~~MIE~~ Q. $X_1, X_2, \dots X_n$ are independent & Identically distributed random variable with $X_i \sim F$.
Design F. [σ is a bijection which g arranges the values in ascending order]

$$F_n^*(x) = \begin{cases} 0 & \text{if } x < X_{\sigma(1)} \\ K/n & \text{if } X_{\sigma(K)} \le x < X_{\sigma(K+1)} \\ 1 & \text{if } X_{\sigma(n)} \le x \end{cases}$$

~~How could~~

fix $x \in \mathbb{R}$, $F_n^*(x)$ is a random variable $\in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$

$P(F_n^*(x))$ = exactly K points lie before x

$$I_{\{x_j \le x\}} = 1 \quad \text{if } x_j \le x$$
$$= 0 \quad \text{o. w.}$$

$$\therefore \left\{P(F_n^*(x)) = \frac{K}{n}\right\} = \left\{\sum_{j=1}^{n} I_{\{x_j \le x\}} = K\right\} = \binom{n}{K} P(r.v \le x)^K (1 - P(r.v) \le x)^{n-K}$$

$\because$ All are independent.

$\boxed{\star}$ Theorem = Let $g: \mathbb{R}^n \to \mathbb{R}^m$ be a Borel measureable function; that is, if $\mathbb{B} \in \mathbb{B}^m$, then $g^{-1}(B) \in \mathbb{B}^n$. If $X = (X_1, \dots X_n)$ is an n dimensional RV $(n \geq 1)$, then $g(X)$ is an m dimensional RV

Proof = For $B \in \mathbb{B}^m$,

$$(g(X_1, X_2, \dots, X_n) \in \mathbb{B}) = ((X_1, X_2 \dots X_n) \in g^{-1}(B))$$

since $g^{-1}(B) \in \mathbb{B}_n$, it follows that $\{(X_1, X_2 \dots X_n) \in g^{-1}(B)\}$ which concludes the proof.

$= \binom{n}{k} (F(x))^{k} (1-F(x))^{n-k}$ Binomial Distribution.

example $X_1 \; X_2 \; X_3 \sim \exp(\lambda)$ for $\lambda > 0$

for $\omega_1 \in \Omega$

let's assume $X_1(\omega_1) = 0.3$ $X_2(\omega_1) = 0.2$ $X_3(\omega_1) = 4$

consider $x = 0.5$

$X_{\sigma_1}(\omega_1) = 0.2$

$X_{\sigma_2}(\omega_1) = 0.3$

$X_{\sigma_3}(\omega_1) = 4$

$\therefore F_n^*(x, \omega) = 2/3$

Take another $\omega_2 \in \Omega$

$X_1(\omega_2) = 0.6$ $X_2(\omega_2) = 0.9$ $X_3(\omega_2) = 0.1$

$F_n^*(x, \omega) = 1/3$

$$E[F_n^*(x, \omega_2)] = \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{1}{n} \sum_{k=0}^{n} (k) \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \left(\frac{1}{n}\right)(np) = p = F_X(x)$$

$$\operatorname{var}(F_n^*(x)) = \frac{F_X(x)\,(1-F_X(x))}{n} \leq \frac{1}{4n}$$

$\therefore$ if $n \uparrow \infty$ $\operatorname{Var}(F_n^*(x)) \downarrow 0$

weak law of large Numbers:

$P(|F_n^*(x) - F(x)| > \varepsilon) \to 0$ as $n \uparrow \infty$ $\forall \varepsilon > 0$

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Parameter Point estimation

setup:

Given $X_1, X_2 \cdots X_n$ i.i.d (independent & identically distributed) s.t $X_1 \sim F$ with functional form of F is known. However, exact parameter values are unknown.

Statistic:

A function $T: \mathbb{R}^n \to \mathbb{R}^k$ is called a statistic if a r.v $T(X_1 \cdots X_n)$ does not contain any unknown parameters

$T_1(X_1 \cdots X_n) = \frac{1}{n} \sum_{k=1}^{n} (x_k - \mu)^2$ is not a statistic

$T_2(X_1 \cdots X_n) = \frac{1}{n} \sum_{k=1}^{n} x_k^2 - \left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)^2$ T_2 is a statistic

Let distr. F has parameters $\vec{\theta}$

Θ denotes parameter space

- Exponential $\theta = \lambda$ $\Theta = \mathbb{R}^+$
- ~~Poi~~ Gaussian $\theta = (\mu, \sigma^2)$ $\Theta = \mathbb{R} \times \mathbb{R}^+$
- Poisson $\theta = (p, n)$ $\Theta = [0,1] \times \mathbb{N}$
- Geometric $\theta = p$ $\Theta = [0,1]$.

Def^n: A statistic $\delta(\bar{x})$ is said to be point estimator of $\vec{\theta}$ iff $\delta: \mathbb{R}^m \to \Theta$

$\bar{x} = [x_1\ x_2\ \cdots\ x_n]$

A singular unknown parameter (scalar θ) out constant.

$E_\theta |\delta(\bar{x}) - \theta|^2 = MSE_\theta(\delta) = P_\theta(|\delta(\vec{x}) - \theta| \le \varepsilon)$

we have to minimise this.

$MSE =$

eg $X_1\ X_2 \dots X_n \sim \exp(1/\lambda)$

$$\delta(\bar{x}) = \frac{1}{n}\sum_{k=1}^{n} X_k$$

Assume $\lambda = 1$

$$E_\theta[\,|\,\delta\bar{x} - \theta|^2] = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n\text{-integrals}} \left[\frac{1}{n}\sum (x_k - \lambda)\right]^2 \cdot \prod_{k=1}^{n} e^{-\lambda}\, dx_1 \dots dx_n$$

Desired Properties

- for given θ, MSE_θ should be as small as possible
- want estimator δ to have small $MSE_\theta(\delta)$ for every $\theta \in \Theta$

I want δ^* ST $\theta \in \Theta$

$$MSE_\theta(\delta^*) \le MSE_\theta(\delta) \quad \text{for any other } \delta.$$

Designing such a δ is very difficult

$\therefore$ we try for

$$\sup_{\theta \in \Theta} MSE_\theta(\delta) \le \sup_{\theta\in\Theta} MSE_\theta(\delta^*)$$

We can have some δ such that $E_\theta[\delta(\bar{x})] = \theta$

example in Bernoulli samples ($\theta = p$)

$$\Theta = [0,1]$$

$$\delta(\vec{x}) = \frac{1}{n}\sum_{k=1}^{n} x_k \quad [\,\delta(\bar{x}) \text{ is a statistic}\,]$$

$$E_p[\delta(\bar{x})] = \frac{1}{n}\sum_{k=1}^{n} E_p[X_k]$$

$$= p \quad \forall\ p \in [0,1]$$

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$MSE_\theta(\delta) = E_\theta[|\delta(\bar{x}) - \theta|^2] = E_\theta[|\{\delta(\bar{x}) - E_\theta[\delta(\bar{x})]\} + \{E_\theta[\delta(\bar{x})] - \theta\}|^2]$

$\because E_\theta[\delta(\bar{x})] = \theta$

$MSE_\theta(\delta) = E_\theta[|\delta(\bar{x}) - E_\theta[\delta(\bar{x})]|^2] = Var_\theta(\delta(\vec{x}))$

for any other θ

$$MSE_\theta(\delta) = Var_\theta(\delta(\bar{x})) + [\underbrace{E_\theta[\delta(\bar{x})] - \theta}_{bias}]^2$$

$\therefore$ unbiased estimator may not give the least $MSE_\theta(\bar{\delta})$