Independence

$X \perp\!\!\!\perp Y$

$\uparrow$ X is independent of Y.

σ- field generated by a r.v.

$X^{-1}(\mathcal{B}) = \{A \mid A \in \mathcal{F}$ & $\exists B \in \mathcal{B}$ s.t. $X(A) = B\}$

$X^{-1}(\mathcal{B}) \subseteq \mathcal{F}$.

X is independent of Y if $X^{-1}(\mathcal{B})$ and $Y^{-1}(\mathcal{B})$ are independent collections.

$$\Rightarrow \forall A_x \in X^{-1}(\mathcal{B}) \ \& \ A_y \in Y^{-1}(\mathcal{B})$$

$$P(A_x \cap A_y) = P(A_x)\, P(A_y)$$

$X \perp\!\!\!\perp Y$ if

$$F_{XY}(x,y) = F_X(x)\, F_Y(y) \quad \forall\, x, y \in \mathbb{R}$$

$$P(X \le x, Y \le y) = P(X \le x)\, P(Y \le y) \quad -(1)$$

The Borel sets considered above are of the form $(-\infty, \lambda]$.

Checking (1) ensures that all the ~~elem~~ events in $X^{-1}(\mathcal{B})$ are independent of all the events in $Y^{-1}(\mathcal{B})$

Also (1) is valid only when the inverse image is applied & on a Borel σ field.

* If the density fn can be split into product of two ~~separate~~ distinct fns of distinct variables then the events are independent.

Fundamental Th. of Probability -

1. Law of large numbers - Let $X_1, X_2 \ldots$ be a seq of independent and identically distributed r.v.'s s.t. $E[|X_1|] < \infty$

~~Weak Law~~

$$X_1, X_2 \ldots \leftrightarrow F$$

$$E[|X_1|] = \int_{-\infty}^{\infty} |x| f_X(x)\, dx$$

$$E[|X_2|]$$

$$S_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

Coin Toss Exp -

$$\Omega = \{0, 1\}^\infty$$

$$\omega = \{b_1, b_2, b_3 \ldots\}$$

$$X_1(\omega) = b_1 \qquad P(X_1 = 1) = P(\text{First cointoss is heads})$$

$$X_2(\omega) = b_2$$

$$\vdots$$

$$X_K(\omega) = b_K$$

S_n will give the fraction of heads in 1^{st} n coin tosses.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$P(\omega \mid |S_n(\omega) - p| > \epsilon) \longrightarrow 0 \text{ as } n \to \infty \quad \forall \epsilon > 0.$$

$p_n(\epsilon) \rightarrow$ Probability of bad ω's in first n tosses.

Weak convergence -

$$P(\omega \mid |S_n - E[X]| > \epsilon) \longrightarrow 0 \text{ as } n \to \infty \quad \forall \epsilon > 0$$

Convergence in probability.

$$P(\{\omega \mid \lim_{n \to \infty} S_n(\omega) \longrightarrow E[X_1]\}) = 1$$

$\uparrow$ Strong Convergence / Almost Sure Convergence.

Strong Convergence $\Rightarrow$ Weak convergence.

$Var(X_1) < \infty$ (Assumption)

$$P(|S_n - E[X_1]| > \epsilon) = P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - E[X_1]\right| > \epsilon\right)$$

$$= P\left(\left|\sum_{k=1}^{n} (X_k - E[X_k])\right| > n\epsilon\right)$$

$$= P(Y > n\epsilon) \leq \frac{E[Y^2]}{n^2 \epsilon^2}$$

$$E[Y^2] = E\left[\left(\sum_{k=1}^{n} (X_k - E[X_k])\right)^2\right]$$

$$= \sum_{k=1}^{n} E[(X_k - E[X_k])^2] \cancel{+ \sum_{i \neq j} E[X_i - E[X_i]] \cdot E[X_j - E[X_j]]}$$

$$+ \sum_{i \neq j} E[(X_i - E[X_i])(X_j - E[X_j])]$$

$$= n Var(X_1) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_i - E[X_i])(x_j - E[X_j]) f_{X_i X_j}(x_i, x_j)\, dx_i\, dx_j$$

$$= n Var(X_1) + \int_{-\infty}^{\infty} (x_i - E[X_i]) f_{X_i}(x_i)\, dx_i \cdot \int_{-\infty}^{\infty} (x_j - E[X_j]) f_{X_j}(x_j)\, dx_j$$

x_i & x_j are independent

$$= n Var(X_1) + E[X_i - E[X_i]] \cdot E[X_j - E[X_j]]$$

$$= n Var(X_1)$$

$$P(Y > n\epsilon) \leq \frac{Var(X_1)}{n \epsilon^2} \rightarrow 0 \text{ as } n \to \infty \quad \forall \epsilon >$$

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~~Central Limit Th.~~

Recap-

Fundamental th of prob $X_1, X_2 \ldots$ independent and identically distributed random variables with $E[X^2]$

Then, $\forall \epsilon > 0$

① $P\left(\left| \frac{1}{n}\sum_{k=1}^{n} X_k - E[X] \right| > \epsilon \right) \to 0$ as $n \to \infty$

② $P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k \to E[X] \right) = 1$

If $S_n = \frac{1}{n^\beta}\sum_{k=1}^{n} X_k$

$$S_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$E[S_n] = E[X] \quad \forall n$$

$$Var(S_n) = E\left[(S_n - E[S_n])^2 \right]$$

$$= E\left[\left(\frac{1}{n}\sum_{k=1}^{n} X_k - E[X] \right)^2 \right]$$

$$= E\left[\frac{1}{n} \left(\sum_{k=1}^{n} X_k - E[X] \right)^2 \right]$$

$$= \frac{1}{n^2} E\left[\left(\sum_{k=1}^{n} (X_k - E[X]) \right)^2 \right]$$

Let $Y_1, Y_2, \ldots$ be zero mean random variables the $Var(\sum Y_i) = \sum Var(Y_i)$

$$\frac{1}{n^2}(n \, Var(X_k)) = \frac{1}{n} \cdot Var(X_k) \Rightarrow \frac{1}{n^{2\beta-1}} Var(X_k)$$

$\beta > 0$

$\Rightarrow \beta > \frac{1}{2}$

$$\lim_{n\to\infty} Var(S_n) = 0$$

If a r.v has its var = 0 and expectation is const. then it is a const. r.v.

S_n is a const. r.v.

Central Limit Th.

$$\frac{\sum_{K=1}^{n} X_K - n E[X]}{\sqrt{n}\,\sigma} \xrightarrow{n \uparrow \infty} G(0,1)$$

in distribution