Solutions of Quiz and Evalutaion Scheme

1. (i) The function is differentiable only at the origin.

Let $f = u + \imath v$. Then $u(x,y) = e^{x^2+y^2}$ and $v(x,y) = 0$ for all (x,y). Since $u_x = 2e^{x^2+y^2}x$, $u_y = 2e^{x^2+y^2}y$ and $v_x = v_y = 0$, f does not satisfy C-R equation except origin. Thus f is not differentiable on $\mathbb{C} \setminus \{0\}$. [2 Marks]

On the other hand,

$$\lim_{z\to 0} \frac{e^{|z|^2} - 1}{z} = \lim_{z\to 0} \frac{e^{|z^2|} - 1}{|z|^2} \lim_{z\to 0} \frac{|z|^2}{z} = 0.$$

[1 Mark]

(ii) Note that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0$. Since the limit exists, the radius of convergence of the series $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$. [2 Marks]

2. Given $u(x,y) = x^3 - kxy^2 + 12xy - 12x$, it is easy to check that $u_{xx} = 6x$ and $u_{yy} = -2kx$. Now if u is a real part of a holomorphic function then u is harmonic. This leads to $(6 - 2k)x = 0$ for all x and therefore $k = 3$. [2 Marks]

For $k = 3$, $u(x,y) = x^3 - 3xy^2 + 12xy - 12x$. Let v be a real valued function such that $u + \imath v$ is holomorphic. Then by C-R

$$v_x = 6xy - 12x \quad \text{and therefore } v(x,y) = 3x^2y - 6x^2 + \phi(y)$$

for some function ϕ. Again by C-R we have

$$3x^2 - 3y^2 + 12y - 12 = 3x^2 + \phi'(y) \Rightarrow \phi(y) = -y^3 + 6y^2 - 12y + c,$$

for some constant c. From the above display equations we get $v(x,y) = 3x^2y - 6x^2 - y^3 + 6y^2 - 12y + c$. Thus u is real part of the holomorphic function

$$f(x,y) = x^3 - 3xy^2 + 12xy - 12x + \imath(3x^2y - 6x^2 - y^3 + 6y^2 - 12y + c).$$

[2 Marks]

Note that $f'(x,y) = u_x(x,y) + \imath v_x(x,y) = (3x^2 - 3y^2 + 12y - 12) + \imath(6xy - 12x)$, $f''(x,y) = 6x + \imath(6y - 12)$ and $f'''(x,y) = 6$. Then $f(0) = \imath c$, $f'(0) = -12$, $f''(0) = -12\imath$, $f'''(0) = 6$ and $f^{(n)}(0) = 0$ for all $n \geq 4$. Thus we get $f(z) = \imath c - 12z - 6\imath z^2 + z^3$.

[2 Marks]

3. Let O be the origin, and let A and B be the other vertices of the triangle on the X-axis and Y-axis respectively. Since the imaginary part of z is 0 on the line segment $\overrightarrow{OA}$, then $\int_{\overrightarrow{OA}} \text{Im} z \, dz = 0$. [1 Mark]

By considering the parametrization $\{z(t) = (1-t) + \imath t : 0 \leq t \leq 1\}$ of the line segment $\overrightarrow{AB}$,

$$\int_{\overrightarrow{AB}} \mathrm{Im} z \; dz = \int_0^1 t(-1+\imath)dt = -1/2 + \imath/2.$$

[1 Mark]

Finally, using the parametrization $\{z(t) = 0 + \imath(1-t) : 0 \leq t \leq 1\}$ of the line segment $\overrightarrow{BO}$, we get

$$\int_{\overrightarrow{BO}} \mathrm{Im} z \; dz = \int_0^1 -\imath(1-t)dt = -\imath/2.$$

[1 Mark]

Thus the value of the integral is $-1/2$. [1 Mark]