PART-A

1

Angle b/w Normal

$\hookrightarrow \vec{n}' = \frac{\nabla\phi}{|\nabla\phi|}$.

Angle b/w Surfaces,

$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$.

given equations,

$\phi = x^2+y^2+z^2=9$

$\bar{\psi} = z = x^2+y^2-3$

Point $(2,-1,2)$.

$\phi = x^2+y^2+z^2-9=0$

$\bar{\psi} = x^2+y^2-z-3=0$.

$\cancel{\nabla\phi = i\partial\phi}$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$.

$= (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$ $\cancel{\rightarrow 1}$

at $(2,-1,2)$

$= 4\hat{i} - 2\hat{j} + 4\hat{k} \rightarrow 1$

$\cancel{\frac{\partial\psi}{\partial}}$ $\nabla\psi = i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z}$

$= i(2x) + (2y)\hat{j} + \bar{k}(-1)$

$= (2x)\hat{i} + (2y)\hat{j} + (-1)\hat{k}$ $\cancel{\rightarrow 2}$

at $(2,-1,2)$

$= 4\hat{i} - 2\hat{j} - \hat{k} \rightarrow 2$

Angle b/w Surfaces,

$$\frac{\partial\phi \cdot \partial\psi}{|\partial\phi||\partial\psi|}$$

$$= \frac{(\cancel{4}\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{\cancel{16}\ 8}{3\ \cancel{6}\sqrt{21}} = \frac{8}{3\sqrt{21}}$$ //.

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

2

Angle b/w Normals.

Points $\rightarrow$ $(1,1,1)$ & $(2,4,1)$.

$$\phi = x^2 - yz.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (2x)\hat{i} + (-z)\hat{j} + (-y)\hat{k} \longrightarrow (1)$$

at $(1,1,1)$

$$\overrightarrow{\nabla\phi_1} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\hat{n_1} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{4+1+1}} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$$

substitute $(2,4,1)$ in eqn ①,

$$\nabla\phi_2 = 4\hat{i} - \hat{j} - 4\hat{k}$$

$$\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{4\hat{i}-\hat{j}-4\hat{k}}{\sqrt{16+1+16}}$$

$$= \frac{4\hat{i}-\hat{j}-4\hat{k}}{\sqrt{33}}$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$= \frac{2\hat{i}-\hat{j}-\hat{k}}{\sqrt{6}} \cdot \frac{4\hat{i}-\hat{j}-4\hat{k}}{\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{6}\sqrt{33}} \;//.$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{6}\sqrt{33}}\right) //.$$

③

Prove, $\operatorname{div}(\operatorname{grad} r^m) = m(m+1)r^{m-2}$.

Sol

given,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2},$$

$$r^2 = x^2+y^2+z^2,$$

Partially diff w.r.t x, y, z,

$$\frac{\partial r^2}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2) + \frac{\partial}{\partial y}(x^2+y^2+z^2) + \frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$2r.\frac{\partial r}{\partial x} = 2x,$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \Big| \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \Big| \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\nabla(r^m). = \frac{\partial (r^m)}{\partial x} = m.r^{m-1}\frac{\partial r}{\partial x}$$

$$= m.r^{m-1}.\frac{x}{r}$$

$$= m.\frac{r^{m-1}}{r^1}.x$$

$$= m.r^{m-1-1}.x$$

$$= m r^{m-2}.x,$$

~~Similarly~~,

$$= \cancel{> m.r^{m-2}\nabla x}.$$

~~div (r^m ∇ r^m) :~~

$$\cancel{\frac{\partial}{\partial x}(m r^{m-1}}$$

Now ,

$$\frac{\partial^2 (r^m)}{\partial x^2} = m\left(r^{m-2} + (m-2) r^{m-3} \times \frac{\partial r}{\partial x}\right)$$

$$= m\left(r^{m-2} + (m-2) r^{m-4} . x^2\right)$$

Similarly,

$$\frac{\partial^2 (r^m)}{\partial y^2} = m\left(r^{m-2} + (m-2) r^{m-4} . y^2\right)$$

$$\frac{\partial^2 (r^m)}{\partial z^2} = m\left(r^{m-2} + (m-2) r^{m-4} . z^2\right)$$

$$\therefore \frac{\partial^2 (r^m)}{\partial x^2} + \frac{\partial^2 (r^m)}{\partial y^2} + \frac{\partial^2 (r^m)}{\partial z^2}$$

$$= m\left(3 r^{m-2} + (m-2) r^{m-4} . (x^2+y^2+z^2)\right)$$

$$= m\left(3 r^{m-2} + (m-2) r^{m-4} . r^2\right)$$

$$= m\left(3 r^{m-2} + (m-2) r^{m-2}\right)$$

$$\text{div}(\text{grad } r^m) = m(m+1) r^{m-2}$$

Hence Proved//

(9) $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

$\bar{F}$ is Irrotational,

$\nabla \times \bar{F} = 0$.

$$\text{Curl } \bar{f} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$\bar{i}\left[\partial/\partial y\,(z^2 - xy) - \partial/\partial z\,(y^2 - zx)\right] - \bar{j}\left[\partial/\partial x\,(z^2 - xy) - \partial/\partial z\,(x^2 - yz)\right] + \bar{k}\left[\partial/\partial x\,(y^2 - zx) - \partial/\partial y\,(x^2 - yz)\right]$$

$$= \bar{i}[\cancel{-x} + \cancel{x}] - \bar{j}[\cancel{-y} + \cancel{y}] + \bar{k}[\cancel{-z} + z]$$

$$= 0.$$

$\bar{F}$ is Irrotational, there exists scalar function.

$$\bar{f} = \nabla \phi$$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}\right)$$

Scalar Potential $\rightarrow \bar{f} = \nabla\phi$.

$$(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x^2 - yz$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy$$

Integrating on BS,

$$\phi = \int (x^2 - zy)\,dx + c \rightarrow (1)$$

$$\phi_1 = \frac{x^3}{3} - yzk + c$$

$$\phi = \int (y^2 - zk)\,dy + c$$

$$\phi = \frac{y^3}{3} - zxy + c \rightarrow (2)$$

$$\phi = \int (z^2 - xy) + c$$

$$\phi = \frac{z^3}{3} - zxy + c \rightarrow (3)$$

from (1), (2), (3),

$$3\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - 3xyz + c \;//$$

5.

$$\bar{F} = xy^3z^2 - 4.$$

point = $(-1,-1,2)$.

$$\nabla \bar{F} = \frac{\partial \bar{f}}{\partial x} i + \frac{\partial \bar{f}}{\partial y}\hat{j} + \frac{\partial \bar{f}}{\partial z}\hat{k}$$

$$= (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (xy^3 2z)\hat{k}$$

at point $(-1,-1,2)$ $= -4\hat{i} - 12\hat{j} + 4\hat{k}$

Unit vector : $\dfrac{\nabla \bar{F}}{|\nabla \bar{F}|}$

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = \frac{\cancel{4}(-\hat{i} - 12\hat{j} + \hat{k})}{\cancel{4}\sqrt{11}}$$

$$= \frac{(-\hat{i} - \cancel{12}^{3}\hat{j} + \hat{k})}{\sqrt{11}}$$

//

6.

given,

$$\phi(x,y,z) = x^2 y z + 4xz^2.$$

$$\cancel{\nabla\phi = (x^2yz+4x}$$

$$\nabla\phi = i\frac{\partial}{\partial x}(x^2yz+4xz^2) + j\frac{\partial}{\partial y}(x^2yz+4xz^2)$$

$$+ k\frac{\partial}{\partial z}(x^2yz+4xz^2)$$

$$\nabla\phi = (2xyz+4z^2)i + (x^2z)j + (x^2y+8x)\hat{k}$$

at point $(1,-2,-1)$

$$= \cancel{z}(\cancel{4}+(-\cancel{4})\hat{i} + (-1)\hat{j} + (7)\hat{k}$$

$$\nabla\phi = -\hat{j}+7\hat{k}$$

→ direction of normal to surface

$$f(x,y,z) = x\log z - y^2.$$

$$\nabla f = (\log z)\hat{i} + (-2y)\hat{j} + \left(x\cdot\frac{1}{z}\right)\hat{k}$$

subs $(-1,+2,+1)$ $\Rightarrow (\log 1)\hat{i} + (-2(-2))\hat{j} + (1/-1)\hat{k}$

$$= \cancel{0+4\hat{j}-\hat{k}}. \quad = -4\hat{j}+\hat{k}$$

$$\nabla \bar{f} = -4\hat{j} + \bar{k}.$$

directional derivative $= \dfrac{\nabla\phi \cdot \nabla\bar{f}}{|\nabla\bar{f}|}$

$$= \frac{-\hat{j} + 7k \cdot -4\hat{j} + \bar{k}}{\sqrt{16+1}}$$

$$= \frac{11}{\sqrt{11}} \quad //.$$

Q/7 prove that $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$.

Sol let $\bar{r} = x\hat{i} + y\hat{j} + z\bar{k}$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r = \sqrt{x^2+y^2+z^2}$$

SOBS,

$$r^2 = x^2+y^2+z^2$$

differentiating,

$$\cancel{2}r\,\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \Big| \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\nabla f(r) = \bar{i}\frac{\partial f(r)}{\partial x} + \bar{j}\frac{\partial f(r)}{\partial y} + \frac{\partial f(r)}{\partial z}$$

$$= \frac{f'(r)}{r}(x,y,z) = \frac{f'(r)}{r}\cdot\bar{r}$$

Prove that

Q

$$\nabla r^n = n \cdot r^{n-2} \bar{r} .$$

$$\nabla f(r) = f'(r) \frac{\bar{r}}{r} . \longrightarrow (1)$$

$$f(r) = r^n$$

$$\boxed{f'(r) = n r^{n-1}}$$

put value in eqn(1),

$$\nabla f(r) = n \cdot r^{n-1} \times \frac{\bar{r}}{r}$$

$$= n \cdot \frac{r^{n-1}}{r^1} \cdot \bar{r}$$

$$= n \cdot r^{n-1-1} \cdot \bar{r}$$

$$\nabla f(r) = n \cdot r^{n-2} \cdot \bar{r}$$

$$r^n = n \cdot r^{n-2} \bar{r}$$

Hence Proved

Part-A

9. $\bar{F} = xy^2 + yz^2 + zx^2.$

tangent to curve,

$x = t$
$y = t^2$
$z = t^3$
| point (1,1,1)

given,

$$\bar{f} = xy^2 + yz^2 + zx^2$$

$$\text{grad } \bar{f} = \left(i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \right)$$

$$= \bar{i}\,(y^2 + 2zx) + \bar{j}(z^2 + 2yx) + \bar{k}\,(2yz + x^2)$$

at (1,1,1)

$$= i(1+2) + j\,(2+1) + \bar{k}\,(2+1)$$

$$= 3\hat{i} + 3\hat{j} + 3\bar{k}$$

let $\bar{r}$ be any position vector on curve,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= t\hat{i} + t^2\hat{j} + t^3\hat{k}.$$

to find normal vector, $\frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}.$

at (1,1,1)

$$\bar{i} + 2\bar{j} + 3\bar{k}.$$

Unit vector, $\bar{n} = \dfrac{(i+2j+3k)}{\sqrt{1+4+9}}$

$$= \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$$

$$DD = \left(\frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}\right)(3\hat{i}+3\hat{j}+3\hat{k})$$

$$= \frac{3+6+9}{\sqrt{14}}$$

$= \dfrac{18}{\sqrt{14}}$ is the DD //.

10.

$$t = xy + yz + zx.$$

$$\nabla t = \left(\frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}\right)$$

$$= (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

at point (1,1,1)

$$= 2\hat{i} + 2\hat{j} + 2\hat{k}$$

Magnitude $= |\nabla t| = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$

$\therefore$ The greatest rate of change is $2\sqrt{3}$.

PART-B

Q① Prove that $\bar{F} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)\bar{k}$ is solenoidal, determine contants a, b, c is $\bar{F}$ is Irrotational

Sol

given,

$$\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

$\bar{F}_1 = x+2y+az$

$\bar{F}_2 = bx-3y-z$

$\bar{F}_3 = 4x+cy+2z$

Solenoidal,

↳ $\nabla \cdot \bar{f} = 0$.

Irrotational

↳ $\nabla \times \bar{f} = 0$.

$$\frac{\partial f_1}{\partial x} = 1 \quad \Bigg| \quad \frac{\partial f_2}{\partial y} = -3 \quad \Bigg| \quad \frac{\partial f_3}{\partial z} = 2$$

$$\operatorname{div} \bar{f} = \nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1 - 3 + 2$$

$$= 3 - 3$$

$$= 0 //$$

The given $\bar{F}$ is solenoiddl

$\bar{F}$ is Irrotational $\rightarrow$ given,

Curl $\bar{f}$ = $\nabla \times \bar{f} = 0$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$\bar{i}\left[\partial/\partial y\,(4x+cy+2z) - \partial/\partial z\,(bx-3y-z)\right]$$

$$-\bar{j}\left[\partial/\partial x\,(4x+cy+2z) - \partial/\partial z\,(x+2y+az)\right]$$

$$+\bar{k}\left[\partial/\partial x\,(bx-3y-z) - \partial/\partial y\,(x+2y+az)\right]$$

$$:\bar{i}\,[c-1] - \bar{j}\,[4-a] + \bar{k}\,[b-2] = 0$$

$$\bar{i}\,[c+1] + \bar{j}\,[+a-4] + \bar{k}\,[b-2] = 0$$

given, $\bar{F}$ is Irrotational, $\nabla \times \bar{f} = 0$,

$c+1=0$	$a-4=0$	$b-2=0$
$\boxed{c=-1}$	$\boxed{a=4}$	$\boxed{b=2}$

Values of a,b,c are 4, 2, -1

Scalar Potential

$$\hookrightarrow \bar{i} = \nabla \phi$$

$$(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\frac{\partial \phi}{\partial x} = (x+2y+4z)$$

$$\frac{\partial \phi}{\partial y} = 2x-3y-z$$

$$\frac{\partial \phi}{\partial z} = 4x-y+2z$$

Integrate on BS,

$$\phi = (x+2y+4z)dx$$
$$= 1$$

$$\phi = (2x-3y-z)dy$$
$$= -3$$

$$\phi = (4x-y+2z)dz$$
$$= 2$$

$$\phi = 1-3+2$$
$$= 3-3$$
$$= 0 //.$$

Q
2. finding constants a & b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

$$\phi = ax^2 - byz - (a+2)x.$$

$$\psi = 4x^2y + z^3 - 4$$

$$\nabla\phi = (2ax - (a+2))\hat{i} - (bz)\hat{j} - (by)\hat{k}$$

$$= (2ax - a - 2)\hat{i} - (bz)\hat{j} - (by)\hat{k}$$

at $(1,-1,2)$

$$\nabla\phi = (2a - a - 2)\hat{i} - (2b)\hat{j} + b\hat{k}$$

$$\nabla\phi = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$$

$\longrightarrow$ (1)

$$\nabla\psi = (8xy)\hat{i} + (4x^2)\hat{j} + 3z^2\hat{k}$$

at $(1,-1,2)$ $\nabla\psi = -8\hat{i} + 4\hat{j} + 12\hat{k}$

$\longrightarrow$ (2)

given, $\bar{a}$ & $\bar{b}$ are orthogonal to eachother,

$$\bar{a} \cdot \bar{b} = 0,$$

orthogonal, $\cos 90° = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$

$$0 = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$= \bar{a}\cdot\bar{b} \,//.$$

$= (a-2)\cdot -8 = 0$

$= -8a-16=0$

$= -2(4a-8)=0$

$= 4a=8$

$\boxed{a=2}$

$-2b\cdot 4=0$

$-8b=0$

$\boxed{b=1/8}$

$(a-2)\cdot(-8)+(-2b)(4)+b(12)=0$

$-8a-16-8b+12b=0$

$-8a+16+4b=0$

$-8a \quad +4b = -16$

$$\boxed{-2a + b = -4} \longrightarrow (3)$$

put $(1,-1,2)$ in $\quad ax^2 - byz = (a+z)x$

$$= \cancel{a}(1) + 2b = (\cancel{a} + 2)$$

$$2b = 2$$

$$\boxed{b = 1}$$

subs $b=1$ in (3),

$$-2a + 1 = -4$$

$$\cancel{-}2a = \cancel{+}5$$

$$\boxed{a = +5/2}$$

$$\nabla \bar{f} = -4\hat{j} + \frac{\bar{k}}{2}.$$

directional derivative $= \dfrac{\nabla\phi \cdot \nabla\bar{f}}{|\nabla \bar{f}|}$

$$= \frac{\cancel{\hat{j}+10\bar{k}}\ \cancel{\hat{j}+7k} \cdot -4\hat{j}+\hat{k}/2}{\sqrt{16+1/4}} = \frac{-8\hat{j}+k}{\sqrt{65}}$$

$$= \cancel{\frac{11}{\sqrt{11}}} \ //.$$

Part-B

$$DD = \frac{1}{\sqrt{65}}(-8\hat{j}+k)(8i-j-10k)$$

3.

$$= \frac{8-10}{\sqrt{65}} = \frac{-2}{\sqrt{65}} //.$$

given, $A = 3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k.$

given, solenoidal,

$\nabla \cdot \bar{f} = 0$

$$\bar{F} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j}\ \cancel{k} - 3x^2y^2k$$

$$\bar{f_1} = 3y^4z^2$$

$$\bar{f_2} = 4x^3z^2$$

$$\bar{f_3} = -3x^2y^2$$

$$\text{div}\,\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 0 + 0 + 0$$

$$= 0 \,//.$$

$\bar{F}$ is solenoidal, $\nabla\cdot\bar{f} = 0 \,//.$

4. $\bar{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)k.$

Irrotational,

$\nabla\times\bar{f} = 0$

$\bar{F}_1 = 6xy+z^3$

$\bar{F}_2 = 3x^2-z$

$\bar{F}_3 = 3xz^2-y$

$$\text{Curl} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$\bar{i}\left[\partial/\partial y(3xz^2-y) - \partial/\partial z(3x^2-z)\right] - \bar{j}\left[\partial/\partial x(3xz^2-y) - \partial/\partial z(6xy+z^3)\right] + \bar{k}\left[(\partial/\partial x(3x^2-z) - \partial/\partial y(6xy+z^3)\right]$$

$$= i[(\cancel{1}-\cancel{1})] - j[\cancel{3z^2} - \cancel{3z^2}] + k[\cancel{6x} - \cancel{6x}]$$

$$= 0 //.$$

$$\nabla \times \bar{f} = 0.$$

The given $\bar{F}$ is Irrotational.

Scalar function, $\bar{f} = \nabla \phi$.

$$(6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k} = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 6xy+z^3$$

$$\frac{\partial \phi}{\partial y} = 3x^2-z$$

$$\frac{\partial \phi}{\partial z} = 3xz^2-y$$

Integrate on BS,

$$\phi = \int (6xy+z^3)\,dx + c$$

$$= \frac{x^2}{2}6y + z^3x + c \quad \rightarrow (1)$$

$$\phi = \int (3x^2-z)\,dy + c$$

$$= 3x^2y - zy + c \quad \rightarrow (2)$$

$$\phi = \int (3xz^2 - y)\,dz + c$$

$$= \frac{z^3}{\cancel{3}} \cdot \cancel{3}x - zy + c \quad \text{--- } (3)$$

from (1), (2), (3),

$$3\phi = 3x^2y + z^3x + 3x^2y - zy + z^3x - zy + c$$

$$\cancel{3}\phi = 6x^2y + 2z^3x - 2zy + c$$

$$\phi = \frac{6x^2y + 2z^3x - 2zy + c}{3}$$

$$\phi = 2x^2y + \frac{2}{3}z^3x - \frac{2}{3}zy + c$$

5. given,

$$\nabla f = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}.$$

$$F(1,0,1) = 8.$$

$$\int (y^2 - 2xyz^3)\,dx = xy^2 - 2yz^3 + c_1$$

$$\int (3 + 2xy - x^2z^3)\,dy = 3y + \cancel{2}xy^2 - x^2yx^3 + c_2$$

$$\int (6z^3 - 3x^2yz^2)\,dz = \frac{6}{4}z^4 - \frac{\cancel{3}x^2yz^3}{\cancel{3}} + c_3$$

$$f(x,y,z) = xy^2 - x^2yz^3 + 3y + xy^2 - x^2yz^3 + \frac{3}{2}z^4 - x^2yz^3 + c$$

$$\Rightarrow \quad 2xy^2 - 3x^2yz^3 + 3y + \frac{3}{2}z^4 + c.$$

We know,

$$f(1,0,1) = 8.$$

$$8 = 2(1)(0)^2 - 3(1)^2(0)(1)^3 + 3(0) + \frac{3}{2}(1)^4 + c$$

$$8 = \frac{3}{2} + c$$

$$c = 8 - \frac{3}{2}$$

$$c = \frac{13}{2} \;//.$$

$$f = 2xy^2 - 3x^2yz^3 + 3y + \frac{3}{2}z^4 + \frac{13}{2} \;//.$$

6.

given,

$$\bar{V} = e^{xyz}(\bar{i} + \bar{j} + \bar{k})$$

$$\bar{V} = e^{xyz}\bar{i} + e^{xyz}\bar{j} + e^{xyz}\bar{k}$$

Curl $\bar{V} \Rightarrow \nabla \times \bar{f} =$

$$\nabla \times \bar{q} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$$

$$i \left[\partial/\partial y (e^{xyz}) - \partial/\partial z (e^{xyz}) - j \left(\partial/\partial x (e^{xyz}) - \partial/\partial z (e^{xyz}) \right] \right.$$

$$+ \bar{k} \left[\partial/\partial x (e^{xyz}) - \partial/\partial y (e^{xyz}) \right]$$

$$= (xz.e^{xyz} - xy.e^{xyz})\hat{i} - \{ yz.e^{xyz} - xy.e^{xyz} \} \hat{j}$$

$$+ (yze^{xyz} - xze^{xyz})k$$

at point (1,2,3)

$$= (3.e^6 - 2e^6)\hat{i} - (6.e^6 - 2e^6)\hat{j} + (6e^6 - 3e^6)k$$

$$= (e^6)\hat{i} - (4e^6)\hat{j} + (3e^6)\hat{j}$$

8. $= e^6 (\hat{i} - 4\hat{j} + 3\hat{j})$

given,

$$\bar{f} = x^2yz,.$$

$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) f$$

$$= \bar{i}(2xyz) + \bar{j}(x^2z) + \bar{k}(x^2y).$$

$$\bar{g} = xy - 3z^2,$$

$$\nabla g = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) g$$

$= \bar{i}(y) + \bar{j}(x) + k(-6z)$

$= y\bar{i} + x\bar{j} - 6z\bar{k}.$

$\nabla f \cdot \nabla g = (2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}) \cdot (y\hat{i} + x\hat{j} - 6z\hat{k})$

$= 2xy^2z + x^3z - 6x^2yz.$

$\nabla(\nabla f \cdot \nabla g) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(\nabla f \cdot \nabla g)$

$= i(2y^2z + 3x^2z - 12xyz) + \bar{j}(4xyz - 6x^2z) + k(2xy^2 + x^3 - 6x^2y)$ //.

9.

given,

$\phi_1 = x^2 + y^2 + z^2 - 29$

$\phi_2 = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47.$

$\nabla\phi_1 = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$

$\nabla\phi_2 = (2x+4)\hat{i} + \hat{j}(2y-6) + k(2z-8)$

$\nabla \phi_1$ at $(4,3,2)$

$\hookrightarrow \quad 8\hat{i} + 6\hat{j} + 4k$

$\nabla \phi_2$ at $(4,3,2)$

$\hookrightarrow \quad 12\hat{i} - 4\hat{k}$

$$\cos\theta = \frac{(8\hat{i} + 6\hat{j} + 4\hat{k})(12\hat{i} - 4\hat{k})}{\sqrt{116} \cdot \sqrt{160}}$$

$$\cos\theta = \frac{80}{\sqrt{116} \cdot \sqrt{160}}$$

$$\theta = \cos^{-1}\left(\frac{80}{\sqrt{116}\sqrt{160}}\right).$$

10.

given,

$$xy = z^2,$$

$$\phi = xy - z^2.$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial k}\right)\phi$$

$$= y\hat{i} + x\hat{j} - 2z\hat{k}$$

at $(4,1,2)$

$$= \hat{i} + 4\hat{j} - 4\hat{k}$$

at $(3,3,-3)$.

$$3\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\cos\theta = \frac{(\hat{i}+4\hat{j}-4\hat{k})\cdot(3\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{16+16+1}\ \sqrt{9+9+36}}$$

$$= \frac{3+12-24}{\cancel{\sqrt{33}\sqrt{3}}\ \sqrt{33}\ \sqrt{54}}$$

$$= \frac{-3}{\sqrt{33}\ 3\sqrt{6}}$$

$$\cos\theta = \frac{-3}{\sqrt{33}\ \sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{33}\sqrt{6}}\right)$$

11.

given,

$$\phi = 2x^2 + 3y^2 - 5z$$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= 4x\hat{i} + 6y\hat{j} - 5\hat{k}$$

at $(2,-2,4)$,

$$= 8\hat{i} - 12\hat{j} - 5\hat{k}$$

at $(-1,-1,1)$

$$= -4\hat{i} - 6\hat{j} - 5\hat{k}$$

$$\cos\theta = \frac{(8\hat{i}-12\hat{j}-5\hat{k})\cdot(-4\hat{i}-6\hat{j}-5\hat{k})}{|8\hat{i}-12\hat{j}-5\hat{k}||-4\hat{i}-6\hat{j}-5\hat{k}|}$$

$$= \frac{-36+72+25}{\sqrt{233}\sqrt{67}}$$

$$\cos\theta = \frac{61}{\sqrt{233}\sqrt{67}}$$

$$\theta = \cos^{-1}\left(\frac{61}{\sqrt{233}\sqrt{67}}\right)$$ //.

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find DD,

$\phi = xy^2 + yz^3$

Point = $(1,-2,-1)$.

$$\nabla\phi_1 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= 2xy\,\hat{i} + (2yx + z^3)\hat{j} + 3yz^2\,\hat{k}$$

at $(1,-2,-1)$

$$\Rightarrow -4\hat{i} + (-5)\hat{j} + (-6)\hat{k}$$

$$= -4\hat{i} - 5\hat{j} + 6\hat{k}$$

~~Cos φ =~~

$$\cos\theta = \frac{(-4\hat{i} - 5\hat{j} + 6\hat{k})\cdot(-4\hat{i} - 5\hat{j} + 6\hat{k})}{\sqrt{16 + 25 + 36}}$$

$$= \frac{16 + 25 + 36}{\sqrt{77}}$$

$$= \frac{77}{\sqrt{77}}$$

$$= \sqrt{77}$$

$$\theta = \cos^{-1}(\sqrt{77})$$

13 given,

$$\bar{A} = (2x+3y+az)\hat{i} + (bx+2y+3z)\hat{j} + (2x+cy+3z)\hat{k}$$ is Irrotational,

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f} = 0$$
↳ Irrotational.

$$\nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$$= i\left[\partial/\partial y(2x+cy+3z) - \partial/\partial z(bx+2y+3z)\right] - \bar{j}\left[\partial/\partial x(2x+cy+3z) - \partial/\partial z(2x+3y+az)\right] + \bar{k}\left[\partial/\partial x(bx+2y+3z) - \partial/\partial y(2x+3y+az)\right]$$

$$= i[c-3] - \bar{j}[2-a] + \bar{k}[b-3].$$

Irrotational, $\bar{f} = 0$,

$c-3=0$	$-2-a=0$	$b-3=0$
$\boxed{c=3}$	$\boxed{a=+2}$	$\boxed{b=3}$

Scalar function, $\bar{f} = \nabla \phi$.

$$(2x+3y+2z)\hat{i} + (3x+2y+3z)\hat{j} + (2x+3y+3z)\hat{k}$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2x+3y+2z$$

$$\frac{\partial \phi}{\partial y} = 3x+2y+3z$$

$$\frac{\partial \phi}{\partial z} = 2x+3y+3z$$

Integrate on BS,

$$\cancel{\phi = \int 2x\,dx + c}$$

$$\cancel{= \frac{2x^2}{2} + c} \rightarrow \cancel{1}$$

$$\phi = \int (2x+3y+2z)\,dx + c$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} + 3xy + 2xz + c \rightarrow 1$$

$$\phi = \int (3x+2y+3z)\,dy + c$$

$$= 3xy + \frac{\cancel{2}y^2}{\cancel{2}} + 3zy + c \rightarrow 2$$

$$\phi = \int (2x+3y+3z)\, dz + c$$

$$= 2xz + 3yz + \frac{3z^2}{2} + c$$

$$\phi = x^2 + 3xy + 2xz + 3xy + y^2 + 3zy + 2xz$$
$$+ 3yz + \frac{3z^2}{2} + c$$

$$= x^2 + y^2 + \frac{3z^2}{2} + 6xy + 4xz + 6yz + c$$

$$= 2x^2 + 2y^2 + 3z^2 + 12xy + 8xz + 12yz + 2c //.$$

14.

given,

$$\bar{f} = xyz^2\hat{i} + yzx^2\hat{j} + 2xy^2\hat{k}.$$

Curl $= \nabla \times \bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & yzx^2 & 2xy^2 \end{vmatrix}$$

$$= \bar{i}\left[\partial/\partial y\,(2xy^2) - \partial/\partial z\,(yzx^2)\right] - \bar{j}\left[\partial/\partial x\,(2xy^2) - \partial/\partial z\,(xyz^2)\right] + \bar{k}\left[\partial/\partial x\,(yzx^2) - \partial/\partial y\,(xyz^2)\right]$$

$$= i\left[2yzx - x^2y\right] - \bar{j}\left[y^2z - 2zxy\right] + \bar{k}\left[2xyz - z^2x\right]$$

$= i[2yzx - x^2y] - j[y^2z - 2xyz] + k[2xyz - z^2x]$

at point $(1, 2, 3)$,

$i[12 - 2] - j[12 - 12] + k[12 - 9]$

$= 10\hat{i} + 3\hat{k}$ //.

15.

given,

$\bar{A} = (bxy - z^3)\hat{i} + (b-2)x^2\hat{j} + (1-b)xz^2k$

curl $\bar{f} = \nabla \times \bar{f}$, $\boxed{b = ?}$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bxy - z^3 & (b-2)x^2 & (1-b)xz^2 \end{vmatrix}$$

$\bar{i}[\partial/\partial y\,(1-b)xz^2 - \partial/\partial z\,(b-2)x^2] - \bar{j}[\partial/\partial x\,((1-b)$

$xz^2 - \partial/\partial z\,(bxy - z^3)] + \bar{k}[\partial/\partial x\,(b-2)x^2 - \partial/\partial y\,(bxy-z$

$i[0 - 0] + j[(1-b)z^2 + 3z^2] + \bar{k}[(b-2)2x - bx]$

$$= \hat{j}\left[z^2 - z^2 b + 3z^2\right] + \hat{k}\left[2xb - 4x - bx\right]$$

given,

Curl $\bar{f} = 0$,

$$z^2 - z^2 b + 3z^2 = 0,$$

$$z^2(1 - b + 3) = 0$$

$$4 - b = 0$$

$$\boxed{b = 4}$$

$$2xb - 4x - bx = 0$$

$$x(2b - 4 - b) = 0$$

$$b - 4 = 0,$$

$$\boxed{b = 4}$$

17.

given,

$$a = x + y + z$$

$$b = x^2 + y^2 + z^2$$

$$c = xy + yz + zx$$

$$\nabla a = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla b = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

$$\cancel{\nabla c = y\hat{i} + z\hat{j} + x\hat{k}}$$

$$\nabla c = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ & & \\ & & \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$\cancel{1(2y(x+z) - 1(2x}$

$1(2y(x+y) - 2z(x+z)) - \bar{j}(2x(x+y) - 2z(y+z)) +$
$1(2x(x+z) - 2y(y+z))$.

$= \cancel{2xy} + \cancel{2y^2} - \cancel{2zx} - \cancel{2z^2} - \cancel{2x^2} - \cancel{2xy} + \cancel{2yz} + \cancel{2z^2}$
$+ \cancel{2x^2} + \cancel{2zx} - \cancel{2y^2} - \cancel{2yz}$

$= 0$ //.

19.

Surface, $xy^2z = 3x + z^2$

$xy^2z - 3x - z^2 = 0.$

point $(1, -2, 1)$.

$\bar{f} = 2x^3y^2z^4$

$$\nabla \bar{f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

$$\nabla \bar{f} = (6x^2y^2 \cdot z^4)\hat{i} + (4x^3yz^4)\hat{j} + (8x^3y^2z^3)\hat{k}$$

at point $(1,-2,1)$,

$$\nabla f = (6(1)^2(-2)^2 \cdot (1)^4)\hat{i} + (4(1)^3(-2)(1)^4)\hat{j}$$
$$+ (8(1)^3(-2)^2(1)^3)\hat{k}.$$

$$= -24\hat{i} - 8\hat{j} + 32\hat{k}$$

$$= -8(3\hat{i} + \hat{j} - 4\hat{k}).$$

$$\nabla f = 3\hat{i} + \hat{j} - 4\hat{k}$$

$$\frac{\partial f_1}{\partial x} = 3$$

$$\frac{\partial f_2}{\partial y} = 1$$

$$\frac{\partial f_3}{\partial z} = -4$$

$$\text{div } \bar{f} = \nabla \cdot (\nabla f) =$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 3 + 1 - 4$$

$$= 0 //.$$