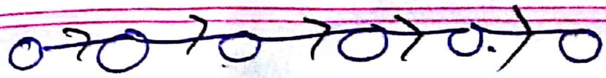
Homework-2

2(a) <u>Homomorphism of graphs</u> A graph homomorphism is a mapping between two graphs G and H defined by a function $f: N_G \to N_H$ such that
for each edge $e \in E_G$
$\exists \; e' \in E_H$
$f(s(e)) = s(e')$
and $f(t(e)) = t(e')$ i.e. every edge from G is mapped to H
(N_G = nodes of graph G
N_H = nodes of graph H
E_G = edges of G, E_H = edges of H)

(b) <u>Line graph L_n -</u>
(L_n)
A line graph is a set of ordered n nodes and (n-1) edges such that $s(e_1) = n_1$
$(n_1, n_2 \ldots n_n)$ $(e_1, e_2 \ldots e_n)$ $t(e_1) = n_2$
~~$s(e_n)$~~ $\vdots$

i.e. for $i \in \{1, 2 \ldots n-1\}$ $s(e_{n-1}) = n_{n-1}$
$t(e_{n-1}) = n_n$
$s(e_i) = n_i$
& $t(e_i) = n_{i+1}$


L_6

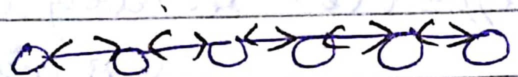
E = set of edges in graph

($\overline{L_n}$)

An undirected line graph is a

set of ordered n nodes and ~~n-1~~ $2(n-1)$ edges
such that $\forall e \in E \ \exists e' \in E$
s.t. $s(e) = t(e')$
and $t(e) = s(e')$

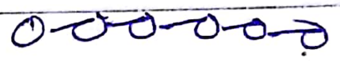
In other words, it resembles a line graph L_n with extra edges such that for each edge in line graph L_n, $\exists e'$ in L_n' with $s(e)=t(e')$ & $t(e)=s(e')$



$\overline{L_6}$

or

L_6



(c) Cycle graph C_n

Cycle graph C_n is a set of ordered n nodes and n edges such that

$s(e_1) = n_1$ ---- $s(e_n) = n$

$t(e_1) = n_2$ ---- $t(e_n) = 1$

i.e. it follows the rule $s(e_i) = n_i$

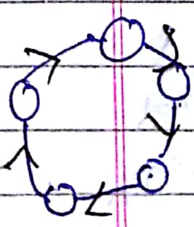
$t(e_i) = n_{i+1}$

$\forall i \in \{1, 2, \ldots n-1\}$

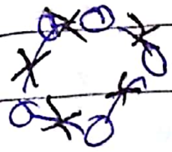
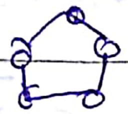
and $s(e_n) = n$

$t(e_n) = 1$

C_5



An undirected cycle graph $\bar{C_n}$ is a set of ordered nodes and edges as defined in C_n but with $2n$ edges such that $\forall e \in E, \exists \cdot e' \in E$ such that $s(e) = t(e')$ & $t(e) = s(e')$

 or  $\bar{G}$

The edges can be written as
$e_1 —— e_n, e_1' —— e_n'$

$\int_{i \in \{1, \dots n-1\}}$ $s(e_i) = n_i$, $t(e_i) = n_{i+1}$, $s(e_i') = n_{i+1}$, $t(e_i') = m_i$

$s(e_n) = m_n$, $t(e_n) = n_1$, $s(e_n') = n_1$, $t(e_n') = n_n$

(d) Directed path in a graph G is said to exist between two nodes (n & m) $m_1 \to m_2$ such that there is an injective homomorphism f b/w L_k and G
$L_k \xrightarrow{f} G$
such that $f(n_1) = m_1$ and $f(n_k) = m_2$ $\quad n_i = i$th node of L_k

(e) An undirected path in a graph G b/w two nodes m_1 and m_2 ($m_1 \sim m_2$) is said to be exist s.t. $\exists$ an injective homomorphism f b/w $\bar{L_k}$ and G
$\bar{L_k} \to G$
s.t. $f(n_1) = m_1$ and $f(n_k) = m_2$ $\quad n_i = i$th node of $\bar{L_k}$

$E = \{\text{edges of } G\}$

(8) An undirected graph $\bar{G}$ is such that for every $e \in E$, $\exists\, e' \in E$ such that $s(e) = t(e')$ and $t(e) = s(e')$

(8) A graph G is said to be strongly connected if $\exists$ a directed path $u \rightsquigarrow v$ for any set of nodes u & $v \in N_G$

(9) If G is a graph, and $\bar{G}$ be the smallest undirected graph containing all edges of G

i.e. $\forall\, e \in E_G \Rightarrow e \in \bar{E}_G$ and $\exists\, e' \in \bar{E}_G$ s.t. $s(e) = t(e')$, $s(e') = t(e)$

then G is connected if $\bar{G}$ is strongly connected

(i) A tree is an undirected (nodes ≥ 3) cycle free graph, i.e. $\exists$ no injective homomorphism f b/w C_n and G

(a) b/w any two nodes of G $(m, 2m)$

such that iff $\forall\, u, v \in N_G$ $\exists$ unique path $u \rightsquigarrow v$

(ii) An undirected cycle in a graph G is said to exist if there is an injective homomorphism f b/w $\bar{C}_n$ and G $(\bar{C}_n \xrightarrow{f} G)$

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Q) A directed cycle in a graph G is said to exist if there is an injective homomorphism f b/w C_k and G $(C_k \xrightarrow{f} G)$

a) A spanning tree T of an undirected graph G is a subgraph which is a tree s.t. ~~and~~ iff $n \in N_G$, $n \in N_T$,
i.e. $\exists$ a homomorphism f b/w a tree graph T and graph G such that number of nodes in T is n.

Spanning trees are not unique for a graph but contain the same number of edges. Every connected graph has a spanning tree.

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a. ~~f is a function that maps nodes to edges~~

$S: R^N \to R^E$

Domain Range

R^N is set of N dimension vectors consisting of vectors of form $\{f(n_1), f(n_2) \ldots f(n_N)\}$

$n_i \in$ set nodes of G (N_G) where $f: N_G \to R$

similarly R^E is a set of E dimensional vectors of form $\{g(e_1), g(e_2) \ldots g(e_E)\}$

$e_i \in E_G$

where $g: E_G \to R$

Also δ s.t: $(\delta(V))_{e_i} = V_{s(e_i)} - V_{t(e_i)}$

$e_i \in E_G \quad i \in \{1, 2, \ldots E\}$

<u>δ is a linear map</u>

<u>Proof:</u>

$\mathbb{R}^N$ and $\mathbb{R}^E$ are defined over field $\mathbb{R}$

Let $v_1, v_2 \in \mathbb{R}^N$

s.t. $\delta(v_1) = w_1 \quad w_1, w_2 \in \mathbb{R}^E$

$\delta(v_2) = w_2$

$a, b \in \mathbb{R}$

$$\delta(v_1)_{e_i} = (v_1)_{s(e_i)} - (v_1)_{t(e_i)}$$

$$\delta(v_2)_{e_i} = (v_2)_{s(e_i)} - (v_2)_{t(e_i)}$$

$$\delta(av_1 + bv_2)_{e_i} = (av_1 + bv_2)_{s(e_i)} - (av_1 + bv_2)_{t(e_i)}$$

$$= (av_1)_{s(e_i)} + (bv_2)_{s(e_i)} - (av_1)_{t(e_i)} - (bv_2)_{t(e_i)}$$

$$= a\left[v_{1(s(e_i))} - v_{1,t(e_i)}\right] + b\left[(v_2)_{s(e_i)} - (v_2)_{t(e_i)}\right]$$

$$= a\,\delta(v_1) + b\,\delta(v_2) \quad \text{— I}$$

δ maps 0 in $\mathbb{R}^N$ to 0 in $\mathbb{R}^E$

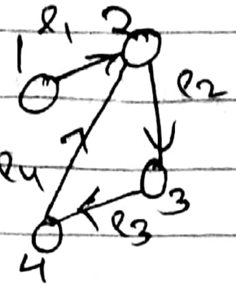
if all nodes are assigned value 0

$$\delta(v)_{e_i} = \underset{0}{\underset{\downarrow}{(v)_{s(e_i)}}} - \underset{0}{\underset{\downarrow}{(v)(t_{e(i)})}}$$

$$= 0 - 0 = 0 \quad \text{— II}$$

$\therefore$ I & II $\Rightarrow$ δ is a linear map for $(V_n)_{n\in N} \longmapsto (V_{s(e)} - V_{t(e)})_{e\in E_G}$

∴ δ is fixed for each graph



$\delta(V) = \nu$

$v_{e_1} = V_1 - V_2$ $\quad$ $v_{e_3} = V_3 - V_4$

$v_{e_4} = V_4 - V_2$

$v_{e_2} = V_2 - V_3$

∴ M_δ: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$M_\delta V = \nu$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} v_{e_1} \\ v_{e_2} \\ v_{e_3} \\ v_{e_4} \end{bmatrix}$$

* δ^T is a map corresponding $\mathbb{R}^E \xrightarrow{\delta} \mathbb{R}^N$

and gives us the accumulation of current in each node.

$$(i_e)_{e\in E} \longmapsto \left(+ \sum_{(l,m)\in E} i_{lm} - \sum_{(m,n)} i_{mn} \right)_{n\in N}$$

(l,m) = one ~~node~~ edge

(m,n) = successive edge

m = node at which accumulation happens.

∴ Domain → $\mathbb{R}^E$

Codomain/Range → $\mathbb{R}^N$

(3) ▵ W_{KVL} is the image of the map δ and hence a subset of $\mathbb{R}^E$

$$W_{KVL} = \operatorname{im}\delta = \{ u \in \mathbb{R}^E \mid \exists V \in \mathbb{R}^N \text{ with } u = \delta(V) \}$$

▵ W_{KCL} is the null space of the map δ^T

$$= \{ i \in \mathbb{R}^E \mid \delta^T(i) = 0 \}$$

Proof $v \in W_{KVL}$ iff v satisfies KVL

Forward direction.

If $u \in W_{KVL}$,

$\exists$ a $V \in \mathbb{R}^N$ s.t. $\delta V = u$

$$(u)_e = V_{s(e)} - V_{t(e)}$$

$e \in E_G$

Consider 2 nodes N_1 & N_2

Consider a path in the graph b/w these two nodes ~~...~~ with k nodes in the path (without loss e.g. n_1)

label the nodes as $n_1 \sim n_k$

$\therefore V_{n_1} = V_{N_1}$ & $V_{n_k} = V_{N_2}$

$\exists$ a injective homomorphism f with L_k

$f(m_1) = n_1$

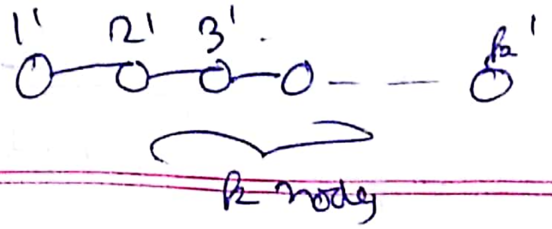
$f(m_k) = n_k$

$$u_{e_1} + u_{e_2} + \dots + u_{e_{k-1}}$$

$$= \{ V_{s(e_1)} - V_{t(e_1)} \} +$$

$m_i \in N_{L_k}$

L_k



Hence $e_1, e_2 \ldots e_{k-1}$ can be mapped to $e_1', e_2' \ldots e_{k-1}'$ respectively.

Note that now same V can be assigned to nodes of L_k also

$\therefore$

$v_{e_1} + v_{e_2} + \ldots + v_{e_{k-1}}$

$= \{V_{s(e_1)} - V_{t(e_1)}\} + \ldots + \{V_{s(e_{k-1})} - V_{t(e_{k-1})}\}$

$= \{V_{s(e_1')} - V_{t(e_1')}\} + \ldots + \{V_{s(e_{k-1}')} - V_{t(e_{k-1}')}\}$

$= (V_1' - V_2') + (V_2' - V_3') + \ldots + (V_{k'} - V_{k-1}')$

$= V_1 - \cancel{V_2} + \cancel{V_2} - V_3 + \ldots + V_k - \cancel{V_{k-1}}$

$= V_1 - V_k$ → independent of path & k

$= V_{N_1} - V_{N_2}$

Hence along any path, KVL satisfied of voltage drop = sum.

$\therefore$ if $V \in W_{KVL} \Rightarrow V$ satisfy KVL

If we go around a loop, sum $= V_{N_1} - V_{N_1} = 0$ $(N_1 = N_2)$

$\therefore$ KVL satisfied.

e_i' = edges of L_k

$s(e_i') = m_i$ $\quad i \in \{1, \ldots k-1\}$

$t(e_i') = m_{i+1}$

Backward direction

Claim- If v satisfies KVL, $v \in W_{KVL}$

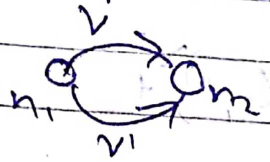
Proof- $v \in W_{KVL}$ if $\exists$ a u s.t. $\delta u = v$

Given v satisfies KVL, voltage drop/gain are path independent

We can also say, in a cycle, voltage drop is 0

Since if γ & γ' are two paths to reach a node n_2 from n_1

$v_\gamma = v_{\gamma'}$,



$v_\gamma - v_{\gamma'} = 0$

$\gamma - \gamma' =$ loop

$= v_{\gamma - \gamma'} = 0$

So Consider any loop with 0 voltage drop in a fixed orientation $e_1, \ldots, e_k$

we know $\sum_i^k v_{e_i} = 0$

Fix a node n_1 w.l.o.g on edge e_1 and e_k

Let ~~$u_1 = v_{e_1} - v_{e_k}$~~ ~~$u_{k-1} = v_{e_{k-1}} - v_{e_{k-2}}$~~

~~$u_2 = v_{e_2} - v_{e_1}$~~ ~~$u_k = v_{e_k} - v_{e_{k-1}}$~~

To create ~~kernel~~ $v(e_i) = u_{s(e_i)} - u_{t(e_i)}$

$u_1 - u_2 = v(e_1)$

$u_2 - u_3 = v(e_2)$

$\vdots$

$u_k - u_1 = v(e_k)$

n eqns, n variable — soln gives u such that $\delta u = v$ $\quad u = \{u_1, u_2, \ldots, u_k\}$

For every edge they can be assumed.

If no cycle is there, $v(e) = u(s(e)) - u(t(e))$

δ can be defined for each edge as 1 in position of $s(e)$ & -1 in position of $t(e)$

$= v \in W_{KVL}$

$\therefore$ Proved.

4. $\delta V = v \quad \delta^T i = 0$

$\sum_{e \in E} i_e v_e = \langle v, i \rangle = \langle \delta V, i \rangle$

i is in Null space of δ^T

If $m_1, m_2 \text{---} m_N$ be the Column vectors of δ,

$m_j \cdot i = 0 \quad \forall j \in \{1, \text{---} n\}$

$$\begin{bmatrix} m_1 \text{---} m_{j_1} & m_N \\ m_{j_2} & \\ | & \\ m_{j_E} & \end{bmatrix} \qquad i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_E \end{bmatrix}$$

$$\sum m_{j_k} * i_k = 0$$

$$\delta V = \begin{bmatrix} m_1 & m_{j_1} & m_N \\ | & | & \\ & m_{j_E} & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ | \\ V_N \end{bmatrix} = (V^T \delta^T)^T$$

If A & B are 2 vectors of size k $\quad A \cdot B = \begin{bmatrix} a_1 \\ | \\ a_k \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ | \\ a_n \end{bmatrix} = \sum_i^k a_i b_i$

$= A^T B$

(Matrix mult. is associative)

$= \langle \delta V, i \rangle = (V^T \delta^T) i = V^T (\delta^T i)$

$= V^T \cdot 0 = 0 \quad \therefore \sum v_e i_e = 0$

5- a) To prove $a \Rightarrow c$

$a \Rightarrow c$

G is a tree, hence $\exists$ a unique undirected path $n_1 \sim n_2$ b/w the nodes n_1 & n_2

Let G have a cycle $\bar{C}$ containing n_1 & n_2 without l.o.g. & having k nodes ($k \geq 3$)

Since $\bar{C}$ is a cycle, $\exists$ an ordered arrangement of edges & nodes (m) (say p nodes & p-1 edges) (e) such that

Let n_1 be node 1 in

$s(e_1) = n_1$ ... $s(e_{p-1}) = m_p$

$t(e_1) = m_1$ ... $t(e_{p-1}) = n_2$

Hence there exists an injective ~~for the remaining k-p nodes $\exists$ exist a~~ ~~sequence of edges & nodes~~ homomorphism b/w L_{p+2} and C

Hence a path $n_1 \overset{L_{p+2}}{\sim} n_2$ exists.

Similarly for remaining k-p nodes, $\exists$ an injective homomorphism b/w L_{k-p+2} and C

Hence another different path $n_1 \overset{L_{k-p+2}}{\sim} n_2$ exists

$\therefore$ it is not a unique path which is a contradiction to G is a tree.

$\therefore$ G has no undirected cycles and $a \Rightarrow c$

$\Leftarrow a$

G has no undirected cycles and is connected.
~~So~~ ~~even~~ $\forall$ nodes $u, v \in N_G$, a path $u \sim v$ exists $\because$ G is connected.
Suppose unique path doesn't exist b/w u & v.
Then $\exists L_1$ (surjective homomorphisms with L_2) b/w L_1 & G, L_2 & G)
$L_1 \neq L_2$
Then L_1 corresponds to path
$(u, n_1, n_2, \ldots n_p, v)$ // Both exist. $n_i \in N_G$
and L_2 corresponds to path $\neq n_i$
$(u, m_1, m_2 \ldots m_k, v)$

Consider the arrangement:

$(u, m_1, m_2 \ldots m_k, v, n_p, \ldots n_2, n_1, u)$
Label nodes as $t_1, t_2 \ldots t_{k+p+2}$
A homomorphism f exists b/w C_{k+p+2} and G such that

$f(s(e_i)) = t_i$ $\quad i \in \{1, \ldots k+p+1\}$ $\quad e_i \in E_G$
$f(t(e_i)) = t_{i+1}$

$f(s(e_{k+p+2})) = t_{k+p+2}$
& $f(t(e_{k+p+2})) = t_1$

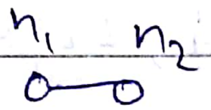
Hence $\exists$ an undirected cycle in G which is a contradiction $\therefore$ unique path b/w any 2 $u, v \in G$ exists $\therefore$ G is a tree (connected & unique path)

To prove a ⟹ b

a ⟹ b

Induction Proof:-

Assume case n (no. of nodes) = 2

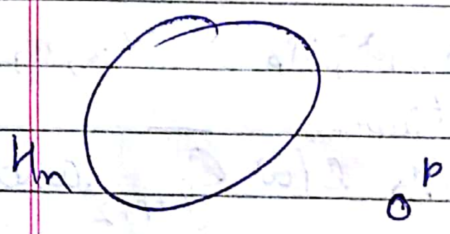


n = 2 (trivial case)

no. of edges = 1

Graph = tree because its connected and unique path exists b/w n_1 & n_2

Now consider a tree of n nodes say having n-1 edges (G_n)



To create a tree of n+1 nodes, we need to add a node p

→ If no edge is added between p and n_i ($i = \{1 \ldots n\}$),

note that G_n can be any arrangement.

Graph is not connected since no such e exists s.t. $s(e) \in N_G$ and $t(e) = p$

due to which an surjective homomorphism

with $\overline{T}$ cannot be formed.

→ If >1 edges are added, take any two ~~with same node~~ edges added e_1, e_2

Let the nodes connected with them be n_1, n_2.

Since $n_1 \sim n_2$, a unique path exists b/w them of arrangement $(n_1, u_1, u_2 - u_k, n_2)$

Consider the arrangement

$$\left(n_1, u_1, u_2 - u_k, \underbrace{n_2, p}_{\text{through } e_2}, \underbrace{n_1}_{\text{through } e_1}\right)$$

∃ exists a homomorphism now b/w T_{k+3} and G such that

$$f(s(e_i)) = t_i \quad i \in \{1, \_ k+2\}$$
$$f(t(e_i)) = t_{i+1}$$
$$\& \; f(s(e_1)) = p, \; f(t(e_1)) = n_1$$
$$f(s(e_2)) = n_2, \; f(t(e_2)) = n_2$$

∴ A cycle is made & T_{n+1} is not a tree.

→ If exactly one edge is added.

Let the edge e be added b/w p & n, $n \in A_n$

n can be any node.

A unique path must exists b/w m_1 & $m_2 \in A_G$

$m_1 \neq n, m_2 \neq n$

because p is connected to n only.

Consider path γ' from n to p
(any homo. with L_2 exists)

from any node m, $\exists$ a unique path γ to n
$m \overset{\gamma}{\rightsquigarrow} n$

Consider the path $\gamma + \gamma'$, its a unique path b/w any node m & p
$\therefore$ G is connected with unique path existing b/w any two nodes
$\therefore$ G is a tree.
with $n-1+1 = n$ edges
~~Hence~~ $H_{n+1} = h$ $\therefore$ ~~...~~ proved.

b $\Rightarrow$ a

G is connected and has $n-1$ edges. $-$ (α)
Since G is connected, a path exists b/w any two nodes n_1 & $m \in N_G$

If G has a cycle, G contains at least one extra path between at least 2 nodes n_1 & n_2 (w.l.o.g)
Remove all extra edges so that $\exists$ a unique path b/w every n_1 & n_2
(new graph $= H$)
H is a tree

a $\Rightarrow$ b implies H has $n-1$ edges.
So G had more than $n-1$ edges as at least one edge had been removed.

$\Leftarrow$ G is a tree.

$\therefore b \Rightarrow a$

* Since $a \Rightarrow b$ & $b \Leftrightarrow c$, $b \Leftrightarrow c$

& $a \Leftrightarrow b \Leftrightarrow c$

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6. $v \in W_{KVL}$, $i \in \mathbb{R}^E$

$$\sum_{e \in E} i_e v_e = 0$$

$\sum i_e v_e = \langle v, i \rangle = \langle \delta V, i \rangle = V^T(\delta^T i) = 0$

$\therefore V^T$ & $(\delta^T i)$ are orthogonal vectors

and they needn't be 0.

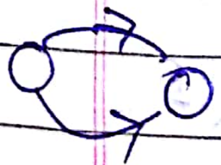
i.e. i needn't be in null space of δ^T

$\therefore$ i needn't be ~~KCL~~ $\in W_{KCL}$.

eg. take $\delta = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ ~~ka~~

$n = 2$ i as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ v as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$l = 2$



Then $V^T(\delta^T i) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

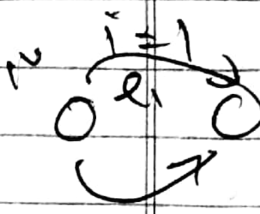
$= 0$

it satisfies all conditions, $\mu = \delta V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_{KVL}$, $\sum i_e v_e = 0$

Not all $i \in \omega_{KCL}$ ∵ $i \notin \ker\{\delta^T\}$

as $\delta^T i \neq [0]$

∴ Counter example given



∴ <u>Disproved</u>

(KCL not satisfied)

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7. Let $x \in \omega_{KCL}$, $x \in \omega_{KVL}$

then $\exists$ a v s.t. $Dx = \delta v$ —①

also $\delta^T x = 0$ —②

$\langle x, x \rangle = 0$

∵ $x \in \omega_{KVL}$ & $x \in \omega_{KCL}$ (Q4)

∴

$x^T x = 0$ $\quad x \in \mathbb{R}^E$

or $x_1^2 + x_2^2 \text{ — } + x_E^2 = 0$

∴ $x_1 = x_2 = \text{—} = x_E = 0$ $\quad x = (x_1, x_2 \text{ — } x_E)$

∴ $x = 0$ vector in $\mathbb{R}^E$

∴

$\omega_{KCL} \cap \omega_{KVL} = \{0\}$ in $\mathbb{R}^E$

First 9th question

Q9. $\delta : \mathbb{R}^{N} \to \mathbb{R}^{E}$ & ω_{KVL} = image of δ

$\dim(\omega_{KVL}) = \text{rank}(\delta)$

$\dim(\ker(\delta))$

ker δ is v s.t. $\delta v = 0$

In a connected graph, $\Rightarrow$ v is such for any edge $v_{s(e)} - v_{t(e)} = 0$

$\Rightarrow v_{s(e)} = v_{t(e)}$ for all edges & since it's connected, all nodes have same potential.

So for a connected component,

$v =$ like $\begin{pmatrix} a \\ a \\ a \\ a \\ \vdots \\ a \end{pmatrix}$ and hence $\dim = 1$

$a \in \mathbb{R}$

Similarly for k connected components with n_i nodes in each component,

$v = \begin{pmatrix} a_1 \\ a_1 \\ a_1 \\ \vdots \\ a_1 \\ a_2 \\ a_2 \\ a_2 \\ \vdots \\ a_2 \\ \vdots \\ a_k \\ a_k \\ \vdots \end{pmatrix}$ with a_1 repeated for n_1 terms, a_2 for n_2 terms, …, a_k for n_k terms

& $\dim(v) = k$

or

$\dim(\ker(\delta)) = k$

By rank nullity theorem

$$\dim(S) + \dim(\ker(S)) = \dim(\mathbb{R}^n)$$

$S : \mathbb{R}^n \to \mathbb{R}^E$

$\Rightarrow \dim(S) =$

$$\dim(W_{KVL}) = n - k$$

$$\dim(W_{KCL}) + \dim(W_{KVL}) = \dim(\mathbb{R}^E)$$

$= W_{KCL} \perp W_{KVL}$ (Both W_{KVL} & W_{KCL} are defined on $\mathbb{R}^E$ same space)

$\Rightarrow \dim(W_{KCL}) + n - k = E$

$$\Rightarrow \dim(W_{KCL}) = E - n + k$$

A8. Using result from Q9,

$\dim(W_{KCL}) = 4 - 4 + 1$ $(n = 4, E = 4, k = 1)$

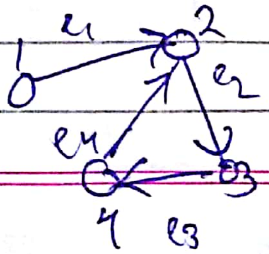
$= 1$

$\dim(W_{KVL}) = 4 - 1 = 3$

Basis of V

$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

(keeping V_4 grounded $\Rightarrow V_4 = 0$ always)

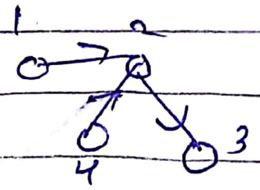


one node is grounded so voltage levels are relative & grounding makes no difference.

$\boxed{\text{For loss of } W_{KVL},}$

fix a V and observe v

Make a spanning tree

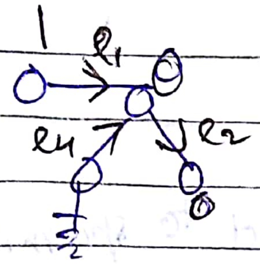


Keep 4 at gnd.

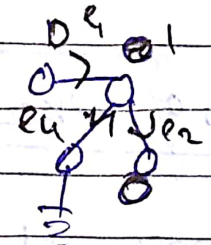
& assign voltage to nodes 1,2 & 3 one by one.

& others at zero

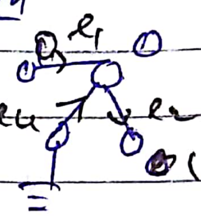
I



II



III



e3 is dependent on e_1, e_2 & e_4 (as its not on spanning tree) & can be written as

$$e_3 = -e_4 - e_2$$

I so

$v_{e_1} = v_1 - v_2 = 1 - 0 = 1$

$v_{e_2} = v_2 - v_3 = 0$

$v_{e_4} = v_4 - v_2 = 0$

$e_3 = 0$

one vector in kernel $= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

II

$v_{e_1} = v_1 - v_2 = -1$

$v_{e_2} = v_2 - v_3 = 1$

$v_{e_4} = v_2 - v_4 = 1$

$v_{e_3} = -1 - 1 = -2$

another vector in kernel $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -2 \end{pmatrix}$

III

$v_{e_1} = v_1 - v_2 = 0$

$v_{e_2} = v_2 - v_3 = -1$

$v_{e_4} = v_2 - v_4 = 0$

$v_{e_3} = 1$

& another vector $= \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

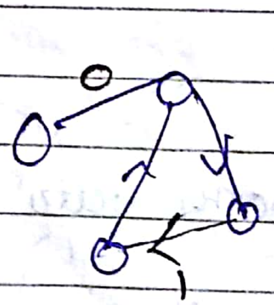
$$\text{Basis of } W_{KVL} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

This basis can be used to construct ~~the any~~ element ~~of~~ of W_{KVL} for any given arrangement of node voltages V

$\boxed{\text{Basis of } W_{KCL}}$

For edges not connected to spanning tree, run current of 1A one by one.

Since only one edge is introduced at a time, there is only cycle & hence current in all others is 0 by KCL



$i_{e1} = 0$
$i_{e2} = 1$
$i_{e3} = 1$
$i_{e4} = 1$

$$\text{Basis of } W_{KCL} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

This means any current in the graph will be of a scalar multiple of $\begin{pmatrix}0\\1\\1\end{pmatrix}$, i.e. current in e_1 is always 0 and currents in e_2, e_3, e_4 are equal $= c \in \mathbb{R}$.

This is true ~~irrespective~~ applied.

for ~~any~~ indepent of node voltage V

$\therefore$ configuration of S:- fixed & if i follows KCL, $S^T i = 0$.

fixed = null space is fixed

(Q8 + Q10)

For no. of eqns in node variable analysis, draw a spanning tree.

Since voltages are relative, ground any one node in the spanning tree.

~~So~~ So only $n-1$ nodes now have to be specified a voltage.

Since the tree is connected, $(n-1)$ KCL eqn ~~along~~ will be obtained on each of these $n-1$ nodes ~~will be sufficient~~.

Hence the system becomes solvable.

Greater than $n-1$ eqns are not needed because there are $n-1$ variables (unknowns) only.

Lesser than $n-1$ eqns won't solve the system.

$\therefore$ In node variable analysis $\boxed{n-1 \text{ eqns}}$ are needed and $n-1$ eqns are available on the spanning tree.

For k connected components, draw a spanning tree in each of the component. For such k spanning trees, $n_k - 1$ eqns are needed for each component with k nodes

Total eqns needed $= \sum n_k - 1$

$= n - k$

$(\sum n_k = n)$

For <u>loop variable analysis</u>,

we have already proved in GS that each cycle having independent current in an "extra" edge is a basis for ~~the total~~ specifying the current on each edge.

There are total E edges with $n-1$ edges being shared by at least 2 cycles.

$=$ no. of edges in span. tree

So excluding these edges gives us min. no. of loop eqs

$= E - (n-1) = E - n + 1$

For k connected components, make a spanning tree separately.

So loop var. eqs $= \sum (e_k - n_k + 1)$

$= \sum e_k - \sum n_k + \sum 1$

$= E - n + k$ loop variable eqs.

Q8 No. of node variable eqs $= n-1 = 4-1 = 3$

No. of loop variable eqs $= E-n+1 = 4-3 = 1$

$\therefore$ better to solve.