20/08

Independence

$X \perp\!\!\!\perp Y$

↑ X is independent of Y.

σ- field generated by a random variable

$$X^{-1}(\mathcal{B}) = \{ A : A \in \mathcal{F}, \exists B \in \mathcal{B} \text{ s.t } X(A) = B \text{ i.e } A = X^{-1}(B) \} \quad A \subset \Omega$$

$Y^{-1}(\mathcal{B})$

~~$Y^{-1}(\mathcal{B})$~~

X and Y are said to be independent if ~~$X^{-1}(\mathcal{B})$ &~~ $Y^{-1}(\mathcal{B})$ & $X^{-1}(\mathcal{B})$ are independent collection

$\Rightarrow \forall\ A_x \in X^{-1}(\mathcal{B})$ & $A_y \in Y^{-1}(\mathcal{B})$

$$P(A_x \cap B_y) = P(A_x) . P(B_y).$$

$X \perp\!\!\!\perp Y$ if

$$F_{XY}(x,y) = F_X(x) . F_Y(y) \quad \forall\ x, y \in R_e$$

$$P(X \leq x, Y \leq y) = \underset{[-\infty, x]}{P(X \leq x)} . \underset{[-\infty, y]}{P(Y \leq y)}$$

$$f_{XY}(x,y) = f_X(x) . f_Y(y)$$

(if we can seperate $f_{XY}(x,y)$ into $g(x) . h(y)$ ie seperate functions of x and y then X and Y are independent.)

Date

Fundamental theorems of Probability

1) Law of large numbers

Let $x_1, x_2, \ldots$ be a sequence of independent and identically distributed random variable s.t $E[|X_1|] < \infty$

(since $X_1, X_2 \ldots$ are identically distributed,

$$E[|X_1|] < \infty \Rightarrow E[|X_k|] < \infty \quad \cancel{\forall k}, \ \forall k)$$

$$\left\{ X_1, X_2, \ldots \sim f \right.$$

$$\left. \begin{aligned} E[|X_1|] &= \int_{-\infty}^{\infty} |x| f(x)\, dx \\ &= E[|X_2|] \end{aligned} \right\}$$

$$S_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$\Omega = \{0,1\}^{\infty}$$

$$\omega = \{b_1, b_2, b_3, \ldots\}$$

$$x_1(\omega) = b_1$$

$$x_2(\omega) = b_2 \quad \ldots \quad x_k(\omega) = b_k$$

~~E.g~~ $P(X_1 = 1) = P($first coin toss is heads$)$

$$P(\omega : |S_n(\omega) - EX| > \varepsilon) \to 0 \text{ as } n \uparrow \infty \quad \forall \varepsilon > 0$$

Weak convergence / convergence in prob.

$$P(\{\omega : \lim_{n \uparrow \infty} S_n(\omega) \to EX_1\}) = 1$$

strong convergence / almost sure convergence.

$Var(X_1) < \infty$ (assume).

$$P(|S_n - EX_1| > \varepsilon) = P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\right| > \varepsilon\right)$$

$$= P\left(\left|\sum_{k=1}^{n}(X_k - EX_k)\right| > n\varepsilon\right)$$

$$= P(Y > n\varepsilon) \le \frac{E[Y^2]}{n^2\varepsilon^2}$$

$$= E\left[\left(\sum_{k=1}^{n}(X_k - EX_k)\right)^2\right]$$

$$= \sum_{k=1}^{n} E[(X_k - EX_k)^2] + \sum_{i \neq j} E[(X_i - EX_i)(X_j - EX_j)]$$

$\Rightarrow Var(X_1)$

$$E[(X_i - EX_i)(X_j - EX_j)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - Ex_i)(x_j - Ex_j) f_{X_i X_j}(x_i, x_j)\, dx_i\, dx_j$$

$$= \int_{-\infty}^{\infty} (x_i - Ex_i) f_{X_i}(x_i)\, dx_i$$

$$\cdot \int_{-\infty}^{\infty} (x_j - Ex_j) f_{X_j}(x_j)\, dx_j$$

$$= E[x_i - Ex_i] \cdot E[x_j - Ex_j]$$

$$= 0 \cdot 0 = \underline{\underline{0}}$$

$$\therefore E[Y^2] = n\, var(X_1)$$

$$\therefore P(Y > n\varepsilon) \le \frac{n\, var(X_1)}{n^2 \varepsilon^2} = \frac{var(X_1)}{n\varepsilon^2}.$$

$\frac{var(X_1)}{n\varepsilon^2} \to 0$ as $n \uparrow \infty$ $\forall\ \varepsilon > 0$.

<u>Central limit theorem</u>

21/08

Fundamental theorem of prob.

$X_1, X_2, \ldots$ independant and identically distributed random variables with $E[x^2] < \infty$. Then, $\forall \ \varepsilon > 0$

1) $P\left(\left| \frac{1}{n} \sum_{k=1}^{n} X_k - EX \right| > \varepsilon \right) \to 0$ as $n \to \infty$.

2) $P\left(\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \to E_X \right) = 1$

$$S_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$E[S_n] = EX \quad \forall n$$

$$var(S_n) = \frac{1}{n} var(X_k)$$

$$\lim_{n \uparrow \infty} var(S_n) = 0.$$

Let $Y_1, Y_2,$ be zero mean random variables, then $var(\sum Y_k) = \sum var(Y_k)$

<u>Central limit theorem</u>

$$\frac{\sum_{k=1}^{n} X_k - nEX}{\sqrt{n}\,\sigma} \xrightarrow{n \uparrow \infty} G(0,1)$$

in distribution

Scanned by CamScanner

Theorem

Let X be a continuous R.V and let g be a differentiable function. Then

$$f_Y(y) = \sum_{k=1}^{\infty} \frac{f_X(x_k)}{|g'(x_k)|}$$

where $x_1, x_2, \ldots$ are roots of $g(x) = y$ (atmost countably many)
i.e.

$$g(x_i) = y \quad \forall i$$

Ex. $g(x) = ax^2$

$$Y = aX^2$$

roots $\sqrt{\frac{y}{a}}, -\sqrt{\frac{y}{a}}$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{2\sqrt{ay}} + \frac{f_X\left(-\sqrt{\frac{y}{a}}\right)}{|-2\sqrt{ay}|}$$

Expectation

- $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$E[X] = \sum_{k=1}^{\infty} a_k P(\{X = a_k\})$$

X may not take the value $E[X]$

$E[(X-c)^2] \rightarrow$ minimize this $\Rightarrow c = E[X]$

↳ mean square error

Scanned by CamScanner

23/08

$f_{XY}(x,y) = e^{-x}$ for $0 \le y \le x < \infty$

$E[Y|X]$

$E[Y|X=x] = g(x)$

$E[Y|X] = \int_{-\infty}^{\infty} \underbrace{E[Y|X=x]}_{g(x)} \, \delta(X-x) \, dx$

$= g(X)$

$E[Y|X=x] = \int_{-\infty}^{\infty} y \, f_Y(y|X=x) \, dy$

$\cancel{f_Y}(\quad f_Y(y|X=x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dy$ for $x \ge 0$

$= \int_0^x e^{-x} \, dy = x e^{-x}$

$f_Y(y|X=x) = 0$ when $x < 0$

$= \dfrac{e^{-x}}{x e^{-x}} = \dfrac{1}{x}$ if $y \le x$

$E[Y|X=x] = \dfrac{x}{2} = g(x)$

$\Rightarrow \cancel{E[X|Y]} \; E[Y|X] = \dfrac{X}{2}$

$Z = E[Y|X]$ is a random variable.

$Z(\omega) = E[Y|X = X(\omega)]$

$\cancel{E[Y|X=x] = \frac{x}{2}}$, $\cancel{Z(\omega) = \frac{X(\omega)}{2} \quad \forall \omega \in \Omega}$

$Z = X/2$

$$E[E[Y|X]] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} E[Y|X=x]\, f_X(x)\, dx$$

$$= \int_{x:-\infty}^{\infty} \int_{y:-\infty}^{\infty} y \underbrace{f_Y(y|X=x)}_{\frac{f_{XY}(x,y)}{f_X(x)}} dy\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\, dy\, dx$$

$$E[Y|X \le 0] = \int_{-\infty}^{\infty} y\, f_Y(y|X \le 0)\, dy$$

$$f_Y(y|X \le 0) = \frac{d}{dy} F_Y(y|X \le 0)$$

$$= \frac{d}{dy} \frac{P(Y \le y, X \le 0)}{P(X \le 0)}$$

Scanned by CamScanner

$$= \frac{1}{F_X(0)} \frac{d}{dy} \int_{-\infty}^{y} \int_{-\infty}^{0} f_{XY}(x,u)\, dx\, du$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} \left[\frac{d}{dy} \int_{-\infty}^{y} f_{XY}(x,u)\, du \right] dx$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} f_{XY}(x,y)\, dx$$

$E[Y \mid X \le 0]$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{\infty} y \int_{-\infty}^{0} f_{XY}(x,y)\, dx\, dy$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} \int_{-\infty}^{\infty} y \overbrace{\frac{f_{XY}(x,y)}{f_X(x)}}^{f_Y(y|x)} dy\, f_X(x)\, dx$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} E[Y \mid X = x]\, f_X(x)\, dx$$

$X_1, X_2, \ldots, X_n$ are independant and identically distributed random variable with $X_i \sim F$

$$F_{x_{k_1}, x_{k_2}}(x_1, x_2) = F(x_1)\, F(x_2)$$

$$F_n^*(x) = \begin{cases} 0 & \text{if } x < X_{\sigma(1)} \\ k/n & \text{if } X_{\sigma(k)} \le x < X_{\sigma(k+1)} \\ 1 & \text{if } X_{\sigma(n)} < x \end{cases}$$

Scanned by CamScanner

Date

for $x \in R$, $F_n^*(x)$ is a random variable $\in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$

$$P\left(F_n^*(x) = \frac{k}{n}\right)$$

$\Rightarrow$ exactly k random variables have values $\le x$, others have value $> x$.

$$I_{\{x_j \le x\}} = \begin{cases} 1 & \text{if } x_j \le x \\ 0 & \text{o.w.} \end{cases}$$

$$\left\{F_n^*(x) = \frac{k}{n}\right\} = \left\{\sum_{j=1} I_{\{x_j \le x\}} = k\right\}$$

Scanned by CamScanner

Date

Tut 2

1) (1) X - discrete random variable
prob. mass func^n -

$$P(X = x) = \begin{cases} kx & \text{for } x = 1, 2, 4 \\ k(x-1) & \text{for } x = 3, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

a) $k(1+2+4) + k(2+4+5) = 1$

$\Rightarrow k = 1/18$

b) $E[X]$

$= k(1+4+16) + k(6+20+30)$

$= k(21+56) = \dfrac{77}{18}.$

$E[X^2] = \cancel{k(1) + k}$

$k(1+8+64) + k(9.2 + 25.4 + 36.5)$

$= k(73 + 18 + 100 + 180)$

$= \dfrac{371}{18}$

(2) X - Gaussian Random variable

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x\, e^{-(x-\mu)^2/2\sigma^2} dx$$

Scanned by CamScanner

(3) X – Poisson random variable

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2, \ldots \quad \lambda > 0.$$

$$\bar{X} = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda.$$

$$\mathrm{Var}(X) = \sum (x - \bar{x})^2 P(X=k)$$

$$= \sum_{k=0}^{\infty} (k-\lambda)^2 \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \sum_{k=0} \frac{k^2 \lambda^k e^{-\lambda}}{k!} - 2\lambda^2 + \lambda^2$$

$$= \sum_{k=0} \frac{k^2 \lambda^k e^{-\lambda}}{k!} - \lambda^2.$$

$e^{-x^2} dx$

$$\frac{d}{d\lambda}\left(\frac{\lambda^k e^{-\lambda}}{k!}\right) = \frac{k\lambda^{k-1} e^{-\lambda} - \lambda^k e^{-\lambda}}{k!}$$

$$= 0 \Rightarrow \lambda = k \quad \therefore \mu$$

$$\frac{d^2}{d\lambda^2}\left(\quad \right) = \frac{k(k-1)\lambda^{k-2} e^{-\lambda} - k\lambda^{k-1} e^{-\lambda} + \lambda^k e^{-\lambda} - k\lambda^{k-1} e^{-\lambda}}{k!}$$

$$= 0 \Rightarrow k(k-1) - 2k\lambda + \lambda^2 \cancel{k\lambda} = 0$$

Scanned by CamScanner

Page No.
Date

5) $X \sim F_X$

$Y = I_{\{X>0\}}$

$$Y = \begin{cases} 1 & X > 0 \\ 0 & x \le 0 \end{cases}$$

$$F_Y(y) = P(Y \le y) = \begin{cases} 0 & y < 0 \\ P(X \le 0) = F_X(0) & 0 \le y < 1 \\ P(X>0) + P(X \le 0) = 1 & y \ge 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(0) & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

(2) (1) X - uniform random variable on $[0, 1]$

$Y = -\log X$

$Y \le y \Rightarrow -\log X \le y$

$\Rightarrow \log X \ge -y \Rightarrow x \ge e^{-y}$

$X \in [e^{-y}, 1]$

↳ Borel set ∴ Y is a r.v.

Scanned by CamScanner

Date

(1) $F_Y(y) = P(Y \le y)$

$= P(-\log X \le y)$

$= P(X \ge e^{-y})$

$= \int_{e^{-y}}^{1} f_X(x)\,dx = \underline{1 - e^{-y}}$

$(f_X(x) = 1)$

(2) X - continuous random variable with f_X

$Y = a f_X + b$.

$Y \le y \Rightarrow a f_X \le y - b$

$\Rightarrow f_X \le \frac{y-b}{a} \quad a > 0 \Rightarrow x \in (-\infty, x_0] \in B$

$f_X \ge \frac{y-b}{a} \quad a < 0 \Rightarrow x \in [x_1, \infty) \in B$

$\therefore Y$ is a r.v

$F_Y(y) = P(Y \le y)$

$= P(a f_X + b \le y) \cancel{\Rightarrow P(}$

$= \begin{cases} P\left(f_X \le \frac{y-b}{a}\right) & a > 0 \\ P\left(f_X \ge \frac{y-b}{a}\right) & a < 0 \end{cases}$

Scanned by CamScanner

(3) $X \geq 0$

$E[X] = \eta$

$$P(X > \varepsilon) \leq \frac{E X^k}{\varepsilon^k}$$

$$P(X > \sqrt{\eta}) \leq \frac{E X}{\sqrt{\eta}} = \frac{\eta}{\sqrt{\eta}} = \sqrt{\eta}$$

$$\Rightarrow P(X > \sqrt{\eta}) \leq \sqrt{\eta}$$

(4) X – Gaussian random variable with $\mu = 0$, $\sigma^2 = 4$, $Y = 3X^2$

mean & variance of Y

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{8\pi}} e^{-x^2/8}$$

$Y = 3X^2$

$$E[Y] = E[3X^2]$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} 3x^2 \cdot \frac{e^{-x^2/8}}{\sqrt{8\pi}} \cdot dx$$

$$\mathrm{var}(X) = E[(X - E[X])^2] = E[X^2] = 4$$

$$E[Y] = 3E[X^2] = 3 \times 4 = 12$$

Scanned by CamScanner

$$\mathrm{var}(Y) = E[(Y - E[Y])^2]$$

$$= \int_{-\infty}^{\infty} (3x^2 - 12)^2 \cdot \frac{e^{-x^2/8}}{\sqrt{8\pi}} \, dx$$

$$= \int_{-\infty}^{\infty} (9x^4 + 144 - 72x^2) \frac{e^{-x^2/8}}{\sqrt{8\pi}} \, dx$$

$$= 144 - (72 \times 4) + \frac{9}{\sqrt{8\pi}} \int_{-\infty}^{\infty} x^4 \frac{e^{-x^2/8}}{\sqrt{8\pi}} \, dx$$

$$= -144 + 48 \times 9$$

$$= \underline{288}$$

$$\int_{-\infty}^{\infty} x^4 \frac{e^{-x^2/8}}{\sqrt{8\pi}} \, dx \rightarrow \text{using Moment generating function.}$$

$$M(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2 X^2}{\lfloor 2} + \dots\right]$$

$$\int_{-\infty}^{\infty} x^4 f_X(x) \, dx = E[X^4] = M^{IV}(0)$$

Moment generating function of Gaussian distribution

$$M(t) = E[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\mu = 0, \sigma^2 = 4$$

$$M(t) = e^{2t^2}$$

$$M'(t) = 4t \, e^{2t^2} \qquad M''(t) = 4e^{2t^2} + 16t^2 e^{2t^2}$$

$$M'''(t) = 16t \, e^{2t^2} + 32t \, e^{2t^2} + 64t^3 e^{2t^2}$$

$$M^{IV}(0) = 16 + 32 = 48$$

Scanned by CamScanner

Page No.
Date

$$E\left[\left[g(x) - E[g(x)]\right]^2\right]$$

$$= E\left[\left(3x^2 - E[3x^2]\right)^2\right] = E[9x^4 + 144 - 72x^2]$$

$$= 9 \times 48 - 72 \times 4 + 144 = \underline{288}$$

(5) $X, Y \rightarrow$ two continuous random variables

$$E[\log f_X(X)] \geq E[\log f_Y(X)]$$

$\rightarrow \log x \leq x - 1 \quad \forall x > 0$

$$\Rightarrow \log \frac{f_Y(X)}{f_X(X)} \leq \frac{f_Y(X)}{f_X(X)} - 1$$

$$E\left[\log \frac{f_Y(X)}{f_X(X)}\right] \leq E\left[\frac{f_Y(X)}{f_X(X)}\right] - 1$$

$$E\left[\frac{f_Y(X)}{f_X(X)}\right] = 1$$

$$E\left[\log \frac{f_Y(X)}{f_X(X)}\right] \leq 0$$

$$\Rightarrow E[\log f_Y(X)] \leq E[\log f_X(X)]$$

→ Jensen's inequality -

$$E[g(X)] \geq g[E(X)] \quad \text{if } g \text{ is convex}$$

$y = -\log x \rightarrow$ convex

$$g(X) = -\log\left(\frac{f_Y(X)}{f_X(X)}\right)$$

$$E\left[-\log \frac{f_Y X}{f_X X}\right] \geq -\log\left[E\left(\frac{f_Y X}{f_X X}\right)\right]$$

$$E\left[\frac{f_Y X}{f_X X}\right] = 1$$

$$\therefore E\left[-\log \frac{f_Y X}{f_X X}\right] \geq 0$$

$$\Rightarrow E[\log f_X X - \log f_Y X] \geq 0$$

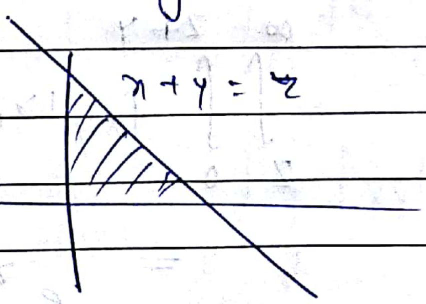
$$\Rightarrow E[\log f_X(X)] \geq E[\log f_Y(X)]$$

(3) (1) X, Y be random variables

$$f_{XY}(x, y) = e^{-(x+y)}$$

$$x, y > 0$$

$\boxed{X+Y}$



$$F_Z(z) = P(Z \leq z)$$

$$= P(X+Y \leq z)$$

$$= \int_0^z \int_0^{z-x} f_{XY}(x,y)\,dy\,dx = \int_0^z \int_0^{z-x} e^{-(x+y)}\,dy\,dx$$

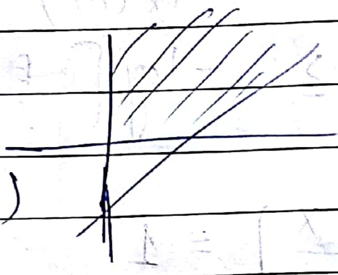
$$\int_0^z e^{-x}\int_0^{z-x} e^{-y}\,dy\,dx = \int_0^z e^{-x}\left[-e^{-y}\right]_{z-x}^{0} dx$$

$$= \int_0^z e^{-x}\left(1-e^{x-z}\right)dx = \int_0^z e^{-x} - e^{-z}\,dx$$

$$= \left[-e^{-x} - x e^{-z}\right]_0^z = \left(-e^{-z} - z e^{-z}\right) - (-1)$$

$$= 1 - e^{-z} - z e^{-z}$$

$\boxed{X - Y}$



$z > 0$

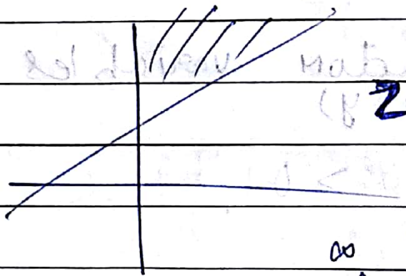
$$F_Z(z) = P(Z \le z)$$

$$= P(X - Y \le z) = P(X \le z + Y)$$

$$= \int_0^\infty \int_0^{z+y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^{z+y} e^{-(x+y)}\,dx\,dy = 1 - \frac{e^{-z}}{2}$$

$(z > 0)$



$z < 0$

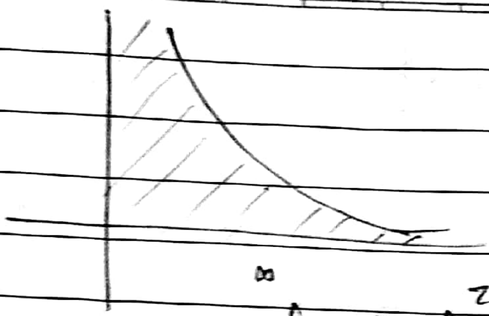
$$\int_z^\infty\int_0^{z+y} f_{XY}(x,y)\,dx\,dy$$

$$= e^{z}/2, \quad z < 0$$

$$F_Z(z) = \begin{cases} 1 - e^{-z}/2, & z > 0 \\ e^{z}/2, & z < 0 \end{cases}$$

Scanned by CamScanner

$\boxed{XY}$



$P(Z \le z) = P(XY \le z)$

$$\int_0^\infty \int_0^{z/y} f_{XY}(x,y)\,dx\,dy.$$

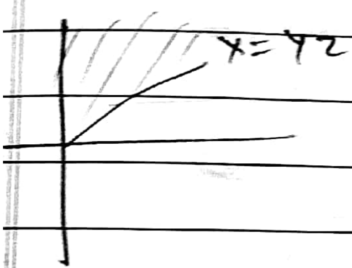
$$= \int_0^\infty e^{-y} \int_0^{z/y} e^{-x}\,dx\,dy$$

$$= \int_0^\infty e^{-y}\left(1 - e^{-z/y}\right) dy$$

$\boxed{X/Y}$

$F_Z(z) = P(Z \le z) = P(X/Y \le z)$

$= P(X \le zY)$



$$= \int_0^\infty \int_0^{yz} e^{-(x+y)}\,dx\,dy$$

$$= \frac{z}{1+z}$$

$\boxed{\min\{X, Y\}}$

$F_Z(z) = P(\min\{X, Y\} \le z)$

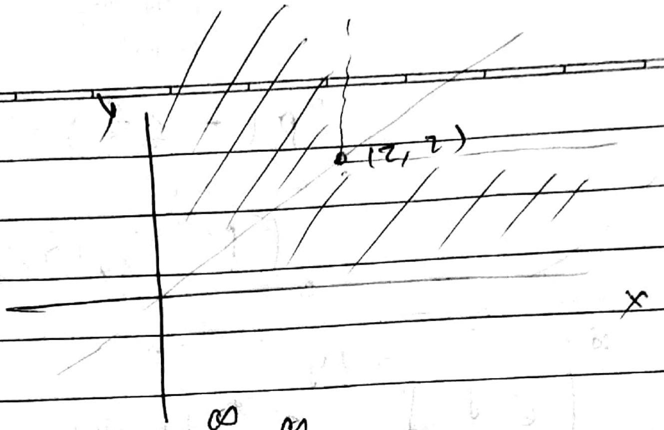
$$= \cancel{P} \begin{cases} P(X \le z) + \ ; & X \le Y \\ P(Y \le z) ; & Y \le X \end{cases}$$

$$= \int_0^z \int_x^\infty f_{XY}(x,y)\,dy\,dx$$

$$+ \int_0^z \int_y^\infty f_{XY}(x,y)\,dx\,dy$$

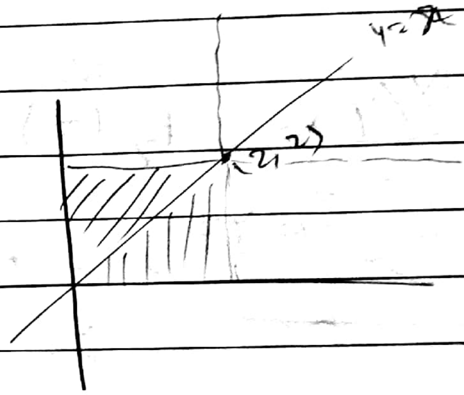
$$= 1 - e^{-2z}$$

Scanned by CamScanner



$$F_Z(z) = 1 - \int_z^{\infty}\int_z^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$\boxed{\text{Max}\{X, Y\}}$$



$Z = \text{Max}\{X, Y\}$

$F_Z(z) = P(\text{Max}\{X, Y\} \le z)$

$= P(X \le z,\ X \ge Y) + P(Y \le z,\ Y > X)$

$$= \int_0^z \int_0^x f_{XY}(x,y)\,dy\,dx + \int_0^z \int_0^y f_{XY}(x,y)\,dx\,dy$$

$= (1 - e^{-z})^2$.

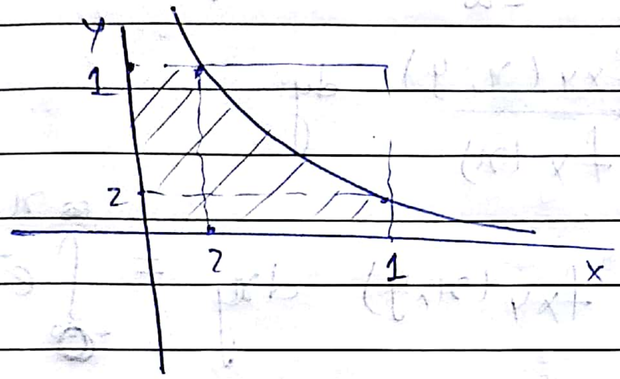
2) $f_{XY}(x,y) = 2(1-x) \quad \forall\ (x,y) \in [0,1]^2$

$= 0 \quad$ o.w

$f_Z, \quad Z = XY.$

$f_Z = \frac{\partial}{\partial z} F_Z(z).$

$F_Z(z) = P(Z \le z) = P(XY \le z)$



$$= \int_0^z \int_0^1 f_{XY}(x,y)\, dy\, dx + \int_z^1 \int_0^{1/x} f_{XY}(x,y)\, dy\, dx$$

$$= z^2 - 2z \ln z$$

$$f_Z = \frac{\partial}{\partial z} F_Z(z) = 2z - 2 - 2\ln z.$$

3) $g: R \to R$ be monotone increasing function

$Y = g(X)$,

$F_{XY} = P(X \le x, Y \le y)$

$= P(X \le x, X \le g^{-1}(y))$

$= P(X \le x, g(X) \le y)$

$= P(X \le \min\{x, g^{-1}(y)\})$

Scanned by CamScanner

Page No.
Date

4) $f_{XY}(x,y) = e^{-x} \quad 0 \le y \le x < \infty$

$= 0 \quad o.w$

$E[Y|X]$

$$E[Y|X] = \int_{-\infty}^{\infty} E[Y|X=x]\,\delta(X-x)\,dx$$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y \cdot f_Y(y|X=x)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{f_{XY}(x,y)}{f_X(x)}\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{0}^{x} e^{-x}\,dy$$

$$= x e^{-x}.$$

$$\int_{0}^{x} y \cdot \frac{e^{-x}}{x e^{-x}}\,dy = \frac{x}{2}$$

$$E[Y|X] = \int_{-\infty}^{\infty} \frac{x}{2}\,\delta(X-x)\,dx = \frac{X}{2}.$$

(5) $$E[Y|X \le 0] = \frac{1}{F_X(0)} \int_{-\infty}^{0} E[Y|X]\,f_X(x)\,dx$$

$$E[Y|X \le 0] = \int_{-\infty}^{\infty} y \cdot f_Y(y|X \le 0)\,dy$$

$$f_Y(y|X \le 0) = \frac{\partial}{\partial y} F_Y(y|X \le 0)$$

Scanned by CamScanner

$$= \frac{\partial}{\partial y} \frac{P(Y \le y, X \le 0)}{P(X \le 0)}$$

$$= \lim_{\Delta y \downarrow 0} \frac{P(X \le 0, y \le Y \le y + \Delta y)}{P(X \le 0) \quad \Delta y}$$

$$= \frac{1}{F_X(0)} \lim_{\Delta y \downarrow 0} \frac{1}{\Delta y} \int_{-\infty}^{0} \int_{y}^{y+\Delta y} f_{XY}(u, v)\, dv\, du .$$

$$= \frac{1}{F_X(0)} \lim_{\Delta y \downarrow 0} \frac{1}{\Delta y} \int_{-\infty}^{0} f_{XY}(u, y)\, du \cdot \Delta y$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} f_{XY}(u, y)\, du .$$

$$E[Y \mid X \le 0] = \int_{-\infty}^{\infty} y \cdot \frac{1}{F_X(0)} \int_{-\infty}^{0} f_{XY}(u, y)\, du \cdot dy$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{\infty} y \cdot \int_{-\infty}^{0} f_{XY}(u, y)\, du\, dy .$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} \int_{-\infty}^{\infty} y\, f_{XY}(u, y)\, du\, dy .$$

$$E[Y \mid X = x] = \int_{-\infty}^{\infty} y\, f_Y(y \mid X = x)\, dy$$

$$= \int_{-\infty}^{\infty} y \frac{f_{XY}(x, y)}{f_X(x)}\, dy$$

$$\cancel{E[Y|X] = \int_{-\infty}^{0}} = \frac{1}{F_X(0)} \int_{-\infty}^{0} \int_{-\infty}^{\infty} \frac{y\, f_{XY}(x, y)}{f_X(x)} \cdot f_X(x)\, dx\, dy$$

$$= \frac{1}{F_X(0)} \int_{-\infty}^{0} E[Y \mid X = x]\, f_X(x)\, dx .$$

Scanned by CamScanner

Date

4) X, Y - r.v defined on (Ω, F, P)

$X+c$, $\max\{X,Y\}$, $\min\{X,Y\}$, $X+Y$,
XY, X/Y are r.v.

- $Z = X+c$. $\quad Z \le z \Rightarrow X+c \le z$

$$X \le z - c$$

$$X \in (-\infty, z-c] \in \mathcal{B}$$

Z is a r.v

- $Z = \max\{X,Y\}$

$$= \begin{cases} X, & X \ge Y \\ Y, & Y \ge X \end{cases}$$

$$Z \le z \Rightarrow \begin{cases} X \le z, \; X \ge Y \Rightarrow x \in (-\infty, z] \; \& \; X \ge Y \\ Y \le z, \; Y \ge X \Rightarrow y \in (-\infty, z] \; \& \; Y \ge X. \end{cases}$$

Z is a r.v

- $Z = \min\{X, Y\}$

$$= \begin{cases} X, & X \le Y \\ Y, & Y \le X \end{cases}$$

$$Z \le z \Rightarrow \begin{cases} X \le z, \; X \le Y \Rightarrow X \in (-\infty, z] \; \& \; X \le Y \\ Y \le z, \; Y < X \Rightarrow \cancel{y} \Rightarrow Y \in (-\infty, z] \; \& \; Y < X. \end{cases}$$

Z is a r.v

Scanned by CamScanner

Page No.
Date

- $Z = X + Y$

$X + Y \le Z$

$X \le Z - Y$

let $q \in \mathbb{Q}$ s.t

$X(\omega) < z - q$

& $Y(\omega) < q$

$\therefore \{\omega \in \Omega : Z(\omega) < z\}$

$= \bigcup_{q \in \mathbb{Q}} \{\omega \in \Omega : \cancel{X}(\omega) \le z - q, \; Y(\omega) \cancel{\le} q\}$

$= \bigcup_{q \in \mathbb{Q}} \{\omega \in \Omega : Y(\omega) \le z - q, \; X(\omega) \le q\}$

$\in \mathcal{F}$

Since set of rationals is countable, a countable union of sets from $\mathcal{F}$ should also be in $\mathcal{F}$.

$\therefore Z$ is a r.v

- $Z = XY$

$XY \le z$

$X \le z/Y$

$\{\omega \in \Omega : Z(\omega) \le z\} = \bigcup_{q \in \mathbb{Q}} \left(\{\omega \in \Omega : X(\omega) \le \frac{z}{q}\} \cap \{\omega : Y(\omega) \le q\} \right)$

countable union of sets.

• $Z = X/Y$

$X/Y \leq z$

$\Rightarrow \quad X \leq zY$

$$\{\omega \in \Omega : Z(\omega) \leq z\} = \bigcup_{q \in \mathbb{Q}} \left(\{\omega \in \Omega : X(\omega) \leq zy\} \cap \{\omega \in \Omega : Y(\omega) \leq y\} \right)$$

$\in \mathcal{F}$.

Z is a r.v.

Scanned by CamScanner

Variance $Var(X) = E[(X-\mu)^2]$
$= E[X^2] - (E[X])^2$

Page No.
Date

Eg $E[g(X,Y)] = \sum_y \sum_x g(x,y)\, p(x,y)$
(discrete)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$
(continuous)

$Var(aX+b) = a^2\, Var(X)$

$Var(X+X) = 4\, Var(X)$
$Var(X+Y) = Var(X) + Var(Y) \rightarrow$ when X & Y are independent

Covariance
$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y$

$= E[XY] - E[X]E[Y]$

$Cov(X,Y) = Cov(Y,X)$
$Cov(X,X) = Var(X)$
$Cov(aX,Y) = a\, Cov(X,Y)$

$Cov(X+Z, Y) = Cov(X,Y) + Cov(Z,Y)$

$Cov\left(\sum_{i=1}^{n} X_i, Y\right) = \sum_{i=1}^{n} Cov(X_i, Y)$

$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} Cov(X_i, X_j)$

Scanned by CamScanner

$X, Y \rightarrow$ independant
$\Rightarrow \text{Cov}(X, Y) = 0$

for independant $X_1, X_2, \dots, X_n$

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$

Chebyshev's inequality

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

27/08

Recap
- Emperical distribution function

setting : $X_1, X_2, \dots, X_n$ iid
with $X_1 \sim F$

Define :
$$F_n^*(x) = \begin{cases} 0 & \text{if } x < X_{\sigma(1)} \\ k/n & \text{if } X_{\sigma(k)} \leq x < X_{\sigma(n)} \\ 1 & \text{if } X_{\sigma(n)} \leq x. \end{cases}$$

Fix $x \in R_e$

If we have ω; we define $X_1(\omega), X_2(\omega), \dots X_n(\omega)$.
correspondingly, we will have
$X_{\sigma 1}(\omega), X_{\sigma(2)}\omega, \dots X_{\sigma(n)}\omega$

and these are also random variables.
thus, -

$F_n^*(x)$ is a random variable
$F_n^*(x)$ is a discrete random variable
taking the values -

For any x, $F_n^*(x)$ is a function of ω.
(i.e fix $x \in \mathbb{R}$, $F_n^*(x)$ is a function of ω)

$$F_n^*(x) \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\}$$

$$P\left(F_n^*(x) = \frac{k}{n}\right) = P\left(X_{\sigma(k)} \le x < x_{\sigma(k+1)}\right)$$

$= P\{$ ~~excat~~ exactly k random variables take value $\le x$ & remaining $> x\}$

$$= \binom{n}{k} P(r.v \le x)^k \left(1 - P(r.v \le x)\right)^{n-k}$$

$$= \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}$$

$X_1, X_2, X_3 \sim \exp(\lambda)$ for $\lambda > 0$
fix $\omega_1 \in \Omega$
lets assume
$X_1(\omega_1) = 0.3$, $X_2(\omega_1) = 0.2$, $X_3(\omega_1) = 4$.
consider $x = 0.5$
$X_{\sigma(1)}(\omega_1) = 0.2$ $X_{\sigma(2)}(\omega_1) = 0.3$
$X_{\sigma(3)}(\omega_1) = 4$

$$F_n^*(x, \omega) = \frac{2}{3}$$

Take another $\omega_2 \in \Omega$ and suppose
$X_1(\omega_2) = 0.6$, $X_2(\omega_2) = 0.9$, $X_3(\omega_2) = 0.1$

$$F_n^*(x, \omega_2) = 1/3$$

$p = F(x)$
$$P\left(F_n^*(x) = \frac{k}{n}\right) = \binom{n}{k} p^k (1-p)^{n-k}$$

Scanned by CamScanner

$$E[F_n^*(x)] = \sum_{k=0}^{n} \frac{k}{n} \cdot \binom{n}{k} p^k (1-p)^k$$

$$= \frac{1}{n} \underbrace{\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}}_{np}$$

$$= p$$

$$\therefore E[F_n^*(x)] = p = F(x)$$

$$var(F_n^*(x)) = \frac{F(x)(1-F(x))}{n} \leq \frac{1}{4n}$$

$$var(X) = E[X^2] - E^2[X]$$

— x — x — x — x —

Weak law of large numbers

$$P(|F_n^*(x) - F(x)| > \varepsilon) \to 0 \quad \text{as } n \uparrow \infty$$
$$\forall \varepsilon > 0$$

Now, consider setting $X_1, X_2, X_3, \ldots, X_n$ i.i.d with $X_1 \sim F$, such that functional form of F is known.

~~$F \sim (0, \sigma^2)$~~ $F = \text{Gaussian}(0, \sigma^2)$

$= \text{Exponential}(\lambda)$ not known

$= \text{Bernoulli}(p)$

$= \text{Binomial}(n, p)$

Scanned by CamScanner

$X \sim \exp(\lambda)$

$E[X] = \frac{1}{\lambda}$

$\frac{1}{n} \sum X_i$ is a good estimate for $E[X]$

Hence, estimate λ as

$$\lambda = 1 / \tfrac{1}{n} \sum X_i$$

30/08

Recap

- Emperical dist. function $F_n(x)$.
- $E[F_n(x)] = F(x) \quad \forall x \in \mathbb{R}_e$
- $var[F_n(x)] \leq 1/4n \quad \forall x \in \mathbb{R}_e$
- $F_n(x) \to F(x)$ weakly as $n \uparrow \infty \quad \forall x \in \mathbb{R}_e$.

Parameter Point estimation

Ex:

- $X \sim \exp(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x} \quad \forall x \in \mathbb{R}_{e+}$$
$$= 0 \quad o.w$$

Exponential dist. has a single parameter $\lambda > 0$

- Poisson r.v

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad \forall k = 0, 1, 2, \ldots$$

parameter $\lambda > 0$.

- Gaussian r.v

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2}$$

parameter (μ, σ^2)

$\mu \in \mathbb{R}_e$, $\sigma^2 \in \mathbb{R}_{e+}$

Scanned by CamScanner

Date

- Binomial

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\forall\ k = 0, 1, \ldots, n$$

$$(p, n) \in [0,1] \times \{1, 2, \ldots\}$$

parameter pt. est.

setup:

~~identical~~ ~~distributio~~ independant

independently identically distributed

Given: $X_1, X_2, \ldots, X_n$ i.i.d s.t

$X_1 \sim f$ with functional form of f is known. However, exact parameter values are unknown.

Statistic:

Defⁿ - A function $T: R_n \to R_k$ is called a statistic if a r.v $T(X_1, \ldots, X_n)$ does not contain any unknown parameters.

$$T_1(X_1, X_2, \ldots, X_n) = \frac{1}{n} \sum_{k=1}^{n} (X_k - \mu)^2$$

$$T_2(X_1, X_2, \ldots, X_n) = \frac{1}{n} \sum_{k=1}^{n} X_k^2 - \left(\frac{1}{n} \sum_{k=1}^{n} X_k\right)^2$$

$$\to E[X^2] - (E[X])^2$$

T_1 is not a statistic

T_2 is a statistic.

Let dist. f has parameters $\vec{\theta}$. Θ denotes parameter space.

- Exponential $\theta = \lambda$, $\Theta = R_{e+}$

- Gaussian $\vec{\theta} = (\mu, \sigma^2)$, $\Theta = R_e \times R_{e+}$
- Binomial $\vec{\theta} = (p, n)$, $\Theta = [0,1] \times \mathbb{N}$.
- Geometric $\theta = p$, $\Theta = [0,1]$.

Def^n: A statistic $\delta(\vec{X})$ is said to be a point estimator of $\vec{\theta}$ if $\delta: R^n \to \Theta$

~~$\vec{X} = [x_1, x_2, \ldots, x_n]$~~

$\vec{X} = [x_1, x_2, \ldots, x_n]$

A single unknown parameter (scalar θ)

$$\cancel{|\delta(\vec{X}) - \theta|^2}$$

Suppose we know θ, and still try to estimate it

$$E_\theta[|\delta(\vec{X}) - \theta|^2]$$

$x_1, \ldots, x_n \sim exp(1/\lambda)$ i.i.d

$$\delta(\vec{X}) = \frac{1}{n}\sum_{k=1}^{n} x_k$$

assume, $\lambda = 1$

$$E_\lambda[|\delta(\vec{X}) - \lambda|^2]$$

$$= \int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \left[\frac{1}{n}\sum_{k=1}^{n} x_k - 1\right]^2 \prod_{k=1}^{n} e^{-x_k}\, dx_n \ldots dx_1$$

for $\lambda = 2$

$$= \int_{x_1=0}^{\infty} \cdots \int_{x_n=0}^{\infty} \left[\frac{1}{n}\sum_{k=1}^{n} x_k - 2\right]^2 \prod_{k=1}^{n} e^{-x_k/2}\, dx_n \ldots dx_1$$

Scanned by CamScanner

Desired properties
for given θ,
$MSE_\theta(\delta)$ should be as small as possible.

want estimator δ to have small $MSE_\theta(\delta)$ for every $\theta \in \Theta$.
We want δ^* s.t $\forall \theta \in \Theta$
$$MSE_\theta(\delta^*) \le MSE_\theta(\delta)$$
for any other estimator δ.

03/09

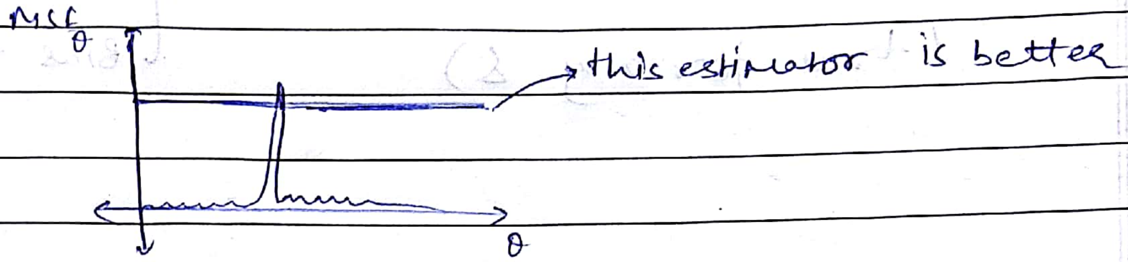
Parameter Point Estimator
Problem statement -
- Given $X_1, X_2, \ldots, X_n$ i.i.d s.t $X_i \sim F$ with parameter θ
- Functional form of F is known, but θ is unknown.
- Find "best" estimate for θ.
- $\theta \in \Theta$, where Θ is parameter space
- $\delta: \mathcal{X}^n \to \Theta$ is called estimator for θ if its a statistic

Possible "goodness" measures
$$-E_\theta[|\delta(\vec{X}) - \theta|^2] = MSE_\theta(\delta)$$
$$-P_\theta(|\delta(\vec{X}) - \theta| \le \varepsilon)$$

Find δ s.t
$$MSE_\theta(\delta) \le MSE_\theta(\delta')$$
$\forall$ estimator δ' & $\forall \theta \in \Theta$

Scanned by CamScanner

$$\sup_{\theta \in \Theta} MSE_\theta(\delta) \leq \sup_{\theta \in \Theta} MSE_\theta(\delta')$$



$$E_\theta[\delta(\vec{X})] = \theta$$

Bernoulli samples (p)

$$\Theta = [0,1]$$

$$\delta(\vec{X}) = \frac{1}{n}\sum_{k=1}^{n} x_k$$

$$E_p[\delta(\vec{X})] = \frac{1}{n}\sum_{k=1}^{n} E_p[x_k]$$

$$= p \quad \forall\ p \in [0,1]$$

$$MSE_\theta(\delta) = E_\theta[|\delta(\vec{X}) - \theta|^2]$$

$$= E_\theta\left[|(\delta(\vec{X}) - E_\theta[\delta(\vec{X})]) + (E_\theta[\delta(\vec{X})] - \theta)|^2\right]$$

$$= E_\theta[(\delta(\vec{X}) - E_\theta[\delta(\vec{X})])^2]$$

$$= var_\theta(\delta(\vec{X}))$$

Scanned by CamScanner

$MSE_\theta(\vec{\delta})$

$= var_\theta(\delta(\vec{X})) + (E_\theta[\delta(\vec{X}) - \theta]^2)^2$

(true for any δ)

Bias function

Scanned by CamScanner