MODULE V (SOLUTIONS)

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December 17, 2024

PART A

PROBLEM SOLVING AND CRITICAL THINKING QUESTIONS

Problem1: Show that $J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer.

Proof: We know

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p}$$

Put $p = -n$. Then,

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n}$$

Since, when $m-n+1 \le 0$ or $m \le n-1$ the gamma function is infinite. Therefore, for $m = 0$ to $n-1$, the expression is zero. Thus,

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\,(m-n)!} \left(\frac{x}{2}\right)^{2m-n}$$

Put $m - n = s$. Then,

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{(s+n)!\,s!} \left(\frac{x}{2}\right)^{2s+n} = (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+n)!\,s!} \left(\frac{x}{2}\right)^{2s+n} = (-1)^n J_n(x)$$

Problem2: Show that $J_n(x) = \frac{1}{\pi}\int_0^\pi cos(n\theta - x\sin\theta)d\theta$ where $J_n(x)$ is Bessel's function, n being an integer.

Proof: Since $\cos(n\theta - x\sin\theta) = cos\, n\theta cos(xsin\theta) + sin\, n\theta sin(xsin\theta)$

$$\int_0^\pi cos(n\theta - x\sin\theta)d\theta = \int_0^\pi [cos\, n\theta cos(xsin\theta) + sin\, n\theta sin(xsin\theta)]d\theta$$

$$= \int_0^\pi cos\, n\theta cos(xsin\theta)d\theta + \int_0^\pi sin\, n\theta sin(xsin\theta)d\theta\,.$$

Now,

$$cos(xsin\theta) = J_0 + 2J_2cos2\theta + 2J_4cos4\theta + \cdots$$

$$sin(x\sin\theta) = 2(J_1sin\theta + J_3sin3\theta + J_5sin5\theta + \cdots).$$

Therefore,

$$\int_0^\pi cos\, n\theta cos(xsin\theta)d\theta = \int_0^\pi cos\, n\theta[J_0 + 2J_2cos2\theta + 2J_4cos4\theta + \cdots]\, d\theta$$

$$= J_0\int_0^\pi cos\, n\theta\, d\theta + 2J_2\int_0^\pi cos\, n\theta cos2\theta d\theta + 2J_4\int_0^\pi cos\, n\theta cos4\theta d\theta + \cdots$$

$$+ 2J_{2m}\int^\pi cos\, n\theta cos2m\theta d\theta + \cdots$$

Continued...

Also,

$$\int_0^{\pi} sin\ n\theta sin(xsin\theta)d\theta = \int_0^{\pi} sin\ n\theta[2(J_1 sin\theta + J_3 sin3\theta + J_5 sin5\theta + \cdots)]\ d\theta$$

$$= 2J_1 \int_0^{\pi} sin\ n\theta\ sin\theta d\theta + 2J_3 \int_0^{\pi} sin\ n\theta sin3\theta d\theta + 2J_5 \int_0^{\pi} sin\ n\theta sin5\theta d\theta + \cdots$$

$$+ 2J_{2m+1} \int_0^{\pi} sin\ n\theta \sin(2m+1)\theta d\theta + \cdots$$

Since, $\int_0^{\pi} \cos x\theta \cos y\theta\ d\theta = \begin{cases} \frac{\pi}{2}, \text{if } \mathrm{x} = \mathrm{y}, \\ 0, \text{if } x \neq y, \end{cases}$ and $\int_0^{\pi} \sin x\theta \sin\ y\theta\ d\theta = \begin{cases} \frac{\pi}{2}, \text{if } \mathrm{x} = \mathrm{y}, \\ 0, \text{if } x \neq y, \end{cases}$

Now, let n is even, i.e., $n = 2m$.

Then,

$$\int_0^{\pi} cos\ 2m\theta cos(xsin\theta)d\theta$$

$$= J_0 \int_0^{\pi} cos\ 2m\theta\ d\theta + 2J_2 \int_0^{\pi} cos\ 2m\theta cos2\theta d\theta + 2J_4 \int_0^{\pi} cos\ 2m\theta cos4\theta d\theta + \cdots$$

$$+ 2J_{2m} \int_0^{\pi} cos\ 2m\theta cos2m\theta d\theta + \cdots = J_0 \times 0 + 2J_2 \times 0 + 2J_4 \times 0 + \cdots + 2J_{2m} \times \frac{\pi}{2}$$

$$= \pi J_{2m}.$$

Continued...

Also

$$\int_0^{\pi} sin\ n\theta sin(xsin\theta)d\theta$$

$$= 2J_1 \int_0^{\pi} sin\ n\theta\ sin\theta d\theta + 2J_3 \int_0^{\pi} sin\ n\theta sin3\theta d\theta + 2J_5 \int_0^{\pi} sin\ n\theta sin5\theta d\theta + \cdots$$

$$+ 2J_{2m+1} \int_0^{\pi} \sin n\theta \sin(2m + 1)\, \theta d\theta + \cdots$$

$$= 2J_1 \times 0 + 2J_3 \times 0 + 2J_5 \times 0 + \cdots + 2J_{2m+1} \times 0 + \cdots = 0.$$

Thus, when n is even

$$\int_0^{\pi} cos(n\theta - x\ sin\ \theta)d\theta = \int_0^{\pi} cos\ n\theta cos(xsin\theta)d\theta + \int_0^{\pi} sin\ n\theta sin(xsin\theta)d\theta = \pi J_n + 0$$

$$\Rightarrow \frac{1}{\pi} \int_0^{\pi} cos(n\theta - x\ sin\ \theta)d\theta = J_n.$$

Similarly, when n is odd.

$$\int_0^{\pi} cos\ n\theta cos(xsin\theta)d\theta = 0, \int_0^{\pi} sin\ n\theta sin(xsin\theta)d\theta = \pi J_n.$$

Therefore, $\frac{1}{\pi} \int_0^{\pi} cos(n\theta - x\ sin\ \theta)d\theta = J_n$.

Combining these two results, we have

$$\frac{1}{\pi} \int_0^{\pi} cos(n\theta - x\ sin\ \theta)d\theta = J_n \text{ for all } n.$$

Problem 3: Show that, Orthogonality relation of **Bessel's function** is:

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \dfrac{a^2}{2} j_{n+1}^2(a\alpha), & \text{if } \alpha = \beta \end{cases}$$

Here α and β are two distinct roots of $J_n(ax) = 0$.

We know that $J_n(\lambda x)$ is the solution of Bessel diferential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0 \quad \text{......................................(1)}$$

Let, $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ be the solutions of (1)

Then, $$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\lambda^2 x^2 - n^2)u = 0 \quad \text{................................(2)}$$

$$x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\lambda^2 x^2 - n^2)v = 0 \quad \text{................................(3)}$$

Now, multiplying (2) by v/x and (3) by u/x, and then substracting , we get

$$\frac{v}{x}\left[x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(\alpha^2x^2-n^2)u\right]-\frac{v}{x}\left[x^2\frac{d^2v}{dx^2}+x\frac{dv}{dx}+(\beta^2x^2-n^2)v\right]=0$$

$$\Rightarrow x\left[v\frac{d^2u}{dx^2}-u\frac{d^2v}{dx^2}\right]+\left[v\frac{du}{dx}-u\frac{dv}{dx}\right]+\frac{uv}{x}\left[\alpha^2x^2-n^2-\beta^2x^2+n^2\right]=0$$

$$\Rightarrow \frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]+uvx\left[\alpha^2-\beta^2\right]=0$$

$$\Rightarrow d\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]=-uvx\left[\alpha^2-\beta^2\right]dx$$

Taking integration both sides from 0 to 1, we get

$$\Rightarrow (\alpha^2-\beta^2)\int_0^1 uvx\,dx=-\int_0^1 d\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]dx$$

$$\Rightarrow \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx=-\frac{1}{(\alpha^2-\beta^2)}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]_0^1$$

$$\Rightarrow \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx=-\frac{1}{(\alpha^2-\beta^2)}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)$$

$$\Rightarrow \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = -\frac{1}{(\alpha^2-\beta^2)} x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Bigg|_0^1$$

$$\Rightarrow \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = -\frac{1}{(\alpha^2-\beta^2)}\left[J_n(\beta x)\alpha J'_n(\alpha x) - J_n(\alpha x)\beta J'_n(\beta x)\right]_0^1$$

$$= -\frac{1}{(\alpha^2-\beta^2)}\left[J_n(\beta)\alpha J'_n(\alpha) - J_n(\alpha)\beta J'_n(\beta)\right]$$

As, $\mathrm{u} = J_n(\alpha x)$, and $\mathrm{v} = J_n(\beta x)$ $\Rightarrow \dfrac{du}{dx} = \alpha J'_n(\alpha x)$ and $\dfrac{dv}{dx} = \beta J'_n(\beta x)$

Again, α and β are the roots of $J_n(x) = 0 \Rightarrow J_n(\alpha) = 0; J_n(\beta) = 0$

CASE1: When $\alpha \neq \beta$

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = -\frac{1}{(\alpha^2-\beta^2)} * 0 = 0$$

CASE2: When $\alpha = \beta$

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{0}{0}\, forms$$

$$= \lim_{\alpha\to\beta} \frac{0 - \beta J'_n(\alpha)J'_n(\beta)}{-2\alpha} = \frac{1}{2}\left[J'_n(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

$$\left[\begin{array}{r}\text{Since, the recurence relation stated that, } J'_n(x) = \dfrac{n}{x}J_n(x) - J_{n+1}(x) \\ \therefore J'_n(\alpha) = -J_{n+1}(\alpha), as J_n(\alpha) = 0\end{array}\right]$$

Problem 4: of integral order is: The **generating function** of Bessel's function

$$\boldsymbol{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}}$$

Then, we can write

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{p=-\infty}^{\infty} J_p(x)t^p$$

GENERATING FUNCTION OF BESSEL FUNCTIONS

The exponential term can be expanded using its series expansion:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} e^{-\frac{x}{2t}}.$$

Using the Taylor expansion for e^u:

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!},$$

we expand each term separately:

$$e^{\frac{xt}{2}} = \sum_{k=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^k}{k!}, \quad e^{-\frac{x}{2t}} = \sum_{m=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^m}{m!}.$$

Thus, the product becomes:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^k}{k!} \cdot \sum_{m=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^m}{m!}.$$

Expand the double sum:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{k+m} t^k(-t^{-m})}{k!m!}.$$

Simplify the powers of t:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{k+m} (-1)^m t^{k-m}}{k!m!}.$$

Change the indices to express it in terms of a single power of t. Let $n = k - m$, so $k = n + m$. Then:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m} (-1)^m}{(n+m)!m!} \right) t^n.$$

The term inside the summation matches the series definition of the Bessel function of the first kind $J_n(x)$:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!(n+m)!}.$$

Thus, the generating function becomes:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Conclusion

We have shown that the exponential $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ expands into a power series in t, where the coefficients are $J_n(x)$, the Bessel functions of the first kind. Therefore, this proves that it is the generating function of the Bessel functions.

Problem 5: Prove that $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$.

Solution: Since

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

So,

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{(2)^{2n+p} \cdot n! \cdot \Gamma(p+n+1)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+2p) x^{2n+2p-1}}{(2)^{2n+p} \cdot n! \cdot (\mathrm{p}+\mathrm{n}) \Gamma(p+n)} \quad \left(\text{since } \Gamma(x+1) = x\Gamma(x)\right)$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+p-1}}{(2)^{2n+p-1} \cdot n! \cdot \Gamma(n+(p-1)+1)}$$

$$= x^p J_{p-1}(x).$$

Problem 6: Prove that $\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$.

Solution: Since $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\cdot\Gamma(p+n+1)}\left(\frac{x}{2}\right)^{2n+p}$

So,

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2)^{2n+p}\cdot n!\cdot \Gamma(p+n+1)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2n x^{2n-1}}{(2)^{2n+p}\cdot n!\cdot \Gamma(p+n+1)} \quad = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2n \cdot x^{2n-1}}{(2)^{2n+p}\cdot n!\cdot \Gamma(n+p+1)}$$

(since the first term is zero for $n = 0$)

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n-1}}{(2)^{2n+p-1}\cdot (n-1)!\cdot \Gamma(n+p+1)}$$

Put $n = s + 1$.

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = \sum_{s=0}^{\infty} \frac{(-1)^{s+1} \cdot x^{2s+1}}{(2)^{2s+p+1}\cdot (n-1)!\cdot \Gamma(s+(p+1)+1)}$$

$$= -x^{p} \sum_{s=0}^{\infty} \frac{(-1)^{s} \cdot x^{2s+(p+1)}}{(2)^{2s+p+}\cdot (n-1)!\cdot \Gamma(s+(p+1)+1)}$$

$$= -x^{p} J_{p+1}(x).$$

Problem 7: Show that $\int J_3(x)dx = -J_2(x) - \frac{2}{x}J_1(x)$ using Bessel's Recurrence relation.

Proof: Let us consider the integral

$$\int J_3(x)dx = \int x^2 x^{-2} J_3(x)dx = x^2 \int x^{-2} J_3(x)dx - \int \left[2x \cdot \int x^{-2} J_3(x)dx\right] dx.$$

We know that

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$$

$$\Rightarrow d\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)dx \Rightarrow \int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$$

By putting $n = 1,2$, we get

$$\int x^{-1}J_2(x)dx = -x^{-1}J_1(x), \int x^{-2}J_3(x)dx = -x^{-2}J_2(x)$$

Then,

$$\int J_3(x)dx = x^2\left(-x^{-2}J_2(x)\right) - \int \left[2x \cdot \left(-x^{-2}J_2(x)\right)\right]dx = -J_2(x) + 2\int x^{-1}J_2(x)dx$$

$$= -J_2(x) + 2\left(-x^{-1}J_1(x)\right)$$

$$= -J_2(x) - \frac{2}{x}J_1(x).$$

EXERCISE

Problem 8, 9: Make use of the generating function

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{p=-\infty}^{\infty} J_p(x)t^p \ ,$$

- **Prove:** $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots)$
- **Prove:** $\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta + \cdots)$.
- **Prove:** $\sin(x\cos\theta) = 2(J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots)$.
- **Prove:** $\cos(x\cos\theta) = J_0 - 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 - \cdots$
- **Show that:** $\cos x = J_0 - 2J_2 + 2J_4 - \cdots$
- **Show that:** $\sin x = J_0 - 2J_2 + 2J_4 - \cdots$

Proof: from the generating function, we know that

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{p=-\infty}^{\infty} J_p(x)t^p$$

$$= \cdots + t^{-3}J_{-3}(x) + t^{-2}J_{-2}(x) + t^{-1}J_{-1}(x) + t^0J_0(x) + t^1J_1(x) + t^2J_2(x) \ldots$$

Also, we know that

$$J_{-p}(x) = (-1)^pJ_p(x).$$

Therefore,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \cdots - t^{-3}J_3(x) + t^{-2}J_2(x) - t^{-1}J_1(x) + J_0(x) + t^1J_1(x) + t^2J_2(x) \ldots$$

$$= J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \cdots$$

Let $t = \cos\theta + i\sin\theta$. Then, $\frac{1}{t} = \cos\theta - i\sin\theta$, $t^p + \frac{1}{t^p} = 2\cos p\theta$ and $t^p - \frac{1}{t^p} = 2i\sin p\theta$.

Therefore, $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}x(2i\sin\quad)} = e^{ix\sin}$. Thus,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{ix\sin\theta} = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + \cdots$$

...............(1)

Continued..

$\cos(x \sin\theta) + i sin(x sin\theta) \;\; = J_0(x) + 2i \sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + 2i \sin 3\theta\, J_3(x) + 2\cos 4\theta\, J_4(x) + \cdots$

$$= [J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + \cdots] + i[2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + 2\sin 5\theta\, J_5(x) + \cdots \dots \dots \dots \dots \dots \dots].$$

Equating the real parts, we get the result

$$\cos(x \sin\theta) = J_0 + 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 + \cdots$$
$$\sin(x \sin\theta) = 2(\sin\theta\, J_1 + \sin 3\theta\, J_3 + \sin 5\theta\, J_5 + \cdots)$$

Continued..

Replace θ by $\left(\frac{\pi}{2}-\theta\right)$. Then, from (1), we get

$$e^{ix\sin\left(\frac{\pi}{2}-\theta\right)}$$
$$= J_0(x) + 2i\sin\left(\frac{\pi}{2}-\theta\right)J_1(x) + 2\cos 2\left(\frac{\pi}{2}-\theta\right)J_2(x)$$
$$+ 2i\sin 3\left(\frac{\pi}{2}-\theta\right)J_3(x) + 2\cos 4\left(\frac{\pi}{2}-\theta\right)J_4(x) + \cdots$$
$$\Rightarrow e^{ix\cos\theta}$$
$$= J_0(x) + 2i\cos\theta\, J_1(x) - 2\cos 2\theta\, J_2(x) - 2i\cos 3\theta\, J_3(x)$$
$$+ 2\cos 4\theta\, J_4(x) + \cdots$$

$$\Rightarrow \cos(x\cos\theta) + i\sin(x\cos\theta)$$
$$= [J_0(x) - 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) - \cdots]$$
$$+ i[2\cos\theta\, J_1(x) - 2\cos 3\theta\, J_3(x) + 2\cos 5\theta\, J_5(x) -].$$

Equating the real parts, we get the result

$$\cos(x\cos\theta) = J_0 - 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 - \cdots$$
$$\sin(x\cos\theta) = 2[\cos\theta\, J_1 - \cos 3\theta\, J_3 + \cos 5\theta\, J_5 - \cdots]$$

Continued..

Put, $\theta = 0$, in both the equations

$$\cos(x) = J_0 - 2J_2 + 2J_4 - \cdots$$

$$\sin(x) = 2[J_1 - J_3 + J_5 - \cdots]$$

Problem 10: Show that $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right)$

Proof: We know the recurrence relation:

$$\frac{2p}{x}J_p(x) = [J_{p-1}(x) + J_{p+1}(x)] \Rightarrow J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x).$$

By putting $p = \frac{3}{2}$ and $\frac{1}{2}$ we have

$$J_{\frac{5}{2}}(x) = \frac{3}{x}J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x), J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x).$$

Therefore, we have

$$J_{\frac{5}{2}}(x) = \frac{3}{x}J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) = \frac{3}{x}\left(\frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x).\right) - J_{\frac{1}{2}}(x) = \left(\frac{3}{x^2} - 1\right)J_{\frac{1}{2}}(x) - \frac{3}{x}J_{-\frac{1}{2}}(x).$$

We also know that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,sinx, J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,cosx.$

Finally, we get

$$J_{\frac{5}{2}}(x) = \frac{3-x^2}{x^2}J_{\frac{1}{2}}(x) - \frac{3}{x}J_{-\frac{1}{2}}(x) = \frac{3-x^2}{x^2}\sqrt{\frac{2}{\pi x}}\,sinx - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\,cosx$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}sinx - \frac{3}{x}cosx\right).$$

PART B

LONG ANSWER QUESTIONS

Problem 1: Show that $\int_0^x x^n J_{n-1}(x)dx = x^n J_n(x)$ where $J_n(x)$ is Bessel's function.

Proof: We know the recurrence relation:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow d[x^n J_n(x)] = x^n J_{n-1}(x)dx$$

Integrating both sides from 0 to x, we get

$$\int_0^x d[x^n J_n(x)] = \int_0^x x^n J_{n-1}(x)dx$$

$$\Rightarrow \int_0^x x^n J_{n-1}(x)dx = [x^n J_n(x)]_0^x = x^n J_n(x).$$

Problem 2: Show that $\int_0^x x^{n+1} J_n(x)\,dx = x^{n+1} J_{n+1}(x)$ where $J_n(x)$ is Bessel's function.

Proof: We know the recurrence relation:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

Replace n by $n+1$.

$$\frac{d}{dx}[x^{n+1} J_{n+1}(x)] = x^{n+1} J_n(x)$$

$$\Rightarrow d[x^{n+1} J_{n+1}(x)] = x^{n+1} J_n(x)dx$$

Integrating both sides from 0 to x, we get

$$\int_0^x d[x^{n+1} J_{n+1}(x)] = \int_0^x x^{n+1} J_n(x)dx$$

$$\Rightarrow \int_0^x x^{n+1} J_n(x)dx = [x^{n+1} J_{n+1}(x)]_0^x = x^{n+1} J_{n+1}(x).$$

Problem 3: Show that $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2(x) - (n+1)J_{n+1}^2(x)\right]$ here $J_n(x)$ is Bessel's function.

Proof: Differentiating, we get

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2\right] = 2J_n(x)J_n'(x) + 2J_{n+1}(x)J_{n+1}'(x)$$

We know the recurrence relations:

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}$$

$$\Rightarrow J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}$$

$$\Rightarrow J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}$$

Replace n by $n+1$. Then, we get $J_{n+1}'(x) = -\frac{n+1}{x}J_{n+1}(x) + J_n$

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2\right] = 2J_n(x)\left(\frac{n}{x}J_n(x) - J_{n+1}\right) + 2J_{n+1}(x)\left(-\frac{n+1}{x}J_{n+1}(x) + J_n\right)$$

$$= \frac{2n}{x}J_n^2(x) - 2J_n(x)J_{n+1}(x) - 2\frac{n+1}{x}J_{n+1}^2(x) + 2J_n(x)J_{n+1}(x)$$

$$= \frac{2}{x}\left[nJ_n^2(x) - (n+1)J_{n+1}^2(x)\right]$$

Problem 4: Show that $\frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right] = x\left(J_n^2(x) - J_{n+1}^2(x)\right)$

Proof: Differentiating we get

$$\frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right] = J_n(x)J_{n+1}(x) + xJ_n'(x)J_{n+1}(x) + xJ_n(x)J_{n+1}'(x)$$

We know the recurrence relations:

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \cdots\cdots\cdots (1)$$

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

$$\Rightarrow xJ_{n+1}'(x) = xJ_n(x) - (n+1)J_{n+1}(x) \cdots\cdots\cdots (2)$$

Therefore,

$$\frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right]$$

$$= J_n(x)J_{n+1}(x) + \left(nJ_n(x) - xJ_{n+1}(x)\right)J_{n+1}(x) + J_n(x)\left(xJ_n(x) - (n+1)J_{n+1}(x)\right)$$

$$= J_n(x)J_{n+1}(x) + nJ_n(x)J_{n+1}(x) - xJ_{n+1}^2(x) + xJ_n^2(x) - (n+1)J_n(x)J_{n+1}(x)$$

$$= x\left(J_n^2(x) - J_{n+1}^2(x)\right)$$

Problem 6: Show that $\int J_3(x)dx = -J_2(x) - \frac{2}{x}J_1(x)$ using Bessel's Recurrence relation.

Proof: Let us consider the integral

$$\int J_3(x)dx = \int x^2 x^{-2} J_3(x)dx = x^2 \int x^{-2} J_3(x)dx - \int \left[2x \cdot \int x^{-2} J_3(x)dx\right] dx.$$

We know that

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$\Rightarrow d\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)dx \Rightarrow \int x^{-n} J_{n+1}(x)dx = -x^{-n} J_n(x)$$

By putting $n = 1,2$, we get

$$\int x^{-1} J_2(x)dx = -x^{-1} J_1(x), \int x^{-2} J_3(x)dx = -x^{-2} J_2(x)$$

Then,

$$\int J_3(x)dx = x^2\left(-x^{-2} J_2(x)\right) - \int \left[2x \cdot \left(-x^{-2} J_2(x)\right)\right]dx = -J_2(x) + 2\int x^{-1} J_2(x)dx$$

$$= -J_2(x) + 2\left(-x^{-1} J_1(x)\right)$$

$$= -J_2(x) - \frac{2}{x} J_1(x).$$

EXERCISE

Problem 7, 8: Make use of the generating function

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{p=-\infty}^{\infty} J_p(x)t^p \ ,$$

- **Prove:** $\cos(x\,sin\,\theta) = J_0 + \ 2J_2cos2\theta + 2J_4cos4\theta + \cdots)$
- **Prove:** $\sin(x\,sin\,\theta) = \ 2(J_1sin\theta + J_3sin3\theta + J_5sin5\theta + \cdots)$.
- **Prove:** $\sin(x\,cos\,\theta) = \ 2(J_1cos\theta - J_3cos3\theta + J_5cos5\theta - \cdots)$.
- **Prove:** $\cos(x\cos\theta) = J_0 - 2\,cos\,2\theta\,J_2 + 2\,cos\,4\theta\,J_4 - \cdots$

Proof: from the generating function, we know that

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{p=-\infty}^{\infty} J_p(x)t^p$$

$$= \cdots + t^{-3}J_{-3}(x) + t^{-2}J_{-2}(x) + t^{-1}J_{-1}(x) + t^{0}J_0(x) + t^{1}J_1(x) + t^{2}J_2(x) \ldots$$

Also, we know that

$$J_{-p}(x) = (-1)^p J_p(x).$$

Therefore,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \cdots - t^{-3}J_3(x) + t^{-2}J_2(x) - t^{-1}J_1(x) + J_0(x) + t^{1}J_1(x) + t^{2}J_2(x) \ldots$$

$$= J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \cdots$$

Let $t = \cos\theta + i\sin\theta$. Then, $\frac{1}{t} = \cos\theta - i\sin\theta$, $t^p + \frac{1}{t^p} = 2\cos p\theta$ and $t^p - \frac{1}{t^p} = 2i\sin p\theta$.

Therefore, $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}x(2i\,\mathrm{si}\quad)} = e^{ix\sin\quad}$. Thus,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{ix\sin\theta} = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + \cdots$$

……………(1)

Continued..

$\cos(x\sin\theta) + i sin(x sin\theta) \ = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + 2i\sin 3\theta\, J_3(x) + 2\cos 4\theta\, J_4(x) + \cdots$

$$= [J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + \cdots] +$$

$i[2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + 2\sin 5\theta\, J_5(x) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots].$

Equating the real parts, we get the result

$\cos(x\sin\theta) = J_0 + 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 + \cdots$

$\sin(x\sin\theta) = 2(\sin\theta\, J_1 + \sin 3\theta\, J_3 + \sin 5\theta\, J_5 + \cdots)$

Problem 9: Show that $J_n(-x) = (-1)^n J_n(x)$ where n is an integer.

Proof: We know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

Then,

$$J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{-x}{2}\right)^{2m+n}$$

Suppose n is an even integer. Let $n = 2s$.

Then,

$$(-x)^{2m+n} = (-1)^{2m+2s}(x)^{2m+2s} = (x)^{2m+2s} = x^{2m+n}$$

$$J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} = J_n(x).$$

Again, if n is an odd integer. Let $n = 2s - 1$.

Then,

$$(-x)^{2m+n} = (-1)^{2m+2s-1}(x)^{2m+2s-1} = -(x)^{2m+2s-1}$$
$$= -x^{2m+n}$$

$$J_n(-x) = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} = -J_n(x).$$

Combining these two results, we get

$$J_n(-x) = (-1)^n J_n(x)$$

Problem 10: Prove that $J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$

or

$$xJ_p'(x) = pJ_p(x) - xJ_{p+1}(x)$$

Solution: We know from recurrence relation II that

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$\Rightarrow x^{-p} J_p'(x) - p x^{-p-1} J_p(x) = -x^{-p} J_{p+1}(x)$$

$$\Rightarrow J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

$$\Rightarrow J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x).$$

Problem 11: Prove that $\frac{d}{dx}\left[J_p(x)\right] = J_{p-1}(x) - \frac{p}{x}J_p(x)$ or $xJ'_p(x) = xJ_{p-1}(x) - pJ_p(x)$

Solution: We know from recurrence relation I that

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

$$\Rightarrow x^p J'_p(x) + p x^{p-1} J_p(x) = x^p J_{p-1}(x)$$

$$\Rightarrow J'_p(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

$$\Rightarrow J'_p(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

$$\Rightarrow xJ'_p(x) = xJ_{p-1}(x) - pJ_p(x)$$

Problem 12: Show that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right)$.

Proof: We know the recurrence relation

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

Put $p = \frac{1}{2}$. Then, we get

$$J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x) = \frac{2 \cdot \frac{1}{2}}{x} J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) \Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

Now, we also know that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \text{ and } J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Therefore,

$$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x = \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right).$$

Problem 14: Show that $\int xJ_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Proof: We know the recurrence relation

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x).$$

By putting $p = 0$, we have

$$\frac{d}{dx}[J_0(x)] = -J_1(x).$$

Integrating by parts, we have

$$\int xJ_0^2(x)dx = \frac{x^2}{2}J_0^2(x) - \int \frac{x^2}{2} 2\cdot J_0(x)J_0'(x)dx = \frac{x^2}{2}J_0^2(x) - \int x^2 J_0(x)\cdot -J_1(x)\,dx$$

$$= \frac{x^2}{2}J_0^2(x) + \int x^2 J_0(x)J_1(x)\,dx$$

Now,

$$\frac{d}{dx}\left(\frac{x^2}{2}J_1^2(x)\right) = \frac{2x}{2}J_1^2(x) + \frac{x^2}{2}2J_1(x)J_1'(x) = xJ_1^2(x) + x^2J_1(x)J_1'(x).$$

Again, we know

$$xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x) \Rightarrow J_p'(x) = J_{p-1}(x) - \frac{p}{x}J_p(x)$$

By putting $p = 1$, we get $J_1'(x) = J_0(x) - \frac{1}{x}J_1(x)$.

Continued...

Therefore,

$$\frac{d}{dx}\left(\frac{x^2}{2}J_1^2(x)\right) = xJ_1^2(x) + x^2J_1(x)J_1'(x)$$

$$= xJ_1^2(x) + x^2J_1(x)\left(J_0(x) - \frac{1}{x}J_1(x)\right) = xJ_1^2(x) + x^2J_1(x)J_0(x) - xJ_1^2(x)$$

$$= x^2J_1(x)J_0(x) \Rightarrow d\left(\frac{x^2}{2}J_1^2(x)\right) = \left(x^2J_1(x)J_0(x)\right)dx$$

Integrating, we have

$$\frac{x^2}{2}J_1^2(x) = \int\left(x^2J_1(x)J_0(x)\right)dx$$

Then,

$$\int xJ_0^2(x)dx = \frac{x^2}{2}J_0^2(x) + \int x^2J_0(x)J_1(x)\,dx = \frac{x^2}{2}J_0^2(x) + \frac{x^2}{2}J_1^2(x)$$

$$= \frac{x^2}{2}[J_0^2(x) + J_1^2(x)].$$

Problem 15: Prove that $\frac{d}{dx}\left[J_p(x)\right] = J_{p-1}(x) - \frac{p}{x}J_p(x)$ or $xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$

Solution: We know from recurrence relation I that

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

$$\Rightarrow x^p J_p'(x) + p x^{p-1} J_p(x) = x^p J_{p-1}(x)$$

$$\Rightarrow J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

$$\Rightarrow J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

$$\Rightarrow xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$$

Problem 16, 17: Show that $J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots) = 1$

Proof: We know that

$$\frac{d}{dx}[J_n^2(x) + J_{n+1}^2] = \frac{2}{x}[nJ_n^2(x) - (n+1)J_{n+1}^2(x)]$$

Put $n = 0,1,2, \dots$

$$\frac{d}{dx}[J_0^2(x) + J_1^2(x)] = \frac{2}{x}[-J_1^2(x)]$$

$$\frac{d}{dx}[J_1^2(x) + J_2^2(x)] = \frac{2}{x}[J_1^2(x) - 2J_2^2(x)]$$

$$\frac{d}{dx}[J_2^2(x) + J_3^2(x)] = \frac{2}{x}[2J_2^2(x) - 3J_3^2(x)]$$

and so on.

By adding those, we get

$$\frac{d}{dx}[J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) \dots)] = 0$$

$$\Rightarrow d\left[[J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) \dots)]\right] = 0.$$

Integrating, we have

$$[J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) \dots)] = C \text{ (integration constant)}$$

Continued...

Since

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!}\left(\frac{x}{2}\right)^{2m+n} = \left(\frac{x}{2}\right)^n \left[1 - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

Then, $J_0(0) = 1, J_1(0) = 0, J_2(0) = 0, \dots$

Therefore,

$$[J_0^2(0) + 2(J_1^2(0) + J_2^2(0) + J_3^2(0) \dots)] = C$$
$$\Rightarrow C = 1.$$

Thus,

$$[J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) \dots)] = 1.$$

Problem 18: Prove that $J_p'(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right]$

Solution: We know from recurrence relation III and IV that

$$J_p'(x) = J_{p-1}(x) - \frac{p}{x}J_p(x)$$

$$J_p'(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$$

By adding these two relations, we get

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$$

$$\Rightarrow J_p'(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right]$$

Problem 19: Show the Bessel's recurrence relation $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

Proof: We know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

Therefore,

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{1}{2}+1\right)} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}$$

Expanding the series, we have

$$J_{\frac{1}{2}}(x) = \frac{(-1)^0}{0!\,\Gamma\left(\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}} + \frac{(-1)^1}{1!\,\Gamma\left(1+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2+\frac{1}{2}} + \frac{(-1)^2}{2!\,\Gamma\left(2+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{4+\frac{1}{2}} \ldots$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{\Gamma\left(\frac{5}{2}\right)} \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma\left(\frac{7}{2}\right)} \left(\frac{x}{2}\right)^4 \ldots\right]$$

Since $\Gamma(x+1) = x\Gamma(x)$, we have

Continued...

Therefore,

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^2 + \frac{1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^4 - \cdots\right] = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}}\frac{1}{\frac{\sqrt{\pi}}{2}}\left[1 - \frac{1}{\frac{3}{2}}\frac{x^2}{4} + \frac{1}{\frac{5}{2}\frac{3}{2}}\frac{x^4}{16} - \cdots\right]$$

$$= \frac{x^{\frac{1}{2}}}{x * 2^{\frac{1}{2}}}\frac{2}{\sqrt{\pi}}\left[x - \frac{1}{3}\frac{x^3}{2} + \frac{1}{5\cdot 3}\frac{x^5}{4} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x\,.$$

Problem 20: Show the Bessel's recurrence relation $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, cosx.$

Proof: We know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+}$$

Therefore,

$$J_{-\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m-\frac{1}{2}+1\right)} \left(\frac{x}{2}\right)^{2m \quad \frac{1}{2}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2m \quad \frac{1}{2}}$$

Expanding the series, we have

$$J_{\frac{1}{2}}(x) = \frac{(-1)^0}{0!\,\Gamma\left(\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{-\frac{1}{2}} + \frac{(-1)^1}{1!\,\Gamma\left(1+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2-\frac{1}{2}} + \frac{(-1)^2}{2!\,\Gamma\left(2+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{4-\frac{1}{2}} \dots$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\Gamma\left(\frac{3}{2}\right)} \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma\left(\frac{5}{2}\right)} \left(\frac{x}{2}\right)^4 \dots\right]$$

Since $\Gamma(x+1) = x\Gamma(x)$, we have

$(3 \quad\quad (1 \quad\quad) \quad 1 \quad (1) \quad \sqrt{\pi} \quad\quad (5) \quad\quad 3\,1 \quad\quad (7) \quad\quad 5\,3\,1$

Continued...

Therefore,

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}}\left[\frac{1}{\sqrt{\pi}} - \frac{1}{\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^2 + \frac{1}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^4 - \cdots\right] = \frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\left[1 - \frac{1}{\frac{1}{2}}\frac{x^2}{4} + \frac{1}{\frac{3}{2}}\frac{x^4}{16} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2} + \frac{1}{4 \cdot 3}\frac{x^4}{2} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cos x\,.$$

❑ Relate $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Proof: We know the recurrence relation:

$$\frac{2p}{x}J_p(x) = [J_{p-1}(x) + J_{p+1}(x)] \Rightarrow J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x).$$

By putting $p = 2$, we have

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x).$$

Similarly, by putting $p = 1$, we have

$$J_2(x) = \frac{4}{x}J_1(x) - J_0(x).$$

Therefore, we have

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\left(\frac{4}{x}J_1(x) - J_0(x)\right) - J_1(x)$$

$$= \frac{16}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x) = \left(\frac{16}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$$

$$= \frac{16 - x^2}{x^2}J_1(x) - \frac{4}{x}J_0(x).$$

EXERCISE

- Relate $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$

THANK YOU