Quiz 2

Data Analysis and Interpretation (EE 223)

Date: 23/08/2018 Time: 9:30pm - 11:00 pm

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1. [2+4 M] While returning your answer sheets back, instead of calling out your number and giving the corresponding answer sheet back, suppose I distribute the sheets randomly without checking which answer sheet goes to which student. Let Z_n denote the number of students who get their own answer sheet, where n denote the class size. Find $\mathbb{E}[Z_n]$ and $var(Z_n)$.

   Hint: Indicator random variables are indeed useful.

2. Let $X \sim Uniform[0,1]$ and the conditional distribution of Y given X is $Uniform[0, cX]$, where $c > 0$ is a constant.

   (a) [2 M] Find $\mathbb{E}[Y]$.

   (b) [2 M] Find $f_{XY}(x, y)$.

3. [4 M] Time between two consecutive bus arrivals at IIT bus stop is an exponential random variable with mean $1/\lambda$ for given $\lambda > 0$. Given that you arrive at the bus stop s times units after the last but arrival find the distribution of your waiting time until the arrival of the next bus.

   Hint: Think about the conditional probability of waiting time to be at least t.

4. [4 M] Let $X \sim Uniform[0, 1]$. Let $Y = g(X)$ be exponential random variable with 1. Find function $g(\cdot)$.

5. [2 M] Let X be a random variable with moment generating function $\varphi(t)$ defined for all $t \geq 0$. Show that

$$\mathbb{P}(X > x) \leq e^{-tx}\varphi(t) \text{ for all } t \geq 0.$$

Q.1 Define

$$x_i = 1 \quad \text{if } i^{th} \text{ student gets his/her a.s.}$$
$$= 0 \quad \text{o.w.}$$

$$Z_n = \sum_{i=1}^{n} x_i$$

$$E[Z_n] = \sum_{i=1}^{n} E[x_i].$$

$$E[x_i] = \frac{1}{n} \quad \forall i.$$

$$\Rightarrow E[Z_n] = 1.$$

$$\text{var}(Z_n) = E[Z_n^2] - E^2[Z_n]$$

$$E[Z_n^2] = E\left[\left(\sum x_i\right)^2\right]$$

$$= E\left[\sum x_i^2 + \sum_{i \neq j}\sum x_i x_j\right]$$

$$= \sum E[x_i^2] + \sum_{i \neq j}\sum E[x_i x_j)$$

$$= 1 + 1 = 2 \quad \text{as } E[x_i x_j] = \frac{1}{n(n-1)}.$$

$$\text{var}(Z_n) = 2 - 1 = 1.$$

—— x —— x ——

Q.2

(a) $E[Y|X] = \frac{c}{2} X$.

$\Rightarrow E[Y] = E\left[E[Y|X]\right] = \frac{c}{2} E[X] = \frac{c}{4}$.

(b) $f_X(x) = 1 \quad \forall x \in [0,1]$

$f_Y(y|X=x) = \frac{1}{cx} \quad \forall y \in [0, cx]$.

$f_{XY}(x,y) = f_Y(y|X=x) \cdot f_X(x)$

$= \frac{1}{cx}$ if $x \in [0,1]$ & $y \in [0, cx]$

$= 0$ otherwise.

Q.3 Let X be the random variable denoting interarrival time.

We need to find

~~$P(X > t+s \mid X > s$~~

$P(\text{Waiting time} > t \mid X > s)$

$= P(X > t+s \mid X > s)$

$= P(X > t+s, X > s) / P(X > s)$

$= e^{-t\lambda}$

$\Rightarrow$ Waiting time is exponential with mean $\frac{1}{\lambda}$.

——— x ——— x ———

Q.4

$Y \sim \exp(1)$.

$$\Rightarrow F_Y(y) = 1 - e^{-y} \quad \forall y \geq 0.$$
$$= 0 \quad \text{otherwise.}$$

We know that

$$F_Y(Y) \sim \text{Uniform}[0,1].$$

Thus, we can choose

$$X = F_Y(Y).$$

$\because$ F_Y is an invertible function $[0,\infty)$

$$X = 1 - e^{-Y}. \Rightarrow e^{-Y} = 1 - X -$$

$$Y = \log\left(\frac{1}{1-X}\right)$$

——— x ——— x ——— x ———

Q.5

$$1_{\{X > x\}} \leq e^{t(X-x)}$$

$$P(X > x) \leq E\left[e^{t(X-x)}\right]$$

$$= e^{-tx} \underbrace{E\left[e^{tX}\right]}_{\varphi(t)} .$$