~~Take~~ sin & cos to be zero only at real values i.e.
$\sin z = 0$ has solns $n\pi$, $\cos z = (n+\frac{1}{2})\pi$

$$\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta \qquad \bar{z} = \frac{1}{z} \text{ on } |z| = 1.$$

$$\text{so } z + \frac{1}{z} = 2\,\mathrm{Re}(z) = \cos\theta \quad \text{on } |z| = 1$$

$$\int_{|z|=1} (2\cos\theta)^{2n} \frac{dz}{z} \quad \text{taking } z = e^{i\theta}.$$

$$dz = iz\,d\theta.$$

$$\text{So } \int_0^{2n} (2\cos\theta)^{2n}\, i\,d\theta = 2\pi i\, {}^{2n}C_n$$

$$\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = 2\frac{\pi\, {}^{2n}C_n}{4^n}.$$

Tut 6

Q 1) c) $f(z) = \dfrac{1}{\sin(1/z)}$, $z = 0$ is a

$\dfrac{1}{f(z)} = \sin\left(\dfrac{1}{z}\right) \Rightarrow$ zeroes give the singularities of $f(z)$.

$\dfrac{1}{f(z)} = \sin\left(\dfrac{1}{z}\right) = 0$

$$\frac{1}{z} = n\pi \Rightarrow \boxed{z = \frac{1}{n\pi}} \quad \forall n \in \mathbb{Z} - \{0\}$$

If $\lim_{z \to z_0} (z - z_0) f(z)$ exists $f(z)$ ~~is a p~~ has a pole of order 1

$\uparrow$ Not a removable

for $\sin^2$, mostly pole of order 2.

To check at $z = \frac{1}{n\pi}$

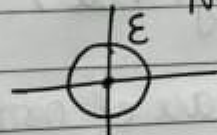
$$\lim_{z \to \frac{1}{n\pi}} \frac{\left(z - \frac{1}{n\pi}\right)}{\sin z} \Rightarrow$$ if it exists $z = \frac{1}{n\pi}$ is a pole of order 1.

$$\Rightarrow \lim_{z \to \frac{1}{n\pi}} \frac{z - \frac{1}{n\pi}}{\sin\left(\frac{1}{z} - n\pi\right)} \text{ for } n \text{ even} = 1$$

$$\text{and } \lim_{z \to \frac{1}{n\pi}} \frac{-\left(z - \frac{1}{n\pi}\right)}{\sin\left(\frac{1}{z} - n\pi\right)} \Rightarrow n \text{ odd} = -1.$$

So at $z = \frac{1}{n\pi}$, we have a pole of order 1.

Non-isolated singularity.



There exists no punctured disc such that there is no singularity inside except the one we're talking about.

for $z = 0$

$$\frac{1}{f(z)} = \sin\left(\frac{1}{z}\right)$$

$\Rightarrow$ taking a disc of radius ε around 0,

$\sin\frac{1}{z} = 0$ has many solutions $\frac{1}{n\pi}$ with $n \geq \frac{1}{\pi\varepsilon}$

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1.4) $\tan\left(\frac{1}{z}\right) \Rightarrow$ do it

Q2) i) $f(z) = \dfrac{1}{(z^4+1)^2}$

To find poles,

$$g(z) = \frac{1}{f(z)} = 0 \Rightarrow (z^4+1)^2 = 0 \Rightarrow z^4+1=0$$

$$\Rightarrow z^4 = -1$$

$$z^4 = e^{i(\pi+2n\pi)} \quad n\in\mathbb{Z}$$

$$z = e^{i\left(\frac{\pi}{4}+\frac{2n\pi}{4}\right)}, n\in\mathbb{Z}.$$

poles are $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{i5\pi/4}, \ldots$

Order of pole is given by order of 0 in $\frac{1}{f(z)}$ (always true).

All the poles are of order 2.

If z_0 is a zero of $f(z)$

$$\left[\begin{array}{l} f(z_0)=0 \\ f'(z_0)=0 \\ \vdots \\ f^{n-1}(z_0)=0 \end{array}\right] \text{ with order } n.$$

Q4) Use prove of Riemann's singularity theorem

Q5) $f(z) = \dfrac{p(z)}{q(z)}$ $p(z)$ & $q(z)$ are differentiable

$p(z_0) \neq 0$

$q(z_0) = 0$

$q'(z_0) \neq 0.$

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$$\int_\gamma f(z)\,dz = \int \frac{b_{-1}}{z}$$

z_0 is a pole of $f(z)$ of order 1.

Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=-1}^{-\infty} a_m (z-z_0)^m$$

series only for isolated singularity.

principle part.

b_{-1} is called residue.

1) If removable, all $b_m = 0$

2) Pole of order m_0, then, all $b_m = 0$ for $m < m_0$

$$\frac{b_{-m}}{z^m} + \frac{b_{-(m+1)}}{z^{m+1}} + \dots \to 0$$

3) essential: principle series part extends to $-\infty$.

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n a_n + \frac{b_{-1}}{z-z_0}$$

$$(z-z_0) f(z) = \sum_{n=0}^{\infty} (z-z_0)^{n+1} a_n + b_{-1}.$$

$$\lim_{z\to z_0} \text{L.H.S.} = \lim_{z\to z_0} b_{-1} = b_{-1}$$

$$\lim_{z\to z_0} (z-z_0)\frac{p(z)}{q(z)} = p(z_0) \lim_{z\to z_0} \frac{z-z_0}{q(z)}$$

$$q'(z_0) = \lim_{z\to z_0} \frac{q(z) - \cancel{q(z_0)}^{0}}{z-z_0}$$

$$\therefore \lim \frac{z-z_0}{q(z)} = \frac{1}{q'(z_0)}$$

$\pi + 2n\pi)$

$n \in \mathbb{Z}$.

$\mathbb{Z}$.

tiable

$$= \frac{p(z_0)}{q'(z_0)}.$$

Q6) i) $\dfrac{1}{z^2 \sin z}$ $\qquad z^2 \sin z = 0$ at $z = 0, n\pi$

$$\lim_{z\to 0} \frac{z^3}{z^2 \sin z} = \lim_{z \to 0} \frac{z}{\sin z} = 1 \text{ exists}$$

So, $f(z)$ has a pole of order 3 at 0.

$$f(z) = \frac{1}{z^2 \sin z} = \sum_{n=0}^{\infty} a_n z^n + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \frac{b_{-3}}{z^3}$$

for getting residue at a pole of order m,

$$g(z) = (z - z_0)^m f(z).$$

$$b_{-1} = \frac{g^{m-1}(z_0)}{(m-1)!}$$

$$g(z) = (z-z_0)^m f(z)$$
$$= (z)^3 f(z).$$

$$b^{-1} = \frac{g^2(z_0)}{2!} = \frac{g^2(0)}{2!}$$

$$g(z) = \frac{z}{\sin z}, \quad g'(z) = \frac{\sin z - z\cos z}{(\sin z)^2}$$

$$g''(z) = \frac{(\sin z)^2 (\cos z - (-z\sin z + \cos z)) - (\sin z - z\cos z)\, 2\sin z \cos z}{(\sin z)^4}$$

$$= \frac{2z(\cos z)^2 \sin z - \sin^3 z \cos z + \sin^3 z\, z}{(\sin z)^4}$$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

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$$= \lim_{z\to 0} \frac{z\sin^3 z}{(\sin z)^4} - \lim_{z\to 0} \frac{2z\cos^2 z\sin z + \sin^2 z\cos z}{(\sin z)^4}$$

$$= \frac{1}{3}.$$

$$\therefore b^{-1} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

Q3) ~~$f(z) = 2\frac{1}{2}.$~~

$$f(z) = \frac{2(z-1)}{z^2-2z-3}$$

$$= \frac{A}{z+1} + \frac{B}{z-3} \quad A = 1, B = 1$$

$|z| < 1$

$$\frac{1}{z+1} = \frac{1}{1-(-z)}$$

$$\frac{1}{z-3} = \frac{-1}{3\left(1-\frac{z}{3}\right)} = -\frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots\right]$$

② $1 < |z| < 3$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\dots\right).$$

as $\frac{1}{|z|} < 1$

$$\frac{1}{z-3} = \frac{1}{3\left(1-\frac{z}{3}\right)} = \frac{1}{3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots\right)$$

as $\left|\frac{z}{3}\right| < 1.$

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(iii) $|z| > 3$

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\left(1+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right)$$

$$\frac{1}{z-3} \cancel{= \frac{1}{3(1-\frac{z}{3})}} \cancel{= \frac{-1}{3}}$$

$$= \frac{1}{z(1-\frac{3}{z})} = \frac{1}{z}\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\dots\right)$$

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