* MODULE - I *

* FIRST ORDER ODE *

Differential equations of 1^{st} order & 1^{st} degree.

Defination:

An Equation involving derivatives of one (or) more dependent variables with respect to one (or) more independent variables is called a differential equation

Types of D.E

These are 2 types of D.E's :-

1) Ordinary d.E's 2) Partial D.E's

(i) A d.E is said to be ordinary if the derivatives in the equation are ordinary derivatives.

Ex:

$$\left[\frac{dy}{dx}\right]^3 - \left[\frac{dy}{dx}\right]^2 + 7y = \cos x.$$

G.S of an O.d.E is $f\left[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0.$

(2) A d.E is said to be partial if the derivatives in the equation have reference to two or more Independent variables.

Ex: $\frac{d^2y}{dx^2} = \frac{1}{c^2}\left(\frac{d^2y}{dx^2}\right)$ [1- dimensional wave equation]

Order & Degree of a D.E:

Order of D.E

The order of the highest order derivative. involved in a d.E is called the order of D.E

A differential equation is said to be of order n, if the n^{th} order derivative is the highest order derivative in that eqⁿ.

Ex: 1) $(x^2+1)\frac{d^2y}{dx^2}+\frac{dy}{dx}+2xy=4x^2$ [order = 2]

2) $\frac{dy}{dx}+2x=4x$ [order = 1]

Degree of D.E:

The degree of a D.E is a highest degree of the highest derivative which occurs in it, after the d.E has been made from radicals and fractions as far as the derivatives are concerned.

Ex: $y=\left(x\frac{d^2y}{dx^2}\right)^3+\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}=1$ [degree = 3]

D.E of the 1^{st} order & of the 1^{st} degree:

An equation of the form $\frac{dy}{dx}=f(x,y)$ is called a d.E of 1^{st} order & of 1^{st} degree.

In general 1^{st} order d.E can be classified as below

1) variables seperable.

2) Homo. eqⁿ & eqⁿ reducible to homo. form.

3) Exact eqⁿs & those which can be made exact by use of integrating factors.

4) linear eqⁿ & Bernoulli's equation.

Procedure.

Variable seperable method:-

1) The given equ $\frac{dy}{dx} = \frac{f(x)}{g(y)} = f(x)\,dx = g(y)\,dy \rightarrow (1)$

2) Integrating b.s & add an arbitary constants of integration to anyone of the two sides.

3) The G.S of (1) is $\int f(x)\,dx = \int g(y)\,dy + c$

(or) $\phi(x, y, c) = 0$.

Homogenous d.e:

Homogenous function:

A function $f(x,y)$ is said to be homogenous function of degree 'n' in 'x' & 'y' if $f(kx, ky) = k^n f(x,y)$ for all values of K where 'n' is a real number.

Ex: Let $f(x, y) = \frac{x^2+y^2}{x^3+y^3}$

$f(kx, ky) = \frac{(kx)^2+(ky)^2}{(kx)^3+(ky)^3} = \frac{k^2x^2+k^2y^2}{k^3x^3+k^3y^3} = \frac{1}{k} f(x,y)$

$= k^{-1} f(x,y)$

$f(kx, ky) = k^{-1} f(x,y)$

$\therefore f(x,y)$ is a homogenous function of degree '-1'

Note:

1) A homogenous function of degree n in x & y. can be expressed as $y^n f\left(\frac{x}{y}\right)$ (or) $x^n f\left(\frac{y}{x}\right)$

2) If $f(x,y)$ is a homogenous function of degree zero then $f(x,y)$ is a function of $\frac{y}{x}$ or $\frac{x}{y}$ alone.

Homogenous d.E

A d.E $\frac{dy}{dx} = f(x,y)$ of 1^{st} order & 1^{st} degree is called homo. d.E in x & y if the function $f(x,y)$ is a homo. function of degree '0' in x & y.

Working rule: 1) let $\frac{dy}{dx} = f(x,y) \rightarrow (1)$ be the homo diff eq

$\because f(x,y)$ is homo. function of degree zero.

we can write $f(x,y) = \phi(y/x) - (2)$ from (1) & (2) we get

$\frac{dy}{dx} = \phi\left(\frac{y}{x}\right) \rightarrow (3)$

2) Put $\frac{y}{x} = v$ (or) $y = vx \rightarrow (4)$

diff (4) w.r.t x we get

$\frac{dy}{dx} = v + x \frac{dv}{dx} \rightarrow (5)$

3) sub (4) & 5 in (3) we get

$v + x\frac{dv}{dx} = \phi(v)$

$x\frac{dv}{dx} = \phi(v) - v$

4) Separating the variables & Integrate b.s we get

$\int \frac{dv}{\phi(v) - v} = \int \frac{dx}{x} + c$ & the g.s can be obtained.

Variable seperable method

① Solve $(y - yx)dx + (x + xy)dy = 0$

Sol Given

$$(y - yx)\,dx + (x + xy)\,dy = 0 \rightarrow ①$$

$$y(1 - x)\,dx = -x(1 + y)\,dy$$

$$\frac{(1 - x)}{x}\,dx = -\frac{(1 + y)}{y}\,dy$$

Integrating on both sides we get

$$\int \frac{1 - x}{x}\,dx = -\int \frac{1 + y}{y}\,dy$$

$$\int \frac{1}{x}\,dx - \int 1\,dx = -\int \frac{1}{y}\,dy - \int 1\,dy$$

$$\log x - x = -\log y - y + c$$

② Solve $x\,dy + 2y\,dx = 2y^2x\,dy$

Sol Given.

$$x\,dy + 2y\,dx = 2y^2x\,dy \rightarrow ①$$

$$2y\,dx = (2y^2x - x)\,dy$$

$$2y\,dx = x(2y^2 - 1)\,dy$$

$$\frac{2}{x}\,dx = \frac{2y^2 - 1}{y}\,dy$$

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Integrating on b.s

$$2\int \frac{1}{x}\,dx = 2\int y\,dy - \int \frac{1}{y}\,dy$$

$$2\log x = 2\frac{y^2}{2} - \log y + c$$

4) Solve $\frac{dy}{dx} = (x+y)^2$

Sol Given:

$$\frac{dy}{dx} = (x+y)^2$$

$$\frac{dy}{dx} = x^2 + y^2 + 2xy$$

Put $x+y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\frac{dt}{dx} - 1 = t^2$$

$$\frac{dt}{dx} = t^2 + 1$$

$$\frac{dt}{dx}$$

$$\frac{dt}{t^2+1} = dx$$

Intergrating on b.s

$$\int \frac{dt}{t^2+1} = \int dx$$

$$\tan^{-1} t = x + c$$

$$\tan^{-1}(x+y) = x + c$$

Homogenous Method

* Solve $(x^2+y^2)\,dx = 2xy\,dy$

Sol Given

$$(x^2+y^2)\,dx = 2xy\,dy \rightarrow (1)$$

$$f(x) = \frac{dy}{dx} = \frac{x^2+y^2}{2xy} \rightarrow (2)$$

$$f(kx, ky) = \frac{k^2x^2+k^2y^2}{2kx\,ky}$$

$$= \frac{\cancel{k^2}[x^2+y^2]}{\cancel{k^2}(2xy)}$$

$$f(kx, ky) = k^0 f(x, y)$$

$\therefore$ Homogenous eqn is satisfied.

Put $y = vx \rightarrow (3)$

differentiate on b.s

$$\frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

$$(2) \Rightarrow v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x(vx)}$$

$$\frac{x\,dv}{dx} = \frac{\cancel{x^2}(1+v^2)}{2(v\cancel{x^2})} - v$$

$$x \frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$x \frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2} dv = \frac{1}{x} dx$$

Integrating on b.s.

$$\int \frac{2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$-\int \frac{-2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$-\log|1-v^2| = \log x + \log c$$

$$-\log\left|1-\frac{y^2}{x^2}\right| = \log(xc)$$

$$\log\left(\frac{x^2-y^2}{x^2}\right) = \log xc$$

Exact differential Equations:

let $M(x,y)dx + N(x,y)dy = 0 \rightarrow (1)$ be a first order and degree d.E where M and N are real value functions for some x, y then the equations $Mdx + Ndy = 0$ is said to be an exact differential equation $\exists$ a function 'f' Such that

$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N.$

Procedure (working rule):

Step 1: Given equation is a d.E of equation, then we can solve $\frac{\partial m}{\partial y}, \frac{\partial n}{\partial x}$ can be solved

Step 2: If $\frac{\partial m}{\partial y}, \frac{\partial n}{\partial x}$ then it is called exact solution. If $\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ then it is said to be non exact form.

Step 3: If it is exact form ~~we can~~ then it is said to be G.S = ~~$\int m dx$~~

$$G.S = \underbrace{\int m\,dx}_{\text{Here 'y' treated as constant}} + \underbrace{\int N\,dy}_{\text{Here 'x' involved then it is 0}} = \int 0$$

(*) Solve the d.E

$$(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0 \rightarrow (1)$$

Sol $M = 2x - y + 1$ | $N = 2y - x - 1$

$\frac{dm}{dy} = 0 - 1 = -1$ | $\frac{\partial N}{\partial x} = 0 - 1 = 0 - 1$

It is an exact form $\left[\because \frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}\right]$

General Solution

$$\int m\,dx + \int N\,dy = \int 0$$

$$\int (2x-y+1)\,dx + \int (2y-x-1)\,dy = \int 0$$

$$\left[\int 2x\,dx - y\int 1\,dx + \int 1\,dx\right] + \left[\int 2y\,dy - \int x\,dy - \int 1\,dy\right] = c$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}} - y(x) + x\right] + \left[\frac{\cancel{2}y^2}{\cancel{2}} - 0 - y\right] = c$$

$$= x^2 - xy + x + y^2 - y = c$$

Solve the d·E

$$(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$$

Sol: $(y^2 - 2xy)\,dx - (x^2 - 2xy)\,dy = 0$.

$$(y^2 - 2xy)\,dx + (2xy - x^2)\,dy = 0$$

$m = y^2 - 2xy$	$N = 2xy - x^2$
$\frac{\partial m}{\partial y} = 2y - x(2)$	$\frac{\partial N}{\partial x} = y(2) - 2x$
$= 2y - 2x$	$= 2y - 2x$.

$$\therefore \frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ It is an exact solution

General Solution

$$\int m\,dx + \int N\,dy = 0$$

$$\int (y^2 - 2xy)\,dx + \int (2xy - x^2)\,dy = \int 0$$

y = constant , $x = 0$.

$$\left[y^2 \int dx - y\int 2x\,dx\right] + \int 2x\,y\,dy - \int x^2\,dy = c$$

$$y^2[x] - y\left[\frac{2x^2}{2}\right] + 0 - 0 = c$$

$$xy^2 - yx^2 = c$$

Solve the d.E

$$(2xy)\,dy - (x^2 - y^2 + 1)\,dx = 0 .$$

Sol:- $M = y^2 - x^2 - 1$ | $N = 2xy$

$$\frac{\partial M}{\partial y} = 2y - 0 - 0 = 2y \qquad \frac{\partial N}{\partial x} = 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then

G.S

$$\int (y^2 - x^2 - 1)\,dx + \int (2xy)\,dy = \int 0$$

$$\left[y^2 \int dx - \int x^2\,dx - \int dx\right] + \left[\int (2xy)\,dy\right] = \int 0$$

$$y^2x - \frac{x^3}{3} - x + 0 = c$$

$$y^2x - \frac{x^3}{3} - x = c$$

Non exact differential Equations

Method - 1 Integrating factors

let $Mdx + Ndy = 0$ is not an exact differential equation. If $Mdx + Ndy = 0$ can be made exact by multiplying it with a suitable function $u(x, y) \neq 0$ then $u(x, y)$ is called an Integrating factor of equation ①

Formulae

1) $d(xy) = x\,dy + y\,dx$.

2) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

3) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

4) $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$

5) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

6) $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$

7) $d\left(\operatorname{Tan}^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$

8) $d\left(\operatorname{Tan}^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$

9) $d(\log(xy)) = \frac{y\,dx + x\,dy}{xy}$

10) $d\left(\log(x^2+y^2)\right) = \frac{2(x\,dx + y\,dy)}{x^2+y^2}$

11) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

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* Solve the differential Equation

$$y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$$

Sol

$$y\,dx - x\,dy + 3x^2y^2e^{x^3}dx \rightarrow (1)$$

dividing $\rightarrow (1)$ by y^2 we get

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0$$

$$\frac{y\,dx - x\,dy}{y^2} = -3x^2e^{x^3}dx$$

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$$d\left(\frac{x}{y}\right) = -3x^2 e^{x^3} dx$$

$$\left[\begin{array}{l} \text{Put } x^3 = t \\ 3x^2 dx = dt \end{array}\right]$$

$$d\left(\frac{x}{y}\right) = -e^t dt$$

Integrating on b.s

$$\int d\left(\frac{x}{y}\right) = -\int e^t dt$$

$$\frac{x}{y} = -e^t + c$$

$$\frac{x}{y} = -e^{x^3} + c$$

* solve the D.E

$$\frac{y(xy + e^x)dx - e^x dy}{y^2} = 0$$

$$\rightarrow \frac{xy^2dx + ye^x dx - e^x dy}{y^2} = 0$$

$$\rightarrow \frac{x\cancel{y^2}dx}{\cancel{y^2}} + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$= \quad x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating on b.s.

$\int x\,dx + \int d\left(\frac{e^x}{y}\right) = \int 0$

$$\frac{x^2}{2} + \frac{e^x}{y} = c$$

Method - 2

To find an integrating factor $m\,dx + n\,dy = 0 \rightarrow (1)$ if equation (1) is a homogenous differential eqⁿ and integrating factor is $\frac{1}{mx+ny}$ is an integrating factor of eqⁿ (1).

Here $mx + ny \neq 0$

* Solve the D.E of $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol Given:

$$x^2y\,dx - (x^3+y^3)\,dy = 0 \longrightarrow (1)$$

$m = x^2y$ | $N = -x^3 - y^3$

$\frac{\partial m}{\partial y} = x^2$ | $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$ is non exact

$$mx + Ny = x(x^2y) + y(-x^3 - y^3)$$

$$= \cancel{x^3y} - \cancel{x^3y} - y^4 \neq 0$$

$$I.F = \frac{1}{mx+ny} = \frac{-1}{y^4}$$

$I.F \times eq$ ①

$$\frac{-1}{y^4}\left(x^2 y\,dx - (x^3+y^3)\,dy\right) = 0$$

$$\frac{-x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{y^3}{y^4}\right)dy = 0$$

$$m_1 = \frac{-x^2}{y^3} \quad \Big| \quad N_1 = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial m_1}{\partial y} = \frac{3x^2}{y^4} \qquad \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4} + 0$$

$$\frac{\partial m_1}{\partial y} = \frac{\partial N_1}{\partial x} \text{ is exact}$$

G.S

$$\underset{y\ const}{\int m_1\,dx} + \underset{x=0}{\int N_1\,dy} = c$$

$$\int \frac{-x^2}{y^3}dx + \int \frac{1}{y}dy = c$$

$$= \frac{-x^3}{3y^3} + \log y = c$$

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* Solve the d.E

$$x\frac{dy^2}{dx} + \frac{y^2}{x} = -y$$

Sol.

$$x\frac{dy}{dx} = y - \frac{y^2}{x}$$

$$x\,dy = \left(\frac{xy - y^2}{x}\right)dx.$$

$$x^2 dy = (xy - y^2)\,dx$$

$$(xy - y^2)\,dx - x^2 dy = 0.$$

$$M\,dx + N\,dy = 0.$$

$M = xy - y^2$	$N = -x^2$
$\frac{\partial M}{\partial y} = x - 2y$	$\frac{\partial N}{\partial x} = -2x.$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ is non exact.}$$

$$Mx + Ny = x(xy - y^2) + y(-x^2)$$

$$= \cancel{x^2y} - xy^2 - \cancel{x^2y}$$

$$= -xy^2 \neq 0$$

$$I.F = \frac{1}{Mx + Ny} = \frac{-1}{xy^2}$$

If $\times$ eq^n (1)

$$\frac{-1}{xy^2}\left((xy-y^2)\,dx - x^2 dy\right)=0$$

$$\left(-\frac{xy}{xy^2}+\frac{y^2}{xy^2}\right)dx + \frac{x^2}{xy^2}\,dy = 0.$$

$$\left(-\frac{1}{y}+\frac{1}{x}\right)dx + \frac{x}{y^2}\,dy = 0$$

$M_1 = -\frac{1}{y}+\frac{1}{x}$ | $N_1 = \frac{x}{y^2}$

$\frac{\partial M_1}{\partial y} = \frac{1}{y^2}$ | $\frac{\partial N_1}{\partial x} = \frac{1}{y^2}$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$. exact

G.S

$$\underset{y\,\text{constant}}{\int M_1\,dx} + \underset{x=0}{\int N_1\,dy} = c$$

$$= \int\left(\frac{-1}{y}+\frac{1}{x}\right)dx + \int 0\,dy = c$$

$$= \frac{-x}{y} + \log x = c$$

* Solve the d.E

$y^2 dx + (x^2 - xy - y^2)\, dy = 0$.

$m = y^2$ | $N = x^2 - xy - y^2$

$\frac{\partial m}{\partial y} = 2y$ | $\frac{\partial N}{\partial x} = 2x - y$

$\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$ is not exact.

$mx + ny = x(y^2) + y(x^2 - xy - y^2)$

$= \cancel{xy^2} + x^2y - \cancel{xy^2} - y^3$

$mx + ny = x^2y - y^3$

$I.F = \frac{1}{x^2y - y^3}$

I.F × eqⁿ ①

$\frac{1}{x^2y - y^3}\left(y^2 dx + (x^2 - xy - y^2)\, dy\right) = 0$

$\frac{1}{x^2y - y^3}(y^2 dx) + \frac{1}{x^2y - y^3}(x^2 - xy - y^2)\, dy$

$\frac{y^2}{x^2y - y^3} dx + \frac{x^2 - xy - y^2}{x^2y - y^3} dy$

$m = \frac{y^2}{x^2y - y^3}$ | $N = \frac{x^2 - xy - y^2}{x^2y - y^3}$

$$m = \frac{y^2}{x^2y - y^3}$$

$$\frac{\partial m}{\partial y} = \frac{y^{\cancel{2}}}{\cancel{y}(x^2-y^2)}$$

$$\frac{\partial m}{\partial y} = \frac{y}{(x^2-y^2)}$$

$$= \frac{x^2-y^2(1) - y(0-2y)}{(x^2-y^2)^2}$$

$$= \frac{x^2-y^2+2y^2}{(x^2-y^2)^2}$$

$$= \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$N = \frac{x^2-xy-y^2}{x^2y-y^3}$$

$$= \frac{(x^2y-y^3)(2x-y) - (x^2-xy-y^2)(2xy)}{(x^2y-y^3)^2}$$

$$\frac{u}{v} = 0.$$
$$\frac{uv' - vu'}{v^2}$$

$$= \frac{2x^3\cancel{y} - x^2y^2 - 2\cancel{xy^3} + y^4 - 2\cancel{x^3y} + 2x^2y^2 + 2\cancel{xy^3}}{(x^2y-y^3)^2}$$

$$= \frac{y^4+x^2y^2}{(x^2y-y^3)^2}$$

$$= \frac{\cancel{y^2}(y^2+x^2)}{\cancel{y^2}(x^2-y^2)^2}$$

G.S

$$= \int m_1 dx + \int n_1 dy = 0$$

$$= \int \frac{y}{x^2-y^2}\,dx + \int \frac{x^2-xy-y^3}{x^2y-y^2}$$

$= y \times \frac{1}{2y} \log \left|\frac{x-y}{x+y}\right| + \int \frac{1}{y} = \log c.$

$= \frac{1}{2} \log \left|\frac{x-y}{x+y}\right| + \log y = \log c$

$= \cancel{\frac{1}{2} \log}$

Method: 3

To find an integrating factor of $Mdx + Ndy = 0$ if the eq and ~~a + a~~ $Mdx + Ndy = 0$ is $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

the $I.F = \dfrac{1}{Mx - Ny}$. Here $Mx \neq Ny$

(*) Solve the d.E

$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$

Sol given:-

$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$

$M = y(x^2y^2+2)$ | $N = x(2-2x^2y^2)$

$\frac{\partial M}{\partial y} = x^2(3y^2)+2$ | $\frac{\partial N}{\partial x} = 2 - 2y^2(3x^2)$

$= 3x^2y^2+2$ | $= 2 - 6x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$I.E = \frac{1}{Mx - Ny} = \frac{1}{y^3x^3 + 2\cancel{x}y - 2\cancel{x}y + 2x^3y^3}$

$$I.F = \frac{1}{3y^3x^3}$$

I.F × eq ① .

$$m_1 = \frac{x^2y^3+2y}{3y^3x^3}$$

$$N_1 = (I.F)N$$

$$= \frac{2x-2x^3y^3}{3y^3x^2}$$

$$m_1 = \frac{1}{3x} + \frac{2}{3x^3y^2}$$

$$N_1 = \frac{2}{3y^3x^2} - \frac{2}{3y}$$

$$\frac{\partial m_1}{\partial y} = 0 + \frac{2}{3x^3}\left(\frac{-2}{y^3}\right)$$

$$= \frac{-4}{3x^3y^3}$$

$$\frac{\partial N_1}{\partial x} = \frac{2}{3y^3}\left(-\frac{2}{x^3}\right)$$

$$= \frac{-4}{3x^3y^3}$$

$$\frac{\partial m_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

∴ It is an exact form.

$$G.S = \int m_1\,dx + \int n_1\,dy = 0$$

$\{x^n = nx^{n-1}$

$$= \int \frac{x^2y^3+2y}{3y^3x^3}\,dx + \int \frac{2x-2x^3y^3}{3y^3x^2}\,dy = 0.$$

$$= \int \frac{1}{3x} + \frac{2}{3x^3y^2}\,dx + \int \frac{2}{3y^3x^2} - \frac{2}{3y}\,dy = 0$$

$$= \frac{1}{3}\log x + \frac{2}{3y^2}\left(\frac{-1}{2x^2}\right) + 0 - \frac{2}{3}\log y = c$$

$$= \boxed{\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c}$$

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Method: 4 :: If their exist a continuous a single variable such that $\frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ then the Integrating factor $= e^{\int f(x)\,dx}$ of eq^n ①.

① Solve the d.E

$$2xy\,dy - (x^2+y^2+1)\,dx = 0$$

$m = -x^2 - y^2 - 1$ | $N = 2xy$

$\frac{\partial m}{\partial y} = -2y$ | $\frac{\partial N}{\partial x} = 2y$

$$\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ It is not an exact form.

$$I.F = e^{\int f(x)\,dx}$$

$$f(x) = \frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-2y - 2y)$$

$$= \frac{1}{\cancel{2}xy}(-\overset{2}{\cancel{4}}y)$$

$$= \frac{-2}{x}$$

$$I.F = e^{\int f(x)\,dx}$$

$$= e^{\int \frac{-2}{x}\,dx}$$

$$= e^{-2\int \frac{1}{x}\,dx}$$

$$= e^{-2\log x}$$

$$I.F = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2} \quad [e^{\log x} = x]$$

$$M_1 = \frac{x^2}{x^2} - \frac{y^2}{x^2} - \frac{1}{x^2} \qquad N_1 = \frac{2xy}{x^2} = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = 2y\left(\frac{-1}{x^2}\right) = -\frac{2y}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial y}$$

$\therefore$ It is an exact form.

$$G.S = \int M_1\,dx + \int N_1\,dy = \int 0$$

$$= \int -1 - \frac{y^2}{x^2} - \frac{1}{x^2}\Big)\,dx + \int \frac{2y}{x}\,dx = c$$

$$= -x - y^2\left(\frac{-1}{x}\right) - \left(\frac{-1}{x}\right) = c$$

$$= -x + \frac{y^2}{x} + \frac{1}{x} = c$$

* Solve the d.E

$(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Sol: $M = 3xy - 2ay^2$ | $N = x^2 - 2axy$

$$\frac{\partial M}{\partial y} = 3x(1) - 2a(2y) = 3x - 4ay \qquad \frac{\partial N}{\partial x} = 2x - 2a(1)y = 2x - 2ay$$

$\partial m \neq \partial n$

It is not an exact form

$$I.F = \frac{1}{mx - ny}$$

$$mx - ny = 3x^2y - 2a\cancel{x}y^2 - x^2y + 2a\cancel{y}^2x = 0$$

$$= 3x^2y - x^2y = 2x^2y$$

$I.F \times eq^n ①$.

$$\cancel{3x^2y - x^2y = 2xy}$$

$$(3xy - 2ay^2 + x^2 - 2axy)$$

$$\left(\frac{3xy}{2x^2y} - \frac{2ay^2}{2x^2y}\right)dx + \left(\frac{x^2}{2x^2y} - \frac{2axy}{2x^2y}\right)dy = 0.$$

$$m = \left(\frac{3}{2x}\right) - \left(\frac{ay}{x^2}\right)$$

$$\frac{\partial m}{\partial y} = 0 - \frac{a(1)}{x^2}$$

$$N = \frac{1}{2y} - \frac{a}{x}\,dy = 0.$$

$$\frac{\partial N}{\partial x} = \frac{-ax^{-1}}{+ax^{-2}}$$

$$= \frac{a}{x^2}$$

G.S

$$\int m\,dx + \int n\,dy = \int 0.$$

$$\int \frac{3}{2x}\,dx - ay\int \frac{1}{x^2}\,dx + \frac{1}{2}\int \frac{1}{y}\,dy - a\int \frac{1}{x}\,dy = \int 0$$

$$\frac{3}{2}\log x - ay\left(\frac{-1}{x}\right) + \frac{1}{2}\log y - 0 = c$$

$$\frac{3}{2}\log x + \frac{ay}{x} + \frac{1}{2}\log y = c \;//$$

Method:5 - If their exist a Continous and differentiable a single variable function $g(y)$ such that $\frac{1}{m}\left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) = g(y)$ then the I.F is $e^{\int g(y)dy}$ in equation ① $(mdx + ndy = 0)$

② Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

$m = xy^3 + y$

$\frac{\partial m}{\partial y} = x(3y^2) + 1$

$N = 2x^2y^2 + 2x + 2y^4$

$\frac{\partial N}{\partial x} = (2x)(y^2) + 2 + 0$

$= 4xy^2 + 2$

$\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ It is non exact form.

$g(y) = \frac{1}{m}\left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) = \frac{1}{xy^3 + y}(4xy^2 + 2 - 3xy^2 - 1)$

$= \frac{1}{xy^3 + y}(xy^2 + 1)$

$= \frac{1}{y\cancel{(xy^2+1)}}\cancel{(xy^2+1)}$

$= \frac{1}{y}$

$I.F = e^{\int g(y)dy}$

$= e^{\int \frac{1}{y}(dy)}$

$= e^{\log y}$

$= y$

$M_1 = (xy^3+y)\,y$ | $N_1 = (2x^2y^2+2x+2y^4)\,y$

$M_1 = xy^4+y^2$ | $N_1 = 2x^2y^3+2xy+2y^5$

$\frac{\partial M_1}{\partial y} = x(4y^3)+2y$ | $\frac{\partial N_1}{\partial x} = \cancel{2x^2}\; 2(2x)y^3+2y(1)+0$

$= 4xy^3+2y$ | $= 4xy^3+2y$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$\therefore$ It is an exact form.

General solutions

$$= \int M_1\,dx + \int N_1\,dy = \int 0$$

$$= \int xy^4+y^2\,dx + \int 2x^2y^3+2xy+2y^5\,dy = 0$$

$$= \frac{x^2}{2}(y^4)+y^2x+2.\frac{y^6}{6} = 0$$

② Solve the d.E $y(2xy+e^x)dx - e^x dy = 0$

$M = 2xy^2+e^x y$ | $N = -e^x$

$\frac{\partial M}{\partial x} = 2x(2y)+e^x(1)$ | $\frac{\partial N}{\partial x} = -e^x$

$= 4xy+e^x$

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial x}$$

It is non exact form.

$$g(y) = \frac{1}{m}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{y(2xy+e^x)}\left(-e^x - 4xy - e^x\right)$$

$$= \frac{-2(\cancel{e^x + 2xy})}{y(\cancel{2xy+e^x})} = \frac{-2}{y}$$

$$I.F = e^{\int g(y)\,dy}$$

$$= e^{\int \frac{-2}{y}\,dy}$$

$$= e^{-2\log y}$$

$$= e^{\log y^{-2}}$$

$$= y^{-2}$$

$$= \frac{1}{y^2}$$

$$M_1 = 2x + \frac{e^x}{y}$$

$$\frac{\partial M_1}{\partial y} = e^x\left(\frac{-1}{y^2}\right)$$

$$N_1 = -\frac{e^x}{y^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{-1}{y^2}(e^x)$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

It is an exact form

G.S:

$\int M\,dx + \int y\,N\,dy = \int 0.$

$= \int 2x\,dx + \int \frac{e^x}{y}\,dx + \int -\frac{e^x}{y^2}\,dy = \int 0$

$= \frac{2x^2}{2} + \frac{e^x}{y} + 0 = c$

** linear differential Equation of first order:-

An equation of the form $\frac{dy}{dx} + p(x)y = Q(x)$ where P and Q are either constant or functions of x only is called linear differential Equation of first order in y.

Working Rule: To solve the linear equation $\frac{dy}{dx} + p(x)y = Q(x)$

1) write the integrating factor. $= e^{\int p(x)dx} = (I.F)$

2) G.S is $y(I.F) = \int Q(I.F)\,dx + c$

Note: An equation of the form $\frac{dx}{dy} + p(y)x = Q(y)$ and treat x has the dependent variable and y has the independent variable. In this case the g.s is given by $x(I.F) = \int Q(I.F)\,dy + c$ where

Integrating factor $= e^{\int p(y)\,dy}$.

Note:- * $\int t e^{t} dt = e^{t}(t-1)+c$

* $\int t e^{-t} dt = -(t+1)e^{-t}+c$

⊛ Solve $x\frac{dy}{dx} + y = \log x$

dividen on b.s

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{\log x}{x}.$$

$$p(x) = \frac{1}{x} \; ; \; Q(x) = \frac{\log x}{x}$$

$$I.F = e^{\int p(x)\,dx}$$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x$$

General solution: $y(I.F) = \int Q(x)\,(I.F)\,dx + c$

$$y(x) = \int \frac{\log x}{x}(x)\,dx$$

$$xy = \int \log x \,dx$$

$$xy = x\log x - x + c$$

⊛ Solve $x\cos x\frac{dy}{dx} + (x\sin x + \cos x)\,y = 1$

$$\frac{dy}{dx} + \left(\frac{x\sin x + \cos x}{x\cos x}\right)y = \frac{1}{x\cos x}$$

P=

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{\sec x}{x}$$

$$P(x) = \tan x + \frac{1}{x} \quad ; \quad Q(x) = \frac{\sec x}{x}$$

$$I.F = e^{\int P(x)\,dx}$$

$$= e^{\int (\tan x + \frac{1}{x})\,dx}$$

$$= e^{(\log \sec x + \log x)}$$

$$= e^{\log \sec x} e^{\log x}$$

$$= \sec x\,(x)$$

$$= x \sec x$$

$$G.S = y\,(I.F) = \int Q(x)\,(I.F)\,dx + c$$

$$y\,(x \sec x) = \int \frac{\sec x}{\cancel{x}} \cancel{x} \sec x \; dx.$$

$$xy \sec x = \int \sec^2 x \, dx$$

$$xy \sec x = \tan x + c$$

③ Solve the d.E $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$$(x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$$

$$x-e^{\tan^{-1}y}=-(1+y^2)\frac{dx}{dy}$$

$$-e^{\tan^{-1}y}=-(1+y^2)\frac{dx}{dy}-x$$

$$(1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

$$\frac{dx}{dy}+\left(\frac{1}{(1+y^2)}\right)x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$I.F=e^{\int p(y)\,dy}$$

$$=e^{\int\frac{1}{1+y^2}dy}$$

$$=e^{\tan^{-1}y}$$

$$G.S=x(I.F)=\int Q(y)(I.F)\,dy+c$$

$$x\left(e^{\tan^{-1}y}\right)=\int\frac{e^{\tan^{-1}y}}{(1+y^2)}e^{\tan^{-1}y}\,dy$$

Put $\tan^{-1}y$

$$\frac{1}{1+y^2}dy=dt$$

$$=\int e^{2t}\,dt$$

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$$= \left(\frac{e^{2t}}{2}\right) + c$$

$$x e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{e} + c$$

(F) $$\frac{dy}{dx} + y = e^{x}$$

$P(x) = 1 \qquad Q(x) = e^{x}$

$$I.F = e^{\int p(x)\, dx}$$

$$= e^{\int 1 \cdot dx}$$

$$= e^{x}$$

$G.S = \cancel{I.F(x)}$

$$y(I.F) = \int Q(x)(I.F)\,dx + c$$

$\cancel{y(e^{x}) = \int}$

$$y(e^{x}) = \int e^{x}(e^{x})\,dx$$

$$y(e^{x}) = \int e^{2x}\,dx$$

$$= \frac{e^{2x}}{2} + c$$

$$2y(e^{x}) = e^{2x} + c$$

(*) Solve $\frac{dy}{dx} + 2y = e^{x} + x \; ; \; y(0) = 1$

$$\frac{dy}{dx} + p(x) \cdot y = Q(x)$$

$$I \cdot F = e^{\int p(x)\, dx}$$

$$= e^{2\int x\, dx}$$

$$= e^{2x}$$

$$G \cdot S = y(I \cdot F) = \int Q \cdot F \, (I \cdot F)\, dx + c$$

$$y(e^{2x}) = \int (e^{x} + x)(e^{2x})\, dx$$

$$= \int (e^{3x} + x e^{2x})\, dx$$

$$y(e^{2x}) = \frac{e^{3x}}{3} + \left[x \frac{e^{2x}}{2} - 1 \frac{e^{2x}}{4} \right] + c$$

$$y e^{2x} = \frac{e^{3x}}{3} + x \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c \rightarrow ①$$

$y(0) = 1 \Rightarrow x = 0, \; y = 1$

Sub in eq^{n} ①

$$① \Rightarrow \quad (1) e^{0} = \frac{e^{0}}{3} + 0 - \frac{e^{0}}{4} + c$$

$$1 = \frac{1}{3} - \frac{1}{4} + c$$

$$c = \frac{1}{1} - \frac{1}{3} + \frac{1}{4}$$

$$= \cancel{\frac{12 - 4 + 3}{12}} = \cancel{\frac{12 - 7}{}} \quad \boxed{c = \frac{11}{12}}$$

Sub in eq^{n} ①

$$ye^{2x} = \frac{e^{3x}}{3} + \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + \frac{11}{12}$$

Bernoulli's Equation:-

An equation of the form $\frac{dy}{dx} + p(x) \cdot y = q(x) \dot{y}^n \to ⑥$

If $n \neq 1$ multiply y^{-n} in eqⁿ ① we get.

$$y^{-n} \cdot \frac{dy}{dx} + p(x) y^{1-n} = Q(x) \to ①$$

Put $y^{1-n} = t$

$$(1-n) y^{1-n-1} \frac{dy}{dx} = \frac{dt}{dx}$$

$$(1-n) y^{-n} \frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dt}{dx}$$

Sub ① we get

$$\frac{1}{1-n} \frac{dt}{dx} + p(x) t = Q(x)$$

$$\frac{dt}{dx} + (1-n) p(x) t = (1-n) Q(x)$$

$$G.S = t\,(I.F) = \int Q(x)\, I.F\, dx + c$$

$$I.F = e^{\int p(x) dx} + c$$

① Solve the d.E

$$x\frac{dy}{dx} + y = x^3 y^6 \rightarrow (1)$$

Sol:
$$\frac{dy}{dx} + \frac{1}{x}(y) = \frac{x^3 y^6}{x}$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2 \rightarrow (2)$$

Put $y^{-5} = t$

$$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}$$

Eq^n (2) becomes.

$$\frac{-1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$$\frac{dt}{dx} - \frac{5}{x}t = -5x^2$$

$$P(x) = \frac{-5}{x} \qquad Q(x) = -5x^2$$

$$I.F = e^{\int p(x)dx}$$
$$= e^{-5\int \frac{1}{x}dx}$$
$$= e^{-5\log x}$$
$$= e^{\log x^{-5}}$$
$$= x^{-5} = \frac{1}{x^5}$$

$G.S = \cancel{y(I.F}$

$= t\,(I.F) = \int Q(x)\, I.F\, dx + c$

$y^{-5}\left(\frac{1}{x^5}\right) = \int -5x^{\cancel{x}} \left(\frac{1}{\cancel{x^5}_3}\right) dx$

$$\frac{1}{x^5 y^5} = -5\left[\frac{x^{-2}}{-2}\right] + c$$

$$\therefore \frac{1}{x^5 y^5} = \frac{5}{2x^2} + c$$

(*) Solve the d.E

$(1-x^2)\frac{dy}{dx} + xy = y^3 \sin^{-1}x$.

Sol

$$\frac{dy}{dx} + \frac{x\,\cancel{y}}{(1-x^2)}y = \frac{y^3 \sin^{-1}x}{(1-x^2)}.$$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{(1-x^2)}\frac{1}{y^2} = \frac{\sin^{-1}x}{1-x^2}.$$

Put $\frac{1}{y^2} = t$

$y^{-2} = t$

$$-2 \cdot y^{-3}\frac{dy}{dx} = \frac{dt}{dx}.$$

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} \cdot \frac{1}{y^3} = -\frac{1}{2}\frac{dt}{dx}.$$

$$-\frac{1}{2}\frac{dt}{dx} + \frac{x}{1-x^2} \cdot t = \frac{\sin^{-1}x}{(1-x^2)}$$

$$\frac{dt}{dx} - \frac{2x}{1-x^2} t = \frac{-2\sin^{-1}x}{1-x^2}$$

$$p(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{-2\sin^{-1}x}{1-x^2}$$

$$I.F = e^{\int p(x)\,dx}$$

$$= e^{\int \frac{-2x}{1-x^2} dx}$$

$$= e^{\log(1-x^2)}$$

$$= (1-x^2)$$

$$G.S = t(I.F) = \int Q(x)(I.F)\,dx + c$$

$$y^{-2}(1-x^2) = \int \frac{-2\sin^{-1}x}{\cancel{(1-x^2)}} \cancel{(1-x^2)}\,dx$$

$$= 2\int \sin^{-1}x\,dx$$

$$\frac{(1-x^2)}{y^2} = -2\left[x\sin^{-1}x + \sqrt{1-x^2}\right] + c$$

⊛ Solve the D.E

$$e^x \frac{dy}{dx} = 2xy^2 + ye^x \rightarrow (1)$$

Sol $$e^x \frac{dy}{dx} - ye^x = 2xy^2$$

$$\frac{dy}{dx} - \cancel{\frac{e^x}{e^x}} y = \frac{2x}{e^x} y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x} \quad \cancel{y}$$

$$\cancel{\frac{1}{y^2}} \quad y^{-1} = t$$

$$-1y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$$

$$\frac{dt}{dx} + t = \frac{-2x}{e^x}$$

$$I.F = e^{\int p(x)dx} = e^{\int 1 dx} = e^x$$

$$G.S = t\,\cancel{e^x} = \int \frac{-2x}{\cancel{e^x}}\,\cancel{e^x}\,dx$$

$$\frac{e^x}{y} = \frac{-\cancel{2}x^2}{\cancel{2}} + c$$

$$\frac{e^x}{y} = -x^2 + c$$

Applications of first order Differential Equations of first order.

There are 4 types

1. Orthogonal trajectories ×
2. Newton law of cooling
3. Neulton law of Growth
4. Newton law of Cooling.

Newton law of Cooling:- The rate of change of temperature of a body is proportional to the difference of temperature of the body and ~~its~~ that of the surroundings medium. Let θ be the temperature of the body at time 't' and 'θ_0' temperature of its surroundings medium (Usually be the temp of its surroundings air) by newton law of cooling.

$$\frac{d\theta}{dt} \propto (\theta - \theta_0) \rightarrow (1)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0) \quad [\because K = \text{positive constant}]$$

Here temperature decreases time increases

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

Integrating on both sides.

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$$\log(\theta - \theta_0) = -kt + \log c$$

$$\frac{\theta - \theta_0}{c} = e^{-kt}$$

$$\theta - \theta_0 = c e^{-kt}$$

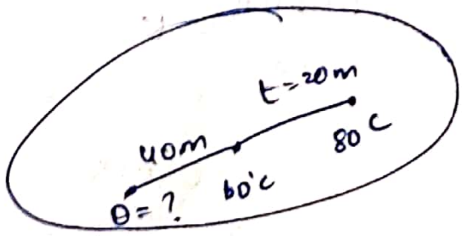
$$\boxed{\theta = \theta_0 + c e^{-kt}} \rightarrow (2) \qquad \boxed{\log(\theta - \theta_0) = -kt + \log c}$$

(*) A body is original at 80°c and cools down to 60°c in 20min if the temp of air is 40°c. & Find the temp of body after 40 mins?

Sol:- Given data :-

let θ be the temp of body at time t

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$



$$\frac{d\theta}{dt} = -kt$$

$$\frac{d\theta}{(\theta - \theta_0)} = -kt$$

Integrating on b.s.

$\theta = \theta_0 + Ce^{-Kt}$ — ①

(i) $\theta = 80,\ t = 0,\ \theta = 40$. sub in eqⁿ ①

$80 = 40 + Ce^{-K(0)}$

$80 = 40 + C$

$\boxed{C = 40}$

(ii) $\theta = 60,\ \theta_0 = 40,\ t = 20,\ C = 40$

$\theta = \theta_0 + Ce^{-Kt}$

$60 = 40 + 40e^{-K(20)}$

$60 = 40 + 40e^{-20K}$

$60 - 40 = e^{-20K}$

$\frac{20}{40} = e^{-20K}$

$\log(0.5) = -20K$

$\frac{-0.301}{-20} = K$

$K = 0.015$

$\frac{1}{2} = e^{-20K}$

$\log\frac{1}{2} = -20K$

$K = \frac{-1}{20}\log\frac{1}{2}$

(iii)

$\theta = ?\ t = 40,\ \theta_0 = 40,\ C = 40$

$K = 0.015$

① $\Rightarrow \theta = 40 + 40e$

$\theta = 40 + 40(e)^{-0.6}$

$= 40(1 + 0.54881)$

$= 40(1.5481)$

$= 61.924$

(iii) $\theta = ?$ $t = 40, \theta_0 = 40, c = 40$

① ⇒ $\theta = 40 + 40e^{\frac{40}{20}\log(\frac{1}{2})}$

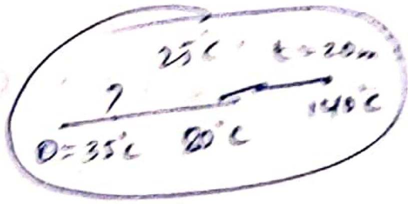
$\theta = 40 + 40e^{2\log(\frac{1}{2})}$

$= 40 + 40e^{\log(\frac{1}{4})}$

$= 40 + \overset{10}{\cancel{40}}\left(\frac{1}{\cancel{4}}\right)$

$= \underline{50°c}$

⊕ A body kept in air with temp 25°c. cools than 140°c to 80°c in 20min. Find when body cools down temperature to 35°c.



Sol: Given:

$\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$\frac{d\theta}{dt} = -kt$

$\frac{d\theta}{(\theta - \theta_0)} = -kt$

Integrating on b.s.

$\theta = \theta_0 + ce^{-kt} \rightarrow$ ① $\quad \left(\log(\theta - \theta_0) = -kt + \log c\right)$

(i) $\theta = 140°c$, $t = 0$, $\theta_0 = 25°c$, sub in ①

$140 = 25 + ce^{-k(0)}$

$\boxed{115 = c}$

(ii) $\theta = 80, \theta_0 = 25, t = 20, C = 115$

Sub in (1)

$$80 = 25 + 115e^{-K(20)}$$

$$55 = 115e^{-20k}$$

$$\frac{55}{115} = e^{-20k}$$

$$\log\left(\frac{11}{23}\right) = \frac{-1}{20}\log\left(\frac{11}{23}\right)$$

(iii) $\theta = 35, t = ?, \theta_0 = 25, C = 115$

$$\theta = \theta_0 + Ce^{-Kt}$$

$$35 = 25 + (115)e^{\frac{t}{20}\log\left(\frac{11}{23}\right)}$$

$$\cancel{10 = 115e^{t}}$$

$$\frac{10}{115} = e^{\frac{t}{20}\log\left(\frac{11}{23}\right)}$$

$$\log\left(\frac{10}{115}\right) = \frac{t}{20}\log\left(\frac{11}{23}\right)$$

$$\cancel{-}1.060 = \frac{t}{20}(\cancel{-}0.32)$$

$$\frac{1.06}{0.32} \times 20$$

$$= 3.3125 \times 20$$

$$= 66.2 \text{ mins}$$

(7) If the temperature of the body is changing from 100°C to 70°C in 15 mins. Find when the temperature will be 40°C. If the temp of air is 30°C ?

sol Given =

30°c ? 15 100
θ=40°c 70

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -kt$$

$$\frac{d\theta}{(\theta - \theta_0)} = -kt$$

I.B.S

$$\theta = \theta_0 + ce^{-kt} \rightarrow (1)$$

(i) $\theta = 100$, $t = 0$, $\theta_0 = 30°c$ sub in (1)

$$\theta = \theta_0 + ce^{-kt}$$

$$100 = 30 + ce^{-k(0)}$$

$$\boxed{70 = c}$$

(ii) $\theta = 70$, $\theta_0 = 30°c$, $t = 15\,min$, $c = 70$

sol

$$\theta = \theta_0 + ce^{-kt}$$

$$70 = 30 + 70e^{-k(15)}$$

$$40 = 70e^{-15k}$$

$$\frac{40}{70} = e^{-15k} \Rightarrow \log\left(\frac{40}{70}\right) = \frac{-1}{15}\log\left(\frac{40}{70}\right)$$

(iii) $\theta = 40^\circ c$, $t = ?$, $\theta_0 = 30^\circ c$, $c = 70$

Sol

$$\theta = \theta_0 + ce^{-kt}$$

$$40^\circ = 30 + 70\, e^{\frac{t}{15}\log\left(\frac{40}{70}\right)}$$

$$\frac{10}{70} = e^{\frac{t}{15}\log\left(\frac{4}{7}\right)}$$

$$\log\left(\frac{1}{7}\right) = \frac{t}{15}\log\left(\frac{4}{7}\right)$$

$$t = \frac{\log\left(\frac{1}{7}\right)}{\log\left(\frac{4}{7}\right)} \times 15$$

$$= \frac{0.84}{0.24} \times 15$$

$$= 3.5 \times 15$$

$$= 52.5 \text{ mins}$$

OR

* An object whose temperature is $75^\circ c$ cools in an atmosphere of constant temperature $25^\circ c$. The rate of change of body after 10 mins. The temperature is falls down in $65^\circ c$ find its temperature after 20 mins. And also find time required to cool down to $55^\circ c$.

Sol

Sol:-

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

25°C
20m 10m
t=? 65°C 75°C

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

25°C
t=? 10m
55 65 75

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

Integrating on b.s.-

$$\boxed{\log(\theta - \theta_0) = -kt + \log c} \rightarrow (1)$$

(i) $\theta_0 = 25$, $\theta = 75°C$, $t = 0$.

$$\log(75-25) = -k(0) + \log c$$

$$\log(50) = \log c$$

$$\boxed{c = 50}$$

(ii) $\theta_0 = 25$, $\theta = 65$, $t = 10$, $c = 50$

$$\log(65-25) = -k(10) + \log 50$$

$$\log\left(\frac{40}{50}\right) = -10k$$

$$k = \frac{-1}{10}\log\left(\frac{4}{5}\right)$$

(iii) $\theta_0 = 25$, $\theta = ?$, $t = 20$, $c = 50$

$$\log(\theta - 25) = \frac{20}{10}\log\left(\frac{4}{5}\right) + \log 50$$

$$\log\left(\frac{\theta-25}{50}\right) = \log\left(\frac{16}{25}\right)$$

$$\theta - 25 = \frac{16 \times \cancel{50}^{2}}{\cancel{25}}$$

$$\theta = 32 + 25$$

$$\theta = 57°C$$

(iv) $\theta_0 = 25$, $\theta = 55$, $t = ?$, $c = 50$

$$\log(55-25) = \frac{t}{10}\log\left(\frac{4}{5}\right) + \log 50$$

$$\log\left(\frac{3}{5}\right) = \log\left(\frac{4}{5}\right)^{t/10}$$

$$\log\left(\frac{3}{5}\right) = \frac{t}{10}(-0.09)$$

$$-0.22 = -0.09\,\frac{t}{10}$$

$$\frac{t}{10} = \frac{0.221}{0.09}$$

$$\frac{t}{10} = 2.2$$

$$t = 22.9 \text{ mins}$$

⊛ If temp of the air is 20°c and temp of the body drops from 100°c to 80°c. in 10mins. what will be its temp after 20 mins.

sol

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$20°C$ 10m
200 — 100
$\theta > ?$ 80

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

I.B.S

$$\boxed{\log(\theta - \theta_0) = -kt + \log c}$$

(i) $\theta_0 = 20$, $\theta = \cancel{80-20}$, $t = 10$.
$= 100$

$$\log(100 - 20) = -k(0) + \log c$$

$$\log(80) = \log c$$

$$\cancel{\log}\ \boxed{c = 80}$$

(ii) $\theta_0 = 20$, $\theta = 80$, $t = 10$, $c = 80$

$$\log(80 - 20) = -k(10) + \log 80.$$

$$\log\frac{(60)}{(80)} = -10k$$

$$k = \frac{-1}{10}\log\left(\frac{3}{4}\right)$$

(iii) $\theta_0 = 20$, $\theta = ?$, $t = 20$, $c = 80$

$$\log(\theta-20) = \frac{20}{10}\log\left(\frac{3}{4}\right)+\log 80$$

$$\log\frac{(\theta-20)}{80} = \log\left(\frac{3}{4}\right)^2$$

$$\frac{\theta-20}{80} = \frac{9}{16}$$

$$\theta-20 = \frac{\cancel{80}^{5}\times 9}{\cancel{16}}$$

$$\theta - 20 = 45$$

$$\theta = 45+20$$

$$\theta = 65\,^{\circ}C$$

(*) A murder victim is discovered and Lieutenant from the forensic ~~sign~~ labatory is ~~summon~~ summoned
Science
to estimate - the time of death. The body is located in a room that is kept at constant temp of 68°F, the lieutenant arrived at 9:40pm and measure the body temp as 94.4°F at the time. Another measurements body temperature at 11 pm. is 89.2F. Find the estimate time of death?

Sol: $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

ok!!
9:40 pm
11 pm

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\frac{d\theta}{\theta - \theta_0} = -k\,dt$

I.B.S

$\log(\theta - \theta_0) = -kt + \log c \rightarrow (1)$

(i) $\theta_0 = 68\,F$, $\theta = 94.4$, $t = 0$.

$\log(94.4 - 68) = -k(0) + \log c$

$\log(26.4) = \log c$

$c = 26.4$

(ii) $\theta_0 = 68$, $\theta = 89.2$, $t = 80\,min$ $c = 26.4$ [9:40 to 11pm = 80m]

$\log(89.2 - 68) = -80k + \log(26.4)$

$\log(21.2) = -80k + \log(26.4)$

$\log\left(\frac{21.2}{26.4}\right) = -80t$

$k = \frac{-1}{80} \log\left(\frac{21.2}{26.4}\right)$

(iii) $\theta_0 = 68\,F$, $\theta = 98.6\,F$, $t =$ $c = 26.4$

$$\log(98.6-68) = \frac{t}{80}\log\left(\frac{21.2}{26.4}\right) + \log(26.4)$$

$$\log(21.6) = \frac{t}{80}\log(-0.095)$$

$$0.064 = \frac{t}{80}(-0.0953)$$

$$0.064 = t(-0.0953)$$

$$5.12 = t(-0.0953)$$

$$t = \frac{5.12}{-0.0953}$$

$$t = -53.8 \text{ mins.}$$

$t = \underset{\sim 54m}{53.8}$ (9:40pm)

$t = 8:\cancel{4}6$ pm

⊛ Law of Natural ~~decay~~ Growth / decay :-

Let $x(t)$ be the amount of a substance at time t and let the substance be getting converted chemically. A law of chemical convertion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance ~~available~~ ~~aa~~ available at that time

Decay: Disintegrating

$\frac{dx}{dt} \propto x$ [∵ Disintegrating at any instant is proportional to amount of material presence]

$\frac{dx}{dt} = -kx$

$\frac{dx}{dt} = -k\,dt$

I.B.S

$\int \frac{dx}{x} = -k \int dt$

$\log x = -kt + \log c \longrightarrow (1)$

$x = ce^{-kt}$

Growth ~~Decay~~: $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$

$\frac{dx}{x} = k\,dt$

I.B.S.

$\int \frac{dx}{x} = \int k\,dt$

$\log x = kt + \log c$

$\boxed{x = ce^{kt}}$

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* The no N of bacteria in a cultral grow at a Rate of proportional to N the value of N was initially 100 and increased to 332 in one hour. what was the value of 'N' after 1 1/2 hour.

Sol:- <u>Given data</u>:-

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

Integrating on b.s

$$\int \frac{1}{N}\,dN = k\int dt$$

$$\log N = kt + \log c \rightarrow (1)$$

(i) $N = 100$, $t = 0$.

$$\Rightarrow \log 100 = k(0) + \log c$$

$$\log 100 = \log c$$

$$\boxed{c = 100}$$

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(ii) $N = 332, \ t = 60\,m$

$$\log\left(\frac{332}{100}\right) = 60k$$

$$k = \frac{1}{60}\log\left(\frac{332}{100}\right)$$

(iii) $$\log N = \frac{90}{60}\log\left(\frac{332}{100}\right) + \log(100)$$

$$\log\left(\frac{N}{100}\right) = \frac{3}{2}\log\left(\frac{332}{100}\right)$$

$$\frac{N}{100} = \left(\frac{332}{100}\right)^{3/2}$$

$$N = 6.049 \times 100$$

$$N = 604.9$$

$$N \simeq 605$$

* The Bacteria in a either closed exponentially show that the initial no has doubled in 3hrs. How many times the initial no will be present after 9 hrs?

Sol Given data:

$$\frac{dA}{dt} \propto N$$

$$\frac{dA}{dt} = kN$$

$\frac{dN}{N} = k\,dt$

I.B.S

$$\int \frac{1}{N}\,dN = k\int dt$$

$\log N = kt + \log c \rightarrow ?$

(i) $N = A$; $t = 0$

$\log A = k(0) + \log c$

$\boxed{c = A}$

(ii) $N = 2A$; $t = 3$

$\log(2A) = 3k$

$\log\left(\frac{2A}{A}\right) = 3k$

$\log 2 = 3k$

$k = \frac{1}{3}\log(2)$

(iii) ~~$\log N \rightarrow$~~ $A = x$ $t = 9$ hrs

$\log x = \frac{9}{3}\log 2 + \log A$

$\log\left(\frac{x}{A}\right) = \log 8$

$\frac{x}{A} = 8$

$\boxed{x = 8A}$

$\therefore$ After 9 hrs 8 times Bacteria increased

Rate of decay:

(*) If 30% of a radioactive substance disappears in 10 days how long will ~~be~~ it take for 90% of it to disappear.

Sol:. $\frac{dN}{dt} \propto N$

$$\frac{dN}{dt} = -kN$$

$$\frac{dN}{N} = -k\,dt$$

I.B.S.

$$\int \frac{1}{N}\,dN = -k\int dt$$

$$\log N = -kt + \log c.$$

~~(i) $N = 70$; $t = 10$~~

~~$\log 30 = -k(10) + \log c$~~

~~$\log 30 = -10k + \log c$~~

(i) $t = 0$; $r = 100\%$.

$$\log 100 = -k(0) + \log c$$

$$c = 100$$

(ii) $t = 10$ days, $r = 30\%$. disappear

app :- $r = 100 - 30$

$$= 70\%$$

(i) $\Rightarrow \log 70 = -k(10) + \log 100$

$\log\left(\frac{7}{10}\right) = -10k$

$k = \frac{-1}{10}\log\left(\frac{7}{10}\right)$

(iii) $t = ?$ $r = 90\%$

app $- r = 100 - 90 = 10\%$

$\Rightarrow \log 10 = \frac{k}{10}\log\left(\frac{7}{10}\right) + \log 10$

$\log\left(\frac{1}{10}\right) = \frac{t}{10}(-0.15)$

$-1 = t(-0.0154)$

$t = 64.9$

$\underline{t \cong 65 \text{ days}}$

~~⊛ In a chemical reaction the given substance being converted into another at a rate proportional to the amount of the substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 mins. How much time will be required to transform 1 1/2~~

Sol: $\frac{dN}{dt} \propto N$

let x gram be the amount of remaining substance after t mins.

$$\frac{dx}{dt} \propto x.$$

$$\frac{dx}{dt} = -kx$$

$$\frac{dx}{x} = -k\,dt$$

I.B.S

$$\int \frac{1}{x}\,dx = -k\int dt$$

$$\log x = -kt + \log c \rightarrow (1)$$

(i) ~~t = 0, x =~~ $t=0 \quad x=5$

$$\log(5) = -k(t) + \log c$$

$$c = 5$$

$t = 4$ mins $\quad C = 5 \quad x = 5$

(ii) $\log\left(x\,\frac{4}{5}\right) = -k(4) + \log x$

$$k = \frac{-1}{4}\log\left(\frac{4}{5}\right)$$

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(i) $t = 0, x = 5$

$$\log 5 = -k(0) + \log c$$

$$\boxed{c = 5}$$

(ii) $t = 4$ min ; $c = 5$, $x = 5 - \frac{5}{5}$

$$= \frac{5s - s}{5}$$

$$= \frac{4s}{5}$$

$$\log\left(\frac{4s}{5}\right) = -k(4) + \log 5$$

$$\log\left(\frac{\frac{4\cancel{8}}{5}}{\cancel{8}}\right) = -4k$$

$$\frac{-1}{4}\log\left(\frac{4}{5}\right) = k$$

(iii) $t = ?$ $x = 5 - \frac{5}{2} = \frac{5}{2}$

$$\log\left(\frac{5}{2}\right) = \frac{t}{k}\log\left(\frac{4}{5}\right) + \log 5$$

$$\log\left(\frac{1}{2}\right) = \frac{t}{4}\log\left(\frac{4}{5}\right)$$

$$-0.3010 = \frac{t}{4}(-0.0969)$$

$t = 12.425$ $\quad$ $\underline{t = 12.4 \text{ mins}}$ $\Rightarrow t \simeq \underline{13 \text{ mins}}$

(A) If radio active carbon 14 has Half life of 5750 yrs what will remain of 1gm after 3000 yrs? (decay).

Sol: ~~integra~~

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = KN$$

$$\frac{dN}{N} = K\,dt$$

I.B.S

$$\int \frac{1}{N}\,dN = K\int dt$$

$$\log N = -Kt + \log C \rightarrow (1).$$

(i) $t=0$, $x=1$ of carbon-14 grams.

$$\log 1 = -K(0) + \log C$$

$$\boxed{C=1}$$

(ii) $t=5750$, $C=1$, $x=\frac{1}{2}$

$$\log\left(\tfrac{1}{2}\right) = -5750K + \log 1$$

$$\frac{1}{5750}\log\left(\tfrac{1}{2}\right) = K$$

(iii). $t=3000$, $x=?$

$$\log x = \frac{3000}{5750}\log\left(\tfrac{1}{2}\right) \rightarrow \log x = 0.521\log(0.5)$$

$$= \log(0.5)^{0.521}$$

$$\log x = \log(0.690)$$

$$\boxed{x = 0.696}$$

After 3000 yrs carbon 14 0.696 is disappeared

Eqn of 1st order but not of the first degree :-

Definitions :-

General form of the first order D.E of degree $n>1$, is

$$P_0\left[\frac{dy}{dx}\right]^n + P_1\left[\frac{dy}{dx}\right]^{n-1} + P_2\left[\frac{dy}{dx}\right]^{n-2} + \cdots\cdots + P_{n-1}\left[\frac{dy}{dx}\right] + P_n = 0 \rightarrow (1)$$

where $P_0, P_1, P_2 \ldots P_{n-1}, P_n$ are functions of x & y.

It will be convenient for us to denote $\frac{dy}{dx}$ by p and so (1) becomes.

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots\cdots + P_{n-1}p + P_n = 0$$

Such equations can be solved by various methods given in this chapter. In each of these methods, the given problem is reduced to that of solving one or more equations of the first order and the first degree (already discussed in chapter 1)

Method : 1 - Eqn Solvable for p :-

Let $P_0p^n + P_1p^{n-1} + P_2p^{n-2} + \cdots\cdots + P_{n-1}p + P_n = 0 \rightarrow (1)$

be the given differential eqn of the first order and degree $n>1$

Resolving the left hand side of (1) into n linear factors, we have

$$[p - f_1(x,y)][p - f_2(x,y)] \cdots\cdots [p - f_n(x,y)] = 0 \rightarrow (2)$$

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Equating each factor ~~equation~~ equal to zero, we obtain n equations of the first order and the first degree, namely.

$$P = \frac{dy}{dx} = f_1(x,y), \; P = \frac{dy}{dx} = f_2(x,y), \ldots, P = \frac{dy}{dx} = f_n(x,y) \quad \to (3)$$

Let the solutions of these equations will be of the type.

$$F_1(x,y,c_1) = 0, \; F_2(x,y,c_2) = 0, \ldots F_n(x,y,c_n) = 0$$

where $c_1, c_2, \ldots, c_n$ are the arbitary constant of integration.

We have already started that ~~Contains~~ Constant occuing in the D.E can be made to take any value whatover we like Hence there is no loss of genearality of we replace all the ~~Contents~~ Contants $c_1, c_2, \ldots, c_n$ by another canstant c.

Hence the 'n' Solutions become

$$F_1(x,y,c) = 0, \; F_2(x,y,c) = 0, \ldots, F_n(x,y,c) = 0$$

Note:- since the given eq^n 1 is of the first order its general solutions cannot have more than one arbitary Constant.

① Solve the following D.E's

(i) $p^2 = ax^3$, where $p = \frac{dx}{dy}$

Sol:- The given D.E can be written as

$$P = \pm (ax^3)^{1/2} = \pm a^{1/2} x^{3/2}$$

Here $p = \frac{dy}{dx} = \pm a^{1/2} x^{3/2}$

Seperating the variables, we get

$$dy = \pm a^{1/2} x^{3/2} dx$$

$\therefore$ the Sol^n is $\int dy = \pm a^{1/2} \int x^{3/2} dx$

$\left[\text{Apply } \int x^n dx = \frac{x^{n+1}}{n+1}\right]$

$$y = \pm a^{1/2} \left[\frac{x^{3/2+1}}{3/2+1}\right] + c = \pm \frac{2}{5} a^{1/2} x^{5/2} + c$$

$$= \pm \frac{1}{5}\left(2a^{1/2} x^{5/2} + 5c\right)$$

$$\Rightarrow 5(y+c) = \pm 2a^{1/2} x^{5/2}$$

Squaring on b.s, we get

$$25(y+c)^2 = \left(\pm 2a^{1/2} x^{5/2}\right)^2 = 4ax^5$$

which is the required solution.

(ii) $p^2 - 7p + 12 = 0$

Sol:- Given D.E is

$$p^2 - 7p + 12 = 0$$

$$(p-3)(p-4) = 0$$

$p-3=0$ and $p-4=0 \Rightarrow p=3$ and $p=4$

so $\frac{dy}{dx} = 3$ and $\frac{dy}{dx} = 4$

$dy = 3dx$ and $dy = 4dx$

Integrating, $y = 3x + c$ and $y = 4x + c$

$y - 3x - c = 0$ and $y - 4x - c = 0$

Hence the required combined solution is

$(y - 3x - c)(y - 4x - c) = 0$, c being an arbitary constant.

(2) Solve

(i) $y\left[\frac{dy}{dx}\right]^2 + (x - y)\frac{dy}{dx} - x = 0$

Sol Given eq^n can be written as

$yp^2 + (x - y)p - x = 0 \rightarrow (1)$

which is quadratic in p.

so (1) can be resolved into linear factors

$yp^2 + xp - yp - x = 0$

i.e $yp(p-1) + x(p-1) = 0$

$(p-1)(yp + x) = 0$

Its components are $p - 1 = 0$ and $yp + x = 0$

or $\frac{dy}{dx} = 1$ and $y\frac{dy}{dx} = -x$

or $dy = dx$ and $y\,dy = -x\,dx$

Integrating we get

$\int dy = \int dx + c$ and $\int y\,dy = -\int x\,dx + c$

i.e $y = x + c$ and $\frac{y^2}{2} = -\frac{x^2}{2} + c$

(or) $y - x - c = 0$ and $x^2 + y^2 - c = 0$

Hence the G.S is

$(y - x - c)(x^2 + y^2 - c) = 0$

3) $p(p+y) = x(x+y)$

$p^2 + py - x^2 - xy = 0$

$(p^2 - x^2) + py - xy = 0$

$(p+x)(p-x) + y(p-x) = 0$

$(p-x)[p+x+y] = 0$

$p - x = 0$

$p = x$

$\frac{dy}{dx} = x$

$dy = x\,dx$

$y = \frac{x^2}{2} + c$

$p + x + y = 0$

$\frac{dy}{dx} = -(x+y)$

$\frac{dy}{dx} + y = -x$

$e^{\int 1\,dx} = e^x$

G.S $\Rightarrow y(e^x) = \int -x e^x\,dx + c$

$ye^x = -[xe^x - e^x] + c$

Method - 3 :- Eq^n Solvable for y :-

If the given eq^n is solvable for y, then we can express 'y' explicity as a function of x and p. Thus, an eq^n solvable for 'y' can be put in the form

$$y = f(x, p) \rightarrow ①$$

① Solve the following DE

ii) $y = (x-a)p - p^2$

Sol Given eq^n is

$$y = (x-a)p - p^2 \rightarrow ①$$

Differentating ① w.r.t x and denoting $\frac{dy}{dx}$ by p we get

$$\frac{dy}{dx} = p = p + (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$0 = (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}(x-a-2p) = 0 \Rightarrow \frac{dp}{dx} = 0$$

Integrating we get

$$p = 2 \rightarrow ②$$

Eliminating p btw ① & ② we get

$$y = (x-a)c - c^2$$

which is the required Solution.

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(iii) $y = x + a\tan^{-1}p$

Given eq^n is

$$y = x + a\tan^{-1}p \rightarrow (1)$$

Eq^n of first order but not of the first ~~order~~ Degree

Differentiating (1) w.r.t x and denoting $\frac{dy}{dx}$ by p,

we get.

$$\frac{dy}{dx} = p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$= \frac{a}{1+p^2}\frac{dp}{dx} = p-1 \Rightarrow \frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

Seperating the variables, we get

$$\frac{dx}{a} = \frac{dp}{(p-1)(1+p^2)} = \left[\frac{A}{p-1} + \frac{Bp+C}{p^2+1}\right]dp,$$

using partial functions

Here $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$

$$\frac{dx}{a} = \frac{dp}{2(p-1)} - \frac{p+1}{2(p^2+1)}dp$$

(or)

$$dx = \frac{a}{2}\left(\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right)dp$$

Integrating,

$$x = \frac{a}{2}\left\{\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}p + \log c\right]$$

$$= \frac{a}{2}\left[\log\left[\frac{c(p-1)}{\sqrt{p^2+1}}\right] - \tan^{-1}p\right] \rightarrow (2)$$

Putting for x in ①, we get

$$y = \frac{a}{2}\left[\log\left\{\frac{c(p-1)}{\sqrt{p^2+1}}\right\} - \tan^{-1}p\right] + a\tan^{-1}p$$

$$= \frac{a}{2}\left[\log\left\{\frac{c(p-1)}{\sqrt{p^2+1}}\right\} + \tan^{-1}p\right] \rightarrow ③$$

Eq^n ② & ③ together give the required soln in parametric form.

Method - III: Eq^n solvable for x:

If the given eq^n is solvable for x, then we can express 'x' explicity as a function of y & p. Thus the eq^n of this type can be written as

$$x = f(y, p) \rightarrow ①$$

① Solve the following D.E

(i) $x = 3y - \log p$

Sol Given $x = 3y - \log p$

$$p = \frac{dy}{dx} \rightarrow ①$$

Differentating ① wrt x & writting $\frac{1}{x}$ for $\frac{dp}{dy}$ we get

$$\frac{dx}{dy} = \frac{1}{p} = 3 - \frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} + \frac{1}{p}\frac{dp}{dy} = 3$$

$$\frac{1}{p}\left[1 + \frac{dp}{dy}\right] = 3$$

$$1 + \frac{dp}{dy} = 3p$$

$\Rightarrow \frac{dp}{dy} = 3p-1 \Rightarrow dy = \frac{dp}{3p-1}$ (variable seperable)

Integrating, we get

$$\int dy = \int \frac{dp}{3p-1} + c$$

$$y = \frac{1}{3}\log(3p-1) + \log c$$

$$= \log(3p-1)^{1/3} c$$

$$(\text{or})\ e^y = c(3p-1)^{1/3}$$

Cubing, $ce^{3y} = 3p-1$ (or) $p = \frac{1+ce^{3y}}{3} \rightarrow (2)$

Eliminating p b/w (1) & (2) we get

$$x = 3y - \log\left[\frac{1+ce^{3y}}{3}\right]$$

which is the required solution.

(2)
(i) $y = 2px + p^3y^2$

Given eq^n can be written as

$$2x = \frac{y}{p} - y^2p^2$$

Diff w.r.t y & writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$2\frac{dx}{dy} = \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}$$

$$= \frac{2}{p} - \frac{1}{p} + 2yp^2 = \frac{-y}{p}\frac{dp}{dy}\left[\frac{1}{p} + 2yp^2\right]$$

$$\left[\frac{1}{p} + 2yp^2\right] + \frac{y}{p}\frac{dp}{dy}\left[\frac{1}{p} + 2yp^2\right] = 0$$

$$=\left[\frac{1}{p}+2yp^2\right]\left[1+\frac{y}{p}\frac{dp}{dy}\right]=0$$

Neglecting the first order factor which does not involve $\frac{dp}{dy}$, the above eq^n reduces to

$$1+\frac{y}{p}\frac{dp}{dy}=0 \Rightarrow y\frac{dp}{dy}=-1 \Rightarrow \frac{dp}{dy}=-\frac{p}{y}$$

$$\Rightarrow \frac{dp}{p}=-\frac{dy}{y}$$

Integrating, $\log p=-\log y+\log c$ or

$$\log p+\log y=\log c$$

(or)

$$py=c$$

$$\therefore p=\frac{c}{y}$$

Putting for p the given eq^n, we get

$$y=2\cdot\frac{c}{y}\cdot x+y^2\left(\frac{c^3}{y^3}\right)=\frac{2cx+c^3}{y}$$

(or)

$$y^2=2cx+c^3$$

which is the required Solution.

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