Home Work 1

Q1

(a) A graph is a set of Nodes N and Edges E along with two functions $\underbrace{s}_{\text{source}} : E \to N$, $\underbrace{t}_{\text{target}} : E \to N$

(b) An Abelian group consists of a set $\mathcal{G}$, an identity element 0, an operator $+_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, ~~the elements of which commute.~~ and an inverse map $-_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$

satisfying

$$g_1 + g_2 = g_2 + g_1 \quad \text{(commutativity)}$$

$$g_1 + (g_2 + g_3) = (g_1 + g_2) + g_3 \quad \text{(assosiativity)}$$

$$g_1 + 0 = 0 + g_1 = g_1$$

$$g_1 + (-g_1) = (-g_1) + g_1 = 0$$

($-g_1$ is inverse element of g_1 with respect to $+_{\mathcal{G}}$)

~~Denoted as ($\mathcal{G}$, +, 0, Additive Inverse)~~

(c) A Field is a set F equipped with elements 0 & 1 in F, such that $0 \neq 1$, operations $+_F : F \times F \to F$, $* : F \times F \to F$, Inverse maps on $+_F$ and $*_F$, satisfying commutativity, associativity and distributivity

1) $(F, 0, +, \text{Additive Inverse})$ OR $(F, 0, +, -)$ is an abelian group

2) $(F \setminus \{0\}, 1, *, \text{Multiplicative Inverse})$ OR $(F \setminus \{0\}, 1, *, ()^{-1})$ is an abelian group

3) $0 * f = 0 \ \forall \ f \in F$

4) $a * (b + c) = a * b + a * c \ \forall a, b, c \in F$ (Distributivity)

(d) A vector space consists of an Abelian group $(V, 0_V, +_V, -)$
a Field F $(F, 0_F, 1_F, +_F, *_F, -_F, ()^{-1}_F \ldots)$
and Scalar Multiplication $\circ : F \times V \to V$ Satisfying

1) $0_F \cdot \vartheta = 0_V$
2) $1_F \cdot \vartheta = \vartheta$
3) $c \cdot (\vartheta +_V \omega) = (c \circ \vartheta) +_F (c \circ \omega)$
4) $(a *_F b) \circ \vartheta = a \circ (b \circ \vartheta)$
5) $c \cdot 0_V = 0_V$

(e) A function $L : V \to W$ where V & W are vector spaces over the same field
is a linear map if and only if
$$L(a \circ_V \vartheta_1 +_V b \circ_V \vartheta_2) = a \circ_W L(\vartheta_1) +_W b \circ_W L(\vartheta_2)$$

(f) ~~A Homomorphism~~ ~~A map $L : V \to W$ is a homomorphism if $L(ab) = L(a) L(b)$~~

A homomorphism the set of mappings $L : U \to V$ such that L is a linear map
The set is called $\mathrm{Hom}(U, V)$. (Note $\mathrm{Hom}(U,V)$ is a vector space)

(g) ~~An isomorphism~~

Two Vector spaces V and W are isomorphic $(V \cong W)$
if and only if there exists Linear map $L : V \to W$
and linear map $\hat{L} : W \to V$ such that $\hat{L} \circ L = \mathrm{Identity}_V$
$(\hat{L}(L(\vartheta)) = \vartheta)$

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(h) The set of Linear Maps $L: V \to \mathbb{R}e$ is V^* which the dual vector space of V.

(i) Inner Product $\langle , \rangle : V \times V \to \mathbb{R}$ is a set satisfying

1) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
2) $\langle v_1, v_2 \cdot a + b \cdot v_3 \rangle = a \langle v_1, v_2 \rangle + b \langle v, w \rangle$
3) $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0$ only if $v = 0_v$

(j) Definition of Linear Dependence - On a Field F, Vector Space V over F,

A subset $X \subseteq V$ is "linearly dependent"

if and only if there exists $n \in \mathbb{Z} > 0$, there exists

$c_1, c_2, \cdots c_n \in F \setminus \{0\}$

there exists $v_1, v_2, v_3 \cdots v_n \in X$ all distinct

such that $\sum_{i=1}^{n} c_i v_i = 0_v$

<u>Linearly</u> <u>Independent</u> $X \subseteq V$ is "linearly indepent" if X is not linearly dependent.

(k) $X \subseteq V$ is spanning if and only if for all $v \in V$, there exists $n \in \mathbb{Z} > 0$, there exists $c_1, c_2 \cdots c_n \in F$ there exists $v_1, v_2, v_3 \cdots v_n \in X$ such that

$v = \sum_{i=1}^{n} c_i v_i$

(l) A subset $B \subseteq V$ is a basis if and only if B is linearly independent and spanning.

~~(Note: If B is finite~~

(m) If $B \subseteq V$ is a basis, then Dimension of vector space V

$\dim(V) = |B|$ i.e. (cardinality of B)

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(n) $S^{\perp}$ is called Orthogonal Space of S if

For all $s' \in S^{\perp}$, for all $s \in S \subseteq V$,

$$\langle s', s \rangle = 0$$

(i.e. Inner product of any element in $S^{\perp}$ and any element in $S = 0$)

2 (a) To Prove that $V \cong V^*$

Let $\{V_1, V_2 \ldots V_n\}$ be the basis of V

claim:

Given any arbitrary set of constants $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ where $\alpha_i \in \mathbb{R}$ $\forall$ $i \in \mathbb{N}$ and $1 \leq i \leq n$, there exists a unique linear map ~~such that~~ $L_1 : V \to \mathbb{R}$ such that

$$L_1(V_i) = \alpha_i \quad \forall \; i = 1, 2 \ldots n \qquad \ldots\ldots (1)$$

Proof of claim:

Assume $\exists$ $L_2 \neq L_1$ s.t. $L_2(V_i) = \alpha_i$ $\forall i = 1, 2 \ldots n$

For any vector $v \in V$, we can write

$v = a_1 V_1 + a_2 V_2 \ldots a_n V_n$ ($\because$ V_is are basis)

$$\begin{aligned}
L_1(v) &= L_1(a_1 V_1 + a_2 V_2 \ldots + a_n V_n) \\
&= a_1 L_1(V_1) + a_2 L_1(V_2) \ldots a_n L_1(V_n) \\
&= a_1 \alpha_1 + a_2 \alpha_2 + \ldots a_n \alpha_n \\
L_2(v) &= L_2(a_1 V_1 + \ldots a_n V_n) \\
&= a_1 L_2(V_1) + a_2 L_2(V_2) + \ldots a_n L_2(V_n) \\
&= a_1 \alpha_1 + a_2 \alpha_2 + \ldots a_n \alpha_n \\
&= L_1(v)
\end{aligned}$$

Thus $L_2(v) = L_1(v)$ for all $v \in V$

$\Rightarrow\Leftarrow$

From (1) we can say that $\exists$ a unique set $\{L_1, L_2, \ldots L_n\}$ of linear maps with the following property:

$$L_1(V_i) = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}$$

$$L_2(V_i) = \begin{cases} 1 & i = 2 \\ 0 & i \neq 2 \end{cases}$$

and so on...

More generally $L_j(V_i) = \delta_{ij}$

$\therefore$ L_1 maps the basis to e_1

L_2 maps the basis to e_2 ...

Claim: This unique set $\{L_1, L_2 \cdots L_n\}$ is a basis for V^*

Proof: Take any linear map $L \in V^*$
let $L(V_i) = \beta_i \quad i = 1, 2, \cdots n$

$$L(V_i) = \beta_i = \beta_1 L_1(V_i) + \beta_2 L_2(V_i) + \beta_3 L_3(V_i) + \cdots \beta_n L_n(V_i)$$

Now take any $V \in V$

$$V = a_1 V_1 + a_2 V_2 + \cdots a_n V_n \quad , a_i \in \mathbb{R}$$

$$L(V) = a_1 L(V_1) + a_2 L(V_2) \cdots a_n L(V_n)$$

$$= a_1 [\beta_1 L_1(V_1) + \beta_2 L_2(V_1) \cdots \beta_n L_n(V_1)]$$

$$+ a_2 [\beta_1 L_1(V_2) + \cdots \beta_n L_n(V_2)]$$

$$\vdots$$

$$+ a_n [\beta_1 L_1(V_n) + \cdots \beta_n L_n(V_n)]$$

$$= \beta_1 [a_1 L_1(V_1) + a_2 L_1(V_2) \cdots a_n L_1(V_n)]$$

$$+ \beta_2 [a_1 L_2(V_1) + a_2 L_2(V_2) + \cdots a_n L_2(V_n)]$$

$$\vdots$$

$$+ \beta_n [a_1 L_n(V_1) + a_2 L_n(V_2) \cdots a_n L_n(V_n)]$$

$$= \beta_1 L_1(a_1 V_1 + a_2 V_2 \cdots a_n V_n) + \beta_2 L_2(a_1 V_1 + \cdots a_n V_n) \cdots + \beta_n L_n(a_1 V_1 + \cdots a_n V_n)$$

$$\cancel{\Sigma \beta_i L_i V_i} = \beta_1 L_1(V) + \beta_2 L_2(V) + \cdots \beta_n L_n(V)$$

$\therefore$ ~~Ay~~ Any linear map L can be written as a ~~[illegible]~~ linear combination of $L_1, L_2 \cdots L_n$.
Hence proved that it is spanning
If $\beta_1 L_1(V) + \beta_2 L_2(V) \cdots + \beta_n L_n(V) = 0 \quad \forall V \in V$ implies $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ Then L_is are linearly independent.

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Put $v = V_1$

$LHS = \beta_1 L_1(V_1) = 0 \quad \therefore \beta_1 = 0$

Put $V = V_2$

$LHS = \beta_2 L_2(V_2) = 0 \quad \therefore \beta_2 = 0$

similarly for $V = V_i$

$LHS = \beta_i L_i(V_i) = \beta_i = 0$

$\therefore \beta_i = 0 \quad \forall \; i = 1, 2 \ldots n$

$\therefore$ L_is are linearly independent

$\Rightarrow \{L_1, L_2 \ldots L_n\}$ is a basis set for V^*

$\therefore \dim(V^*) = n = \dim(V)$

$\Rightarrow V \cong V^*$

(b) False.

Counter-example: $F = \{0, 1\}$.

Define $+$ as: $0+0=0$

$0+1=1$

$1+0=1$

$1+1=0$

Define $\times$ as: $0\times 0=0$

$0\times 1=0$

$1\times 0=0$

$1\times 1=1$

Verify that this satisfies all axioms of a Field.

(c) Every finite dimensional vector space has an inner product.

<u>Proof</u>:

Let $\{V_1, V_2, \dots V_n\}$ be basis of V

We can define an inner product as follows:

For vectors u and $w \in V$ such that

$u = \sum_{i=1}^{n} a_i V_i$, $w = \sum_{i=1}^{n} b_i V_i$; $a_i, b_i, b_i', c, d \in \mathbb{R}$ $\forall i = 1, 2, \dots, n$

$$\langle u, w \rangle = \sum_{i=1}^{n} a_i b_i$$

This definition satisfies all the axioms of inner product:

(i) $\langle u, w \rangle = \langle w, u \rangle$ ✓

(ii) Let $w_1 = \sum_{i=1}^{n} b_i V_i$, $w_2 = \sum_{i=1}^{n} b_i' V_i$

~~Then~~ $\cancel{\langle u, cw_1 + dw_2 \rangle}$

$$cw_1 + dw_2 = c \sum_{i=1}^{n} b_i V_i + d \sum_{i=1}^{n} b_i' V_i$$

$$= \sum_{i=1}^{n} (cb_i + db_i') V_i$$

$$\langle u, cw_1 + dw_1 \rangle = \sum_{i=1}^{n} a_i (c b_i + d b_i')$$

$$= c \sum_{i=1}^{n} a_i b_i + d \sum_{i=1}^{n} a_i b_i'$$

$$\cancel{= c \langle u_1, w_1 \rangle}$$

$$= c \langle u, w_1 \rangle + d \langle u, w_2 \rangle$$

(iii) $\langle u, u \rangle = \sum_{i=1}^{n} a_i a_i = \sum_{i=1}^{n} a_i^2 \geq 0$

$\sum a_i^2 = 0$ iff $a_i = 0 \ \forall \ i = 1, 2 \cdots n$

$\Leftrightarrow \langle u, u \rangle = 0$ iff $u = 0_v$

Therefore every finite dimensional vector space will have an inner product.

2(d) $\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$

Therefore for a given $v \in V$

$$\langle \, , v \rangle : V \longrightarrow \mathbb{R}$$

Lets call this function f

$$\langle \alpha_1 \omega_1 + \alpha_2 \omega_2 , v \rangle = \alpha_1 \langle \omega_1 , v \rangle + \alpha_2 \langle \omega_2 , v \rangle$$

(Using properties of inner product)

$$\Rightarrow \quad f(\alpha_1 \omega_1 + \alpha_2 \omega_2) = \alpha_1 f(\omega_1) + \alpha_2 f(\omega_2)$$

$$\forall \; \omega_1 , \omega_2 \in V$$

$\therefore$ f is a linear map from $V \rightarrow \mathbb{R}$

$\therefore$ $\langle \, , v \rangle \in V^*$

2(e) Let $\{V_1, V_2, \ldots V_n\}$ be basis of V

Define inner product as in 2.(c), that is,

for any $u = \sum_{i=1}^{n} a_i V_i$, $w = \sum_{i=1}^{n} b_i V_i$

$$\langle u, w \rangle = \sum_{i=1}^{n} a_i b_i$$

We proved in 2(d) that $\langle \ , v \rangle \in V^*$ $\forall v \in V$

$\therefore \langle \ , \ \rangle : V \to V^*$

<u>Claim</u>: $\{\langle \ , V_1 \rangle, \langle \ , V_2 \rangle \ldots, \langle \ , V_n \rangle\}$ is a basis for V^*

<u>Proof</u>: (1) To show it is spanning:

Take any $L : V \to \mathbb{R} \in V^*$

Let $L(V_i) = \alpha_i$, $i = 1, 2 \ldots, n$

Let $w = \sum_{i=1}^{n} \alpha_i V_i$

Then $L \equiv \langle \ , w \rangle$

Because for ~~as~~ any $v = \sum_{i=1}^{n} \beta_i V_i$, $L(V) = L\left(\sum_{i=1}^{n} \beta_i V_i\right)$

$= \sum_{i=1}^{n} \beta_i L(V_i) = \sum_{i=1}^{n} \alpha_i \beta_i = \langle V, w \rangle$ as defined above.

$\therefore L = \langle \ , w \rangle = \langle \ , \sum_{i=1}^{n} \alpha_i V_i \rangle$

$$= \sum_{i=1}^{n} \alpha_i \langle \ , V_i \rangle$$

Hence proved it is spanning.

(2) To prove linearly independent:

Let $L = \sum_{i=1}^{n} c_i \langle\ , v_i \rangle$

If $L = 0 \Rightarrow c_i = 0$ then its done $\forall i = 1, 2 \ldots n$

($L = 0$ is a linear map which maps every element to 0)

$L(v) = 0 \Rightarrow \sum_{i=1}^{n} c_i \langle v, v_i \rangle = 0 \quad \forall \ v \in V$

Put $v = v_1$

Then $\sum_{i=1}^{n} c_i \langle v_1, v_i \rangle = c_1 \langle v_1, v_1 \rangle$

$= c_1 \Rightarrow c_1 = 0$

similarly for any $v = v_k$

$\sum_{i=1}^{n} c_i \langle v_k, v_i \rangle = c_k \langle v_k, v_k \rangle$

$= c_k = 0$

$\therefore L = 0 \Rightarrow c_i = 0 \quad \forall \ i = 1, 2 \ldots n$

Hence proved it is linearly independent.

Thus for basis $\{v_1, v_2 \ldots v_n\}$ of V

$\{\langle\ , v_1 \rangle, \langle\ , v_2 \rangle, \ldots \langle\ , v_n \rangle\}$ are basis of V^*

~~For same~~ For linear map between same ~~diffe~~ dimensional vectorspaces, it is ~~sufficient~~ sufficient describe where the basis are mapped to describe the entire map.

$\delta_{\langle,\rangle} : V \to V^*$, $\delta(v_i) = \langle\ , v_i \rangle \quad \forall \ i = 1, 2 \ldots n$

$\delta^{-1}_{\langle,\rangle} : V^* \to V$, $\delta^{-1}(\langle\ , v_i \rangle) = v_i \quad \forall \ i = 1, 2 \ldots n$

Thus inner product defines an isomorphism

$V \cong V^*$

(f) Let $\{V_1, V_2 \ldots V_n\}$ be basis for V

Let $\{L_1, L_2 \ldots L_n\}$ be the basis for V^* where L_i are ~~stated~~ defined as in proof of 2(a) i.e. $L_j(V_i) = \delta_{ij}$

Define $F : V \to V^*$ as

$$F(V_i) = L_i \quad \forall \; i = 1, 2 \ldots n$$

and this is a linear map.

~~Since F is a~~ $\therefore$ $F(V) = F\left(\sum_{i=1}^{n} a_i V_i\right)$

$$= \sum_{i=1}^{n} a_i F(V_i)$$

$$= \sum_{i=1}^{n} a_i L_i$$

Let $w = \sum_{j=1}^{n} b_j V_j$

$$(F(V))(w) = \left(\sum_{i=1}^{n} a_i L_i\right)(w)$$

$$= a_1 L_1(w) + a_2 L_2(w) \ldots a_n L_n(w)$$

$$= a_1 L_1\left(\sum_{j=1}^{n} b_j V_j\right) + a_2 L_2\left(\sum_{j=1}^{n} b_j V_j\right) \ldots$$

$$= a_1 b_1 + a_2 b_2 + \ldots$$

$$= \sum_{i=1}^{n} a_i b_i$$

$$\therefore \; G(v, w) = \sum_{i=1}^{n} a_i b_i$$

This satisfies the axioms of inner product (shown in 2(c))

3. $(G, 0, +, ()^{-1})$; $g, h, k \in G$

Given : $g + h = 0$

$h + k = 0$

$0 + g = g$ (Property of elements)

$\Rightarrow (h+k) + g = g$ ($h + k = 0$, given)

$\Rightarrow (k+h) + g = g$ (commutativity)

$\Rightarrow k + (h+g) = g$ (associativity)

$\Rightarrow k + 0 = g$ ($\because h + g = 0$ given $\Rightarrow g+h=0$, commutativity)

$\Rightarrow k = g$ ($\because k + 0 = k$)

4. L is an isomorphism

$$L : V \to W$$

then Let $\hat{L}_1 : W \to V$ and $\hat{L}_2 : W \to V$

be 2 distinct inverses of L

$$\hat{L}_1 = \hat{L}_1 \circ Id_W$$

$$= \hat{L}_1 \circ (L \circ \hat{L}_2)$$ (Property of inverse)

$$= (\hat{L}_1 \circ L) \circ (\hat{L}_2)$$

$$= Id_V \circ \hat{L}_2$$ (Property of inverse)

$$= \hat{L}_2$$

$\Rightarrow \hat{L}_1 = \hat{L}_2$ $\Rightarrow\Leftarrow$

Therefore L has a unique inverse.

For a matrix A, B is its inverse iff

$$AB = BA = I$$

Now if C is also inverse of A

$$AC = CA = I$$

$$CA = I \Rightarrow (CA)B = IB = B$$

$$\Rightarrow C(AB) = B \Rightarrow C = B$$

5. $Y \subseteq V$ is spanning and $X \subseteq V$
s.t. $|Y| \leq |X|$

Let $Y = \{y_1, y_2, \dots y_m\}$
$X = \{x_1, x_2 \dots x_n\}$

$m \leq n$

$\because$ Y is spanning $\therefore \exists\ \alpha_{ij}$ s.t.

$x_1 = \alpha_{11} y_1 + \alpha_{12} y_2 \dots \alpha_{1m} y_m$

$\vdots$

$x_n = \alpha_{n1} y_1 + \dots\dots \alpha_{nm} y_m$

$$\underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{n\times m}{\begin{bmatrix} \alpha_{11} & \cdots & \\ \vdots & \ddots & \\ & \cdots & \alpha_{nm} \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}$$

Rank of matrix $\begin{bmatrix} \alpha_{11} & \cdots \\ & \ddots \\ & & \alpha_{nm} \end{bmatrix} \leq \min(n, m)$

~~Rank~~ $\Rightarrow$ Rank $\leq m$

$\Rightarrow$ Dimension of row space $\leq m$

$\Rightarrow$ Maximal linearly ~~indepenent~~ independent set has number of elements $\leq m$

~~$\Rightarrow$ Any set of elemen~~

$\Rightarrow$ Any subset of row space with more than m elements is linearly dependent.

$\Rightarrow$ The set with rows of the matrix as elements in linearly dependent.

$\Rightarrow \exists\ \beta_1, \beta_2 \dots \beta_n$ s.t.

$$\beta_1 [\alpha_{11} \dots \alpha_{1m}] + \beta_2 [\alpha_{21} \dots \alpha_{2m}] + \dots \beta_n [\alpha_{n1} \dots \alpha_{nm}] = 0$$

and not all β_i are zero.

$\Rightarrow \exists\ \beta_1, \beta_2 \dots \beta_n$ s.t. $[\beta_1\ \beta_2 \dots \beta_n] \begin{bmatrix} \alpha_{11} & \cdots & \\ & \ddots & \\ & & \alpha_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = 0$

and not all β_i are 0.

$\Rightarrow \exists\ \beta_1, \beta_2 \dots \beta_n$ s.t. $[\beta_1\ \beta_2 \dots \beta_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$

and not all β_i are 0

$\Rightarrow \{x_1, x_2 \dots x_n\}$ is linearly dependent

Exercises III, §5

1. What is the dimension of the following spaces (refer to Exercises 11 through 16 of the preceding section):
   (a) 2×2 matrices (b) $m \times n$ matrices
   (c) $n \times n$ matrices all of whose components are 0 expect possibly on the diagonal.
   (d) Upper triangular $n \times n$ matrices.
   (e) Symmetric 2×2 matrices.
   (f) Symmetric 3×3 matrices.
   (g) Symmetric $n \times n$ matrices.

2. Let V be a subspace of $\mathbf{R}^2$. What are the possible dimensions for V? Show that if $V \neq \mathbf{R}^2$, then either $V = \{O\}$, or V is a straight line passing through the origin.

3. Let V be a subspace of $\mathbf{R}^3$. What are the possible dimensions for V? Show that if $V \neq \mathbf{R}^3$, then either $V = \{O\}$, or V is a straight line passing through the origin, or V is a plane passing through the origin.

III, §6. The Rank of a Matrix

Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

be an $m \times n$ matrix. The columns of A generate a vector space, which is a subspace of $\mathbf{R}^m$. The dimension of that subspace is called the **column rank** of A. In light of Theorem 5.4, the column rank is equal to the maximum number of linearly independent columns. Similarly, the rows of A generate a subspace of $\mathbf{R}^n$, and the dimension of this subspace is called **the row rank**. Again by Theorem 5.4, the row rank is equal to the maximum number of linearly independent rows. We shall prove below that these two ranks are equal to each other. We shall give two proofs. The first in this section depends on certain operations on the rows and columns of a matrix. Later we shall give a more geometric proof using the notion of perpendicularity.

We define the **row space** of A to be the subspace generated by the rows of A. We define the **column space** of A to be the subspace generated by the columns.

Consider the following operations on the rows of a matrix.

Row 1. Adding a scalar multiple of one row to another.

Row 2. Interchanging rows.

Row 3. Multiplying one row by a non-zero scalar.

These are called the row operations (sometimes, the elementary row operations). We have similar operations for columns, which will be denoted by **Col 1**, **Col 2**, **Col 3** respectively. We shall study the effect of these operations on the ranks.

First observe that each one of the above operations has an inverse operation in the sense that by performing similar operations we can revert to the original matrix. For instance, let us change a matrix A by adding c times the second row to the first. We obtain a new matrix B whose rows are

$$B_1 = A_1 + cA_2,\ A_2, \dots, A_m.$$

If we now add $-cA_2$ to the first row of B, we get back A_1. A similar argument can be applied to any two rows.

If we interchange two rows, then interchange them again, we revert to the original matrix.

If we multiply a row by a number $c \neq 0$, then multiplying again by c^{-1} yields the original row.

Theorem 6.1. *Row and column operations do not change the row rank of a matrix, nor do they change the column rank.*

Proof. First we note that interchanging rows of a matrix does not affect the row rank since the subspace generated by the rows is the same, no matter in what order we take the rows.

Next, suppose we add a scalar multiple of one row to another. We keep the notation before the theorem, so the new rows are

$$B_1 = A_1 + cA_2,\ A_2, \dots, A_m.$$

Any linear combination of the rows of B, namely any linear combination of

$$B_1,\ A_2, \dots, A_m$$

is also a linear combination of $A_1,\ A_2, \dots, A_m$. Consequently the row space of B is contained in the row space of A. Hence by Theorem 5.6, we have

$$\text{row rank of } B \leqq \text{row rank of } A.$$

Since A is also obtained from B by a similar operation, we get the reverse inequality

$$\text{row rank of } A \leqq \text{row rank of } B.$$

Hence these two row ranks are equal.

Third, if we multiply a row A_i by $c \neq 0$, we get the new row cA_i. But $A_i = c^{-1}(cA_i)$, so the row spaces of the matrix A and the new matrix obtained by multiplying the row by c are the same. Hence the third operation also does not change the row rank.

We could have given the above argument with any pair of rows A_i, A_j $(i \neq j)$, so we have seen that row operations do not change the row rank.

We now prove that they do not change the column rank.

Again consider the matrix obtained by adding a scalar multiple of the second row to the first:

$$B = \begin{pmatrix} a_{11} + ca_{21} & a_{12} + ca_{22} & \cdots & a_{1n} + ca_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let $B^1,\dots,B^n$ be the columns of this new matrix B. We shall see that the relation of linear dependence between the columns of B are precisely the same as the relations of linear dependence between the columns of A. In other words:

A vector $X = (x_1,\dots,x_n)$ gives a relation of linear dependence

$$x_1B^1 + \cdots + x_nB^n = O$$

between the columns of B if and only if X gives a relation of linear dependence

$$x_1A^1 + \cdots + x_nA^n = O$$

between the columns of A.

Proof. We know from Chapter II, §2 that a relation of linear dependence among the columns can be written in terms of the dot product with the rows of the matrix. So suppose we have a relation

$$x_1B^1 + \cdots + x_nB^n = O.$$

This is equivalent with the fact that

$$X \cdot B_i = 0 \qquad \text{for} \quad i = 1,\dots,m.$$

Therefore

$$X \cdot (A_1 + cA_2) = 0, \quad X \cdot A_2 = 0, \quad \dots, \quad X \cdot A_m = 0.$$

The first equation can be written

$$X \cdot A_1 + cX \cdot A_2 = 0.$$

Since $X \cdot A_2 = 0$ we conclude that $X \cdot A_1 = 0$. Hence X is perpendicular to the rows of A. Hence X gives a linear relation among the columns of A. The converse is proved similarly.

The above statement proves that if r among the columns of B are linearly independent, then r among the columns of A are also linearly independent, and conversely. Therefore A and B have the same column rank.

We leave the verification that the other row operations do not change the column ranks to the reader.

Similarly, one proves that the column operations do not change the row rank. The situation is symmetric between rows and columns. This concludes the proof of the theorem.

Theorem 6.2. *Let A be a matrix of row rank r. By a succession of row and column operations, the matrix can be transformed to the matrix having components equal to 1 on the diagonal of the first r rows and columns, and 0 everywhere else.*

$$\begin{array}{c c}
 & \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{r} \\
r\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \\
 &
\end{array}
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.$$

In particular, the row rank is equal to the column rank.

Proof. Suppose $r \neq 0$ so the matrix is not the zero matrix. Some component is not zero. After interchanging rows and columns, we may assume that this component is in the upper left-hand corner, that is this component is equal to $a_{11} \neq 0$. Now we go down the first column. We multiply the first row by a_{21}/a_{11} and subtract it from the second row. We then obtain a new matrix with 0 in the first place of the second row. Next we multiply the first row by a_{31}/a_{11} and subtract it from the third row. Then our new matrix has first component equal to 0 in the third row. Proceeding in the same way, we can transform the matrix so that it is of the form

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
0 & a_{m2} & \cdots & a_{mn}
\end{pmatrix}.$$

Next, we subtract appropriate multiples of the first column from the second, third, ..., n-th column to get zeros in the first row. This transforms the matrix to a matrix of type

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Now we have an $(m-1) \times (n-1)$ matrix in the lower right. If we perform row and column operations on all but the first row and column, then first we do not disturb the first component a_{11}; and second we can repeat the argument, in order to obtain a matrix of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

Proceeding stepwise by induction we reach a matrix of the form

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{ss} & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with diagonal elements $a_{11}, \ldots, a_{ss}$ which are $\neq 0$. We divide the first row by a_{11}, the second row by a_{22}, etc. We then obtain a matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus we have the unit $s \times s$ matrix in the upper left-hand corner, and zeros everywhere else. Since row and column operations do not change the row or column rank, it follows that $r = s$, and also that the row rank is equal to the column rank. This proves the theorem.