Tut 5

1) $f: \mathbb{C} \to \mathbb{C}$ is entire. Show f is const.

(i) $\exists$ $c>0$ s.t. $|f(z)| > c > 0$ $\forall z \in \mathbb{C}$

$$g(z) = \frac{1}{f(z)} \to \text{entire}$$

$$0 < |g| \leq \frac{1}{c} \to g \text{ is entire \& bounded}$$

$\Rightarrow g$ is const.

$\to$ Reciprocal of entire is also entire.

(ii) $\underbrace{\text{Re}(f(z)) \geq 0}_{\text{convert}}$ to $g(z) > c > 0$.

$\therefore$ $g'(z) = 1 + f(z) \Rightarrow \underbrace{\text{Re}(g') \geq 1}$ same argument as above.

$0 < \left| \frac{1}{\sqrt{\text{Re}(g)^2 + \text{Im}(g)^2}} \right| \leq 1 \Rightarrow g'$ - entire & bounded

$g \Rightarrow$ const.

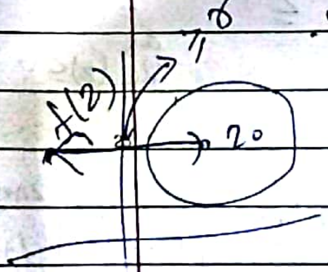
(iii) $\overline{f(\mathbb{C})}$: closure of f^n. closed set also with limit pts of f.

$\overline{f(\mathbb{C})} \neq \mathbb{C}$ i.e. $\mathbb{C} \setminus \overline{f(\mathbb{C})} \neq \phi$

$\exists$ z_0 in $\left(\overline{f(\mathbb{C})}\right)^c \Rightarrow$ open & $|z - z_0| < r$ in

i.e. z_0 - has no preimage.

$\therefore$ $h(z): \mathbb{C} \to \mathbb{C} = \frac{1}{f(z) - z_0} \to$ well defined



$$\therefore |h| = \frac{1}{|f(z) - z_0|} \leq \frac{1}{r} < \frac{2}{r}.$$

$\downarrow$ entire & has upper bound

2) $f(1/n) = \begin{cases} 1/n & n \text{ is even} \\ -1/n & n \text{ is odd} \end{cases}$

$\downarrow$ let it be holom.

$\Rightarrow f'(0,1)$ has infinite set of pts where it is zero. By identity thm. $g(z) = 0$. Also valid for $\mathbb{C}$.

It is not holom.

$f(0) \therefore \quad \frac{z^N f^N(0)}{N!}$

Method 2:

$f'(0) = \lim_{h \to 0} \frac{f(h)}{h}$ → different along diff. sequences

$= \lim_{n \to \infty} n \cdot f(1/n)$

3. $\xrightarrow{\text{entire}}$ ~~$f(1/n^2) = 1/n \quad \forall n \in \mathbb{N}$~~ $g(z^2) = z$ → diff

$g(z) = z^{1/2}$ – not entire – has 2 branches

↓

In a branch, by identity thm $= f$. but g is not entire. so f isn't entire.

e^{tz}

te^{tz}

$\therefore f \to$ ~~doesn't~~ exist.

4. $f(z) = f(0) + z f'(0) + \dots \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N f^{N+1}(tz)\,dt$

a) $\left| e^z - \sum_{i=0}^{N} \frac{z^i}{i!} \right| = \frac{|z|^{N+1}}{(N+1)!} \left| \int_0^1 (1-t)^N \cancel{e} f^{N+1}(tz)\,dt \right|$

$\leq \frac{|z|^{N+1}}{(N+1)!} \int_0^1 \underbrace{|1-t|^N}_{\leq 1} \underbrace{|e^{tz}|}_{\leq 1} \, dt$

as $Re(z) \leq 0$

$\leq \frac{|z|^{N+1}}{(N+1)!}$

b) $\left| \cos z - \sum_0^N (-1)^n \frac{z^{2n}}{2n!} \right| = \left| \frac{z^{2n+2}}{(2n+2)!} \int_0^1 (1-t)^{2N+1} f^{2N+1}(\dots) \right|$

$f^{2n+1} = 0$

$f \leq \left| \frac{z^{2n+2}}{(2n+2)!} \right| \int_0^1 |(1-t)^{2N}| \, | \cancel{\sin} \cos tz | \, d\dots$

$Im(z) \leq R \qquad \cos(tz) = \frac{e^{itz} + e^{-itz}}{2}$

$\exp\left(\frac{\cdots}{2\pi}\right)$ $\quad + i(\theta + 2n\pi)$ $\quad |z-p| < |z_0 - p|$

$i\theta,\ i(\theta + 2\pi)$

$$|\cos(tz)| \leq \frac{|e^{itx - ty}|}{2} + \frac{|e^{-itx + ty}|}{2} \leq \frac{e^{R} + e^{-R}}{2}$$

$\downarrow$ $\cosh R.$

$$f \leq \left|\frac{z^{2n+2}}{(2n+2)!}\right| \cosh R.$$

5) $$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}}\, dz =$$

$$g = (z^2+1)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 z^2 + \dots {}^{2n}C_n z^{2n} + {}^{2n}C_{n+1} z^{2n-2}$$

$$g^{2n}(0) = {}^{2n}C_n \cdot 2n!$$

$$I = {}^{2n}C_n \cdot (2n)! \cdot \frac{(2\pi i)}{(2n)!} = {}^{2n}C_n \cdot (2\pi i)$$

$$\int_0^{2\pi} 2^{2n} \cdot \frac{(\cos t)^{2n}}{e^{it}}\, i e^{it}\, dt = {}^{2n}C_n (2\pi i)$$

$$\int_0^{2\pi} (\cos t)^{2n}\, dt = \frac{{}^{2n}C_n \cdot 2\pi}{4^n}$$

Laurent series: expand holom. f^n and isolated singular pt.

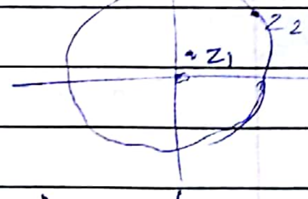
$$L = \sum_{-\infty}^{\infty} a_j (z-p)^j$$

$\downarrow$

converges if $\sum_0^{\infty} a_j (z-p)^j$ & $\sum_1^{\infty} a_{-j}(z-p)^{-j}$ converges.

$\rightarrow$ At any pt p where the power series converge $\forall\ |z| < |p|$ also it converges

$\therefore$ If $\sum_0^{\infty} a_i (z-p)^i$ converges at z_2



$\therefore$ For $\sum_1^{\infty} a_j \left(\frac{1}{z-p}\right)^j$

$$\left|\frac{1}{z-p}\right| < R^* \Rightarrow |z-p| > \frac{1}{R^*}$$

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Tb

1) i) $\frac{1}{\sin(1/z)} \rightarrow$ def $\forall$ $\sin\left(\frac{1}{z}\right) \neq 0$ & $\frac{1}{z}$ ~~$\neq 0$~~ defined

$z \neq \frac{1}{n\pi}$, $n \in \mathbb{Z}-\{0\}$, $z \neq 0$

In any neighbourhood and $z=0$. $\exists$ singularity.

$\therefore$ $z=0 \Rightarrow$ non-isolated

$z = \frac{1}{n\pi} \Rightarrow \lim_{z \to \frac{1}{n\pi}} \frac{(z - 1/n\pi)}{\sin(1/z)} = 1$ (exist

$\therefore$ pole of order one.

2) (i) $\frac{1}{(z^4+1)^2} \rightarrow \frac{1}{(z^2-i)(z^2+i)} = \pm\sqrt{i}, \pm\sqrt{-i}$

$z = e^{\pm i\pi/4}, e^{\pm i 3\pi/4}$

$\therefore f = \prod_{j=0}^{3} \frac{1}{\left(z - e^{i\left(\frac{\pi}{4}\right)(2j+1)}\right)^2}$

$\therefore$ ~~$\not\exists$~~ $\lim_{z \to z_0} (z-z_0)^2 \cdot f \rightarrow$ exist

$\hookrightarrow$ pole of order 2.

5) p, q diff at z_0 $p(z_0) \neq 0$; $q(z_0) = 0$
$q'(z_0) \neq 0$.

z_0 is simple pole.

$f = \frac{p}{q}$

$\therefore \lim_{z \to z_0} \frac{q(z)}{z-z_0} = q'(z_0)$.

$\lim_{z \to z_0} (z-z_0)^1 \frac{p}{q} = \lim_{z \to z_0} \frac{p}{\left(\frac{q-0}{z-z_0}\right)}$

$= \frac{\lim_{z \to z_0} p(z)}{\lim_{z \to z_0} \left(\frac{q-0}{z-z_0}\right)} = \frac{p(z_0)}{q'(z_0)} = \mathrm{Res}(f, z_0)$

as $\underset{m \neq 0}{\mathrm{Res}}[f, z_0] = \frac{1}{(m-1)!} \left[\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)\right]$

$\oint \frac{z+1}{\sin z}$ $\quad$ $\frac{2!}{2\pi i}\int_\gamma \frac{1}{z^2 \sin z}\,dz$

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6 (i) $\frac{1}{z^2 \sin z}$ $\rightarrow$ sing pts $= 0$. $z = n\pi$ $\rightarrow$ all isolated.

Order of $0 \Rightarrow 3$

" " $n\pi$ $(n \neq 0) \Rightarrow 1$

$$\text{Res}(f, n\pi) = \frac{(-1)^n}{(n\pi)^2}$$

$$\text{Res}(f, 0) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{z^3}{z^2 \sin z}\right)$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d}{dz}\left[\frac{\sin z - z\cos z}{\sin^2 z}\right]$$

$$= \frac{1}{2!}\left[\frac{-\cos z}{\sin^2 z} - \frac{(-z\sin z + \cos z)}{\sin^2 z} + \frac{z\cos^2 z(2\sin z)}{\sin^2 z}\right]$$

$$= \frac{1}{6}$$

3. $f = \frac{2(z-1)}{z^2 - 2z - 3} \rightarrow (-1, 3) = \frac{1}{z+1} + \frac{1}{z-3}$

(i) $0 < |z| < 1$ $\quad f = \frac{1}{1-(-z)} - \frac{1}{3}\left(\frac{1}{1 - z/3}\right)$

$$= \sum_0^\infty (-1)^i z^i + \left(\sum_0^\infty \frac{(-1)^{i+1}}{3^{i+1}} z^i\right)$$

$$= \sum_0^\infty z^i (-1)^i \left(1 - \frac{1}{3^{i+1}}\right)$$

(ii) $1 < |z| < 3$

$$f = \frac{1}{z}\left(\frac{1}{1-\left(-\frac{1}{z}\right)}\right) - \frac{1}{3}\left(\frac{1}{1 - z/3}\right)$$

$$= \frac{1}{z}\sum_0^\infty (-1)^i \left(\frac{1}{z^i}\right) + \sum_0^\infty \frac{(-1)^{i+1}}{3^{i+1}} z^i$$

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T-6

1- a) $\sin\left(\frac{1}{z}\right) \Rightarrow$ essential.

(b) $\frac{z^2 + z + 1}{z^3 - 11z + 13} \Rightarrow$ roots of $(z^3 - 11z + 13)$
$\downarrow$
simple poles.

(d) $\tan(1/z) = \frac{\sin(1/z)}{\cos(1/z)}$ not defined at $z = 0$.

& $\cos\left(\frac{1}{z}\right) = 0$ at $\frac{1}{z} = \frac{(2n+1)\pi}{2} \Rightarrow z = \frac{2}{(2n+1)\pi}$

$\lim_{z \to \frac{2}{(2n+1)\pi}} \frac{\left(z - \frac{2}{(2n+1)\pi}\right) \sin\frac{1}{z}}{\left(\cos\frac{1}{z}\right)}$

$$\lim_{z \to \frac{2}{(2n+1)\pi}} \frac{\left(z - \frac{2}{(2n+1)\pi}\right) \cdot \sin(1/z)}{\cos 1/z}$$

$$= \frac{\sin 1/z + \left(z - \frac{2}{(2n+1)\pi}\right)\left(-1/z^2\right)\cos 1/z}{\sin 1/z \cdot (-1/z^2)}$$

$$= -\left(\frac{2}{(2n+1)\pi}\right)^2 \Rightarrow \lim \text{ exist.}$$

$\therefore$ simple pole.

and $D(0, \varepsilon)$, $\exists$ $z' = \frac{2}{(2n+1)\pi}$ inside the disc: $\frac{2}{(2n+1)\pi} < \varepsilon\pi$

$$\frac{2n+1}{2} > \frac{2}{\pi\varepsilon} - 1$$

$$n > \frac{1}{\pi\varepsilon} - \frac{1}{2}$$

$\therefore$ zero · non-isolated - singularity at pts near zero.

(ii) $\frac{1}{z^2+z-1}$ $\quad z = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow$ poles of order 1.

6) (i) $\frac{1}{z^2 \sin z}$ singularity $z = 0$: order 3

$z = n\pi$ order 1

(ii) $\frac{1}{z(1-z)^2}$ $\quad z = 0, 1$

order 1   order 2.

$\text{Res}(f; 0) = 1$

$$\text{Res}(f : 1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} (z-1)^2 \cdot \frac{1}{z(z-1)^2}$$

$$= -1$$

(iii) $\left(\frac{z+1}{z-1}\right)^3 \Rightarrow z = 1$ of order 3

$$\text{Res}(f : 1) = \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} (z+1)^3$$

$$= \frac{1}{2}(3)(2)(2) = 6$$

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Picard's thm:

Big Pic. : if z_0 is essential sing. of $f(z)$. Then, in any punctured neigh of z_0 → allowed to miss atmost one pt.

Q) If non-const entire f^n misses one pt c, it is of form $e^{f(z)} + c$ for some entire f^n & $f(z)$

$\frac{g'}{g} = f'$

Tut 7.

3) $\alpha \in \mathbb{D}$; $\psi_\alpha : \overline{\mathbb{D}} \to \mathbb{C}$ by $\psi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$

(i) if $|z| = 1$, $|\psi_a(z)| = 1$

$\left[\psi_a(z) \cdot \overline{\psi_a(z)}\right] = 1 \Rightarrow |\psi_a(z)| = 1$

$z \neq 0$ & $z \neq \alpha$

(ii) $1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(1-\bar{\alpha}z)(1-\alpha\bar{z})} \cdot \frac{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2}{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2}$

(ii) $\psi_a(\mathbb{D}) \subseteq \mathbb{D}$.

$\psi_a(z) - \bar{\alpha}\,\psi_a(\bar{z}) \cdot z = z - \alpha$.

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$$\frac{\varphi_a(z)+\alpha}{1+\bar{\alpha}\,\varphi_a(z)} = z$$

$\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D} \iff$ codomain on $\mathbb{D}$ is $\mathbb{D}$.

$\iff |Q_\alpha(z)| < 1 \quad \forall\ |z|<1$

given: $|Q_\alpha(z)| = 1 \quad \forall\ |z|=1$

By MMT, any value in $\mathbb{D} = Q_\alpha(z)$ less than that of boundary $\Rightarrow |Q_\alpha(z)| < 1$

$\varphi_\alpha'(z) \Rightarrow$ holom. as $z = \frac{1}{\bar{\alpha}}$ outside $\bar{\mathbb{D}}$

(iii) $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ is invertible.

$\varphi_\alpha \circ \varphi_{-\alpha}(z) = z = \varphi_{-\alpha} \circ \varphi_\alpha(z)$

$\forall\ z \in \mathbb{D}$.

$$\varphi_\alpha(\varphi_{-\alpha}(z)) = \cancel{\varphi_\alpha\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right)}$$

$$= \frac{\varphi_{-\alpha}(z)-\alpha}{1-\bar{\alpha}\,\varphi_{-\alpha}(z)} = \frac{\frac{z+\alpha}{1+\bar{\alpha}z}-\alpha}{1-\bar{\alpha}\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right)} = \frac{z(1-|\alpha|^2)}{1-\alpha\bar{\alpha}}$$

$= z$

Automorphisms: $\mathbb{D} \to \mathbb{D}$: $\varphi_\alpha(z)$

$A \to A$: $K \cdot \varphi_\alpha(z)$.

4) f is analytic on $\mathbb{D}$ with $|f| < M$, $f(a) = 0$

show: $|f| \le M \left|\frac{z-a}{1-\bar{a}z}\right| \quad \forall\ z \in \mathbb{D}$.

let $g(z) = \frac{1}{M}(f \circ \varphi_a)(z)$

$\therefore g(0) = \frac{1}{M} f(+a) = 0$.

By Schwarz lemma, $|g(z)| \le |z| \quad \forall\ z \in \mathbb{D}$.

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$$\therefore g(\varphi_a(z)) = \frac{f(z)}{M}$$

$$\therefore |g(\varphi_a(z))| \le \left|\frac{z-a}{1-\bar{a}z}\right|$$

$$\therefore |f(z)| \le M\left|\frac{z-a}{1-\bar{a}z}\right| \quad \forall z \in \mathbb{D}.$$

5: $f : \mathbb{D} \to \mathbb{D}$, holom. & $f(a_i) = b_i$ ~~t~~.

$$\left|\frac{b_2-b_1}{1-\bar{b}_1 b_2}\right| \le \left|\frac{a_2-a_1}{1-\bar{a}_1 a_2}\right|$$

$$g(z) = \left(\psi_{+b_1} \circ f \circ \psi_{-a_1}\right)^2$$

$\mathbb{D} \to \mathbb{D}$ $\quad$ $\mathbb{D} \to \mathbb{D}$ $\quad$ $\mathbb{D} \to \mathbb{D}$

$$g(0) = \psi_{+b_1}(b_1) = 0.$$

$$\therefore |g(z)| \le 1$$

By S.L., $|g(z)| \le |z| \quad \forall z \in \mathbb{D}$

$$|\psi_{b_1} \circ f(a_2)| \le \left|\frac{a_2-a_1}{1-\bar{a}_1 a_2}\right|$$

$$\left|\frac{b_2-b_1}{1-\bar{b}_1 b_2}\right| \le \text{" \quad "}$$

2. $e^{i\theta} = z$ $\quad$ ~~$\theta = \ln z$~~ $\qquad \dfrac{d\theta}{a+1-2a\cos\theta}$

$$e^{i\theta} \cdot i \cdot d\theta = dz$$

$$d\theta = \frac{dz}{iz}$$

$$\frac{2\pi}{1-a^2} \Leftarrow \oint_{C_1} \frac{dz}{iz\left[a+1 - a(z+\bar{z})\right]} \qquad \text{where } a$$

poles: $z = 0$ $\qquad \cos\theta = \dfrac{a+1}{2a}$

$= \dfrac{1}{2} +$

1) $\dfrac{\pi}{6}$

~~$a < 0$~~

~~$\frac{1}{a} <$~~

$0 < a < 1$

$\frac{1}{a} > 1$

$\frac{1}{2a} > \frac{1}{2} \Rightarrow$ (1)

$\frac{a+1}{a} = z + \bar{z}$ $f(z) = \oint$ $Re(z) = \frac{a+1}{2a}$ $-3a(a-1) - 1$

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$= 2\pi i \; Res(f:0)$

$= 2\pi i \left(\frac{1}{i}\right) \cdot \frac{1}{(a+1)} = \frac{2\pi}{1+a}$

$+ 2\pi i \left(\frac{1}{i}\right) \cdot \cancel{\frac{(2a)}{a+1}}$

$d\theta = \frac{dz}{iz} : \oint_{C_1} \frac{dz}{iz\left[(a+1) - a\left(z+\frac{1}{z}\right)\right]}$

$= \oint_{C_1} \frac{dz}{i\left[(a+1)z - az^2 - a\right]}$

$= \oint_{C_1} \frac{i\,dz}{(a \cdot z^2 - (a+1)z + a)}$

$z = \frac{(a+1) \pm \sqrt{(a+1)^2 - 4a^2}}{2a}$

$= \frac{(a+1) \pm \sqrt{-3a^2 + 2a + 1}}{2a}$

$= \frac{(a+1) \pm \sqrt{-3a^2 + 3a - a + 1}}{2a}$

$= \frac{a+1}{2a} + \frac{1}{2a}\sqrt{(a-1)(-3a-1)}$

$= \frac{a+1}{2a} \pm \frac{1}{2a}\sqrt{(1-a)(3a+1)}$

Q2 $\Rightarrow I = \oint_{C_1} \frac{dz}{i((a^2+1)z - az^2 - a)} = \oint_{C_1} \frac{dz}{i(-az(z-a) + 1(z-a))}$

$e^{i\theta} = z$

$\Downarrow$

$d\theta = \frac{dz}{iz}$

$= \oint_{C_1} \frac{dz}{ai(z-a)\left(\frac{1}{a} - az\right)} = \oint_{C_1} \frac{i\,dz}{a(z-a)\left(z-\frac{1}{a}\right)}$

$= 2\pi i \left(\frac{i}{a}\right)\left[\frac{1}{a-\frac{1}{a}}\right]\cancel{\left[\frac{1}{\frac{1}{a}-a}\right]} = \frac{2\pi}{1-a^2}$

$= 2\pi i \; Res(f:a) =$

$(a < 1 \text{ in } C_1 \;\&\; \frac{1}{a} > 1 \text{ outside})$

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Q1) $\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx$

$\int_\gamma \frac{e^{iz}\,dz}{(1+z^2)^2} = \int_{C_R} \frac{e^{iz}\,dz}{(1+z^2)^2} + \int_{-R}^{R} \frac{e^{iz}\,dz}{(1+z^2)^2}$

$\lim_{R\to\infty} \frac{1}{1+R^2} \to 0$

$= \int_{-R}^{R} \frac{\cos z\,dz}{(1+z^2)^2}$

LHS: $\int_\gamma \frac{e^{iz}\,dz}{(1+z^2)^2} = 2\pi i \cdot \text{Res}(f; i)$.

$2\pi i \cdot \frac{1}{(2-1)!} \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 \cdot \frac{e^{iz}}{(z-i)^2 (z+i)^2}\right)$

$= 2\pi i \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)$

$= 2\pi i \left[\frac{i e^{iz}(z+i)^2 - e^{iz}(2)(z+i)}{(z+i)^4}\right]_i$

$= 2\pi i \left[\frac{i e^{-1}(4)(-1) - e^{-1}(4i)}{2^4}\right]$

$= 2\pi i \left[\frac{-8e^{-1} i}{2^4}\right] = \frac{\pi}{e}$.

T6

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$\lim_{z\to \frac{2}{(2n+1)\pi}}$ $\cancel{\left(z - \frac{2}{(2n+1)\pi}\right) \sin 1/z}$

$\cancel{(z - 1)}$