Network Theory Homework 1

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Homework is not to be submitted. If you prepare a solution and upload it to Moodle, you may be eligible for extra credit, as per the course rules.

If not specified, you may assume that V is a finite dimensional vector space over the real numbers.

1. Define the following:
   (a) A graph
   (b) An Abelian group
   (c) A field
   (d) A vector space
   (e) A linear map
   (f) $Hom(U, V)$ where U, V are vector spaces over the same field
   (g) An isomorphism $V \cong W$ of vector spaces
   (h) Dual vector space V^*
   (i) Inner product $\langle , \rangle$ for a real vector space
   (j) Linearly independent set $X \subseteq V$
   (k) Spanning subset $X \subseteq V$
   (l) Basis $B \subseteq V$
   (m) Dimension of a vector space
   (n) The orthogonal space $S^{\perp}$ of $S \subseteq V$.

2. Prove or disprove: Fix a vector space V over $\mathbb{R}$.
   (a) $V \cong V^*$
   (b) Every field F is infinite
   (c) Every finite dimensional vector space has an inner product.
   (d) If $\langle , \rangle$ is an inner product, and $v \in V$ is a vector then $\langle , v\rangle$ is an element of V^*.
   (e) An inner product $\langle , \rangle$ defines an isomorphism $V \cong V^*$.

(f) Fix an isomorphism $F : V \to V^*$. Define a map $G : V \times V \to \mathbb{R}$ by $G(v, w) := (F(v))(w)$. That is, take v to $F(v) \in V^*$ using F, and then apply this element of V^* to $w \in V$. Then G is an inner product.

3. Prove or disprove: In an Abelian group $(G, 0, +, -())$, if $g, h, k \in G$ are such that $g + h = h + k = 0$ then $g = k$.

4. Prove/ disprove: if a linear map L is an isomorphism then it has a unique inverse. Equivalently, an invertible matrix has a unique inverse.

5. Prove/ disprove: If $Y \subseteq V$ is spanning, and $X \subseteq V$ is such that $|Y| \leq |X|$ then X is linearly dependent.

6. This question revises some concepts you should know from you first year linear algebra. In case they are unfamiliar to you, please refer to a standard Linear Algebra book like "Introduction to Linear Algebra" by Serge Lang. Define row rank and column rank of a (possibly rectangular) matrix. Define row operations and column operations on a matrix. Show that row operations do not change row rank. Show that row operations do not change column rank. Show that with row and column operations every matrix can be put into diagonal form. Hence argue that row rank of a matrix equals column rank of a matrix.

7.