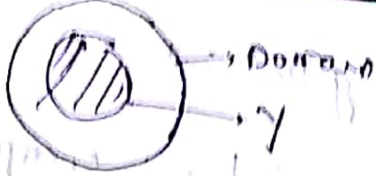
Tutorial - 4.

cauch's ~~theorem~~ Integral form



$f(z) \Rightarrow$ holomorphic on γ which is inside Ω
(Simply connected Ω).

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

z_0 is point where it is not holomorphic.

cauchy's theorem

$$\int_\gamma f(z)\,dz = 0$$

γ is curve

$f(z)$ is holomorphic on and inside Ω.

Generalised cauch's Integral form

$$f^n(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

1) $f(z) = \sum_{i=0}^{\infty} \alpha_i z^i$

) $f(\bar{z}) = \overline{f(\bar{z})}$

$g(z) = f(\bar{z}) - \overline{f(\bar{z})} = 0$

$\sum_{i=1}^{\infty} \alpha_i \bar{z}^i = \sum \bar{\alpha} \bar{z}^i$

$g(z) = \sum (\alpha_i - \bar{\alpha}_i)\bar{z} = 0$ it is also holomorphic function

$g(z) = 0$ So it is analytic function

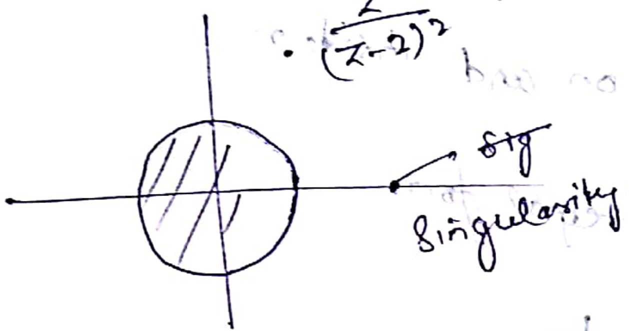
$g^n(z) = 0$ $\qquad (\alpha_1 - \bar{\alpha})^{2} = \quad \dfrac{g'(z)}{} \quad n!$

$(\alpha_1 - \bar{\alpha}) =$ compare

$\alpha_1 - \bar{z}_1 = 0$

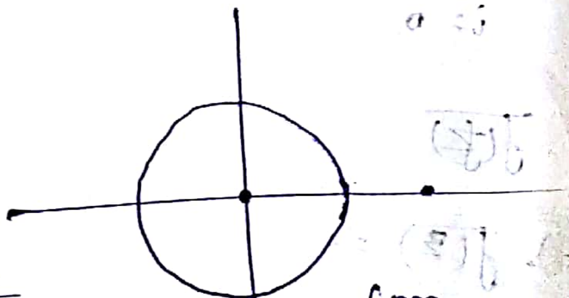
$\alpha_1 = \bar{\alpha}_1$ So g is real

2) (i) $\displaystyle\int_{|z|=1} \frac{z}{(z-2)^2} dz$



If singularity lies inside the countor then used Integral form of cauchy theorem.

(ii) $\displaystyle\int_{|z|=2} \frac{e^{z^2}}{z(z-3)} dz$



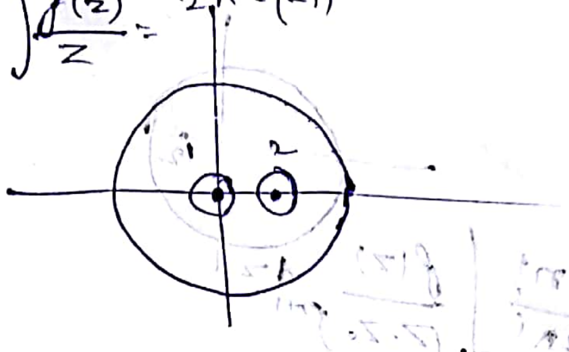
$f(z_0)$, $f(z) = \dfrac{e^{z^2}}{(z-3)}$

$f(0) = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(z)\, dz}{(z-0)}$ $\qquad f(0)\, 2\pi i = \displaystyle\int \frac{e^{z} dz}{z(z-3)}$

(iii) $$\int_{|z|=2} \frac{e^z}{z(z-1)}dz = \int_1 \frac{e^z}{z(z-1)}dz + \int_2 \frac{e^z}{z(z-1)}dz$$

$f(z) = \frac{e^z}{(z-1)}$

$\frac{1}{2\pi i}\int \frac{f(z)}{z-0} = f(0) \qquad \int \frac{f(z)}{z} = 2\pi i(-1)$



$f(z) = \frac{e^z}{z}$

$\frac{1}{2\pi i}\int \frac{f(z)}{(z-1)} = f(1)$

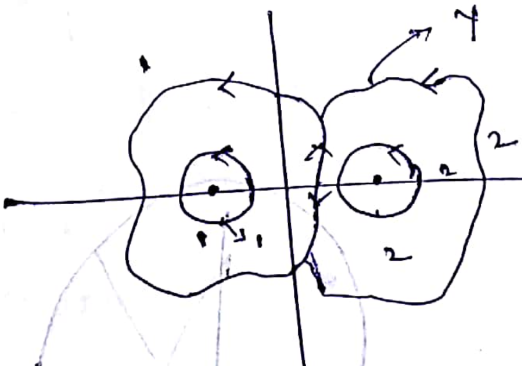
$\int \frac{f(z)}{(z-1)} = (2\pi i e)$

$\int_\gamma = 1+1 = -2\pi i + 2\pi i e$

$2\pi i(e-1)$ Ans

Standard Procedure

1) first make countour.
2) find the singularity Point (general form)
3) Think about ~~Int~~ cauchy Integral and cauchy theorem.



$\int_\gamma = \int_1 + \int_2$

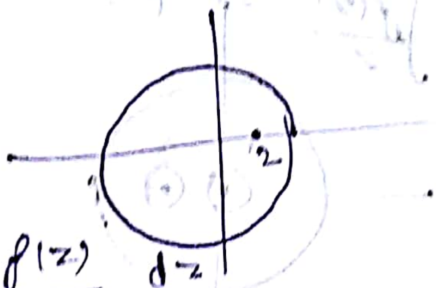
(iv) $\displaystyle\int_{|z|=4} \frac{\sin z}{(z-2)^2}\,dz$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$\frac{d \sin}{dz} = \frac{e^{iz}\, i + e^{-iz}\, i}{2i}$

$\frac{e^{iz} + e^{-iz}}{2}$

Cos.



$f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$f(z) = \sin z$ $\quad n=1$

$f'(z) = \cos z$

z_0 $z=2$

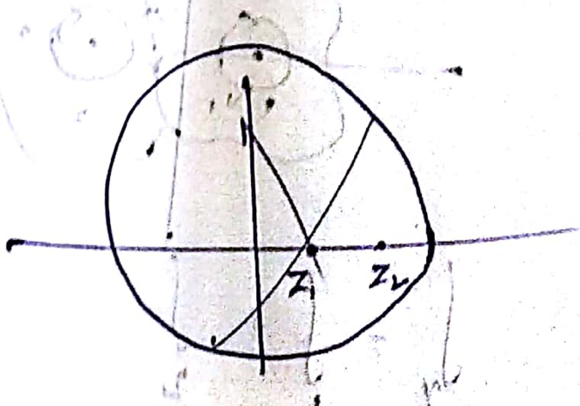
$f'(z_0) = \frac{1!}{2\pi i}\int \frac{\sin z}{(z-2)^2}\,dz$

$f'(z_0)\, 2\pi i = \int \frac{\sin z}{(z-2)^2}\,dz$

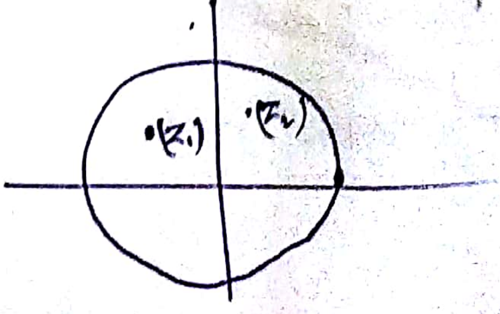
$\cos(2)\, 2\pi i$

3) $R > \max\{|z_1|, |z_2|\}$

$\displaystyle\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}$



$= \frac{1}{(z_2-z_1)}\int \frac{(z_2-z_1)\, f(z)}{(z-z_1)(z-z_2)}$



$$\int \frac{1}{z_2-z_1} = \frac{(z-z+z_2-z_1)f(z)}{(z-z_1)(z-z_2)}$$

$$(z-z_1)+(z_2-z$$

$$g(z)=\frac{f(z)}{z-z_0}$$

$$\int_1 \frac{g(z)}{z-z_1}dz = 2\pi i\, g(z_0) = 2\pi i \frac{f(z_1)}{z_1-z_2}$$

$$g(z)=\frac{f(z)}{z-z_1}$$

$$\int_2 \frac{g(z)}{z-z_2}dz = 2\pi i\, g(z_2) = \frac{f(z_2)}{z_2-z_1} \times 2\pi i$$

Add 1 and 2

4) f and $g \rightarrow$ holomorphic

$f(z)=g(z)$ on γ.

$g(z)=f(z)-g(z)$ holomorphic

$g(z)=0$ on γ.

$$\int \frac{g(z)\,dz}{(z-z_0)^{n+1}}=0 \quad \text{for all } n \text{ as } g(z)=0 \text{ on } \gamma$$

z_0 inside γ

$$\int \frac{g(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} g(z_0)$$

$f(z)=0$ on all point z_0 inside γ so $f(z)=g(z)$ for all z inside γ