EE 223

- Parameter Estimation
- Hypothesis Testing
- Regression

- 6 quizzes - 20 marks best 5/6
- 10:30 - 12 am

Representation of Data

- 1D - value
- value - frequency - no. of times a value appears in dataset

relative frequency $= \dfrac{f_i}{\sum_{j=1}^{n} f_j}$

Sample mean : $(\bar{x})$ $\sum_{i=1}^{n} x_i / n = \dfrac{\sum_{j=1}^{k} f_j z_j}{\sum_{j=1}^{k} f_j}$

Sample variance : $\bar{s}^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Sample standard variance : $\bar{s} = +\sqrt{\bar{s}^2}$

Median

Let $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$

permutation function (bijective) $\quad x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} \dots < x_{\sigma(n)}$

Median is $x_{\sigma((n+1)/2)}$ if n is odd

$\dfrac{x_{\sigma(n/2)} + x_{\sigma(n/2+1)}}{2}$ if n is even

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Mode :

$$j^* \in \underset{j=1,\dots,K}{\arg\max}\,(f_j)$$

$$f_{j^*} = \max_{j=1,\dots,K}(f_j) \qquad \text{Mode} = z_{j^*}$$

Percentile

p-percentile for a value x_i is p% of all the values is less than x_i.

Chebyshev's inequality

$$S_K = \{ i : |x_i - \underbrace{\bar{x}}_{\text{sample mean}}| < K \underbrace{\bar{s}}_{\text{sample std. deviation}} \}$$

$$\frac{1}{K^2} \le \frac{|S_K^c|}{n}$$

$$\frac{n}{K^2} \le |S_K^c|$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \bar{s} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

S.T.

$$\underbrace{\frac{|S_K|}{n}}_{\text{cardinality of } S_K} \ge \cancel{1} \; 1 - \frac{1}{K^2} \quad (K > 1)$$

$$(n-1)\bar{s}^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad |x_i - \bar{x}|^2 < \frac{K^2}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \sum_{i \in S_K}(x_i - \bar{x})^2 + \sum_{i \in S_K^c}(x_i - \bar{x})^2 \qquad \frac{n-1}{K^2}|x_i - \bar{x}|^2 < \sum_{i=1}^{n}(x_i - \bar{x}$$

$$n\bar{s}^2 \ge |S_K^c| K^2 \bar{s}^2$$

$$|S_K^c| \le \frac{n}{K^2} \Rightarrow \frac{|S_K|}{n} \ge 1 - \frac{1}{K^2}$$

Axiomatic Probability Theory

$(\Omega, \mathcal{F})$

Ω → set of all possible outcomes

$\mathcal{F}$ – event space: collection of subsets of Ω

$A \in \mathcal{F} \Rightarrow A$ is an event.

Properties of $\mathcal{F}$

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. If $A_1, A_2, A_3 \dots \in \mathcal{F}$

   $\bigcup A_i \in \mathcal{F}$

If any collection $\exists$ of sets that satisfy ①, ② and ③, then it is called σ field on Ω.

$|A| = |B|$

if there exists one-one $f: A \to B$ and one-one $g: B \to A$

Real no. in $[0,1]$ to naturals

$g: [0,1] \to \mathbb{N}$ one to one

↳ no such g is possible

$\mathbb{N} \rightarrow$ cauntably infinite

$\mathbb{R} \rightarrow$ uncountably infinite

$(\Omega, \mathcal{F}) \rightarrow$ measurable space

Probability measure

$$P : \mathcal{F} \rightarrow [0,1]$$

1) $P(\Omega) = 1$

2) Suppose $A_1, A_2, \dots \in \mathcal{F}$ and mutually disjoint then

σ-additivity

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

To prove $P(\phi) = 0$ use $\Omega \cup \phi$

→ If $A \subseteq B$, then $P(A) \leq P(B)$

$(B \sim A) \cap (A) = \phi$

$(B \sim A) \cup (A) = B$

$B \setminus A = B \cap A^c$

⇓

This belongs

$P(B) = P(B \sim A) + P(A)$

$\Rightarrow P(A) \leq P(B)$

$A_1, A_2 \in \mathcal{F} \Rightarrow A_1^c, A_2^c \in \mathcal{F}$

$A_1^c \cup A_2^c \in \mathcal{F}$

$\Rightarrow (A_1 \cap A_2)^c \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$

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$A = \{A_1, \ldots, A_n\}$

$\sigma(A)$ - the smallest σ field that contains A

ω: $\omega \in \Omega$ is an outcome

A: $A \subset \Omega$ is an event

event A happened if outcome $\in A$

$\mathcal{F}$ is collection of events satisfying certain ~~ring~~ conditions.

Union Bound

$$P\left(\bigcup_{n=1}^{N} A_n\right) \le \sum_{n=1}^{N} P(A_n)$$ → here N can $\to \infty$

Consider

$B_1 = A_1$ $\quad$ $B_3 = A_3 \setminus (A_2 \cup A_1)$

$B_2 = A_2 \setminus A_1$ $\quad$ $B_n = A_n \setminus \left(\bigcup_{n-1} A_i\right)$

$$\sum P(B_n) \le \sum P(A_n)$$

Continuity of Prob. measure

$\boxed{f: R \to R}$

Continuity from below:-

Suppose $A_1, A_2, \ldots \in \mathcal{F}$ S.T. $A_n \subseteq A_{n+1}$

Then P is said to be cont. from below if

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

$$A_n = \bigcup_{k=1}^{n} A_k$$

$$\bigcup_{k=1}^{\infty} A_k = \lim_{n\to\infty} \bigcup_{k=1}^{n} A_k = \lim_{n\to\infty} A_n$$

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Proof: Define

$B_1 = A_1$

$B_2 = A_2 \setminus A_1$ $\quad$ B_n's are mutually disjoint

$B_3 = A_3 \setminus A_2$

$\vdots$

$$P\left(\lim_{n\to\infty} \bigcup_{k=1}^{n} A_k\right) = P\left(\lim_{n\to\infty} \bigcup_{k=1}^{n} B_k\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$= \lim_{n\to\infty} P(A_n)$$

Conditional Probability

If $D \in \mathcal{F}$ s.t. $P(D) > 0$ then $\forall A \in \mathcal{F}$

$$P(A|D) = \frac{P(A \cap D)}{P(D)}$$

$P_D : \mathcal{F} \to [0,1]$

P_D is also a probability measure of $[\Omega, \mathcal{F}]$

Baye's theorem

$$P(B/A) = P(A/B)\frac{P(B)}{P(A)}$$

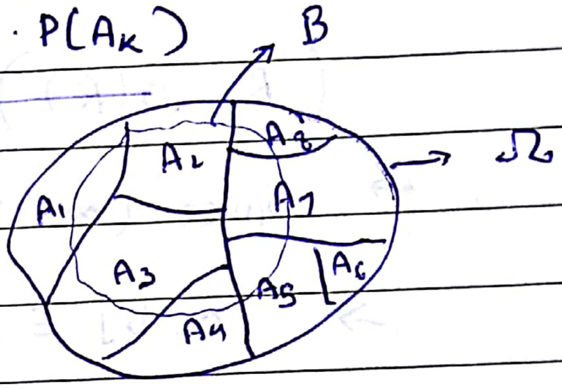
Law of total Probability

Let $\{A_1, \dots, A_n\}$ denote a partition of Ω

$\Rightarrow$ mutually disjoint

and $\bigcup_{i=1}^{n} A_i = \Omega$

$$\checkmark \; P(B) = \sum_{K=1}^{n} P(B|A_K) \cdot P(A_K)$$



Independence

Events A and B said to be independent if

$$P(A \cap B) = P(A)\, P(B)$$

Independence is symmetric relation and is a function of probability measure.

~~$\{A_1, \dots, A_n\}$ said to be independent for any collection if~~

A	B
C	D

$$\prod_{i=1}^{n} P(A_i) = P(A_1 \cap \dots \cap A_n)$$

$A \cup B$

$$P(A_i \cap A_j) = P(A)\; P(B)$$

$A \cup C$

$\rightarrow$ Both ① and ② don't imply each other

$A \cup D$

A collection of events $\{A_1, \dots, A_n\}$ is said to be independent if for any subcollection $\{A_{n(K)}\}$

$$P\left(\bigcap A_{n(K)}\right) = \prod P(A_{n(K)})$$

Random Variable :-

→ Denoted by $X : \Omega \to R$ → mapping set of all possible outcomes to set of real numbers.

Borrel σ-field ($\mathcal{B}(R)$)

→ The smallest σ field containing $\{(-\infty, x], x \in R\}$

$(R, \mathcal{B}(R))$ is a measurable space on R

→ Since $(-\infty, x] \in \mathcal{B}(R) \Rightarrow (x, \infty) \in \mathcal{B}(R)$

→ $(a, b] \in \mathcal{B}(R) \ \forall \ a < b$

$X : \Omega \to R$

** $P_X(B) = P(\{\omega : X(\omega) \in B\})$

$B \in \mathcal{B}(R)$

$\underbrace{\{\omega : X(\omega) \in B\}}_{X^{-1}(B)} \in \mathcal{F}$ → X is Borel measurable

$\subseteq \Omega$

Dice throw example

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F}$ = powerset of Ω $\qquad P(\{\omega\}) = \frac{1}{6}$

$X_1(\omega) = \omega \ \forall \ \omega \subset \Omega \qquad X_2(\omega) = 2 \ \forall \ \omega \subset \Omega$

$B_1 = (1.5, 3)$

$X_1^{-1}(B_1) = \{2\} \qquad X_2^{-1}(B_1) = \Omega$

Suppose $\mathcal{F} = \{\phi, \Omega, \{1, 2, 3\}, \{4, 5, 6\}\}$

$X_1(\omega) = \omega \ \forall \ \omega \subset \Omega$

$B_1 = [0, 2] \Rightarrow X_1^{-1}(B_1) = \{1, 2\} \notin \mathcal{F}$

- X is said to be Borel measurable if $X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}$

Any Borel measurable real valued fn is called RANDOM VARIABLE

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$$

$$\Rightarrow X^{-1}((-\infty, x]) \in \mathcal{F} \quad \forall x$$

Define $P_X(B) = P(X^{-1}(B))$

Distribution fn

$$F_X : \mathbb{R} \to [0,1]$$

$$F_X(x) = P(X \le x)$$

$$= P_X((-\infty, x]) = P(X^{-1}((-\infty,x]))$$

$$= P(\{\omega : X(\omega) \le x\})$$

Properties of F_X

① Monotone non-decreasing

Pf: Let $x_1 < x_2$

$$F_X(x_1) = P(\{\omega : X(\omega) \le x_1\})$$

$$F_X(x_2) = P(\{\omega : X(\omega) \le x_2\})$$

$$\Rightarrow F_X(x_1) < F_X(x_2)$$

② $F_X(+\infty) = 1$

③ $F_X(-\infty) = 0$

④ F_X can have atmost countable number of jumps.

$J_n = \{ x : \text{jump at } x \geq \frac{1}{n} \}$

$|J_n| \leq n$

$J = \bigcup_{n=1}^{\infty} J_n \rightarrow$ countable

⑤ <u>F_X is a cadlag fn</u>

→ F_X is continuous from right always

$\Rightarrow \lim_{n \uparrow \infty} F_X(x_n)$ S.T $x_n \downarrow x$ $_{n \to \infty}$ $= F_X(x)$

$\lim_{n \uparrow \infty} F_X(x_n) = \lim_{n \uparrow \infty} P(X \leq x_n) = \lim_{n \uparrow \infty} P_X((-\infty, x_n])$

$B_n \supseteq B_{n+1}$

$= P_X(\lim_{n \uparrow \infty} (-\infty, x_n])$

$= P_X(\bigcap_{n=1}^{\infty} (-\infty, x_n])$

$= P_X((-\infty, x])$

$\lim_{n \uparrow \infty} F_X(x'_n) = \lim_{n \uparrow \infty} P(X \leq x'_n)$

$\boxed{\lim_{n \uparrow \infty} x'_n \uparrow x}$ $= \lim_{n \uparrow \infty} P_X((-\infty, x'_n]) = F_X(x)$

$= P_X(\lim_{n \uparrow \infty} (-\infty, x'_n])$

$= P_X((-\infty, x))$

Random variables

discrete: F_X is a staircase fn

continuous: F_X is a continuous fn

- Any random variable can be written as a linear combination of the above 2 types (discrete and continuous)

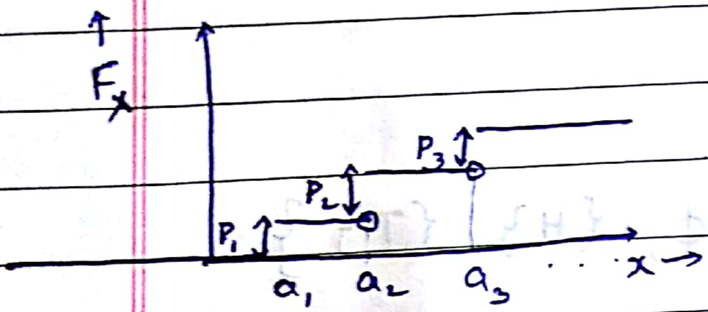
→ • <u>Density fn</u>

A density fn $f_X(x)$ is any function S.T. $\forall$ $x \in R$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$ → for continuous random variables (valid for discrete also after dirac-delta)

$$f_X \; R \rightarrow R^+$$

→ <u>Probability mass function</u>



$$P_K = F_X(a_K) - F_X(a_K^-)$$

$\{P_1, P_2, \ldots\}$ → probability mass function

$$P(X = a_K) = P_K$$

$$f_X(x) = \sum_{K=1}^{\infty} P_K\, \delta(u - a_K)$$

$$\int_{-\infty}^{x} f_X(u)\, du = \int_{-\infty}^{x} \sum_{K=1}^{\infty} P_K\, \delta(u - a_K)$$

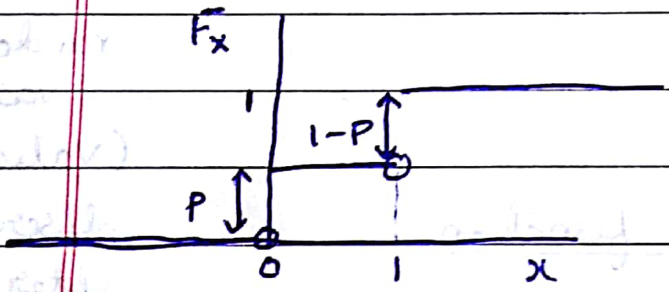
$$= \sum_{K=1}^{\infty} \int_{-\infty}^{x} \delta(u - a_K)\, du$$

$$= \sum_{K : a_K \le x} P_K$$

Examples

Discrete distributions

1) Bernoulli distribution (p)



Coin toss:

$\Omega = \{H, T\}$

$\mathcal{F} = \{\Omega, \phi, \{H\}, \{T\}\}$

$P(\{H\}) = p \qquad P(\{T\}) = 1 - p$

Define $X : \Omega \to R \quad X(H) = 0 \quad X(T) = 1$

$$F_{X}(x) = \begin{cases} 0 & x < 0 \\ p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

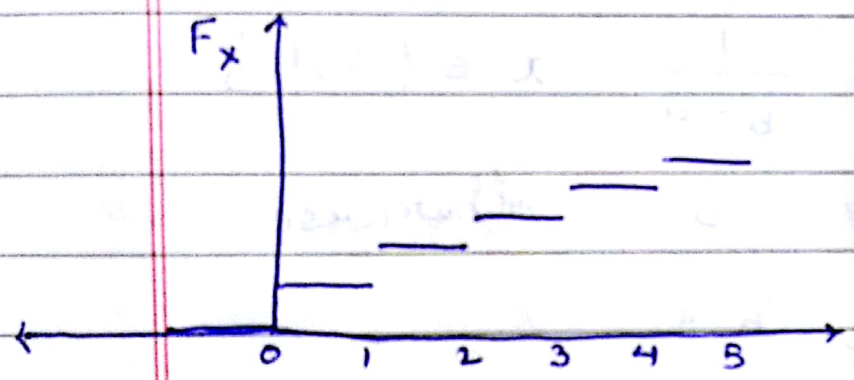
Define a measurable space $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$

$$X(\omega) = 1 \quad \omega \in A$$
$$= 0 \quad \omega \notin A$$

Indicator random variable i_A or I_A

2) <u>Binomial distribution</u> (n, p)

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, \dots, n$$

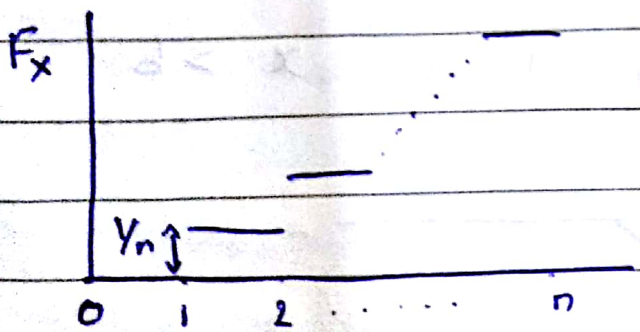


p – success probability

n – no. of trials

k – no. of successful trials.

3) <u>Uniform distribution</u> (n)



$$P(X = k) = \frac{1}{n}$$
$$\forall \; k = 1, \dots, n$$

4) <u>Geometric Distribution (p)</u>

$$P(X=K) = p(1-p)^{K-1} \quad \text{for } K=1,2,\ldots$$

5) <u>Poisson ~~Random~~ Distribution (λ)</u>

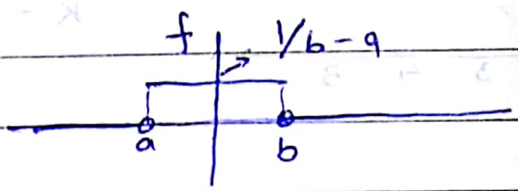
$$P(x=K) = e^{-\lambda} \frac{\lambda^K}{K!} \quad \forall K=0,1,2,\ldots$$

<u>Continuous distributions</u>

1) <u>Uniform [a, b]</u>

$$f_X(x) = \frac{1}{b-a} \quad x \in [a,b]$$

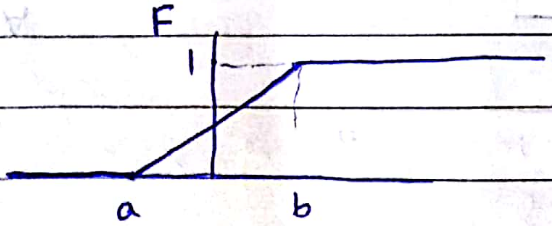
$$= 0 \quad \text{otherwise}$$



$$F_X(x) = 0 \quad x < a$$

$$= \frac{x-a}{b-a} \quad x \in [a-b]$$

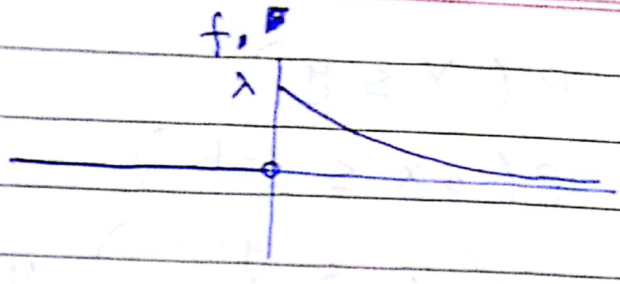
$$= 1 \quad x > b$$



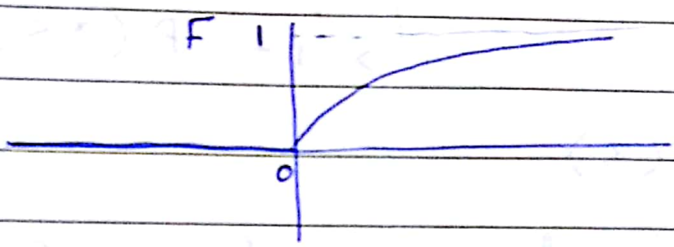
2) Exponential (λ) $\lambda > 0$

$$f_X(x) = \lambda e^{-\lambda x} \text{ if } x \geq 0$$

$$= 0 \quad \text{otherwise}$$

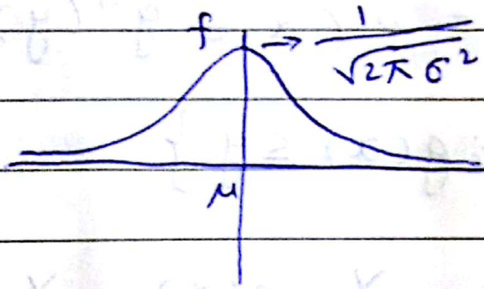


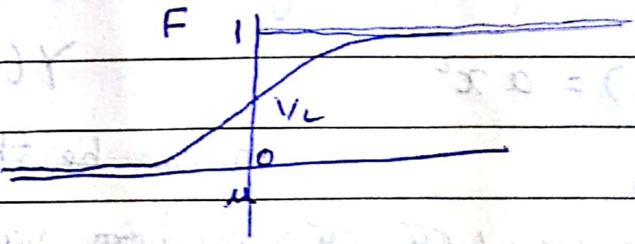
$$F_X(x) = 0 \qquad x < 0$$
$$= 1 - e^{-\lambda x} \qquad x \geq 0$$



3) <u>Gaussian Distribution</u> (μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$
$$\forall x \in R$$





<u>Function of Random Variables</u>

$Y = aX + b$

$X \sim F_X$ Find F_Y

$$F_Y(y) = P(Y \le y)$$
$$= P(aX \le y-b)$$
$$= P\left(X \le \frac{y-b}{a}\right) \text{ if } a>0$$

$F_X\left(\frac{y-b}{a}\right)$

$$P\left(X \ge \frac{y-b}{a}\right) \text{ if } a<0$$
$$\hookrightarrow 1 - P\left(X < \frac{y-b}{a}\right)$$

$Y = g(X)$

g is Borel measurable from R to R

$g : R \to R$

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(x \in \tilde{g}^{-1}(y))$$

$$\tilde{g}^{-1}(y) = \{x : g(x) \le y\}$$

For example:- $Y = aX^2$ $\quad Y = g(x)$ $\quad Y : \Omega \to R$

$Y(\omega) = g(X(\omega))$

$g(x) = ax^2$

$\hookrightarrow F_Y(y)$

$= P\left(X \in \left[-\sqrt{\frac{y}{a}}, \sqrt{\frac{y}{a}}\right)\right)$

g can be though as random variable from $(R, \mathcal{B}) \to (R, \mathcal{B})$