MA 205 Complex Analysis

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Zero's of holomorphic functions

Theorem

If f is a non-zero holomorphic function on a domain Ω, then each zero of f has finite multiplicity; i.e., there exists m such that $f(z) = (z - z_0)^m g(z)$ locally with $g(z_0) \neq 0$.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a domain Ω. Let z_0 be a point in Ω at which f vanishes. Then either f is identically zero or there exists a neigborhood of z_0 in which f has no other zero.

Theorem

Zeroes of a holomorphic function are isolated.

Zeros are Isolated

Theorem (Identity theorem)

If f and g are holomorphic in Ω, then $f \equiv g$ iff there exists a non-constant sequence $\{z_n\} \subseteq \{z \in \Omega \mid f(z) = g(z)\}$ such that $\lim_{n\to\infty} z_n = z_0 \in \Omega$.

Example: You cannot have two distinct holomorphic functions on a domain containing 0 which agree on all $\frac{1}{n}$.

Example: $\exp(z)$ is the only holomorphic function which agrees with e^x on the real line. Similarly, for $\sin z, \cos z$ etc.

Example: Identities like $\sin^2 z + \cos^2 z = 1$ follow without any further computation since they hold true over reals.

Example: $\exp(z + w) = \exp(z)\exp(w)$ now has another proof since this is true over reals!

One could ask if the converse is true; namely given a discrete set of points going to infinity does there exist a holomorphic function with the property that it vanishes exactly on this set ? Indeed this holds. In fact one can find a holomorphic function vanishing exactly on any discrete set with prescribed vanishing multiplicities at each of those points. This is called the **Weierstrass product theorem**.

(Karl Weierstrass (1815-1897) was a very important mathematician of the 19th century. He was responsible for giving rigorous foundations to analysis. The precise $\epsilon - \delta$ definition of limit was formulated by Weierstrass).

Failure of the property in the C^∞ case

Note that this property fails for real differentiable functions. It however works equally well for real analytic functions as well (same proof as above). The function

$$f(x) = e^{-1/x} \text{ for } x > 0$$

$$= 0 \text{ for } x < 0$$

is infinitely differentiable but vanishes along the entire negative real line.

Once we prove that holomorphic functions are analytic, the above fact will hold for holomorphic functions as well.

MA 205 Complex Analysis: Singularities and Laurent Seires

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Recall

We have studied zeroes of holomorphic functions. For a non-zero holomorphic function, the multiplicity of it's zeroes are finite. We have also proved that zeroes of a non-zero holomorphic function are isolated. In other words, zeroes of a non-zero holomorphic function on domain Ω can not have a limit point inside Ω but it may have a limit point on the boundary of Ω. We have also seen the following rigidity theorem known as identity theorem: If two holomorphic functions f and g on Ω are identically equal iff they are same on a sequence of points in Ω which has limit in Ω. Today, we will look at points where a function is not holomorphic and study the function on a small neighborhood of such a point.

Many times, one has a situation where Ω is an open set and f is a holomorphic function on the complement of a certain subset. The points of this subset are called **singularities** of the function. Given the rigid nature of holomorphic functions, we can get a lot of information on the nature of the singularities; essentially by looking at the function in small punctured neighborhoods of those points. Let us see this in more detail.

Definitions

Singularity of a function: The set of points in Ω where f is not defined or not holomorphic are called the singularities of Ω.
For example $1/z$ has a singularity at 0.

Singularities are of 2 types, isolated and non-isolated singularities.
A singular point is said to be isolated if the function is holomorphic in a punctured disc around that point.
For example $1/z$ is holomorphic in any punctured disc around 0.
$\frac{1}{z(z-1)}$ has 2 singular points 0 and 1, both of which are isolated singularities; the function is holomorphic in a punctured disc of radius 1 around both of them.

A singularity is non-isolated if it is not isolated ! That is, in no punctured neighborhood of the singularity is the function holomorphic.
For example $f(z) = |z|$ has all points as singularities and hence no point is an isolated singularity.

Removable and Non-Remonavle Singularities

Isolated singularities are of three types; removable singularity, pole and essential singularity.

If an isolated singularity can be removed by defining a certain value at that point, we say that the singularity is removable. For instance, the function $f(z) = \frac{sin(z)}{z}$ has a removable singularity at the origin. By redefining the function to be $f(z) = \frac{sin(z)}{z}$ for $z \neq 0$ and 1 for $z = 0$, we get a function which is holomorphic even at 0.

Note that if an isolated singularity at z_0 is removable, then $\lim_{z \to z_0} f(z)$ exists. The converse is also true and that is the Riemann's Removable Singularity Theorem.

Riemann's Removable Singularity Theorem

<u>**Theorem:**</u> An isolated singularity $z_0 \in \Omega$ of f is a removable singularity iff $\lim_{z \to z_0} f(z)$ exists.

Proof: Clearly removable singularity implies the limit exists. For the converse, suppose this limit exists. Then $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Then define g on a small open disc at z_0 by

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

If f is analytic in a punctured neighbourhood of z_0, then clearly g is analytic throughout that neighbourhood. Write

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots.$$

Note that $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$. Thus,

$$g(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \dots.$$

If we define $f(z_0) = c_2$, then f is holomorphic throughout. i.e., z_0 is a removable singularity.

Pole

Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z \to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $g(z) = \frac{1}{f(z)}$ is holomorphic at z_0 with $g(z_0) = 0$ (Why ?). Since $g(z)$ is not identically equal to zero, it follows that there exists a positive integer m such that $g(z) = (z - z_0)^m h(z)$ for some holomorphic function $h(z)$ defined in a neighborhood of z_0 with $h(z_0) \neq 0$. Note that such an m and therefore such a $h(z)$ is uniquely defined. Thus for all z in a punctured neighborhood of z_0, $f(z) = (z - z_0)^{-m} \frac{1}{h(z)} = (z - z_0)^{-m} f_1(z)$ for some holomorphic function $f_1(z)$. In this case, m is called the order of the pole and is a measure of how fast the function blows up at z_0. If m is one, we say that the pole is a **simple pole**.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

<u>Theorem:</u> If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

The if part if obvious. For the only if part, suppose f has an essential singularity. Let a be any complex number. Suppose f does not attain values arbitrarily close to a, then $\lim_{z \to z_0} (z - z_0) \dfrac{1}{(f(z) - a)} = 0$. Hence by Riemann's theorem above, it has a removable singularity at z_0.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0. In either case we have a contradiction.

For example, the function $e^{1/z}$ has an essential singularity at 0. (Check !)

MA 205 Complex Analysis: Laurent Seires and Cauchy Residue Theorem

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Recall

Recall that in the last class we have studied isolated singularities of a holomorphic functions. They are of three different type; namely, Removable singularity, Pole and Essential Singularity. We have also seen different methods to determine these singularities. Today, we will find a series expansion of a holomorphic function around it's isolated singular points.

We would like to expand a holomorphic function around an isolated singular point, much like the power series expansion of a holomorphic function around a point. A laurent series expansion around a point P is an expression of the form

$$\sum_{-\infty}^{\infty} a_j(z-p)^j.$$

Such a laurent series converges if both the series $\sum_0^\infty a_j(z-p)^j$ and $\sum_1^\infty a_{-j}(z-p)^{-j}$ converges. A Laurent series typically converges on an annulus $\{z : r < |z| < R\}$ for some $0 \leq r < R$.

Recall how we derived the power series representation of a holomorphic function on a disc centered around z_0. We used

$$f(z) = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{w-z}dw,$$

and manipulated $\frac{1}{w-z}$ as

$$\frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

Now suppose z_0 is an isolated singularity for f. Consider an annulus with radii $R > r$ centered at z_0 such that f is holomorphic on $\overline{D(z_0, R)} \setminus \{z_0\}$. CIF takes the form:

$$f(z) = \frac{1}{2\pi\imath} \int_{|w-z_0|=R} \frac{f(w)}{w-z} dw - \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw.$$

The first integral gives rise to $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ with

$$a_n = \frac{1}{2\pi\imath} \int_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}} dw,$$

exactly as before.

Laurent Series

In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}}.$$

Note that $\left|\frac{w-z_0}{z-z_0}\right| < 1$ for all w with $|w-z_0| = r$.

Expand to get $\displaystyle\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ with :

$$a_{-n} = \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{-n+1}} dw.$$

We write both together as $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. This is the Laurent series around the isolated singularity z_0. The negative part of the series is called the **Principal part of the Laurent series**.

Singularity using Laurent series expansion

Suppose z_0 is an isolated singularity for f and $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent series expansion of f on $r < |z-z_0| < R$. Note that the singularity at z_0 is

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z - z_0| < R$ for some R. The corresponding Laurent expansion is called the Laurent expansion around z_0. Consider the -1-st coefficient of this Laurent series.

$$a_{-1} = \frac{1}{2\pi\imath} \int_\gamma f(z) dz.$$

If you integrate a Laurent series, only a_{-1} remains; other terms vanish. What remains is usually called a residue.

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Often a_{-1} is easy to compute from $f(z)$ and if that's the case integration has become easy.

Cauchy Residue Theorem

Suppose f is given and γ is given. Suppose there are finitely many isolated singularities of f inside γ; say $z_1, z_2, \ldots, z_n$. What's $\int_\gamma f(z)dz$?

Theorem (Cauchy Residue Theorem)

$$\int_\gamma f(z)dz = 2\pi\imath \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

Thus integral of a function on a closed curve is zero not just when the function is holomorphic throughout; isolated singularities inside are okay, provided residues are zero.

How to compute residue?

1. If z_0 is a removable singularity of f, then $\text{Res}(f; z_0) = 0$.
2. If z_0 is a simple pole of f, then $\text{Res}(f; z_0) = \lim_{z \to z_0}(z - z_0)f$.
3. If z_0 is a pole of order m, then

$$\text{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d}{dz^{m-1}}[(z - z_0)^m f(z)].$$

4. If z_0 is a simple pole of $f = \frac{f_1}{f_2}$ with f_1 and f_2 are holomorphic at z_0, then $\text{Res}(f; z_0) = \frac{f_1(z_0)}{f'(z_0)}$. (Will be proved in the tutorial).

Examples

Consider $f(z) = \frac{e^z}{z^3}$.

e^z has a Taylor series expansion $\displaystyle\sum_0^\infty \frac{z^n}{n!}$. Then the Laurent series for $f(z)$ is given by $\displaystyle\sum_0^\infty \frac{z^{n-3}}{n!}$.

Hence the residue of $f(z)$, which is the coefficient of z^{-1} is given by $1/2$.

Alternatively, we note that $f(z)$ has a pole of order 3 at $z = 0$, so we can use the general formula for the residue at a pole:
$res(f; 0) = \frac{1}{2!}[\frac{d^2}{dz^2}(z^3 f(z))]_{z=0} = \frac{1}{2}[e^z]_{z=0} = \frac{1}{2}$.

Example

Lets compute the residues of $f(z) = \frac{1}{sinh(\pi z)}$ at its singularities. $\frac{1}{sinh(\pi z)}$ has a simple pole at ni for all $n \in \mathbb{Z}$ (Note : To check this show that $\lim_{z \to ni} \dfrac{z - ni}{sinh(\pi z)}$ is a non-zero number). Thus the residue at ni is given by:

$res(f; ni) = \lim_{z \to ni} \dfrac{z - ni}{sinh(\pi z)}$

By L'Hospital's rule $= \lim_{z \to ni} \dfrac{1}{\pi cosh(\pi z)}$

$= \frac{1}{\pi cosh(n\pi i)}$

$= \frac{1}{\pi cos(n\pi)}$

$= \frac{(-1)^n}{\pi}$

Example

$f(z) = \frac{1}{sinh^3(z)}$
We have seen that $sinh^3(z)$ has a pole of order 3 at πi with Taylor series:
$sinh^3(z) = -(z-\pi i)^3 - \frac{1}{2}(z-\pi i)^5 + ...$
Thus, $\frac{1}{sinh^3(z)} = -(z-\pi i)^{-3}(1 + \frac{1}{2}(z-\pi i)^2 + ...)^{-1}$
$= -(z-\pi i)^{-3}(1 - \frac{1}{2}(z-\pi i)^2 + ...)$
The coefficient of $(z-\pi i)^{-1}$ in the above expression is $1/2$ which is therefore residue of f at πi.

Example

Compute $\int_{|z|=2} \frac{(z-4)}{(z^2+2)^2} dz$.
Recall : If $f(z)$ has a pole at z_0 of order m, then the residue of f at z_0 can be computed as :
$res(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d}{dz^{m-1}} [(z - z_0)^m (f(z))]$

Therefore in the given example,

$res(f, \sqrt{2}i) = \frac{d}{dz}[(z - \sqrt{2}i)^2 \frac{(z-4)}{(z^2+2)^2}]_{z=\sqrt{2}i}$ and

$res(f, -\sqrt{2}i) = \frac{d}{dz}[(z + \sqrt{2}i)^2 \frac{(z-4)}{(z^2+2)^2}]_{z=-\sqrt{2}i}$
Adding the avove values, we get the final answer. I'll leave the details of the computation to you.

MA 205 Complex Analysis: Singularity at ∞ and Real integral

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Singularity at ∞

Isolated Singularity at Infinity: $f(z)$ is said to have an isolated singularity at ∞ if f is holomorphic outside a disc of radius R for some R. Equivalently, $f(1/z)$ has an isolated singularity at 0.
If f has an isolated singularity at ∞, we can talk about the nature of singularity at ∞.
Definition: f is said to have a zero (resp. removable singularity, pole, essential singularity) at ∞ if $f(1/z)$ has a zero (resp. removable singularity, pole, essential singularity) at 0.

Examples:

- Entire functions has isolated singularity at ∞.
- Constant function has removable singularity at ∞.
- Polynomials have pole at ∞.
- e^z has an essential singularity at ∞.
- If f is an entire function which has a zero at ∞, then f is identically zero. (Why ??)
- There are plenty of meromorphic functions which have a zero at ∞, for example $1/z$.

Theorem

An entire functions from $\mathbb{C}$ to $\mathbb{C}$ has a pole at ∞ if and only if it is a non-constant polynomial.

Computing Real Integrals

One of the important applications of Complex Analysis is computation of real integrals.

Let $f : [0, \infty] \to \mathbb{R}$ be a function such that $\int_0^R f(x)dx$ exists for each $R \geq 0$. One then defines the Improper integral $\int_0^\infty f(x)$ to be $\lim\limits_{R\to\infty} \displaystyle\int_0^R f(x)dx$.
Similarly if $f : [-\infty, \infty] \to \mathbb{R}$ is a function such that $\int_{-a}^b f(x)dx$ exists for each $a, b \geq 0$, then the improper integral $\int_{-\infty}^\infty f(x)$ is defined as $\lim\limits_{a,b\to\infty} \displaystyle\int_{-a}^b f(x)dx$. If f is integrable, then its integral can be computed as $\lim\limits_{R\to\infty} \displaystyle\int_{-R}^R f(x)dx$.

Improper Integral

For instance the function $\frac{1}{1+x^2}$ is integrable on $\mathbb{R}$ while the integral $\int_{-\infty}^{\infty} sin(x)dx$ does not exist. Intuitively, for such an improper integral to exist, the function has to decay to zero sufficiently rapidly outside a"small set ". (Note that it need not quite tend to zero as $|x| \to \infty$).
Often, instead of a real variable, the function $f(z)$ with the complex variable is holomorphic outside some discrete set. This allows us to exploit Cauchy's residue formula to compute the real integral as follows.

Consider a close contour $\gamma_R \cup C_R$ where γ_R being a line segment along the real axis between $-R$ and R and C_R is the semicircle of radius R around 0. We can then evaluate $\int_{\gamma_R \cup C_R} f(z)dz$ by means of residue theorem, and show that the integral over the extra "added" part of γ_R, namely C_R asymptotically vanishes as $R \to \infty$. Thus taking the contour integral over γ_R and allowing R to tend to ∞, we get the desired answer.

Example

Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.
You might have seen the computation of this integral in MA 105 but now lets work out this computation using using MA 205 !
The idea is to compute $\int_{-r}^{r} \frac{x^2}{1+x^4} dx$ and take limit as $r \to \infty$. Fix $r > 1$. Let γ_r be $[-r, r]$ together with C_r, the upper part of the circle $|z| = r$ oriented counterclockwise. Take $f(z) = \frac{z^2}{1+z^4}$. Then f has two poles inside γ. Now,

$$\frac{1}{2\pi\imath} \int_{\gamma} f(z) dz = \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \frac{-\imath}{2\sqrt{2}}.$$

This is same as

$$\frac{1}{2\pi\imath} \int_{-r}^{r} \frac{x^2}{1+x^4} dx + \frac{1}{2\pi\imath} \int_{C_r} \frac{z^2 dz}{1+z^4}.$$

Example

By changing to polar coordinates, the second integral becomes,

$$\frac{1}{2\pi}\int_0^{\pi}\frac{r^3e^{3\imath t}}{1+r^4e^{4\imath t}}dt.$$

Note that,

$$\left|r^3\int_0^{\pi}\frac{e^{3\imath t}}{1+r^4e^{4\imath t}}dt\right|\le\frac{\pi r^3}{r^4-1}.$$

Thus, in the limit, this integral is zero. Therefore,

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx=\frac{\pi}{\sqrt{2}}.$$

Useful Estimate

If $P(z)/Q(z)$ is a rational function such that with deg $Q(z) \geq$ deg $P(z) + 2$. Then there exists a constant C such that for $|P(z)/Q(z)| \leq C/|z^2|$ for $|z|$ sufficiently large. Thus for a large real number R, $|P(z)/Q(z)| \leq \frac{C}{R^2}$ on the circle of radius R.

Example

To compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$, we consider a contour γ to be the union of the segment from $-R$ to R along with the upper half semicircle C_R of radius R, oriented positively. There is just one pole inside γ which is $\imath$. Compute $\mathrm{Res}(f;\imath)$, where $f(z) = \frac{1}{(1+z^2)^n}$. This is given by $\frac{g^{(n-1)}(\imath)}{(n-1)!}$, where $g(z) = \frac{1}{(z+\imath)^n}$. Check:

$$\mathrm{Res}(f;\imath) = \frac{-\imath}{2^{2n-1}} \binom{2n-2}{n-1}.$$

By the earlier remark, there exists a constant C , such that $|\frac{1}{(1+z^2)^n}| \leq \frac{C}{R^2}$ on C_R for large enough R. Then by ML Lemma $\int_{C_R} \frac{dz}{(1+z^2)^n}$ tends to zero as $R \to \infty$. Thus, the value of the real integral is $\frac{\pi}{4^{n-1}} \binom{2n-2}{n-1}$.

Theorem (Jordan's Lemma)

Let f be a continuous function defined on the semicircular contour $C_R = \{Re^{\imath\theta} \mid \theta \in [0, \pi]\}$ of the form

$$f(z) = e^{\imath az} g(z),$$

where $g(z)$ is a continuous function and with $a > 0$. Then,

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0,\pi]} |g(Re^{\imath\theta})|.$$

Proof:

$$\int_{C_R} f(z)dz = \int_0^{\pi} g(Re^{\imath\theta})e^{\imath aR(\cos\theta+\imath\sin\theta)}\imath Re^{\imath\theta}\,d\theta.$$

Therefore,

$$\begin{aligned}
\left|\int_{C_R} f(z)dz\right| &\leq R\int_0^{\pi}\left|g(Re^{\imath\theta})e^{aR(\imath\cos\theta-\sin\theta)}\imath e^{\imath\theta}\right|d\theta \\
&= R\int_0^{\pi}\left|g(Re^{\imath\theta})\right|e^{-aR\sin\theta}d\theta \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta \quad \text{where } M_R = sup|g(Re^{i\theta})| \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{\frac{-2aR\theta}{\pi}}d\theta = \frac{\pi}{a}(1-e^{-aR})M_R \leq \frac{\pi}{a}M_R,
\end{aligned}$$

since $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \frac{\pi}{2}]$.

Example

Compute $\int_0^\infty \frac{\sin x}{x} dx$.
We'll consider the function

$$f(z) = \frac{e^{\imath z}}{z}.$$

Let γ be the boundary of the upper part of the annulus $A(0; r, R)$.
Then, $\int_\gamma f(z)dz = 0$, by Cauchy's theorem.

Example

But,

$$\int_\gamma f(z)dz = \int_r^R \frac{e^{\imath x}}{x}dx + \int_{\gamma_R} \frac{e^{\imath z}}{z}dz + \int_{-R}^{-r} \frac{e^{\imath x}}{x}dx + \int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Now,

$$\begin{aligned} \int_r^R \frac{\sin x}{x}dx &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x} - e^{-\imath x}}{x}dx \\ &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x}}{x}dx + \frac{1}{2\imath}\int_{-R}^{-r} \frac{e^{\imath x}}{x}dx. \end{aligned}$$

Thus, we only need to compute

$$\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{\imath z}}{z}dz \quad \& \quad \lim_{r\to 0}\int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Example

Now,

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{\imath z}}{z}dz = 0,$$

by Jordan's lemma. On the other hand, note that $\frac{e^{\imath z}-1}{z}$ has a removable singularity at 0. Thus, there is $M > 0$ such that

$$\left|\frac{e^{\imath z}-1}{z}\right| \le M,$$

for $|z| \le 1$. Thus,

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\imath z}-1}{z}dz = 0,$$

by appealing to ML inequality.

Example

Therefore,

$$\lim_{r \to 0} \int_{\gamma_r} \frac{e^{\imath z}}{z} dz = -\pi \imath.$$

Thus,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example

Show that $\int_0^\infty \frac{\log x}{1+x^2} dx = 0$.
We'll work with γ as in the previous problem. We take

$$f(z) = \frac{\log z}{1+z^2},$$

where $\log z$ is a branch of the logarithm which is defined on the x-axis, so that $\int_r^R$ and $\int_{-R}^{-r}$ make sense. For instance, we can take the branch with negative y-axis as the branch cut. Then,

$$\log x = \begin{cases} \log x & \text{if } x > 0, \\ \log|x| + \imath\pi & \text{if } x < 0. \end{cases}$$

Example

Now,

$$\int_\gamma \frac{\log z}{1+z^2}dz = \int_r^R \frac{\log x}{1+x^2}dx + \int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx + \int_{\gamma_r} \frac{\log z}{1+z^2}dz.$$

LHS is $2\pi\imath \cdot \mathrm{Res}(f;\imath) = 2\pi\imath \cdot \frac{\log \imath}{2\imath} = \frac{\pi^2\imath}{2}$. Also,

$$\begin{aligned} &= \int_r^R \frac{\log x}{1+x^2}dx + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx \\ &= 2\int_r^R \frac{\log x}{1+x^2}dx + \imath\pi \int_r^R \frac{dx}{1+x^2} \\ &= 2\int_r^R \frac{\log x}{1+x^2}dx + \frac{\pi^2\imath}{2}. \end{aligned}$$

(In the Limit)

Thus,

$$\int_r^R \frac{\log x}{1+x^2}dx = -\frac{1}{2}\left[\int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{\gamma_r} \frac{\log z}{1+z^2}dz\right].$$

Note that

$$\begin{aligned}
\left|\int_{\gamma_\rho} \frac{\log z}{1+z^2}dz\right| &= \left|\rho\int_0^\pi \frac{\log \rho + \imath\theta}{1+\rho^2 e^{\imath\theta}} e^{\imath\theta}\, d\theta\right| \\
&\leq \frac{\rho|\log\rho|}{|1-\rho^2|}\int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|}\int_0^\pi \theta d\theta \\
&= \frac{\pi\rho|\log\rho|}{|1-\rho^2|} + \frac{\rho\pi^2}{2|1-\rho^2|}.
\end{aligned}$$

This is zero in the limit if $\rho \to 0+$ or $\rho \to \infty$. Thus, the given integral is zero.

MA 205 Complex Analysis: Real integral and Maximum modulus principle

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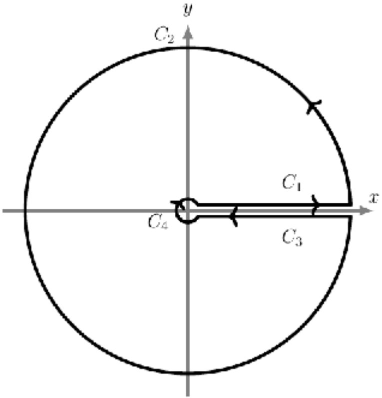
Recall

Last time we began by understanding the notion of an isolated singularity at ∞. Much the same way as isolated singularity at a point in $\mathbb{C}$, we can classify isolated singularities at infinity into removable, pole and essential singularity.
We then looked at various examples of computing residues and contour integrals. Let us begin by looking at some more today.

Show that $\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.
We'll integrate

$$f(z) = \frac{z^{-c}}{1+z},$$

where z^{-c} is the branch corresponding to branch cut being the positive real axis. Consider the contour $\gamma = C_1 \cup C_2 \cup C_3 \cup C_4$:



By residue theorem,

$$\int_\gamma \frac{z^{-c}}{1+z}dz = 2\pi\imath e^{-\imath\pi c}.$$

Note that

$$\int_r^R \frac{t^{-c}}{1+t}dt = \lim_{\delta\to 0}\int_{C_1}\frac{z^{-c}}{1+z}dz.$$

Similarly,

$$\lim_{\delta\to 0}\int_{C_3}\frac{z^{-c}}{1+z}dz = -e^{-2\pi\imath c}\int_r^R \frac{t^{-c}}{1+t}dt.$$

Let γ_ρ be the ρ radius of circle. Then,

$$\left|\int_{\gamma_\rho} \frac{z^{-c}}{1+z} dz\right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi\imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

Real Integral

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

By residue theorem, $\int_\gamma \frac{e^{z/2}dz}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.
Now $|cosh(L + iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq \frac{1}{2}(|e^{L+iy}| - |e^{-L-iy}|) = (e^L - e^{-L})/2 \geq e^L/4$ (for L sufficiently large)
From this it follows from the ML-inequality that as L tends to ∞, the integral along the right vertical side tends to zero. Similarly one checks that the integral along the left vertical side also tend to zero.

Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_{L}^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x + i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x/2}dx}{coshx}$$

Taking L tending to ∞ , we see that
$I = \frac{2\pi e^{i\pi/4}}{(1+e^{i\pi/2})} = \frac{\pi}{cos(\pi/4)} = \pi\sqrt{2}.$

Choice of contour for integration

It might be a bit of a mystery as to which contour one should consider for a given contour integral. Here is a general recipe. Suppose the improper integral is of the form $\int_{-\infty}^{\infty} f(x)dx$. The general idea of course is to find a contour which contains the real line as part of the contour in the "limit". The choice should be made so that by residue theory one knows the integral over the full contour and such that the integal over the extra added part goes to zero in the limit.

I. For instance suppose there exists a constant C such that $|f(z)| \leq \frac{C}{|z^r|}$ for sufficiently large $|z|$ and for some $r > 1$ (here $f(z)$ is an extension of $f(x)$ to a function of the complex variable). Note that this happens for instance in the case when $f(x) = P(x)/Q(x)$ where $\deg Q(x) \geq \deg P(x) + 2$. Then close up the interval with a semicircle into the upper half plane and integrate along the contour and take limit as the radius of semicircle goes to infinity. Use ML inequality to show that the integral along the semicircle goes to zero as radius goes to ∞.

Choice of contour for integration

In case the integral is from 0 to ∞ , try and relate it to some integral from $-\infty$ to ∞. For instance the function may have a natural continuation to the negative reals. In case this is not possible, often because $f(z)$ has a singularity at origin; usually a pole, then try using a half annular region $A(0; r, R)$ like we have done in earlier examples. This will avoid the pole and then show that in the limit as R tends to ∞ and r tends to zero, the integrals along the larger and smaller circle tend to zero. Hence in the limit we will get integral over the real line. Then try and relate the integral over the positive real to that over the negative reals. See the example of $\int_0^\infty \frac{log(x)}{1+x^2}$ as an illustration of this.

Choice of contour for integration

II. If the integrand is of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} sin(x) dx$, where $P(x)$ and $Q(x)$ are polynomials with $degQ(x)$ atleast one more than that of $P(x)$, close up the interval by the semicircular region in the upper half plane and use Jordan's lemma to show that the integral over the semicircle goes to zero using Jordan's lemma. (We have seen this when we integrated $sin(x)/x$).

III. If the integrand is of the type $\int_0^{2\pi} P(cos(t), sin(t)) dt$, set $z = e^{it}$ and use $cos(t) = \frac{z+z^{-1}}{2}$ and $sin(t) = \frac{z-z^{-1}}{2i}$. dt becomes $\frac{dz}{iz}$ and then the integral assumes the form $\int_{|z|=1} P(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}) \frac{dz}{iz}$ which can then be computed by using residue theorem.

IV. If the integrand has infinitely many poles going to infinity, you are usually better off using a rectangular contour which emcompasses only finitely many poles.

As before one tries to show that in the limit, the integral over the extra added vertical sides goes to zero in the limit and the intergals over the two horizontal sides are related ; usually proportional to each other. Thus taking limit as the length of the rectangular sides goes to infinity, one gets the desired answer.

V. In case the function involves a branch cut, choose a contour which avoids (goes around) the branch cut like in the earlier example.

An important theorem in Complex Analysis states that a non-constant holomorphic function on an open connected domain never attains its maximum modulus at any point in the domain. This is called the maximum modulus theorem. Once again, this is vastly different from what happens to real differentiable functions; in fact even for real analytic functions. Real analytic functions can achieve maximum anywhere inside the interval. We'll use CIF and the identity theorem to prove MMT.

Maximum Modulus Theorem

Proof: Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Then we'll prove that f is a constant. Let γ be a small circle around z_0 with radius r such that the closed disc with boundary γ is contained in Ω. CIF gives,

$$f(z_0) = \frac{1}{2\pi \imath} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\imath\theta}) d\theta.$$

Hence,

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{\imath\theta})| d\theta \leq |f(z_0)|,$$

since $|f(z_0)|$ is assumed to be the maximum value.

Thus,

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{\imath\theta})| \right] dt = 0.$$

Note that the integrand is non-negative. Therefore it has to be zero; i.e., $|f(z_0)| = |f(z_0 + re^{\imath\theta})|$ for all θ. Since this is true for each small r, we see that $|f(z)|$ is a constant on a small disc around z_0. This means that $f(z)$ is a constant, say c, on this small disc. (Why?) This implies that $f \equiv c$ on Ω by the identity theorem, since a disc has limit points!

A nice consequence of the Maximum modulus principle is the following lemma of Schwartz.

Schwarz Lemma : Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbb{D}$.
Then, $|f(z)| \leq |z| \ \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.

Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$.

Moreover, suppose $|f(z)| = |z|$ for some non-zero z in $\mathbb{D}$, or $|f'(0)| = 1$.Then, $|g(z)| = 1$ at some point of $\mathbb{D}$. Hence by Maximum Modulus Principle, $g(z)$ is a constant , say a with $|a| = 1$. Therefore, $f(z) = az$, as desired.

Open Mapping Theorem

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem: Any non-constant holomorphic function defined on a domain $\Omega \subseteq \mathbb{C}$ is open; i.e, maps open subsets of $\mathbb{C}$ contained in Ω to open subsets of $\mathbb{C}$.

The theorem has an interesting proof which unfortunately we will skip due to lack of time.

MA 205 Complex Analysis: Counting Zeros and Poles

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September 3, 2018

Multiplicity of a zero

Recall that a holomorphic function f on Ω has a zero at z_0 of multiplicity m if m is the least positive integer with $f^{(m)}(z_0)$ is non-zero. This is also equivalent to the fact that $f(z) = (z - z_0)^m h(z)$ where h is a holomorphic function on some small neighborhood of z_0 and $h(z_0) \neq 0$. Here $h(z)$ can be taken to be holomorphic on Ω (why?). Thus if f has finite number of zeros $z_1, \ldots, z_n$ inside Ω with multiplicities $m_1, \ldots, m_n$ respectively, then

$$f(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} H(z) \ (z \in \Omega)$$

for some holomorphic function H on Ω which does not vanish on Ω.

Counting zeros

Theorem

Let f be a holomorphic function on Ω and $\bar{D}(P, r) \subset \Omega$. Suppose that f does not vanish on $\{z : |z - P| = r\}$ and that $z_1, \ldots, z_n$ are the zeros of f in $D(P, r)$ with multiplicities $m_1, \ldots, m_n$ respectively. Then

$$\frac{1}{2\pi\imath} \int_{|z-p|=r} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{n} m_i.$$

Proof: Let

$$f(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} H(z) \ (z \in \Omega)$$

for some holomorphic function H on Ω which does not vanish at $z_1, \ldots, z_{n-1}$ and z_n.

Proof Cont..

Then

$$f'(z) = \prod_{i=1}^{n}(z-z_i)^{m_i} H'(z) + \sum_{i=1}^{n} m_i(z-z_i)^{m_i-1} \prod_{1\leq j\leq n, j\neq i} (z-z_j)^{m_j} H(z)$$

and therefore,

$$\frac{f'(z)}{f(z)} = \frac{H'(z)}{H(z)} + \sum_{i=1}^{n} \frac{m_i}{z-z_i}.$$

Since $\frac{H'(z)}{H(z)}$ is holomorphic on an open set containing $\bar{D}(P,r)$,

$$\frac{1}{2\pi\imath}\int_{|z-P|=r} \frac{f'(z)}{f(z)} dz = 0 + \sum_{i=1}^{n} m_i.$$

Counting poles

Recall that if f is a meromorphic function on Ω with poles $z_1, \ldots, z_n$ of orders $m_1, \ldots, m_n$ respectively. Then

$$H(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} f(z)$$

becomes an holomorphic function on Ω.

Theorem

Let f be a meromorphic function on Ω with poles $z_1, \ldots, z_n$ of orders $m_1, \ldots, m_n$, respectively. Suppose $\bar{D}(P, r) \subset \Omega$ contains all the poles of f and f does not vanish on $\bar{D}(P, r)$. Then

$$\frac{1}{2\pi\imath} \int_{|z-P|=r} \frac{f'(z)}{f(z)} dz = -\sum_{i=1}^{n} m_i.$$

Proof: Define

$$H(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} f(z).$$

Then H is an holomorphic function on an open set containing $\bar{D}(P, r)$ and does not vanish on $\bar{D}(P, r)$. Note that for $z \in \Omega \setminus \{z_1, \dots, z_n\}$,

$$\frac{H'(z)}{H(z)} = \frac{f'(z)}{f(z)} + \sum_{i=1}^{n} \frac{m_i}{z - z_i}.$$

Then by integrating on $|z - P| = r$ we get

$$\frac{1}{2\pi\imath} \int_{|z-P|=r} \frac{f'(z)}{f(z)} dz = 0 - \sum_{i=1}^{n} m_i.$$

Argument principle

Combining the above results, we get a variant of the residue theorem and is known as the argument principle. It is used to count zero's and poles of a meromorphic function on a domain.

Theorem (Argument Principle)

Let f be a meromorphic function on Ω, and let γ be a closed contour contained in Ω such that γ does not pass through any of the zeros and poles of $f(z)$. Suppose, inside γ, f has zeros at $z_1, \ldots, z_n$ with multiplicities $m_1, \ldots, m_n$ respectively and has poles at $w_1, \ldots, w_k$ of orders $\ell_1, \ldots, \ell_k$ respectively. Then

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{n} m_i - \sum_{j=1}^{k} \ell_j.$$

A nice and useful corollary of the argument principle is the following theorem:

Theorem (Rouche's Theorem)

Let γ be a simple closed contour and let $f(z)$ and $g(z)$ be two functions holomorphic on an open set containing γ and its interior. Suppose $|f(z) - g(z)| < |f(z)|$ at all points on γ. Then γ encloses the same number of zero's, counting multiplicities, of $f(z)$ and $g(z)$.

Proof: Let $h(z) = \frac{g(z)}{f(z)}$. Then h is an meromorphic function on an open set containing γ. Note that h does not have any zeros or poles on γ. Since $|h(z) - 1| < 1$ for all z on γ,

$$\int_\gamma \frac{h'}{h} dz = 0.$$

Thus number of zeros and poles of h, counting multiplicities, inside γ are same.

Example

The proof of the theorem follows easily from the fact that, after canceling common factors, the zeros of g (resp. f) are the zero's (resp. poles) of h.

Let us compute the number of zero's of
$f(z) = z^6 + 11z^4 + z^3 + 2z + 4$ inside the unit disc.
Take $g(z) = 11z^4$. Then $|g(z) - f(z)| < |g(z)|$ on the unit circle. Hence $g(z)$ has the same number of roots as $f(z)$ inside the unit circle. But the number of roots of $g(z)$ inside unit circle is 4 (counting mutiplicity) which therefore equals number of roots of $f(z)$.

Example

Lets count number of roots of $f(z) = e^z - 2z - 1$ inside the unit circle.
Let us consider $g(z) = -2z$. Then

$$|g(z) - f(z)| = |e^z - 1| = |\sum_{1}^{\infty} \frac{z^n}{n!}| \leq \sum_{1}^{\infty} \frac{|z^n|}{n!} = e - 1 < |g(z)|$$

on the unit circle. Hence by Rouche's theorem $f(z)$ and $g(z)$ have equal number of roots in the unit circle, namely 1.

FTA

Here's another quick and pretty proof of FTA using Rouche's theorem.

Let $f(z) = a_0 + a_1 z + \cdots + z^n$ be a non-constant polynomial. Take $g(z) = z^n$. Then on a sufficiently large circle around 0 of radius R, $|f(z) - g(z)| < |f(z)|$. Hence $f(z)$ and $g(z)$ have same number of zero's in the disc of radius R. Since $g(z)$ has n zero's, so does $f(z)$!

Picard's theorem

I now restate another absolutely spectacular theorem in complex analysis called Picard's theorem on the values taken by a holomorphic function.

Theorem (Big Picard's Theorem)

Let z_0 be an essential singularity of $f(z)$. Then in any punctured neighborhood of z_0, the image of $f(z)$ can miss atmost one point.

This theorem is called the Big Picard Theorem in view of what comes next.

Theorem (Little Picard theorem)

Any non-constant entire function can miss atmost one point.

The little Picard Theorem can be seen to be a corollary of the Big Picard Theorem as follows.

Picard's Theorem

Recall the following fact mentioned earlier: An entire function has a pole at infinity if and only if it is a non-constant polynomial.

Let $f(z)$ be a non-constant entire function. We wish to show it misses atmost one point. If $f(z)$ is a polynomial, then it is surjective by FTA. If $f(z)$ is not a polynomial, then it has an essential singularity at infinity (WHY ?). That is $f(\frac{1}{z})$ has an essential singularity at 0. Thus by Big Picard theorem, in any punctured neighborhood of 0, say of radius r, $f(\frac{1}{z})$ misses atmost one point. But this implies that in the complement of the circle of radius $1/r$, $f(z)$ misses atmost one point. This is what we wanted.

Exercise: If a non-constant entire function misses one point c, show that it is of the form $e^{f(z)} + c$ for some entire function $f(z)$.

Picard (1856-1941); Wiki

Picard was a top rate mathematician who did fundamental work in many disciples; analysis, function theory, differential equations, and analytic geometry to name a few. In physics he worked on elasticity, heat and electricity. Hadamard wrote about his teacher Picard:- A striking feature of Picard's scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known.

It is a remarkable fact that between 1894 and 1937 he trained over 10000 engineers who were studying at the cole Centrale des Arts et Manufactures.

Conformal Mappings and Riemann Mapping Theorem

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Motivation

Today we look at conformal mappings. Roughly speaking a conformal map between two subsets U and V of $\mathbb{R}^n$ is a differentiable mapping that preserves magnitude and orientation of angles between directed curves.
A more general class of mappings which only preserve magnitude of angles between directed curves but not neccessarily their orientation are called <u>isogonal mappings</u>. We will focus our attention on studying conformal mappings between open subsets of $\mathbb{C}$.

Preservation of Angles

Let Ω be a domain in $\mathbb{C}$ and let $f(z)$ be a holomorphic function on Ω. C be a smooth parametrized curve in Ω represented by the equation $z(t); a \leq t \leq b$. Consider the image of C under f , say γ. It is parametrized by $w = f[z(t)]$. Suppose C passes through $z_0 = z(t_0);\ a < t_0 < b$ at which $f(z)$ is analytic and $f'(z_0) \neq 0$. Then by chain rule,

$$w'(t_0) = f'[z(t_0)]z'(t_0)$$

and hence $\arg\ w'(t_0) = \arg\ f'[z(t_0)] + \arg\ z'(t_0)$

Thus if the directed tangent to C at z_0 makes an angle θ with the x-axis, then γ makes an angle $\theta + \arg f'(z_0)$ with the x-axis.Consequently if C_1 and C_2 are two smooth, parametrized (hence also directed) curves passing through z_0 and intersecting at an angle ϕ at z_0 (meaning their tangents at z_0 make an angle ϕ), then their images also make an angle ϕ at $f(z_0)$.
Because of this angle preserving property, a holomorphic function $w = f(z)$ is said to be conformal at z_0 if $f'(z_0)$ is non-zero.

Examples

The mapping $w = e^z$ is conformal at all points since the derivative is everywhere non-vanishing. Consider two lines $C_1 = \{x = c_1\}$ and $C_2 = \{y = c_2\}$ in the domain; the first one directed upwards and the second one directed to the right. These lines intersect at the point (c_1, c_2) at right angles. Under this transformation, these lines get mapped to a positively oriented circle around origin and a ray from the origin respectively.Thus the images also intersect at right angles. Also note that the orientation is respected under the mapping; the angle between C_1 and C_2 and well as their images is $-90°$.

Counterexample

The mapping $z \to \bar{z}$ which is reflection about real line is isogonal but not conformal.

The mapping $z \to z^2$ is not conformal at 0 and does not preserve angles; the images of the real and imaginary axis are the real axis and the real axis with opposite orientation resp. Thus the angle between curves through 0 gets doubled. This is true for any two smooth curves passing through 0. This is a special case of the following more general fact:

If z_0 is a point at which first $m-1$ derivatives vanish, then then angle between two smooth curves passing through z_0 gets multiplied by m.

Inverse Function Theorem (Special Case)

Let us understand holomorphic mappings in a neighborhood of a conformal point. Recall the basic fact from calculus: If $f \in C^1(\mathbb{R})$ and $f'(x_0) \neq 0$, then in a neighborhood of x_0, $f(x)$ is either strictly increasing or strictly decreasing. In particular it is injective in a neighborhood of x_0. Converse is not true : Even if the derivative vanishes at a point, the function could be injective in a neighborhood (Example ?).It is natural to ask for the analogues statement for functions of a complex variable. A special case of the inverse function theorem provides the answer:

Let Ω be a domain in $\mathbb{C}$ and let $f(z)$ be a holomorphic function on Ω such that for some $z_0 \in \Omega$, $f'(z_0) \neq 0$. Then in a neighborhood of z_0, $f(z)$ is injective.

In this setting even the converse is true: If z_0 is a point such that $f'(z_0) = 0$ then in no neighborhood of z_0 is $f(z)$ injective.
For example note that while $x \to x^3$ is injective in a neighborhood of 0, $z \to z^3$ is not injective in any neighborhood.

Policy

- Two marks will be deducted if you don't write Tutorial Batch/Division in the answer booklet. Note that Tutorial Batch and Division are the same for this course.
- Use only results which are discussed in the class or given in the tutorial sheet.
- Please write short answers. Short and correct answers would be given high value.
- Answers with missing justification/argument/step will loose marks.

Biholomorphism

A very important subclass of conformal mappings are what are called Biholomorphisms.

Definition: If U and V are open subsets of $\mathbb{C}$ (not necessarily domains), a biholomorphism from U to V is a mapping $f : U \to V$ which is bijective and holomorphic.
If such a mapping exists U and V are said to be biholomorphic.
Note in particular by the earlier remark that such an $f(z)$ is conformal at all points in U. An easy exercise show that the inverse mapping is automatically holomorphic.
Thus a biholomorphism is a bijective map, holomorphic both ways.
Clearly composite of biholomorphisms is a biholomorphism and the inverse of a biholomorphism is a biholomorphism. In view of the special case of the inverse function theorem stated earlier, a holomorphic map which is conformal at a point z_0 is a biholomorphism in a neighborhood of z_0 (what's called a local biholomorphism).

Motivation for this notion

The motivation for this definition is simple: If two open subsets are biholomorphic, then (loosely speaking) studying complex analysis on one of them is equivalent to studying it on the other. For example if $f : U \to V$ is a biholomorphism, then $g : V \to \mathbb{C}$ is holomorphic on V if and only if $g \circ f$ is a holomorphic function on U.

Examples

1. The identity map $f : U \to U$ is clearly a biholomorphism. More generally, multiplication by a non-zero scalar defines a biholomorphism from $\mathbb{C}$ to $\mathbb{C}$. Similarly open discs of any two radii are biholomorphic.

2. The mapping from the open unit disc to the upper half plane $\mathbb{D} \to \mathbb{H}$ given by $z \to i\frac{1-z}{1+z}$ is a biholomorphism as one can check.

If $U \subseteq \mathbb{C}$ is open, then a biholomorphism from U to U is called an automorphism.

3. A basic fact is that the only automorphisms of $\mathbb{C}$ are of the form $az + b$ with $a \neq 0$. This is because biholomorphisms can be easily seen to have poles at infinity and hence polynomial. But the only injective polynomial functions are linear polynomials with non-zero linear coefficient !

Examples

4. As a consequence of Schwartz lemma, one can show that the automorphisms of the unit disc are

$$z \to \lambda \frac{z - a}{1 - \bar{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$.

I now state the deep, fundamental and spectacular theorem of Riemann:

Riemann Mapping Theorem: Any open, simply-connected subset of $\mathbb{C}$ other than $\mathbb{C}$ is biholomorphic to the open unit disc.

Note that by Lioville's theorem, the plane and the disc are not biholomorphic.
Given the rigid nature of holomorphic functions, the theorem is hugely surprising and beautiful. This theorem was conjecture by Riemann in 1851 in his thesis. He gave an incomplete proof based on Dirichlet principle stated roughly as : Minimizer of a certain energy functional is a solution to Poisson's equation.
Weierstrass found an error in the proof. The first complete proof was due to Constantin Carathodory in 1922 and simplified by Paul Koebe 2 years later. Here is a link to the history of the Riemann mapping theorem.
https://www.math.stonybrook.edu/ bishop/classes/math401.F09/GrayRMT.pdf

MA 205 Complex Analysis

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Introduction

Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it enjoyable. Complex analysis is one of the most beautiful areas of pure mathematics; in fact quite a few mathematicians regard it as the most beautiful area of mathematics. At the same time it is an important and powerful tool in the physical sciences and engineering. Here's a very brief summary of what we shall study in the next two months.

Introduction

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable in the "complex sense". If f fails to be differentiable at some points, we'll also investigate such failure.

Thus, in a way,

> *in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.*

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely beautiful and charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105, but no MA 305 or 405!

Furthermore complex analysis introduces techniques which are often useful in answering questions which a priori dont have anything to do with complex numbers. We will see examples of this later.
Today, I'll try to illustrate the beauty of complex analysis in two distinct ways; I'll give you two surprises!

Motivation

God made integers, all else is the work of man
- Leopold Kronecker

So let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the basic "algebra test", namely division. $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! , namely Cauchy sequences of rational numbers need not converge to a rational number.

$\mathbb{R}$ passes both these tests and hence analysis over $\mathbb{R}$ is rich and exciting. But it fails another "algebra test" namely obtaining roots of polynomials.

Theorem

Every non-constant polynomial with complex coefficients has a complex root.

This theorem fails over all the other "number systems "we know, namely $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Today this theorem has more than hundred proofs, many of them using complex analysis. We will see at least one proof of this in this course.

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. In other words, you are <u>imagining</u> an $\imath$, which has this property that $\imath^2 = -1$. And then we can write $x^2 + 1 = (x - \imath)(x + \imath)$.

We imagine $\mathbb{C}$ as

$$\mathbb{R} + \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$\begin{aligned}(a + \imath b)(c + \imath d) &= ac + a\imath d + \imath bc + (\imath b) \cdot (\imath d) \\ &= (ac - bd) + \imath(ad + bc)\end{aligned}$$

which is an another complex number.

Incidentally, if $z = x + \imath y \in \mathbb{C}$, we call x to be $\mathrm{Re}(z)$ and y to be $\mathrm{Im}(z)$. So coming back to the fundamental theorem of algebra, it is interesting that just adding one root of one real polynomial, namely $X^2 + 1$ gives you all the roots of all the complex polynomials !

For $z, z_0 \in \mathbb{C}$, $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$.
For $\delta > 0$

$$\mathcal{B}(z_0, \delta) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

is the open disc of radius δ and center at z_0 (Also known as δ-neighbourhood of z_0). Let $\Omega \subseteq \mathbb{C}$. We say that Ω is an open subset of $\mathbb{C}$ if given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that $\mathcal{B}(z_0, \delta) \subseteq \Omega$.

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them, i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain. An arbitrary open set in $\mathbb{C}$ is a disjoint union of domains. In this course we will be mostly interested in complex-valued functions defined over domains.

Limit

Let's start with recalling the notions of limit, continuity, and differentiation. Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\lim_{x \to a} f(x) = L$$

$$\iff$$

values of $f(x)$ can be made as close to L as we like,
by taking x close enough to a, on either side of a,
but not equal to a.

$$\iff$$

$|f(x) - L|$ can be made as small as we like by taking
$|x - a|$ sufficiently small, for $x \neq a$.

Limit

$\iff$

for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f as $z \mapsto z_0$, $z_0 \in \mathbb{C}$, if for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this!

Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ and $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then,

1. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$
2. $\lim_{z \to z_0} [cf(z)] = c \lim_{z \to z_0} f(z)$
3. $\lim_{z \to z_0} [f(z)g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$
4. $\lim_{z \to z_0} \frac{1}{f(z)} = \frac{1}{\lim_{z \to z_0} f(z)}$, *if* $\lim_{z \to z_0} f(z) \neq 0$.

Continuity

In the real case:

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain if it is continuous at every point in the domain.

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is said to be **differentiable**, at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

Differentiation

We say that f is holomorphic on an open set Ω if f is differentiable at each point of Ω. f is holomorphic (also called complex analytic) at z_0 if it is holomorphic in some neighbourhood of z_0. Similarly, holomorphicity on a non-empty open set.

Remark: Note that differentiability at a point does not imply holomorphicity at that point.

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable, always differentiable.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case. f' needn't even be continuous. Examples?

Can someone venture a guess as to why identical definitions lead to such extreme scenarios?

Syllabus

Definition and properties of analytic functions.
Cauchy-Riemann equations, harmonic functions.
Power series and their properties.
Elementary functions.
Cauchys theorem and its applications.
Taylor series ans Laurent expansions.
Residues and the Cauchy residue formula.
Evaluation of improper integrals.
Conformal mappings.
Inversion of Laplace transforms.

References

1. R. V. Churchill and J. W. Brown, Complex variables and applications (7th Edition), McGraw-Hill (2003).
2. J. M. Howie, Complex analysis, Springer-Verlag (2004).
3. M. J. Ablowitz and A. S. Fokas, Complex Variables-Introduction and Applications, Cambridge University Press, 1998 (Indian Edition).
4. E. Kreyszig, Advanced engineering mathematics (8th Edition), John Wiley (1999).

References

More advanced references :

1. Lars Ahlfors - Complex Analysis
2. John Conway - Functions of a Complex Variable
3. Serge Lang - Complex Analysis

MA 205 Complex Analysis: CR Equations

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In the last class, we introduced complex numbers and studied complex valued functions defined on a domain in $\mathbb{C}$. We stated the fact that every polynomial with complex coefficients has a complex roots. This is called the fundamental theorem of algebra. We introduced complex-differentiability of a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}$. We also stated the fact that if f is once differentiable in Ω, then it is infinitely many times differentiable in Ω.

Cauchy-Riemann Equations

Today, first we'll derive the so called Cauchy-Riemann equations. There are two Cauchy-Riemann equations, and these are partial differential equations; i.e., equations containing partial derivatives. If f is complex differentiable at a point $z_0 = a + \imath b$, then these two equations will be satisfied at the point (a, b).

Cauchy-Riemann Equations

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Thus,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists. In the last class, we have stressed the point that the existence of this complex limit means a lot; the limit exists as z approaches z_0 along any path. To derive the CR equations, we'll in particular look at the existence of this limit as $z \to z_0$ along the x-direction and the y-direction.

Let $z = x + \imath y$ and $f(z) = u(x, y) + \imath v(x, y)$. Now, as $z \to z_0$ in the x-direction:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
$$= \lim_{h \to 0} \left[\frac{u(a + h, b) - u(a, b)}{h} + \imath \frac{v(a + h, b) - v(a, b)}{h} \right]$$
$$= \lim_{h \to 0} \frac{u(a + h, b) - u(a, b)}{h} + \imath \lim_{h \to 0} \frac{v(a + h, b) - v(a, b)}{h}$$
$$= u_x(a, b) + \imath v_x(a, b).$$

In writing the limit of a sum as the sum of the limits, we have used the fact that the individual limits exist. Why is this true in our situation?

Similarly, in the y-direction, we get

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0 + \imath k) - f(z_0)}{\imath k} = v_y(a, b) - \imath u_y(a, b).$$

Thus, differentiability of $f = u + \imath v$ at $z_0 = a + \imath b$ implies that u_x, u_y, v_x, v_y exist at (a, b) and they satisfy

$$u_x = v_y \ \& \ u_y = -v_x$$

at (a, b). These are the CR equations. If CR equations are not satisfied at a point, then f is not differentiable at that point.

Example: Consider $f(z) = |z|^2$. Here, $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. Thus CR equations are satisfied only at the point $(0, 0)$. We conclude that f is not differentiable at any point other than $(0, 0)$. Can we conclude that f is differentiable at $(0, 0)$? Well, we need to check; CR equations give only one direction. In other words, real and imaginary parts of f satisfying CR equations at a point is necessary but not sufficient for f to be differentiable at that point. In this example:

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2}{z} = 0.$$

Example:

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Here,

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

$$v(x, y) = \begin{cases} \frac{-3x^2y + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Check that CR equations are satisfied at $(0, 0)$. You'll get $u_x = v_y = 1$ and $u_y = -v_x = 0$ at $(0, 0)$.

But, f is not differentiable at 0.

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{\bar{z}^2}{z^2}=\lim_{(x,y)\to(0,0)}\frac{(x-\imath y)^2}{(x+\imath y)^2}.$$

If $(x, y) \to (0, 0)$ via either of the axes, this limit is 1. If $(x, y) \to (0, 0)$ via $y = x$, this limit is -1. So limit does not exist.

Cauchy-Riemann Equations

If $z = x + \imath y$, then,

$$x = \frac{z + \bar{z}}{2}, \; y = \frac{z - \bar{z}}{2\imath}.$$

Suppose for a moment that z and $\bar{z}$ are independent variables! Formally applying chain rule:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2\imath} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath \frac{\partial f}{\partial y}\right).$$

Similarly,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \imath \frac{\partial f}{\partial y}\right).$$

Cauchy-Riemann Equations

Motivated by this, we introduce the symbols:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right);\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right).$$

Note that CR equations now can be written as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Cauchy-Riemann Equations

We can of course view $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ as a function of two real variables;

$$f(x, y) = (u(x, y), v(x, y)).$$

For such functions, in MA 105, you have seen the notion of the total derivative. Recall: f is differentiable at (a, b) if there exists a 2×2 matrix $Df(a, b)$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(a+h, b+k) - f(a, b) - Df(a, b) \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

Of course, if total derivative exists, then all the partial derivatives exist, and

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Existence of partial derivatives does not imply the existence of total derivative, but existence of partial derivatives which are continuous throughout the domain does imply the existence of total derivative.

Exercise

Exercise: Show that if f is complex differentiable, then f is real differentiable; i.e., f has a total derivative as a function of two real variables. Show that the converse is not true.
(At the moment solve this exercise assuming the continuity of the first partial derivatives of u and v. We shall see later that this assumption can be removed (it is automatic)).

Thus, complex differentiability implies:

- real differentiability
- real and imaginary parts satisfy CR.

What if we assume both these? Can we then say f is complex differentiable? <u>And the answer is Yes</u>.

Proof: Since $f = u + \imath v$ is real differentiable,

$$\lim_{(x,y)\to(a,b)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \right\|}{\|(x-a,\ y-b)\|} = 0.$$

Note that the numerator is nothing but

$$|f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)|,$$

where $\alpha = u_x + \imath v_x$, $\beta = u_y + \imath v_y$.

Define

$$\eta(z) = \frac{f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)}{z - z_0}.$$

Observe that

$$\lim_{z \to z_0} \eta(z) = 0.$$

Thus,

$$f(z) - f(z_0) = \alpha(x-a) + \beta(y-b) + \eta(z)(z - z_0),$$

with $\eta(z) \to 0$ as $z \to z_0$.

Then

$$f(z) - f(z_0) = \frac{\alpha - \imath\beta}{2}(z - z_0) + \frac{\alpha + \imath\beta}{2}\overline{z - z_0} + \eta(z)(z - z_0).$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{z - z_0}}{z - z_0} + \eta(z).$$

Question is whether the lhs limit exists as $z \to z_0$. This exists if and only if the rhs limit exists. Note that

$$\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}$$

does not exist (why?) and $\lim_{z \to z_0} \eta(z)$ exists.

Thus the rhs limit exists if and only if

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

i.e., CR equations are satisfied at z_0. Also, if this is the case,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0),$$

since $\lim_{z \to z_0} \eta(z) = 0$.

Corollary: Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be such that the partial derivatives exists in a neighborhood of z_0 and continuous at z_0. If they satisfy the CR equations at z_0, f is differentiable at z_0. (Proof?)

The assumptions in the statement of the corollary can be weakened. In fact, the following is true:

Theorem

Let f be continuous on Ω. Suppose the partial derivatives exist and satisfy the Cauchy-Riemann equations at every point in Ω. Then f is holomorphic in Ω.

We shall not prove this theorem.

Exercise

Exercise: Show that $f(z) = e^x(\cos y + \imath \sin y)$ is holomorphic throughout $\mathbb{C}$.
Note that $f'(z) = f(z)$. This is the complex exponential function.
Exercise: Show that the CR equations take the form

$$u_r = \frac{1}{r} v_\theta \ \& \ v_r = -\frac{1}{r} u_\theta$$

in polar coordinates.

MA 205 Complex Analysis: Power Series

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In the last lecture, you saw holomorphic functions in some detail. If $f = u + \imath v$ is holomorphic in Ω, then (i) both u and v satisfy CR equations, and (ii) $f(x, y) = (u(x, y), v(x, y))$ is real differentiable. We also saw that though neither (i) nor (ii) is sufficient to guarantee holomorphicity, both (i) and (ii) together do guarantee holomorphicity of f. Today, we'll discuss the so called <u>harmonic</u> and <u>analytic</u> functions

A real valued function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if it is twice continuously differentiable and satisfies $u_{xx} + u_{yy} = 0$ on U. If $f = u + \imath v$ is holomorphic on Ω, then both u and v are harmonic on Ω. Indeed, CR equations tell us that

$$u_x = v_y \ \& \ u_y = -v_x.$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0.$$

Similarly for v.

Suppose u and v are harmonic functions on Ω. We say that v is a harmonic conjugate of u if $f = u + \imath v$ is holomorphic in Ω.
Example: $v(x, y) = 2xy$ is a harmonic conjugate of $x^2 - y^2$ in any domain. Indeed, $f(z) = z^2$ is holomorphic everywhere.
Note that v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v! In fact:
Exercise: If u and v are harmonic conjugates of each other, show that they are constant functions.

Here's a general method to find a harmonic conjugate: given a harmonic u, find u_x. Equate $v_y = u_x$ and integrate wrt y. You'll get $v(x, y) = \int u_x dy + \phi(x)$. Now $v_x = \ldots + \phi'(x)$. Equate this to $-u_y$ to find $\phi(x)$. That gives you v.
This might give you an impression that you can always find a harmonic conjugate, but this is not so.

Unfortunately this method fails in general. Try and think of the reason !

But if Ω is "nice", then every harmonic u on Ω has a harmonic conjugate. Conversely, if every harmonic u on Ω has a harmonic conjugate, then Ω has to be "nice". Thus, the question in analysis: "does every harmonic function has a harmonic conjugate?" is answered by geometry: "answer depends on the shape of the domain". It's relevant at this point to recall from MA 105 that curl of grad is always zero but curl free is certainly a grad of something only when the domain is "nice" (for example $\mathbb{C}$ or a disc in $\mathbb{C}$). Remember this !

Polynomials

Now, we'll discuss the so called analytic functions. To warm up, let's first look at the simplest of all functions. What's the most trivial function? $f(z) = a_0$, i.e, constant functions. The next easiest class of functions are polynomials:
$f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$. The same polynomial $f(z)$ can be expanded along any point z_0. That is, $f(z)$ can be written as

$$b_0 + b_1(z - z_0) + \ldots + b_n(z - z_0)^n.$$

A smarter way to calculate b_i would be $b_i = \frac{f^{(i)}(z_0)}{i!}$.

A polynomial, by definition, is a "finite" polynomial; i.e., it comes with a finite degree. As the next class of functions, we consider functions defined by their power series; i.e.,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

or more generally, $\sum_{i=0}^{\infty} a_i (z - z_0)^i$. Of course one has to be careful; there are convergence issues. For example, $f(z) = 1 + z + z^2 + \ldots$ makes sense for all z such that $|z| < 1$, but not when $|z| > 1$. (Why?)

It's a beautiful fact that the radius of convergence exists for any power series; i.e., there exists R such that $\sum_{i=0}^{\infty} a_i (z - z_0)^i$ converges when $|z - z_0| < R$, and diverges when $|z - z_0| > R$. In other words, the radius of convergence is the largest R such that the given power series converges inside a disc of radius R. We'll soon give a formula for R in terms of the coefficients of a given power series.

We write $a = \sum_{n=1}^{\infty} a_i$, $a_i \in \mathbb{C}$, if $\lim_{n\to\infty} s_n = a$ where $s_n = a_1 + \ldots + a_n$. The series $\sum_{n=1}^{\infty} a_i$ is said be absolutely convergent if $\sum_{n=1}^{\infty} |a_i|$ is convergent.

Exercise: 1. Absolute convergence $\Rightarrow$ convergence.

2. (Comparison Test) If $\sum_{n=1}^{\infty} b_i$ is absolutely convergent, and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum_{n=1}^{\infty} a_i$ is absolutely convergent.

Easy Observation: If a power series $\sum_{i=0}^{\infty} a_i z^i$ converges for $z = z_0$, then it converges absolutely for any z with $|z| < |z_0|$.

Recall **Supremum**: Let $\{x_n\}$ be a sequence of real numbers. We say that a real number M is the supremum of this sequence if every term of the sequence is less than or equal to M and there exists terms of the sequence which are arbitrarily close to M. Equivalently it is the smallest real number having the property that it is greater than or equal to all the terms of the sequence. The supremum may or may not be equal to any of the terms of the sequence.

Upper Limit/ Limit Supremum: For a sequence of real numbers $x_1, x_2, \ldots$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \ldots\}$. Then the sequence $y_1, y_2, \ldots$ is a monotonically decreasing sequence. The limit of $\{y_n\}$ is called the upper limit (also called limit superior, denoted lim sup) of the sequence $\{x_i\}$. It can be ∞. If limit exists, then the upper limit coincides with the usual limit.

Examples:

1. the sequence $1, 2, 3, \ldots$ has upper limit ∞.
2. the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has upper limit 0.
3. the sequence $1, -1, 1, -1, \ldots$ has upper limit 1.

Tutors

Tutorial Batch	Tutor	Venue
T1	Deep Karman Pal Singh	LT1
T2	Archiki Prasad	LT2
T3	Shourya Pandey	LT3
T4	Shubhang Bhatnagar	LT4
T5	Reebhu Bhattacharyya	LT5

Theorem (Cauchy's Root Test)

For a series $\sum_{n=1}^{\infty} a_i$ *of complex numbers, let* $C = \limsup_{i\to\infty} \sqrt[i]{|a_i|}$. *Then the series converges absolutely if* $C < 1$ *and it diverges if* $C > 1$.

The test is indecisive for $C = 1$

Proof: If $C < 1$, then we can choose a k such that $\sqrt[i]{|a_i|} < k < 1$ after a stage (by the definition of the upper limit). Thus, after a stage, $|a_i| < k^i < 1$.

Now $\sum_{n=1}^{\infty} k^i$ converges absolutely, and therefore $\sum_{n=1}^{\infty} a_i$ is absolutely convergent (by comparison test). If $C > 1$, then for infinitely many i, $\sqrt[i]{|a_i|} > 1$. Hence $|a_i|$ is bigger than 1 for infinitely many i.

Thus, $\lim_{i \to \infty} a_i \neq 0$. So $\sum_{n=1}^{\infty} a_i$ diverges. (Why?)

Theorem (Ratio Test)

For a series $\sum_{n=1}^{\infty} a_i$, *let* $L = \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. *Then, if* $L < 1$, *the series converges absolutely. The series diverges if there exists* N *such that* $\left|\frac{a_{i+1}}{a_i}\right| > 1$ *for* $i \geq N$.

Remark $L > 1$ in the above test doesn't imply that the series diverges. (Exercise !)

Proof: Let $L < 1$. Let r be such that $L < r < 1$. Then after a stage, say for $i \geq N$, $|a_{i+1}| < r|a_i|$. So $|a_{i+k}| < r^k|a_i|$. Now

$$\sum_{n=1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{N+1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{1}^{\infty}|a_{N+i}|$$

$$< \sum_{0}^{N}|a_i| + |a_N|\sum_{1}^{\infty} r^i = \sum_{0}^{N}|a_i| + |a_N|\frac{r}{1-r} < \infty.$$

In the other case, $|a_{i+1}| > |a_i|$ for all large enough i, so $\lim_{n\to\infty} a_i \neq 0$. Therefore the series diverges.

Theorem (Existence of Radius of Convergence)

For the power series $\sum_{i=1}^{\infty} a_i(z-z_0)^i$*, let* $R = \dfrac{1}{\limsup\limits_{i\to\infty} \sqrt[i]{|a_i|}}$*. Then the power series converges absolutely if* $|z-z_0| < R$ *and diverges if* $|z-z_0| > R$*.*

<u>Proof</u>: Apply root test.

Remark: (i) If $\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$ exists, then by applying the ratio test $R = \lim_{i\to\infty}\left|\frac{a_i}{a_{i+1}}\right|$.
(ii) If a series converges by the ratio test, then it converges by the root test as well. But not conversely. Thus the root test is better than the ratio test. But the ratio test is often easier to use whenever it succeeds. In fact:

$$\limsup_{i\to\infty}\sqrt[i]{|a_i|} \leq \limsup_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$$

Examples:

1. $\sum_n \frac{z^n}{n!}$. Apply ratio test. $\lim_{i\to\infty}\left|\frac{a_i}{a_{i+1}}\right| = \lim_{i\to\infty} i = \infty$; i.e., the series converges everywhere.

2. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$ Radius of convergence is 1. Both the tests apply here.

3. $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \ldots$. Check that the ratio test fails. Apply root test to show that the radius of convergence is $\frac{1}{\sqrt{2}}$.

MA 205 Complex Analysis: Exponential Function

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July 26, 2018

So what did we do in the last class? We looked at power series, saw the existence of radius of convergence for any power series, remember that this was given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

and let me repeat once again, it's limsup and not the limit in the above and thus it always exists. If

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists, then this too gives R.

Power series can be added, subtracted, and multiplied in the obvious way. It can also be differentiated term by term, in its domain of convergence. Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, then,

$$\begin{aligned}
&\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\
&= \lim_{z\to z_0} \sum_n a_n \left(\frac{(z-a)^n - (z_0-a)^n}{z-z_0} \right) \\
&= \sum_n a_n \left(\lim_{z\to z_0} \frac{(z-a)^n - (z_0-a)^n}{z-z_0} \right) \\
&= \sum_n n a_n (z_0-a)^{n-1}.
\end{aligned}$$

Apply root test to check that the radius of convergence of $\sum_n n a_n (z-a)^{n-1}$ is same as the given power series.

Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is said to be **analytic** if it is locally given by a convergent power series; i.e., every $z_0 \in \Omega$ has a neighbourhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighbourhood. Analytic functions are infinitely differentiable; you only have to differentiate the power series term by term. Also, if $f(z) = \sum_{n=1}^{\infty} a_i (z - z_0)^i$, then $a_i = \frac{f^{(i)}(z_0)}{i!}$. Thus, an analytic function is given by its Taylor series. We'll later prove:

$$\textit{holomorphic} \implies \textit{analytic.}$$

This would prove our statement from Lecture 1 that once differentiable is always differentiable!

Just as in the complex case, power series and analytic functions can be defined in the real case too. But unlike in the complex case, differentiable does not mean real analytic. In fact, even infinitely differentiable does not mean real analytic. For example, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable but not real analytic. In this example, $f^{(i)}(0) = 0$ for all i, and thus the Taylor series of f is the zero function.

Now, we'll use our knowledge of power series to construct a few basic functions. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots.$$

We've seen that its radius of convergence is ∞; i.e., this function is well-defined for any $z \in \mathbb{C}$. This function will keep our company throughout this course. So let's befriend it a bit more ! By term by term differentiation, one observes that

$$f'(z) = f(z),$$

and $f(bz)' = bf(bz)$. We'll denote this function by $\exp(z)$, also denoted e^z.

Now consider the function

$$h(z) = \exp(z) \cdot \exp(-z).$$

This is defined throughout $\mathbb{C}$. What's $h'(z)$?

$$h'(z) = \exp(z) \cdot (-\exp(-z)) + \exp(-z) \cdot \exp(z) = -h + h = 0.$$

Therefore $h(z) \equiv c$ and since $h(0) = 1$, it is identically equal to 1.
Thus, we have proved two things:
(i) $\exp(z)$ is never vanishing.
(ii) $\exp(-z) = \frac{1}{\exp(z)}$.

Note that the derivative of $f(z) = a\exp(bz)$ is $f'(z) = bf(z)$. Interestingly, the converse is also true. Thus,

$$f(z) = a\exp(bz) \text{ for } a, b \in \mathbb{C} \iff f'(z) = bf(z).$$

Proof: Assume $f'(z) = bf(z)$ for $b \in \mathbb{C}$. Now consider

$$h(z) = f(z)\exp(-bz).$$

Then, $h'(z) = -bh + bh = 0$, for all z in the domain. So, $h(z) \equiv a$ for some $a \in \mathbb{C}$. Therefore,

$$f(z) = \frac{a}{\exp(-bz)} = a\exp(bz),$$

by what we already know.

Corollary: $f' = f$ and $f(0) = 1$ characterizes the exponential function. The function

$$f(z) = e^x(\cos y + \imath \sin y),$$

is holomorphic throughout $\mathbb{C}$ and $f' = f$. Clearly, $f(0) = 1$ as well. Thus,

$$\exp(z) = e^x(\cos y + \imath \sin y).$$

Remark: e^x here is e to the power of x, and e is the number that you know from MA 105 (base of natural logarithm). We'll try and reconstruct everything about logarithm and exponential function from scratch.

By now we know that exp is defined throughout $\mathbb{C}$ and that 0 is not in the range of $\exp(z)$. , exp is a map from $\mathbb{C} \to \mathbb{C}^\times$. Now exp has this wonderful property that it takes the "correct" operation in $\mathbb{C}$ to the "correct" operation in $\mathbb{C}^\times$.

$$\exp(w + z) = \exp(w) \cdot \exp(z).$$

Thus in the language of group theory exp is a homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$.
Proof: Fix $w \in \mathbb{C}$. Then the function $f(z) = \exp(w + z)$ is holomorphic in $\mathbb{C}$ and $f'(z) = f(z)$. So, $f(z) = a \exp(z)$ for some constant a. By evaluating f at 0, see that $a = \exp(w)$. Thus, $f(z) = \exp(w) \cdot \exp(z)$.

This property of converting addition into multiplication also characterizes $\exp(z)$.

Let $0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is such that f is differentiable at 0 and $f(0) \neq 0$. Suppose $f(w+z) = f(w)f(z)$ whenever $w, z, w+z \in \Omega$. Then, $f(z) = \exp(bz)$, where $b = f'(0)$.

Trigonometric Functions

Recall the Taylor expansions:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots = \frac{\exp(\imath y) - \exp(-\imath y)}{2\imath}$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots = \frac{\exp(\imath y) + \exp(-\imath y)}{2}.$$

Motivated by this, we define complex trigonometric functions:

$$\sin z = \frac{\exp(\imath z) - \exp(-\imath z)}{2\imath} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = \frac{\exp(\imath z) + \exp(-\imath z)}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots.$$

Hyperbolic Functions

Define hyperbolic sine and hyperbolic cosine by:

$$cos(hz) = \frac{\exp(z) + \exp(-z)}{2}.$$

Its power series is given by

$$coz(hz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

Simillarly,

$$sin(hz) = \frac{\exp(z) - \exp(-z)}{2}.$$

Its power series is given by

$$sin(hz) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

<u>Exercise</u>:
(i) Define other trigonometric functions.
(ii) Show that $\exp : \mathbb{C} \to \mathbb{C}^{\times}$ is onto. Is it one-to-one ? (Show that $\exp(z) = \exp(z + 2\pi i)$)
(iii) Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. In particular, note the difference with real sine and cosine which were bounded by 1.
(iv) Show that $\sin^2 z + \cos^2 z = 1$,
$\sin(z + w) = \sin z \cos w + \cos z \sin w$,
$\cos(z + w) = \cos z \cos w - \sin z \sin w$.

We have seen that exp is not a 1-1 function. Hence its inverse is not defined everywhere. Nevertheless we would like to construct an analytic inverse function, called logarithm on certain subsets, i.e, a analytic function $g(z) = log(z)$ such that $\exp(g(z)) = z$. As remarked before, log will be undefined at 0. Let z be any complex number. Then
$z = |z|(cos(\theta) + isin(\theta)) = |z| \exp(i\theta) = \exp(log|z| + i\theta)$. Here $|z|$ is the magnitude and θ is the argument. Note that θ is defined only upto an integer multiple of $2\pi i$. Then the solutions of $\exp(g(z)) = z$ are given by $g(z) = log|z| + i(\theta + 2\pi n)$.

Definition

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f(z)$ be a continuous function on Ω such that $\exp(f(z)) = z \ \forall z \in \Omega$. Then f is called a branch of the logarithm.

Logarithm cont...

Lemma

Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a branch of the logarithm. Then any other branch of the logarith differs from f by a constant multiple multiple of $2\pi i$.

Proof.

Let $g(z)$ be a branch of the logarithm. Then $f(z) - g(z)$ is a constant multiple of $2\pi i$ for all $z \in \Omega$. Since Ω is connected, and $f(z) - g(z)$ is continuous while integral multiples of $2\pi i$ is a discrete set, it follows that $f(z) - g(z)$ is a constant multiple of $2\pi i$

$\square$

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let $\Omega \subset \mathbb{C}$ be the open subset defined by $\mathbb{C}$ minus the negative real line. For any $z \in \Omega$, $z = |z|e^{i\theta} : -\pi < \theta < \pi = re^{i\theta}$, define $f(z) = logr + i\theta = log|z| + iArg(z)$.

One checks that $f(z)$ is a branch of $log(z)$ on Ω. In fact, f is analytic and $f'(z) = \frac{1}{z}$.

MA 205 Complex Analysis: Logarithm and Integration

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July 30, 2018

Introduction

So last lecture, almost all the time we spent discussing the exponential function. We defined it by a power series which converges everywhere. We characterized the exponential function in two distinct ways. It's the only function which is invariant under differentiation modulo the normalization that it takes the value 1 at the point 0. It's essentially the only function with the property of converting addition in $\mathbb{C}$ to multiplication in $\mathbb{C}^\times$. We checked that, for a real variable, the exponential function matches with real exponential function e^x. e^x is monotonic increasing, hence one-to-one, hence invertible. This inverse is the logarithm.

Introduction

To define complex logarothm, we tried to construct an analytic function $g(z) = log(z)$ such that $\exp(g(z)) = z$. Solution of the above equation is a multi-valued function

$$g(z) = log|z| + i(\theta + 2\pi n) \quad (z \neq 0, n \in \mathbb{Z}),$$

where θ is the argument of z. So we need to choose values so that g become single-valued and continuous (hence analytic) function.

Definition

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f(z)$ be a continuous function on Ω such that $\exp(f(z)) = z \; \forall z \in \Omega$. Then f is called a branch of the logarithm.

Logarithm cont...

Lemma

Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a branch of the logarithm. Then any other branch of the logarith differs from f by an integral multiple of $2\pi i$.

Proof: Let $g(z)$ be a branch of the logarithm. Then $f(z) - g(z)$ is an integral multiple of $2\pi i$ for all $z \in \Omega$. Since Ω is connected, and $f(z) - g(z)$ is continuous while integral multiples of $2\pi i$ is a discrete set, it follows that $f(z) - g(z)$ is an integral multiple of $2\pi i$.

Logarithm cont..

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let $\Omega \subset \mathbb{C}$ be the open subset defined by $\mathbb{C}$ minus the negative real line. For any $z \in \Omega$, $z = |z|e^{i\theta} : -\pi < \theta < \pi$, define $f(z) = \log r + i\theta = \log|z| + iArg(z)$.

One checks that $f(z)$ is a branch of $\log(z)$ on Ω. In fact, f is analytic and $f'(z) = \frac{1}{z}$.

Integration - MA 105

Now that we've differentiated enough, it's time to integrate. Recall integration from MA 105. You first integrated real valued functions on an interval. Remember Riemann sum, Riemann integration, area under a curve, etc? Integrals had nice properties: well behaved under addition and scalar multiplication. In the language of MA 106, integral is a linear functional from the vector space of integrable functions to $\mathbb{R}$. Integral also respected monotonicity. But in spite of all the nice properties, one struggled hard to integrate. To effortlessly integrate, we needed the fundamental theorem.

So that was a quick summary of real integration from MA 105. Now to complex integration in MA 205. First we integrated $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$. Now let's start with a continuous function $f : [a, b] \to \mathbb{C}$. Let $f(t) = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

where both these integrals are defined to be the usual Riemann Integrals.

Some basic properties:

$$1. Re \int_a^b f(t)dt = \int_a^b Ref(t)dt = \int_a^b u(t)dt$$

$$2. Im \int_a^b f(t)dt = \int_a^b Imf(t)dt = \int_a^b v(t)dt$$

$$3. \int_a^b (c_1 f_1(t) + c_2 f_2(t))dt = c_1 \int_a^b f_1(t) + c_2 \int_a^b f_2(t)dt$$

$$4. \quad |\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$$

Proof: f is complex valued and so $\int_a^b f(t)dt \in \mathbb{C}$, say w_0. Let $c = \frac{|w_0|}{w_0}$. Thus, $|c| = 1$ and $\mathrm{Re}(cf(t)) \leq |cf(t)| = |f(t)|$. Thus,

$$\begin{aligned} \left|\int_a^b f(t)dt\right| &= c\int_a^b f(t)dt \\ &= \int_a^b \mathrm{Re}(cf(t))dt \\ &\leq \int_a^b |f(t)|dt. \end{aligned}$$

Do we have a complex version of the fundamental theorem?

Theorem (Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{C}$ be a continuous function. Then,

$$x \mapsto \int_a^x f(t)dt$$

is an anti-derivative (or a primitive) of f. If F is any anti-derivative of f, then for any $a \leq r < s \leq b$,

$$\int_r^s f(t)dt = F(s) - F(r).$$

What about the proof? It's easy. Just apply the fundamental theorem from MA 105 to real and imaginary parts of f.

We now discuss the complex analogue of line integrals form calculus.

We say a curve $\gamma(t) = x(t) + iy(t)$ is C^1 if both $x(t)$ and $y(t)$ are C^1 functions of t. A **contour** is a curve consisting of a finite number of C^1 curves joined end to end. A curve is said to be **simple** if the parametrization map is one to one except possibly at the end-points. (Intuitively it means that the curve does not cross itself). It is said to be **closed** if the initial and end-point are the same. i.e, $\gamma(a) = \gamma(b)$.

Jordan Curve Theorem

Any simple closed curve in $\mathbb{R}^2$ seperates the plane in to two connected components. The curve is the common boundary of both of them. Exactly one of the components is bounded.

The theorem was first discovered by Camille Jordan in 1887 although his proof was not rigorous. The first rigorous proof was due to Oswald Veblenin 1905. Although intuitively very believable the proof of this theorem is non-trivial. We shall not prove this here.

Let $f : \Omega \to \mathbb{C}$ be a complex function defined on a domain Ω and let $\gamma(t) = x(t) + iy(t), t \in [a, b]$ be a contour. The integral of f along γ is defined as

$$\int_\gamma f(z)dz \stackrel{def}{=} \int_a^b f(\gamma(t))\gamma'(t)dt$$
$$= \int_a^b [(u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$
$$+i \int_a^b [(u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]dt,$$

where $f(z) = u(x, y) + iv(x, y)$.

Properties

The usual properties of real line integrals get carried over to the complex analogues:

1. This integral is independent of parametrization.
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$ where $-C$ is the opposite curve, i,e curve with the opposite parametrization.
3. $\int_{C_1 \cup C_2 \cup \ldots C_n} f(z)dz = \int_{C_1} f(z)dz + \ldots + \int_{C_n} f(z)dz$
4. $|\int_C f(z)dz| \leq \int_C |f(z)|dz$

Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0} dz$ where C is any circle around z_0.

We can parametrize C as $z(t) = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$.
Then $\int_C f(z)dz = \int_C \frac{1}{z-z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i$

Note that the integral is independent of the circle chosen around z_0.

We will show that a function f defined on a domain Ω has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned}\int_C f(z)dz = \int_C F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(\gamma(t))\right] dt \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points of the path and not the path itself. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$. How do we go about getting such an F? Intuitively, something whose derivative is the given function should be an integral of that function! To get a function of z, we'll integrate "up to" z. Fix z_0. Let z be an arbitrary point in Ω. Choose <u>any</u> path joining z_0 to z; this exists since Ω is path connected.

Proof of Path Independence

Thus, our candidate for the primitive is

$$F(z) = \int_{\gamma(z_0,z)} f(z)dz.$$

This function is well defined because of the hypothesis of independence of integral on the path. We have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. Let $h \in \mathbb{C}$ be such that the straight line $z + th$, $(t \in [0, 1])$, joining z and $z + h$ lie in Ω.

Proof of Path Independence

Then,

$$
\begin{aligned}
& \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \\
= & \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\
= & \lim_{h\to 0} \frac{1}{h}\int_{\gamma(z,z+h)} f(w)dw \\
= & \lim_{h\to 0} \frac{1}{h}\int_0^1 f(z+ht)hdt \\
= & f(z).
\end{aligned}
$$

This finishes the proof.

MA 205 Complex Analysis: Cauchy Integral Theorems

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August 2, 2018

Recall

Last time we defined Principle Branch of Logarithm and also remarked that a Branch of logarithm can be defined by removing a ray originating from the origin. We have defined integral of complex valued functions defined on an interval. Also studied line integral of a complex valued function.
Today, we will see one of the most important theorems in Complex Analysis known as Cauchy's theorem and its beautiful consequences.

Recall that for a function $f : \Omega \to \mathbb{C}$ defined on a domain Ω and a parametrized C^1 curve $\gamma(t) = x(t) + iy(t), t \in [a, b]$. We define

$$\int_\gamma f(z)dz \stackrel{def}{=} \int_a^b f(\gamma(t))\gamma'(t)dt$$
$$= \int_a^b [(u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$
$$+ i \int_a^b [(u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]dt,$$

where $f(z) = u(x, y) + iv(x, y)$.

The usual properties of real line integrals get carried over to the complex analogues:

1. This integral is independent of parametrization.
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$ where $-C$ is the opposite curve, i,e curve with the opposite parametrization.
3. $\int_{C_1 \cup C_2 \cup \ldots \cup C_n} f(z)dz = \int_{C_1} f(z)dz + \ldots + \int_{C_n} f(z)dz$
4. $|\int_C f(z)dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt$

Path independence

We will show that a function f defined on a domain Ω has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned} \int_C f(z)dz = \int_C F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(\gamma(t))\right] dt \\ &= F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points of the path and not the path itself. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$. How do we go about getting such an F? Intuitively, something whose derivative is the given function should be an integral of that function! To get a function of z, we'll integrate "up to" z. Fix z_0. Let z be an arbitrary point in Ω. Choose <u>any</u> path joining z_0 to z; this exists since Ω is path connected.

Thus, our candidate for the primitive is

$$F(z) = \int_{\gamma(z_0, z)} f(z)dz.$$

This function is well defined because of the hypothesis of independence of integral on the path. We have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. Let $h \in \mathbb{C}$ be such that the straight line $z + th$, $(t \in [0, 1])$, joining z and $z + h$ lie in Ω.

Proof of Path Independence

Then,

$$
\begin{aligned}
& \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \\
= & \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\
= & \lim_{h\to 0} \frac{1}{h}\int_{\gamma(z,z+h)} f(w)dw \\
= & \lim_{h\to 0} \frac{1}{h}\int_0^1 f(z+ht)hdt \\
= & f(z).
\end{aligned}
$$

This finishes the proof.

We now come to the most important, central theorem in this subject on which most of complex analysis depends, namely Cauchy's theorem.

Theorem

Let C be a simple closed contour and let f be a holomorphic function defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

Remark Note that by Jordan curve theorem, interior of C makes sense.

Cauchy's theorem

Proof: Let $f(z) = u(x, y) + iv(x, y)$. The proof uses Green's theorem.

Theorem

If P and Q are two real valued functions with continuous first partial derivatives on an open set containing C and its interior, then

$$\int_C (Pdx + Qdy) = \int\int_\Omega (Q_x - P_y)dxdy$$

Note that a priori we do not have the hypothesis to guarantee continuity of the first partial derivatives of u and v since we do not know whether $f'(z)$ is continuous. However it is a fact that if $f(z)$ is holomorphic, then $f'(z)$ is continuous.

This is called Goursat's theorem. We will assume this theorem without proof.
Thus by Green's theorem,

$$\begin{aligned}
& \int_C f(z)dz \\
= & \int_C (udx - vdy) + i \int_C (vdx + udy) \\
= & \int \int_\Omega (-v_x - u_y)dxdy + i \int \int_\Omega (u_x - v_y)dxdy \\
= & \; 0 \quad \text{(By CR equations).}
\end{aligned}$$

Definition

An open subset $\Omega \subseteq \mathbb{C}$ is said to be **simply connected** if every simple closed curve in Ω has all its interior points belonging to Ω.

Examples: $\mathbb{C}$, any open disc, $\mathbb{C}$ minus negative reals etc. Open annulus i.e, area between two concentric circle is **NOT** simply connected.

Theorem

(Cauchy's theorem for simply connected domain) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then* $\int_C f(z)dz = 0$

Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0} dz$ where C is any positively oriented circle around z_0.

We can parametrize C as $z(t) = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$.
Then $\int_C f(z)dz = \int_C \frac{1}{z-z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i$

Note that the integral is independent of the circle chosen around z_0. This is an instance of the following more general situation:

Theorem

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other and f is a holomorphic function on Ω, then $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.*

Remark: A similar computation shows that $\int_C \frac{1}{(z-z_0)^m} = 0$ for all $m \neq 1$. This follows from the fact that $\frac{1}{(z-z_0)^m}$ admits a primitive in $\mathbb{C} - \{z_0\}$ when $m \neq 1$. Note that for $m = 1$, $\frac{1}{(z-z_0)^m}$ does not admit a primitive; $log(z - z_0)$ does not define a holomorphic function on any open set in $\mathbb{C}$ containing C.

We'll now use Cauchy's theorem to prove another very important, closely related theorem in this subject, namely the Cauchy Integral Formula.

Theorem (Cauchy Integral Formula)

Let f be holomorphic everywhere within and on a simple closed curve γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{f(z)dz}{z - z_0}.$$

We use the following fact in the proof, often called ML inequality : If γ is a contour of length L and $|f(z)| \leq M$ on γ, then $|\int_{\gamma} f(z)dz| \leq ML$.

MA 205 Complex Analysis: Cauchy Integral Formula and its Beautiful Consequences

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August 6, 2018

Recall

We saw several versions of Cauchy's theorem in the last class.

Theorem (Cauchy's theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let C be a simple closed contour and let f be holomorphic on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

Theorem

(Cauchy's theorem for simply connected domain) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then $\int_C f(z)dz = 0$*

Theorem

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other and f is a holomorphic function on Ω, then $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.*

Examples

Let C_1 be the line segment joining -1 and i and let C_2 be the arc of the unit circle with initial point -1 and end point i.

$$C_1 : z_1(t) = -1 + (1+i)t = (-1+t) + it : 0 \leq t \leq 1$$

$-C_2 : z_2(t) = e^{it} : \pi/2 \leq t \leq \pi$.
Then

$$\begin{aligned} \int_{C_1} |z|^2 dz &= \int_0^1 ((-1+t)^2 + t^2)(1+i)dt \\ &= (1+i)\int_0^1 (2t^2 - 2t + 1)dt \\ &= \frac{2(1+i)}{3}. \end{aligned}$$

$$\int_{-C_2} |z|^2 dz = \int_{\pi/2}^{\pi} ie^{it} dt = -1 - i.$$

Hence the results do not agree and this is consistent with the fact that $|z|^2$ is not holomorphic.

Cauchy integral formula

Theorem (Cauchy Integral Formula)

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let f be holomorphic everywhere within and on a simple closed contour γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

Recall that $\int_C \frac{1}{z-z_0} dz = 2\pi\imath$ for any positivily oriented curve C with z_0 is interior to C.
If $|f| < M$ on C and $\gamma : [a, b] \to \mathbb{C}$ is a parametrization of C, then

$$|\int_C f(z)dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq M \int_a^b |\gamma'(t)|dt = M\ell(C).$$

Proof

Proof: We need to show that

$$\int_\gamma \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{f(z_0)}{z - z_0} dz,$$

since the latter integral is $2\pi\imath \cdot f(z_0)$. Thus, we need to show that

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

Proof cont...

Since f is continuous at z_0, given $\epsilon > 0$, there is $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Choose $r < \delta$ and consider the circle $C_r : |z - z_0| = r$. By Cauchy's theorem applied to γ and C_r, we get

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

Now,

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

Thus, $\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$ can be made arbitrarily small; i.e., it is zero.

MA 205 Complex Analysis

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August 9, 2018

Recall

We have seen two important theorems in Complex Analysis, namely; Cauchy's theorem and Cauchy integral formula. Today we will consider some examples to demonstrate how these theorems can be use to evaluate line integrals and will see some beautiful consequences of these theorems.

Example:

(i) $\int_{|z|=2} \frac{e^z dz}{(z+1)(z-3)^2} = \int_{|z|=2} \frac{f(z)dz}{z+1}$, where $f(z) = \frac{e^z}{(z-3)^2}$. So by CIF, answer is $2\pi\imath f(-1) = \frac{\pi\imath}{8e}$.

(ii) $\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2} \int_{|z|=3} \left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1}\right] dz = 0$

OR

$$\begin{aligned}\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz &= \int_{|z-1|=\varepsilon} \frac{\frac{\cos \pi z}{z+1}}{z-1} dz + \int_{|z+1|=\varepsilon} \frac{\frac{\cos \pi z}{z-1}}{z+1} dz \\ &= 0.\end{aligned}$$

Example

Here is an example where the Cauchy Integral formula can be used to compute a seemingly hard real integral.
Let k be a real constant. Show that $\int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)d\theta = 0$ and $\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)d\theta = 2\pi$.
Applying CIF to $\int_{|z|=1} \frac{e^{kz}}{z}dz = (2\pi i)e^{k.0} = 2\pi i$.

Hence

$$\begin{aligned} 2\pi i &= \int_0^{2\pi} \frac{e^{k(\cos\theta + i\sin\theta)}}{e^{i\theta}} ie^{i\theta}d\theta \\ &= i\int_0^{2\pi} e^{k\cos\theta}[\cos(k\sin\theta) + i\sin(k\sin\theta)]d\theta \end{aligned}$$

Equating the real and imaginary parts gives us the answer.

Summing Up

We saw two very important theorems, namely Cauchy's theorem and the Cauchy integral formula. The first said that the integral along a closed curve of a function is zero if the function is holomorphic on and within the curve. The second said:

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz$$

if f is holomorphic on and within the simple closed curve γ. We derived Cauchy's theorem by appealing to Green's theorem after assuming Goursat's theorem. We derived CIF from Cauchy's theorem by making use of the computation $\int_\gamma \frac{dz}{z-z_0} = 2\pi\imath$. In fact, Cauchy's theorem is equivalent to CIF. (Try and prove this little fact).

Let f be holomorphic on Ω and $z_0 \in \Omega$. Let $R > 0$ be such that f is holomorphic in $|z - z_0| < R$. Let γ be a circle of radius r with $r < R$ centered at z_0. CIF gives us:

$$f(z) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{f(w)}{w - z} dw,$$

for any z such that $|z - z_0| < r$. Note that we have chosen $r < R$ so that f is holomorphic on and within γ. The aim is to prove that f is analytic at z_0; i.e., to show that f can be expanded as a power series around z_0 in a neighborhood of z_0. The idea is to get a power series in $(z - z_0)$ from the rhs of CIF. The term in CIF which looks amenable to some manipulation is

$$\frac{1}{w - z}.$$

Also always keep in mind that the only series that we know well is the geometric series! Let's look at it closely.

Now,

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0}\cdot\frac{w-z_0}{w-z}\\ &= \frac{1}{w-z_0}\cdot\frac{1}{1-\left[\frac{z-z_0}{w-z_0}\right]}\\ &= \frac{1}{w-z_0}\cdot\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots\right]\end{aligned}$$

since $\left|\frac{z-z_0}{w-z_0}\right| < 1$ for every $w \in \gamma$. We plug this in CIF.

Holomorphic $\Longrightarrow$ Analytic

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z_0}\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\ldots\right]dw \\
&= \left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)}dw\right]+\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^2}dw\right](z-z_0) \\
&\quad +\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^3}dw\right](z-z_0)^2+\ldots \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n,
\end{aligned}$$

for $|z-z_0|<r$ where

$$a_n = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw.$$

Remark: Integral of sum needn't be sum of integrals in general, but in the previous slide it can be justified. The key word is "uniform convergence". We'll skip the details.

Holomorphic $\implies$ Analytic

Thus, we have proved that if f is holomorphic in the disc $|z - z_0| < R$, then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi\imath} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} dw,$$

for any $r < R$. Since the power series converges to $f(z)$ for $|z - z_0| < r$, the radius of convergence is at least r. We also know that

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

whenever $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. In particular, a_n does not depend on r. Any $r < R$ gives same a_n. Thus the radius of convergence is at least R.

Holomorphic $\implies$ Analytic

Examples:
(i) $f(z) = \frac{e^z}{\sin z + \cos z}$ expanded as a power series centered at 0 has radius of convergence $= \frac{\pi}{4}$.
(ii)

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

expanded as a power series centered at 0 has radius of convergence $= 2\pi$.
Thus, we have proved:

holomorphic $\implies$ analytic.

Since analytic functions are infinitely differentiable (term by term differentiation), we have proved :

A holomorphic function is infinitely differentiable - a long-awaited claim !

Cauchy's Estimate

We have also concluded that if $f : \Omega \to \mathbb{C}$ is holomorphic, and if $\{z \mid |z - z_0| \leq r\} \subset \Omega$, then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where γ is $|z - z_0| = r$. Now suppose f is holomorphic in $|z - z_0| < R$ and suppose f is bounded by $M > 0$ there. Can apply ML inequality in the above formula to get

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Since this is true for any $r < R$, we get, $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$. This is called Cauchy's estimate.

Liouville's Theorem

A function defined all over $\mathbb{C}$ is called entire if it is holomorphic everywhere in $\mathbb{C}$. Examples? Polynomials, $\exp(z)$, $\sin z$, $\cos z$, etc. Clearly, sums and products of entire functions are entire. The fact that the function is defined and holomorphic everywhere puts strong restrictions on the function. For instance, we have the so called Liouville's theorem, which says:

a bounded entire function is a constant.

A non-constant entire function has to be unbounded. As we have seen $\exp(z)$ takes all values in $\mathbb{C}$ except 0, sin and cos are surjective, in particular these are all unbounded. Polynomials are also clearly unbounded (Why?).

Proof of Liouville's theorem: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We need to show that f is a constant. We'll show this by showing that $f' \equiv 0$. It's enough to show $|f'(z)|$ can be made arbitrarily small. By Cauchy's estimate.

$$|f'(z)| \leq \frac{M}{R},$$

if f is holomorphic in a disc with center z and radius R. But R can be as large as we want, since f is entire. So, $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is a constant.

Fundamental Theorem of Algebra

Recall that the fundamental theorem of algebra asserts that every non-constant polynomial with complex coefficients has a complex root.

Proof: We first show that $|f(z)|$ tends to ∞ as $|z|$ tends to ∞.
If $f(z) = a_0 + a_1 z + \cdots a_n z^n$, then

$$\begin{aligned} |f(z)| &= |z|^n(|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + ... + a_n|) \\ &\geq |z^n| \left(\Big||a_n| - |\frac{a_0}{z^n} + ... + \frac{a_{n-1}}{z}|\Big|\right) \end{aligned}$$

As $|z|$ tends to ∞, $|\frac{a_0}{z^n} + ... + \frac{a_{n-1}}{z}|$ tends to zero and hence the above quantity clearly tends to infinity.
Suppose $f(z)$ does not have any zero, then $\frac{1}{f(z)}$ defines a holomorphic function which, by the above computation, is bounded and hence by Lioville's theorem is constant. A contradiction.

MA 205 Complex Analysis

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Recall

In the last class, we have seen that holomorphic functions and analytic functions are same using CIF. We have derived CIF for derivatives

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n \in \mathbb{N}).$$

Using Cauchy estimate we have proved a very important property of an entire function, i.e. Every bounded entire function is constant. This is known as Liouville's theorem and it gives us a proof of the FTA. We have studied many analytic properties of holomorphic functions. Today, we will see some geometric properties.

Examples

Example 1:

$$\begin{aligned}\int_{|z|=2} \frac{z^2}{z-1} dz &= 2\pi i [z^2]|_{z=1} \\ &= 2\pi i.\end{aligned}$$

Example 2:

$$\begin{aligned}\int_{|z|=2} \frac{e^z}{z^2(z-1)} dz &= \int_{|z|=\epsilon} \frac{e^z/(z-1)}{z^2} dz + \int_{|z-1|=\epsilon} \frac{e^z/z^2}{z-1} dz \\ &= 2\pi i \left[\frac{d}{dz}\left(\frac{e^z}{z-1}\right)\right]_{z=0} + 2\pi i \left[\frac{e^z}{z^2}\right]_{z=1} \\ &= -4\pi i + (2\pi i)e \\ &= 2\pi i(e-2)\end{aligned}$$

Example 3:

$$\begin{aligned}
\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1}dz &= \int_{|z-1|=1} \frac{z^2-z-1-3z+4}{z^2-z-1}dz \\
&= \int_{|z-1|=1} [1-\frac{3z-4}{z^2-z-1}]dz \\
&= 0-\int_{|z-1|=1} \frac{3z-4}{(z-\frac{1+\sqrt{5}}{2})(z-\frac{1-\sqrt{5}}{2})} \\
&= -\int_{|z-1|=1} \frac{\left(\frac{3z-4}{z-\frac{1-\sqrt{5}}{2}}\right)}{(z-\frac{1+\sqrt{5}}{2})} \\
&= -[(3z-4)/(z-(1-\sqrt{5})/2)]_{z=(1+\sqrt{5})/2} \\
&= (\sqrt{5}-3)/2.
\end{aligned}$$

Exercise

Exercise: Let f be an entire function such that there exists a real constant C such that for all $z \in \mathbb{C}$, $|f(z)| \leq C|z|^n$, then $f(z)$ is a polynomial of degree less than or equal to n.

Logarithm Revisited

Theorem

Let Ω be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F(z)$ on Ω, (denoted $\log(z)$) such that:
1. $F(1) = 0$ and $F'(z) = 1/z$
2. $e^{F(z)} = z \ \ \forall z \in \Omega$
3. $F(r) = \ln(r)$ when r is a positive real number close to 1. (With the usual definition of ln for real numbers)

Proof:
Since $1 \in \Omega$ and $0 \notin \Omega$, define the function $F(z) = \int_1^z \frac{1}{w} dw$.Since Ω is simply-connected, it follows by Cauchy's theorem, that this function is well defined.We have seen before that this defines a holomorphic function on Ω. Clearly $F(1) = 0$ and $F'(z) = \frac{1}{z}$ proving 1.

Proof cont ..

One checks that the function $ze^{-F(z)}$ has its derivative identically vanishing and hence is a constant. Substituting $z = 1$, this constant is seen to be 1. This proves 2. For proving 3, take a straight path joining 1 and r, for a small real number r. Then $F(r) = \int_1^r \frac{1}{t} dt = ln(r)$.

Note that such a function is unique. (Why ?)

Quiz policy and Seating arrangement

Policy: Two marks will be deducted if you don't write Tutorial Batch/Division in the answer booklet. Note that Tutorial Batch and Division are the same for this course.

Tutorial Batch/Division	**Venue**	**Block** (From the stage)
T1	LA 301	Left Block
T2	LA 301	Middle Block
T3	LA 301	Right Block
T4	LA 302	Left Block
T5	LA 302	Middle Block

Find a seat according to the above assignment and make sure that there is a gap between you and the student next to you.

Zero's of Homolorphic Functions

Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a complex analytic function defined on Ω. This means f can be expressed by a power series expanded around any point in Ω. Let z_0 be a point in Ω at which f vanishes. We will show that either f is identically zero or there exists a neigborhood of z_0 in which f has no other zero. Assume the contrary. Then f is a non-zero function for which there exists a sequence of points $\{z_n\}$ converging to z_0 such that f vanishes along this sequence. We show that $f^k(z_0) = 0$ for all $k \geq 0$. Without loss of generality, we can assume that this open set contains 0 and $z_0 = 0$. Let n be the largest natural number such that $f^i(0) = 0$ for all $0 \leq i \leq n$. Then f can be expanded in a neighborhood as

$$f(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(0) + \frac{z^{n+2}}{(n+2)!} f^{(n+2}(0) + \ldots$$

$$= z^{n+1} \left(\frac{f^{(n+1)}(0)}{(n+1)!} + \frac{z}{(n+2)!} f^{n+2}(0) + \ldots \right)$$

Now as $z \to 0$ along the sequence $\{z_n\}$, we see that the lhs is identically zero. Hence the rhs also vanishes identically along this sequence. Hence the term inside the bracket vanishes along $\{z_n\}$ and hence by continuity, vanishes at the limit namely 0, thereby showing that $f^{(n+1)}(0) = 0$. This contradicts the assumption on n. Now consider the set

$$A = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

$A \neq \emptyset$ since $z_0 \in A$. We'll show that A is both open and closed, which shows that $A = \Omega$. This of course would mean $f \equiv 0$.

Zero's of holomorphic functions

To show that A is closed, we need to show that A contains all its limit points. If z is a limit point of A, let $z_k \in A$ be such that $\lim z_k = z$. Since $f^{(n)}$ is continuous, it follows that $f^{(n)}(z) = 0$; i.e., $z \in A$.
To show that A is open, we need to show that every $a \in A$ has a neighborhood which is contained in A. Since Ω is open, there is a neighborhood of a which is contained in Ω. On this neighborhood, if we write $f(z) = \sum a_n(z-a)^n$, then $a_n = \frac{1}{n!} f^{(n)}(a) = 0$ for each $n \geq 0$. Thus, $f \equiv 0$ for all z in this neighborhood. Therefore, this neighborhood is in fact contained in A.

Corollary [Identity Theorem]: If f and g are holomorphic in Ω, then $f \equiv g$ iff there exists a non-constant sequence $\{z_n\} \subseteq \{z \in \Omega \mid f(z) = g(z)\}$ such that $\lim_{n\to\infty} z_n = z_0 \in \Omega$.