MC

Module – 4

1) a) odd integer

- let the odd integer be in the form $2n+1$ and form algebraic system 'O'

- for multiplication $(*, O)$

$$= (2n+1)(2n+1)$$

$$= 4n^2+4n+1$$

$$= 4(n^2+n)+1$$

$$= 2(2n^2+2n)+1$$

$$= 2s+1$$

The product of two odd number results odd, so satisfies properties of binary system under '*'.

- for addition $(+, O)$

$$= (2n+1)+(2n+1)$$

$$= 4n+2$$

$$= 2(2n+1)$$

$$= 2s$$

The sum of two odd numbers is resulting even so doesn't satisfy properties of binary system under '+'.

b) All positive integers.

let all the positive integers (1, 2, 3 ..) form algebraic system Z^+

— under multiplication

let n be a positive integer

$= n * n$

$= n^2$

multiplication of two positive integer results positive, so satisfies properties under binary operations of multiplication

— under addition

let n be a positive integer

$= n+n$

$= 2n$

addition of two positive integer results positive, so satisfies properties of binary operation of addition

Solve that (Z,*) is an abelian group where Z is a set of integers and the binary erations * is defined as a*b = a+b-3.

2) $(Z,*)$ – abelian group, where Z is set of integers

$*$ is defined as $a*b=a+b-3$

i) closure axiom: since a,b and -3 are integers, $a+b-3$ is also integer

$\therefore a*b \in Z \ \forall a,b \in Z$

ii) Associative axiom: let $a,b,c * G$.

$$(a*b)*c = (a+b-3)*c$$
$$= (a+b-3)+c-3$$
$$= a+b+c-6$$

$$a*(b*c) = a*(b+c-3) = a+(b+c-3)-3.$$
$$= a+b+c-6$$

$\Rightarrow a*(b*c) = (a*b)*c$

Thus associative axiom is true [• in doubt]

iii) Identity axiom:

let e be the identity axiom.

by defination of e, $a*e=a$.

by the defination of $*$, $a*e = a+e-3$

$\Rightarrow a+e-3=a$

$\Rightarrow e=3$

since $3 \in Z$ The identity axiom is true

iv) Inverse axiom:

let $a \in G$ and $\bar{a}'$ be the inverse element of a.

By defination of $\bar{a}'$, $a*\bar{a}' = e = +3$

By defination of $*$, $a*\bar{a}' = a+\bar{a}'-3$

$\Rightarrow a+\bar{a}'+(-3) = +3$

$\bar{a}' = -a+6$

The Inverse axiom is true.

$\therefore (Z,*)$ is a group.

v) commutative property

let $a,b \in G$

$a*b = a+b-3 = b+a-3 = b*a$

$\therefore *$ is commutative.

$\therefore (Z,*)$ is an abelian group also Z is an infinite set

$\therefore Z$ is an infinite abelian group.

3) If o is an operation on Z defined by xoy=x+y+1, Prove that <Z,o> is an abelian group.

3) o is operation of Z
defined by $x o y = x+y+1$
T.P $\Rightarrow$ (Z, o) is abelian group.

since $x, y \in Z \Rightarrow x+y+1 \in Z$, Therefore Z is closed under o

$\forall x, y, w \in Z$

$$w o (x o y) = (x+y+1) o w$$
$$= w + (x+y+1)+1$$

$$(w o x) o y = (w+x+1) o y$$
$$= (w+x+1)+y+1$$
$$= w+(x+y+1)+1$$

Since o is associative

$$w o (x o y) = (w o x) o y$$

also o is commutative

$$x o y = x+y+1 = y+x+1 = y o x$$

If $x \in Z$, -1 is an identity element for o, then

$$x o (-1) = x+(-1)+1 = x = [(-1) o x]$$

again $x \in Z$, $\Rightarrow -x-2 \in Z$ and

$$x o (-x-2) = x+(-x-2)+1$$
$$= -1$$
$$= (-x-2) o x$$

4) On the set Q of all rational numbers, the operation * is defined by a * b= a+ b- ab. Show that under this operation Q forms a commutative monoid.

4) Q — all rational numbers

$*$ defined by $a * b = a + b - ab$

since ~~x, y ∈~~ $a, b \in Q$,

since $a, b, c \in Q$

$(a * b) * c = (a + b - ab) * c$

$= c + (a + b - ab) - ab$

$a * (b * c) = a + (b + c - ab) - ab$

$= c + (a + b - ab) - ab$

since $*$ is associative

$a * (b * c) = (a * b) * c$

also $*$ is commulative

$a * b = a + b - ab = b + a - ab = b * a$

If $a \in Q$, ab an identity element for $*$ then

by defination of e, $a * e = a$

by defination of $*$, $a * e = a + e - ab$

$a + e - ab = a$

$e = ab$

since $a, b \in Q$ then $ab \in Q$. The identity axiom is true

— since its commutative, associative and have identity element, forms commutative monoid.

Prove that if $(ab)^{-1} = a^{-1}b^{-1}$, for all a, b ϵG then G is abelian

Proof:

Given $(ab)^{-1} = a^{-1}.b^{-1}$........................(1), for all a, b ϵG

Since G is a group, we have $(ab)^{-1} = b^{-1}.a^{-1}$ ....(2), for all a, b ϵ G

From (1) and (2), $b^{-1}a^{-1} = a^{-1}b^{-1}$

Taking the inverse

$(b^{-1}a^{-1})^{-1} = (a^{-1}\,b^{-1})^{-1}$

$(b^{-1})^{-1}(a^{-1})^{-1} = (a^{-1})^{-1}(b^{-1})^{-1}$ [since $(a^{-1})^{-1} = a$, for all a ϵG]

$\Rightarrow ba = ab$

 G is an abelian group.

6)Choose the number of row Of 6 Americans, 7 Mexicans And 10 Canadians in which An American invariably Stands between a Mexican And a Canadian never standing Side by side.

7) From the words. (a) TALLAHASSEE (b) MISSISSIPPI How many arrangements can be made such that, (a) No two letters A of TALLAHASSEE appear together Number of 4 letter words for both the given words.

3) **Find how many integers Between 1 and 10^4 contain Exactly one 8 and one 9.**

8) exactly one 8 and one 9 between 1 & 10^4

to place 8 in above 4 places — $^4C_1$ ways
to place 9 after that — $^3C_1$ ways
to fill other 2 places with remaining digits = 8×8 ways
total no of ways to do this $= {}^4C_1 \times {}^3C_1 \times 8 \times 8$
$= 4 \times 3 \times 8 \times 8$
$= 768$

There are 4 places to put the 8, 3 to put the 9 after that and 8×8 ways to assign the other two digits. Thus there are $4 \times 3 \times 8 \times 8 = 768$ admissible numbers.

9)Choose how many integers between 10^5and10^6 ,(i) Have no digit other tha
Have no digit other than 0,2,5or 8.

9) between ~~10,000~~ 1,00,000 and 10,00,000

6 empty space

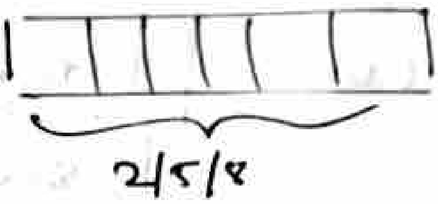
i) no other digits than 2, 5, 8

so we have 3 choices for first place, 3 for second and so on

$\Rightarrow 3 \times 3 \times 3 \times 3 \times 3 \times 3$

$\Rightarrow (3)^6$

$\Rightarrow 729$



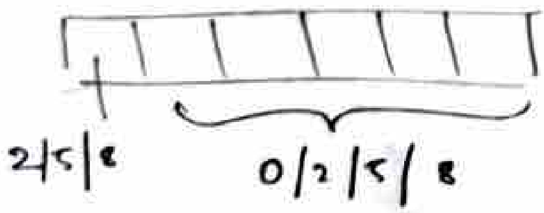
ii) no digit other than 0, 2, 5 or 8

– 0 cannot be filled in first place but other 3 can be

so first box has 3 choices and remaining all other have 4 choices

$\Rightarrow 3 \times 4 \times 4 \times 4 \times 4 \times 4$

$\Rightarrow 3072$



10) MISSISSIPPI

arrangement of word MISSISSIPPI with no two pair of consecutive same letter

$\Rightarrow$ Total $-$ arrangement of consecutive same letter

$$\text{Total} = \frac{11!}{4!\,4!\,2!}$$

arrangement of consecutive same letter

$$\Rightarrow \underbrace{\underset{\text{1 unit}}{\boxed{M}}\ \underset{\text{1 unit}}{\boxed{IIII}}\ \underset{\text{1 unit}}{\boxed{SSSS}}\ \underset{\text{1 unit}}{\boxed{PP}}}_{\text{4 units}} \quad \text{— } 4! \text{ ways.}$$

$$\Rightarrow \frac{11!}{(4!)^2\,2!} - 4!$$

$\Rightarrow$ 34624

Definition 14.1. *A **ring** is a nonempty set R equipped with two operations $\oplus$ and $\otimes$ (more typically denoted as addition and multiplication) that satisfy the following conditions. For all $a, b, c \in R$:*

(1) *If $a \in R$ and $b \in R$, then $a \oplus b \in R$.*
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
(3) $a \oplus b = b \oplus a$
(4) *There is an element 0_R in R such that*

$$a \oplus 0_R = a \quad , \quad \forall\, a \in R \quad .$$

(5) *For each $a \in R$, the equation*

$$a \oplus x = 0_R$$

has a solution in R.
(6) *If $a \in R$, and $b \in R$, then $ab \in R$.*
(7) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.
(8) $a \otimes (b \oplus c) = (a \otimes b) \oplus (b \otimes c)$

Definition 14.2. *A **commutative ring** is a ring R such that*

(14.1) $$a \otimes b = b \otimes a \quad , \quad \forall\, a, b \in R \quad .$$

Definition 14.3. *A **ring with identity** is a ring R that contains an element 1_R such that*

(14.2) $$a \otimes 1_R = 1_R \otimes a = a \quad , \quad \forall\, a \in R \quad .$$

2) Let G be the set of all non-zero real numbers and let a*b= ½ ab. Show that <G,*> is an abelian Group.

1. First verify * is a binary operatin.

If a and b $\in$ G then $a * b\left(= \frac{ab}{2}\right)$ is a nonzero real number,

$\therefore a * b \in G$

2. Associativity : Since

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4}$$

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4}$$

$$= \frac{(ab)c}{4}$$

$\therefore$ $*$ is associative.

The number 2 is the identity in G and $a \in G$ then

$$a * 2 = \frac{(a)(2)}{2} = a = \frac{(2)(a)}{2} = 2 * a$$

Finally, if $a \in G$ then $a' = \frac{4}{a}$ is an inverse of a.

$$\because a * a' = a * \frac{4}{a} = \frac{a(4/a)}{2}$$

$$= 2\frac{(4/a)(a)}{2} = \frac{4}{a} * a = a * a'$$

Since $a * b = b * a \forall a, b \in G$ we conclude that G is an abelian group.

$a + b + ab = -1$

$b + ab = -1 - a$

$b(1+a) = -1(1+a)$

$b = -1$ (since $a \neq -1, 1+a \neq 0$)

This is impossible, because $b \neq -1$.

$\therefore$ our assumption is wrong

$\therefore$ $a * b \neq -1$ and hence $a*b \in G$

$\therefore$ closure axiom is true.

ii) associative axiom:

let $a, b, c \in G$

$a * (b*c) = a * (b+c+bc)$

$= (\cancel{a+b+bc})$.

$= a + (b+c+bc) + a(b+c+bc)$

$= a+b+c+bc+ab+ac+abc$

$(a*b)*c = (a+b+ab)*c$

$= (a+b+ab)+c+(a+b+ab)c$

$= a+b+c+ab+ac+ab+abc$

$(a*b)*c = a*(b*c)$

$\therefore$ its true

iii) Identity axiom:

let e be identity element by defination

$\Rightarrow e*a = a$.

By defination of $*$, $a*e = a+e+ae$.

$a + e + ae = a$

$e(1+a) = 0$

$e = 0$ since $a \neq -1$

$e = 0 \in G$

Identity axiom is satisfied

iv) Inverse axiom:

Let a^{-1} be the inverse of $a \in G$

by defination $a * a^{-1} = e = 0$

by defination of $*$, $a * a^{-1} = a + a^{-1} + a a^{-1}$

$a + a^{-1} + a a^{-1} = 0$

$a^{-1}(1+a) = -a$

$a^{-1} = \dfrac{-a}{1+a} \in G$ since $a \neq -1$

The inverse axiom is satisfied $\therefore (G, *)$

v) Commutative axiom: For any $a, b \in G$

$a * b = a + b + ab = b + a + ba = b * a$

$\therefore$ $*$ is commutative in G and hence $(G, *)$ is an abelian group.

and thus the relation becomes

$$(ab)(ab) = a^2b^2.$$

Equivalently, we can express it as

$$abab = aabb.$$

Multiplying by a^{-1} on the left and b^{-1} on the right, we obtain

$$a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1}.$$

5) show that If A ={1,-1,I,-1} are the fourth roots of unity. Show that (A,*) forms a group.

a.1) First I need to show that G is indeed closed under the operation *

we have $1 * 1 = 1$ where $1 \in G$

we have $-1 * -1 = 1$ where $1 \in G$

we have $1 * -1 = -1$ where $-1 \in G$ and $-1 * 1 = -1 \in G$

we have $1 * i = i$ and $i * 1 = i$ where $i \in G$

we have $-1 * i = -i$ and $i * -1 = -i$ where $-i \in G$

let $k \in \mathbb{N}$ then $i^{2k} = -1$ where $-1 \in G$

Finally let $k \in \mathbb{N}$ then we have $i^{2k+1} = -i$ where $-i \in G$

Okay so I have shown that all possible outcomes from every combination of multiplication between any elements yields an element in G

a.2) To show that this is a group it must be associative. We may assume that since the roots of unity is associative that G is therefore associative.

a.3) Since $1 \in G$ we can see that:

$i * 1 = i, \ -i * 1 = -i, \ 1 * 1 = 1, \ -1 * 1 = -1$
This shows that every element has an identity element

a.4) Does every element have an inverse? well we have
$1 * 1 = e, -1 * -1 = e,$
$i * i * -1 = e, -i * -i = e$ Yes

Since it has all the properties forms a group

Part.-B

6) Given, no. of boys = 6, no. of girls = 5

$\Rightarrow$ Total no. of objects = 6+5 = 11

~~We~~ Total no. of ways of selecting 11 people is 2^{11}.

[$\because$ We can select 0 people, 1 person, 2 people, 3 people, —— 11 people. There is no restriction that we should select only a particular no. of people. So, total no. of selections are :-

$^{11}C_0 + {^{11}C_1} + {^{11}C_2} + {^{11}C_3}$ —— —— $+ {^{11}C_{11}} = 2^{11}$]

7) Given, no. of trousers = 4, no. of shirts = 3.

Total no. of ways of selecting a pair :- $^4C_1 \times {^3C_1}$

$\Rightarrow {^4C_1} \times {^3C_1} = 4 \times 3 = 12$

8) A person can travel from Hyderabad to Chennai in 4 ways and ~~that~~ can travel from Chennai to Banglore in 3 ways. So, the total no. of ways of travelling from Hyderabad to Banglore via Chennai is 4×3 = 12 ways.

9) Given, ~~Total~~ No. of numbers (or) elements = 5. We can form a 3 digit number from a given five numbers without repetition is

$^5P_3 = {^5C_3} \times 3!$

$= 10 \times 3! = 10 \times 6$

$= 60$

We can form three digit numbers ~~with rep~~ which can be repeated ~~or not~~ cannot be repeated ~~is in~~ by using 5 numbers (or) elements in

$5^3 = 125$ ways.

~~11) We can select upto 7 people because the total no. of people to select is 7. We ~~ ~~cannot select 8 people and 9 people.~~

~~Total~~ no. of selections

~~=2^7~~.

~~$= {}^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3$~~

9) We cannot select 9 people since the total no. of people to be selected is 7.

10) Given, No. of boys = 5,
No. of girls = 4

~~Now,~~
We should fix the first position and the last position for ~~any~~ boys and we can arrange any two boys in these two positions. This can be done in 2! i.e 2 ways.

We can arrange the remaining boys, i.e., ~~5-3~~ 5-2 = 3 boys and 4 girls in ~~5~~ (3+4)! = 7! = 5040 ways.

So, the total no. of ways of ~~arranging~~ arrangement is ~~5~~ $2! \times 7! = 2 \times 5040 = 10080$.

11) Given, No. of Indians = 5,
No. of Russians = 4

We must form a committe of 5 members in which at least 3 ~~are~~ are from Indians.

The no. of ways of doing this is :

Indians	Russians	Total
3	2	~~5~~
4	1	5
5	0	5

~~$= {}^5C_3 \times 5$~~

${}^5C_3 \times {}^4C_2 + {}^5C_4 \times {}^4C_1 + {}^5C_5 \times {}^4C_0$

$= 10 \times 6 + 5 \times 4 + 1$

$= 60 + 20 + 1$

$= 80 + 1$

$= 81$

12) Total no. of ways of forming a 4 letter word from the word MIXTURE in which atleast one letter is repeated, is :

Total no. of words with repetition − Total no. of words without repetition

~~$= 7^4 - 7$~~

$= 7^4 - {}^7P_4$

$= 2401 - 840$

$= 1561$

13) Number of ways of distributing n identical ...

places is ${}^{n-1}C_{r-1}$.

$n=12$, $r=4$

$\Rightarrow {}^{n-1}C_{r-1} = {}^{12-1}C_{4-1}$

$= {}^{11}C_3 = 165$

14) Same as ~~Q-13~~ ~~for~~ the previous one

19) Given, no. of friends $= 5$, no. of envelops $= 5$.

No. of ~~ways of~~ ~~decor~~ arrangements $=$

$5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$

$= 5!\left(1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right)$

$= 44$

20) a) Given word :- ENGINEERING.
In this, the letter E is repeated for 3 times, G for 2 times, N for 3 times, I for 2 times.
So, the total no. of ways of arranging this word is:-

$\frac{11!}{3!\times 2!\times 3!\times 2!} = \frac{39,916,800}{144}$

$= 277,200$

b) Number of ways in which a person can go P to Q and return by a different bus is ${}^{25}P_2 = 600$

19) a) Given, no. of friends $= 5$, no. of envelops $= 5$.
No. of derangements as :-

$5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = 44$

b) Total no. of ~~ways~~ arrangements $= 5! = 120$

No. of ways of arranging zero letters in wrong envelops $= 1$.

~~No.~~
~~We cannot put one~~
It is impossible to put only one letter in a wrong envelop since we have to put in another letter into a wrong envelop to make this happened.
No. of ways of placing one letter in wrong envelop is zero.

~~∴ Total~~
∴ No. of ways of ~~keeping~~ placing atleast two letters in wrong envelops is $120 - 1 = 119$