EXAMPLE 11

Consider the network of Fig. 3-36. For this example, let us write the Kirchhoff voltage equations in chart form, where the first row of the chart is equivalent to the equation

$$0 = 4i_1 - i_2 + 0i_3 - i_4 + 0i_5 + 0i_6 + 0i_7 + 0i_8 + 0i_9 \quad (3\text{-}57)$$

			Coefficient of								
Eq.	Voltage		i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9
1	0	=	4	−1	0	−1	0	0	0	0	0
2	1	=	−1	5	−1	0	−1	0	0	0	0
3	0	=	0	−1	4	0	0	−1	0	0	0
4	−1	=	−1	0	0	5	−1	0	−1	0	0
5	0	=	0	−1	0	−1	4	−1	0	−1	0
6	0	=	0	0	−1	0	−1	5	0	0	−1
7	1	=	0	0	0	−1	0	0	4	−1	0
8	0	=	0	0	0	0	−1	0	−1	5	−1
9	0	=	0	0	0	0	0	−1	0	−1	4

Again, this chart may be constructed in a very simple manner. All terms of the principal diagonal of the chart are found as the summation of resistance around each of the nine loops. The terms off the diagonal are all negative, and are all the value of the resistance common to the two loops being considered, identified by the row number (equation number) and the column number (subscript of the current).

From the chart, or from the corresponding matrix of the form of Eq. (3-48), observe: (1) The elements on the principal diagonal are all positive; all others are negative or zero. (2) There is symmetry about the principal diagonal. This symmetry and the sign rule always apply when loops are drawn in the same clockwise or counterclockwise direction. These observations for one example turn out to describe the general case, in the absence of controlled sources.

What about mutual inductance and controlled sources? Mutual inductance will present no problems as shown by the examples of the last section, and the symmetry observations will also hold since $M_{ij} = M_{ji}$. The presence of controlled sources is another matter. Such sources will give rise to terms of the form $v_j = ki_k$ which will appear in the summation around the loop containing v_j but not in the loop defining i_k. Writing the equations will not be a problem, but the rules of symmetry and sign that we have observed will usually not hold (exceptions exist) in the presence of controlled sources. This topic will be explored in greater depth in Chapter 9 in connection with our study of reciprocity.

3-6. NODE VARIABLE ANALYSIS

Consider a network with n nodes and only one part. As discussed in Section 3-3, there are $n - 1$ independent node pairs. Of the many possibilities for node-pair variables, we will select the node-to-datum voltages as our variables exclusively. The form of the voltages for the branch connecting node j to node k with node j positive will be $v_j - v_k$ (from Kirchhoff's voltage law). For each of the $n - 1$ nodes at which the Kirchhoff current law will be formulated, we will assume that currents are directed *out* of the node to be consistent with the voltage sign assignment we have just made. We recall from our earlier discussion that this is an arbitrary choice, and that selecting the other alternative is equivalent to multiplying the resulting equations by -1.

We will follow the practice of converting all voltage sources into equivalent current sources as preparation of the network preceding the writing of the equations. Let us postpone consideration of mutual inductance and controlled sources, and consider a passive network made up of resistors, capacitors, and inductors. Note first that for elements connected as shown in Fig. 3-37, the elements may be replaced by an equivalent system made up as follows: (1) all parallel capacitances replaced by an equivalent capacitance of value $C_{kj} = C_1 + C_2 + \ldots$; (2) an equivalent resistance found by adding conductances as $G_{kj} = 1/R_{kj} = G_1 + G_2 + \ldots$; and (3) an equivalent inductance of value L_{kj}, where $1/L_{kj} = 1/L_1 + 1/L_2 + \ldots$. Applying this network simplification to the elements from node k to all other nodes from $j = 1$ to $j = N$, we have the equation

$$\sum_{j=1}^{N}\left(G_{kj} + C_{kj}\frac{d}{dt} + \frac{1}{L_{kj}}\int dt\right)v_j = i_k, \qquad k = 1, 2, \ldots, N \tag{3-58}$$

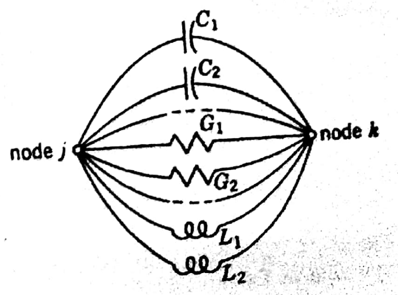

Fig. 3-37. Elements connecting nodes j and k. The three kinds of elements may be combined to give an equivalent parallel *RLC* network between nodes j and k.

The expansion of ... with a's replaced ... and v's by the loop case, Eq. (3-47), with a's replaced ... i's.

In applying this equation to networks, it is not necessary to simplify the network by combining elements. At node j, the capacitance C_{jj} is the sum of the capacitance *connected* to node j or the capacitance from node j to ground with all other nodes grounded. The value of C_{kj} is the sum of the capacitances *connected between* node j and node k or the capacitance from node j to node k with all other nodes grounded. Similar instructions hold for inverse inductance $1/L$ and for conductance $G = 1/R$. Coefficients can thus be found by inspection by simply noting which elements are "hanging on" or "hanging between" the various nodes.

If the same convention for positive current is maintained in formulating all node equations for a network, the sign of b_{kj} will be positive when $k = j$, and negative when $k \neq j$.

EXAMPLE 12

A network with two independent node pairs is shown in Fig. 3-38. For this network, Kirchhoff's current law is

$$\sum_{j=1}^{2} b_{kj} v_j = i_k, \qquad k = 1, 2 \tag{3-61}$$

or

$$b_{11}v_1 + b_{12}v_2 = i_1, \qquad b_{21}v_1 + b_{22}v_2 = i_2 \tag{3-62}$$

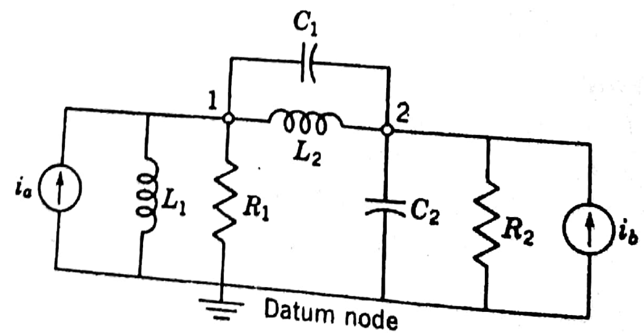

Fig. 3-38. Network with two independent node-pair voltages analyzed in Example 11.

In the form of a matrix equation, we have

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3-63}$$

Values for the operator coefficients are summarized in chart form as follows.

Eq.	Current	Coefficient of v_1	Coefficient of v_2
1	i_a	$G_1 + C_1\frac{d}{dt} + \left(\frac{1}{L_1}+\frac{1}{L_2}\right)\int dt$	$-C_1\frac{d}{dt} - \frac{1}{L_2}\int dt$
2	i_b	$-C_1\frac{d}{dt} - \frac{1}{L_2}\int dt$	$+G_2 + (C_1+C_2)\frac{d}{dt} + \frac{1}{L_2}\int dt$

EXAMPLE 13

Consider the resistive network shown in Fig. 3-39. For this network, the six node variable equations may be routinely written in the following chart form:

Eq. for node:	Current	Coefficient of v_a	v_b	v_c	v_d	v_e	v_f
a	0	$\frac{5}{2}$	-1	0	0	0	$-\frac{1}{2}$
b	0	-1	2	-1	0	0	0
c	0	0	-1	$\frac{5}{2}$	-1	0	$-\frac{1}{2}$
d	0	0	0	-1	2	-1	0
e	1	0	0	0	-1	$\frac{5}{2}$	$-\frac{1}{2}$
f	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$

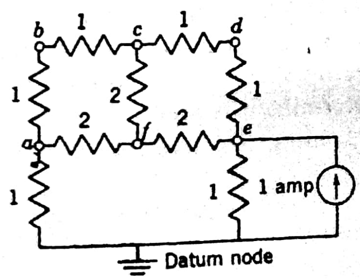

Fig. 3-39. Network of Example 13. Element values are in ohms.

In the form of a matrix equation, we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -1 & 0 & 0 & 0 & -\frac{1}{2} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & \frac{5}{2} & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \\ v_f \end{bmatrix} \qquad (3\text{-}6$$

Such equations can be written by inspection, using the "hanging on" and "hanging between" rules and the sign convention for the i_j and G_{kj} entries. Note that all terms of the *principal* diagonal are positive and that symmetry exists with respect to the principal diagonal.

Special problems are encountered in the nodal analysis of networks containing mutual inductance, and a good working rule is to bypass the problem by always analyzing such networks on the loop basis. Should nodal analysis be required, one approach is to replace the coupled coils by an equivalent network without mutual inductance.[3] The presence of controlled sources in the network to be analyzed creates no special problems but generally results in a nonsymmetrical matrix of the form given in Eq. 3-64.

Assuming that we can now write network equations in the two representations, the next problem is to be able to solve the sets of equations, which will require a knowledge of determinants.

3-7. DETERMINANTS: MINORS AND THE GAUSS ELIMINATION METHOD

The array of quantities enclosed by straight-line brackets

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\ldots} & & & & \end{vmatrix}$$

is known as

EXAMPLE 14

For a certain three-loop network, the following equations are iven.

$$\begin{aligned} 5i_1 - 2i_2 - 3i_3 &= 10 \\ -2i_1 + 4i_2 - 1i_3 &= 0 \\ -3i_1 - 1i_2 + 6i_3 &= 0 \end{aligned} \tag{3-78}$$

rom Cramer's rule we write the solution for i_1 as

$$i_1 = \frac{D_1}{\Delta} = \frac{10\begin{vmatrix} 4 & -1 \\ -1 & 6 \end{vmatrix} - 0\begin{vmatrix} -2 & -3 \\ -1 & 6 \end{vmatrix} + 0\begin{vmatrix} -2 & -3 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & -3 \\ -2 & 4 & -1 \\ -3 & -1 & 6 \end{vmatrix}} = \frac{230}{43} \tag{3-79}$$

imilarly,

$$= \frac{-(+10)\begin{vmatrix} -2 & -1 \\ -3 & 6 \end{vmatrix}}{\Delta} = \frac{150}{43}, \qquad i_3 = \frac{+(10)\begin{vmatrix} -2 & 4 \\ -3 & -1 \end{vmatrix}}{\Delta} = \frac{140}{43} \tag{3-80}$$

When the order of the determinant becomes larger than 4 or 5, he *Gauss elimination method* or its variants offers advantages over xpansion by minors, requiring only $n^3/3$ multiplications rather than $\times\, n!$. The Gauss elimination method is a systematic way of eliminatg variables, which will be introduced by the example of Eqs. (3-78) hich we have just solved. Note that both sides of an equation may be ultiplied by a constant without changing the equation. If we multiply e first equation in (3-78) by $\frac{2}{5}$ and then add the first and second quations, we have

$$0i_1 + \tfrac{16}{5}i_2 - \tfrac{11}{5}i_3 = 4 \tag{3-81}$$

ext we multiply the first equation by $\frac{3}{5}$ and add it to the third quation giving

$$0i_1 - \tfrac{11}{5}i_2 + \tfrac{21}{5}i_3 = 6 \tag{3-82}$$

ow if we multiply Eq. (3-81) by $\frac{11}{16}$, we may eliminate i_2 by adding e resulting equation to Eq. (3-82), giving

$$\tfrac{215}{5}i_3 = 140 \tag{3-83}$$

he three equations

$$\begin{aligned} 5i_1 - 2i_2 - 3i_3 &= 10 \\ 0i_1 + \tfrac{16}{5}i_2 - \tfrac{11}{5}i_3 &= 4 \\ 0i_1 + 0i_2 + 43i_3 &= 140 \end{aligned} \tag{3-84}$$