x = 50; \quad z = \frac{x-\mu}{\sigma} = \frac{50-35}{5} = 3$

$P(z > 3) = 0.5 - A(3)$

$= 0.5 - 0.499$

$= 0.01$

$\therefore$ No. of students $= 1000 \times 0.001 = \frac{1}{2}$

Verified

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Asst-C