$< \frac{\pi^2}{6} n^2$

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emperical distn'n fn setting $x_1, x_2 \dots x_n$ with $x_1 \sim F$.

Defn $F_n^*(x) = \begin{cases} 0 & \text{if } x < X_{\sigma(1)} \\ k/n & X_{\sigma(k)} \le x < X_{\sigma(k+1)} \\ 1 & X_{\sigma(n)} \le x \end{cases}$

if ω: $x_1(\omega), x_2(\omega) \dots x_n(\omega)$ fixed $\Rightarrow$ $X_{\sigma(1)}(\omega), x_{\sigma 2}(\omega) \dots$ fixed.

then $F_n^*(x)$ is a discrete random variable.

$\hookrightarrow \{0, \frac{1}{n}, \dots 1\}$

complete f

R.V. for a given fixed $x \in \mathbb{R}$.

$P\left(F_n^*(x) = \frac{k}{n}\right) = P\left(X_{\sigma(k)} \le x < X_{\sigma(k+1)}\right)$

$= P$ [Exactly k R.V. take value $\le x$ & rest $> x$)

$= \binom{n}{k} P(\text{r.v.} \le x)^k (1 - P(\text{r.v} \le x))^{n-k}$

$= \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}$.

$= {}^nC_k\, p^k (1-p)^{n-k}$.

eg: $X_1, x_2, X_3 \sim \exp(\lambda)$ for $\lambda > 0$

for fixed $\omega_1 \in \Omega$. $x_1(\omega_1) = 0.3$ $x_2(\omega) = 0.2$, $x_3(\omega) = 4$.

let $x = 0.5$

$\therefore X_{\sigma(1)}(\omega_1) = 0.2$, $x_{\sigma(2)}(\omega_1) = 0.3$, $X_{\sigma(3)}(\omega_1) = 4$

$F_n^*(x, \omega_1) = \frac{2}{3}$.

$x_1(\omega_2) = 0.6$, $x_2(\omega_2) = 0.9$, $x_3(\omega_2) = 0.1 \Rightarrow$

$F_n^*(x, \omega_2) = \frac{1}{3}$.

$\therefore E[F_n^*(x)] = \cancel{n F(x)} \sum_{k=0}^{n} \frac{k}{n} \cdot {}^nC_k \cdot p^k (1-p)^k$

$= p = F(x)$.

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$$var(F_n^*(x)) = \sum \left(\frac{k}{n}\right)^2 \binom{n}{k} p^k (1-p)^{n-k} - (E[\cdot])^2$$

$$= \frac{1}{n^2}\left(\sum k^2 \binom{n}{k} p^k (1-p)^{n-k}\right)$$

$$= \frac{F(x)(1-F(x))}{n} \leq \frac{1}{4n}$$

$\therefore$ For large n, $var \to 0$.

$\therefore$ By weak law of large numbers.

$$P(|F_n^*(x) - F(x)| > \varepsilon) \to 0 \text{ as } x \uparrow \infty \quad \forall\ \varepsilon > 0.$$

→ If functional form of F is known

eg: F ~ Gaussian $(0, \sigma^2)$ — not known!

→ Parameter ~~local~~ Point estimation.

Eg: $X \sim exp(\lambda)$ ie: $f_X(x) = \lambda e^{-\lambda x}\ \forall\ x \in R^+$
$= 0$ otherwise

- Exponential dist. has single parameter $\lambda > 0$.

Poisson R.V. $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}\ \forall\ k = 0, 1, 2, 3, \ldots$

Parameter $\lambda > 0$.

→ Gaussian R.V.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Parameters (μ, σ^2) where $\mu \in \mathbb{R}$ & $\sigma^2 \in \mathbb{R}^+$

→ Binomial R.V.

$$P(X=k) = {}^nC_k\, p^k (1-p)^{n-k}$$

Parameters (p, n) where $p \in [0,1]$ & $n \in \mathbb{N}$

If moment gen. func^n exist, distribu^n can be found out.

eg: All moments of Gaussian are $f(\mu, \sigma^2)$.

Parameter point estimation

setup: given: $X_1, X_2 \ldots X_n$ i.i.d s.t. $X_i \sim F$ with functional form of F is known. However, exact parameter value are unknown.

Statistic:

Def^n: A fn $T: \mathbb{R}_n \to \mathbb{R}_k$ is called a statistic if a R.V. $T(X_1, X_2 \ldots X_n)$ doesn't contain any unknown parameter.

eg: $T_1(X_1, \ldots X_n) = \frac{1}{n} \sum_{k=1}^{n} (X_k - \mu)^2$ (var) → Not a statistic.

$T(X_1, \ldots X_n) = \frac{1}{n} \sum_{k=1}^{n} X_k^2 - \left(\frac{1}{n} \sum_{k=1}^{n} X_k\right)^2$ - statistic.

$\to E[X^2] - (E[X])^2$

Let distr. F has parameters $\vec{\theta}$.

Θ denotes parameters space

eg: exponential $\vec{\theta} = \lambda$ $\quad \Theta = \mathbb{R}^+$

Gaussian $\vec{\theta} = (\mu, \sigma^2)$ $\quad \Theta = \mathbb{R} \times \mathbb{R}^+$

Binomial $\vec{\theta} = (p, n)$ $\quad \Theta = [0,1] \times \mathbb{N}$

Geometric $\vec{\theta} = p$ $\quad \Theta = [0,1]$

Uniform $\vec{\theta} = [a, b]$ $\quad \Theta = \mathbb{R} \times \mathbb{R}$

Def^n: A statistic $\delta(\vec{X})$ is said to be a point estimator of $\vec{\theta}$ if $\delta: \mathbb{R}^m \to \Theta$

$\vec{X} = [X_1\ X_2 \ldots X_n] \to$ R.V.s

Ideally $\delta(\vec{X})$ should be the actual parameter value.

closeness of δ to actual parameter ⇒

$|\delta(\vec{X}) - \theta|^2 \to$ R.V.

Fixed unknown

$$\Rightarrow E_\theta\left[\,|\delta(\vec{x}) - \theta|^2\,\right] = MSE_\theta(\delta)$$

$X_1, X_2 \dots X_n \sim \exp(1/\lambda)$ i.i.d.

$$\delta(\vec{x}) = \frac{1}{n}\sum_{k=1}^{n} x_k$$

For $\lambda = 1$

$$E_\lambda\left[\,|\delta_{\vec{x}} - \lambda|^2\,\right] = \cancel{\int_0^\infty \cdots \int_0^\infty \frac{1}{n} \sum_{k=1}^{n} x_k}$$

$$= \int_{x_1=0}^{\infty} \cdots \int_{x_n=0}^{\infty} \left[\frac{1}{n} \sum_{k=1}^{n} x_k - 1\right]^2 \prod_{k=1}^{n} 1 \cdot e^{-1(x_k)} \, dx_n \dots dx_1$$

Likewise check for diff samples

∴ For given θ,
$MSE_\theta(\delta)$ shud be as small as possible.

∴ Want estimator δ to have small $MSE_\theta(\delta)$ for every $\theta \in \Theta$. I want δ^* s.t. $\forall\, \theta \in \Theta$ $MSE_\theta(\delta^*) \le MSE_\theta(\delta)$ for any other estimator δ.

not satisfying this

eg: $\Theta = \mathbb{R}_0$

$\delta_1(\vec{x}) = 1$ — this estimator minimizes MSE for $\theta = 1$

$\delta_2(\vec{x}) = 2$ — minimize for $\theta = 2$.

∴ try minimise $MSE_\theta(\delta)\ \forall \theta \in \Theta$

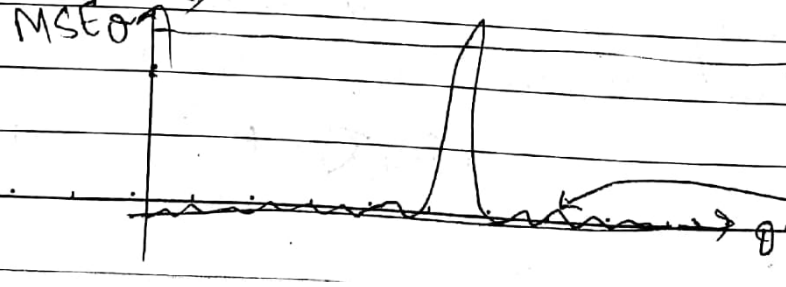
(or) $\cancel{\sup_{\theta \in \Theta} MSE_\theta(}$

~~Thus, restrict the class~~ of estimators.

Find δ s.t. $MSE_\theta(\delta) \le MSE_\theta(\delta')$
$\forall$ estimator δ' & $\forall\, \theta \in \Theta$

$$\Rightarrow \sup_{\theta \in \Theta} MSE_\theta(\delta) \le \sup_{\theta \in \Theta} MSE_\theta(\delta')$$

eg:



This'd be preferred than

$[\theta-\varepsilon \quad \theta \quad \theta+\varepsilon]$

Other criteria: $P_\theta\left(|\delta(\vec{x}) - \theta| \le \varepsilon\right)$

Else: take θ as R.V.: its distribu^n given by Θ - distribu^n

Atleast, $E_\theta[\delta(\vec{x})] = \theta$.

eg: Bernoulli R.V. $\Rightarrow \Theta = [0,1]$.

let $\delta(\vec{x}) = \frac{1}{n}\sum_{k=1}^{n} x_k$ → statistic.

$$\therefore E_\theta[\delta(\vec{x})] = \int_0^1 \theta \cdot \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) d\theta$$

$$E_p[\delta(\vec{x})] = \frac{1}{n}\sum_{k=1}^{n} E_p[x_k] = p. \quad \forall\, p \in [0,1]$$

Since $\delta(\vec{x})$ is estimator $[\mathbb{R}^n \Rightarrow \Theta]$.

$\delta(\vec{x}) = \frac{1}{n}\sum x_k^2$ → also satisfies

$\cancel{\frac{1}{n}\sum_{k=1}^{n} x_k^2} =$

$$MSE_\theta(\delta) = E_\theta\left[|\delta(\vec{x}) - \theta|^2\right]$$

$$= E_\theta\left[|\delta(\vec{x}) - E_\theta(\delta(\vec{x})) + E_\theta[\delta(\vec{x})] - \theta|^2\right]$$

$$= E_\theta\left[(\delta(\vec{x}) - E_\theta[\delta(\vec{x})])^2\right]$$

$$= Var_\theta(\delta(\vec{x}))$$

$\therefore$ Best estimator minizes the variance

General case: $MSE_\theta(\vec{\delta}) = (Var_\theta(\delta(\vec{x}))) + \underbrace{\left(E_\theta[\delta(\vec{x})] - \theta\right)^2}_{\text{Bias } f^n}$

True for any δ

Unbiased estimator can've small var.