MODULE - IV

~~CIE - I~~ PART - A

① Given; $\phi_1 = x^2 + y^2 + z^2 - 9$

$\phi_2 = x^2 + y^2 - z - 3$

point $(2, -1, 2)$.

let; θ be the angle between the surfaces ϕ_1 and ϕ_2, then $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$.

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + K\frac{\partial\phi_1}{\partial z}$$

$$\nabla\phi_1 = i\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + j\frac{\partial}{\partial y}(x^2+y^2+z^2-9) + K\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$$

$$\nabla\phi_1 = i(2x) + j(2y) + K(2z)$$

$\nabla\phi_1$ at point $(2, -1, 2)$ is given by
$\quad\quad\quad\quad\quad\quad\; x \;\; y \;\; z$

$$\nabla\phi_1 = i(2(2)) + j(2(-1)) + K(2(2))$$

$$\nabla\phi_1 = 4i - 2j + 4K$$

$$|\nabla\phi_1| = \sqrt{(4)^2 + (-2)^2 + (4)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\nabla\phi_2 = i\,\frac{\partial\phi_2}{\partial x} + j\,\frac{\partial\phi_2}{\partial y} + k\,\frac{\partial\phi_2}{\partial z}.$$

$$\nabla\phi_2 = i\,\frac{\partial}{\partial x}(x^2+y^2-z-3) + j\,\frac{\partial}{\partial y}(x^2+y^2-z-3) + k\,\frac{\partial}{\partial z}(x^2+y^2-z-3)$$

$$\nabla\phi_2 = i\,(2x) + j\,(2y) + k\,(-1)$$

$\nabla\phi_2$ at point $(2,-1,2)$ is given by

$$\nabla\phi_2 = i(2(2)) + j\,(2(-1)) + k\,(-1)$$

$$\nabla\phi_2 = 4i - 2j - k$$

$$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$$

Now;

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$\cos\theta = \frac{(4i-2j+4k)\cdot(4i-2j-k)}{6\cdot\sqrt{21}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

2) let, the angle between the normals to the surfaces be 'θ'.

Given; surface $\rightarrow$ $x^2 = yz$ i.e $f(x,y,z) = x^2 - yz$.

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at $(1,1,1)$ and $(2,4,1)$ respectively.

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\cdot\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{i}\frac{\partial}{\partial x}(x^2 - yz) + \bar{j}\frac{\partial}{\partial y}(x^2 - yz) + \bar{k}\frac{\partial}{\partial z}(x^2 - yz)$$

$$\nabla f = \bar{i}(2x) + \bar{j}(-z) + \bar{k}(-y)$$

$$\nabla f = 2x\bar{i} - z\bar{j} - y\bar{k}$$

$\bar{n}_1 = \nabla f$ at $(1,1,1) = 2(1)\bar{i} - 1\cdot\bar{j} - 1\cdot\bar{k}$

$\bar{n}_1 = 2\bar{i} - \bar{j} - \bar{k}$

$|\bar{n}_1| = \sqrt{4+1+1} = \sqrt{6}$

$\bar{n}_2 = \nabla f$ at $(2,4,1) = 2(2)\bar{i} - 1\cdot\bar{j} - 4\bar{k}$

$\bar{n}_2 = 4\bar{i} - \bar{j} - 4\bar{k}$

$|\bar{n}_2| = \sqrt{16+1+16} = \sqrt{33}$

Now; $\cos\theta = \dfrac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \dfrac{(2\bar{i}-\bar{j}-\bar{k})\cdot(4\bar{i}-\bar{j}-4\bar{k})}{\sqrt{6}\cdot\sqrt{33}}$

$$\cos\theta = \frac{8+1+4}{3\sqrt{22}} = \frac{13}{3\sqrt{22}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)}$$ ✓

③ Given; $\operatorname{div}(\operatorname{grad} r^m) = m(m+1) r^{m-2}$

where; $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

let; $r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$ — ①

P.D ① w r to x $\rightarrow$ $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly; $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Consider; $\nabla r^m = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) \cdot r^m$

$\nabla r^m = i\left(m r^{m-1} \frac{\partial r}{\partial x}\right) + j\left(m r^{m-1} \frac{\partial r}{\partial y}\right) + k\left(m r^{m-1} \frac{\partial r}{\partial z}\right)$

$\nabla r^m = m r^{m-1} \left(\frac{xi + yj + zk}{r}\right)$

$\nabla r^m = m r^{m-2} (xi + yj + zk)$

Now;

$\nabla(\nabla r^m) = m\nabla(r^{m-2} x i + r^{m-2} y j + r^{m-2} z k)$

$\nabla(\nabla r^m) = m\left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right)\left(r^{m-2} x i + r^{m-2} y j + r^{m-2} z k\right)$

$\nabla(\nabla r^m) = m\left(\frac{\partial}{\partial x}(\underset{u}{r^{m-2}} \cdot \underset{v}{x}) + \frac{\partial}{\partial y}(r^{m-2} \cdot y) + \frac{\partial}{\partial z}(r^{m-2} \cdot z)\right)$

$\nabla(\nabla r^m) = m\Big((m-2) r^{m-3} \cdot x \cdot \frac{\partial r}{\partial x} + r^{m-2} \cdot 1 + (m-2) r^{m-3} \cdot y \cdot \frac{\partial r}{\partial y}$

$+ r^{m-2} \cdot 1 + (m-2) r^{m-3} \cdot z \cdot \frac{\partial r}{\partial z} + r^{m-2} \cdot 1\Big)$

$\nabla(\nabla r^m) = m\left(3r^{m-2} + (m-2) r^{m-3}\left(x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r}\right)\right)$

$$\nabla(\nabla r^m) = m\left(3r^{m-2} + (m-2)r^{m-3}\,\frac{x^2+y^2+z^2}{r}\right)$$

$$\nabla(\nabla r^m) = m\left(3r^{m-2} + \underbrace{(m-2)r^{m-3}\cdot\frac{r^{\cancel{2}}}{\cancel{r}}}_{r^{m-2}}\right)$$

$$\nabla(\nabla r^m) = m\left(3r^{m-2} + m\cdot r^{m-2} - 2r^{m-2}\right)$$

$$\nabla(\nabla r^m) = m\left(m\cdot r^{m-2} + 1\cdot r^{m-2}\right)$$

$$\nabla(\nabla r^m) = m(m+1)r^{m-2}.$$

Hence proved.

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(4) Given; $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

(i) curl $\bar{f} = 0$ (prove करने का)

(ii) scalar potential gradient (?)

Given; $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= i\left(-x-(-x)\right) - j\left(-y-(-y)\right) + k\left(-z-(-z)\right)$$

$$= \vec{0}i + 0j + 0k$$

curl $\bar{f} = \vec{0}$ → The vector is irrotational.

(4) Since $\bar{f}$ is irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$

$$\Rightarrow (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

comparing w.r.to components we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \qquad \frac{\partial\phi}{\partial y} = y^2 - zx \qquad \frac{\partial\phi}{\partial z} = z^2 - xy$$

$$\partial\phi = (x^2 - yz)\,\partial x \qquad \partial\phi = (y^2 - zx)\,\partial y \qquad \partial\phi = (z^2 - xy)\,\partial z$$

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$$\phi = \frac{x^3}{3} - xyz \qquad \phi = \frac{y^3}{3} - xyz \qquad \phi = \frac{z^3}{3} - xyz$$

from (1), (2) and (3)

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + \text{Constant}$$

$\therefore$ The scalar potential (ϕ) is given by
$\frac{1}{3}(x^3 + y^3 + z^3) - xyz + \text{Constant}$

(5) Given surface $xy^3z^2 - 4 = \phi$

at the point $(-1, -1, 2)$
$\quad\quad\quad\quad\quad\quad\quad x \;\; y \;\; z$

If ϕ is the surface, then the unit normal vector to the surface ϕ at the point (x, y, z) is given by $N = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i\frac{\partial}{\partial x}(xy^3z^2-4) + j\frac{\partial}{\partial y}(xy^3z^2-4) + k\frac{\partial}{\partial z}(xy^3z^2-4)$$

$$\nabla\phi = i(y^3z^2) + j(3xy^2z^2) + k(2xy^3z)$$

point $(-1, -1, 2)$
$\quad\quad\;\; x \;\; y \;\; z$

$$\nabla\phi = i(\underbrace{(-1)^3}_{-1}\underbrace{(2)^2}_{4}) + j(3\underbrace{(-1)}_{-1}\underbrace{(-1)^2}_{1}\underbrace{(2)^2}_{4}) + k(2\underbrace{(-1)}_{-1}\underbrace{(-1)^3}_{-1}2)$$

$$\nabla\phi = -4i - 12j + 4k$$

$$|\nabla\phi| = \sqrt{16+144+16} = \sqrt{176} = 4\sqrt{11}$$

$$N = \frac{-4i-12j+4k}{4\sqrt{11}} \Rightarrow N = \frac{-i-3j+k}{\sqrt{11}}$$

$\therefore$ A unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$ is $\dfrac{-i-3j+k}{\sqrt{11}}$

(b) Given: $\phi(x,y,z) = x^2yz + 4xz^2$ at point $(1,-2,-1)$

and $f(x,y,z) = x\log z - y^2$ at point $(-1,2,1)$.

* The directional derivative of a scalar ϕ, in the direction of a vector f is $N \cdot \nabla\phi$.

where $N = \dfrac{\nabla f}{|f|}$,

Now;
$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \bar{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) +$$
$$\bar{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$\nabla\phi = \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y + 8xz)$$

Now; $\nabla\phi$ at $(1,-2,-1)$ is given by

$$\nabla\phi = \bar{i}\,(2(1)(-2)(-1) + 4(-1)^2) + \bar{j}\,(1^2(-1)) + \bar{k}\,(1^2(-2) +$$
$$8(1)(-1))$$

$$\nabla\phi = 8\bar{i} - \bar{j} - 10\bar{k}$$

Now; unit normal vector f is $N = \dfrac{\nabla f}{|\nabla f|}$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{i}\frac{\partial}{\partial x}(x\log z - y^2) + \bar{j}\frac{\partial}{\partial y}(x\log z - y^2) + \bar{k}\frac{\partial}{\partial z}(x\log z - y^2)$$

$$\nabla f = (\log z)\,\bar{i} - (2y)\,\bar{j} + \left(\frac{x}{z}\right)\bar{k}$$

(7) <u>To prove:</u> $\nabla r^n = n r^{n-2} \bar{r}$.

Let; $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and let $r = |\bar{r}|$

$$r^2 = x^2 + y^2 + z^2$$

Differentiating w.r.to x partially, we have

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly; $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now; $\nabla r^n = \sum \bar{i} \frac{\partial}{\partial x}(r^n)$

$$\nabla r^n = \sum \bar{i}\, n r^{n-1} \cdot \frac{\partial r}{\partial x}$$

$$\nabla r^n = \sum \bar{i}\, n r^{n-1} \cdot \frac{x}{r}$$

$$\nabla r^n = n r^{n-2} \sum \bar{i} x$$

$$\nabla r^n = n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\nabla r^n = n r^{n-2} \cdot \bar{r}$$

Hence proved.

(8) To prove: $\nabla f(r) = \frac{f'(r)}{r} . \bar{r}$ where

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and let $r = |\bar{r}|$

Given; $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ ~~and let $r = |\bar{r}|$~~

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$$

Differentiating partially with respect to x, we get

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly; $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

now; $$\nabla(f(r)) = \bar{i}\frac{\partial f(r)}{\partial x} + \bar{j}\frac{\partial f(r)}{\partial y} + \bar{k}\frac{\partial f(r)}{\partial z}$$

$$\nabla(f(r)) = f'(r)\left\{\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right\}$$

$$\nabla(f(r)) = f'(r)\left\{\frac{x\bar{i}}{r} + \frac{y\bar{j}}{r} + \frac{z\bar{k}}{r}\right\}$$

$$\nabla(f(r)) = f'(r)\left\{\frac{x\bar{i}+y\bar{j}+z\bar{k}}{r}\right\}$$

$$\nabla(f(r)) = \frac{f'(r)}{r} . \bar{r}$$

Hence proved.

(9) Given; $\phi = xy^2 + yz^2 + zx^2$.

curve $\rightarrow$ $x = t$; $y = t^2$; $z = t^3$ at point $(1, 1, 1)$

$\rightarrow$ Directional vector — '?'

Now; $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\nabla\phi = \bar{i}\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \bar{j}\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2) +$
$\bar{k}\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$

$\nabla\phi = \bar{i}(y^2 + 2xz) + \bar{j}(2xy + z^2) + \bar{k}(x^2 + 2yz)$

$\nabla\phi$ at $(1, 1, 1)$ is given by.

$\nabla\phi = \bar{i}(1^2 + 2(1)(1)) + \bar{j}(2(1)(1) + 1^2) + \bar{k}(1^2 + 2(1)(1))$

$\nabla\phi = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve $x = t$, $y = t^2$, $z = t^3$. Then

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$\frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$

at $(1, 1, 1)$

$\frac{d\bar{r}}{dt} = \bar{i} + 2\bar{j} + 3\bar{k}$

We know that $\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve. Therefore unit vector along the tangent $N = \frac{\bar{t}}{|\bar{t}|}$

$$N = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{1+4+9}} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}}$$

Now; Directional derivative along the tangent is given by $= N \cdot \nabla\phi$

$$= \frac{(\bar{i} + 2\bar{j} + 3\bar{k})}{\sqrt{14}} \cdot (3\bar{i} + 3\bar{j} + 3\bar{k})$$

$$= \frac{3+6+9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

(10) Given; $t = xy + yz + zx$; point $(1,1,1)$

The greatest ~~rate~~ rate of increase of t at any point is given in magnitude and direction by ∇t.

$$\nabla t = \bar{i}\frac{\partial t}{\partial x} + \bar{j}\frac{\partial t}{\partial y} + \bar{k}\frac{\partial t}{\partial z}$$

$$\nabla t = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

at $(1,1,1)$

$$\nabla t = 2\bar{i} + 2\bar{j} + 2\bar{k}$$

$|t|$ = magnitude of this vector $= \sqrt{4+4+4}$

$\Rightarrow \sqrt{12} = 2\sqrt{3}$

Hence, at the point $(1,1,1)$, the temperature changes most rapidly in the direction given by the vector $2\bar{i}+2\bar{j}+2\bar{k}$ and the greatest rate of increase $= 2\sqrt{3}$

PART - B

(1) Given; $\bar{f} = (x+2y+az)\,\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$ —— (1)

Given; $\bar{f}$ is irrotational i.e curl $\bar{f} = \bar{0}$

To find (i) a, b, c values ?

(ii) Scalar potential ?

(i) curl $\bar{f} = \bar{0}$ ($\because \bar{f}$ is irrotational)

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow \bar{i}\,(c-(-1)) - \bar{j}\,(4-a) + \bar{k}\,(b-2) = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$c+1=0$
$c=-1$

compare with respective components.

We get the values of a, b, c are 4, 2, -1 respectively.

substitute $\bar{a}$, b, c values in (1)

(ii) $\bar{f} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\vec{K}$

$\because$ curl $\bar{f} = \bar{0} \rightarrow$ irrotational $\rightarrow$ scalar potential $\downarrow$ $\phi = ?$

ϕ exists:-

scalar potential function $\bar{f} = \nabla\phi$

$f_1 i + f_2 j + f_3 k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$f_1 = \frac{\partial\phi}{\partial x}$; $f_2 = \frac{\partial\phi}{\partial y}$; $f_3 = \frac{\partial\phi}{\partial z}$

$\therefore$ Total derivative is

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$d\phi = f_1 dx + f_2 dy + f_3 dz$$

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$$\int d\phi = \int(x+2y+4z)dx + \int(2x-3y-z)dy + \int(4x-y+2z)dz$$

$$\cancel{\phi = \frac{x^2}{2} + 2xy + 4xz + 2xy - 3\cdot\frac{y^2}{2} - zy + 4xz - yz + 2\frac{z^2}{2}}$$

$$\cancel{\phi = \frac{x^2-3y^2+2z^2}{2} + 4xy - 2yz + 8xz + const}$$

$$\cancel{\phi = \frac{1}{2}(x^2 + 2z^2 - 3y^2) + 2(2xy - yz + 4xz) + const}$$

$$\phi = \int x\,dx + \int 2z\,dz - \int 3y\,dy + \int 2(y\,dx + x\,dy) + \int 4(x\,dz + z\,dx) - \int(z\,dy + y\,dz)$$

$$\phi = \frac{x^2}{2} + 2\cdot\frac{z^2}{2} - \frac{3y^2}{2} + 2xy + 4xz - yz$$

(2) Surface (ϕ_1) $\to$ $ax^2 - byz = (a+2)x$ is orthog...

to surface (ϕ_2) $\to$ $4x^2y + z^3 = 4$

at point $(1,-1,2)$

To find: value of a & b?

let: $\phi_1 = ax^2 - byz - (a+2)x$ — (1)

$\phi_2 = 4x^2y + z^3 - 4$ — (2)

point $(1,-1,2)$
$x \; y \; z$

Substitute point in (1), we get

$$a + 2b - (a+2) = 0$$

$$\boxed{b = 1}$$

Now; $\nabla\phi_1 = \bar{i}\dfrac{\partial\phi_1}{\partial x} + \bar{j}\dfrac{\partial\phi_1}{\partial y} + \bar{k}\dfrac{\partial\phi_1}{\partial z}$

$\nabla\phi_1 = \bar{i}[2ax - (a+2)] + \bar{j}[-bz] + \bar{k}[-by]$

at point $(1,-1,2)$; $b=1$
$x \; y \; z$

$\nabla\phi_1 = \bar{i}[\underbrace{2a(1) - (a+2)}_{2a-a-2}] + \bar{j}[-1(2)] + \bar{k}[-1(-1)]$

$\nabla\phi_1 = (a-2)\bar{i} - 2\bar{j} + \bar{k}$

Now; $\nabla\phi_2 = \bar{i}\frac{\partial\phi_2}{\partial x} + \bar{j}\frac{\partial\phi_2}{\partial y} + \bar{k}\frac{\partial\phi_2}{\partial z}$

$\nabla\phi_2 = \bar{i}\left[12x^2y\right] + \bar{j}\left[4x^3\right] + \bar{k}\left[3z^2\right]$

at point $(1,-1,2)$.
$\quad\quad\quad\quad x\,y\,z$

$\nabla\phi_2 = \bar{i}\left[12(1)^2(-1)\right] + \bar{j}\left[4(1)^3\right] + \bar{k}\left[3(2)^2\right]$

$\nabla\phi_2 = \bar{i}(-12) + \bar{j}(4(1)) + \bar{k}(3(4))$

$\nabla\phi_2 = -12\bar{i} - 4\bar{j} + 12\bar{k}$

Since ϕ_1 and ϕ_2 are orthogonal to each other at point $(1,-1,2)$. $\Rightarrow$ $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$\Rightarrow ((a-2)\bar{i} - 2\bar{j} + \bar{k}).(-12\bar{i} - 4\bar{j} + 12\bar{k}) = \bar{0}$

$\Rightarrow -12(a-2) + 8 + 12 = 0$

$\Rightarrow -12a + 24 + 8 + 12 = 0$

$\Rightarrow 12a = 44$

$$\boxed{a = \frac{11}{3}}$$

$\therefore$ The values of a and b are $\frac{11}{3}$ and 1 respectively.

(3) Given; $\bar{A} = 3y^4z^2\bar{i} + 4x^3z^2\bar{j} - 3x^2y^2\bar{k}$

To prove : $\bar{A}$ is Solenoidal.

let's find div $\bar{A}$ i.e

$$\text{div}\,\bar{A} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)(\bar{A})$$

~~$\bar{i}\frac{\partial \bar{A}}{\partial x} + \bar{j}$~~

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(3y^4z^2\bar{i} + 4x^3z^2\bar{j} - 3x^2y^2\bar{k})$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial y}(-3x^2y^2)$$

Since $\bar{i}\cdot\bar{i} = 1$; $\bar{j}\cdot\bar{j} = 1$; $\bar{k}\cdot\bar{k} = 1$

$$= 0 + 0 + 0$$

$$= 0$$

$\therefore$ div $\bar{A} = 0$

$\therefore$ $\bar{A}$ is solenoidal

Hence proved

(4) Given $\bar{A} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$

To prove: $\bar{A}$ is irrotational

To find: scalar function.

let's find curl $\bar{A}$ i.e

$$\text{curl } \bar{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \vec{A}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \bar{i}(-1 - (-1)) - \bar{j}(3z^2 - 3z^2) + \bar{k}(6x - 6x)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$\therefore$ curl $\bar{A} = \bar{0}$ $\therefore$ $\bar{A}$ is irrotational

Hence proved.

To find scalar function. $\rightarrow \bar{f} = \nabla \phi$

$$f_1\bar{i} + f_2\bar{j} + f_3\bar{k} = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$f_1 = \frac{\partial \phi}{\partial x} \quad f_2 = \frac{\partial \phi}{\partial y} \quad f_3 = \frac{\partial \phi}{\partial z}$$

Total derivative is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\phi = f_1 dx + f_2 dy + f_3 dz$$

$$d\phi = (6xy + z^3) dx + (3x^2 - z) dy + (3xz^2 - y) dz$$

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$$\int d\phi = \int (6xy + z^3) dx + \int (3x^2 - z) dy + \int (3xz^2 - y) dz$$

$$\phi = 6y \cdot \frac{x^2}{2} + z^3 \cdot x + 3x^2 y - yz + 3x \frac{z^3}{3} - yz$$

$$\phi = 6x^2 y + 2xz^3 - 2yz$$

$$\int d\phi = \int (6xy + z^3) dx + \int (3x^2 - z) dy + \int (3xz^2 - y) dz$$

$$\phi = \frac{6x^2}{2} y + z^3 (x) + 3x^2 y - z(y) + 3x \frac{z^3}{3} - yz$$

$$\phi = (3x^2 y + xz^3 - yz) \times 2 + \text{constant}$$

↓

our required Scalar function

(5) Given; $\nabla f = (y^2 - 2xyz^3)\bar{i} + (3 + 2xy - x^2z^3)\bar{j} + (6z^3 - 3x^2yz^2)\bar{k}$; $f(1,0,1) = 8$, $f = ?$

$$\nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix}$$

$$= \bar{i}(-3x^2z^2 + 3x^2z^2) - \bar{j}(-6xyz^2 + 6xyz^2) + \bar{k}(2y - 2xz^3 - 2y + 2xz^3)$$

$$= 0\bar{i} + 0\bar{j} + 0\vec{k}$$

$$= \bar{0}$$

curl $\bar{f} = \bar{0}$ $\rightarrow$ irrotational vector and Hence scalar function exists.

Total derivative $\Rightarrow df = f_1\,dx + f_2\,dy + f_3\,dz$

$f_1 = \frac{\partial f}{\partial x}$; $f_2 = \frac{\partial f}{\partial y}$; $f_3 = \frac{\partial f}{\partial z}$

$$df = (y^2 - 2xyz^3)\,dx + (3 + 2xy - x^2z^3)\,dy + (6z^3 - 3x^2yz^2)\,dz$$

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$\int dH = \int (y^2 - 2xyz^3)\,dx + \int (3 + 2xy - x^2z^3)\,dy + \int (6z^3 - 3x^2yz^2)\,dz$

$f = xy^2 - x^2yz^3 + 3y + xy^2 - x^2yz^3 + 6\frac{z^4}{4} - x^2yz^3$

$\therefore$ Scalar potential $f = xy^2 - x^2yz^3 + \frac{6z^4}{4} + 3y$.

Now; $\underset{x\,y\,z}{f(1,0,1)} = (1)(0)^2 - (1)^2(0)(1)^3 + 6\frac{(1)^4}{4} + 3(0)$

$= 0 - 0 + \frac{6}{4} + 0$.

$= \frac{3}{2}$

$= 1.5$

(6) $\bar{V} = e^{xyz}(\bar{i} + \bar{j} + \bar{k})$ point $(1, 2, 3)$

curl $\bar{V} = ?$

$$\text{curl } \bar{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$$

$$= e^{xyz}\{(xz - xy)\bar{i} - (yz - xy)\bar{j} + (yz - xz)\bar{k}\}$$

at point $(1, 2, 3)$
x, y, z

$$= e^{1(2)(3)}\{(1(3) - 1(2))\bar{i} - ((2)(3) - 1(2))\bar{j} + ((2)(3) - (1)(3))\bar{k}\}$$

$$= e^{6}\{(3-2)\bar{i} - (6-2)\bar{j} + (6-3)\bar{k}\}$$

$$= e^{6}(\bar{i} - 4\bar{j} + 3\bar{k}).$$

(1) $f(r)$ is differentiable ; $r = (x^2+y^2+z^2)^{1/2}$

$r^2 = x^2+y^2+z^2$

let ; $r = |\bar{r}|$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Partially differentiate w r to x.

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$

To prove : $f(r)\,\bar{r}$ is irrotational.

Now ; $\text{curl}\{f(r).\bar{r}\} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$

$= \bar{i}\left\{f'(r).\frac{\partial r}{\partial y}.z - y.f'(r)\frac{\partial r}{\partial z}\right\} - \bar{j}\left\{z.f'(r).\frac{\partial r}{\partial x} - x.f'(r)\frac{\partial r}{\partial z}\right\} + \bar{k}\left\{y.f'(r)\frac{\partial r}{\partial x} - x.f'(r)\frac{\partial r}{\partial y}\right\}$

$= \bar{i}\left\{z.\frac{y}{r}\cancel{f'(r)} - y.\frac{z}{r}\cancel{f'(r)}\right\} - \bar{j}\left\{z.\frac{x}{r}\cancel{f'(r)} - x\frac{z}{r}\cancel{f'(r)}\right\} + \bar{k}\left\{y.\frac{x}{r}\cancel{f'(r)} - x.\frac{y}{r}\cancel{f'(r)}\right\}$

$\Rightarrow 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$

$\therefore \text{curl}\,(f(r).\bar{r}) = \bar{0}$

$\therefore f(r).\bar{r}$ is irrotational.

(8) $f = x^2yz$; $g = xy - 3z^2$

$\nabla(\nabla f \cdot \nabla g) = ?$

$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$\nabla f = \bar{i}(2xyz) + \bar{j}(x^2z) + \bar{k}(x^2y)$.

$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$\nabla g = \bar{i}(y) + \bar{j}(x) + \bar{k}(-6z)$.

$\nabla f \cdot \nabla g = ((2xyz)\bar{i} + (x^2z)\bar{j} + (x^2y)\bar{k}) \cdot (y\bar{i} + x\bar{j} - 6z\bar{k})$

$\nabla f \cdot \nabla g = 2xy^2z + x^3z - 6x^2yz$

$\because \bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$\nabla(\nabla f \cdot \nabla g) = \bar{i}\frac{\partial}{\partial x}(2xy^2z + x^3z - 6x^2yz) + \bar{j}\frac{\partial}{\partial y}(2xy^2z + x^3z - 6x^2yz) + \bar{k}\frac{\partial}{\partial z}(2xy^2z + x^3z - 6x^2yz)$

$\nabla(\nabla f \cdot \nabla g) = (2y^2z + 3x^2z - 12xyz)\bar{i} + (4xyz - 6x^2z)\bar{j}$
$(2xy^2 + x^3 - 6x^2y)\bar{k}$.

(9) let 'ϕ' be the angle of intersection between the given spheres. Let $f = x^2+y^2+z^2-29$ and $g = x^2+y^2+z^2+4x-6y-8z-47$. point $(4,-3,2)$

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\text{grad } f = 2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}$$

$$\bar{n}_1 = (\text{grad } f)_{(4,-3,2)} = 8\bar{i} - 6\bar{j} + 4\bar{k}$$

$$\text{grad } g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

$$\text{grad } g = (2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$$

$$\bar{n}_2 = (\text{grad } g)_{(4,-3,2)} = 12\bar{i} - 12\bar{j} - 4\bar{k}$$

$$|\bar{n}_1| = \sqrt{8^2+6^2+4^2} = \sqrt{116} \quad \& \quad |\bar{n}_2| = \sqrt{144+144+16} = \sqrt{304}$$

$$\cos\phi = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(8\bar{i}-6\bar{j}+4\bar{k})(12\bar{i}-12\bar{j}-4\bar{k})}{\sqrt{116}\sqrt{304}}$$

$$\cos\phi = \frac{152}{\sqrt{116}\sqrt{304}}$$

$$\boxed{\phi = \cos^{-1}\left(\frac{152}{\sqrt{116}\sqrt{304}}\right)}$$

(10) Given surface is $f(x,y,z) = xy - z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at $(4,1,2)$ and $(3,3,-3)$ respectively.

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\text{grad } f = \bar{i}(y) + \bar{j}(x) + \bar{k}(-2z)$$

$$\text{grad } f = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$\bar{n}_1 = (\text{grad} f)_{(4,1,2)} = \bar{i} + 4\bar{j} - 4\bar{k} \rightarrow |\bar{n}_1| = \sqrt{33}$$

$$\bar{n}_2 = (\text{grad } f)_{(3,3,-3)} = 3\bar{i} + 3\bar{j} + 6\bar{k} \rightarrow |\bar{n}_2| = \sqrt{54}$$

Let 'θ' be the angle between two normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(\bar{i}+4\bar{j}-4\bar{k})\cdot(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{33}\sqrt{54}}\right)$$

(11) Given Surface; $f(x,y,z) = 2x^2+3y^2-5z$

let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at $(2,-2,4)$ and $(-1,-1,1)$ respectively.

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\text{grad } f = \bar{i}(4x) + \bar{j}(6y) + \bar{k}(-5)$$

$$\text{grad } f = 4x\bar{i} + 6y\bar{j} - 5\bar{k}$$

$$\bar{n}_1 = (\text{grad} f)_{(2,-2,4)} = 8\bar{i} - 12\bar{j} - 5\bar{k} \rightarrow |\bar{n}_1| = \sqrt{233}$$

$$\bar{n}_2 = (\text{grad} f)_{(-1,-1,1)} = -4\bar{i} - 6\bar{j} - 5\bar{k} \rightarrow |\bar{n}_2| = \sqrt{57}$$

let 'θ' be the angle between two normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(8\bar{i}-12\bar{j}-5\bar{k})\cdot(-4\bar{i}-6\bar{j}-5\bar{k})}{\sqrt{233}\sqrt{57}}$$

$$\cos\theta = \frac{-32+72+25}{\sqrt{233}\sqrt{57}} = \frac{65}{\sqrt{233}\sqrt{57}}$$

$$\theta = \cos^{-1}\left(\frac{65}{\sqrt{233}\sqrt{77}}\right)$$

(13) Given: $\bar{A} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$

$\bar{A}$ is irrotational

$$\text{curl}\,\bar{A} = \bar{0} \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = \bar{0}$$

$$\Rightarrow \bar{i}(c-3) - \bar{j}(2-a) + \bar{k}(b-3) = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Compare w.r.to components.

$\Rightarrow$ $c-3=0$, $\boxed{c=3}$ | $2-a=0$, $\boxed{a=2}$ | $b-3=0$, $\boxed{b=3}$

$\therefore$ The values of a, b, c constants are 2, 3, 3 respectively.

$$\bar{A} = (2x+3y+2z)\bar{i} + (3x+2y+3z)\bar{j} + (2x+3y+3z)\bar{k}$$

To find scalar potential $\rightarrow$ $\bar{A} = \nabla\phi$

$$\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

compare w.r.to components.

$$A_1 = \frac{\partial\phi}{\partial x} \; ; \; A_2 = \frac{\partial\phi}{\partial y} \; ; \; A_3 = \frac{\partial\phi}{\partial z},$$

Total derivative is

$$d\phi = A_1 dx + A_2 dy + A_3 dz$$

$$d\phi = (2x+3y+2z)dx + (3x+2y+3z)dy + (2x+3y+3z)dz$$

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$$\int d\phi = \int (2x+3y+2z)dx + \int (2x+2y+3z)dy + \int (2x+3y+3z)dz$$

$$\cancel{\phi = 2\cdot\frac{x^2}{2} + 3xy + 2zx + 2xy + 2\cdot\frac{y^2}{2}} + 3yz +$$
$$\cancel{2xz + 3yz + 3\cdot\frac{z^2}{2}} + C.$$

$$\phi = \cancel{\frac{2x^2+2y^2+3z^2}{2}} + 5xy + 6yz + 2zx + \text{constant.}$$

~~is a scalar function.~~

$$\phi = \int 2x\,dx + \int 2y\,dy + \int 3z\,dz + \int 3y\,dx + \int 2x\,dy +$$
$$\underbrace{\int 3z\,dy + \int 3y\,dz}_{3yz} + \underbrace{\int 2z\,dx + \int 2x\,dz}_{2zx}$$

$$\phi = 2\cdot\frac{x^2}{2} + 2\cdot\frac{y^2}{2} + 3\cdot\frac{z^2}{2} + \_\_\_\_ + 3yz + 2zx + \text{constant}$$

is a scalar function.

(14) $\bar{f} = xyz^2\,\bar{i} + yzx^2\,\bar{j} + zxy^2\,\bar{k}$ (Say)

curl $\bar{f}$ at point $(1, 2, 3)$?
$\quad\quad\quad\quad\quad\quad\quad\quad x\ y\ z$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & yzx^2 & zxy^2 \end{vmatrix}$$

$$= \bar{i}\{2xyz - yx^2\} - \bar{j}\{zy^2 - 2xyz\}$$
$$+ \bar{k}\{2xyz - zz^2\}$$

Substitute the point

$$= \bar{i}\{2(1)(2)(3) - 2(1)^2\} - \bar{j}\{3\underbrace{(2)^2}_{4} - 2(1)(2)(3)\}$$
$$+ \bar{k}\{2(1)(2)(3) - 1(3)^2\}$$

$$= \bar{i}\{12 - 2\} - \bar{j}\{12 - 12\} + \bar{k}\{12 - 9\}$$

$$= 10\,\bar{i} - 0\,\bar{j} + 3\,\bar{k}$$

$b - (+3) = 0$

$b - 2 = 0$

(15) $\bar{A} = (bxy - z^3)\bar{i} + (b-2)x^2\bar{j} + (1-b)xz^2\bar{k}$

curl $\bar{A} = 0$; find $b = ?$

$$\text{curl } \bar{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bxy - z^3 & (b-2)x^2 & (1-b)xz^2 \end{vmatrix} = \bar{0}$$

$\Rightarrow \bar{i}\{0 - 0\} - \bar{j}\{(1-b)z^2 - 3z^2\} + \bar{k}\{2x(b-2) - bx\} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Compare w.r.to components.

$(1-b)z^2 - 3z^2 = 0$

$z^2 - bz^2 - 3z^2 = 0$

$\cancel{z^2 - (b+3)z^2 = 0}$

$-bz^2 - 2z^2 = 0$

$\boxed{b = -2}$

$2x(b-2) - bx = 0$

$2bx - 4x - bx = 0 \rightarrow bx = 4x$ (b=4)

$2b\cancel{x} = (4+b)\cdot\cancel{x}$

$2b = 4 + b$

$\boxed{b = 4}$ ✓

$\therefore$ The values of 'b' are -2 or 4.

(16) $\text{div}\,\bar{f} = ?$ $\quad \bar{f} = r^n \bar{r}$

If $\bar{f}$ is solenoid, find n?

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

let $r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$ — (1)

P.D (1) w.r to x, y, z respectively, we get

$\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = z/r$.

$\bar{f} = r^n \bar{r} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$\text{div}\,\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}\right)$

$= \frac{\partial}{\partial x}(\underset{u\ v}{r^n x}) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z).$

$uv' + vu'$

Since $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$= n r^{n-1} \cdot \frac{\partial r}{\partial x} \cdot x + r^n(1) + n r^{n-1} \frac{\partial r}{\partial y} \cdot y + r^n(1)$
$\quad + n r^{n-1} \frac{\partial r}{\partial z} \cdot z + r^n(1)$

$= n r^{n-1} \left\{ \frac{x}{r}\cdot x + \frac{y}{r}\cdot y + \frac{z}{r}\cdot z \right\} + r^n + r^n + r^n$

$= n r^{n-1} \left\{ \frac{\cancel{x^2 + y^2 + z^2}}{\cancel{r}} \right\} + 3r^n$

$= n r^{n-1} \cdot r + 3r^n$

$\operatorname{div} \bar{f} = n r^n + 3 r^n$

$\boxed{\operatorname{div} \bar{f} = (n+3) r^n}$

If $\bar{f}$ is solenoid $\rightarrow$ $\operatorname{div} \bar{f} = 0$

$(n+3) r^n = 0$

$n + 3 = 0$

$\boxed{n = -3}$

(17) Given

$a = x + y + z$

$b = x^2 + y^2 + z^2$

$c = xy + yz + zx$

To prove: $[\text{grad } a \ \ \text{grad } b \ \ \text{grad } c] = 0$

$\text{grad } a = \nabla a = \bar{i}\frac{\partial a}{\partial x} + \bar{j}\frac{\partial a}{\partial y} + \bar{k}\frac{\partial a}{\partial z}$

$\text{grad } a = \bar{i}(1) + \bar{j}(1) + \bar{k}(1)$

$\text{grad } a = \bar{i} + \bar{j} + \bar{k}$

$\text{grad } b = \nabla b = \bar{i}\frac{\partial b}{\partial x} + \bar{j}\frac{\partial b}{\partial y} + \bar{k}\frac{\partial b}{\partial z}$

$\text{grad } b = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$

$\text{grad } b = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$\text{grad } c = \nabla c = \bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \bar{k}\frac{\partial c}{\partial z}$

$\text{grad } c = \bar{i}(y+z) + \bar{j}(z+x) + \bar{k}(x+y)$

Now:

$$[\text{grad } a \ \ \text{grad } b \ \ \text{grad } c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

From cyclic determinents concept

[grad a grad b grad c] = $\underline{0}$

Hence proved.

(18) let; $\bar{f} = (y^2 - z^2 + 3yz - 2x)\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2xz + 2z)\bar{k}$

(i) solenoidal $\rightarrow$ $\operatorname{div} \bar{f} = 0$ ✓

$$\nabla \bar{f} = \left(\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} \right) \bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \right) \cdot \left((y^2 - z^2 + 3yz - 2x)\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2zx + 2z) \right)$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \partial/\partial z(3xy - 2zx + 2z)$$

since $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$

$$= \cancel{-2} + \cancel{2x} - \cancel{2x} + \cancel{2}$$

$$= 0$$

$\therefore$ since $\operatorname{div} \bar{f} = 0$, Hence $\bar{f}$ is a Solenoidal vector

(ii) Irrotational $\rightarrow$ curl $\bar{f} = 0$

$\nabla \bar{f}$ = curl $\bar{f}$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$\text{curl } \bar{f} = \bar{i}\left\{\cancel{3x} - \cancel{3x}\right\} - \bar{j}\left\{\cancel{3y} - \cancel{2z} - (-\cancel{2z} + \cancel{3y})\right\}$$
$$+ \bar{k}\left\{\cancel{3z} + \cancel{2y} - (\cancel{2y} + \cancel{3z})\right\}$$

$$\text{curl } \bar{f} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$\text{curl } \bar{f} = \bar{0}$$

$\therefore$ Since curl $\bar{f} = 0$; Hence $\bar{f}$ is a irrotational vector.

Therefore; $(y^2 - z^2 + 3yz - 2x)\,\bar{i} + (3xz + 2xy)\,\bar{j} + (3xy - 2xz + 2z)\,\bar{k}$ is both solenoidal and irrotational.

Hence proved.

(a) $f = 2x^3y^2z^4$ Point $(1,-2,1)$

$\nabla\cdot\nabla f = \text{div}(\text{grad } f)$

$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$\text{grad } f = \bar{i}(6x^2y^2z^4) + \bar{j}(4x^3yz^4) + \bar{k}(8x^3y^2z^3)$.

$\text{div}(\text{grad } f) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\bar{i}(6x^2y^2z^4) + \bar{j}(4x^3yz^4) + \bar{k}(8x^3y^2z^3)\right)$

Since $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$\nabla\cdot\nabla f = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$

$\nabla\cdot\nabla f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

let ; $\nabla\cdot\nabla f = \phi_1$

surface $\phi_2 = xy^2z - 3x - z^2 = 0$ → given.

* The directional derivative of ϕ_1 in the direction of normal to the surface ϕ_2

is: $N\cdot\nabla\phi_1$

$N = \frac{\nabla\phi_2}{|\nabla\phi_2|}$ (unit normal vector).

$\phi_2 = xy^2z - 3x - z^2 = 0$

$\nabla \cdot \phi_2 = \bar{i}\frac{\partial \phi_2}{\partial x} + \bar{j}\frac{\partial \phi_2}{\partial y} + \bar{k}\frac{\partial \phi_2}{\partial z}$

$\nabla \cdot \phi_2 = \bar{i}(y^2z - 3) + \bar{j}(2xyz) + \bar{k}(xy^2 - 2z)$

~~point~~

point $(1, -2, 1)$
$x \; y \; z$

$\phi_1 = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\nabla\phi_1 = \bar{i}\frac{\partial \phi_1}{\partial x} + \bar{j}\frac{\partial \phi_1}{\partial y} + \bar{k}\frac{\partial \phi_1}{\partial z}$

$\nabla\phi_1 = \bar{i}(12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2) + \bar{j}(24xyz^4 + 48x^3yz^2) + \bar{k}(48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)$

$\nabla\phi_1$ at $(1, -2, 1)$ is
$x \; y \; z$

$\nabla\phi_1 = \bar{i}(12\underbrace{(-2)^2}_{4}(1)^4 + 12(1)^2(1)^4 + 72(1)^2\underbrace{(-2)^2}_{4}(1)^2) + \bar{j}(24(1)(-2)(1)^4 + 48(1)^3(-2)(1)^2) + \bar{k}(48(1)\underbrace{(-2)^2}_{4}(1)^3 + 16(1)^3(1)^3 + 48(1)^3\underbrace{(-2)^2}_{4}(1))$

$\nabla\phi_1 = \bar{i}(48 + 12 + 288) + \bar{j}(-48 - 96) + \bar{k}(192 + 16 + 192)$

$\nabla\phi_1 = 348\bar{i} - 144\bar{j} + 400\bar{k}$

$|\nabla\phi_1| = \sqrt{348^2 + 144^2 + 400^2}$ → Not necessarily needed

$\therefore$ The directional derivative is $N \cdot \nabla\phi_1$

$N = \dfrac{\nabla\phi_2}{|\nabla\phi_2|}$

$\nabla\phi_2 = (y^2z - 3)\bar{i} + (2xyz)\bar{j} + (xy^2 - 2z)\bar{k}$

$(1, -2, 1)$
$x \quad y \quad z$

$\nabla\phi_2 = (\underbrace{(-2)^2(1)}_{4} - 3)\bar{i} + (\underbrace{2(1)(-2)(1)}_{-4})\bar{j} + (\underbrace{(1)(-2)^2}_{4} - 2(1))\bar{k}$

$4 - 3 \qquad -4 \qquad 4-2$

$\nabla\phi_2 = \bar{i} - 4\bar{j} + 2\bar{k}$

$|\nabla\phi_2| = \sqrt{1+16+4} = \sqrt{21}$

$N = \dfrac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{21}}$

$N \cdot \nabla\phi_1 = \dfrac{(\bar{i} - 4\bar{j} + 2\bar{k})}{\sqrt{21}} \cdot (348\bar{i} - 144\bar{j} + 400\bar{k})$

$\Rightarrow \dfrac{348 + 576 + 800}{\sqrt{21}} = \dfrac{1724}{\sqrt{21}}$

(20) $\nabla^2 f = \nabla \cdot \nabla f = \text{div}(\text{grad } f)$

$f = 3x^2 z - y^2 z^3 + 4x^3 y + 2x - 3y - 5$

$\text{grad } f = \bar{i}\frac{\partial}{\partial x} f + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$ $\quad \begin{matrix}(1, 1, 0)\\ x\ y\ z\end{matrix}$

$= \bar{i}(6xz + 12x^2 y + 2) + \bar{j}(-2yz^3 + 4x^3 - 3)$
$+ \bar{k}(3x^2 - 3y^2 z^2)$

Now;

$\text{div}(\text{grad } f) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right) \cdot \left(\bar{i}(6xz + 12x^2y + 2)\right.$
$\left. + \bar{j}(4x^3 - 2yz^3 - 3) + \bar{k}(3x^2 - 3y^2 z^2)\right)$

Since $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$= \frac{\partial}{\partial x}(6xz + 12x^2 y + 2) + \frac{\partial}{\partial y}(4x^3 - 2yz^3 - 3) +$
$\frac{\partial}{\partial z}(3x^2 - 3y^2 z^2)$

$= 6z + 24xy \cancel{+} - 2z^3 - 6y^2 z$

point $\begin{matrix}(1, 1, 0)\\ x\ y\ z\end{matrix}$

$= 6(0) + 24(1)(1) - 2(0)^3 - 6(1)^2(0)$

$= 0 + 24 - 0 - 0$

$\boxed{\therefore \nabla^2 f = 24.}$

Note

✓

5Q) ⟶ $f(1,0,1) = 3/2$ (Question mistake)

12Q) ⟶ Insufficient data

Another surface is not given. Hence, Directional derivative cannot be calculated

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Verified!!