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Tutorial-3

Warming up

1. $E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

$F_X(x) = \begin{cases} 1, & x > 1 \\ 1-p, & 0 < x < 1 \\ 0, & x < 0 \end{cases}$

$f_X(x) = p\,\delta(x-1) + (1-p)\,\delta(x)$

$E[e^{tX}] = p e^{t \times 1} + (1-p) e^{t \times 0} = pe^t + (1-p)$

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$E[X] = 1 \times p + 0 \times (1-p) = \underline{p}$

$var(X) = E[X^2] - E[X]^2$

$= 1 \times p + 0 \times (1-p) - p^2$

$= \underline{p(1-p)}$

b) $E[X] = \sum_{i=0}^{n} i \, {}^nC_i \, p^i (1-p)^{n-i}$

$1 = \sum {}^nC_i \, p^i (1-p)^{n-i}$

$0 = \sum {}^nC_i \, i p^{i-1} (1-p)^{n-i}$

$- \sum {}^nC_i (n-i)(1-p)^{n-i-1} p^i$

$\frac{\sum {}^nC_i \, i \, p^i (1-p)^{n-i}}{p} = \frac{n \sum {}^nC_i (1-p)^{n-i} p^i}{1-p} - \frac{\sum {}^nC_i \, i (1-p)^{n-i} p^i}{1-p}$

$\sum {}^nC_i \, i (1-p)^{n-i} p^i \left(\frac{1}{p} + \frac{1}{1-p}\right) = \frac{n}{1-p}$

$E[X] = \sum {}^nC_i \, i (1-p)^{n-i} p^i = \underline{np}$

$var[X] = E[X^2] - E[X]^2$

$E[X^2] = \sum_{i=0}^{n} i^2 \, {}^nC_i \, p^i (1-p)^{n-i}$

$\sum {}^nC_i \, i (1-p)^{n-i} p^i = np$

$\Rightarrow \sum {}^nC_i \, i \left(i p^{i-1} (1-p)^{n-i} - (n-i)(1-p)^{n-i-1} p^i\right) = n$

$\Rightarrow \sum {}^nC_i \, i \left(\frac{i p^i (1-p)^{n-i}}{p} + \frac{i (1-p)^{n-i} p^i}{1-p}\right) = n + n \sum i \, {}^nC_i \frac{(1-p)^{n-i} p^i}{1-p}$

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$$\sum i^2 \, {}^nC_i p^i (1-p)^{n-i} \left(\frac{1}{p(1-p)}\right) = n + \frac{n}{1-p} \times np$$

$$= \frac{n(1-p) + n^2 p}{1-p}$$

$$\sum i^2 \, {}^nC_i p^i (1-p)^{n-i} = np(1-p) + n^2 p^2$$

$$\therefore var[x] = \underline{\underline{np(1-p)}}$$

$$E[e^{tx}] = \sum e^{tk} \, {}^nC_k p^k (1-p)^{n-k}$$

$$= \sum {}^nC_k (e^t p)^k (1-p)^{n-k}$$

$$= \underline{\underline{(e^t p + 1 - p)^n}}$$

c) $$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-1)!}$$

$k-1 = t$

$$= \sum_{t=0}^{\infty} \frac{e^{-\lambda} \lambda^{t+1}}{t!} = \lambda \sum_{t=0}^{\infty} \frac{e^{-\lambda} \lambda^t}{t!} = \underline{\underline{\lambda}}$$

$$E[X^2] = \sum_{k=1} k^2 \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= \sum k \frac{e^{-\lambda} \lambda^k}{(k-1)!} = \sum (k-1+1) \frac{e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \sum_{k=2} \frac{e^{-\lambda} \lambda^k}{(k-2)!} + \sum_{k=1} \frac{e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$$

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$$E[e^{tX}] = \sum e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \frac{\sum (\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^t} = \underline{e^{\lambda e^t}}$$

d) $P(X=k) = p(1-p)^{k-1} \quad k = 1, 2, 3 \ldots$

$$E[X] = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1}$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} k(1-p)^{k}$$

$$S = (1-p) + 2(1-p)^2 + 3(1-p)^3 \ldots$$

$$(1-p)S = (1-p)^2 + 2(1-p)^3 \ldots$$

$$(1-1+p)S = (1-p) + (1-p)^2 + (1-p)^3 \ldots$$

$$pS = (1-p)\left(\frac{1}{1-(1-p)}\right) = \frac{1-p}{p}$$

$$S = \frac{1-p}{p^2}$$

$$E[X] = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \underline{\frac{1}{p}}$$

$$\cancel{E[X^2] = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1}}$$

$$E[e^{tX}] = \sum^{\infty} e^{tk} p(1-p)^{k-1}$$

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$$\frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{\Gamma(n)}, x>0$$

$$= \frac{p}{1-p} \sum e^{tk}(1-p)^k = \frac{p}{1-p} \sum_{k=1} \left(e^t(1-p)\right)^k$$

$$= \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)}$$

$$= \frac{pe^t}{1-e^t(1-p)}$$

e) $$E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(-t+\lambda)x} dx = \frac{\lambda}{\lambda - t}$$

f) $$E[e^{tX}] = \int_0^\infty \frac{e^{tx} \lambda e^{-\lambda x}(\lambda x)^{n-1}}{\Gamma(n)} dx$$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty e^{-(-t+\lambda)x} x^{n-1} dx$$

$(-t+\lambda)x = u$

$$= \frac{\lambda^n}{\Gamma(n)} \frac{1}{(-t+\lambda)} \int_0^\infty e^{-u} \frac{u^{n-1}}{(\lambda-t)^{n-1}} du$$

$$= \frac{\lambda^n}{\Gamma(n)(\lambda-t)^n} \Gamma(n) = \frac{\lambda^n}{(\lambda-t)^n}$$

g) $$E[e^{tX}] = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

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h) $E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$

$$tx - \frac{(x^2+\mu^2-2\mu x)}{2\sigma^2}$$

$$\frac{(2\sigma^2 t + 2\mu)x - x^2 - \mu^2}{2\sigma^2}$$

$$= \frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{(2\sigma^2 t + 2\mu)x}{2\sigma^2}} e^{-x^2/2\sigma^2} \, dx$$

$$= \underline{\underline{e^{\mu t + \sigma^2 t^2/2}}}$$

j) $E[e^{tX}] = \int_{1}^{\infty} e^{tx} C_\alpha x^{-\alpha} \, dx$

$$= \frac{C_\alpha}{t} \int_{1}^{\infty} e^{tx} x^{-\alpha} \, dx$$

$$E[X] = \int_{1}^{\infty} x \, C_\alpha x^{-\alpha} \, dx = C_\alpha \left(\frac{x^{2-\alpha}}{2-\alpha}\right)_1^{\infty}$$

$$= \underline{\underline{\frac{C_\alpha}{\alpha-2}}}$$

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2.

$$E[e^{tS_n}] = E_n[e^{t\sum x_k}]$$

$$= \prod_{i=1}^{n} E[e^{tX_i}]$$

$$= \underline{(E[e^{tX}])^n}$$

a) $X \to$ exp

$$E[e^{tX}] = \frac{\lambda}{\lambda - t}$$

$$\therefore E[e^{tS_n}] = \underline{\frac{\lambda^n}{(\lambda-t)^n}} \to \text{gamma}$$

b) $X \to$ gaussian

$$E[e^{tX}] = e^{\mu t + \sigma^2 t^2/2}$$

$$\therefore E[e^{tS_n}] = \underline{e^{(n\mu)t + \left(\frac{n\sigma^2}{2}\right)t^2}} \to \text{gaussian}$$

c) $X \to$ ~~binomial~~ bernoulli

$$E[e^{tX}] = pe^t + 1 - p$$

$$E[e^{tS_n}] = \underline{(pe^t + 1 - p)^n} \to \text{binomial}$$

d) $X \to$ poisson

$$E[e^{tX}] = e^{\lambda(e^t - 1)}$$

$$E[e^{tS_n}] = \underline{e^{n\lambda(e^t-1)}} \to \text{poisson}$$

Getting more into it

1. $f_X(x) = 1$

$F_Y(y) = P(Y \leq y)$

$= P(-2\log X \leq y)$

$= P(\log X \geq -y/2) = P(X \geq e^{-y/2})$

$= \int_{e^{-y/2}}^{1} 1\, dx = 1 - e^{-y/2}$

$= \chi^2(2)$

2. p = probability bus has enough space $= \frac{B}{a+1}$

Let me get into k^{th} bus.

$P(N=k) = (1-p)^{k-1} p$

Let W denote waiting time.

$W = X_1 + X_2 \cdots + X_N = \sum^N X_i$

$F_W(w) = P(W \leq w) = P(\sum^N X_i \leq w)$

$= \sum_{n=1}^{\infty} P(\sum^N X_i \leq w, N=n)$

$= \sum_{n=1}^{\infty} P(\sum^n X_i \leq w, N=n)$

$= \sum_{n=1}^{\infty} P(\sum^n X_i \leq w) P(N=n)$

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$$E[e^{t\sum^{N} X_k}] = \int_0^\infty e^{tx} f_W(w)\,dw$$

$$= \int_0^\infty e^{tx} \frac{d}{dw}\left(\sum_{n=1}^{\infty} P\left(\sum^{n} X_i \le w\right) P(N=n)\right) dw$$

$$= \sum_{n=1}^{\infty} P(N=n) \int_0^\infty e^{tx} \frac{d}{dw}\left(F_{\sum X_i}(w)\right) dw$$

$$= \sum_{n=1}^{\infty} P(N=n)\, E[e^{t\sum_{i=1}^{n} X_i}] \xrightarrow{\text{exponential}}$$

$$= \sum_{n=1}^{\infty} (1-p)^{n-1} p \cdot \frac{\lambda^n}{(\lambda-t)^n}$$

$$= \frac{p}{1-p} \sum_{n=1}^{\infty} \left(\frac{\lambda(1-p)}{(\lambda-t)}\right)^n$$

$$= \frac{p}{\cancel{1-p}} \cdot \frac{\lambda\cancel{(1-p)}}{\lambda-t} \times \frac{1}{1-\frac{\lambda(1-p)}{\lambda-t}}$$

$$= \frac{p\lambda}{\cancel{\lambda-t}} \cdot \frac{\cancel{\lambda-t}}{\cancel{\lambda}-t-\cancel{\lambda}+p\lambda}$$

$$= \frac{\lambda p}{p\lambda - t} = \frac{\lambda p}{\lambda p - t}$$

exponential with parameter λp.

$\cancel{3.\ F_Y(y) = P(Y \le y)}$

$\cancel{X_1 + X_2 \cdots + X_Y}$

4. $F^*(x) = \begin{cases} 0, & x < X_{(1)} \\ k/n, & X_{(k)} \le x \quad X_{(k+1)} \\ 1, & x \geq X_{(n)} \end{cases}$

$P\left(F^*(x) = k/n\right) =$ exactly k r.vs are less than equal to x & rest are greater

$= {}^nC_k \, P(r.v \le x)^k \, P(r.v > x)^{n-k}$

$= {}^nC_k \left(F_X(x)\right)^k \left(1 - F_X(x)\right)^{n-k}$

5. $E[P(X>1|Y)] = \int_0^\infty f_Y(y) \, P(X>1|Y) \, dy$

$\left[f_Y(y) = \int_0^\infty \frac{e^{-x/y} e^{-y}}{y} \, dx = \frac{e^{-y}}{y} \left(-y e^{-x/y}\right)_0^\infty = e^{-y} \right]$

$= \int_0^\infty \cancel{e^{-y}} \, \frac{P(X>1, Y)}{\cancel{P(Y)}} \, dy \qquad = \cancel{e^{-y}}$

$= \int_0^\infty P(X>1, Y) \, dy$

$= P(X - Z \leq y)$

6. $f_Y(y) = \int_{y}^{\infty} f_Y(y \mid X = x) f_X(x)\, dx \quad 0<y<x$

$= \int_{y}^{\infty} \frac{1}{\cancel{x}} \cdot \lambda^2 \cancel{x} e^{-\lambda x}\, dx$

$= \lambda^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right)_y^{\infty} = \underline{\underline{\lambda e^{-\lambda y}}}$