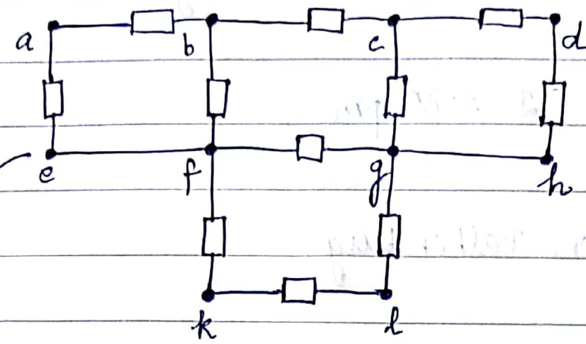
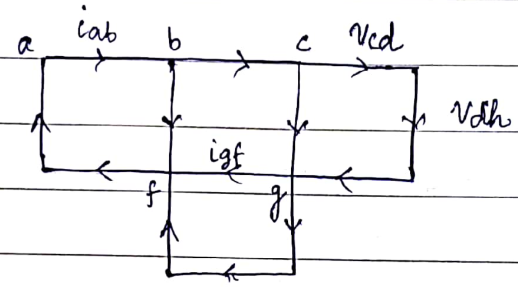
18.7.18

Tellegen's Theorem



Vertices/ nodes

Graph



orientation for every edge
↓
direction of current flow.

Net power dissipation = 0.

$$\begin{pmatrix} i_{ab} \\ \vdots \\ \vdots \\ i_{ek} \end{pmatrix} \cdot \begin{pmatrix} V_{ab} \\ \\ \\ V_{ek} \end{pmatrix} = 0$$

satisfies KCL

satisfies KVL

Even if we replace the elements, the dot product remains 0.

Vector space of the currents is perpendicular to the vector space of voltages.

Set :

Relation : S, T

$$S \times T = \{ (s,t) \mid s \in S, t \in T \}$$

$$R \subseteq S \times T$$

Consider $R \subseteq S \times S$

1) **Reflexive**

R is reflexive iff $\Delta = \{ (s,s) \mid s \in S \} \subseteq R$

2) **Symmetric**

R is symmetric iff for all $s,t \in S$, if $(s,t) \in R$, then $(t,s) \in R$.

3) **Transitive**

R is transitive iff $(s,t) \in R, (t,u) \in R \Rightarrow (s,u) \in R$.

4) **Equivalence**

R is an equivalence iff R is reflexive, symmetric and transitive.

GRAPH :

N : nodes

E : edges.

FUNCTION is a relation that is not one-to-many

source

function $s : E \to N$

$t : E \to N$

target.

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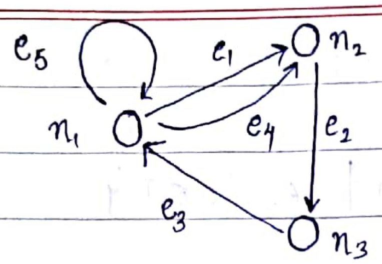
ex.



DIRECTED MULTIGRAPH → orientation

multiple edges b/w two nodes

$N = \{n_1, n_2, n_3\}$

$E = \{e_1, e_2, e_3\}$

$s(e_1) = n_1$ $\quad t(e_1) = n_2$

$s(e_2) = n_2$ $\quad t(e_2) = n_3$

$s(e_3) = n_3$ $\quad t(e_3) = n_1$

$s(e_4) = n_1$ $\quad t(e_4) = n_2$

$s(e_5) = t(e_5) = n_1$

★ If S, T are sets, then $\{f : S \to T\}$ will be denoted by T^S

set of all functions from $S \to T$

→ $\{f : E \to \mathbb{R}\}$ "voltage" $v : E \to \mathbb{R}$

$V_{KVL} \subseteq \mathbb{R}^E \supseteq V_{KCL}$ "current" $i : E \to \mathbb{R}$

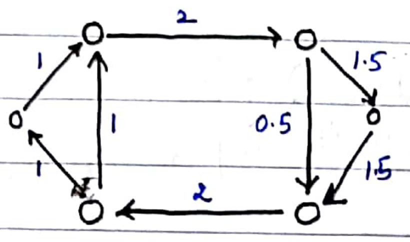
Tellegen's Theorem

$V_{KCL} \perp V_{KVL}$

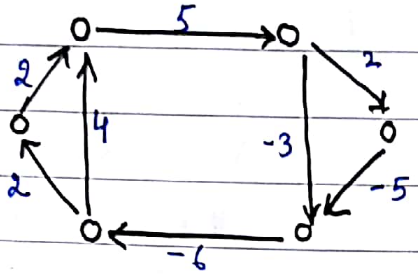
$V_{KCL} \oplus V_{KVL} = \mathbb{R}^E$

Lecture 2.

e.g.

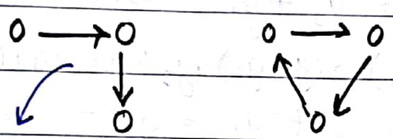


$$\begin{pmatrix}1\\1\\1\\2\\2\\0.5\\1.5\\1.5\end{pmatrix}\begin{pmatrix}2\\2\\4\\5\\-6\\-3\\2\\-5\end{pmatrix} = 0$$



Dot product = 0.

* The whole graph need not be connected.



Theorem still applies!

Any voltage can be prescribed.

LINEAR ALGEBRA

$$v = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \in \mathbb{R}^3$$

$c.v \in \mathbb{R}^3 \qquad c \in \mathbb{R}$

$w+v = v+w \in \mathbb{R}^3$

$\bar{0} + v = v + \bar{0} = v$

$c.(v+w) = c.v + c.w$.

$1.v = v \qquad 0.v = \bar{0}$

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v vectorspace
$(v, \bar{0}, +_v, \underbrace{(F, 0, 1)}_{(+_F, \times_F)}, \underset{\text{map}}{\cdot})$

$\cdot : V \times F \to v$

Abelian Group:
Set G
Element 0
Operation $+_g : g \times g \to g$

~~Parallelogram~~ (Parallelogram Law)
$g_1 +_g g_2 = g_2 + g_1$ COMMUTATIVE
$g_1 + (g_2 + g_3) = (g_1 + g_2) + g_3$ ASSOCIATIVE
$g_1 +_G 0 = 0 + g_1 = g_1$

Inverse operation: $G \to G$
$g \to (-g)$

$g + (-g) = 0$.

Field:
generalises properties of a scalar

$(\mathbb{R}, 0, +, \text{Add. inverse.})$
e.g. of an Abelian group.

non-zero real numbers
$(\mathbb{R} \setminus \{0\}, 1, \times, ()^{-1}) \to$ Abelian group
identity

$f_1 \times f_2 = f_2 \times f_1$
$f_1 \times 1 = 1 \times f_1 = f_1$
Inverse exists

$a \times (b + c) = a \times b + a \times c$

Field consists of a set F, elements $0, 1 \in F$, $0 \neq 1$

operations $+ : F \times F \to F$, $\times : F \times F \to F$

1) $(F, 0, +, \text{inverse})$ is an Abelian Group

2) $(F \setminus \{0\}, 1, \times, ()^{-1})$ is an Abelian Group

3) $0 \cdot f = 0$ for all $f \in F$

4) $a \times (b+c) = a \times b + a \times c$. DISTRIBUTIVE

e.g. $F = \{0, 1\}$

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

We choose the values post operations

1+1 has to be 0 otherwise additive inverse of one will not exist.

1 has a multiplicative inverse

e.g. $F = \{0, 2\}$

2 will be the multiplicative inverse.

From a given set, we choose additive inverse and multiplicative inverse to satisfy the conditions for a field.

* check for distributivity as well.
deleting multiple sol.^{n}

e.g. $F = \{0, 1, 2\}$

$$
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1
\end{array}
\qquad
\begin{array}{c|ccc}
\times & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 1
\end{array}
$$

if $a + b = 0$
$a + c = 0$
$b = c$

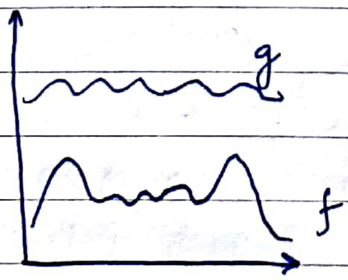
inverse is unique
$a + b + c = c$
$b = c$
using associativity & commutativity.

VECTOR SPACE

1) Abelian group $(V, 0_V, +_V, \overset{(-)}{\text{inverse}})$
2) Field $(F, 0_F, 1_F, +_F, \times_F, -_F, ()_F^{-1})$
set of scalars
3) $\cdot : F \times V \to V$.

1) $0_F \cdot v = 0_F$
2) $1_F \cdot v = v$
3) $c.(v+w) = c \cdot v + c \cdot w$ DISTRIBUTIVITY OF SCALAR MULTIPLICATION
4) $(a \times_F b) \cdot v = a \cdot (b \cdot v)$
5) $c \cdot 0_V = 0_V$.

✓ 0
✓ f+g



set of all functions on an interval
↓
VECTOR SPACE.

$(\{f : [0,1] \to \mathbb{R}\}, \bar{0}, +, -)$

LINEAR MAP

V, W : vector spaces over F

$L : V \to W$ is a linear map.

linear mapping

iff $L(a \cdot_v v_1 +_v b \cdot_v v_2) = a \cdot_w L(v_1) +_w b \cdot_w L(v_2)$

LINEAR SUPERPOSITION

e.g. $\mathbb{R}^3 \to \mathbb{R}^2$

$$(\quad)_{2\times 3}\begin{pmatrix} \\ \\ \end{pmatrix} \to \begin{pmatrix} \\ \\ \end{pmatrix}$$

Matrix multiplication is a linear map.

Q. Is there a linear map from $\mathbb{R}^3 \to \mathbb{R}^2$ such that there does not exist a matrix M satisfying $Lv = Mv \quad \forall v \in \mathbb{R}^3$? NO.

★ POST BASIS CONCEPT.

Applying linear map to basis vectors e_1, e_2, e_3, we look at the coefficients of the matrix given by $L(e_1), L(e_2), L(e_3)$

linear maps & matrices are same

$\downarrow$

depend on a basis.

essence of a diagonal matrix.

Category Theory
Postmodern Algebra.

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Lecture 3.

LINEAR DEPENDENCE

Fix field F, vector space V over F

A subset $X \subseteq V$ is 'linearly dependent' iff $\exists n \in \mathbb{Z} > 0$, $c_1, c_2 \ldots c_n \in F \setminus \{0\}$, $\exists v_1, \ldots, v_n \in X$ distinct such that

$$\sum_{i=1}^{n} c_i v_i = 0_V$$

at least one of them is non-zero

can be defined using LINEAR COMBINATION

e.g. $v_1 = (3,1)$ $v_2 = (-2,2)$ $v_3 = (-1,-2)$

$X = (v_1\ v_2\ v_3)$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ -2 \end{pmatrix} = 0$$

$3c_1 - 2c_2 = 0$ $\quad$ $3c_1 - 2c_2 - c_3 = 0$

$c_1 + 2c_2 = 0$ $\quad$ $c_1 + 2c_2 - 2c_3 = 0$

★ A maximum of 2 vectors can be linearly independent. Linearly dependent.

$4c_1 = 3c_3$

$c_1 = 3c_3/4$

- Also, three vectors are always linearly dependent.

$3c_3 + 8c_2 - 8c_3 = 0$

$c_2 = 5c_3/8$

Note: If $0_V \in X$ then $c \cdot 0_V = 0_V$ $\forall c \in F$ implies X is linearly dependent.

★ X is LINEARLY INDEPENDENT $:=$ X is not linearly independent that is $\sum_{i=1}^{n} c_i v_i = 0_V \Rightarrow c_i = 0$ for all i.

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→ cardinality..

Cantor's diagonalisation

classmate

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* If $X \subseteq \mathbb{R}^2$ is linearly independent then $|X| = 0, 1, 2$

SPANNING :

$X \subseteq V$ is spanning iff $\forall\, v \in V$, $\exists\, n \in \mathbb{Z} > 0$,
$\exists\, c_1, c_2, c_3 \ldots c_n \in \mathbb{F}$,
$\exists\, v_1, v_2, \ldots, v_n \in X$ s.t. $v = \sum_{i=1}^{n} c_i v_i$

Note :

If $X \subseteq V$ is linearly independent & $Y \subseteq V$ is spanning then $|X| \leq |Y|$

Proof : $X = \{x_1, \ldots\ldots x_m\}$

$Y = \{y_1, \ldots\ldots y_n\}$

To show $m \leq n$

$$x_1 = \sum_{i=1}^{n} c_{1i} y_i$$

rank (M) = row rank (M)

column rank (M) $\quad x_n = \sum_{i=1}^{n} c_{ni} y_i$

$$x_m = \sum_{i=1}^{n} c_{mi} y_i \qquad C = (c_{ji})_{n \times n}$$

INVERTIBLE

Proof : C is invertible

C is not invertible.

Then rank (C) < n

Then $\exists\, (a_1 \ldots\ldots a_n) \neq 0$

s.t. $(a_1 \ldots\ldots a_n)\, C = \bar{0}$

Then $\sum a_i x_i = aCy = 0$

CONTRADICTION,

since X is linearly independent

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = C^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$x_{n+1} = \sum_{i=1}^{n} d_i y_i$ (Y is spanning)

$x_{n+1} = \sum_{i=1}^{n} e_i x_i$

non-zero

$\Rightarrow\!\!\!\!\times$ since X is linearly independent

BASIS

A subset $B \subseteq V$ is a basis iff B is linearly independent and spanning.

If B is finite then V is finite dimensional and $\dim(V) = |B|$

→ (a) Every maximal linearly independent set is a (b) basis. & vice versa

→ Every minimal spanning set is a basis. & vice versa

→ If $B_1, B_2 \subseteq V$ are basis, then $|B_1| = |B_2|$.
& finite dimensional.

Lecture 4:

Proof: without loss of generality, suppose $m < n$.
Since u is spanning, from (note), v is linearly dependent CONTRADICTION

→ Let $A \subseteq V$.

(i) If A is a basis, it satisfies a & b.

Proof: * consider $A \cup \{v\}$, some $v \notin A$.
But $v = \sum_{v_i \in A} c_i v_i$ because A is spanning
$\therefore$ A' is linearly dependent

* If we remove an element from A, then we won't be able to express that vector in terms of the leftover vectors.

EXTREMAL ARGUMENTS

(i) If A is maximal linearly independent set, than A is a basis.

(ii) If A is a minimum spanning set, then A is a basis.

Proof: If A is not spanning, let $v \in V$ s.t. $v \notin span(A)$.

let $A' = A \cup \{v\}$.

A' is linearly dependent since A is maximal linearly independent set

$\therefore \sum_{v_i \in A} c_i v_i + cv = 0$ for some $c_i, c \in \mathbb{R}$, not all 0. $c \neq 0 \because A$ is linearly independent $\therefore v = -\sum_{v_i \in A} \frac{c_i}{c} v_i$

$\Rightarrow v \in span(A)$ CONTRADICTION.

COROLLORY : Every vector space has a basis.

LINEAR MAP CONT.

$Hom(V, W) = \{L : V \to W\}$ L linear

HOMOMORPHISM

→ $Hom(V, W)$ is a vector space.

Pf: $(0, +, \cdot,$ Additive inverse$)$.

$0 : V \to W \quad v \mapsto 0_W$

$(L_1 + L_2)v = L_1(v) +_W L_2(v)$

$(-L)(v) = -(L(v))$.

$L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$

$L_1 + L_2 = L_2 + L_1$

$(c.L)(v) = c.(L(v))$

ISOMORPHISM

for finite dimensional cases.

one-one & onto

We say $V \simeq W$ iff

$\exists \; V \underset{\hat{L}}{\overset{L}{\rightleftarrows}} W$

such that $\hat{L} \cdot L = Id_V$, i.e. $\hat{L}(L(v)) = v \;\; \forall v \in V$

and $L \cdot \hat{L} = Id_V$

$\Leftrightarrow \dim V = \dim W$.

DUAL VECTOR SPACE

$V^* = \{ L : V \to \mathbb{R} \mid L \}$ linear map.

$V \simeq V^*$ isomorphic.

$L \in V^*, \; v \in V$

$\Rightarrow L(v) \in \mathbb{R}$

INNER PRODUCT

$(\mathbb{R}^2)^* = \{ L : \mathbb{R}^2 \to \mathbb{R} \}$

$\{ (u_1, u_2) : u_1, u_2 \in \mathbb{R} \}$

$V \cdot W = \sum v_i w_i$

$\langle \, , \, \rangle : V \times V \to \mathbb{R}$

IP 1) $\langle v, w \rangle = \langle w, v \rangle$

IP 2) $\langle u, av + bw \rangle = a \langle v, v \rangle + b \langle v, w \rangle$

IP 3) $\langle v, u \rangle \geq u$ with $\langle u, u \rangle = 0$ only if $u = 0_V$.

Lecture 5

$\langle v, \_ \rangle : V \to \mathbb{R}$

$\langle v, \_ \rangle \in V^*$

Basis $B = \{v_1, \dots, v_n\}$ for V

$B^* = \{\langle v_1, \_\rangle, \langle v_2, \_\rangle, \dots, \langle v_n, \_\rangle\}$

<u>Claim</u> B^* is a basis for V^*

(rho) $\rho_{\langle , \rangle} : V \to V^*$

$v_i \longmapsto v_i^* = \langle v_i, \_ \rangle$

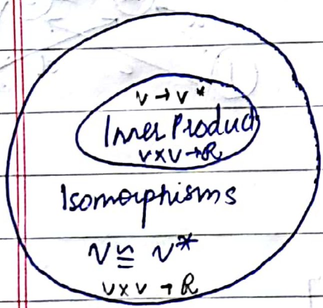
$v = \sum c_i v_i \longmapsto \sum c_i v_i^*$

An inner product gives an isomorphism $V \cong V^*$

Q Suppose $f : V \to V^*$ is an isomorphism.

$\langle , \rangle_f : V \times V \to \mathbb{R}$

$(v_1, v_2) \longmapsto (f(v_1))(v_2)$.



→ Transpose of a matrix.

$V \xrightarrow{L} W \xrightarrow{w^*} \mathbb{R}$ $\qquad v^* = w^* \circ L$

$w^* \in W^* \xrightarrow{L^*} V^*$ → adjoint

$\to w^* \circ L$

Define $L^* : W^* \to V^*$ by
$L^*(w^*) := w^* \circ L$

$W \xrightarrow{L^T} V$. ← Applying isomorphism from $V^* \to V$.

★ Back to graphs :-

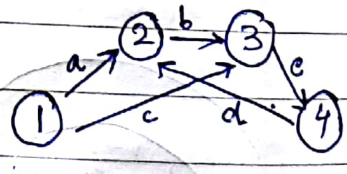
$G = (N, E \qquad s: E \to N)$
$\qquad t: E \to N$.

$\mathbb{R}^E = \{ f : E \to \mathbb{R} \}$

Vector space. $\qquad$ edges – a, b, c, d, e.

$V_{KVL} = \{$

$\dim(\mathbb{R}^E)$
$= |E|$.



Representing graph as a matrix.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ 3 & -1 & -1 & 0 & 1 \\ 4 & 0 & 1 & -1 & 0 \end{array}$$

NODE X NODE matrix

SUBGRAPH -

$G' \subseteq G = (N, E)$ s t

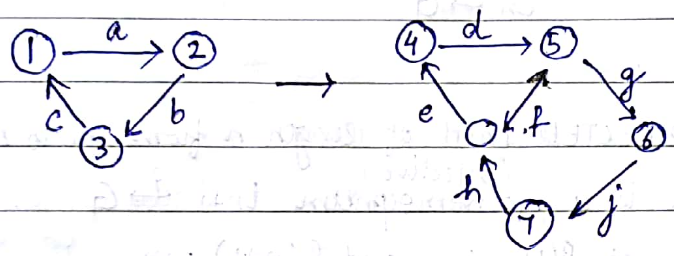
$G' = (N' \subseteq N, E' \subseteq E)$

$s|_{E'}$ $t|_{E'}$

* ~~Trees~~ :

~~Def~~: A ~~cycle~~ is a ~~subgraph~~

GRAPH HOMOMORPHISM



function from nodes to nodes sending edges to edges.

If G, H are graphs, a graph homomorphism

$f: G \to H$ is a function $f: N_G \to N_H$

s.t. $\forall e \in E_G$, $\exists e' \in E_H$.

s.t. $f(s(e)) = s(e')$

$f(t(e)) = t(e')$

Line graph L_n

Nodes 1 2 . . . n

Edges $1 \to 2 \to 3 \dots \to n$ $(n-1)$ edges.

Cycle graph C_n.

Nodes 1, 2 . . . n

Edges $1 \to 2 \to \dots \dots \to n$ (n) edges.

no. of options

$|N| = |\{L_1 \to G\}|$

$|E| = |\{L_2 \to G\}|$ provided at most one edge b/w a pair of nodes

Undirected graph

G is undirected if for every $e \in E$, there exists $e' \in E$ s.t. $s(e) = t(e')$ and $t(e) = s(e')$.

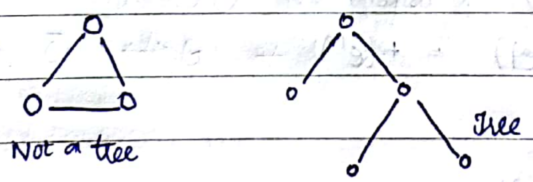
Def: A DIRECTED CYCLE (size n) is a injective homomorphism

$$C_n \to G$$

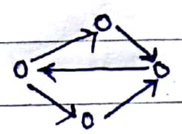
Def: A DIRECTED PATH of length n from m_1 to m_2 in a graph is a (injective) homomorphism $L_{n+1} \xrightarrow{f} G$ s.t. $f(1) = m_1$ and $f(n+1) = m_2$

Tree

A tree is an undirected cycle-free graph $[n \geq 3]$



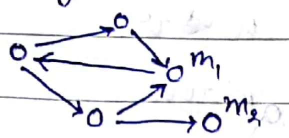
Def: A graph is STRONGLY CONNECTED if for every $m, n \in N_G$, there is a directed path from m to n.



cture6.

For a graph G, let $m, n \in N_G$

1) $m \rightsquigarrow n$: There is directed path from m to n is G, if there are edges arranged from head to tail leading m to n.



(injective) $\exists$ homomorphism $L_K \xrightarrow{f} G$

s.t. $f(1) = m_1$, $f(K) = m_2$.

2) $m \frown n$: Iff $\exists$ injective homomorphism $L_K \xrightarrow{f} G$

s.t. $f(1) = m_1$

$f(K) = m_2$

3) If G is a graph, let $\bar{G}$ be the smallest undirected graph containing (all edges of) G

i.e. if $e \in E_G$ then $e \in E_{\bar{G}}$, $e' \in E_{\bar{G}}$

where $s(e') = t(e)$

$t(e') = s(e)$

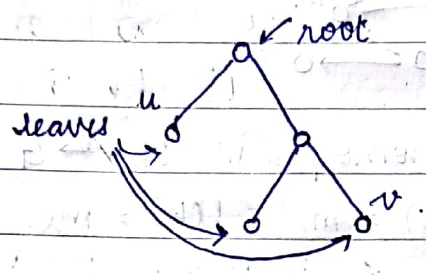
4) G is 'strongly connected' iff $u \rightsquigarrow v$ $\forall\, u, v \in N_G$.

5) G is 'connected' iff $\bar{G}$ is strongly connected.

UNDIRECTED TREE

An undirected graph G is a ~~tree~~ iff $\forall\ u, v \in N_G$

$\exists$ unique path $u \rightsquigarrow v$.



Claim

Let G be an undirected graph : The following are equivalent :

without self loops and without multi-edges.

$s(e) \neq t(e)\ \forall\ e \in E_G$ $\quad$ $s(e) = s(f)$, $t(e) = t(f)$ then $e = f$ $\ \forall e, f \in E_G$.

1) G is connected and has $|N| - 1$ undirected edges.

2) G is a tree

3) G is connected and "cycle-free".
i.e. $\bar{C}_k \not\hookrightarrow G\ \ \forall\ k \geq 3$.

Proof: $1 \Leftrightarrow 2$

Suppose G is connected and has $|N| - 1$ edges and has a cycle.

Inductive hypothesis $\quad$ o——o $\rightarrow$ tree

$|N-1| - 1 \rightarrow$ tree

$|N| - 1 \rightarrow$ tree
no cycle.

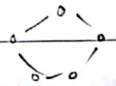
$2 \Rightarrow 1$

Since G is a tree, we claim $\exists\, v \in N_G$ s.t. v is a leaf. Suppose not: Then every node has at least 2 edges incident on it. And using pigeonhole, it'll meet somewhere.

$2 \Rightarrow 3$

G is a tree. suppose G has a cycle $\bar{C_k}$

But $\exists$ 2 dinstinct paths $1 \sim k$ in $\bar{C_k}$



$$\begin{tikzcd} \bar{L_k} \arrow[r, "T_d"] & \bar{C_k} \arrow[r, "h"] & G \\ \bar{L_2} \arrow[u, "f"'] & & \end{tikzcd}$$

$f(1) = 1$
$f(2) = k$.

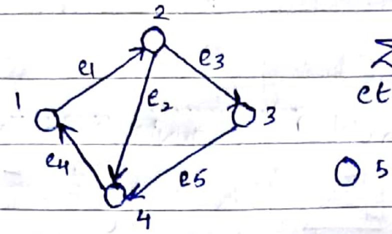
$3 \Rightarrow 2$

node repeat, cycle, no unique path

To show $\exists!\, u \sim v$. Since G is connected $\exists\, u \sim v$.
Suppose not unique if $(u, u_1, u_2 \ldots \ldots u_k, v)$ & $(u, v_1, v_2 \ldots v_l, v)$ are two paths let x be the first vertex, common to both paths, after u, we get $(u, u_1, u_2 \ldots x \ldots v_2, v_1, u)$ a cycle

CONTRADICTION.

TELLEGEN'S THEOREM



$$\sum_{e \in E} v_e . i_e = 0$$

$R^N : \{ f : N \to \mathbb{R} \}$

$R^E : \{ g : E \to R \}$ $\quad \delta : \mathbb{R}^N \longrightarrow \mathbb{R}^E$

$$(V_n)_{n \in N} \longmapsto (V_{s(e)} - V_{t(e)})_{e \in E}$$

Q. Is δ a linear map?
yes, it sends $0 \to 0$, satisfies linear superposition, vector space $\to$ vector space.

can be represented with a matrix.

$$M_\delta \quad \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]_{E \times N} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}_{N \times 1} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}_{E \times 1}$$

INCIDENCE MATRIX $\quad \underline{+1} V(s(e)) \; + \underline{(-1)} V(t(e))$

$\hookrightarrow$ source $(+1)$, target (-1)

All the information about the graph in the linear map

δ^T, M_δ^T

"Accumulation" $\delta^T : R^E \to R^N$.

R^E ↓ current $\quad R^N$ ↓ acc. of current $\quad$ [current coming in − going out]

$$(i_e)_{e \in E} \longmapsto \left(+ \sum_{(l,m) \in E} i_{l,m} - \sum_{(m,n) \in E} i_{m,n} \right)$$

W_{KCL} by definition is $\perp$ to W_{KVL}.

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Def 1 $W_{KVL} = \operatorname{im} \delta = \{ v \in \mathbb{R}^E \mid \exists\, V \in \mathbb{R}^N \text{ with } v = \delta(V) \}$.
$\subseteq \mathbb{R}^E$

linear combination of columns of δ.
COLUMN SPAN

2 $W_{KCL} = \ker \delta^T := \{ i \in \mathbb{R}^E \mid \delta^T(i) = 0 \}$
null space

perpendicular to a bunch of vectors.

CLAIM

1) W_{KVL} is a vector space.
2) W_{KCL} is a vector space

usual checks for vector space.

3) $v \in \mathbb{R}^E$ satisfies KVL iff $v \in W_{KVL}$.

2 $\to$ $\delta^T(i_1) = 0$, $\delta^T(i_2) = 0$

$\delta^T(a i_1 + b i_2)$
$= a\,\delta^T(i_1) + b\,\delta^T(i_2)$
$= a \cdot 0 + b \cdot 0 = 0$.
$\Rightarrow a i_1 + b i_2 \in W_{KCL}$.

3 $\to$ (a) suppose $1 \to 2 \to 3 \to \cdots \to n \to 1$ is a cycle in G.

Then $v_{(1,2)} + v_{(2,3)} + \cdots + v_{(n,1)}$

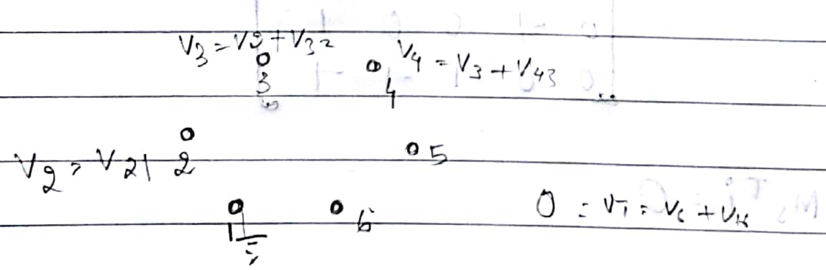
$= V_1 - V_2 + V_2 - V_3 + V_3 - \cdots + V_n - V_1$

$= 0$.

$v \to$ exact
gradient of a potential

(b) Suppose $v \in \mathbb{R}^E$ satisfies KVL. Then $v \in W_{KVL}$.

i.e. $\exists\, V \in \mathbb{R}^N$ st $v = \delta(V)$.



Pf continued

→ Tellegen's Theorem :

$(W_{KVL})^{\perp} = W_{KCL}$

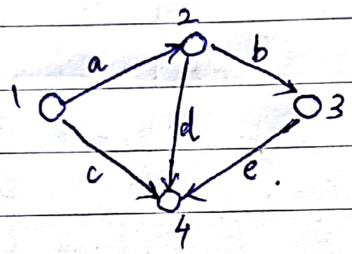
Pf: Suppose $v \in N_{KVL}$, $i \in W_{KCL}$.

Then $v . i = < \delta V, i >$

$(M_\delta V).(i) = (i^T M_\delta) V$

$= (M_\delta^T i). V$

$< \delta V, i > = < V, \delta^T i > = < V, 0 > = 0$.



$$M_\delta = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} |E| \times |N|.$$

$$M_\delta^T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

$M_\delta^T i = 0$.

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Pf: We will show that if $v \in W_{KVL}$, $i \in W_{KCL}$ then $v.i = 0$.

Since $v \in W_{KVL}$, $v = \sum_{l \in N} c_l m_l$.

$$\therefore v.i = \sum_{l \in N} c_l (\cancel{m_l \cdot i}^{\,0})$$

$$v.i = \langle S V, i \rangle = \langle V, S^T i \rangle = \langle V, 0 \rangle = 0.$$

2) To show: If $i \in W_{KVL}^{\perp}$ then $i \in W_{KCL}$.

Pf: suppose $i \in W_{KVL}^{\perp}$. Then since $m_1, m_2, m_n \in W_{KVL}$, $i.m_l = 0$

$$\Rightarrow i \in Ker(M_S^T) \longrightarrow i \in W_{KCL}.$$

→ Network equations.

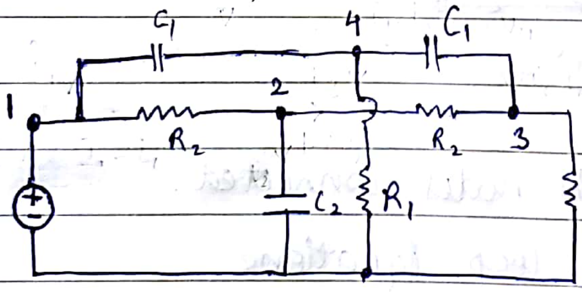
$V = IR$

$I = C\, dV/dt$

$V = L\, dI/dt$

$C = \frac{Q}{V}$

ex4.



$V_1 - (I - i_1) R_2 - V_2 = 0$ $\qquad i_2 = C_2 dV_2/dt$ $\qquad i_1 = C_1 dV_3/dt$

$V_1 - (I - i_1) R_2 - (I - i_1 - i_2) R_2 - (I - i_2 - i_3) R_L = 0$ $\qquad i_1 - i_3 = C_1 dV_4/dt$

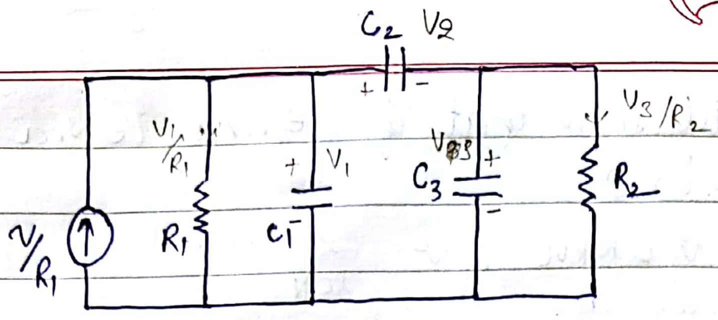
V_1

$CV = Q$

$I = C\,dV/dt$

II

Ex 8.

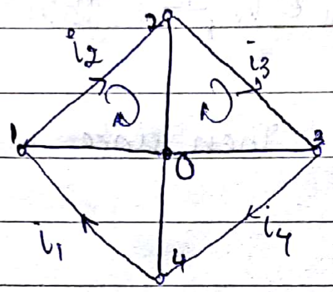


$V_1 = V_2 + V_3$

$$\frac{V}{R_1} = \frac{V_1}{R_1} + C_1\frac{dV_1}{dt} + C_2\frac{dV_2}{dt} + C_3\frac{dV_3}{dt} + \frac{V_3}{R_2}$$

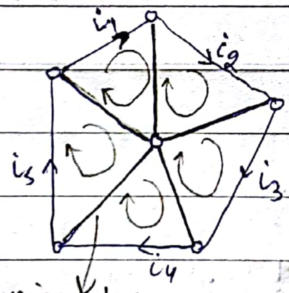
I.

We draw a graph.



$v_e(t)$, $i_e(t)$, dv_e/dt, di_e/dt.

Loop variable Analysis



1) Spanning tree

2) For each "branch", introduce one variable.

E edges, N nodes, connected.

Q How many loop equations.

Spanning tree : N nodes. (N-1) edges.

Branch edges : $E - (N-1)$

$= E - N + 1$

$(N-1)$

KCL.

$$\dim(\underbrace{W_{KVL}^{\subseteq \mathbb{R}^E}}_{N-1}) + \dim(\underbrace{W_{KCL}^{\subseteq \mathbb{R}^E}}_{E-N+1}) = E$$

Node Variable Analysis

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n- dimensional Boolean hypercube

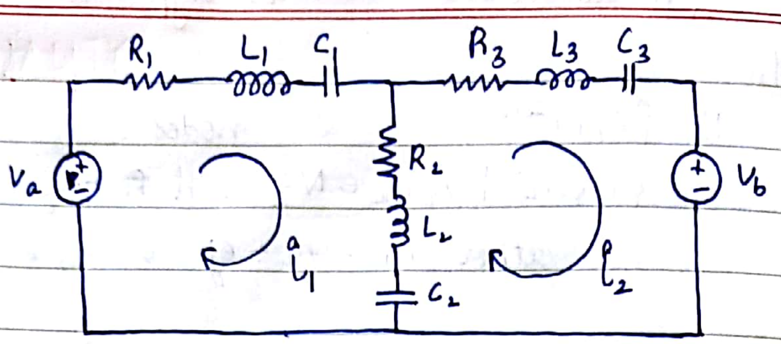
H_n

HW. $N = \{0,1\}^n$ 2^n nodes.

$E = \{(n_1, n_2) \mid n_1, n_2 \in N$ and $\exists\ e_j = (00100)$

s.t. either $n_1 = n_2 + e_j$ or $n_2 = n_1 + e_j \}$

Ch3. Ex 10 :



$$V_a = \left(R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int_{-\infty}^{t} i_1\,dt\right) + R_2(i_1-i_2) + L_2\frac{d}{dt}(i_1-i_2) + \frac{1}{C_2}\int_{-\infty}^{t}(i_1-i_2)\,dt$$

$$-V_b = \frac{1}{C_2}\int_{-\infty}^{t}(i_2-i_1)\,dt + L_2\frac{d(i_2-i_1)}{dt} + R_2(i_2-i_1) + R_3 i_2 + L_3\frac{di_2}{dt} + \frac{1}{C_3}\int_{-\infty}^{t} i_2\,dt$$

$$\begin{pmatrix} V_1 = V_a \\ V_2 = -V_b \end{pmatrix} = \begin{pmatrix} \text{i_1 contr. to Loop 1} & \text{common \& due to i_2} \\ \text{common due to i_1} & \text{i_2 cont. to loop 2} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

i_2

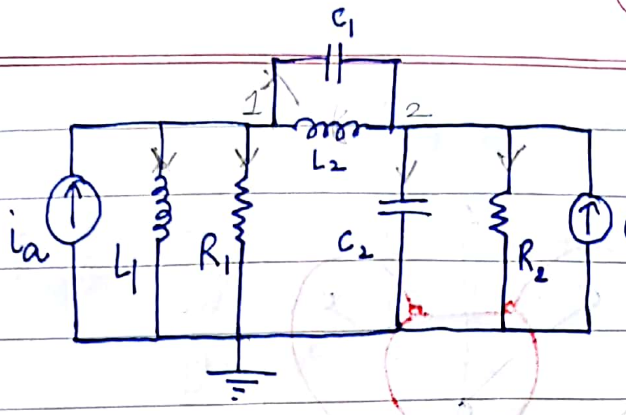
Ohm's law

$V = iR.$

$$\underbrace{\begin{pmatrix} R_1 + L_1\frac{d}{dt} + \frac{1}{C_1}\int + R_2 + L_2\,d/dt + 1/C_2\int & -R_2 - L_2\frac{d}{dt} - \frac{1}{C_2}\int \\ -R_2 - L_2\frac{d}{dt} - \frac{1}{C_2}\int & R_3 + L_3\frac{d}{dt} + 1/C_3\int + R_2 + L_2\,d/dt + \frac{1}{C_2}\int \end{pmatrix}}$$

Z = impedance matrix.

Ex 12



currents going out positive

currents coming in negative

$$i_a = \left(\frac{1}{R_1} + \frac{1}{L_1}\int + \frac{1}{L_2}\int + C_1\frac{d}{dt}\right)V_1 - \left(\frac{1}{L_2}\int + C_1\frac{d}{dt}\right)V_2$$

$$i_b = \left(\frac{1}{R_2} + C_2\frac{d}{dt} + \frac{1}{L_2}\int\right)V_2 - \left(\frac{1}{L_2}\int + C_1\frac{d}{dt}\right)V_1$$

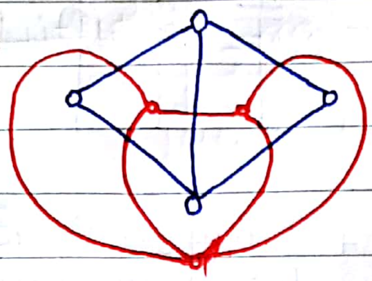
$$\begin{pmatrix} i_a \\ i_b \end{pmatrix} = \begin{pmatrix} \frac{1}{R_1} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int + C_1\frac{d}{dt} & -\frac{1}{L_2}\int - C_1\frac{d}{dt} \\ -\frac{1}{L_2}\int - C_1\frac{d}{dt} & \frac{1}{R_2} + (C_1+C_2)\frac{d}{dt} + \frac{1}{L_2}\int \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Y = Admittance matrix.

$$Z_{jk} = \frac{V_j}{i_k}\Bigg|_{i_j = 0 \text{ if } j \neq k.}$$

- Study Van Valkenburg.

DUALITY



$S: R^N \to R^E$

INCIDENCE MATRIX

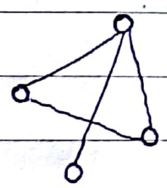
$$M_S = \begin{array}{c|c} & n_1 \;\dots\; n_k \\ \hline \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array} \quad |E| \times |N|.$$

$$M_S^T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \quad |N| \times |E|$$

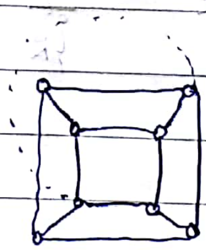
Nodes ~ Cycle loops

Def. Dual (G) is the graph with incidence matrix $M_{S,G}^T$ where $M_{S,G}$ is the incidence matrix of G.

PLANAR GRAPH



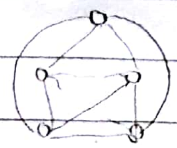
Planar



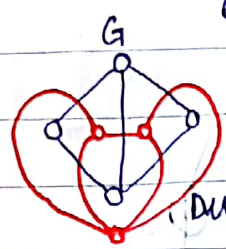
cube

★ Minimum no. of nodes required to get a non-planar graph.

5.



e.g. Planar drawing of a planar graph.



Dual(G).

No. of nodes = No. of cycles + 1
No. of edges remain same

$(N, E, E-N+1) \longleftrightarrow (E-N+2, E, N-1)$.

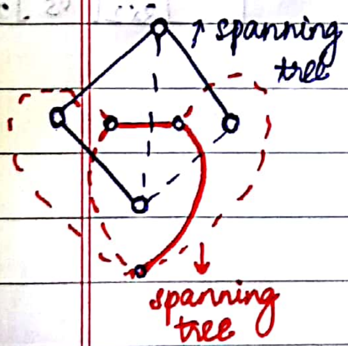
$\updownarrow$

Dual is unique for every graph.

$(N, E, E-N+1)$. original graph.

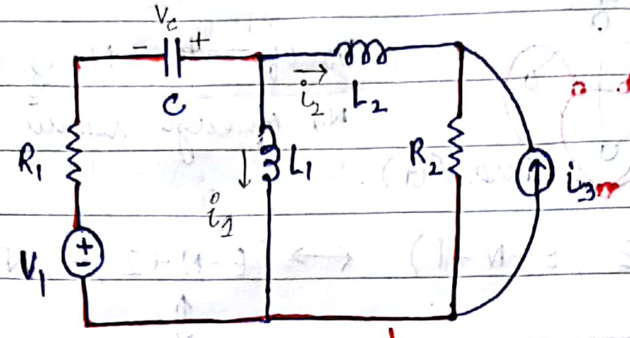
PLANAR NETWORK. underlying graph is planar
no wire cross

Original	Dual
R Ω	1/R Ω
L H	L F
C F	C H
v(t) V	v(t) A
i(t) A	i(t) A
loop variable	Node variable
Node Variable	loop variable



STATE SPACE EQUATIONS

* Ex 14



spanning tree.

$i = C\dfrac{dV}{dt}$ $\quad$ $V = L\dfrac{di}{dt}$

1) $C\dfrac{dV_c}{dt} = -i_1 - i_2$

2) $L_1\dfrac{di_1}{dt} = -i_1R_1 + V_c + V_1 - i_2R_1$

3) $L_2\dfrac{di_2}{dt} = -(R_1+R_2)i_2 + V_c - R_2 i_s + V_1 - i_1R_1$

$$\frac{d}{dt}\begin{bmatrix} V_c \\ i_1 \\ i_2 \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} \; & \; & \; \\ & & \\ & & \end{bmatrix}}\begin{bmatrix} V_c \\ i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0 \\ V_1/L_1 \\ V_1/L_2 - \frac{i_sR_2}{L_2} \end{bmatrix}$$

state variables.

$$\underset{3\times2}{\begin{bmatrix} \; & \; \\ & \\ & \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} V_1 \\ i_s \end{bmatrix}}.$$

* C → charge / voltage

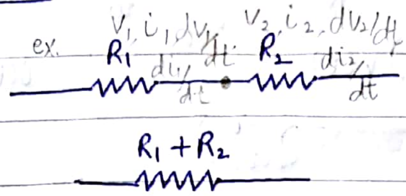
L → current / magnetic flux.

$$\boxed{\frac{d\bar{x}}{dt} = A\bar{x} + \bar{b}}$$

EQUIVALENCE

nothing changes

ex.



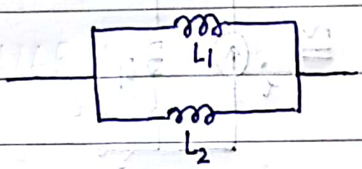
$i_1 = i_2 = i$

$V_1 = R_1 i \quad V_2 = R_2 i$

$V = (R_1 + R_2) i$ → linear relation Equivalent.

$v_1 \; i_1 \; \frac{dv_1}{dt} \; \frac{di_1}{dt}$ $\qquad$ $V_2 \; i_2 \; \frac{dV_2}{dt} \; \frac{di_2}{dt}$.

Fig 3.6. 1.



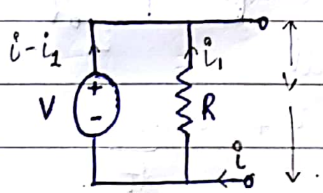
$$V = L_1 \frac{di_1}{dt} = \left(\frac{L_1 L_2}{L_1 + L_2}\right) \frac{di}{dt}$$

$$L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} \qquad i_1 + i_2 = i$$

$$L_1 \frac{di_1}{dt} = L_2 \frac{d(i - i_1)}{dt} \Rightarrow \frac{(L_1 + L_2)}{L_2} \frac{di_1}{dt} = \frac{di}{dt}.$$

$$\frac{di_2}{dt} = \left(\frac{L_1}{L_1 + L_2}\right) \frac{di}{dt}$$

2.



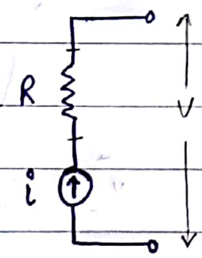
$V = i_1 R \qquad i_1 = V/R.$

$dv/dt = R \, di_1/dt$

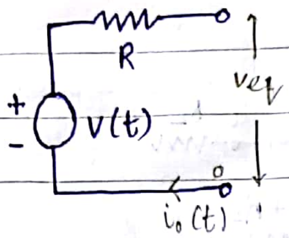
The final circuit will also have v.

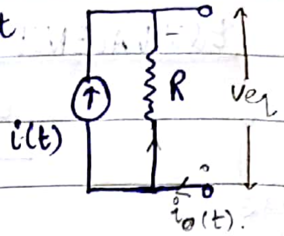
Only a voltage source.

3.



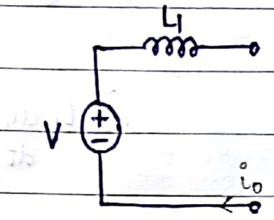
Only a current source

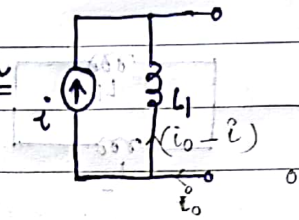
4. When is 

equivalent to 

$+v(t) - i_0 R(t) = V_{eq}$ $\quad$ $v(t) = R\,i(t)$.

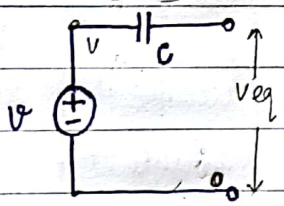
$-R(i_0(t) - i(t)) = v_{eq}$

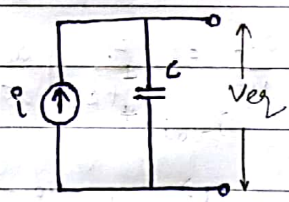
5. 

$\cong$ 

$+V - L_1 \frac{di_0}{dt} = -L_1 \frac{di_0}{dt} + L_1 \frac{di}{dt}$

$V = L_1\, di/dt$

6. 

$\cong$ 

$CV = Q$
$C\frac{dv}{dt} = i$.

$C\frac{d(V_{eq} - V)}{dt} = i + \frac{C\,dV_{eq}}{dt}$

$i = C\,dV/dt$.

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Tutorial / Revision :

★ Claim : Let V be a finite dimensional vector space over $\mathbb{R}$. Then $V \cong V^*$ where V^* = the dual vector space of V = $\{ L : V \to \mathbb{R} \mid L \text{ linear map} \}$.

Proof: $B = \{e_1, e_2, \ldots, e_n\} \to V$.

Define $B^* = \{f_1, f_2, \ldots, f_n\} \subseteq V^*$ by

$$f_i(v) = ?$$

$$v = \sum_{j=1}^{n} c_j e_j \quad (\text{since } e_j \text{ form a basis}).$$

$$\therefore f_i(v) = \sum_{j=1}^{n} c_j f_i(e_j) \quad [\text{linear superposition}]$$

$$f_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(1) B^* is spanning. Pf: consider $L \in V^*$.

To show $L = \sum d_i f_i$ where $d_i \to$ real no.s.

$$L(\sum c_i e_i) = \sum c_i L(e_i) \to \text{Linear superposition}$$

$$= \sum L(e_i) f_i(v)$$

$$= \sum \left(L(e_i) f_i \right) v.$$

L is being expressed as a linear combination.

(2) B^* is linearly independent.

To show If $\exists\, d_i \in \mathbb{R}$ s.t.

$$\sum_{i=1}^{n} d_i f_i = 0 \quad \text{then } d_1 = d_2 = \cdots = d_n = 0$$

$$\underset{\mathbb{R}}{\overset{\Downarrow}{0}} = 0(e_j) = \left(\sum d_i f_i\right)(e_j) = d_j.$$

All d_is. $= 0$