MECHANICS OF SOLIDS

B. Tech III semester (Autonomous IARE BT23)

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Whenever a load is attached to a thin hanging wire, it elongates and the load moves downwards (sometimes through a negligible distance).

The amount, by which the wire elongates, depends upon the amount of load and the nature as well as cross- sectional area of the wire material.

It has also been observed that the process of deformation stops when the force of resistance is equal to the external force (i.e., the load attached).

When, the force of resistance, offered by the molecules, is less than the external force. In such a case, the deformation continues until failure takes place.

The resistance by which the material of the body opposes the deformation is known as **strength of the materials**

MECHANICS OF SOLIDS – COURSE OVERVIEW

Mechanics of solids deals with deformable solids, requires basic knowledge of principles of mechanics from Engineering Mechanics course and acts as a pre-requisite to the advanced courses on Aircraft structures and Analysis of aircraft structures.

This course introduces the concepts of simple stresses, strains and principal stresses on deformable solids and focuses on the analysis of members subjected to axial, bending and tensional loads.

In a nutshell, the course aims at developing the skill to solve engineering problems on strength of materials.

Eventually, through this course content, engineers can analyze the response of various structural members under different loading conditions and design the same, satisfying the safety and serviceability conditions

Student will try to learn

I	The concepts of mechanics of deformable solids and their constitutive relations (including stress – strain relations), principal stresses and strains, and resilience produced under various loading conditions used in predicting the strength of aircraft structures.
II	The procedure of estimating shear force - bending moment, twisting moment, flexural Stresses, shear stresses in beams subjected to various loadings, for designing the shape, size and material of aircraft components.
III	The methods for determining the slope and deflection of beams and critical load on columns subjected to various loading conditions for determining the stiffness and strength of aircraft structures.
IV	The methods of failures and distribution of stresses in cylinders due to internal pressure.

MECHANICS OF SOLIDS – COURSE OUTCOMES

After successful completion of the course, students should be able to:

CO 1	**Demonstrate** the concepts of stress-strain, material constitutional relationship and strain energy induced in isotropic materials under different loadings.
CO 2	**Make use of** the concepts of shear force and bending moment in beams, and power transmission in shafts for analyzing the strength under different loadings.
CO 3	**Apply** theory of distribution of bending stresses and shearing stress across the beams for the safe design of aircraft components subjected different loadings.
CO 4	**Utilize** Maxwell's reciprocal theorem and double integration for determining the slope and deflections in beams under different types of loadings.
CO 5	**Utilize** Euler's formula for determining the buckling load in columns under different end conditions.
CO 6	**Apply** the concepts of longitudinal and circumferential stresses induced in cylinders for the safe design under inside and outside pressure.

MODULE – I: STRESSES AND STRAINS (09)

Elasticity and plasticity – Types of stresses and strains–Hooke's law– stress – strain diagram for mild steel – Working stress – Factor of safety – Lateral strain, Poisson's ratio, volumetric strain – Elastic module and the relationship between them – Bars of varying section – composite bars – Temperature stresses. Strain energy – Resilience – Gradual, sudden, impact and shock loadings.

MODULE – II: SHEAR FORCE AND BENDING MOMENT (09)

Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, u.d.l., uniformly varying loads and combination of these loads – Point of contra flexure – Relation between S.F., B.M and rate of loading at a section of a beam.

TORSION OF CIRCULAR SHAFTS: Theory of pure torsion – Derivation of Torsion equations: $T/J = q/r = C\theta/L$ – Assumptions made in the theory of pure torsion – Torsional moment of resistance –Polar section modulus – Power transmitted by shafts;

MODULE – III: FLEXURAL STRESSES AND SHEAR STRESSES (09)

Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/y = E/R Neutral axis – Determination bending stresses – section modulus of rectangular and circular sections (Solid and Hollow), I,T,Angle and Channel sections – Design of simple beam sections.

Derivation of formula – Shear stress distribution across various beams sections like rectangular, circular, triangular, I, T angle sections

MODULE –IV: DEFLECTION OF BEAMS AND COLUMNS (09)

Deflection in simply supported beams and cantilever beams with concentrated loads, uniformly distributed loads and their combination using Double integration method and Macaulay 's method,

Columns and Struts: Introduction; Failure of a column; Euler's column theory; assumptions in the Euler's column theory; Sign conventions; types of end conditions of the columns; columns with both ends hinged; column one end fixed and other end free; columns with both ends fixed; columns with one end fixed and other end hinged; Euler's formula and equivalent length of a column; slenderness ratio; limitations of Eluer's formula, Empirical formulae for columns; Rankine's formulae; Johnson's formula for columns; Indian Standard code for columns

MODULE –V: PRINCIPAL STRESSES, STRAINS AND CYLINDERS (09)

Introduction – Stresses on an inclined section of a bar under axial loading – compound stresses – Normal and tangential stresses on an inclined plane for biaxial stresses – Two perpendicular normal stresses accompanied by a state of simple shear – Mohr's circle of stresses – Principal stresses and strains – Analytical and graphical solutions. Thin Cylinders: Thin seamless cylindrical shells – Derivation of formula for longitudinal and circumferential stresses – hoop, longitudinal and volumetric strains – changes in diameter, and volume of thin cylinders due to internal pressure; Thick Cylinders: Lame's equation- cylinders subjected to inside and outside pressures

TEXT BOOKS:

1. R. K Bansal, "Strength of Materials", Laxmi publications, 6th edition, 2018.
2. T. H. G. Megson, "Aircraft Structures for Engineering Students", Butterworth-Heinemann Ltd, 6th edition, 2015.
3. Gere, Timoshenko, "Mechanics of Materials", McGraw Hill, Reprint 2020.

Useful Data - SI UNITS

In this system of units, the fundamental units are metre (m), kilogram (kg) and second (s).

- Density or Mass density : kg/m^3,
- Force (in Newtons) : N (= $kg.m/s^2$)
- Pressure (in Pascals) : Pa (= $N/m^2 = 10^{-6}\ N/mm^2$)
- Stress (in Pascals) : Pa (= $N/m^2 = 10^{-6}\ N/mm^2$)
- Work done (in Joules) : J (= N-m)
- Power (in Watts) : W (= J/s)

Introduction

- The practical application of engineering materials in design and manufacturing engineering depends upon thorough knowledge of the material properties under wide range of conditions.
- The term property is a qualitative or quantitative measure of response of materials to externally imposed conditions like forces and temperature.

Classification Of Material Properties

- Mechanical properties
- Thermal properties
- Optical properties
- Physical properties
- Magnetic properties
- Chemical properties
- Electrical properties

Importance Of Knowledge Of Material Properties

- An engineer must have an intimate knowledge of the properties and behavioral characteristics of the materials that he intends to use.
- While designing a product, the engineer needs to select right materials to create the product.

Engineering Materials – Mechanical Properties

The mechanical properties of the metals are those which are associated with the ability of the material to resist mechanical forces and load.

- The properties of materials that determines its behaviour under applied forces are called mechanical properties.
- They are usually related to the elastic and plastic behaviour of the material.
- The mechanical properties are generally expressed as the function of stress-strain etc.
- Knowledge of mechanical properties provides the basis for predicting behaviour of materials under different load conditions.

Mechanical Properties of Materials are evaluated by the following tests:

1. Tensile test 2. Compressive test 3. Impact test and 4. Fatigue test

Engineering Materials – Mechanical Properties – Classification

1	Elasticity	7	Hardness	13	Fatigue
2	Strength	8	Brittleness	14	Toughness
3	Stiffness	9	Creep	15	Resilience
4	Plasticity	10	Formability	16	Machinability
5	Ductility	11	Castability	17	Thermal Conductivity
6	Malleability	12	Weldability	18	Electrical resistivity
				19	Electrical conductivity

1. ELASTICITY

- It is defined as the property of a material to regain its original shape after deformation when the external forces are removed.
- It is also called as the tensile property of the material.
- This property is desirable for materials used in tools and machines. It may be noted that steel is more elastic.

2. STRENGTH

- Strength is defined as the ability of a material to resist the externally applied forces without breakdown or yielding.
- This property of material, therefore, determines the ability to withstand stress without failure.
- Strength varies according to the type of loading such as tensile, compressive, shearing, and torsional strengths.

3. STIFFNESS

- It is defined as the ability of a material to resist deformation under stress. The resistance of a material to elastic deformation or deflection is called stiffness or rigidity. The modulus of elasticity is the measure of stiffness.
- A material that suffers slight or very less deformation under load has a high degree of stiffness or rigidity, and the higher is the value of Young's modulus.
- In tensile and compressive stress, it is called modulus of stiffness or modulus of elasticity; in shear stress, the modulus of rigidity, and in volumetric distortion, the bulk modulus.

4. PLASTICITY

- Plasticity is defined as the mechanical property of a material that retains the deformation produced under load permanently without rupture or failure. The deformation will take only after the elastic limit is exceeded.
- This property of the material is required in forging, stamping images on coins and ornamental work.
- Materials such as clay, lead, etc. are plastic at room temperature and steel is plastic at forging temperature. This property generally increases with an increase in the temperature of materials.

5. DUCTILITY

- Ductility is termed as the property of a material enabling it to be drawn into the wire with the application of tensile load. A ductile material must be strong and plastic.
- The ductility is usually measured by the terms of percentage elongation and percent reduction in area.
- The materials that possess more than 5% elongation are called as ductile materials.
- The ductile material commonly used in engineering are mild steel, copper, aluminum, nickel, zinc, tin, and lead.
- The materials that possess less than 5% elongation are called as brittle materials, ex: Ceramics.

6. MALLEABILITY

- Malleability is the ability of the material to be flattened into thin sheets under applications of heavy compressive forces without cracking by hot or cold working.
- It is a special case of ductility which permits materials to be rolled or hammered into thin sheets.
- The malleable materials commonly used in engineering practice are lead, soft steel, wrought iron, copper, aluminum, etc.

7. HARDNESS:

- Hardness is defined as the ability of a metal to cut another metal.
- It is a very important property of the metals and has a wide variety of meanings such as resistance to wear, scratching, deformation, and machinability, etc.
- The hardness is usually expressed in numbers be determined by the following tests: (a) Brinell hardness test, (b) Rockwell hardness test, (c) Vickers hardness test and (d) Shore scleroscope.

8. BRITTLENESS

- Brittleness is the property of a material opposite to ductility. It is the property of breaking of a material with little permanent distortion. The materials having less than 5% elongation under loading behavior are said to be brittle materials.
- If a material cannot undergo any deformation (like glass, china-ware, etc.) when some external forces act on it and it fails by rupture, it is called a brittle material.
- Glass, cast iron, brass, and ceramics are considered brittle material.

9. CREEP

- When a metal part when is subjected to a high constant stress at high temperature for a longer period, it will undergo a slow and permanent deformation called creep. This property is considered in designing internal combustion engines, boilers, and turbines.

10. FORMABILITY

- It is the property of metals that denotes the ease in its forming into various shapes and sizes.
- The different factors that affect the formability are crystal structure of the metal, the grain size of metal hot and cold working, alloying element present in the parent metal.

11. CASTABILITY

- Castability is defined as the property of metal, which indicates the ease with it can be cast into different shapes and sizes. Cast iron, aluminum, and brass are possessing good castability.

12. WELDABILITY

- Weldability is defined as the property of a metal which indicates the two similar or dissimilar metals are joined by fusion with or without the application of pressure and with or without the use of filler metal efficiently.
- Metals having weldability in the descending order are iron, steel, cast steels, and stainless steels.

13. FATIGUE

- When a material is subjected to repeated stresses, it fails at stresses below the yield point stresses. This type of failure of material is known as fatigue.

14. TOUGHNESS

- Toughness. It is the property of a material to resist fracture due to high impact loads like hammer blows. The toughness of the material decreases when it is heated.
- It is measured by the amount of energy that a unit volume of the material has absorbed without rupturing.
- This property is desirable in parts subjected to shock and impact loads. The toughness of the material decreases when it is heated.

15. RESILIENCE

- It is the property of a material to absorb energy and to resist shock and impact loads. It is measure by the amount of energy absorbed per unit volume within the elastic limit. This property is essential for spring materials

16 MACHINABILITY.

- It is the property of a material which refers to a relative easy with which material can be cut.
- It measured in a number of ways such as comparing the **tool life** for cutting different materials or **thrust required** to remove the material at some given rate or the **energy required** to remove a unit volume of the material. It may be noted that brass can be easily machined than steel.

17. THERMAL CONDUCTIVITY:

- This is the ability of the material to transmit heat energy by conduction

18. ELECTRICAL RESISTIVITY :

- It is the property of a material by which it resists the flow of electricity through it.

19. ELECTRICAL CONDUCTIVITY:

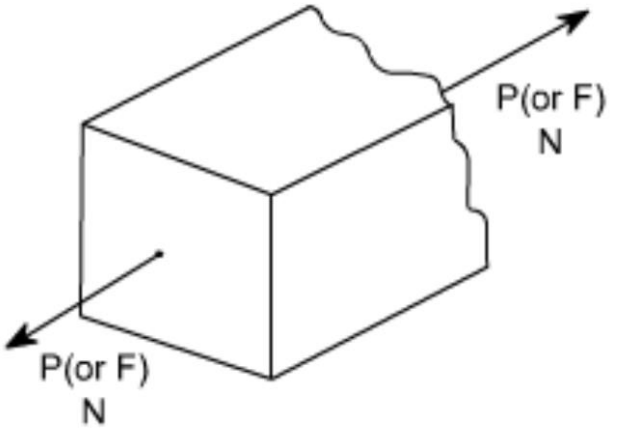
- It is the property of a material due to which it allows the flow of electricity through it.

STRESS - Definition

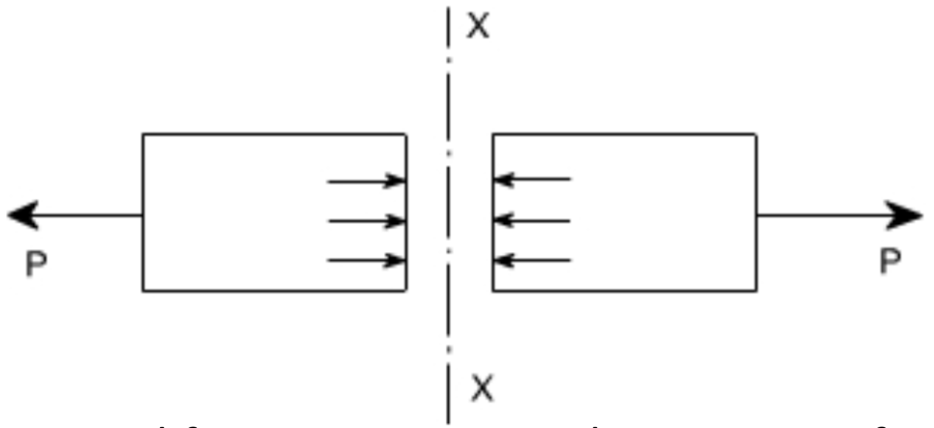
Stress is the internal resistance offered by the body to the external load applied per unit cross sectional area. Stress is represented with σ. $\sigma = \frac{P}{A}$

- **Stresses are normal to the plane to which they act and are tensile or compressive in nature.**

Consider a rectangular rod subjected to an axial pull P.



- Let us imagine that the same rectangular bar is assumed to be cut into two halves at section *XX*.
- Each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section *XX* has been shown.



Units: The basic units of stress in S.I units are N/m^2 (or Pa)

Kilo Pascal, KPa = 10^3 Pa

Mega Pascal, MPa = 10^6 Pa = N/mm^2

Giga Pascal, GPa = 10^9 Pa

US customary unit is pound per square inch (psi).

- These internal forces give rise to the concept of stress.

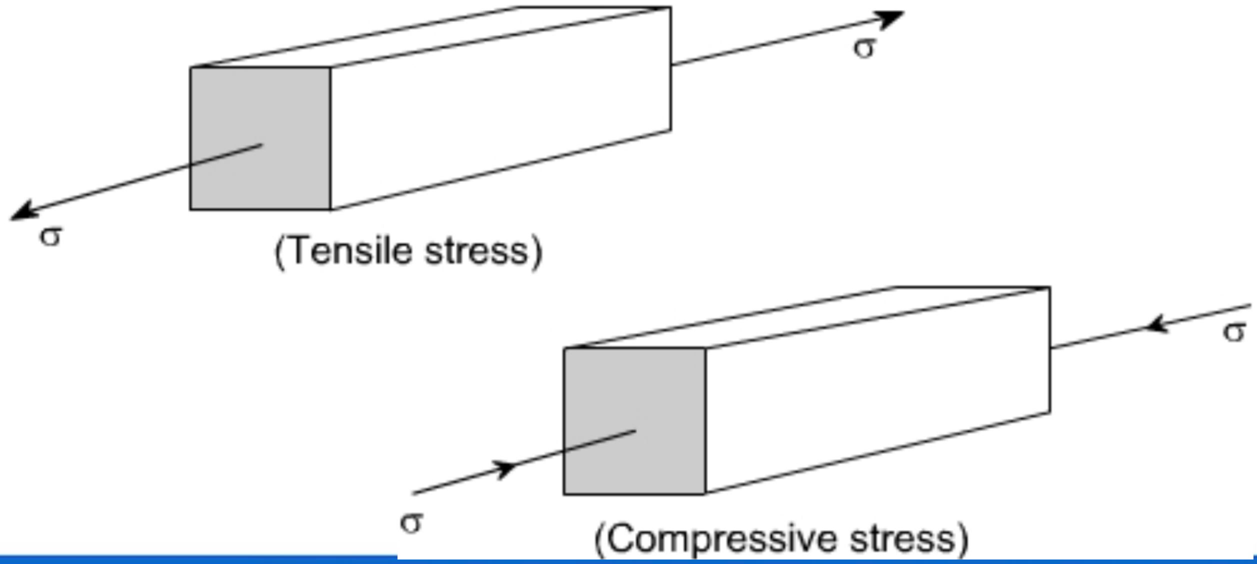
Types of Stresses

Major classification of stresses are:

(i) Normal Stresses

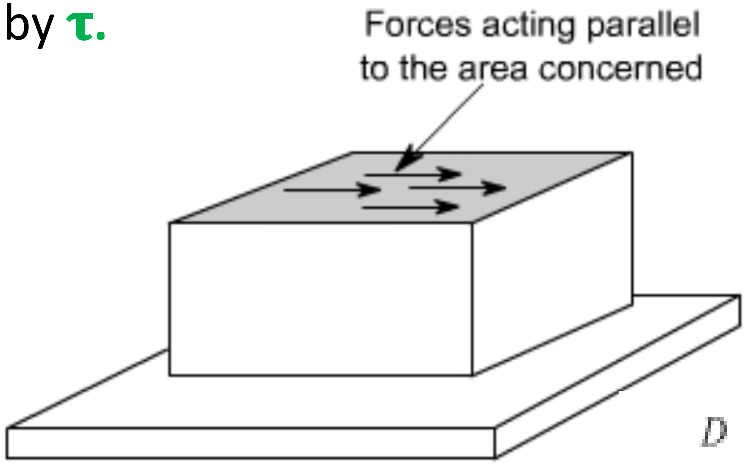
The stresses normal to the area concerned are termed as normal stresses and are generally denoted by **σ.**

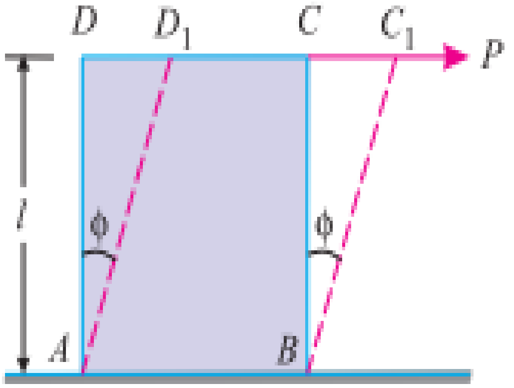
Tensile Stresses and Compressive Stresses:
The normal stresses can be either tensile or compressive based on whether the stresses acts out of the area or into the area



(ii) Shear Stresses

The stresses along (parallel) the area concerned are termed as shear stresses and are generally denoted by **τ.**





Strain - Definition

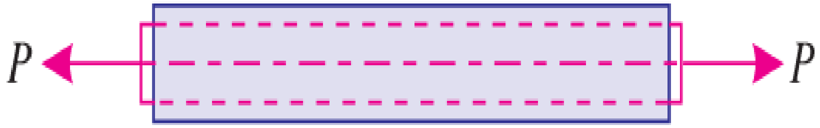
- When a force (or a system of forces) acts on a body, it undergoes deformation.
- This deformation per unit length is known as strain.
- Strain is represented with **ϵ or ε**
- Mathematically, **strain (ϵ) =δl/l.**

  where δl = Change of length of the body, and l = Original length of the body.

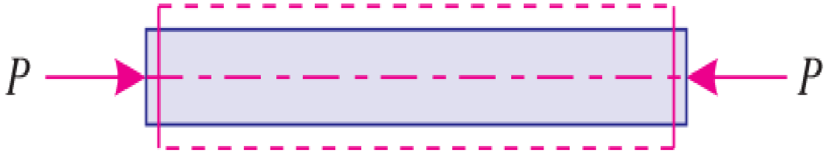
Strain is a unitless quantity

Types of Strains:

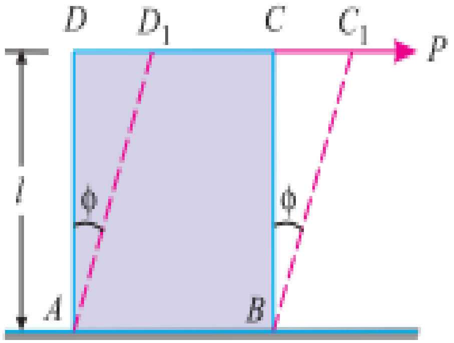
- **Tensile strain**
- **Compressive strain**
- **Shear strain**
- **Volumetric strain**
- **Thermal strain**



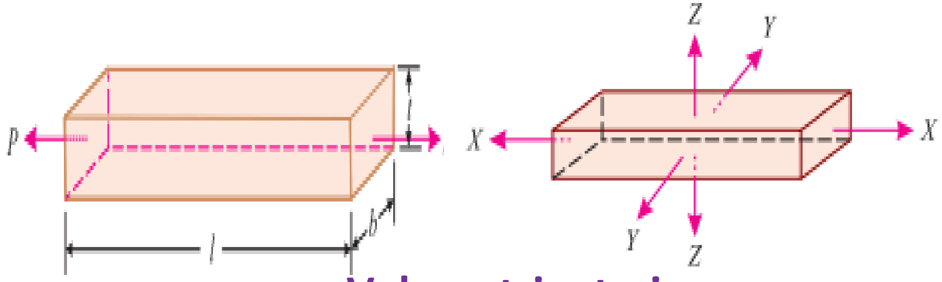
Tensile strain



Compressive strain



Shear strain



Volumetric strain

Hooke's Law

- In 19th-century, English scientist Robert Hooke noticed that many materials exhibited a similar property when the stress-strain relationship was studied. This is known as Hooke's Law.

Hooke's Law states that, "the strain of the material is proportional to the applied stress within the elastic limit of that material".

Mathematically,

$$\sigma \propto \epsilon$$

$$\sigma = E \times \epsilon$$

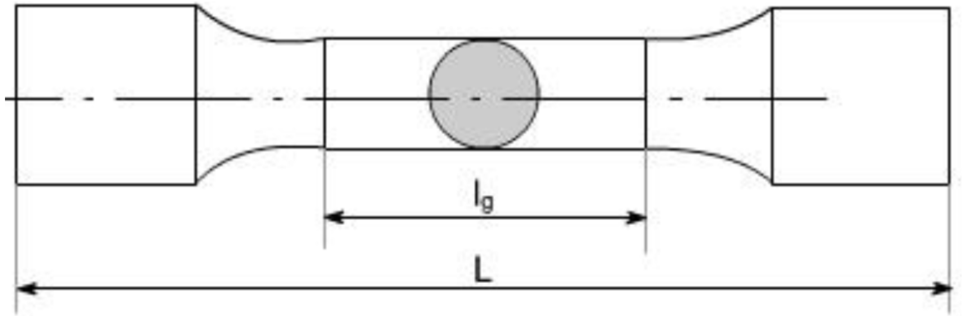
$$\text{or} \quad E = \sigma/\epsilon$$

Where,

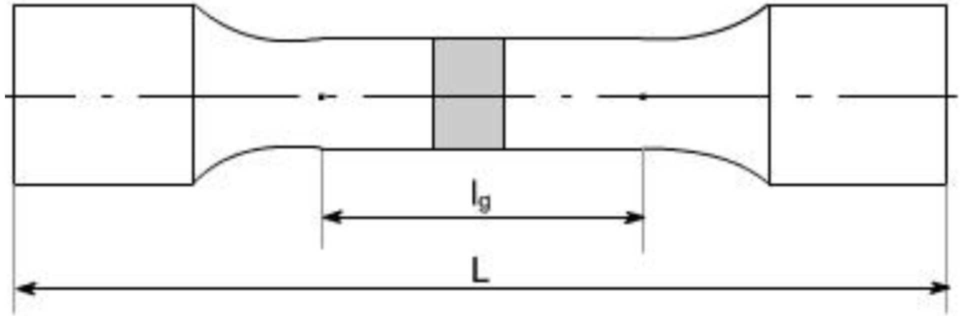
- σ = Stress; ϵ = Strain
- E = Constant of proportionality known as modulus of elasticity (or) Young's modulus.

Uniaxial Tension Test

Standard specimens are used for the tension test.



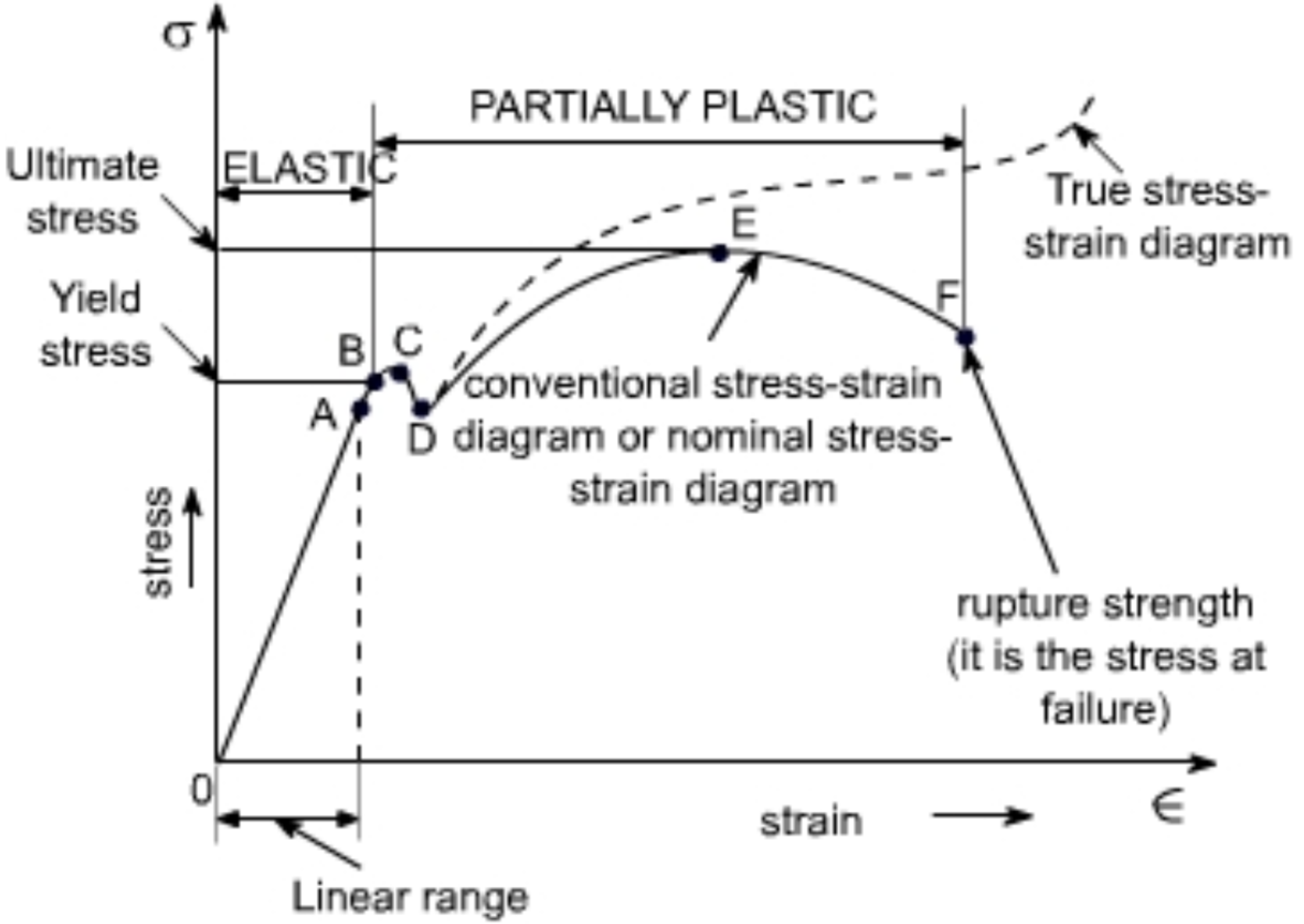
[specimen with circular X-section]



[specimen with rectangular X-section]

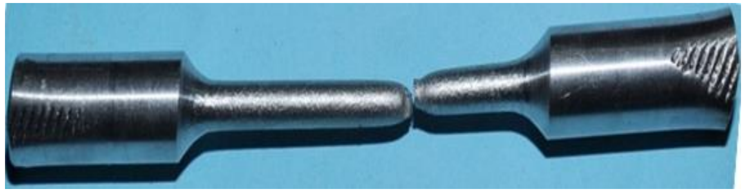
l_g = gauge length i.e. length of the specimen on which we want to determine the mechanical properties

Stress – Strain Curve for Mild Steel



Salient points of the graph

A	Proportional Limit
B	Elastic Limit
C&D	Yield Point
E	Ultimate Stress Point
F	Fracture or Breaking Point

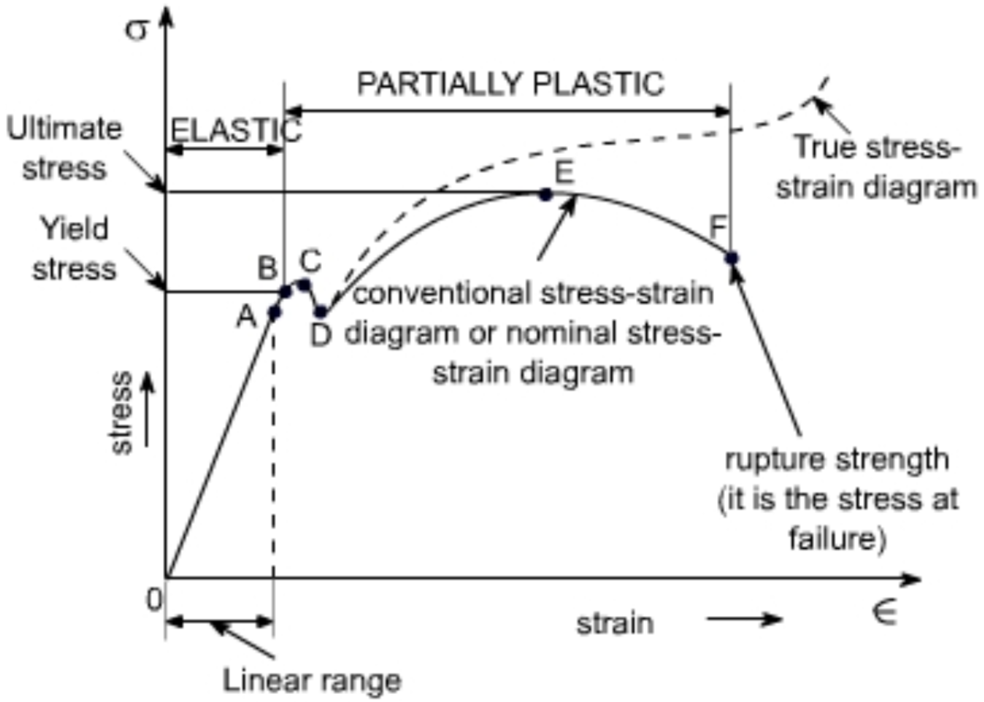


Stress – Strain Curve for Mild Steel

A. Proportional Limit: It is the region in the stress-strain curve that obeys the Hooke's Law. In this limit, the ratio of stress with strain gives proportionality constant known as young's modulus.

B. Elastic Limit: It is the region in the stress-strain curve upto which the material returns to its original shape when the load acting on it is completely removed.
OB is the region of elasticity of the material.

C&D. Yield Point: Beyond the elastic limit, the material will undergo plastic deformation and the strains are not totally recoverable. These two points, C & D, are termed as upper and lower yield points respectively.
The stress at the yield point is called the yield strength.



E. Ultimate Stress Point: It is a point that represents the maximum stress that a material can endure before failure. Beyond this point, failure occurs.

F. Fracture or Breaking Point: It is the point in the stress-strain curve at which the failure of the material takes place.

Working Stress and Factor of Safety

Working Stress:

The stress-strain curve gives a valuable information about the mechanical properties of a metal.

Generally, the safe stress is well below the elastic limit.

This safe stress, which is allowed to be undertaken by the material in designs, is called the working stress.

Factor of Safety: The factor of safety is defined as the ratio of ultimate stress to the working stress.

Mathematically, the Factor of safety is the ratio of material strength to allowable stress.

The factor of the safety equation depends on the type of material:

Factor of safety = Ultimate strength/Working stress -for brittle material (concrete)

Factor of safety = Yield strength/Working stress -for ductile material (steel)

Working Stress and Factor of Safety

Structural steel work in buildings: 4.0 to 6.0

Brittle Material: 1.0 to 6.0

Structural steel work in bridges: 5.0 to 7.0

Structural Members in building services: 2

For structural steel work (when subjected to gradually increasing loads) the factor of safety is taken as the ratio of elastic limit to the working stress; whose value is taken as 2 to 2.5.

But in the case of cast iron, concrete, wood, etc. (or when structural steel work is subjected to sudden loads) the factor of safety is taken as the ratio of ultimate stress to the working stress, whose value is taken as 4 to 6

Problem

A mild steel rod of 12 mm diameter was tested for tensile strength, with the gauge length of 60 mm. Following were the observations :

(a) Final length = 78 mm, (b) Final diameter = 7 mm, (c) Yield load = 34 kN, (d) Ultimate load = 61 kN

Calculate: (a) yield stress, (b) ultimate tensile stress, (c) percentage reduction, and (d) percentage elongation.

Given. Original diameter of rod = 12 mm; Original length = 60 mm;

Final length = 78 mm; Final diameter = 7 mm; Yield load = 34 kN = 3.4×10^4 N

Ultimate load = 61 kN = 6.1×10^4 N.

Original area of cross section = $(\pi/4) \times (12)^2 = 113\ mm^2$,

Final area of cross section = $(\pi/4) \times (7)^2 = 38.5\ mm^2$

Problem

Yield stress

We know that the yield stress

$$= \frac{\text{Yield load}}{\text{Area}} = \frac{3.4\times10^4}{113} \text{ N/mm}^2$$

$$= 300.8 \text{ N/mm}^2 \textbf{ Ans.}$$

Ultimate tensile stress

We know that the ultimate tensile stress

$$= \frac{\text{Ultimate load}}{\text{Area}} = \frac{6.1\times10^4}{113} \text{ N/mm}^2$$

$$= 539.8 \text{ N/mm}^2 \textbf{ Ans.}$$

Percentage reduction

We know that the percentage reduction

$$= \frac{\text{Original area} - \text{Final area}}{\text{Original area}} \times 100$$

$$= \frac{113-38.5}{113} \times 100 = 65.9\% \textbf{ Ans.}$$

Percentage elongation

We also know that the percentage elongation

$$= \frac{\text{Final length} - \text{Original length}}{\text{Final length}} \times 100$$

$$= \frac{78-60}{78} \times 100 = 23\% \textbf{ Ans.}$$

Deformation of a Body Due to Force Acting on it

Consider a body subjected to a tensile stress.

Let

P = Load or force acting on the body,

l = Length of the body,

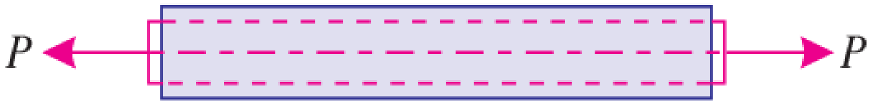
A = Cross-sectional area of the body,

σ = Stress induced in the body,

E = Modulus of elasticity for the material of the body,

ε = Strain, and

δl = Deformation of the body.



We know that the stress

$$\sigma = \frac{P}{A}$$

Strain, $$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

and deformation, $$\delta l = \varepsilon . l = \frac{\sigma . l}{E} = \frac{Pl}{AE}$$

$$\left(\because \sigma = \frac{p}{A}\right)$$

Problem

A steel rod 1 m long and 20 mm x 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if the modulus of elasticity for the rod material is 200 GPa.

Given: Length (l) = 1 m = 1×10^3 mm ; Cross-sectional area (A) = 20 × 20 = 400 mm^2 ; Tensile force (P) = 40 kN = 40×10^3 N and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Elongation of the rod,

$$\delta l = \frac{P.l}{A.E} = \frac{(40 \times 10^3) \times (1 \times 10^3)}{400 \times (200 \times 10^3)} = 0.5\ mm$$

Problem

A hollow cylinder 2 m long has an outside diameter of 50 mm and inside diameter of 30 mm. If the cylinder is carrying a load of 25 kN, find the stress in the cylinder. Also find the deformation of the cylinder, if the value of modulus of elasticity for the cylinder material is 100 GPa.

Given:

Length (l) = 2 m = 2 × 10^3 mm

Outside diameter (D) = 50 mm

Inside diameter (d) = 30 mm

Load (P) = 25 kN = 25 × 10^3 N

Modulus of Elasticity (E) = 100 GPa = 100× 10^3 N/mm^2

Stress in the cylinder

The cross-sectional area of the hollow cylinder

$$A = \frac{\pi}{4}\times(D^2 - d^2) = \frac{\pi}{4}\times[(50)^2 - (30)^2] = 1257\ \text{mm}^2$$

and stress in the cylinder,

$$\sigma = \frac{P}{A} = \frac{25\times10^3}{1257} = 19.9\ \text{N/mm}^2 = 19.9\ \text{MPa}$$

Deformation of the cylinder

$$\delta l = \frac{P.l}{A.E} = \frac{(25\times10^3)\times(2\times10^3)}{1257\times(100\times10^3)}$$

$$= 0.4\ \text{mm}$$

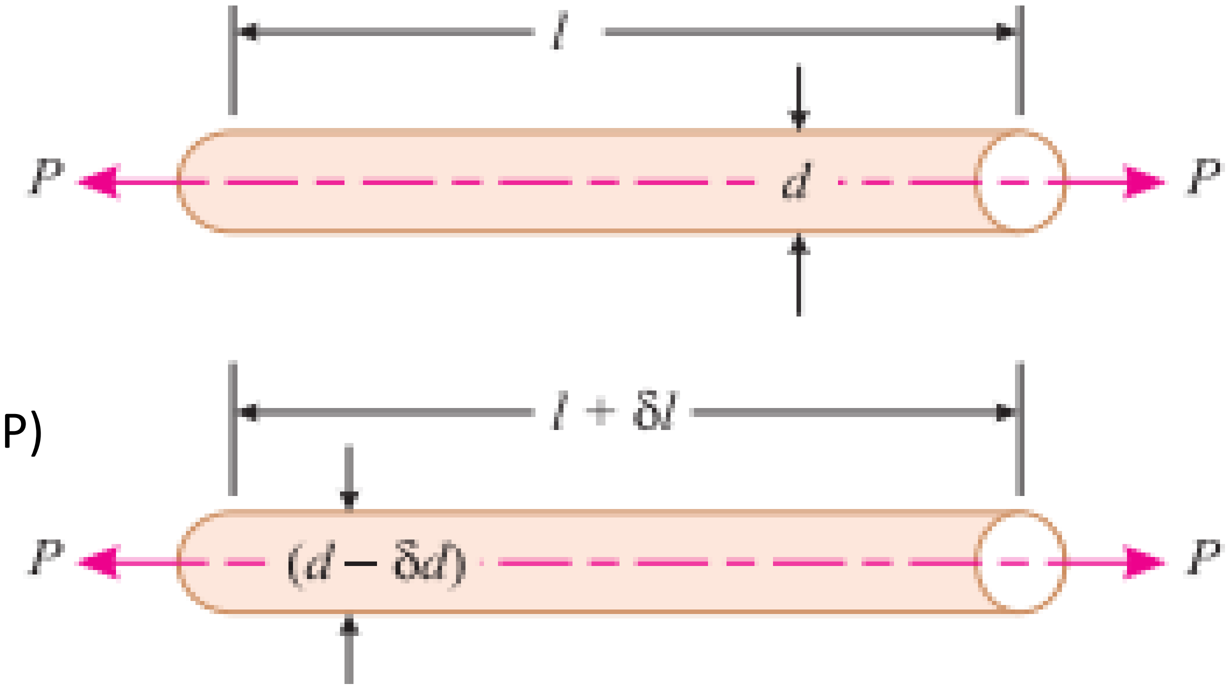
Lateral Strain – Poisson's Ratio

Two types of strains

1. Linear Strain (Primary)
2. Lateral Strain (Secondary)

Linear Strain (Primary) = $\frac{\delta l}{l}$ (deformation along P)

Lateral Strain (Primary) = $\frac{\delta d}{d}$ (deformation normal to P)



Poisson's Ratio (μ or 1/m):

It is the ratio of lateral strain to the linear strain.

It ranges from 0.25 to 0.33 for engineering materials

Problem

A steel bar 2 m long, 40 mm wide and 20 mm thick is subjected to an axial pull of 160 kN in the direction of its length. Find the changes in length, width and thickness of the bar. Take E = 200 GPa and Poisson's ratio = 0.3.

Given: Length $(l) = 2\text{ m} = 2 \times 10^3\text{ mm}$; Width $(b) = 40\text{ mm}$; Thickness $(t) = 20\text{ mm}$;

Axial pull $(P) = 160\text{ kN} = 160 \times 10^3\text{ N}$; Modulus of elasticity $(E) = 200\text{ GPa} = 200 \times 10^3\text{ N/mm}^2$ and

poisson's ratio $= 0.3$.

Change in length,

$$\delta l = \frac{Pl}{AE} = \frac{(160 \times 10^3) \times (2 \times 10^3)}{(40 \times 20) \times (200 \times 10^3)} = 2\text{ mm}$$

Change in width,

$$\text{Linear strain} = \frac{\delta l}{l} = \frac{2}{2 \times 10^3} = 0.001$$

Lateral strain = μ × Linear Strain = 0.3×0.001 = 0.0003

Change in width, $\delta b = b \times \text{Lateral strain} = 40 \times 0.0003 = 0.012\text{ mm}$

Change in thickness,

$$\delta t = t \times \text{Lateral strain} = 20 \times 0.0003 = 0.006\text{ mm}$$

Volumetric Strain

The ratio of change in volume to the original volume is known as volumetric strain.

Mathematically,

$$\epsilon_v = \frac{\delta V}{V}$$

For the volumetric strain, two important force types acting on a body considered are:

1. A rectangular body subjected to an axial force
2. A rectangular body subjected to three mutually perpendicular forces

Volumetric Strain of a Rectangular Body Subjected to an Axial Force

Consider a rectangular cross-sectional bar subjected to an axial tensile force as shown in figure.

Let, l = Length of the bar,

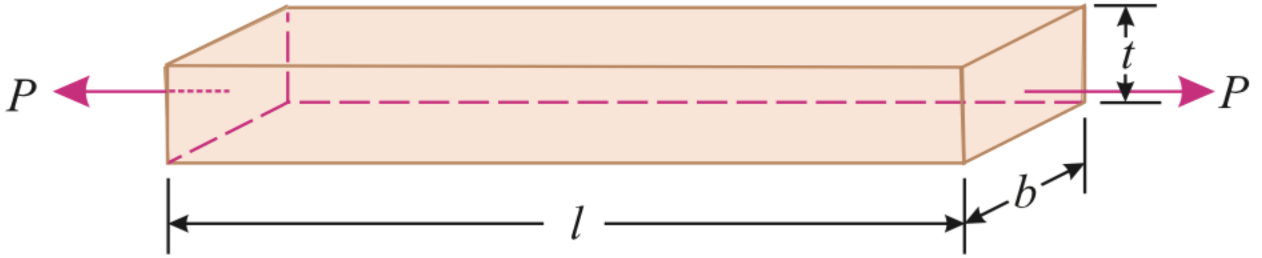
b = Breadth of the bar,

t = Thickness of the bar,

P = Tensile force acting on the bar,

E = Modulus of elasticity and

$\frac{1}{m}$ = Poisson's ratio.



We know that change in length,

$$\delta l = \frac{Pl}{AE} = \frac{Pl}{bt\,E}$$

and linear stress,

$$\sigma = \frac{\text{Force}}{\text{Area}} = \frac{P}{b\,t}$$

$$\therefore \quad \text{Linear strain} = \frac{\text{Stress}}{E} = \frac{P}{bt\,E}$$

and lateral strain $= \frac{1}{m} \times \text{Linear strain} = \frac{1}{m} \times \frac{P}{bt\,E}$

$\therefore$ Change in thickness,

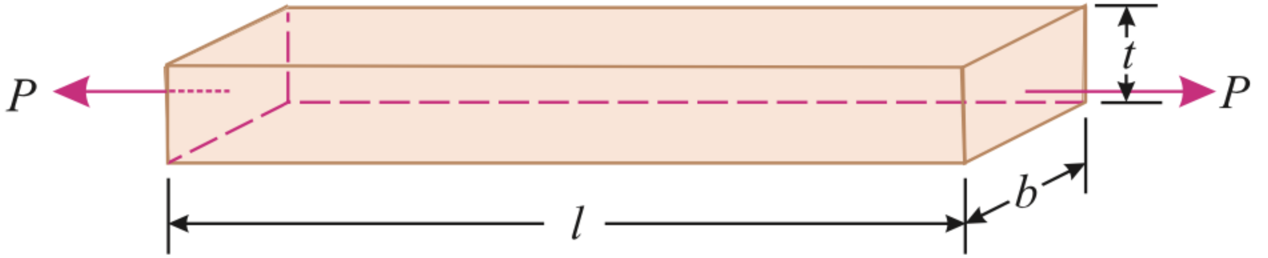
$$\delta t = t \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mb\,E}$$

and change in breadth,

$$\delta b = b \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mt\,E}$$

Volumetric Strain of a Rectangular Body Subjected to an Axial Force

Because of the tensile force,
Let the final length = $l + \delta l$
Final breadth = $b - \delta b$
Final thickness = $t - \delta t$



The original volume of the body,
$V = l.b.t$

and final volume $= (l + \delta l)(b - \delta b)(t - \delta t)$

$$= lbt\left(1 + \frac{\delta l}{l}\right)\left(1 - \frac{\delta b}{b}\right)\left(1 - \frac{\delta t}{t}\right)$$

$$= lbt\left[1 + \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta t}{t} - \left(\frac{\delta l}{l} \times \frac{\delta b}{b}\right) - \left(\frac{\delta l}{l} \times \frac{\delta t}{t}\right) + \left(\frac{\delta b}{b} \times \frac{\delta t}{t}\right) + \left(\frac{\delta l}{l} \times \frac{\delta b}{b} \times \frac{\delta t}{t}\right)\right]$$

$$= lbt\left[1 + \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta t}{t}\right]$$ ...(Ignoring other negligible values)

Change in volume,

$$\delta V = \text{Final volume} - \text{Original volume}$$

$$= lbt\left(1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)-lbt = lbt\left(\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)$$

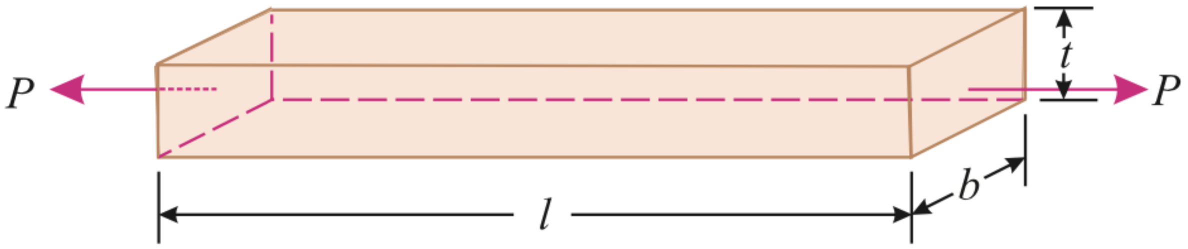
$$= lbt\left[\frac{\frac{Pl}{btE}}{l}-\frac{\frac{P}{mtE}}{b}-\frac{\frac{P}{mbE}}{t}\right] = lbt\left(\frac{P}{btE}-\frac{P}{mbtE}-\frac{P}{mbtE}\right)$$

$$= V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)$$

and volumetric strain,

$$\frac{\delta V}{V} = \frac{V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)}{V} = \frac{P}{btE}\left(1-\frac{2}{m}\right)$$

$$= \varepsilon\left(1-\frac{2}{m}\right)$$



Problem

A steel bar 2 m long, 20 mm wide and 15 mm thick is subjected to a tensile load of 30 kN. Find the increase in volume, if Poisson's ratio is 0.25 and Young's modulus is 200 GPa.

Given: Length $(l) = 2\text{ m} = 2 \times 10^3\text{ mm}$; Width $(b) = 20\text{ mm}$; Thickness $(t) = 15\text{ mm}$;

Tensile load $(P) = 30\text{ kN} = 30 \times 10^3\text{ N}$; Poisson's ratio $\left(\frac{1}{m}\right) = 0.25$ or $m = 4$ and Young's modulus of elasticity $(E) = 200\text{ GPa} = 200 \times 10^3\text{ N/mm}^2$.

Let $\quad \delta V = $ Increase in volume of the bar.

We know that original volume of the bar,

$$V = l.b.t = (2 \times 10^3) \times 20 \times 15 = 600 \times 10^3\text{ mm}^3$$

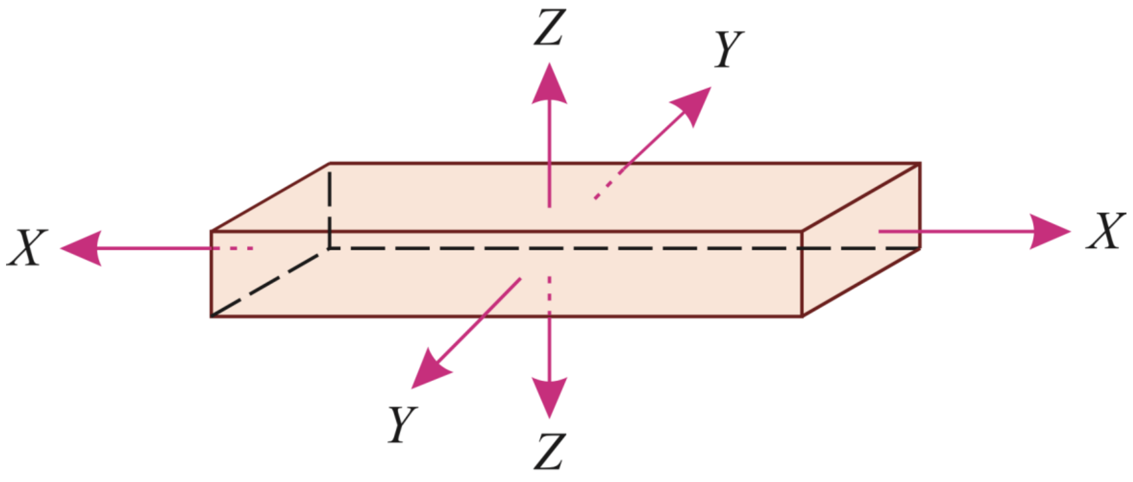
and

$$\frac{\delta V}{V} = \frac{P}{btE}\left(1 - \frac{2}{m}\right) = \frac{30 \times 10^3}{20 \times 15 \times (200 \times 10^3)}\left(1 - \frac{2}{4}\right) = 0.000\,25$$

$\therefore$

$$\delta V = 0.000\,25 \times V = 0.000\,25 \times (600 \times 10^3) = 150\text{ mm}^3 \qquad \textbf{Ans.}$$

Consider a rectangular cross-sectional bar subjected to direct tensile stresses along three mutually perpendicular axes as shown in figure.



Let, σ_x = Stress in x-x direction,

σ_y = Stress in y-y direction,

σ_z = Stress in z-z direction and

E = Young's modulus of elasticity.

$\therefore$ Strain in x-x direction due to stress σ_x,

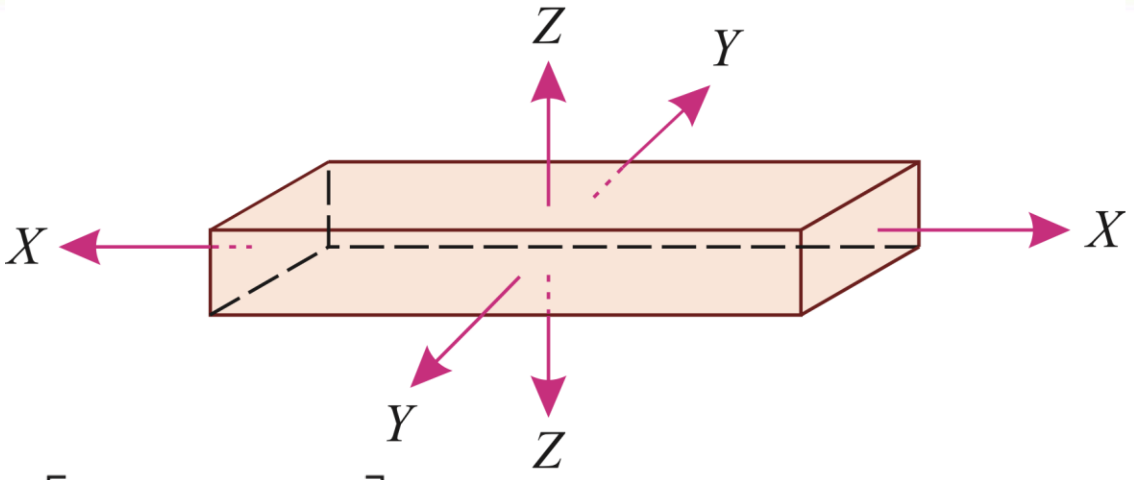
$$\varepsilon_x = \frac{\sigma_x}{E}$$

Similarly, $\varepsilon_y = \dfrac{\sigma_y}{E}$ and $\varepsilon_z = \dfrac{\sigma_z}{E}$

The resulting strains in the three directions, may be found out by the principle of superposition, i.e., by adding algebraically the strains in each direction due to each individual stress.

Volumetric Strain of a Rectangular Body Subjected to Three Mutually Perpendicular Forces

For the three tensile stresses shown in figure, the resultant strain in x-x direction is given as,



$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_x - \frac{\sigma_y}{m} - \frac{\sigma_z}{m}\right]$$

Similarly,

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_y - \frac{\sigma_x}{m} - \frac{\sigma_z}{m}\right]$$

and

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE} = \frac{1}{E}\left[\sigma_z - \frac{\sigma_x}{m} - \frac{\sigma_y}{m}\right]$$

The volumetric strain may then be found by the relation;

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Working Stress and Factor of Safety

Working Stress:

The stress-strain curve gives a valuable information about the mechanical properties of a metal.

Generally, the safe stress is well below the elastic limit.

This safe stress, which is allowed to be undertaken by the material in designs, is called the working stress.

Factor of Safety: The factor of safety is defined as the ratio of ultimate stress to the working stress.

Mathematically, the Factor of safety is the ratio of material strength to allowable stress.

The factor of the safety equation depends on the type of material:

Factor of safety = Ultimate strength/Working stress -for brittle material (concrete)

Factor of safety = Yield strength/Working stress -for ductile material (steel)

Working Stress and Factor of Safety

Structural steel work in buildings: 4.0 to 6.0

Brittle Material: 1.0 to 6.0

Structural steel work in bridges: 5.0 to 7.0

Structural Members in building services: 2

For structural steel work (when subjected to gradually increasing loads) the factor of safety is taken as the ratio of elastic limit to the working stress; whose value is taken as 2 to 2.5.

But in the case of cast iron, concrete, wood, etc. (or when structural steel work is subjected to sudden loads) the factor of safety is taken as the ratio of ultimate stress to the working stress, whose value is taken as 4 to 6

Problem

A mild steel rod of 12 mm diameter was tested for tensile strength, with the gauge length of 60 mm. Following were the observations :

(a) Final length = 78 mm, (b) Final diameter = 7 mm, (c) Yield load = 34 kN, (d) Ultimate load = 61 kN

Calculate: (a) yield stress, (b) ultimate tensile stress, (c) percentage reduction, and (d) percentage elongation.

Given. Original diameter of rod = 12 mm; Original length = 60 mm;

Final length = 78 mm; Final diameter = 7 mm; Yield load = 34 kN = 3.4×10^4 N

Ultimate load = 61 kN = 6.1×10^4 N.

Original area of cross section = $(\pi/4) \times (12)^2 = 113\ mm^2$,

Final area of cross section = $(\pi/4) \times (7)^2 = 38.5\ mm^2$

Problem

Yield stress

We know that the yield stress

$$= \frac{\text{Yield load}}{\text{Area}} = \frac{3.4\times10^4}{113}\ \text{N/mm}^2$$

$$= 300.8\ \text{N/mm}^2\ \textbf{Ans.}$$

Ultimate tensile stress

We know that the ultimate tensile stress

$$= \frac{\text{Ultimate load}}{\text{Area}} = \frac{6.1\times10^4}{113}\ \text{N/mm}^2$$

$$= 539.8\ \text{N/mm}^2\ \textbf{Ans.}$$

Percentage reduction

We know that the percentage reduction

$$= \frac{\text{Original area} - \text{Final area}}{\text{Original area}} \times 100$$

$$= \frac{113-38.5}{113}\times100 = 65.9\%\ \textbf{Ans.}$$

Percentage elongation

We also know that the percentage elongation

$$= \frac{\text{Final length} - \text{Original length}}{\text{Final length}} \times 100$$

$$= \frac{78-60}{78} \times 100 = 23\%\ \textbf{Ans.}$$

Deformation of a Body Due to Force Acting on it

Consider a body subjected to a tensile stress.

Let P = Load or force acting on the body,

l = Length of the body,

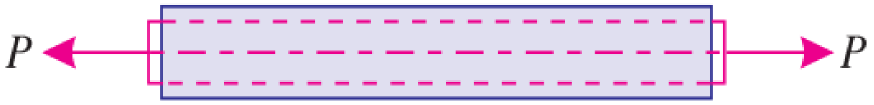
A = Cross-sectional area of the body,

σ = Stress induced in the body,

E = Modulus of elasticity for the material of the body,

ε = Strain, and

δl = Deformation of the body.



We know that the stress

$$\sigma = \frac{P}{A}$$

Strain, $$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

and deformation, $$\delta l = \varepsilon . l = \frac{\sigma . l}{E} = \frac{Pl}{AE}$$

$$\left(\because \sigma = \frac{p}{A}\right)$$

Problem

A steel rod 1 m long and 20 mm x 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if the modulus of elasticity for the rod material is 200 GPa.

Given: Length (l) = 1 m = 1×10^3 mm ; Cross-sectional area (A) = 20×20 = 400 mm^2 ; Tensile force (P) = 40 kN = 40×10^3 N and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Elongation of the rod,

$$\delta l = \frac{P.l}{A.E} = \frac{(40 \times 10^3) \times (1 \times 10^3)}{400 \times (200 \times 10^3)} = 0.5\ mm$$

A hollow cylinder 2 m long has an outside diameter of 50 mm and inside diameter of 30 mm. If the cylinder is carrying a load of 25 kN, find the stress in the cylinder. Also find the deformation of the cylinder, if the value of modulus of elasticity for the cylinder material is 100 GPa.

Given: Length (l) = 2 m = 2×10^3 mm

Outside diameter (D) = 50 mm

Inside diameter (d) = 30 mm

Load (P) = 25 kN = 25×10^3 N

Modulus of Elasticity (E) = 100 GPa = 100×10^3 N/mm^2

Stress in the cylinder

The cross-sectional area of the hollow cylinder

$$A = \frac{\pi}{4}\times(D^2 - d^2) = \frac{\pi}{4}\times[(50)^2 - (30)^2] = 1257\,\text{mm}^2$$

and stress in the cylinder,

$$\sigma = \frac{P}{A} = \frac{25\times10^3}{1257} = 19.9\ \text{N/mm}^2 = 19.9\ \text{MPa}$$

Deformation of the cylinder

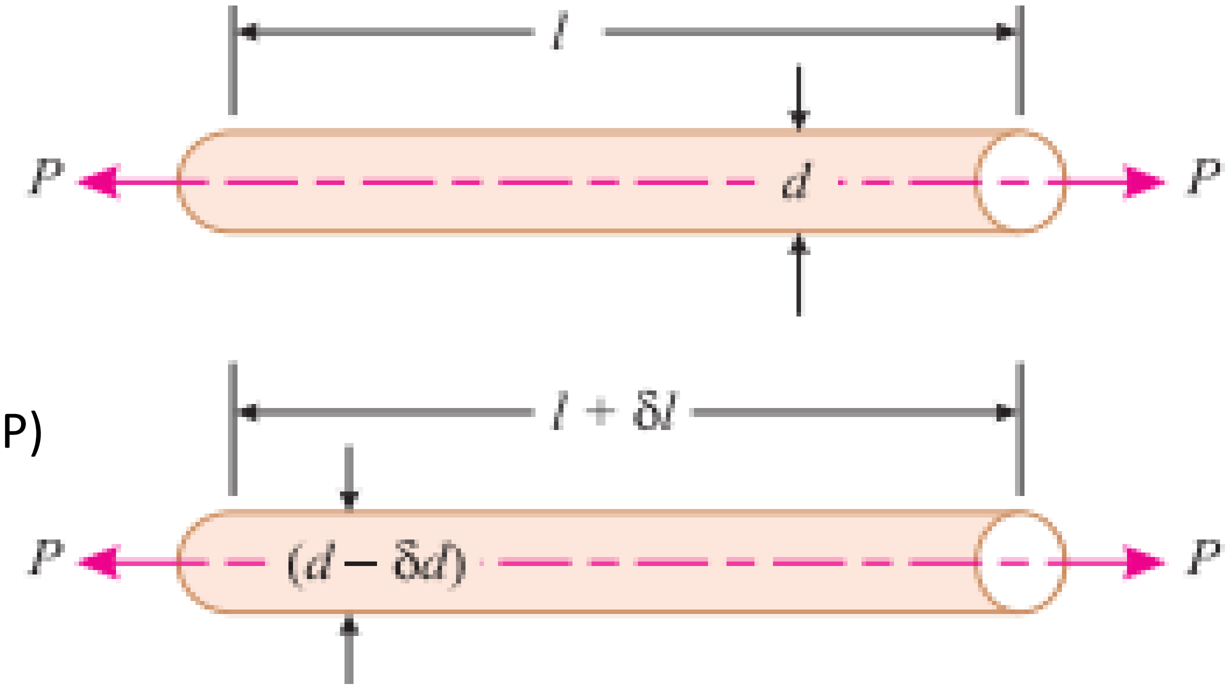
$$\delta l = \frac{P.l}{A.E} = \frac{(25\times10^3)\times(2\times10^3)}{1257\times(100\times10^3)}$$

$$= 0.4\ \text{mm}$$

Lateral Strain – Poisson's Ratio

Two types of strains

1. Linear Strain (Primary)
2. Lateral Strain (Secondary)



Linear Strain (Primary) = $\frac{\delta l}{l}$ (deformation along P)

Lateral Strain (Primary) = $\frac{\delta d}{d}$ (deformation normal to P)

Poisson's Ratio (μ or 1/m):

It is the ratio of lateral strain to the linear strain.

It ranges from 0.25 to 0.33 for engineering materials

Problem

A steel bar 2 m long, 40 mm wide and 20 mm thick is subjected to an axial pull of 160 kN in the direction of its length. Find the changes in length, width and thickness of the bar. Take E = 200 GPa and Poisson's ratio = 0.3.

Given: Length $(l) = 2\text{ m} = 2 \times 10^3\text{ mm}$; Width $(b) = 40\text{ mm}$; Thickness $(t) = 20\text{ mm}$;

Axial pull $(P) = 160\text{ kN} = 160 \times 10^3\text{ N}$; Modulus of elasticity $(E) = 200\text{ GPa} = 200 \times 10^3\text{ N/mm}^2$ and

poisson's ratio $= 0.3$.

Change in length,

$$\delta l = \frac{Pl}{AE} = \frac{(160 \times 10^3) \times (2 \times 10^3)}{(40 \times 20) \times (200 \times 10^3)} = 2\text{ mm}$$

Change in width,

Linear strain $= \dfrac{\delta l}{l} = \dfrac{2}{2 \times 10^3} = 0.001$

Lateral strain = μ × Linear Strain = 0.3×0.001 = 0.0003

Change in width, $\delta b = b \times \text{Lateral strain} = 40 \times 0.0003 = 0.012\text{ mm}$

Change in thickness,

$$\delta t = t \times \text{Lateral strain} = 20 \times 0.0003 = 0.006\text{ mm}$$

Volumetric Strain

The ratio of change in volume to the original volume is known as volumetric strain.

Mathematically,

$$\epsilon_v = \frac{\delta V}{V}$$

For the volumetric strain, two important force types acting on a body considered are:

1. A rectangular body subjected to an axial force
2. A rectangular body subjected to three mutually perpendicular forces

Volumetric Strain of a Rectangular Body Subjected to an Axial Force

Consider a rectangular cross-sectional bar subjected to an axial tensile force as shown in figure.

Let, l = Length of the bar,

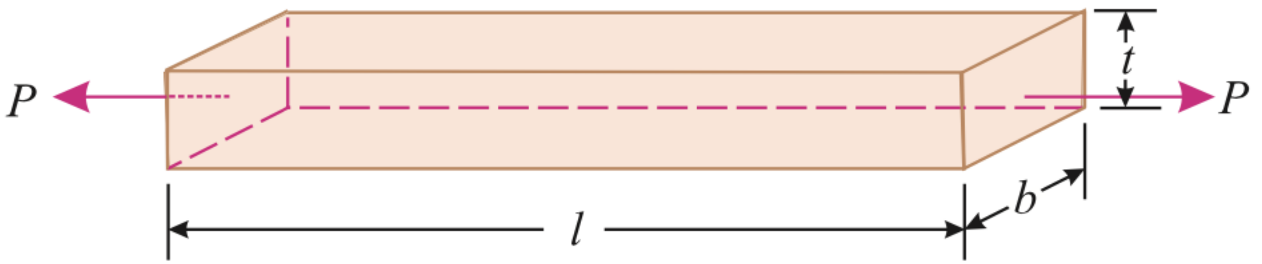
b = Breadth of the bar,

t = Thickness of the bar,

P = Tensile force acting on the bar,

E = Modulus of elasticity and

$\frac{1}{m}$ = Poisson's ratio.



We know that change in length,

$$\delta l = \frac{P\,l}{A\,E} = \frac{P\,l}{bt\,E}$$

and linear stress,

$$\sigma = \frac{\text{Force}}{\text{Area}} = \frac{P}{b\,t}$$

$$\therefore \quad \text{Linear strain} = \frac{\text{Stress}}{E} = \frac{P}{bt\,E}$$

and lateral strain $= \frac{1}{m} \times \text{Linear strain} = \frac{1}{m} \times \frac{P}{bt\,E}$

$\therefore$ Change in thickness,

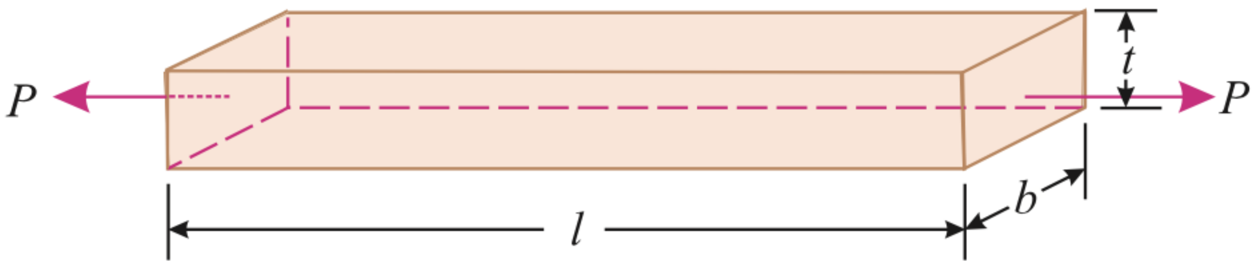
$$\delta t = t \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mb\,E}$$

and change in breadth,

$$\delta b = b \times \frac{1}{m} \times \frac{P}{bt\,E} = \frac{P}{mt\,E}$$

Volumetric Strain of a Rectangular Body Subjected to an Axial Force

Because of the tensile force,
Let the final length = $l + \delta l$
Final breadth = $b - \delta b$
Final thickness = $t - \delta t$



The original volume of the body,
$V = l.b.t$

$$\text{and final volume} = (l + \delta l)(b - \delta b)(t - \delta t)$$

$$= lbt\left(1 + \frac{\delta l}{l}\right)\left(1 - \frac{\delta b}{b}\right)\left(1 - \frac{\delta t}{t}\right)$$

$$= lbt\left[1 + \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta t}{t} - \left(\frac{\delta l}{l} \times \frac{\delta b}{b}\right) - \left(\frac{\delta l}{l} \times \frac{\delta t}{t}\right) + \left(\frac{\delta b}{b} \times \frac{\delta t}{t}\right) + \left(\frac{\delta l}{l} \times \frac{\delta b}{b} \times \frac{\delta t}{t}\right)\right]$$

$$= lbt\left[1 + \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta t}{t}\right]$$

...(Ignoring other negligible values)

Volumetric Strain of a Rectangular Body Subjected to an Axial Force

Change in volume,

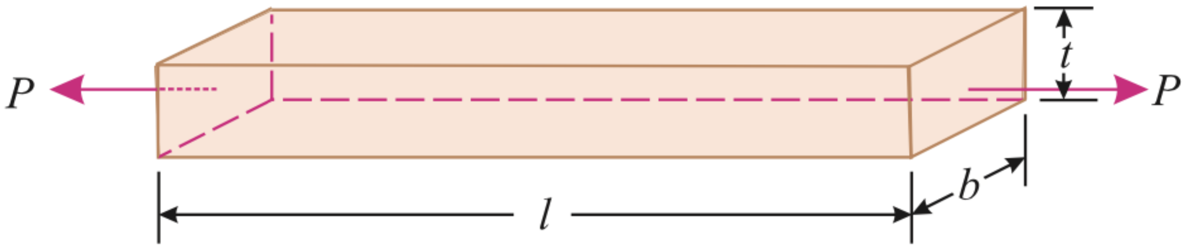
$$\delta V = \text{Final volume} - \text{Original volume}$$

$$= lbt\left(1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)-lbt = lbt\left(\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)$$

$$= lbt\left[\frac{\frac{Pl}{btE}}{l}-\frac{\frac{P}{mtE}}{b}-\frac{\frac{P}{mbE}}{t}\right] = lbt\left(\frac{P}{btE}-\frac{P}{mbtE}-\frac{P}{mbtE}\right)$$

$$= V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)$$

and volumetric strain,



$$\frac{\delta V}{V} = \frac{V\times\frac{P}{btE}\left(1-\frac{2}{m}\right)}{V} = \frac{P}{btE}\left(1-\frac{2}{m}\right)$$

$$= \varepsilon\left(1-\frac{2}{m}\right)$$

Problem

A steel bar 2 m long, 20 mm wide and 15 mm thick is subjected to a tensile load of 30 kN. Find the increase in volume, if Poisson's ratio is 0.25 and Young's modulus is 200 GPa.

Given: Length $(l) = 2\ \text{m} = 2 \times 10^3\ \text{mm}$; Width $(b) = 20\ \text{mm}$; Thickness $(t) = 15\ \text{mm}$;

Tensile load $(P) = 30\ \text{kN} = 30 \times 10^3\ \text{N}$; Poisson's ratio $\left(\frac{1}{m}\right) = 0.25$ or $m = 4$ and Young's modulus of elasticity $(E) = 200\ \text{GPa} = 200 \times 10^3\ \text{N/mm}^2$.

Let δV = Increase in volume of the bar.

We know that original volume of the bar,

$$V = l.b.t = (2 \times 10^3) \times 20 \times 15 = 600 \times 10^3\ \text{mm}^3$$

and
$$\frac{\delta V}{V} = \frac{P}{btE}\left(1 - \frac{2}{m}\right) = \frac{30 \times 10^3}{20 \times 15 \times (200 \times 10^3)}\left(1 - \frac{2}{4}\right) = 0.000\ 25$$

$\therefore$
$$\delta V = 0.000\ 25 \times V = 0.000\ 25 \times (600 \times 10^3) = 150\ \text{mm}^3$$ **Ans.**

Volumetric Strain of a Rectangular Body Subjected to Three Mutually Perpendicular Forces

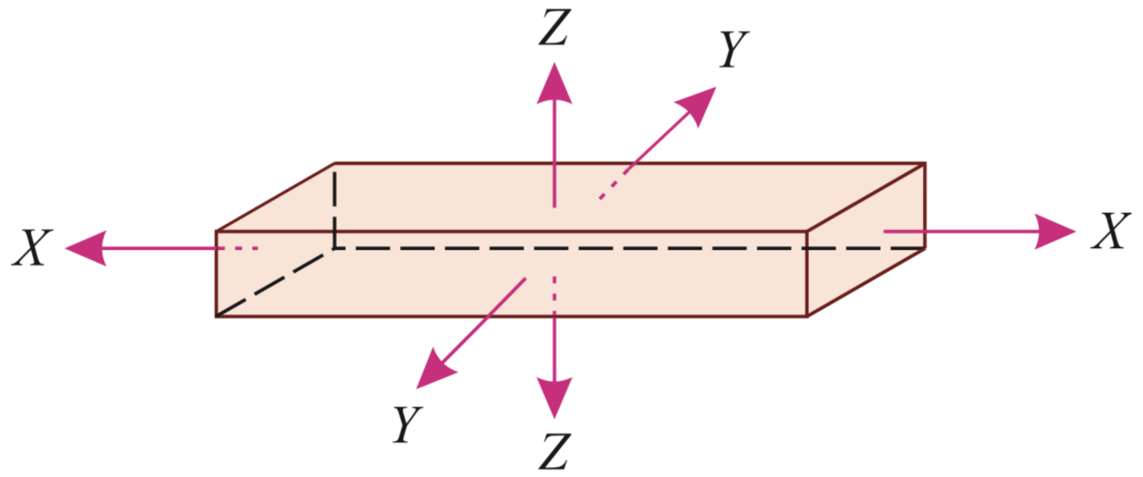
Consider a rectangular cross-sectional bar subjected to direct tensile stresses along three mutually perpendicular axes as shown in figure.

Let, σ_x = Stress in x-x direction,

σ_y = Stress in y-y direction,

σ_z = Stress in z-z direction and

E = Young's modulus of elasticity.



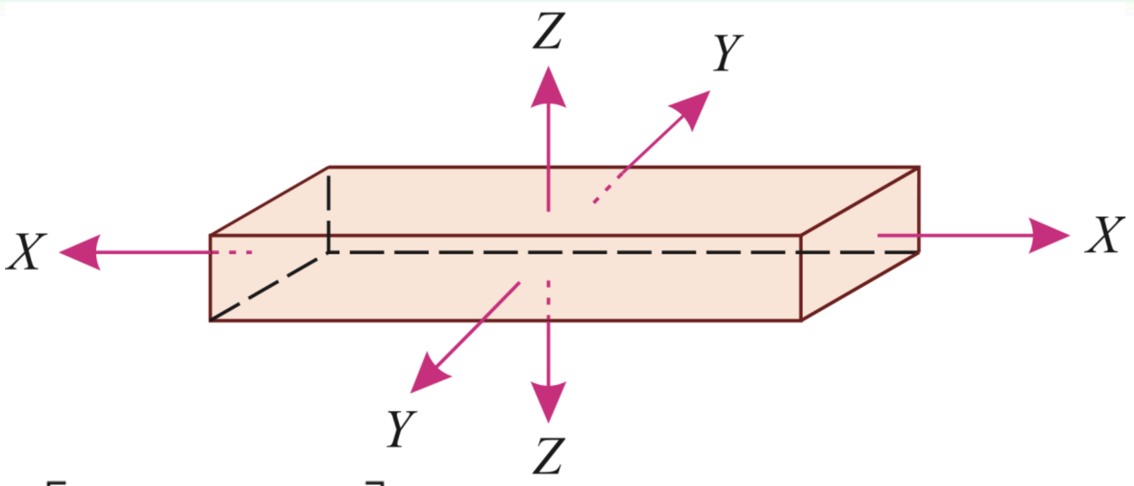
$\therefore$ Strain in x-x direction due to stress σ_x,

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Similarly, $\varepsilon_y = \dfrac{\sigma_y}{E}$ and $\varepsilon_z = \dfrac{\sigma_z}{E}$

The resulting strains in the three directions, may be found out by the principle of superposition, i.e., by adding algebraically the strains in each direction due to each individual stress.

For the three tensile stresses shown in figure (taking tensile strains as +ve and compressive strains as –ve), the resultant strain in x-x direction is given as,



$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_x - \frac{\sigma_y}{m} - \frac{\sigma_z}{m}\right]$$

Similarly,

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE} = \frac{1}{E}\left[\sigma_y - \frac{\sigma_x}{m} - \frac{\sigma_z}{m}\right]$$

and

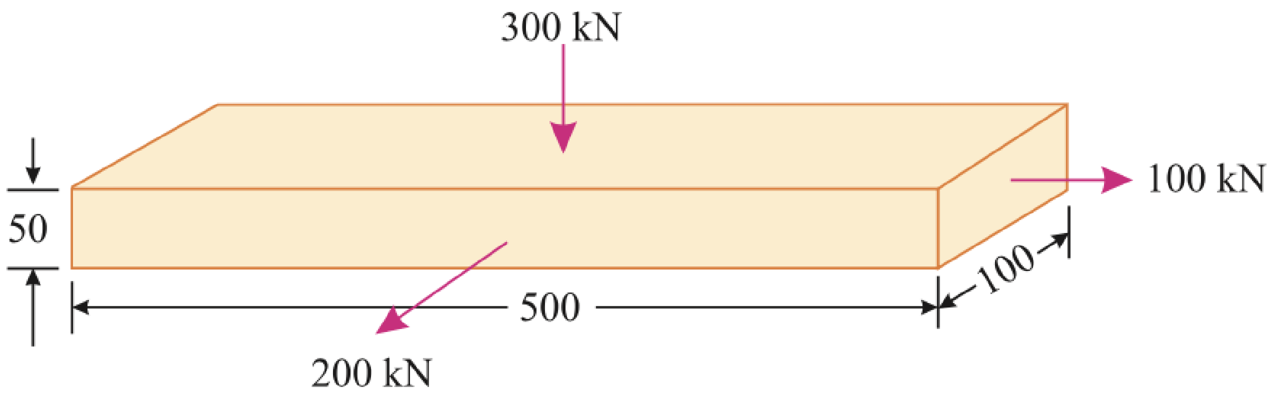
$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE} = \frac{1}{E}\left[\sigma_z - \frac{\sigma_x}{m} - \frac{\sigma_y}{m}\right]$$

The volumetric strain may then be found by the relation;

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Problem

A rectangular bar 500 mm long and 100 mm x 50 mm in cross-section is subjected to forces as shown in fig. What is the change in the volume of the bar? Take modulus of elasticity for the bar material as 200 GPa and Poisson's ratio as 0.25.



Given:

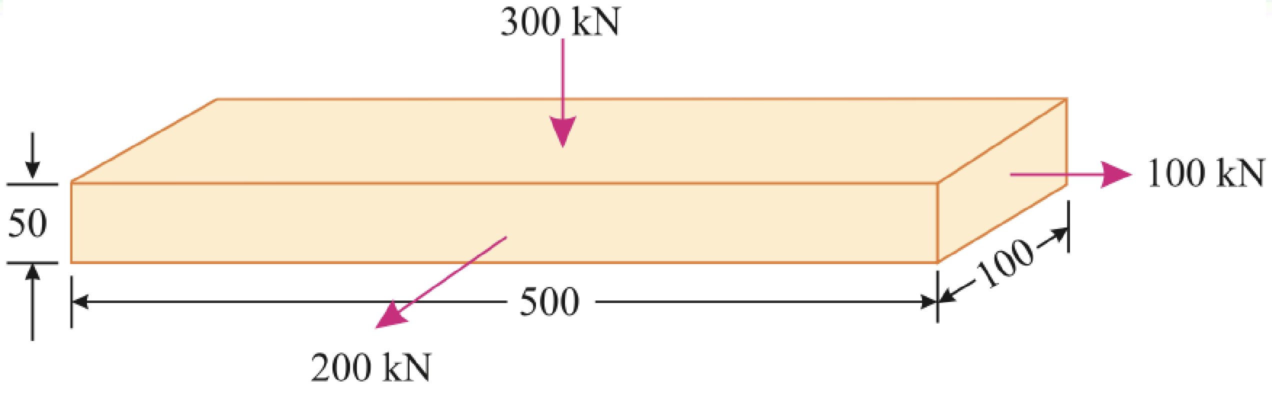
Length (l) = 500 mm; Width (b) = 100 mm; Thickness (t) = 50 mm; Force in x-direction (P_x) = 100 kN = 100×10^3 N (Tension) ; Force in y-direction (P_y) = 200 kN = 200×10^3 N (Tension) ; Force in z-direction (P_z) = 300 kN = 300×10^3 N (Compression) ; Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2 and Poisson's ratio ($1/m$) = 0.25 or $m = 4$.

Let δV = Change in the volume of the bar.

We know that original volume of the rectangular bar,

$$V = l \times b \times t = 500 \times 100 \times 50 = 2.5 \times 10^6 \text{ mm}^3$$

Problem



Stress in x-x direction,

$$\sigma_x = \frac{P_x}{A_x} = \frac{100\times10^3}{100\times50} = 20\ \text{N/mm}^2\ \text{(Tension)}$$

Similarly

$$\sigma_y = \frac{P_y}{A_y} = \frac{200\times10^3}{500\times50} = 8\ \text{N/mm}^3\ \text{(Tension)}$$

$$\sigma_z = \frac{P_z}{A_z} = \frac{300\times10^3}{500\times100} = 6\ \text{N/mm}^2\ \text{(Compression)}$$

The resultant strain in x-x direction considering tension as positive and compression as negative,

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\sigma_y}{mE} + \frac{\sigma_z}{mE} = +\frac{20}{E} - \frac{8}{4E} + \frac{6}{4E} = \frac{19.5}{E}$$

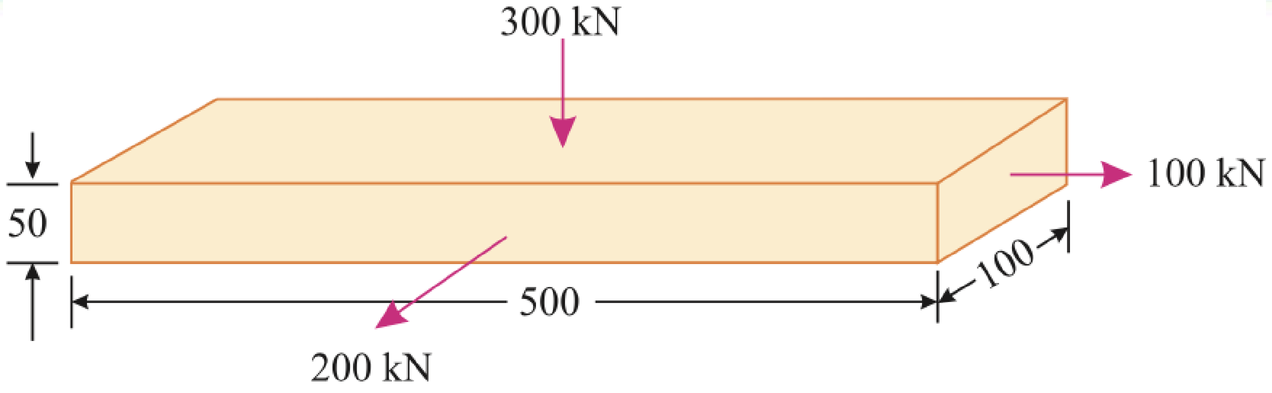
Similarly

$$\varepsilon_y = +\frac{\sigma_y}{E} - \frac{\sigma_x}{mE} + \frac{\sigma_z}{mE} = +\frac{8}{E} - \frac{20}{4E} + \frac{6}{4E} = \frac{4.5}{E}$$

$$\varepsilon_z = -\frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE} = -\frac{6}{E} - \frac{20}{4E} - \frac{8}{4E} = -\frac{13}{E}$$

Problem

Now, volumetric strain,



$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{\delta V}{2.5\times 10^6} = \frac{19.5}{E} + \frac{4.5}{E} - \frac{13}{E} = \frac{11}{E} = \frac{11}{200\times 10^3} = 0.055 \times 10^{-3}$$

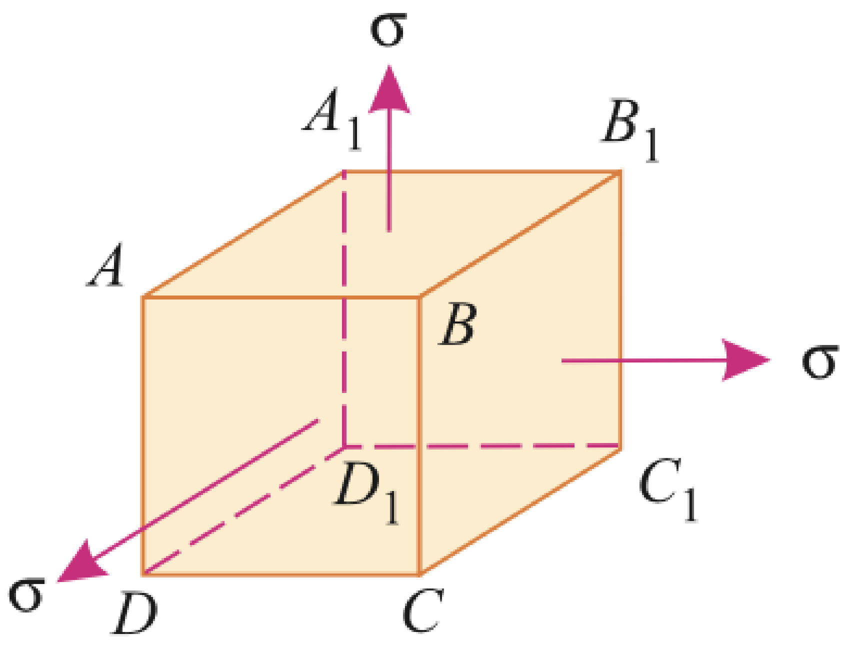
$$\delta V = (0.055 \times 10^{-3}) \times (2.5 \times 10^6) = 137.5 \text{ mm}^3 \qquad \textbf{Ans.}$$

Bulk Modulus

When a body is subjected to three mutually perpendicular stresses of equal intensity, the ratio of direct stress to the corresponding volumetric strain is known as bulk modulus.

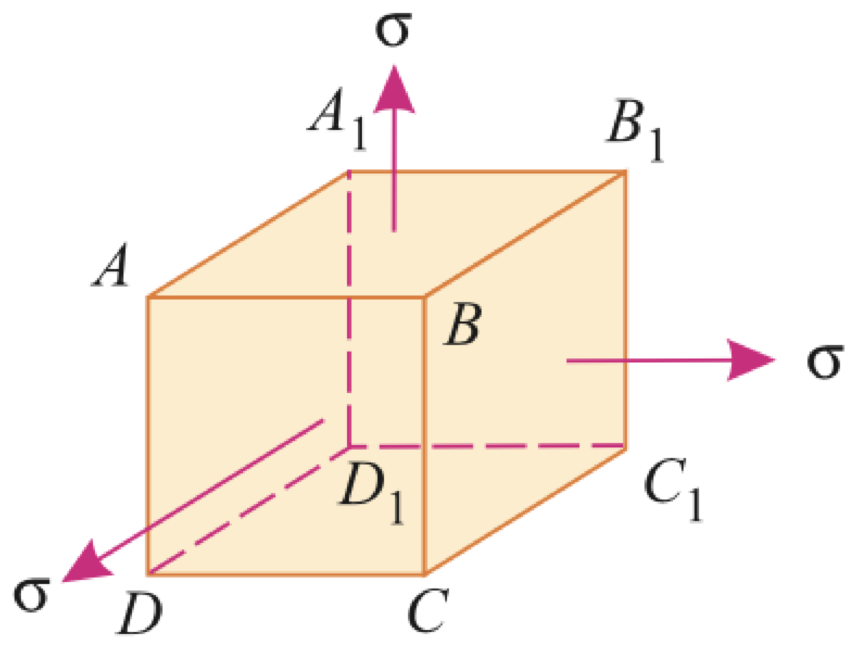
It is denoted by 'K'.

Mathematically, $K = \dfrac{Direct\ Stress}{Volumetric\ Strain} = \dfrac{\sigma}{\delta V / V}$



Relation Between Bulk Modulus and Young's Modulus

Consider a cube as shown in figure. Let the cube be subjected to three mutually perpendicular tensile stresses of equal intensity **σ**. Let the length of each side of the cube is 'l' and 'E' is the Young's modulus for the material of the block.



Consider the deformation of one side of the cube, i.e., AB, under the action of three mutually perpendicular stresses.

Hence, the strains that are experienced by the side AB because of the pair of stresses are:

1. Tensile strain equal to $\frac{\sigma}{E}$ due to stresses on the faces BB_1CC_1 and AA_1DD_1.
2. Compressive lateral strain equal to $\frac{1}{m} \times \frac{\sigma}{E}$ due to stresses on faces AA_1B_1B and DD_1C_1C.
3. Compressive lateral strain equal to $\frac{1}{m} \times \frac{\sigma}{E}$ due to stresses on faces ABCD and $A_1B_1\,C_1D_1$.

Relation Between Bulk Modulus and Young's Modulus

Therefore net tensile strain, which the side AB will suffer, due to these stresses,

$$\frac{\delta l}{l} = \frac{\sigma}{E} - \left(\frac{1}{m} \times \frac{\sigma}{E}\right) - \left(\frac{1}{m} \times \frac{\sigma}{E}\right) = \frac{\sigma}{E}\left(1 - \frac{2}{m}\right) \quad \ldots(i)$$

We know that the original volume of the cube,

$$V = l^3$$

Differentiating the above equation with respect to l,

$$\frac{\delta V}{\delta l} = 3\, l^2$$

$$\delta V = 3\, l^2 . \delta l = 3\, l^3 \times \frac{\delta l}{l}$$

Substituting the value of $\frac{\delta l}{l}$ from equation (i)

$$\delta V = 3\, l^3 \times \frac{\sigma}{E}\left(1 - \frac{2}{m}\right)$$

$$\frac{\delta V}{V} = \frac{3 l^3}{l^3} \times \frac{\sigma}{E}\left(1 - \frac{2}{m}\right) = \frac{3\sigma}{E}\left(1 - \frac{2}{m}\right)$$

$$\frac{\sigma}{\frac{\delta V}{V}} = \frac{E}{3} \times \frac{1}{\left(1 - \frac{2}{m}\right)} = \frac{E}{3} \times \frac{1}{\left(\frac{m-2}{m}\right)}$$

$$K = \frac{m E}{3\,(m-2)}$$

Problem

If the values of modulus of elasticity and Poisson's ratio for an alloy body is 150 GPa and 0.25 respectively, determine the value of bulk modulus for the alloy.

Given: Modulus of elasticity (E) = 150 GPa = 150×10^3 N/mm^2 and Poisson's ratio

$$\left(\frac{1}{m}\right) = 0.25 \qquad \text{or} \qquad m = 4.$$

$$K = \frac{mE}{3\,(m-2)} = \frac{4 \times (150 \times 10^3)}{3\,(4-2)} = 100 \times 10^3 \text{ N/mm}^2$$

$$= 100 \text{ GPa} \qquad \textbf{Ans.}$$

Shear Modulus or Modulus of Rigidity

Experimentally it was found that within the elastic limit, the shear stress is proportional to shear strain.

Mathematically,

$$\tau \propto \phi$$

$$\tau = C \times \phi$$

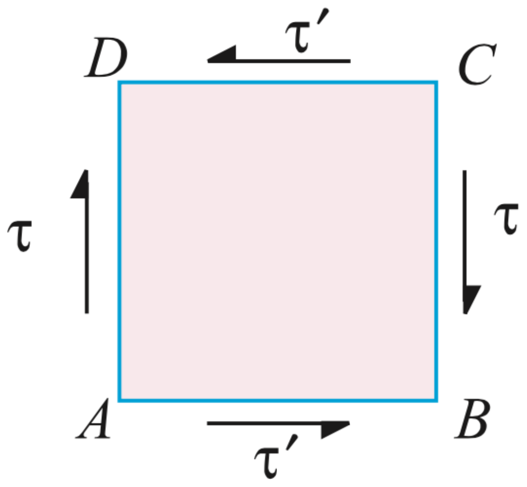
$$\frac{\tau}{\phi} = C(or\ G\ or\ N)$$

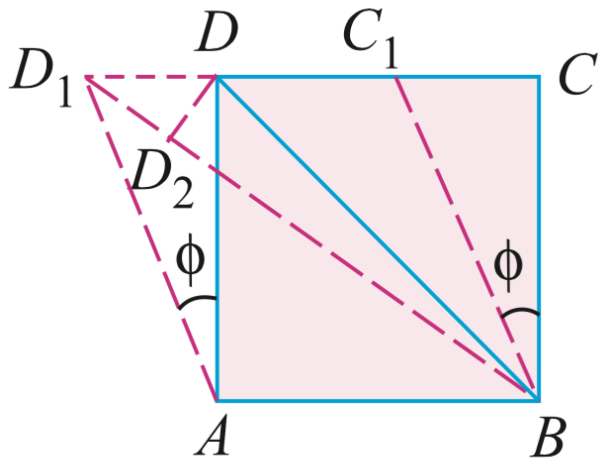
Where, τ = Shear stress,

ϕ = Shear strain

C = A constant, known as shear modulus or modulus of rigidity, denoted by 'G' or 'N'

Consider a square element ABCD subjected to pure shear **τ** as shown in figure.
ABC_1D_1 is the deformed shape due to shear.
Drop perpendicular DD_2 to deformed diagonal BD_1.
Let ϕ be the shear strain, E is modulus of elasticity and G is modulus of rigidity





Strain in diagonal BD $= \dfrac{BD_1 - BD_2}{BD_2}$

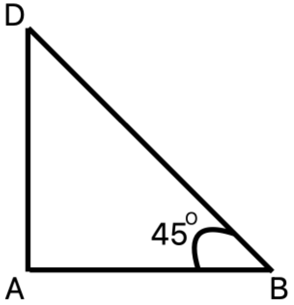
$$= \frac{D_1D_2}{BD}$$

$$= \frac{D_1D_2}{AD\sqrt{2}}$$

$$\because \boldsymbol{Sin45} = \frac{\boldsymbol{AD}}{\boldsymbol{BD}}$$

$$\frac{1}{\sqrt{2}} = \frac{\boldsymbol{AD}}{\boldsymbol{BD}}$$

$$BD = AD\sqrt{2}$$



Relation Between Modulus of Elasticity (E) and Modulus of Rigidity (G or C)

Since angle of deformation is very small, we can assume $\angle DD_1D_2=45^0$, hence, $D_1D_2=DD_1Cos45^0$

Strain in diagonal BD $= \dfrac{D_1D_2}{BD}$

$$= \frac{DD_1Cos45}{AD\sqrt{2}}$$

$$= \frac{ADTan\varphi Cos45}{AD\sqrt{2}}$$

$$= \frac{Tan\varphi}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}Tan\varphi$$

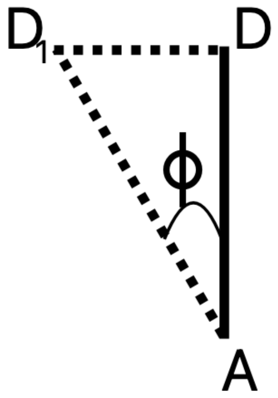
$$= \frac{1}{2}\varphi$$

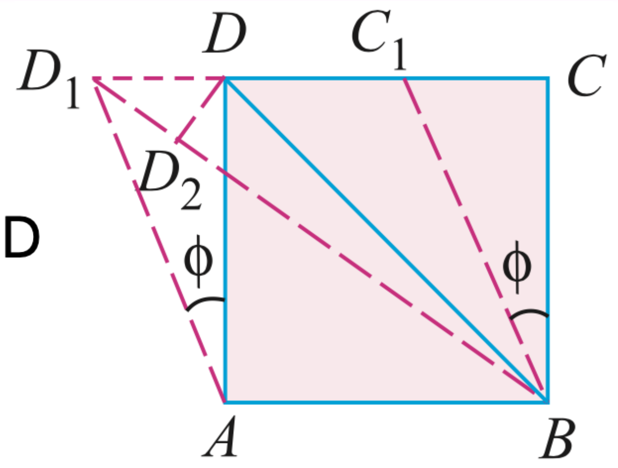
$$= \frac{1}{2} \times \frac{\tau}{G}$$

.......(i)

$$\because Tan\varphi = \frac{DD_1}{AD}$$

$$DD_1 = ADTan\varphi$$





$\because$ φ is very small, $Tan\varphi = \varphi$

$\because$ Shear strain, $\varphi = \dfrac{Shear\ stress}{Modulus\ of\ rigidity}$

Relation Between Modulus of Elasticity (E) and Modulus of Rigidity (G or C)

Due to pure shear, the material experiences axial tensile stress **τ** in the direction of diagonal BD and axial compression **τ** perpendicular to it.

These two stresses cause tensile strain along the diagonal BD.

Tensile strain on BD due to tensile stress on BD = $\frac{\tau}{E}$

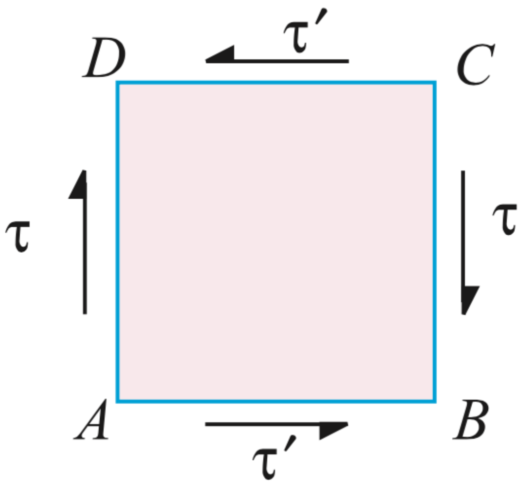
Tensile strain on BD due to compressive stress on BD = $\mu\frac{\tau}{E}$

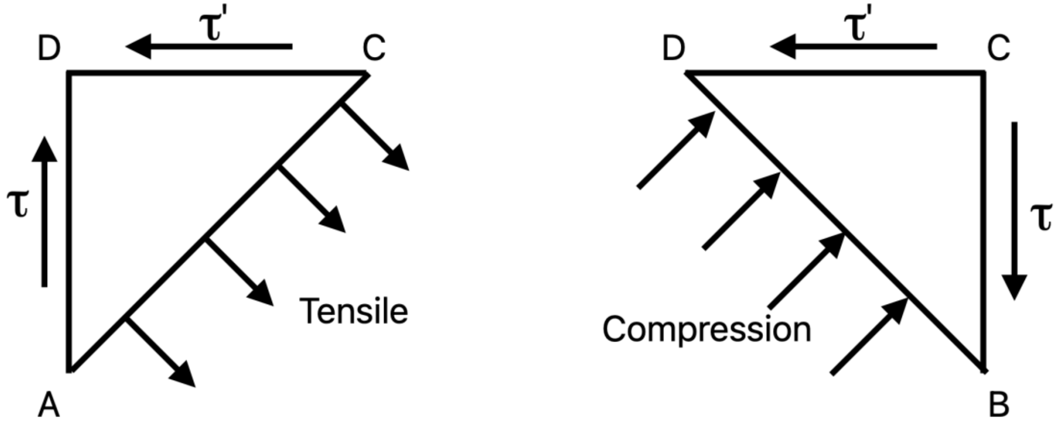
The combined effect of above two stresses on diagonal BD,

$$= \frac{\tau}{E} + \mu\frac{\tau}{E}$$

$$= \frac{\tau}{E}(1+\mu)$$

.......(ii)





From eqn's (i) & (ii)

$$\frac{1}{2}\times\frac{\tau}{G} = \frac{\tau}{E}(1+\mu)$$

$$E = 2G(1+\mu)$$

Or

$$E = 2G\left(1+\frac{1}{m}\right)$$

Problem

An alloy specimen has a modulus of elasticity of 120 GPa and modulus of rigidity of 45 GPa. Determine Poisson's ratio of the material.

Given:

$E = 120 \times 10^3$ N/mm^2

$G = 45 \times 10^3$ N/mm^2

$\mu = ?$

We have, $E = 2G(1 + \mu)$

$$1 + \mu = \frac{E}{2G}$$

$$= \frac{120 \times 10^3}{2 \times 45 \times 10^3} = \frac{4}{3}$$

$$\mu = \frac{4}{3} - 1$$

$$= \frac{1}{3}$$

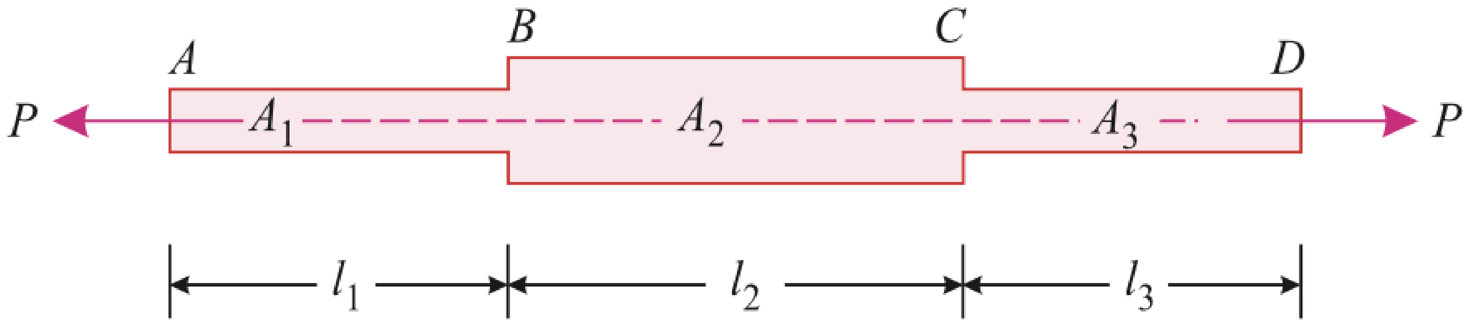
Bars of Varying Sections

1. Bars of different sections
2. Bars of uniformly tapering sections
3. Bars of composite sections

Bars of Different Sections

Bars made up of different lengths having different cross-sectional areas.

- The stresses, strains and changes in lengths for each section is worked out separately.
- Total changes in length is equal to sum of changes of all the individual lengths.



P = Force acting on the body,
E = Modulus of elasticity for the body,
l_1 = Length of section 1,
A_1 = Cross-sectional area of section 1,
l_2, A_2 = Corresponding values for section 2 and so on.

Now, $$\delta l_1 = \frac{Pl_1}{A_1 E}$$

Similarly $$\delta l_2 = \frac{Pl_2}{A_2 E}$$ and so on

$\therefore$ Total deformation of the bar,

$$\delta l = \delta l_1 + \delta l_2 + \delta l_3 + \ldots\ldots\ldots$$

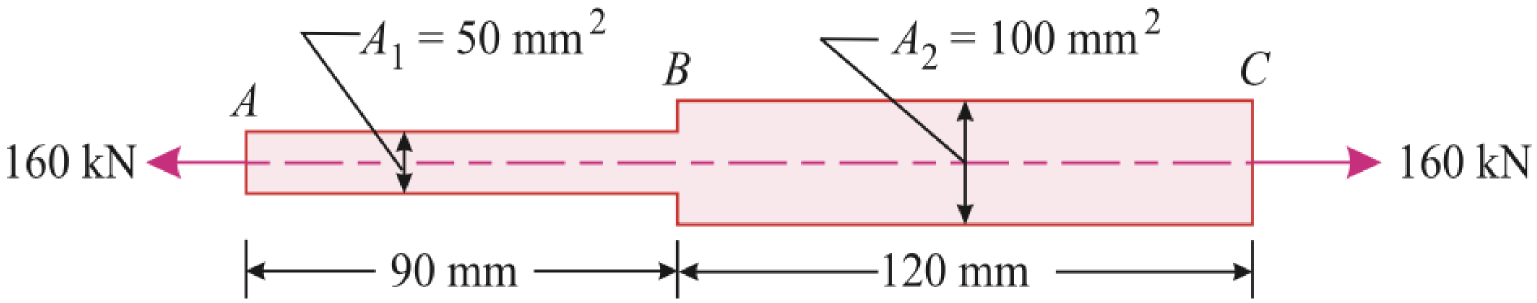
$$= \frac{Pl_1}{A_1 E} + \frac{Pl_2}{A_2 E} + \frac{Pl_3}{A_3 E} + \ldots\ldots\ldots$$

$$= \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} + \ldots\ldots\ldots\right)$$

If modulus of elasticity is different,

$$\delta l = P\left(\frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2} + \frac{l_3}{A_3 E_3} + \ldots\ldots\ldots\right)$$

An automobile component shown in figure is subjected to a tensile load of 160 kN. Determine the total elongation of the component, if its modulus of elasticity is 200 GPa.



Given: Tensile load (P) = 160 kN = 160×10^3 N ; Length of section 1 (l_1) = 90 mm;

Length of section 2 (l_2) = 120 mm ; Area of section 1 (A_1) = 50 mm^2 ; Area of section 2 (A_2) = 100 mm^2 and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

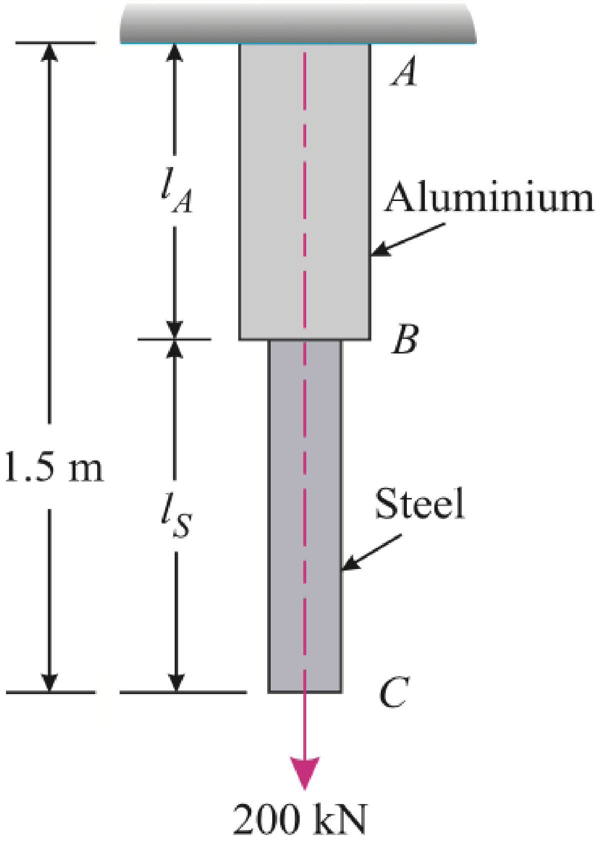
We know that total elongation of the component,

$$\delta l = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2}\right) = \frac{160 \times 10^3}{200 \times 10^3}\left(\frac{90}{50} + \frac{120}{100}\right) \text{ mm}$$

$$= 0.8 \times 1.8 + 1.2 = 2.4 \text{ mm} \qquad \textbf{Ans.}$$

Problem

A compound bar ABC 1.5 m long is made up of two parts of aluminium and steel and that cross-sectional area of aluminium bar is twice that of the steel bar. The rod is subjected to an axial tensile load of 200 kN. If the elongations of aluminium and steel parts are equal, find the lengths of two parts of the compound bar. Take E for steel as 200 GPa and E for aluminium as one-third of E for steel.



Given: Total length $(L) = 1.5 \text{ m} = 1.5 \times 10^3$ mm ;

Cross-sectional area of aluminium bar $(A_A) = 2\,A_S$; Axial tensile load $(P) = 200 \text{ kN} = 200 \times 10^3$ N ; Modulus of elasticity of steel $(E_S) = 200$ GPa $= 200 \times 10^3$ N/mm^2 and modulus of elasticity of aluminium $(E_A) =$

$$\frac{E_S}{3} = \frac{200 \times 10^3}{3} \text{ N/mm}^2.$$

Let, l_A = Length of the aluminium part,

and l_S = Length of the steel part.

We know that elongation of the aluminium part *AB*,

$$\delta l_A = \frac{P . l_A}{A_A . E_A} = \frac{(200 \times 10^3) \times l_A}{2A_S \times \left(\dfrac{200 \times 10^3}{3}\right)} = \frac{1.5\, l_A}{A_S} \qquad ...(i)$$

Problem

and elongation of the steel part *BC*,

$$\delta l_S = \frac{P.l_S}{A_S.E_S} = \frac{(200\times10^3)\times l_S}{A_S\times(200\times10^3)} = \frac{l_S}{A_S} \qquad \ldots(ii)$$

Since elongations of aluminium and steel parts are equal, therefore equating equations (*i*) and (*ii*),

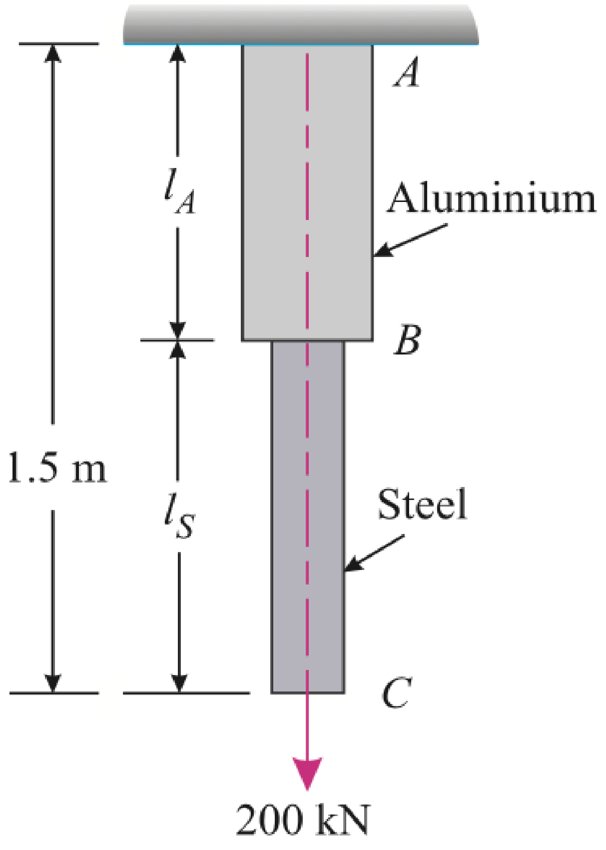
$$\frac{1.5\,l_A}{A_S} = \frac{l_S}{A_S} \qquad \text{or} \qquad l_S = 1.5\,l_A$$

We also know that total length of the bar *ABC* (*L*)

$$1.5\times10^3 = l_A + l_S = l_A + 1.5\,l_A = 2.5\,l_A$$

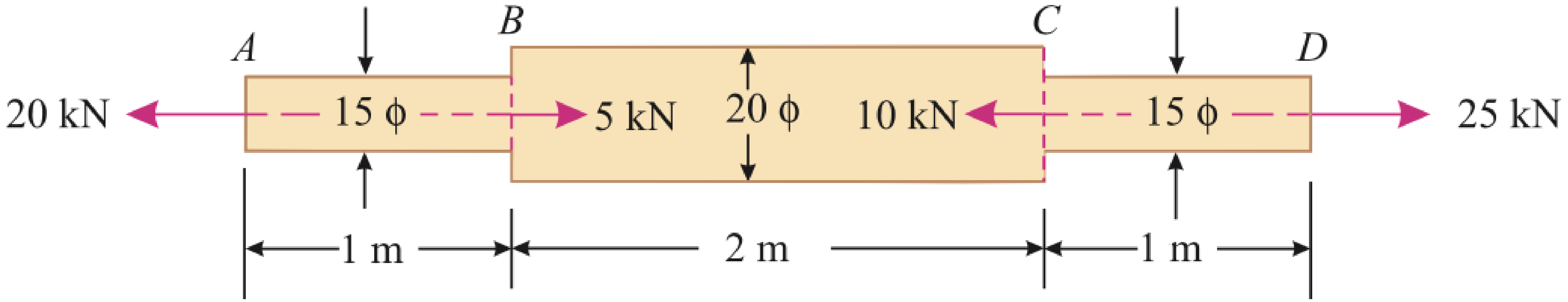
$\therefore$ $$l_A = \frac{1.5\times10^3}{2.5} = 600\text{ mm} \qquad \textbf{Ans.}$$

and $$l_S = (1.5\times10^3) - 600 = 900\text{ mm} \qquad \textbf{Ans.}$$



Problem

A steel bar ABCD 4 m long is subjected to forces as shown in figure. Find the elongation of the bar. Take E for the steel as 200 GPa.

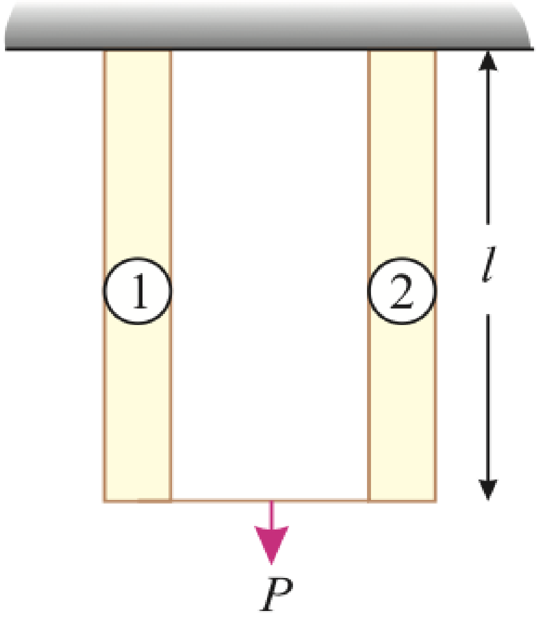


Bars of Composite Sections

A structure made up of two or more different material bars is called a composite bar.

The bars are joined in such a manner that the system extends or contracts as one unit equally, when subjected to tension or compression.

- Strain is equal for all the bars
- Total external load is equal to sum of the loads carried by the different materials of the structure



Consider a composite bar made up of two different materials as shown in figure

Let, P = Total load on the bar

l_1 = Length of the bar 1

l_2 = Length of the bar 2

A_1 = Area of bar 1

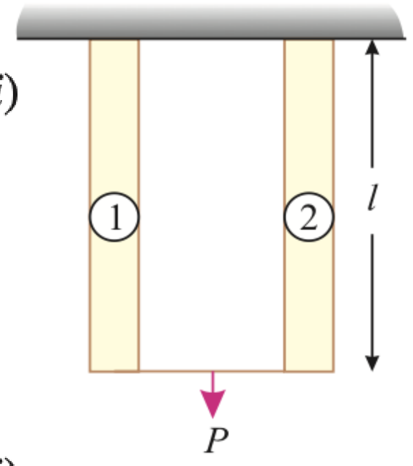
E_1 = Modulus of elasticity of bar 1

P_1 = Load shared by bar 1

A_2, E_2, P_2 = Corresponding values for bar 2

Total load on the bar

$$P = P_1 + P_2 \quad \text{...(i)}$$



$\therefore$ Stress in bar 1,
$$\sigma_1 = \frac{P_1}{A_1}$$

and strain in bar 1,
$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{A_1 E_1}$$

$\therefore$ Elongation,
$$\delta l_1 = \varepsilon_1 . l_1 = \frac{\sigma_1 l_1}{E_1} = \frac{P_1 l_1}{A_1 E_1} \quad \text{...(ii)}$$

Similarly, elongation of bar 2,

$$\delta l_2 = \varepsilon_2 . l_2 = \frac{\sigma_2 l_2}{E_1} = \frac{P_2 l_2}{A_2 E_2} \quad \text{...(iii)}$$

Since both the elongations are equal, therefore equating (*ii*) and (*iii*), we get $\delta l_1 = \delta l_2$

$$\frac{P_1 l}{A_1 E_1} = \frac{P_2 l}{A_2 E_2} \quad \text{or} \quad \frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \quad \text{...(iv)}$$

or
$$P_2 = P_1 \times \frac{A_2 E_2}{A_1 E_1}$$

But
$$P = P_1 + P_2 = P_1 + P_1 \times \frac{A_2 E_2}{A_1 E_1}$$
$$= P_1 \left(1 + \frac{A_2 E_2}{A_1 E_1}\right) = P_1 \left(\frac{A_1 E_1 + A_2 E_2}{A_1 E_1}\right)$$

or
$$P_1 = P \times \frac{A_1 E_1}{A_1 E_1 + A_2 E_2} \qquad \ldots(v)$$

Similarly,
$$P_2 = P \times \frac{A_2 E_2}{A_1 E_1 + A_2 E_2} \qquad \ldots(vi)$$

From these equations we can find out the loads shared by the different materials. We have also seen in equation (*iv*) that

$$\frac{Pl_1}{A_1 E_1} = \frac{Pl_2}{A_2 E_2}$$

Bars of Composite Sections

or $$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \qquad \cdots\left(\because \frac{P}{A} = \sigma = \text{Stress}\right)$$

$\therefore$ $$\sigma_1 = \frac{E_1}{E_2} \times \sigma_2 \qquad \ldots(vii)$$

Similarly, $$\sigma_2 = \frac{E_2}{E_1} \times \sigma_1 \qquad \ldots(viii)$$

From the above equations, we can find out the stresses in the different materials. We also know that the total load,

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

From the above equation, we can also find out the stress in the different materials.

Consider a circular bar AB of uniformly tapering circular section as shown in figure.

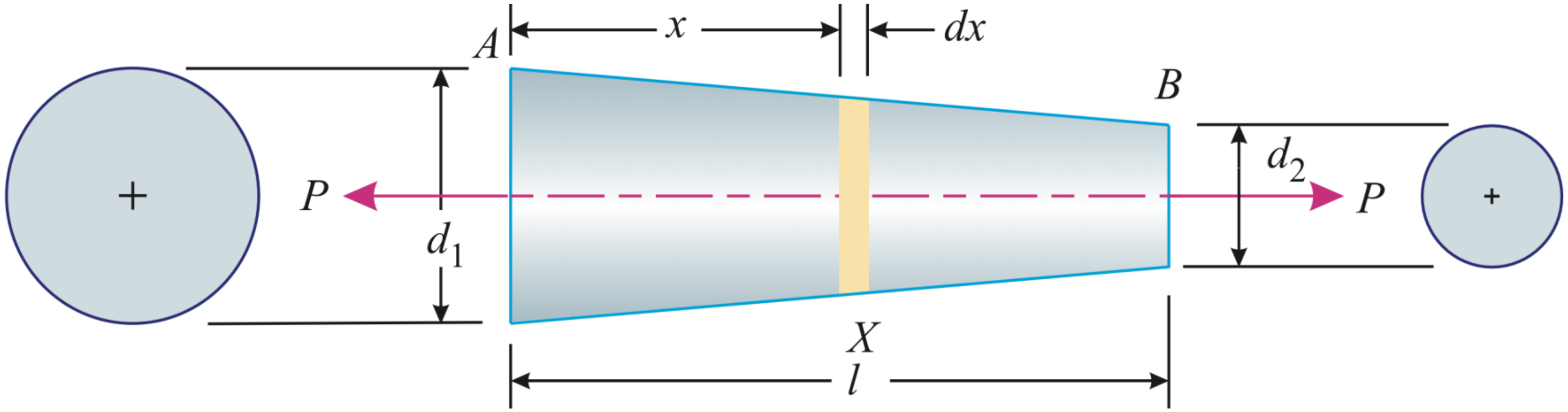
Let,

P = Pull on the bar

l = Length of the bar

d_1 = Diameter of the bigger end of the bar

d_2 = Diameter of the smaller end of the bar



Consider small element of length ‘dx’ at a distance of ‘x’ from bigger end of the bar. Now, for diameter of the bar at this section:

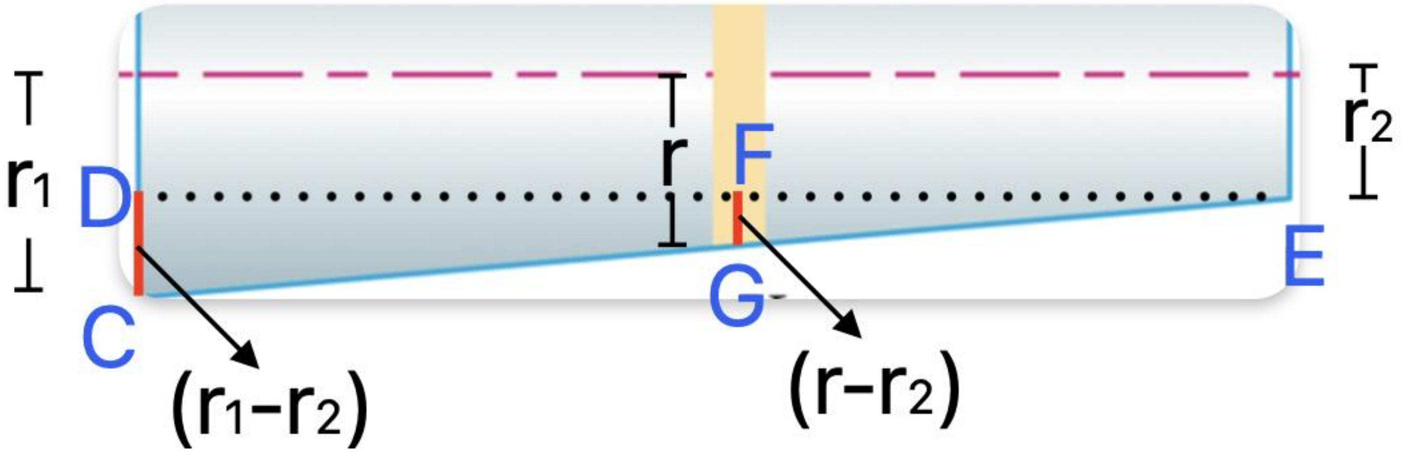
From the figure,

Consider similar △CDE and △GFE

$$\frac{FG}{CD} = \frac{FE}{DE}$$

$$\frac{r - r_2}{r_1 - r_2} = \frac{l - x}{l}$$

$$r - r_2 = \frac{(r_1 - r_2)(l - x)}{l}$$



Bars of Uniformly Tapering Circular Sections

$$r = r_2 + \frac{(r_1 - r_2)(l - x)}{l}$$

$$d = d_2 + \frac{(d_1 - d_2)(l - x)}{l}$$

$$= d_2 + \frac{(d_1 - d_2)}{l} \times l - \frac{(d_1 - d_2)}{l} \times x$$

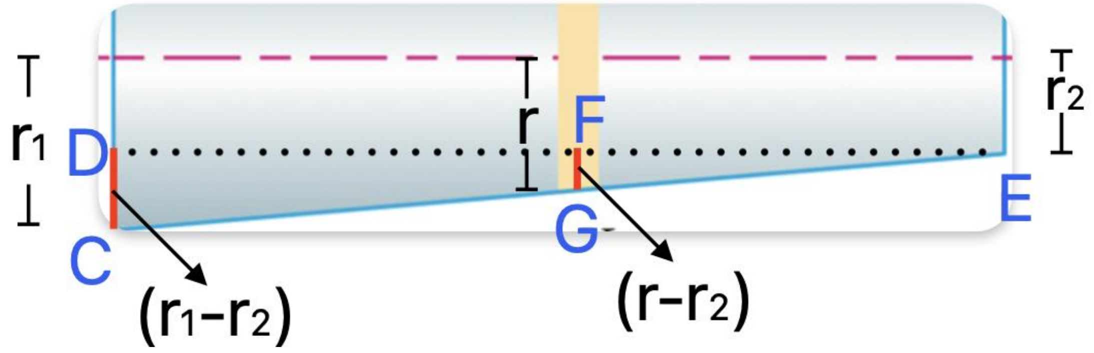
$$d = d_1 - \frac{(d_1 - d_2)}{l} \times x$$

Let, $\frac{(d_1 - d_2)}{l}$ = k

$$\therefore d = d_1 - kx$$

Cross-sectional area of the bar at this section,

$$A_X = \frac{\pi}{4}(d_1 - kx)^2$$



$\therefore$ Stress,

$$\sigma_X = \frac{P}{\frac{\pi}{4}(d_1 - kx)^2} = \frac{4P}{\pi(d_1 - kx)^2}$$

and strain,

$$\varepsilon_X = \frac{\text{Stress}}{E} = \frac{\frac{4P}{\pi(d_1 - kx)^2}}{E} = \frac{4P}{\pi(d_1 - kx)^2 E}$$

Bars of Uniformly Tapering Circular Sections

Elongation of the elementary length

$$= \varepsilon_X . dx = \frac{4P . dx}{\pi (d_1 - kx)^2 E}$$

Total extension of the bar may be found out by integrating the above equation between the limit 0 and l.

$$\delta l = \int_0^l \frac{4P . dx}{\pi (d_1 - kx)^2 E}$$

$$= \frac{4P}{\pi E} \int_0^l \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{4P}{\pi E} \left[\frac{(d_1 - kx)^{-1}}{-1 \times -k} \right]_0^l$$

$$= \frac{4P}{\pi E k} \left[\frac{1}{d_1 - kx} \right]_0^l$$

$$= \frac{4P}{\pi E k} \left[\frac{1}{d_1 - kl} - \frac{1}{d_1} \right]$$

Substituting the value of $\frac{(d_1 - d_2)}{l}$ = k

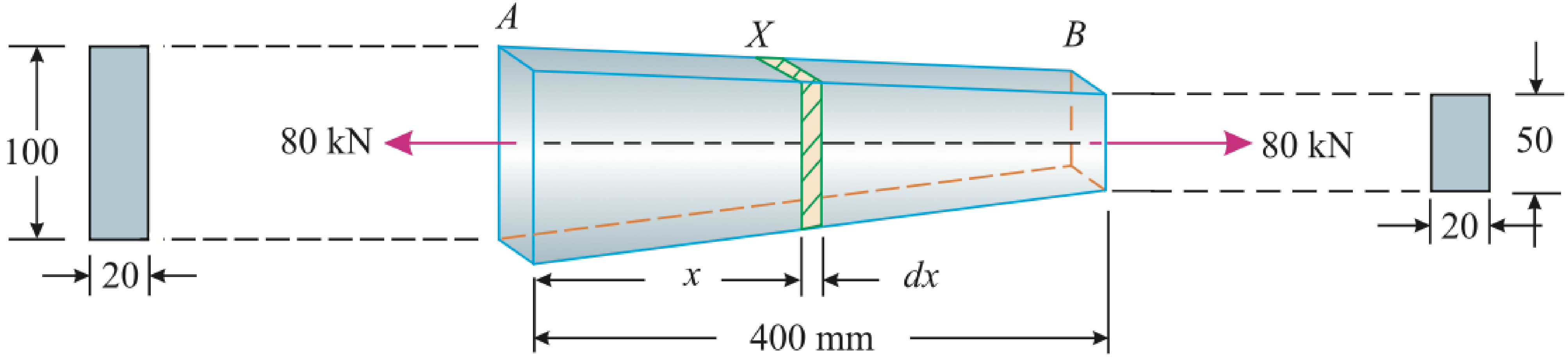
$$\delta l = \frac{4P}{\pi E \frac{(d_1 - d_2)}{l}} \left[\frac{1}{d_1 - \frac{(d_1 - d_2)\, l}{l}} - \frac{1}{d_1} \right]$$

$$= \frac{4Pl}{\pi E (d_1 - d_2)} \left[\frac{1}{d_2} - \frac{1}{d_1} \right] = \frac{4Pl}{\pi E (d_1 - d_2)} \left[\frac{d_1 - d_2}{d_2 d_1} \right]$$

$$\delta l = \frac{4Pl}{\pi E d_2 d_1}$$

Bars of Uniformly Tapering Rectangular Sections

The uniformly tapering section may vary from square section at one end to another square section at the other end, or it may vary from rectangular section at one end to another rectangular section at the other end. In such cases stresses should be found out from the fundamentals.



From the geometry of the figure, the width of the plate at a distance ‘x’ from ‘A’ is

$$= 100 - (100 - 50) \times \frac{x}{400} = 100 - 0.125\,x \ \text{ mm}$$

From this calculate the cross-sectional area, stress, strain and elementary elongation; then integrate the equation with limits applied to get the total elongation of the bar.

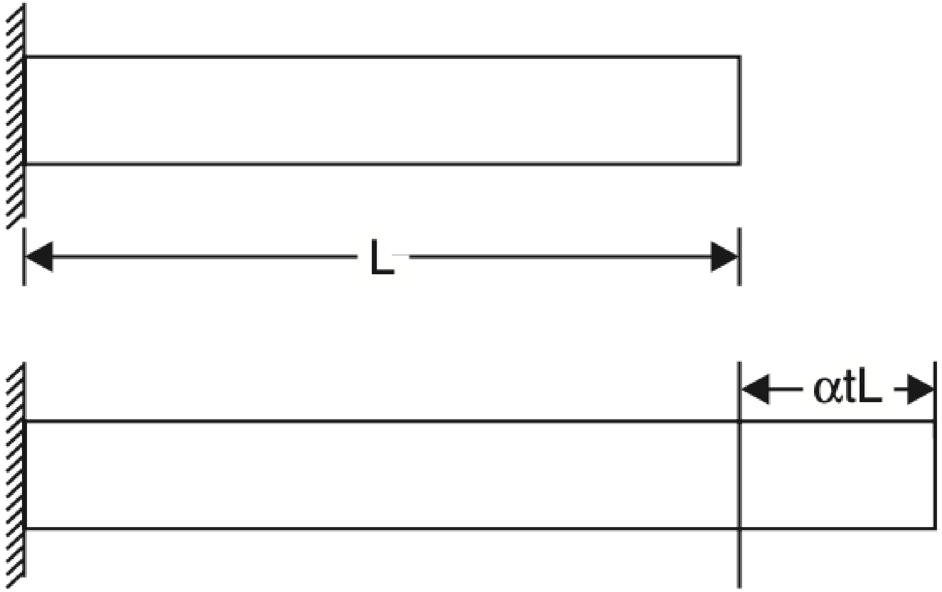
Thermal Stresses

Every material expands when temperature rises and contracts when temperature falls.
It is established experimentally that the change in length 'Δ' is directly proportional to the product of length of the member 'L' and change in temperature 't'.
Thus,

$$\Delta \propto tL$$
$$\Delta = \alpha tL$$

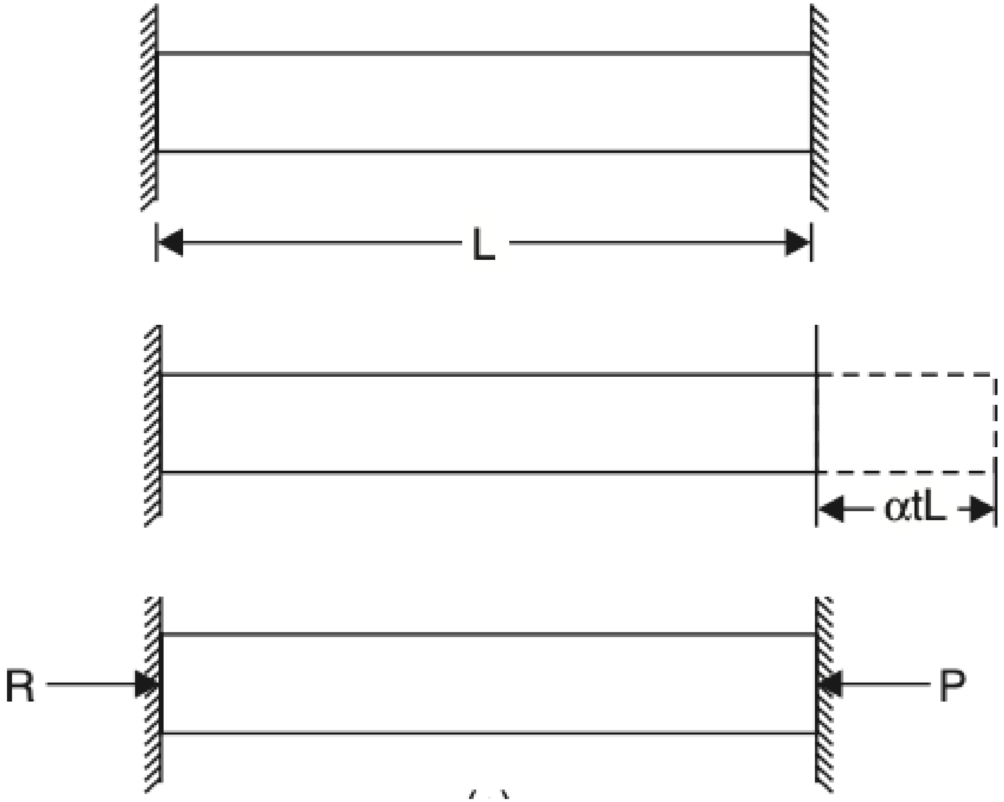
The constant of proportionality 'α' is called coefficient of thermal expansion and is defined as change in unit length of material due to unit change in temperature.

If expansion is freely permitted, no temperature stresses are induced in the material



Thermal Stresses

If the free expansion is prevented fully or partially the stresses are induced in the bar, by the support forces



Fully restricted:

If free expansion is permitted the bar would have expanded by, $\square = \square\, tL$

Since support is not permitting it, the support force P develops to keep it at the original position.
Magnitude of this force is such that contraction is equal to free expansion, i.e.

$$\frac{PL}{AE} = \square\, tL$$

or, $\boldsymbol{\sigma} = E\,\alpha\, t$

Where '**σ**' is the temperature stress and is compressive in nature in this case.

Thermal Stresses

If the free expansion is prevented fully or partially the stresses are induced in the bar, by the support forces

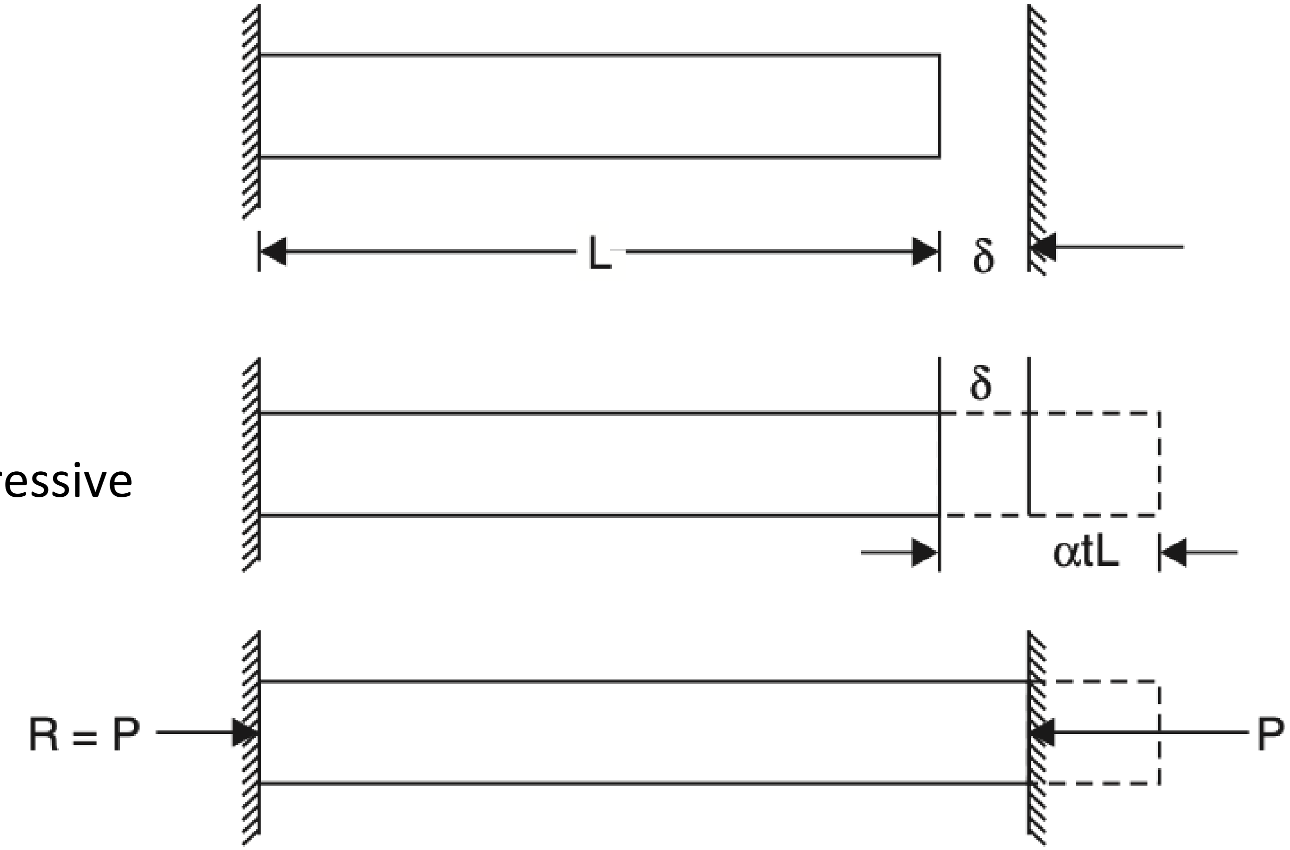
Partially restricted:

If free expansion is permitted the bar would have expanded by, $= \langle\, tL$

Expansion prevented, $= \langle\, tL -$ ™

The expansion is prevented by developing compressive force P at supports, i.e.

$$\therefore \frac{PL}{AE} = \square = \square\, tL - \square$$



Problem

A steel rail is 12 m long and is laid at a temperature of 18^0C. The maximum temperature expected is 40^0C.

(*i*) *Estimate the minimum gap between two rails to be left so that the temperature stresses do not develop.*

(*ii*) *Calculate the temperature stresses developed in the rails, if:*

(*a*) *No expansion joint is provided.*

(*b*) *If a 1.5 mm gap is provided for expansion.*

(*iii*) *If the stress developed is 20 N/mm^2, what is the gap provided between the rails?*

Take $E = 2 \times 10^5$ N/mm^2 and $\alpha = 12 \times 10^{-6}$/°C.

(*i*) The free expansion of the rails

$$= \alpha\, tL = 12 \times 10^{-6} \times (40 - 18) \times 12.0 \times 1000$$

$$= 3.168 \text{ mm}$$

∴ **Provide a minimum gap of 3.168 mm between the rails, so that temperature stresses do not develop.**

Problem

(*ii*) (*a*) If no expansion joint is provided, free expansion prevented is equal to 3.168 mm.

i.e. $$\Delta = 3.168 \text{ mm}$$

$\therefore$ $$\frac{PL}{AE} = 3.168$$

$\therefore$ $$\mathbf{p} = \frac{P}{A} = \frac{3.168 \times 2 \times 10^5}{12 \times 1000} = \mathbf{52.8\ N/mm^2}$$

(*b*) If a gap of 1.5 mm is provided, free expansion prevented $\Delta = \alpha\, tL - \delta = 3.168 - 1.5 =$ 1.668 mm.

$\therefore$ The compressive force developed is given by $\frac{PL}{AE} = 1.668$

or $$\mathbf{p} = \frac{P}{A} = \frac{1.668 \times 2 \times 10^5}{12 \times 1000} = \mathbf{27.8\ N/mm^2}$$

Problem

(*iii*) If the stress developed is 20 N/mm^2, then $p = \dfrac{P}{A} = 20$

If δ is the gap, $\Delta = \alpha\, tL - \delta$

$$\therefore \qquad \frac{PL}{AE} = 3.168 - \delta$$

$$i.e. \qquad 20 \times \frac{12 \times 1000}{2 \times 10^5} = 3.168 - \delta$$

$$\therefore \qquad \boldsymbol{\delta} = 3.168 - 1.20 = \mathbf{1.968\ mm}$$

Thank You