Q1] (a) $\phi(t) = E[e^{tX}] = \sum e^{tx} f_x dx$ $\phi'(0) = EX = p$

$= e^{t(0)}(1-p) + e^{t(1)} p$ Bernoulli

$= (1-p) + pe^{t}$

(b) $\phi(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \, {}^nC_k \, p^k (1-p)^{n-k}$

Binomial

$= \sum_{k=0}^{n} {}^nC_k (pe^{t})^k (1-p)^{n-k}$

$= (1-p+pe^{t})^n$

$\phi'(0) = (n)(p)(1)^{n-1} = np$

(c) $\phi(t) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!}$

Poisson

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{t})^k}{k!}$

$= e^{-\lambda} e^{\lambda e^{t}}$

$\phi'(0) = \lambda$

(d) $\phi(t) = E[e^{tX}] = \sum e^{tk} p (1-p)^{k-1} = \frac{p}{1-p} \sum_{k=0}^{\infty} e^{tk} (1-p)^k$

$= \cancel{(1-p) \sum e^{tk} \frac{p^k}{(1-p)^k}}$ $= \frac{p}{1-p} \cdot \frac{1}{1-e^{t}(1-p)}$

$= \cancel{(1-p) \, p \sum_{k=0}^{\infty} \left(\frac{e^{t}}{1-p}\right)^k}$ Geometric

$= \cancel{(1-p) \cdot \frac{1}{1+\frac{e^{t}}{1-p}}}$

(e) $\phi(t) = E[e^{tX}] = \cancel{\sum_{k=0}^{\infty} e^{tk} \lambda e^{-\lambda k}} = \cancel{\lambda \sum_{k=0}^{\infty} [e^{(t-\lambda)}]^k}$

$= \lambda \frac{1}{1-e^{t-\lambda}}$ Exponential

$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{(t-\lambda)x} dx$

$= \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x}\right]_0^{\infty} = \frac{\lambda}{\lambda-t}$

$$f(n,\lambda) = \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{\Gamma(n)}$$

$$E(e^{tx}) = \int_0^{\infty} \frac{e^{tx} \lambda e^{-\lambda x} (\lambda x)^{n-1}}{\Gamma(n)} dx$$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^{\infty} e^{-(\lambda - t)x} x^{(n-1)} dx$$

$(\lambda - t)x = m$

$(\lambda - t) dx = dm$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^{\infty} e^{-m} \frac{m^{(n-1)}}{(\lambda - t)^{(n-1)}} \frac{dm}{(\lambda - t)}$$

$$= \frac{\lambda^n}{(\lambda - t)^m \Gamma(n)} \int_0^{\infty} e^{-m} m^{(n-1)} dm = \frac{\lambda^n}{(\lambda - t)^m}$$

Notice : Moment. GF of Gamma = [MGF of Exp]^n

(g) Uniform (a,b)

$$E[e^{tx}] = \int_a^b \frac{e^{tx}}{(b-a)} dx = \frac{1}{(t)(b-a)} \left[e^{(b-a)t} - e^{-at} \right]$$

(h) Gaussian (μ, σ^2)

$$E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{(2\sigma^2 t + 2\mu)x - x^2 - \mu^2}{2\sigma^2}} dx$$

$$= \frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{(2\sigma^2 t + 2\mu)x/2\sigma^2} e^{-(x^2)/2\sigma^2} dx$$

After completing squares

$$= e^{\mu t + \frac{\sigma^2}{2} t^2}$$

(i) Chi- square $\chi^2(n)$

$$f(x,n) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{n/2}\,\Gamma(n/2)} & x > 0 \\ 0 & 0.\infty \end{cases}$$

$$E[e^{tx}] = \int_0^\infty \frac{e^{tx}\, x^{n/2-1}\, e^{-x/2}}{2^{n/2}\,\Gamma(n/2)}\, dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{\frac{n}{2}-1}\, e^{-(\frac{1}{2}-t)x}\, dx$$

$(\frac{1}{2}-t)x = m$

$(\frac{1}{2}-t)dx = dm$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)\,(\frac{1}{2}-t)^{\frac{n}{2}-1}} \int_0^\infty (m)^{n/2-1}\, e^{-mx}\, dx$$

$$= \frac{1}{(\frac{1}{2}-t)^{n/2-1}} \int_0^\infty f(x,n)\,dx = \frac{1}{(\frac{1}{2}-t)^{\frac{n}{2}-1}}$$

(ii) $f_X(x) = C_\alpha x^{-\alpha}$ for $x \geq 1$ $\alpha \geq 2$

$$E[e^{tx}] = C_\alpha \int_1^\infty e^{tx}\, x^{-\alpha}\, dx$$

$$= C_\alpha t^{\alpha-1} \int_t^\infty e^m\, m^{-\alpha}\, dm$$

Q2] (a) $S_n = \sum_{k=1}^{n} X_k$

$X_k \sim Exp(\lambda)$

$$f_{\Sigma X_k}(x) = [f_{X_1}(x_1)]\ldots = [\lambda e^{-\lambda x_1}]^n [\lambda e^{-\lambda x_2}]\ldots$$

$$f_{\Sigma X}(x) = \lambda^n e^{-\lambda \Sigma x_n}$$

$$E[e^{tX}] = \frac{\lambda}{\lambda - t}$$

$$E[e^{t\Sigma X}] = \{E[e^{tX}]\}^n = \left[\frac{\lambda}{\lambda-t}\right]^n$$

$\therefore$ the distribution is Gamma

$$f_X(n,\lambda) = \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{\Gamma(n)}$$

(b) Gaussian

$$E[e^{tx}] = e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$$\therefore E[e^{t\Sigma x}] = [E[e^{tx}]]^n = e^{n\mu t + n\frac{\sigma^2}{2}t^2}$$

$$\therefore f_X(x) = \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-(x-(n\mu))^2/2n\sigma^2}$$

(c) $E[e^{tx}] = \cancel{\binom{n}{k}}\ (1-p) + e^t p$

$$\therefore E[e^{t\Sigma x}] = [E[e^{tx}]]^n = (1-p+e^tp)^n$$

which is the M.G.F of Binomial distribution with parameter p

$$\therefore f_X(k) = \binom{n}{k} p^k (1-p)^k$$

(d) Poisson (λ_k)

$$E[e^{tx}] = e^{-\lambda} e^{\lambda e^t}$$

$$E[e^{t\Sigma x}] = [E[e^{tx}]]^n = e^{-n\lambda} e^{n\lambda e^t}$$

which is the MGF of a Poisson distribution with $\lambda = n\lambda_k$

$$\therefore f_X(k) = \frac{(n\lambda)^k e^{-n\lambda}}{k!}$$

Section 2

Q1] $X \sim$ Unif $(0,1)$

$$F_Y(y) = \mathbb{P}(Y \leq y)$$

$$= \mathbb{P}(-2\log X \leq y)$$

$$= \mathbb{P}(X \geq e^{-y/2})$$

$$= \int_{e^{-y/2}}^{1} dx = 1 - e^{-y/2}$$

This is the distribution of a type Chi-square with degree of freedom $= 2$ $\therefore$ $Y \sim \chi^2(2)$.

Q5] $f_{XY}(x,y) = \dfrac{e^{-x/y} e^{-y}}{y}$

$P(Y=y) = \int_0^\infty \dfrac{e^{-x/y} e^{-y}}{y} dx$

$= \dfrac{e^{-y}}{y} \int_0^\infty e^{-x/y} dx$

$\therefore P(Y=y) = -e^{-y}$

$P(X>1, Y=y) = \int_1^\infty \dfrac{e^{-x/y} e^{-y}}{y} dx$

$= \dfrac{e^{-y}}{y} \left[-y e^{-x/y}\right]_1^\infty$

$= -e^{-y} \cdot e^{-1/y}$

$\therefore P(X>1|y) = -e^{-1/y}$

$\therefore E[-e^{-1/y}] = \int_0^\infty e^{-1/y} (-e^{-y}) dy$ //

Q2] $X_1 X_2 \cdots X_n$ iids following $Exp(\lambda)$

$W = X_1 + X_2 + \cdots + X_n$ given we climb on the n^{th} bus

$F_W(w) = P(W<w)$

$= P(\Sigma X < w)$

$= \sum_{n=1}^\infty P(\Sigma X < w, N=n)$

$= \sum_{n=1}^\infty P(\Sigma X < w) \; P(N=n)$

$= \sum_{n=1}^\infty (1-p)^{n-1} p \; P(\Sigma X < w)$

We know that the ΣX_n, where $X \sim Exp(\lambda)$, has a distribution

$F_X(x) = \left(\dfrac{\lambda}{\lambda - t}\right)^n$

$\therefore F_W(w) = \sum_{n=1}^\infty (1-p)^{n-1} (p) \left(\dfrac{\lambda}{\lambda - t}\right)^n$

$= \dfrac{p}{1-p} \sum_{n=1}^\infty \left[\dfrac{(1-p)\lambda}{\lambda - t}\right]^n = \dfrac{p}{1-p} \; \dfrac{1}{1 - \frac{(1-p)\lambda}{\lambda - t}} \; \dfrac{(1-p)\lambda}{(\lambda - t)}$

$= \dfrac{\lambda p}{\lambda p - t}$

$\therefore S_n$ follows an exponential distribution with $\lambda = \lambda p$

Scanned by CamScanner

Q6) $Y \sim \text{Unif}(0,X)$ $X \sim \text{Gamma}(2,\lambda)$ $Z = X-Y$

$$\cancel{F_{YZ}(y,z)} = \mathbb{P}(\cancel{Y \le y}, \cancel{Z \le z})$$

$$= \mathbb{P}(\cancel{Y \le y}, \cancel{X-Y \le z})$$

$$= \cancel{\mathbb{P}}(\cancel{Y \le y}, \cancel{X-z \le Y})$$

$$= \mathbb{P}(\cancel{X-z \le Y \le y})$$

$$= \cancel{F_Y(y)} - \cancel{F_Y(X-z)}$$

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{2-1}}{\Gamma(2)} = \lambda^2 e^{-\lambda x} x$$

$$f_Y(Y|X) = \frac{1}{X}$$

$$f_Z(z) = \int f_X(x)\, f_Y(Y = x-z \mid X = x)\, dx$$

$$= \int f_X(x)\, f_Y(x-z \mid X = x)\, dx \qquad \therefore x \geq z$$

$$\therefore f_Z(z) = \int_z^\infty f_X(x)\, f_Y(x-z \mid X = x)\, dx = \int_z^\infty \frac{1}{x} \lambda^2 e^{-\lambda x} dx = \lambda e^{-\lambda z}$$

Now $f_Y(y) = \int f_X(x)\, f_Y(y \mid X = x)\, dx$

$$= \lambda e^{-\lambda y}$$

We know that $X \sim$ ~~Gaussian~~ Gamma $\therefore E[e^{tX}] = \frac{\lambda^2}{(\lambda-t)^2} = E[e^{t(Z+Y)}]$

we also know that Y & Z follow Exponential distribution

$$\therefore E[e^{tY}] = \frac{\lambda}{\lambda-t} = E[e^{tZ}]$$

$\therefore$ we can see that, $E[e^{t(Z+Y)}] = E[e^{tZ} \cdot e^{tY}] = E[e^{tZ}]\, E[e^{tY}]$

$\therefore$ ~~fx~~ Z & Y are independent.

Scanned by CamScanner

Q3] Suppose we reach the busstop and see exactly n busses

let N be the r.v for the number of buses seen

∴ ~~[illegible]~~

We have found out in Q2.2] that the distribution of total time between for n-buses follows a Gamma distribution.

let n buses consume a time t_0 out of t

$\therefore \quad \Sigma x_i = t_0$

$$\therefore f_{\Sigma x_i}(t_0) = \frac{\lambda e^{-\lambda t_0} (\lambda t_0)^{n-1}}{(n-1)!}$$ ie probability that $\Sigma x_i = t_0$

Also, the next bus should arrive after a total time of $(t - t_0)$

this probability is given by $\because x_i \sim Exp$

$$\int_{t-t_0}^{\infty} \lambda e^{-\lambda y} dy = e^{-\lambda (t-t_0)}$$

∴ Required probability for $t_0 \in [0, t]$

$$= \int_0^t e^{-\lambda(t-t_0)} \frac{\lambda e^{-\lambda t_0} (\lambda t_0)^{n-1}}{(n-1)!} dt_0$$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$ ⇒ Poisson's eqn in (λt)

Q4] $F_{X_{(K)}}(x) = P(X_{(K)} \leq x)$

This means : Probability (atleast K numbers less than equal to x)

$$\therefore F_{X_{(K)}}(x) = \sum_{K=K}^{n} \binom{n}{K} (F_X(x))^K (1 - F_X(x))^{n-K}$$