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Tut 1
MA-205

1. $P(n) = a_n x^{n} + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0$

$P(1) = a_1 x + a_0$ (FTA → Any complex polynomial has 1 roots)

This has 1 root (FTA)

Assume that a polynomial of degree $n-1$ has $n-1$ roots.

Taking $P_n = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

we can factorise this as ~~$(x-\alpha)(b_{n-1}x^{n-1} + b_{n-2}$~~ $(x-\alpha)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_0)$ because

$x = \alpha$ is a root of P_n (By FTA). So P_n has $1 + (n-1) = n$ roots. (By Induction)

2. Real polynomial ~~then~~ is irreducible it has a degree ≤ 2.

$P_n = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ $[a_n, a_{n-1}, \dots, a_0 \in \mathbb{R})$ $n \geq 3$

Prove that $\boxed{P_n \text{ is reducible} \rightarrow \text{real factors of } P_n \text{ exist}}$

Case I → Complex Roots

By FTA → assume 1 roots $= \alpha$

So another root $= \bar{\alpha}$ (as coefficient are real).

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we can factorize it as

$P_n = (x-\alpha)(x-\bar{\alpha})\, q_{n-2}$

{ q_{n-2} is a $(n-2)$ degree polynomial }

$\therefore P_n$ is reducible

as $(x-\alpha)(x-\bar{\alpha}) = [x^2 + |\alpha|^2 - (\alpha+\bar{\alpha})\, x] \to \in R$

Case - 2 → Real root of p_n exist.

then root be $= x_0$

$P_n = (x-x_0)(q_{n-1})$, P_n is reducible.

Ans 3 (iii) $f(z) = z^n$

$$f'(z_0) = \lim_{\substack{h\to 0 \\ h\in C}} \frac{(z_0+h)^n - (z_0)^n}{h}$$

$$\Rightarrow \lim_{h\to 0} (z_0^n) \left(\frac{\left(1+\frac{h}{z_0}\right)^n - 1}{h} \right)$$

$$\Rightarrow \lim_{h\to 0} (z_0^n) \frac{\left(1+\frac{nh}{z_0}\right) - 1}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(z_0^n)(n)\frac{h}{z_0}}{h} = n z_0^{n-1}$$

(differentiable & Holomorphic)

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(iv) $f(z) = Re(z)$

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{Re(z_0+h) - Re(z_0)}{h}$$

If taking a limit along x-axis $\to h \in \mathbb{R}$

$$f'(z_0) = \frac{Re(z_0+h) - Re(z_0)}{h} = 1$$

taking $R \in Im$

$$f'(z_0) = \frac{Re(z_0) - Re(z_0)}{h} = 0 \quad (\text{as } Re(z_0+h) = Re(z_0))$$

So as limit doesn't match, derivative doesn't exist for any z.

(vi) $f(z) = \bar{z}$

$$f'(z_0) = \lim_{h \to 0} \frac{(\overline{z_0+h}) - \bar{z_0}}{h}$$

$$f'(z_0) = \left(\frac{h + z_0 - z_0}{h}\right) = 1 \quad (\text{as } h \in \mathbb{R})$$

$$f'(z_0) = \frac{\bar{h} + \bar{z_0} - \bar{z_0}}{h}$$

$$= \frac{-h}{h} = -1 \quad (\text{as } h \in I)$$

so it is not d...

4 $f(z)$ is real valued, $z \in \mathbb{C}$, $f(z) \in \mathbb{R}$.

$f'(z) = 0$ or doesn't exist.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{R}$$

~~(As $(z-z_0)$)~~ (as $(z-z_0)$ along x-axis)

~~$f'(z) = 0$ or doesn't exist~~.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in I_m$$

(As $f(z) - f(z_0) \in \mathbb{R}$, $z - z_0 \in I_m$)

$f'(z) = 0$ or doesn't exist.

(as only $0 \in \mathbb{R}$ and in both, I & II should be equal to exist).

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

$f(z)$ to be holomorphic

$f'(z) = 0$ everywhere

also any $f(z) = \lim_{z \to z_0} f'(z_0) \times (z_0 - z_0) + f(z_0)$

$= 0 \times (z - z_0) + f(z_0)$.

$f(z) = f(z_0)$

$f(z)$ is constant if $f(z)$ is Holomorphic)

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Tutorial - 2

1. $u_x = v_y$

$u_y = -v_x$

$$f(z) = e^x(\cos y + i\sin y)$$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y$

so as above eq^n are satisfied and u_x, v_y are continuous & holomorphic.

Or

$f(z) = e^z$ with $z = x + iy$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = e^z$$

2. $x = r\cos\theta$

$y = r\sin\theta$

~~$u_r = u_x \cos\theta$~~

$$\frac{du}{dr} = \frac{du}{dx} \cdot \frac{dx}{dr} \qquad \frac{du}{d\theta} = \frac{du}{dy} \cdot \frac{dy}{d\theta}$$

$u_r = u_x \cdot \cos\theta \qquad u_\theta = u_y \cdot r\cos\theta$

$= -u_x \sin\theta$

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$V_\theta = -V_y \cdot r\cos\theta$

$V_r = V_y r\sin\theta$

so $U_r = \frac{1}{r} V_\theta$ & $V_r = \frac{-1}{r} U_\theta$.

3 v is a harmonic conjugate of u.

$\Rightarrow$ $u + iv$ is holomorphic

u is a harmonic conjugate of v

$\Rightarrow$ $v + iu$ is holomorphic

So CR eqⁿ of [Cauchy Reimann]

$u_x = v_y$

$u_y = -v_x$

& CR eqⁿ on 2

$v_x = u_y$

$v_y = -u_x$

So $u_x^2 + v_y^2 = 0 \Rightarrow$ They are constant function.

4/ (i) $u(x,y) = xy + 3x^2y - y^3$

$u_x = v_y$

$u_y = -v_x$

$\Rightarrow$ $u_x = y + 6xy = v_y$

$u_y = x + 3x^2 - 3y^2 = -v_x$

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MA205 Tut-1

$y \neq 0$ as

$iz_0 + \sqrt{1 - z_0^2} \neq 0$

$\Rightarrow -z_0^2 = 1 - z_0^2$

$\Rightarrow 1 = 0$

so $\forall z_0 \ \exists z$

Thus $\sin z$ is surjective.

(b) $\cos z = \dfrac{e^{z} + e^{-iz}}{2} = z_0$

$\Rightarrow y + \dfrac{1}{y} = 2z_0$

$\Rightarrow$

$y = \dfrac{2z_0 \pm \sqrt{4z_0^2 - 4}}{2} = z_0 \pm \sqrt{z_0^2 - 1}$

$y \neq 0$ as assuming $y = 0$

$\Rightarrow -1 = 0$ a contradiction (same as above)

So $\cos z$ is a surjective function as ~~___~~ ~~___~~ $\cos z$ can take any value in complex domain

$e^{iz} = y$.

$iz = \log(y) + 2n\pi i$

$z = -i\log(y) + 2n\pi$

So $\cos z$ & $\sin z$ ~~are~~ have a period of 2π.

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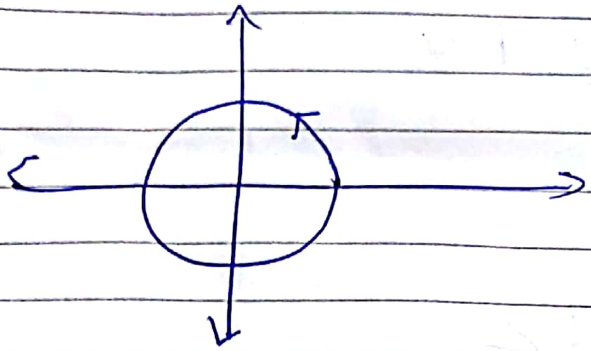
5 (b) $\int \bar{z}^{m}\, dz \quad (m \in \mathbb{Z})$



$z = R.e^{i\theta}$

$\bar{z} = R.e^{-i\theta}$

$\Rightarrow \int R^{m} e^{-im\theta} . dz$

$\boxed{dz = iRe^{i\theta} d\theta}$

$\Rightarrow i \int_0^{2\pi} R^{m+1} . e^{-im\theta} . e^{i\theta} d\theta = i \int_0^{2\pi} R^{m+1} . e^{-i(m-1)\theta} d\theta$

case I $\rightarrow m \neq 1$

$\Rightarrow i \int R^{m+1} . \frac{e^{-i(m-1)\theta}}{-i(m-1)} d\theta$

$= R^{m+1} (e^{-i\, 2\pi (m-1)} - e^{i \times 0})$

$= 0$

case II $\rightarrow m = 1$

$\Rightarrow i \int_0^{2\pi} R^{m+1} . d\theta$

$\Rightarrow R^{m+1} . i . 2\pi$

(c) $\int |z|^{m} dz$

$\Rightarrow \int R^{m} dz$

$\Rightarrow \int_0^{2\pi} (R)^{m} (iR) (e^{i\theta} d\theta)$

$= R^{m+1} . i \int_0^{2\pi} e^{i\theta} d\theta = 0$

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$$\int_{\partial D} \bar{z}\,dz = 2i\ \text{Area}(D)$$

$$\bar{z} = x - iy$$

$$dz = dx + i\,dy$$

$$\Rightarrow \oint_{\partial D} (x-iy)(dx+i\,dy)$$

$$\Rightarrow \oint (x-iy)\,dx + i(x-iy)\,dy$$

using

$$\Rightarrow \iint_D \left(\frac{\partial m}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy$$

$$\Rightarrow \iint (i - (-i))\ dx\,dy$$

$$\Rightarrow \iint (2i)\ dx\,dy$$

$$\Rightarrow (2i)(\text{Area}(D))$$

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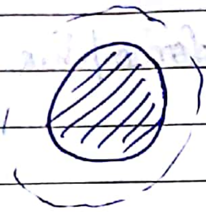
MA 205 Tut-4

★ Cauchy's Theorem

$f(z) \rightarrow$ holomorphic on γ & inside Ω (Simply connected)

$$f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-z_0}$$

$\gamma \rightarrow$ curve inside Ω.



$$\int_\gamma f(z)\,dz = 0$$

$f(z) \rightarrow$ holomorphic on & inside γ.

★ Generalised theorem

$$f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

1. $f(z) = \sum_{i=1}^{\infty} \alpha_i z^i$

$$f(\bar{z}) = \overline{f(z)}$$

$$\sum_{i=1}^{\infty} \alpha_i \bar{z}^i = \sum \bar{\alpha}_i \bar{z}^i$$

$g(z) = \sum_i (\alpha_i - \bar{\alpha}_i)\bar{z}^i = 0$ for all

$(\alpha_i = \bar{\alpha}_i)$

~~for~~ α_i $f'(z)=0$

$f''(z)=0$

so basically α_i is real

2. (i) $$\int_{|z|=1} \frac{z}{(z-2)^2}\,dz$$

Using cauchy's theorem As it is holomorphic inside the circle.

$$\Rightarrow \int_{|z|=1} f(z) = 0.$$

(ii) $$\int_{|z|=2} \frac{e^z}{(z)(z-3)}\,dz$$

Let $f(z) = \frac{e^z}{z-3}$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{f(z)}{z}\,dz = f(z_0) = \frac{-1}{3}$$

so $$\int \frac{e^z}{(z)(z-3)} = \frac{-2\pi i}{3}$$

(iii) $\displaystyle\int_{|z|=2} \frac{e^z}{(z)(z-1)}\, dz$

$\Rightarrow \displaystyle\int_{|z|=2} \left(\frac{e^z}{z-1} - \frac{e^z}{z}\right) dz$

$\Rightarrow f_0(z) = e^z$

$\Rightarrow \displaystyle\frac{1}{2\pi i}\int_{|z|=2} \frac{e^z}{z-1} = f_0(z=1) = e$

$\Rightarrow$ & $\displaystyle\frac{1}{2\pi i}\int_{|z|=2} \frac{e^z}{z} = f_0(z=0) = 1$

Ans $(e-1)\,(2\pi i)$

(iv) $\displaystyle\int_{|z|=4} \frac{\sin z}{(z-2)^2}\, dz$

$\Rightarrow f_0'(z) = \displaystyle\frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\, dz$

$f(z) = \sin z$, $f_0'(z) = \cos(z)$

$\Rightarrow \cos(2) \cdot 2\pi i = \displaystyle\int_{|z|=4} \frac{\sin z}{(z-2)^2}\, dz$

Ans $2\pi i \cos(2)$

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3. $\int \frac{f(z)}{(z-z_1)(z-z_2)}$

$$= \frac{1}{z_2-z_1}\int_{|z|=R} \left(\frac{f(z)}{(z-z_2)} - \frac{f(z)}{(z-z_1)}\right) dz$$

Ans $\Rightarrow \frac{2\pi i}{z_2-z_1}\left(f(z_2) - f(z_1)\right)$

4. f and $g \rightarrow$ holomorphic

$f(z) = g(z)$ on γ.

Let $p(z) = f(z) - g(z) \rightarrow$ (holomorphic function)

$p(z) = 0$ (on γ)

$\int_\gamma \frac{p(z)\,dz}{(z-z_0)^{n+1}} = 0$ for all n.

z_0 inside γ. [as $p(z) = 0$ on γ]

$\int \frac{pz}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} p^n(z_0) = 0$

So $p(z) = 0$ (for all points inside γ)

So $f(z) = g(z)$ for all ~~point~~s z inside γ.

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→ Singularity - A point on which the function is not holomorphic.

→ Isolated
- Removable $\lim_{z \to z_0} f(z)$ exists
- Poles $\lim_{z \to z_0} |f(z)| \to \infty$
- Essential → Not pole, Not Removable

→ Not Isolated

1(c) $f(z) = \frac{1}{\sin(\frac{1}{z})}$ → $z = 0$ and

$\frac{1}{f(z)}$ → Zeroes give the singularities of $f(z)$.

$\frac{1}{f(z)} = \sin\left(\frac{1}{z}\right) = 0$.

$\Rightarrow z = \frac{1}{n\pi}$

to check at $z = 1/n\pi$

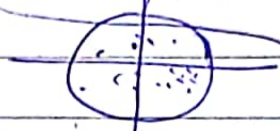
$\lim_{z \to 1/n\pi} \frac{(z - 1/n\pi)}{\sin(\frac{1}{z})}$ → (if it exist then $z = \frac{1}{n\pi}$ is a pole of order 1)

for n = even Limit $= 1$
n = odd Limit $= -1$

So at $z = \frac{1}{n\pi}$, we have a pole of order 1.

$z=0$, $\epsilon \to 0$ $\sin\left(\frac{1}{z}\right)=0$

for $z = 0 \to$



$\frac{1}{f(z)} = \sin\left(\frac{1}{z}\right) \to$ taking a disc of radius $\epsilon \Rightarrow$

$\sin\left(\frac{1}{z}\right) = 0$ has many solution $= \frac{1}{n\pi}$ with $\left(n = \frac{1}{\epsilon}\right)$

2/ (i) $\frac{1}{(z^4+1)^2} = f(z)$

$\Rightarrow \frac{1}{f(z)} = 0$

$\Rightarrow (z^4+1)^2 = 0$

$\Rightarrow z^4 = -1$

$\Rightarrow z^4 = e^{i(\pi + 2n\pi)}, n \in z$

$\Rightarrow z = e^{\frac{n\pi}{2} + \frac{i\pi}{4}}, n \in z$

poles are $z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

order of pole $\to$ is given by order of zero in $\frac{1}{f(z)}$

All the poles are of order $\boxed{2}$.

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order of pole is defined as $f(z_0)=0,\ f'(z_0)=0 \ldots f^{n-1}(z_0)=0$

The number of repeated roots in above eq^n.

5 $f(z) = \dfrac{p(z)}{q(z)}$, $p(z)$ & $q(z)$ are diff

[pole of order 1 at z_0] $p(z_0) \neq 0$
$q(z_0) = 0$
$q'(z_0) \neq 0$

Laurent Series:-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=-1}^{-\infty} b_n (z-z_0)^m$$

f has singularity at z_0

if removable → all $b_m = 0$, pole of order m_0 then all $b_m = 0$ for $m < m_0$

$$\frac{b_{-m}}{z^m} + \frac{b_{-(m+1)}}{z^{m+1}} + \ldots \rightarrow 0$$

$$\oint f_z\, dz = \int \frac{b_{-1}}{z}\, dz$$ [where b_1 = residue term]

$$\lim_{z \to z_0} (z-z_0) f(z) = \left(\sum a_n (z-z_0)^n + \frac{b_{-1}}{z-z_0} \right) z-z_0$$

$$= p(z) \lim_{z \to z_0} \frac{(z-z_0)}{q(z)}$$

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$$q(z_0) = \lim_{z \to z_0} \frac{q(z)}{z-(z_0)}$$

$$\frac{1}{g'(z_0)} = \lim_{z \to z_0} \frac{(z-z_0)}{g(z)}$$

6 (i) $\frac{1}{z^2 \sin z}$ → calculate residue → (b-1)

$$\lim_{z \to 0} \frac{z}{z^2 \sin z} = \lim_{z \to 0} \frac{z}{\sin z} = 1$$

so $\frac{1}{z^2 \sin z}$ has a pole of order 3 at 0.

$$f(z) = \frac{1}{z^2 \sin z} = \sum_{n=0}^{\infty} a_n(z^n) + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \frac{b_{-3}}{z^3}$$

for getting a residue at a pole of order m take $g(z) = (z-z_0)^m f(z)$

the residue is $b_{-1} = \frac{g^{m-1}(z_0)}{(m-1)!}$

$$g(z) = \frac{z^3}{z^2 \sin z} = \frac{z}{\sin z}$$

$$\frac{1}{2!} \frac{d^2(g(z))}{dz^2} = b_{-1} = \frac{1}{2} \frac{d}{dz} \left(\frac{\sin z - z\cos z}{(\sin z)^2} \right)$$

$$= \lim_{z \to 0} \frac{1}{2} \left(\frac{z(\sin z)^3}{(\sin z)^4} - \frac{2\cos(z)\sin z\, z(\sin z - z\cos z)}{(\sin z)^4} \right)$$

$$= \frac{1}{2}\left(1 - \frac{2 \times 1}{3}\right) = \frac{1}{6}$$

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3) $f(z) = \frac{2(z-1)}{z^2-2z-3}$

$= \frac{A}{z+1} + \frac{B}{z-3}$, $A=1$, $B=1$

(i) $|z|<1$

$\frac{1}{z+1} = \frac{1}{1-(-z)} = 1+(-z)+(-z)^2+\ldots$

$\frac{1}{z-3} = \frac{-1}{3\left(1-\frac{z}{3}\right)} = \frac{-1}{3}\left(1+\frac{z}{3}+\frac{z}{3}^2+\ldots\right)$

(ii) $1<|z|<3$

$\frac{1}{z+1} = \frac{1}{(z)\left(1-\frac{1}{z}\right)} = \left(\frac{1}{z}\right)\left(\frac{1}{z}+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)$

$= \frac{1}{|z|}<1$ so we can expand

$= \sum_{n=1}^{\infty} \frac{1}{z^n}$

$\frac{1}{z-3} = \frac{-1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right)$

because $\frac{|z|}{3}<1$

$|z| > 3$

$$\frac{1}{z-1} = \frac{1}{(z)\left(1-\frac{1}{z}\right)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots \quad (1)$$

$$\frac{1}{z-3} = \frac{1}{(z)\left(1-\frac{3}{z}\right)} = \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \ldots$$

as $\frac{3}{|z|} < 1$

Tut-6

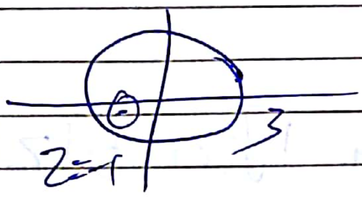
Q3 $f(z) = \frac{1}{z+1} + \frac{1}{z-3}$

(i) $|z| < 1 \rightarrow \frac{1}{1-(-z)} = 1 + (-z) + (-z)^2 + \ldots$

$\frac{1}{z-3} = \frac{-1}{3\left(1-\frac{z}{3}\right)} = \frac{-1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \ldots\right)$

as $\left|\frac{z}{3}\right| < 1$

(ii) $1 < |z| < 3$



$\frac{1}{z+1} \neq 1 + z \ldots\ldots$

$|z| > 1$

$\frac{1}{z+1} = \frac{1}{(z)\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left(1 + \left(\frac{1}{-z}\right) + \left(\frac{-1}{z}\right)^2 + \left(\frac{-1}{z}\right)^3 + \ldots\right)$

(iii) $\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \left(\frac{1}{z}\right)\left(1 + \left(\frac{-1}{z}\right) + \left(\frac{-1}{z}\right)^2 \ldots\right)$

as $\frac{1}{|z|} < 1$

$\frac{1}{z-3} = \frac{1}{(z)\left(1-\frac{3}{z}\right)} = \left(\frac{1}{z}\right)\left(1 + \frac{3}{z} + \frac{9}{z^2} + \ldots\right)$

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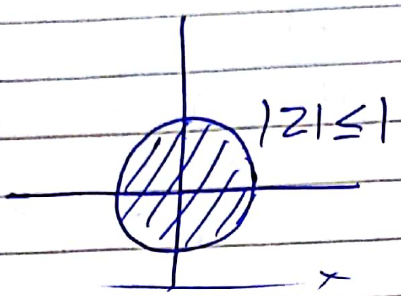
Tut · 7

Max. Modulus Theorem

A non-constant holomorphic function $f(z)$ on a Domain $\bar{D}$ attains its maximum at the boundary points.



$|z| = 1$

⋆ Schwartz Lemma :-

$\to$ unit Disk $|z| \le 1$

If $f(z) \to$ holomorphic on $\bar{D}$

is (i) $f(0) = 0$

(ii) $|f(z)| \le 1$ then $|f(z)| \le |z|$

⋆ Jordan's Lemma.

for as $f(z) = e^{iaz} g(z)$

$$\int_{\gamma_R} f(z)\,dz \le \left(\pi/a\right) \text{Max } |g(z)|$$

$\gamma_R \to$ Semi circular Disk with $R \to \infty$

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Tut-7

1) $I = \int \frac{\cos x}{(1+x^2)^2} dx$

$$I = R_e \left(\int_{-\infty}^{\infty} \frac{e^{iz}}{(1+z^2)^2} dz \right)$$

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{(z-i)^2(z+i)^2} dz + \int_{\gamma_R} \frac{e^{iz}}{(z-i)^2(z+i)^2} dz = \int_C \frac{e^{iz}}{(z-i)(z+i)} dz$$

As using Jordan Lemma :-

$$\int_{\gamma_R} \frac{e^{iz}}{(z-i)^2(z+i)^2} dz \le \pi \times \underset{\text{on } \gamma_R}{\max} \left(\left| \frac{1}{(z^2+1)^2} \right| \right) \to 0 .$$

As $z \to \infty$

So $\int \frac{e^{iz}}{(z-i)^2(z+i)^2} dz = 0$

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{(z^2+1)^2} dz = \int_C \frac{e^{iz}}{(z^2+1)^2} dz = 2\pi i \times \text{Res}\left(\frac{e^{iz}}{(1+z^2)^2} \text{ at } i \right)$$

(using Residue formulae)

$$g(z) = (z-i)^2 \times \frac{e^{iz}}{(1+z^2)^2}$$

$$\text{Res} = g'(z) \Big|_{z=i} = \frac{1 \times e^{-1}}{2i} = \frac{1}{2ei}$$

$$I = 2\pi i \times \frac{1}{2ie} = \frac{\pi}{e}$$

3) $\bar{D} \to |z| \le 1$

$D \to |z| < 1$

$$\phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

(i) for $|z| = 1$

$$\phi_\alpha(z) = \frac{(z-\alpha)\cdot\bar{z}}{(1-\bar{\alpha}z)\cdot\bar{z}} = \frac{(z-\alpha)\bar{z}}{(\bar{z} - \alpha \times z\cdot\bar{z})}$$

$$= \frac{(z-\alpha)(\bar{z})}{\bar{z}-\bar{\alpha}}$$

as $z\cdot\bar{z} = |z|^2 = 1$

$$|\phi_\alpha(z)| = \left|\frac{(z-\alpha)\cdot\bar{z}}{(\bar{z}-\bar{\alpha})}\right| = \left|\frac{(z-\alpha)}{\bar{z}-\bar{\alpha}}\right| |\bar{z}| = 1.$$

(ii) Deduce $\phi_\alpha(D) < 0$

ϕ_α is non-constant

ϕ_α is holomorphic on "D"

$\Rightarrow 1 - \bar{\alpha}z = 0$

$\Rightarrow z = \frac{1}{\bar{\alpha}} \to |z| = \frac{1}{|\bar{\alpha}|} > 1$

so only singularity is outside D.

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Using Maximum Modulus Theorem →

as $|\phi_\alpha(z)| = 1$ on $|z| = 1$

$|\phi_\alpha(z)| \leq 1$ on $|z| < 1$

as 1 is the max $|\phi_\alpha(z)|$

So $\phi_\alpha(D) < 0$

as D is $|z| < 1$

(ii) $\phi_a \circ \phi_{(a)}(z) = z$

$$= \frac{\phi_{-\alpha} - \alpha}{1 - \phi_\alpha \bar{\alpha}} = \frac{\frac{z+\alpha}{1+\bar{\alpha}z} - \alpha}{1 - \frac{(z+\alpha)}{1+\bar{\alpha}z} \times \bar{\alpha}}$$

$$= \frac{(z+\alpha) - \alpha(1+\bar{\alpha}z)}{1+\bar{\alpha}z - z\bar{\alpha} + \alpha\bar{\alpha}}$$

$$= \frac{z + \bar{\alpha}\alpha z}{1+\alpha\bar{\alpha}}$$

$$= \frac{(z)(1+\alpha\bar{\alpha})}{1+\alpha\bar{\alpha}}$$

$$= z$$

ϕ_α is an automorphism

ϕ_α is the only automorphism on the disk

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4/1 $|f| < M$ and $f(a)=0$ and f is analytic.
prove $|f(z)| < M\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right|$

Note → $\phi_a(a) = 0$

$g(z) = \frac{1}{M} \times f \circ \phi_a(z)$ valid because $|\phi_\alpha(z)| < 1$

$g(z) = \left(\frac{1}{M}\right) \times f \circ \phi_{-a}(z)$ valid because $|\phi_\alpha(z)| < 1$

for $g(0) = 0$ as

$f\left(\frac{0-(-a)}{1-\bar{a}\times 0}\right) = f(a) = 0$

$|(g(z))| < 1$ as $\left|f(z) \times \frac{1}{M}\right| < M \times \frac{1}{M} < 1$

* Schwarz Lemma is applicable on $g(z)$ -

$|g(z)| < |z|$

putting →

$z = \phi_\alpha(z)$ (R.H.S for getting)

$g(\phi_a(z)) < \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right|$

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$$\frac{1}{m} f \circ \phi_{-a} \circ \phi_a(z) < \frac{|z-\alpha|}{|1-\bar{\alpha}z|}$$

$$f(z) < m \left| \frac{z-\alpha}{1-\bar{\alpha}z} \right|$$

5) $f(a_i) = b_i$ for $i = 1, 2$ on $D \to D$

prove $\left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| < \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|$

define $g = \phi_{b_1} \circ f \circ \phi_{-a_1}(z)$

$g(0) = \phi_{b_1} \circ f \circ \phi_{-a_1}(0)$

$= \phi_{b_1} \circ f(a_1)$

$= \phi_{b_1} b_1 = 0$

So

$g(0) = 0$

$|g(z)| < 1$

$\Rightarrow |\phi_{b_1}(z)| < 1$ for any z
So putting $z = f \circ \phi_{-a}(z)$

$|g(z)| < 1$

g is holomorphic.

using Schwartz Lemma,

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$|\phi_{b_1} \circ f \circ \phi_{a_1}(z)| < |z|$

So putting $z = \phi_{a_1}(z)$

$$0 < |\phi_{b_1} \circ f \circ (\underbrace{\phi_{-a} \circ \phi_{a_1}(z)}_{z})| < \left|\frac{z - a_1}{1 - z\bar{a}_1}\right|$$

$\Rightarrow$ putting $z = a_2$.

$$\phi_{b_1} \circ f(a_2) < \left|\frac{a_2 - a_1}{1 - \bar{a}_1 a_2}\right|$$

$$\Rightarrow \phi_{b_1}(b_2) < \left|\frac{(a_2 - a_1)}{1 - \bar{a}_1 a_2}\right|$$

$$\Rightarrow \left|\frac{b_2 - b_1}{1 - \bar{b}_2 b_1}\right| < \left|\frac{(a_2 - a_1)}{1 - \bar{a}_1 a_2}\right|$$

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