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Tut 5

Q1 f is entire

(i) $\exists c$ $|f(z)| > c > 0 \quad \forall z$

$g(z) = \frac{1}{f(z)}$ is entire $(|f(z)| > 0$ always$)$

$g(z)$ is bounded

Liouville's Theorem

$g(z) = \text{const}$

$f(z) = \text{const}$

(ii) $\text{Re}\, f(z) \geq 0 \quad \forall z \in \mathbb{C}$

$g(z) = f(z) + c \qquad c \in \mathbb{R},\ c > 0$

$|g(z)| \geq |\text{Re}(g(z))|$

$= |\text{Re}(f(z)) + c| \geq c$

use part (i)

(iii) $f(\mathbb{C})$ = range of f

$\overline{f(\mathbb{C})} \rightarrow$ closure of a set S is the smallest closed set containing S

i.e $S \cup \partial S$

(boundary)

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by definition $\mathbb{C}$ & ϕ are both closed & open

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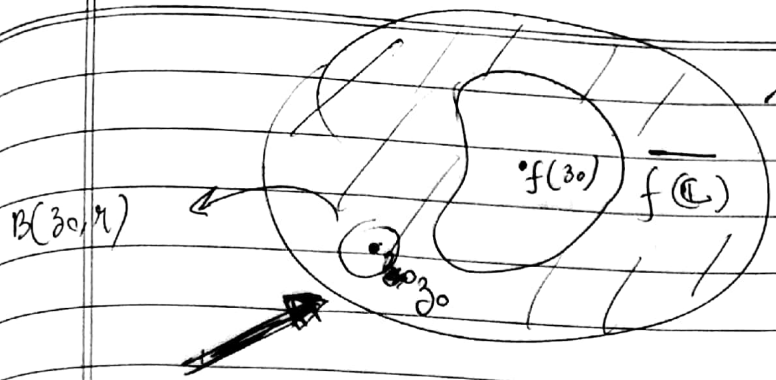

$$\mathbb{C} \cap \left(\overline{f(\mathbb{C})}\right)^c \neq \phi$$

let $z_0 \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$

consider $B(z_0, r) \subseteq \mathbb{C} \setminus \overline{f(\mathbb{C})}$

$$|f(z_0) - z_0| \geq r$$

use part (i)

Q2. $f\left(\frac{1}{n}\right) = \begin{cases} 1/n & n \text{ is even} \\ -1/n & n \text{ is odd} \end{cases}$

Domain $D = B(0,1)$

Identity theorem: If $\exists \{z_n\}$ in D converges to $z_0 \in D$ ... for f & g then $f \equiv g$

$f\left(\frac{1}{2n}\right) = \frac{1}{2n} \implies f(z) = z$

$f\left(\frac{1}{2n-1}\right) = \frac{-1}{2n-1} \implies f(z) = -z$

Q3. f is entire, $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$

$f(z^2) = z \quad \forall\, z \in \mathbb{C}$ by ~~the~~ identity theorem

$2z f'(z^2) = 1$

Q4 (a) $\left| e^z - \sum_0^N \frac{z^n}{n!} \right| \le \frac{|z|^{N+1}}{(N+1)!}$ $\quad \mathrm{Re}(z) \le 0$

$$\text{LHS} = \left| \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N f^{(N+1)}(tz)\, dt \right|$$

$f(z) = e^z$

$f'(z$

$$= \left| \frac{z^{N+1}}{(N+1)!} \right| \left| \int_0^t (1-t)^N e^{tz}\, dt \right|$$

$$\le \frac{|z|^{N+1}}{(N+1)!} \underbrace{\int_0^1 \left|(1-t)^N e^{tz}\right| dt}$$

$f^{(N+1)}(tz)$

$= f^{(N+1)}(z) \Big|_{z = tz}$

$$\le \int_0^1 |e^{tz}|\, dt$$

$$= \int_0^1 |e^{t(x+yi)}|\, dt$$

$$= \int_0^1 |e^{tx}|\, dt \le dt \quad \because \quad \boxed{\begin{array}{c} 0 < t < 1 \\ x < 0 \end{array}}$$

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Q5 $I = \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} dz$

consider $f(z) = (z^2+1)^{2n}$,

apply CIF for derivatives
$z_0 = 0$

$f^{(n')}(z_0) = \frac{n'!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n'+1}} dz$

$n' = 2n$

$\therefore I = \cancel{\frac{(2n)!}{2\pi i}} \frac{2\pi i}{(2n)!} f^{(2n)}(0)$ use binomial theorem, differentiate 2n times

$= \frac{2\pi i}{\cancel{(2n)!}} \cancel{(2n)!}\, {}^{2n}C_n$

now put $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta = i z\, d\theta$

$\frac{dz}{z} = i\, d\theta$

$\therefore I = \int_0^{2\pi} \left(e^{i\theta} + e^{-i\theta}\right)^{2n} i\theta = 2^{2n} i \int_0^{2\pi} (\cos\theta)^{2n} d\theta$

$= 2\pi i\, {}^{2n}C_n.$

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Q4b $\left| \cos z - \sum_{0}^{N} \frac{(-1)^n z^{2n}}{(2n)!} \right| \leq \frac{z^{2(N+1)}}{(2(N+1))!} \left| \int_0^1 (1-t)^{2N+2} f^{(2N+2)}(tz)\, dt \right|$

$$f(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$|f^N(z)| = \left| \frac{e^{iz} + e^{-iz}}{2} \right| \text{ if } N \text{ even}$$

$$\leq \frac{z^{2N+2}}{(2N+2)!} \int_0^1 \left| \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \right|$$

cosh R

since $y \leq R$

$$\int_0^1 \left| \frac{e^{(ix-y)t} + e^{(-ix+y)t}}{2} \right| dt$$

use $|a+b| \leq |a| + |b|$

$$\leq \int_0^1 \underbrace{\frac{e^{-yt} + e^{yt}}{2}}\, dt$$

$\hookrightarrow \cosh yt \quad 0 < t < 1$

$\leq \cosh y$ cosh is increasing.

$\leq \cosh R$

to compute $\displaystyle\int \frac{f(w)}{(w-z_0)^{\pm n+1}}\,dw$

any curve containing z_0 (my singularity)

✡✡ given $f(z) = \displaystyle\sum_{-\infty}^{\infty} b_n (z-z_0)^n$ on an annulus.

Show that b_n's are given by the above formulae.

integrate both sides on curve γ containing z_0

$n \geq 0 \longrightarrow$ holomorphic
$\therefore$ zero

$n < -1$ $\quad \dfrac{1}{(z-z_0)^n}$ has a primitive
$\therefore$ zero

$n = -1$

$$\int_\gamma f(z)\,dz = \int_\gamma b_{-1}(z-z_0)^{-1}\,dz = b_{-1}\cdot(2\pi i)$$

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similarly $\int_{\gamma} (z-z_0)^{m} f(z)\, dz = (2\pi i)\, b_{-m-1}$

$\cancel{\gamma}$
γ_m

$\therefore$ formula of b_n holds ✓

NOTE: Laurent series expansion → unique

$\downarrow$ –ive part → PRINCIPLE PART

Limit exists

$\boxed{z_0 \text{ is removable singularity iff principle part} \equiv 0}$

$\boxed{\text{pole iff " is finite sum}}$ i.e only finite a_{-n} are non zero

$(z-z_0)^m f \longrightarrow$ becomes holomorphic.

smallest $\mathbb{Z}^+$ m → order of the pole

or $f(z) = (z-z_0)^{-m} \times$ (holomorphic f^n)

power series
no –ive part

$\boxed{\text{essential iff principle part } \infty}$

RESIDUE

Laurent series around z_0 in $r < |z - z_0| < R$

$$\boxed{a_{-1} = \text{Res}(f; z_0)}$$

only term that remains $\int_\gamma f(z)\,dz$

$$\boxed{a_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)\,dz}$$

If residue is known, $\int_\gamma f(z)\,dz$

$\boxed{z_0 \text{ is } \cancel{\text{ante}} \text{ any } \cancel{\text{pole}} \text{ isolated singularity}}$

$\boxed{\gamma \to \text{ contains } z_0}$ ★★

obviously

special case CIF

$$g(z) = \frac{1}{2\pi i} \int_\gamma \underbrace{\left(\frac{g(w)}{w-z}\right)}^{f(z)} dw$$

residue of $f(z)$ of form $\boxed{\frac{g(w)}{w-z}}$

restrictive ∵ z is a simple pole.

similarly CIF of derivatives $\to$ pole of order n

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CAUCHY RESIDUE THEOREM

finitely many isolated singularities

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{res}(f, z_i)$$

REMARK: if residues are zero, $\int_\gamma f(z)\,dz = 0$

HOW TO COMPUTE RESIDUES?

* removable singularity : res = 0
* simple pole : res = $\lim_{z \to z_0} (z - z_0) f$

f starts from $a_{-1}(z-z_0)^{-1}$

$(z-z_0)$ in every term

* pole of order m, res = $\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$

on multiplication by $(z-z_0)^{m-1}$,

a_{-1} = coeff of $(z-z_0)^{m-1}$

* z_0 is a simple pole of $f = \frac{f_1}{f_2}$, f_1 & f_2 holomorphic

wrong in slides

$$\operatorname{Res}(f, z_0) = \frac{f_1(z_0)}{f_2'(z_0)}$$

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eg $f(z) = \dfrac{1}{\sinh(\pi z)}$ use $\dfrac{f_1}{f_2}$ formula,

or multiply by $(z - ni)$

$\lim_{z \to ni} \dfrac{z - ni}{\sinh(\pi z)}$ exists (L'hopital rule)

$z - ni$ ∵ simple pole.

∴ this limit itself is the residue

formulae → should become analytic in neighbourhood of singularity

eg $\displaystyle\int_{|z| = 2} \frac{z-4}{(z^2+2)^2} \, dz$

↓

$\displaystyle\int_{|z - \sqrt{2}i| = \varepsilon_1} + \int_{|z + \sqrt{2}i| = \varepsilon_2}$

Tut 6

Q1.

isolated
- removable
- pole
- essential

Non isolated

no ~~isolate~~ punctured disc around the point where holomorphic

(c) $f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$ $z = \frac{1}{n\pi}$ $n = \pm 1, \pm 2, \pm 3, \ldots$

→ poles of ~~or~~ order 1

$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi} \ldots \longrightarrow 0$

$\therefore$ $z = 0$ becomes a, non isolated singularity

(after a stage, all terms of series arbitrarily close → ~~punctured~~ punctured disc not possible)

Q2. poles of $f(z)$ = zeroes of $g(z) = \frac{1}{f(z)} = (z^4+1)^2$

4 roots, each of order 2

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Q3. $f(z) = \dfrac{2(z-1)}{z^2-2z-3} = \dfrac{1}{z-3} + \dfrac{1}{z+1}$

(i) $|z|<1$

$f(z) = \dfrac{1}{1-(-z)} + \dfrac{1}{(-3)\left(1-\frac{z}{3}\right)}$

$|z|<1$ $\qquad\qquad$ $\left|\frac{z}{3}\right|<1$

$= 1 + (-z) + (-z)^2 + (-z)^3 + \cdots$

$\cancel{+}$ $-\frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right)$

(ii) $1<|z|<3$

$f(z) = \dfrac{1}{z-3} + \dfrac{1}{z+1}$

$= \dfrac{1}{-3\left(1-\frac{z}{3}\right)} + \dfrac{1}{z+1}$

$\left|\frac{z}{3}\right|<1$

$\dfrac{1}{z\left(1-\left(-\frac{1}{z}\right)\right)}$ $\to$ $\left|\frac{1}{z}\right|<1$

$= \dfrac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right)$

similarly (iii)

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Q4. Riemann's theorem for removable singularities

z_0 → isolated singularity. If $\lim_{z \to z_0} f(z)$ exists

it is removable singularity

removable

iff $\lim_{z \to z_0} f(z)$ is finite

if $|f(z)| \leq M$ then

$\lim_{z \to z_0} (z - z_0) f(z)$ exists

⋮

see slides.

given z_0 is an isolated singularity

not essential ∵ $|f(z)| \leq M$, ∴ $f(z)$ can never be arbitrarily close to $2M$, say

if $\lim_{z \to z_0} |f(z)| = \infty$ → NOT POSSIBLE since $|f(z)| \leq M$

∴ not a pole

∴ must be removable

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Q5. $f(z) = \frac{p(z)}{q(z)}$ p & q differentiable

$p(z_0) \neq 0, \quad q(z_0) = 0$

$q'(z_0) \neq 0$

to prove $\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$

z_0 is a pole of f : order = 1

(a_{-1} of $f(z) = a_0'$ of $h(z)$)

consider $h(z) = (z - z_0) f(z)$

↓

holomorphic (pole of order 1)

then $\text{Res}(f, z_0) = \lim_{z \to z_0} h(z_0)$ {Laurent series ....}

$$\text{Res}(f, z_0) = \lim_{z \to z_0} f(z)(z - z_0)$$

$$= \lim_{z \to z_0} \frac{p(z)}{\frac{q(z)}{z - z_0}}$$

$$= \lim_{z \to z_0} \frac{p(z)}{\frac{q(z) - \overbrace{q(z_0)}^{\to 0}}{z - z_0}}$$

$$= \frac{p(z_0)}{q'(z_0)}$$

(p is ~~cont~~ continuous at 0, since it was differentiable)

Q6(i) generalize method of Q5.

z_0 : ~~pole~~ pole of order m of $f(z)$

$h(z) = f(z)(z-z_0)^m$

↳ holomorphic in a small neighbourhood around z_0

∵ isolated singularity

~~Res~~ " $h(z) = f(z)(z-z_0)^m$

$\cancel{\Sigma}$ a_{m-1} ←

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$$