Module-3

① $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1,3]$

$\Rightarrow$ f is continuous in $[1,3]$

$\Rightarrow$ f is differentiable $(1,3)$

(iii) $f(a) = f(b) =$

$f(1) = (1)^3 - 6(1)^2 + 11(1) - 6$

$= 1 - 6 + 11 - 6 = 12 - 12 = 0$

$f(3) = (3)^3 - 6(3)^2 + 11(3) - 6$

$= 27 - 54 + 33 - 6 = 0$

So $f(a) = f(b)$ condition satisfies.

(iv) at least one point c exists. $f'(c) = 0$.

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$.

$a = 3$

$b = -12$

$c = 11$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$\frac{12 \pm \sqrt{12}}{6}$$

$$\frac{12 \pm 2\sqrt{3}}{6}$$

$$= \frac{6 \pm \sqrt{3}}{3},$$

$\frac{6 + \sqrt{3}}{3} = 2....$ So it lies in interval $[1,3]$.

Verified Rolle's theorm.

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$\pm\sqrt{ab}$

(2) $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$

$= \log(x^2+ab) - \log(a+b) - \log x$

$f(a) = \log\left(\frac{a^2+ab}{(a+b)a}\right) = \log 1 = 0$

$f(b) = \log\left(\frac{b^2+ab}{(a+b)b}\right) = \log 1 = 0$

$f(a) = f(b)$

$f'(x) = \frac{1}{x^2+ab} \cdot 2x - 0 - \frac{1}{x}$

$\Rightarrow \frac{2x}{x^2+ab} - \frac{1}{x}$

$\frac{2x^2 - x^2 - ab}{x(x^2+ab)} = \frac{x^2 - ab}{x(x^2+ab)}$

$f'(c) = 0$

$\frac{c^2 - ab}{c(c^2+ab)} = 0$

$c^2 - ab = 0$

$c^2 = ab$

$c = \pm\sqrt{ab}$

$0 \in [a, b] \; \sqrt{ab} \in [a, b]$

All least one point satisfies.

(3) $0<x<1$

Let $0<c<x$

$f(x) = \sin^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f'(c) = \frac{1}{\sqrt{1-c^2}}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}x - \sin^{-1}0}{x-0}$

$= \frac{\sin^{-1}x}{x}$

$0<c^2<x^2$

$0>-c^2>-x^2$

$1>1-c^2>1-x^2$

$1>\sqrt{1-c^2}>\sqrt{1-x^2}$

$1<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-x^2}}$

$1<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-x^2}}$

$1<\frac{\sin^{-1}x}{x}<\frac{1}{\sqrt{1-x^2}}$

$= x<\sin^{-1}x<\frac{x}{\sqrt{1-x^2}}$

$x^{1/5-1}$

$\frac{1}{5}^{-1}$ $-1/4$

(4) $\sqrt[5]{245}$

$f(x) = \sqrt[5]{245} \Rightarrow$

$f(x) = \sqrt[5]{x} = x^{1/5}$ & $a = 243 = (3)^5$, $b = 245$

$f'(x) = \frac{1}{5}x^{-4/5}$

$f'(c) = \frac{1}{5}c^{-4/5}$

By LMVT we have,

$\frac{f(b)-f(a)}{b-a} = f'(c)$

$\Rightarrow \frac{f(245)-f(243)}{245-243} = \frac{1}{5}c^{-4/5}$

$f(245) = f(243) + \frac{2}{5} c^{-4/5}$

c lies b/w 243, & 245 take $\boxed{c = 243.}$

$\sqrt[5]{245} = (243)^{1/5} + \frac{2}{5}(243)^{-4/5}$

$f(x) = \sqrt[5]{x} = x^{1/5}$

$f(243) = \sqrt[5]{243} = (243)^{1/5}$

$= (3^5)^{1/5} + \frac{2}{5}(3^5)^{-4/5}$

$= 3 + \left(\frac{2}{5}\right)\left(\frac{1}{81}\right) = \frac{5}{405} = 3.0049.$

⑤ Cauchy's

$f(x) = x^3$

$g(x) = 2-x$ in $[0, 9]$

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$f'(x) = 3x^2$

$g'(x) = -1$

$\Rightarrow \frac{3c^2}{-1} = \frac{b^3-a^3}{2-b-(2-a)}$

$-3c^2 = \frac{b^3-a^3}{a-b}$

$-3c^2 = \frac{(9)^3-(0)^3}{0-9}$

$-3c^2 = \frac{-729}{-9}$

$-3c^2 = -81$

$c^2 = 27$

$\boxed{c = \sqrt{27}}$

Part-B

(1) $f(x) = e^{x} \sin x$
Rolle's theorem

$f'(x) = e^{x}$
$f'(c) = e^{c}$

$\Rightarrow f(a) = f(b)$
$a = 0$
$b = \pi$

$u \cdot v$

$u \frac{dv}{dx} + v \cdot \frac{du}{dx}$

$e^{x} \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^{x})$

$e^{x} \cos x + \sin x \, e^{x}$

$f'(x) = e^{x}(\sin x + \cos x)$

$f'(c) = e^{c}(\sin c + \cos c)$

$\Rightarrow f(a) = f(b)$
$a = 0$
$b = \pi$

$f(a) = e^{0} \sin 0$
$= 1 \times 0 = 0$

$f(b) = e^{\pi} \sin \pi = 0$

$f(a) = f(b) = 0$

(iii) $f'(c) = 0$
$\Rightarrow e^{c}(\sin c + \cos c) = 0$

$\Rightarrow e^{c} = 0$

$\sin c + \cos c = 0$

$\sin c = -\cos c$

$-\frac{\sin c}{\cos c} = 1$

$-\tan c = 1$

$\Rightarrow \left(\frac{-\pi}{4}\right)$

$-\frac{\pi}{4} \notin [0, \pi]$

$\Rightarrow \tan c = -1$

$\tan 135 =$

$\boxed{c = 135}$

(2) $f(x) = e^{-x}[\sin x - \cos x]$

$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

u·v rule.

$u \cdot \frac{d}{dx}(v) + v \cdot \frac{d}{dx}(u)$

$e^{-x} \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \frac{d}{dx}(e^{-x})$

$e^{-x} \frac{d}{dx}(\cos x + \sin x) + (\sin x - \cos x) - (e^{-x})$

$e^{-x}(\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x)$

$e^{-x}(2\cos x)$

$f'(x) = e^{-x}(2\cos x)$

$f'(c) = e^{-c}(2\cos c)$

$\Rightarrow$ (ii) $f(a) = e^{-a}[\sin a - \cos a] = e^{-\frac{\pi}{4}}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right] = 0$

$f(b) = e^{-b}[\sin b - \cos b] = e^{5\pi/4}\left[\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right] = 0$

$f'(c) = 0$

$e^{-c}(2\cos c) = 0$

$e^{-c} = 0$, $2\cos c = 0$

$\cos c = 0$

$c = 90^\circ = \frac{\pi}{2}$

$\Rightarrow$ Rolle's theorm is verified. atleastone points $\boxed{c = \frac{\pi}{2}}$

(3) $f(x) = x^3 - x^2 - 5x + 3$. in interval $[0,4]$

f is continuous, differentiable, derivable,
$[0,4]$ $(0,4)$ $(0,4)$

$f(0) = (0)^3 - (0)^2 - 5(0) + 3$

$= 0 - 0 - \cancel{0} + 3 = 3$

$f(4) = (4)^3 - (4)^2 - 5(4) + 3 = 64 - 16 - 20 + 3$

$= 31$

$f(x) = x^3 - x^2 - 5x + 3$

$f'(x) = 3x^2 - 2x - 5$

$\Rightarrow f'(c) = 3c^2 - 2c - 5$

According to Lagrange's theorem.

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$3c^2 - 2c - 5 = \frac{31 - 3}{4 - 0}$

$3c^2 - 2c - 5 = \frac{\cancel{28}\,7}{\cancel{4}}$

$3c^2 - 2c - 5 - 7 = 0$

$3c^2 - 2c - 12 = 0$

$a = 3,\ b = -2,\ c = -12$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{2 \pm \sqrt{(-2)^2 - 4(3)(-12)}}{2(3)}$

$\frac{2 \pm \sqrt{4 + 144}}{6}$

$\frac{2 \pm \sqrt{148}}{6},\ \frac{2 \pm \sqrt{4 \times 37}}{6}$

$\Rightarrow \frac{2 \pm 2\sqrt{37}}{6}$

$\frac{\cancel{2}(1 \pm \sqrt{37})}{\cancel{6}\,3}$

$= \frac{1 \pm \sqrt{37}}{3},$

$\frac{1 + \sqrt{37}}{3}$ lies interval $[0,4]$

lagrangi's theorm.

(a) $a<b$, S.T

$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

$f(x) = \tan^{-1}x$

$f'(x) = \frac{1}{1+x^2}$

$f'(c) = \frac{1}{1+c^2}$

$f(a) = \tan^{-1}a$

$f(b) = \tan^{-1}b$

According to lagrangi's theorm

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$

Condition, $a<c<b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$

$\frac{b-a}{1+a^2} > \tan^{-1}b - \tan^{-1}a > \frac{b-a}{1+b^2}$

$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$.

$b = \frac{4}{3}$

$a = 1$

$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+(1)^2}$

$\frac{\frac{1}{3}}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{1}{3}}{1+(1)^2}$

$\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$

$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$.

(ii) $\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$

$b=2$

$a=1$

$\frac{2-1}{1+(2)^2} < \tan^{-1}(2)-\tan^{-1}(1) < \frac{2-1}{1+(1)^2}$

$\frac{1}{1+4} < \tan^{-1}(2)-\frac{\pi}{4} < \frac{1}{1+1}$

$\frac{1}{5} < \tan^{-1}(2)-\frac{\pi}{4} < \frac{1}{2}$

$\frac{\pi}{4}+\frac{1}{5} < \tan^{-1}(2) < \frac{\pi}{4}+\frac{1}{2}$

$\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{2\pi+4}{8}$

(5) $f(x)=e^{x}$ $\quad [3,7]$

$g(x)=e^{-x}$ $\quad a=3$, $b=7$

using cauchy mean theorm formula

$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$\Rightarrow f'(x)=e^{x}, \; g'(x)=-e^{-x}$

$f'(c)=e^{c}, \; g'(c)=e^{-c}$

$f(a)=e^{3}, \; f(b)=e^{7}$

$g(a)=e^{-3}, \; g(b)=e^{-7}$

$\frac{e^{c}}{-e^{-c}} = \frac{e^{7}-e^{3}}{e^{-7}-e^{-3}},$

$-e^{2c} = \frac{e^{7}-e^{3}}{\frac{1}{e^7}-\frac{1}{e^3}},$

$-e^{2c} = \frac{e^{7}-e^{3}}{\frac{e^3-e^7}{e^7e^3}}$

$-e^{2c} = \frac{e^{7}-e^{3}}{\frac{-(e^7-e^3)}{e^7e^3}}$

$e^{2c} = e^{10}$

$2c=10, \; \boxed{c=5}$

$c \in [3,7]$

(6) Cauchy's theorm

$f(x) = \sqrt{x}$ in $[a,b]$

$g(x) = \frac{1}{\sqrt{x}}$. $\quad 0<a<b$

$g(x) = \frac{1}{\sqrt{x}}, \; g'(x) = \frac{-1}{2x\sqrt{x}}, \; \boxed{g'(c) = \frac{-1}{2c\sqrt{c}}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\boxed{f'(c) = \frac{1}{2\sqrt{c}}}$

By cauchy's theorm $= \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$\Rightarrow f(a) = \sqrt{a}, \; g(a) = \frac{1}{\sqrt{a}}$

$f(b) = \sqrt{b} \quad g(b) = \frac{1}{\sqrt{b}}$

$\Rightarrow \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{a}-\sqrt{a}}{\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{a}}}$

$\frac{1}{\cancel{2\sqrt{c}}} \times \frac{-\cancel{2\sqrt{c}}c}{1} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} \Rightarrow$

$\Rightarrow \cancel{-}c = \cancel{-}\sqrt{ab}$

$\boxed{c = \sqrt{ab}}$

Since, $a, b > 0$ is their geometric mean and we have

$a < \sqrt{ab} < b$, $c \in (a,b)$

Cauchy's mean value theorm.

(7) Cauchy's theorm

$f(x) = x^2$ in interval $[1,2]$.
$g(x) = x3$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$f'(x) = 2x$
$g'(x) = 3x^2$

$f(a) = (1)^2 = (1)$, $g(a) = (1)^3 = 1$
$f(b) = (2)^2 = 4$, $g(b) = (2)^3 = 8$

$\Rightarrow f'(c) = 2c$
$g'(c) = 3c^2$

$$\frac{2\cancel{c}}{3c^{\cancel{2}}} = \frac{4-1}{8-1}$$

$$\frac{2}{3} \times \frac{1}{c} = \frac{3}{7}.$$

$$c^{-1} = \frac{\frac{3}{7}}{\frac{2}{3}}$$

$$c^{-1} = \frac{3}{7} \times \frac{3}{2}$$

$$c^{-1} = \frac{9}{14}.$$

$$\boxed{c = \frac{14}{9}}$$

$\frac{9)14(1}{9}$
$\overline{5}$

(8) $f(x)$
$= (x-a)^m (x-b)^n$

$f(a) = (a-a)^m (a-b)^n = 0$
$f(b) = (b-a)^m (b-b)^n = 0$

$f'(c) = 0$

$m(x-a)^{m-1}(x-b)^n + n(x-b)^{n-1}(x-a)^m = 0$

$(x-a)^{m-1}(x-b)^{n-1}\, mx - mb + nx - na = 0$

$$\boxed{c = \frac{mb+na}{m+n}} = 0$$

$mb + na = 0$

(9) for $0 < a < b$

$$1 - \frac{a}{b} < \log \frac{b}{a} < \frac{b}{a} - 1$$

deduces:

$$\frac{1}{6} < \log \frac{6}{5} < \frac{1}{5}$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\Rightarrow f(x) = \log x$
$f'(x) = \frac{1}{x}$
$f'(c) = \frac{1}{c}$

$$\frac{1}{c} = \frac{\log b - \log a}{b-a}$$

$a < c < b$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{1}{a} > \frac{\log b - \log a}{b-a} > \frac{1}{b}$$

$$\frac{1}{b} < \frac{\log b - \log a}{b-a} < \frac{1}{a}$$

$$\frac{b-a}{b} < \log b - \log a < \frac{b-a}{a}$$

$b = 6,\ a = 5$

$$\frac{1}{6} < \log \frac{6}{5} < \frac{1}{5}$$

(10) $f(x) = (x-1)(x-2)(x-3)$ in $[0, 4]$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$

$f(a) = (0-1)(0-2)(0-3)$

$= (-1)(-2)(-3)$

$= -6$

$f(b) = (4-1)(4-2)(4-3)$

$= (3)(2)(1) = 6$

By, lagrange's meanvalue theorm

$$\Rightarrow 3c^2 - 12c + 11 = \frac{6+6}{4}$$

$3c^2 - 12c + 11 = 3$

$3c^2 - 12c + 11 - 3 = 0$

$3c^2 - 12c + 8 = 0$

$$c = \frac{6 \pm 2\sqrt{3}}{3}$$

$\Rightarrow \boxed{c = 3.15}$

verified by lagrange's theorm.

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Part-C

1) Let $f(x)$ be a function such that

(i) It is continuous in closed interval $[a,b]$

(ii) It is differentiable in open interval (a,b)

(iii) $f(a) = f(b)$

Then there exists atleast one point 'c' in (a,b) such that $f'(c) = 0$

2) Let $f(x)$ be a function such that

(i) It is continuous in closed interval $[a,b]$

(ii) It is differentiable in open interval (a,b) such that

then there exists atleast one point 'c' in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

3) If $f: [a,b] \to R$

$g: [a,b] \to R$

(i) f, g = continuous in closed interval $[a,b]$

(ii) f, g = differentiable (a,b)

(iii) $g'(x) \neq 0 \;\forall\; x \in (a,b)$

exists atleast one point 'c' in (a,b)

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(4) Let $f:[a,b]\to R$ be a function satisfying the three condition of Rolle's theorm. Then graph

(1) $y=f(x)$ is a continuous curve in $[a,b]$

(2) There exist a unique tangent line at every point $x=c$, where $a<c<b$.

(3) The oridinates $f(a), f(b)$ at the endpoints A, B are equal so that the points A and B equidistant from the x-axis.

(4) By Rolle's theorm - there atleast one point $x=c$ b/w A and B on the curve at which the tangent line is parallel to the x-axis and also it parallel to chor of the curve.

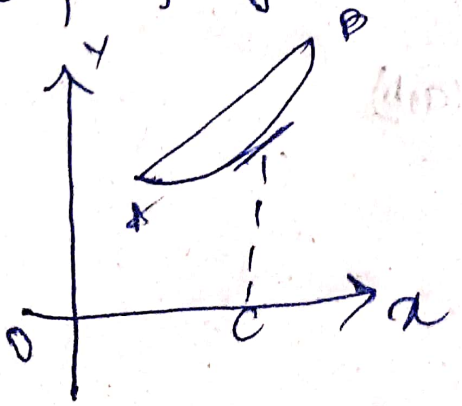
(5) $f:[a,b]\to R$

1) $y=f(x)$ in $[a,b]$

2) every point $x=c$, $a<c<b$, $y=f(x)$, unique tangent to the curve.

3) exist atleast one point C on the curve b/w A and B tangent line ∥el to the chord AB.

Lagrange's graph



(8) $f(x) = \log x$ in $(1, e)$

$\Rightarrow f'(x) = \frac{1}{x}$, $f(a) = \log 1 = 0$.

$f'(c) = \frac{1}{c}$ $\quad f(b) = \log e = \cancel{2.7} \; 1$

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$\frac{1}{c} = \frac{1-0}{e-1} \Rightarrow \frac{1}{c} = \frac{1}{e-1}$

$c = e-1$

Note that $(e-1)$ is in the interval $(1, e)$

Lagrange's mean value theorem is verified.

(9) $f'(x) = -(2x+4)$

for increasing

$f'(x) > 0$

$-(2x+4) > 0$

$2x+4 < 0$

$x < -2$

for decreasing

$f'(x) < 0$

$-(2x) - 4) < 0$

$2x+4 > 0$

$x > -2$