Tutorial I

1. Show that a complex polynomial of degree n has exactly n roots. (Assuming the fundamental theorem of algebra)

2. Show that a real polynomial that is irreducible has degree at most two i.e., if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

3. Check for differentiability and holomorphicity:

   (i) $f(z) = c$, $c \in \mathbb{C}$;

   (ii) $f(z) = z$;

   (iii) $f(z) = z^n, \;\; n \in \mathbb{N}$;

   (iv) $f(z) = \mathrm{Re}(z)$;

   (v) $f(z) = |z|^2$;

   (vi) $f(z) = \bar{z}$;

   (vii) $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

4. If $f(z)$ is a real valued function in a domain $\Omega \subseteq \mathbb{C}$, then show that for any $z \in \Omega$ either $f'(z) = 0$ or $f'(z)$ does not exist. Hence, conclude that a real valued function f defined on a domain is holomorphic if and only if it is constant.

Tutorial II

1. Show that $f(z) = e^x(\cos y + i \sin y)$ is holomorphic throughout $\mathbb{C}$.

2. Show that the CR equations take the form
$$u_r = \frac{1}{r} v_\theta \ \& \ v_r = -\frac{1}{r} u_\theta$$
in polar coordinates.

3. If u and v are harmonic conjugates of each other, show that they are constant functions.

4. Show that following functions are harmonic and find their harmonic conjugate.
   (i) $u(x, y) = xy + 3x^2y - y^3$;
   (ii) $u(x, y) = 3x^2 + 2x - 3y^2 - 1$.

5. Find the radius of convergence of the following power series :
   (i) $\sum_{k=1}^{\infty} kz^k$;
   (ii) $\sum_{p \text{ prime}} z^p$;
   (iii) $\sum_{k=1}^{\infty} \frac{k! z^k}{k^k}$.

6. Give an example of a series which can be shown to be convergent by root test but not by ratio test.

Tutorial III

1. Show that $\exp(z_1) = \exp(z_2)$ if and only if $z_1 - z_2 = 2\pi n i$ for some $n \in \mathbb{Z}$.

2. Prove that

   (i) $\sin z = \sin x \cosh y + i \cos x \sinh y$;

   (ii) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

3. Let γ be the boundary of the triangle $\{0 \leq y \leq 1 - x; 0 \leq x \leq 1\}$ taken with the anticlockwise orientation. Evaluate:

   a) $\int_\gamma \mathrm{Re}(z)\, dz$;

   b) $\int_\gamma z^2\, dz$.

4. Show that $\sin; \cos : \mathbb{C} \to \mathbb{C}$ are surjective. How often does it attain a given value ?

5. Let γ be the circle with radius R around the origin with counter-clockwise orientation. Compute the following integrals :

   a) $\int_\gamma z^m\, dz, m \in \mathbb{Z}$;

   b) $\int_\gamma \bar{z}^m\, dz, m \in \mathbb{Z}$;

   c) $\int_\gamma |z|^m\, dz, m \in \mathbb{Z}$.

6. Show that if D is a bounded domain with C^1 boundary,

$$\int_{\partial D} \bar{z}\, dz = 2i\mathrm{Area}(D).$$

Tutorial IV

1. A power series with center at the origin and positive radius of convergence, has a sum $f(z)$. If it known that $f(\bar{z}) = \overline{f(z)}$ for all values of z within the disc of convergence, what conclusions can you draw about the power series ?

2. Evaluate the following integrals:

   (i) $\int_{|z|=1} \frac{z}{(z-2)^2} dz$;

   (ii) $\int_{|z|=2} \frac{e^z}{z(z-3)} dz$;

   (iii) $\int_{|z|=2} \frac{e^z}{z(z-1)} dz$;

   (iv) $\int_{|z|=4} \frac{\sin z}{(z-2)^2} dz$.

3. Let $z_1, z_2 \in \mathbb{C}$, $R > \max\{|z_1|, |z_2|\}$ and $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that
$$\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)} dz = 2\pi\imath \frac{f(z_2) - f(z_1)}{z_2 - z_1}.$$

4. Let f and g be two holomorphic functions on an open set containg a simple closed contour γ and its interior. If $f(z) = g(z)$ for all z on γ, what can be said about f and g in the interior of γ ?

Tutorial V

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f is constant in the following cases:

   (i) There exists a $c > 0$ such that $|f(z)| > c > 0$ for all $z \in \mathbb{C}$.

   (ii) Re $f(z) \geq 0$ for all $z \in \mathbb{C}$.

   (ii) $\overline{f(\mathbb{C})} \neq \mathbb{C}$, i.e. $\mathbb{C} \setminus \overline{f(\mathbb{C})}$ is a non-empty open set.

2. Does there exists a holomorphic function f on the open unit disc such that

$$f\left(\frac{1}{n}\right) = \begin{cases} 1/n & \text{if } n \text{ is even;} \\ -1/n & \text{if } n \text{ is odd} \end{cases} \quad ?$$

3. Let f be an entire function such that $f(\frac{1}{n^2}) = 1/n$ for all $n \in \mathbb{N}$. What can be said about f?

4. The following identity is called Taylor series with remainder :

$$f(z) = f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+\cdots+\frac{z^N}{N!}f^N(0)+\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N f^{N+1}(tz)\, dt.$$

   Use this to prove the following inequalities :

   a) $\left| e^z - \sum\limits_0^N \frac{z^n}{n!} \right| \leq \frac{|z|^{N+1}}{(N+1)!}$, $\mathrm{Re}(z) \leq 0$;

   b) $\left| \cos z - \sum\limits_0^N (-1)^n \frac{z^{2n}}{2n!} \right| \leq \frac{|z|^{2N+2} \cosh R}{(2N+2)!}$, $\mathrm{Im}(z) \leq R$.

5. By computing $\int_{|z|=1} (z + 1/z)^{2n}\ \frac{dz}{z}$, show that $\int_0^{2\pi} (\cos\theta)^{2n}\ d\theta = \frac{2\pi}{4^n}\frac{2n!}{n!^2}$.

Tutorial VI

1. Locate and classify the singularities of the following :

   a) $\sin(1/z)$;

   b) $\frac{z^2+z+1}{z^3-11z+13}$;

   c) $\frac{1}{\sin(1/z)}$;

   d) $\tan(1/z)$.

2. Find the poles and their orders of the functions

   (i) $\frac{1}{(z^4+1)^2}$, (ii) $\frac{1}{z^2+z-1}$.

3. Find Laurent expansions for the function $f(z) = \frac{2(z-1)}{z^2-2z-3}$ valid on the following sets: (i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$.

4. Let Ω be a domain in $\mathbb{C}$. Suppose that $z_0 \in \Omega$ is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighborhood of z_0 (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $0 < |z - z_0| < r$). Show that $f(z)$ has a removable singularity at z_0.

5. If $f(z) = \frac{p(z)}{q(z)}$ where p, q are differentiable with $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then show that

$$\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

6. Calculate residue at each singular point of the functions

   (i) $\frac{1}{z^2 \sin z}$, (ii) $\frac{1}{z(1-z)^2}$, (iii) $(\frac{z+1}{z-1})^3$.

Tutorial VII

1. Compute the following using residue theory:
$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx.$$

2. Compute the value of $\int_0^{2\pi} \frac{d\theta}{a+1-2a\cos\theta}$, where $a < 1$, by transforming into an integral over the unit circle.

3. Let $\overline{\mathbb{D}}$ be the closed unit disc. For any $\alpha \in \mathbb{D}$, define $\varphi_\alpha : \overline{\mathbb{D}} \to \mathbb{C}$ by
$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

   (i) Show that for all $|z| = 1$, $|\varphi_\alpha(z)| = 1$.

   (ii) Using (i) deduce that $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.

   (iii) Show that $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ is invertible by proving
$$\varphi_\alpha \circ \varphi_{-\alpha}(z) = z = \varphi_{-\alpha} \circ \varphi_\alpha(z) \quad (z \in \mathbb{D}).$$

4. Suppose f is an analytic function on the unit disc $\mathbb{D}$ with $|f| < M$ and $f(a) = 0$ for some $a \in \mathbb{D}$. Show that $|f(z)| \leq M|\frac{z-a}{1-\bar{a}z}|$ for all $z \in \mathbb{D}$.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. If $f(a_i) = b_i$ for all $i = 1, 2$, then show that
$$\left|\frac{b_2 - b_1}{1 - \bar{b_1}b_2}\right| \leq \left|\frac{a_2 - a_1}{1 - \bar{a_1}a_2}\right|.$$