Tutorial III

1. Let $z_1 = i\theta_1$

$z_2 = i\theta_3$

Now.

$$\because e^{z_1} = e^{z_2}$$

$$e^{z_1 - z_2} = 1 = e^{0}.$$

$$\therefore i(\theta_1 - \theta_3) = 0.$$

$$\therefore \theta_1 - \theta_3 = 2\pi n.$$

hence $z_1 - z_2 = 2\pi n i$

2.(i) $\sin z = \cancel{\sin x \cos y + i \cos x \sinh y}$

$$\cancel{= \sin x\left(\frac{e^{iy}+e^{-iy}}{2}\right) + i\cos\theta\left(\frac{e^{iy}-e^{-iy}}{2}\right)}$$

$$\cancel{= \sin x \cos y + i \cos x \sin y}$$

$$\cosh y = \frac{e^{y}+e^{-y}}{2} \qquad \sinh y = \frac{e^{y}-e^{-y}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$= \frac{(\cos x + i \sin y)\, e^{-y} - (\cos y - i \sin y)\, e^{y}}{2}$$

$$= \cos x \left(\frac{e^{-y} - e^{y}}{2i} \right) + i \sin y \left(\frac{e^{-y} + e^{y}}{2} \right)$$

$$= \underline{\underline{\sin y \cosh y + i \cos x \sinh y}}$$

#P.

(ii) $$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

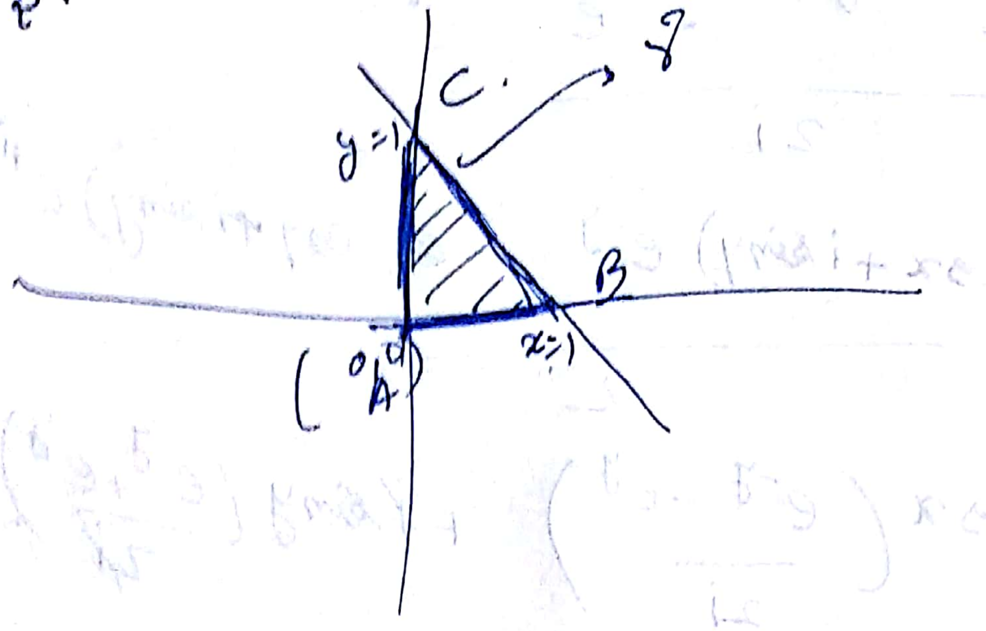
$$= \frac{e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}}{2}$$

$$= \frac{(\cos x + i \sin x) e^{-y} + (\cos x - i \sin x) e^{y}}{2}$$

$$= \frac{\cos x}{2} (e^{-y} + e^{y}) + \sin x \left(\frac{e^{-y} - e^{y}}{2} \right)$$

$$= \cos x \cosh y + i \sin x (-\sinh y)$$

3.



(a) $\int_{\gamma} Re(z)\cdot dz$

$= \int_{AB} Re(z)\,dz + \int_{BC} Re(z)\cdot dz + \int_{CA} Re(z)\cdot dz$

$= \int_{AB} x(dx+idy) + \int_{BC} x(dx+idy) + \int_{CA} x(dx+idy)$

$= \frac{x^2}{2}\Big|_0^1 + \int_1^0 t(dt + i(-dt)) + i\int_1^0 dy \times 0$

$= \frac{1}{2} + \frac{-1}{2} + i(-1) + \cancel{i(-1)}$

$= \cancel{0}$ 1

$z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$$\int_\gamma z^2\,dz = \int_{AB} (x^2 - y^2 + 2xyi)\,dx + \int_{BC} + \int_{CA} (x^2 - y^2 + 2xyi)\,dy$$

$$= \int_0^1 x^2\,dx + i\int_1^0 -y^2\,dy + \int_1^0 \left(t^2 - (1-t)^2 + 2t(1-t)i\right) \begin{matrix} dt \\ +i(-dt) \end{matrix}$$

$$= \frac{1}{3} + \frac{1}{3}i + \int_1^0 (-1 + 2t + 2ti - 2t^2 i)(dt + i\,dt)$$

$$= \frac{1}{3}(1+i) + \int_1^0 (-1 + 2t + 2ti - 2t^2 i)\,dt + i\int (-1 + 2t + 2ti - 2t^2 i)\,dt$$

$$\left(-t + t^2 + t^2 i - \frac{2}{3}t^3 i\right)\Big|_1^0$$

$$\left(-(\cancel{-1} + \cancel{1} + i - \frac{2}{3}i\right.$$

$$-\frac{1}{3}i \quad -i \times -\frac{1}{3}i$$

$$= \frac{1}{3}(1+i) \quad -\frac{1}{3}i \quad -\frac{1}{3}$$

4.

Surjective (onto): i.e each element of codomain is maped by atleast one element of domain

$$\sin z = \frac{e^{iz} + e^{-iz}}{2i}$$

Let $\sin z = z_0$

$$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = z_0.$$

Let $e^{iz} = y$

$$\frac{y - \frac{1}{y}}{2i} = z_0.$$

$$y^2 - 1 = i2yz_0$$

$$y^2 - 2yzi - 1 = 0$$

$$y = \frac{2z_0 i \pm \sqrt{-4z^2+4}}{2}$$

$$y = z_0 i \pm \sqrt{1-z_0^2}.$$

$\because y \neq 0$

assumin $y = 0$

$(z_0^2 \quad 1-z_0^2$

hence.

$e^{iz} = z_0 i \pm \sqrt{1-z_0^2}$

$iz = \ln\left(z_0 i \pm \sqrt{1-z_0^2}\right) + 2n\pi i$

$z = -i \ln\left(z_0 i \pm \sqrt{1-z_0^2}\right) + 2n\pi i$

hence $\sin z$ is surjective.

similarly for $\cos z$.

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = z_0.$

$y^2 + 1 = 2y z_0.$

$y^2 - 2y z_0 + 1 = 0.$

$y = \dfrac{2z_0 \pm \sqrt{4z_0^2 - 4}}{2}$

$y = z_0 \pm \sqrt{z_0^2 - 1}$

$y \neq 0$

assu $y = 0$

$z_0^2 = z_0^2 - 1$

$0 = -1$

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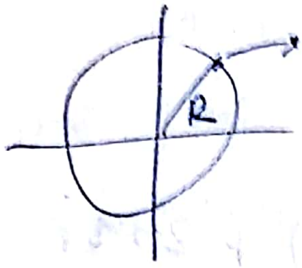
$\therefore z = -i \ln\left(z_0 i \pm \sqrt{z_0^2 - 1}\right) + 2n\pi$

hence, $\cos z$ is surjective.

5.



$(R\cos\theta, R\sin\theta)$

$\theta \in [0, 2\pi)$

$z = R\cos\theta + iR\sin\theta$.

$dz = R(-\sin\theta + i\cos\theta)\, d\theta$.

(a) $= R^{m+1}\int_0^{2\pi} (\cos\theta + i\sin\theta)^m (-\sin\theta + i\cos\theta)\, d\theta$.

$= R^{m+1} \int_0^{2\pi} e^{i\theta \times m} \cdot i e^{i\theta} \cdot d\theta$.

$= iR^{m+1} \int_0^{2\pi} e^{i\theta(m+1)} \cdot d\theta$.

$= \frac{\cancel{i} R^{m+1}}{\cancel{i}(m+1)} e^{i\theta(m+1)} \Big|_0^{2\pi}$.

$= \frac{R^{m+1}}{m+1}(1 - 1) = \underline{\underline{0}}$

(b) $R^{m+1} \int (\cos\theta - i\sin\theta)^m (-\sin\theta + i\cos\theta) \cdot d\theta$.

$= R^{m+1} \int e^{-i\theta m} \cdot i e^{i\theta} \cdot d\theta$

$= iR^{m+1} \int e^{i\theta(1-m)}$

$= \frac{R^{m+1}}{\ldots} e^{i\theta(1-m)} \Big|_0^{2\pi} = \underline{0}$

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(c) ~~R~~ $\int R \cdot R\, ie^{i\theta} \cdot d\theta$

$= R^2 i \int_0^{2\pi} e^{i\theta} \cdot d\theta$

$= \underline{0}$

6. we will be using green's theorem in this question. So statement is:

<u>Greens Theorem</u> :- greens theorem states that a line integral around the boundary of a plane region D can be computed as a double integral over D.

i.e $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Now

$\bar{z} = x - iy \qquad dz = dx + i\,dy$

$\int \bar{z}\, dz = \int (x-iy)(dx + i\,dy)$

$= \int_{\partial D} (x-iy)\,dx + (y+ix)\,dy$

using greens theorem

$= \iint_D (+i + i)\, dx\,dy$

$= 2i \iint_D dx\,dy = \underline{\underline{2i \text{ Area } D}}$ Ans.

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