Tutorial-7

Maximum modulus theorem

Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic f^n on a domain Ω.
Then, $|f|$ does not achieve a maxima inside Ω.

Moreover, if f is defined on the boundary γ of Ω, then

$$\sup_{z \in \Omega} |f(z)| = \sup_{z \in \gamma} |f(z)|$$

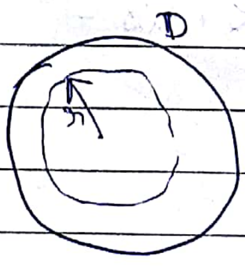
Schwarz lemma

Suppose $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic f^n s.t.
(i) $f(0) = 0$ (ii) $|f(z)| \le 1 \ \forall z \in \mathbb{D}$

Then, $|f(z)| \le |z| \ \forall z \in \mathbb{D}$ and $|f'(z)| \le 1$
Moreover, if $|f(z)| = |z|$ for some ~~z~~ ~~z~~ $z \in \mathbb{D} \setminus \{0\}$
or if $|f'(0)| = 1$, then $f(z) = az \ \forall z \in \mathbb{D}$ for some $a \in \mathbb{C}$, $|a| = 1$.

Proof: Take $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$

P.T. $|g(z)| \le 1 \ \forall z \in \mathbb{D}$



$|g(z)| \le |g(z_0)| \Rightarrow |g(z)| \le \frac{|f(z_0)|}{r}$

$\Rightarrow |g(z)| \le \frac{1}{r} \ \forall |z| < r < 1$ $\quad r > \frac{1}{1+\epsilon}$

[Basically ~~...~~

If $|g(z)| = 1 + \varepsilon$ for some $\varepsilon > 0$, choose $r = \frac{1 + \frac{1}{1+\varepsilon}}{2}$

$|g(z)| > \frac{1}{r} \Rightarrow$ Contradiction

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3. $\varphi_\alpha : \overline{\mathbb{D}} \to \mathbb{C}$ by
$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

(i) Show that $|z| = 1 \Rightarrow |\varphi_a(z)| = 1$

$$|\varphi_\alpha(z)|^2 = \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)\left(\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}}\right)$$
$$= \frac{z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha}}{1+\alpha\bar{\alpha}z\bar{z} - \bar{\alpha}z - \alpha\bar{z}} = 1$$

$$\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = \left|\frac{(z-\alpha)\bar{z}}{\bar{z}(1-\bar{\alpha}z)}\right|$$
$$= \left|\frac{(z-\alpha)}{(\bar{z}-\bar{\alpha})} \cdot z\right|$$
$$= \frac{|z-\alpha|}{|\bar{z}-\bar{\alpha}|} \cdot |z| = 1$$

(ii) P.T. $\varphi_a(\mathbb{D}) \subseteq D$

① $\varphi_a(z)$ is non-constant : $|a|<1$, $\varphi_a(0) = -\alpha$, $\varphi_a(\alpha) = 0$, $\alpha \neq 0$

$\alpha = 0 \Rightarrow \varphi_0(z) = z$

② $\therefore |\varphi_a(z)| = 1 \;\forall\, z \in \partial D$

$\therefore |\varphi_a(z)| < 1 \;\forall\, z \in \mathbb{D}$

(iii) Show that $z = \varphi_\alpha(\varphi_{-\alpha}(z)) = \varphi_{-\alpha}(\varphi_\alpha(z)) \;\forall\, z \in \mathbb{D}$

$$\varphi_\alpha(\varphi_{-\alpha}(z)) = \frac{\left(\frac{z-\alpha}{1+\bar{\alpha}z}\right) - \alpha}{1-\bar{\alpha}\left(\frac{z-\alpha}{1+\bar{\alpha}z}\right)} = \frac{(z\cancel{+\alpha}) - \alpha + \alpha\bar{\alpha}z}{1+\cancel{\bar{\alpha}z} - \cancel{\bar{\alpha}z} + \alpha\bar{\alpha}}$$

$$= \cancel{\frac{(z-2\alpha+\alpha\bar{\alpha}z)\, z}{z(1-2\bar{\alpha}z+\alpha\bar{\alpha})}} = \cancel{z\,\frac{(z-2\alpha+\alpha\bar{\alpha}z)}{(z+\alpha\bar{\alpha}z -}}$$

$$= \frac{z+\alpha\bar{\alpha}z}{1+\alpha\bar{\alpha}} = \boxed{z}.$$

$\varphi_a : \mathbb{D} \to \mathbb{D}$

Möbius transformation
→ holomorphic
→ bijection
→ domain = range
} automorphism

↳ only class of automorphism

wrong sol^n: ~~$f(z)$~~ $h(z) = \frac{f(z)}{M\varphi_a(z)}$ & $h(z) \le$ ~~M~~ 1 @ boundary $\Rightarrow h(z) \le 1$ inside

DOESN'T WORK → may not be defined @ boundary

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Q4. $f : \mathbb{D} \to \mathbb{D}$

$|f(z)| < M, \ \forall z \in \mathbb{D}$

$f(a) = 0$ for some $a \in \mathbb{D}$

$$g(z) = \frac{f(z+a)}{M}$$

we want

$f(g(0)) = 0$ $\qquad \varphi_\alpha = \frac{z-a}{1-\bar{a}z}$

~~$|f(\varphi_\alpha(0))|$~~ $\qquad \frac{f(\varphi_\alpha(0))}{M} = 0$

$$g(z) = \frac{1}{M} f(\varphi_{-a}(z)), \quad g(0) = 0$$

$|g(z)| \le 1 \quad \forall z \in D$

$z = \varphi_a(z)$ So, $|g(z)| \le z \quad \forall z \in \mathbb{D}$

$\Rightarrow |f(\varphi_{-a}(z))| \le$ ~~M~~ $M|z|$

$\Rightarrow$ ~~$|f(\varphi_{-a}(\varphi_a(z)))|$~~ $\le M|\varphi_a(z)|$

$\Rightarrow |f(z)| \le M|\varphi_a(z)| \quad \forall z \in \mathbb{D}$

Q5. $f : \mathbb{D} \to \mathbb{D}$ $\qquad f(a_i) = b_i$ $\quad$ for $i = 1, 2$

TO show $\left|\frac{b_2 - b_1}{1 - \bar{b}_1 b_2}\right| \le \left|\frac{a_2 - a_1}{1 - \bar{a}_1 a_2}\right|$

$a_1 \to \alpha$

$\left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| \to$ hyperbolic distance

$f(a_1) = b_1$

$\varphi_{b_1}(f(a_1)) = 0 = \varphi_{b_1}(b_1)$

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Define $g : \mathbb{D} \to \mathbb{D}$ as

$$g(z) = \varphi_{b_1}(f(\varphi_{-a_1}(z))) \quad \forall z \in \mathbb{D}$$

Then,

$$g(0) = \varphi_{b_1}(f(\varphi_{-a_1}(0)))$$
$$= \varphi_{b_1}(f(a_1)) = \varphi_{b_1}(b_1) = 0$$

$$|g(z)| = |\varphi_{b_1}(f(\varphi_{-a_1}(z)))| \leq 1 \quad \forall z \in \mathbb{D}$$

$$\Rightarrow |g(z)| \leq |z|$$

$$\Rightarrow |\varphi_{b_1}(f(\varphi_{-a_1}(z)))| \leq |z|$$

$$\Rightarrow |\varphi_{b_1}(f(\varphi_{-a_1}(\varphi_{a_1}(z))))| \leq |\varphi_{a_1}(z)|$$

Put $\boxed{z = a_2}$

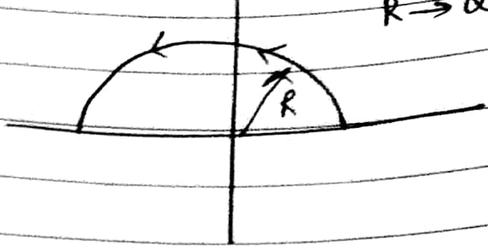
$$\Rightarrow |\varphi_{b_1}(\overset{b_2}{f(a_2)})| \leq |\varphi_{a_1}(a_2)|$$

$$\Rightarrow \left|\frac{b_2 - b_1}{1 - \bar{b}_1 b_2}\right| \leq \left|\frac{a_2 - a_1}{1 - \bar{a}_1 a_2}\right|$$

when $f(a) = b$, in general,

$$\varphi_b(f(\varphi_{-a}(z)))$$

1. $R \to \infty$



2. $z = e^{i\theta}$