Tut 5

Identity Theorem:

Let f be a holomorphic f^n on an open set Ω. Suppose there exists a sequence $a_1, a_2, a_3 \dots$ of points in Ω such that $f(a_i) = 0 \ \forall i \in \mathbb{N}$ and the sequence converges to a point $a \in \Omega$. Then $f(z) = 0 \ \forall z \in \Omega$

Liouville's Theorem: An entire bounded f^n is constant

1. $f : \mathbb{C} \to \mathbb{C}$

(i) $\exists\, c > 0$ such that $|f(z)| > c \ \forall z \in \mathbb{C}$

$$g(z) = \frac{1}{f(z)} \quad \forall z \in \mathbb{C}$$

$\rightarrow$ g is entire

$$|g(z)| = \frac{1}{|f(z)|} < \frac{1}{c}$$

$\therefore$ g is const $\Rightarrow$ f is const.

(ii) $Re(f(z)) \geq 0 \quad \forall z \in \mathbb{C}$

$$g(z) = f(z) + 1$$

$$|g(z)| \geq \sqrt{(Re f(z)+1)^2 + (Im(z))^2}$$

$$|g(z)| \geq \sqrt{(Re f(z)+1)^2}$$

$$|g(z)| \geq |Re f(z)+1|$$

$$|g(z)| \geq 1 \quad c$$

$\therefore g(z)$ is const

$\therefore f(z)$ is const

(iii) $\overline{f(\mathbb{C})} \neq \mathbb{C}$, i.e. $\mathbb{C} \setminus f(\mathbb{C})$ is a non empty open set

2. $f\left(\frac{1}{n}\right) = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

$f : B(0,1) \to \mathbb{C}$

holomorphic

We apply identity Theorem $-(h(x) = f(x) - x)$

$f(x)$ x agree $\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \dots \to 0$ lies in Ω

$\therefore f(x) = x$

~~f(x)~~

~~f(x)~~ $f\left(\frac{1}{3}\right) = -\frac{1}{3}$

CONTRADICTION

Which fails above

$\therefore$ no f^n exists

3. Let f be an ent..

$f : \mathbb{C} \to \mathbb{C}$ entire

$f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ $\forall n \in \mathbb{N}$

$g(z) = f(z^2)$ $\forall z \in \mathbb{C}$

$g(z) = z$ $\forall z = 1, \frac{1}{2}, \frac{1}{3} \dots \to 0$

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By identity Theorem

$$g(z) = z \quad \forall z \in \mathbb{C}$$

$$\therefore f(z^2) = z \quad \forall z \in \mathbb{C}$$

$$2z f'(z^2) = 1 \quad \forall z \in \mathbb{C}$$

Put $z = 0$

$0 = 1$ Contradiction

4. $$f(z) = f(0) + zf'(0) + \frac{z^2 f''(0)}{2} \cdots + \frac{z^N f^{(N)}(0)}{N!} + \frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N f^{(N+1)}(tz)\,dt$$

(a) $$\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \le \frac{|z|^{N+1}}{(N+1)!}, \quad Re(z) \le 0$$

$$\Leftrightarrow \left| \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N f^{N+1}(tz)\,dt \right| \le \frac{|z|^{N+1}}{(N+1)!} \qquad (f(z) = e^z)$$

$$\Leftrightarrow \left| \int_0^1 (1-t)^N f^{N+1}(tz)\,dt \right| \le 1$$

$$f^{N+1}(tz) = \left(f^{N+1}(z)\right)\Big|_{tz}$$

$$\Leftrightarrow \left| \int_0^1 (1-t)^N e^{zt}\,dt \right| \le 1$$

$$\left(\le \left| \int_0^1 e^{zt}\,dt \right| \le \int_0^1 |e^{tz}|\,dt = \int_0^1 e^{tRe(z)} \le \int_0^1 e^{0t} = 1 \right.$$

cause $Re(z) < 0$

$$\left| \cancel{e^z} \right|$$

H.P.)

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5. By comp. $\int_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z}$, s.t.

$$I = \int_0^{2\pi} (\cos\theta)^{2n} d\theta = \frac{2\pi}{4^n} \frac{2n!}{(n!)^2}$$

$z = e^{i\theta}$, $\theta \in [0, 2\pi)$

$dz = ie^{i\theta} d\theta$

$$\int_0^{2\pi} \left(e^{i\theta} + e^{-i\theta}\right)^{2n} \frac{ie^{i\theta} d\theta}{e^{i\theta}}$$

$$\int_0^{2\pi} 4^n (\cos\theta)^{2n} \, i \, d\theta$$

$$= 4^n i \, I$$

$$\int \underbrace{z^{2n-1} + z^{2n-2} \cdots 1}_{} + \underbrace{\frac{()}{z} + \frac{()}{z^2}}_{=0}$$

as they have primitive

$$\int_{|z|=1} \frac{{}^{2n}C_n \, dz}{z}$$

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$

$$4^n i \, I = {}^{2n}C_n \times 2\pi i \qquad \text{H.P.}$$

Alternate $\int \frac{(z^2+1)^{2n}}{z^{2n+1}} dz$ Use CIF

$z_0 = 0, \quad m = 2n$

$$\frac{2\pi i}{(2n!)} \frac{d^{2n}}{dz} (z^2+1)^{2n} \Big|_{z=0}$$

$$\frac{2\pi i}{(2n!)} \, 2n! \, {}^{2n}C_n$$

1. (iii) <u>Limit point</u>

<u>Closure</u>

→ Boundary (Set of all Bound points)

→ Boundary point

→ Interior point

$\exists \, r > 0$ s.t. $B(z_0, r) \subset A$

Closure − Interior = Boundary

→ Interior : set of all interior points

A^o

Limit point

$a_1, a_2 \ldots \to z$
$\in A$

Q − Rational

$\overline{Q} = R$

Closure of Rationals

→ closure : set of limit points

$\overline{A}$

Q → If Closure of $f \neq \mathbb{C}$ then f is const

$$\overline{f(\mathbb{C})} \neq \mathbb{C}$$

For z_0, lim of seq $= z_0$ DNE, ~~Suppose~~ Suppose, then $\exists r$ s.t.

for any z $\quad$ $f(z) \xleftrightarrow{dist} z_0 \geq r$

$$g(z) = f(z_0) - z_0$$

$$|g(z)| \geq \underline{r} > 0$$

So $g(z) = \text{const}$

$\therefore$ $f(z) = \text{const}$