Test — MA 205

$\mathbb{C}$ $\quad \mathbb{R}^2$

$(x_1, x_2) \times (y_1, y_2) = (x_1 y_1, x_2 y_2)$

$(x_1, x_2) \times (y_1, y_2)$ Multiplicative inverse is absent.

$(x_1 y_1 - x_2 y_2, \; x_1 y_2 + x_2 y_1)$

$\forall z = (x, y) \neq 0, \exists z^{-1}$ s.t. $z z^{-1} = 1$

$\mathbb{R}[i] = \mathbb{R} + i\mathbb{R} \qquad i^2 = -1$

$\cong$

$\mathbb{C}$ <u>algebraically closed.</u>

F.T.A. $\rightarrow$ Every $p \in \mathbb{C}[x]$ polynomial having complex coeff. has a root in $\mathbb{C}$.

$\mathbb{C}$ does not have a order that restricts to the usual order on $\mathbb{R}$.

Exercise.

$|z_1 - z_2| = \|(x_1, y_1) - (x_2, y_2)\|$

absolute value on $\mathbb{C}$

Given any seq. $\{z_n\}$ in $\mathbb{C}$

$z_n \to z$ if

$\forall \epsilon > 0 \quad \exists \; N \in \mathbb{N}$

s.t. $\forall \; n \geq N \quad |z - z_n| < \epsilon$.

In complex space if a func^n is once differentiable, it is ∞ differentiable.

Page
Date

Same for continuity as for $\mathbb{R}^2$.

Derivability.

$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - Df(h)}{\|h\|}$$

For Complex, $x, y \in \mathbb{C}$

$$\lim_{y\to 0} \frac{f(x+y) - f(x)}{y} = f'(x)$$

Cauchy Riemann's Eq^n

Say $f: \mathbb{C} \to \mathbb{C}$ differ. at z_0.

then If $f = u + iv$ $u, v: \mathbb{R}^2 \to \mathbb{R}$

then $u_x = v_y$
$u_y = -v_x$.

$$f'(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}$$

$\mathbb{C}$

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \quad \alpha, \beta \in \mathbb{R}$$

$\alpha + i\beta$.

$\to$ Multiplying with same type of matrix gives again same type of matrix.

$$f = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x = 0 \end{cases}$$

Smooth func^n but taylor series at 0 gives 0.

(2) <u>Irreducible in $\mathbb{R}[x]$.</u>

$f(x) \in \mathbb{R}[x]$.

if $\exists$ $g(x), h(x) \in \mathbb{R}[x]$ s.t.

$$f(x) = g(x) \cdot h(x)$$

g, h are not constant.

Consider $f \in \mathbb{R}[x]$ s.t. $\deg(f) > 2$

If f has a real root, say α,

$$f(x) = (x-\alpha)\, g(x)$$

Justify $g \neq$ constant.

If f doesn't have a real root, ~~the~~

but FTA says f ~~hasnt~~ have a root in $\mathbb{C}$, say, z.

~~$f(x) = (x-z)$~~

Let $f(x) = a_n x^n + \ldots + a_0 \quad a_i \in \mathbb{R}$

$$\left.\begin{aligned} f(z) &= 0. \\ \Rightarrow f(\bar{z}) &= 0 \end{aligned}\right\} \text{ Since } a_i \in \mathbb{R}.$$

Since $(x-z)(x-\bar{z}) \mid f$.

$$\Rightarrow (x^2 + |z|^2 - 2\,Re(z)x)$$

Real.

$$\Rightarrow f(x) = (x^2 - |z|^2)\, h(x)$$

Justify $h(x) \neq$ constant and $(x^2 + |z|^2 - 2Re(z)x)$ is real polynomial.

$f = a_n x^n$ $\qquad a_1 x + b$

Page
Date

6

~~fezcucactaib~~

(3) (iii) $f(z) = z^n \quad n \in N$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0}$$

$$\lim_{z \to z_0} \frac{\cancel{(z - z_0)}\,(z^{n-1} + z^{n-2} z_0 + \cdots)}{\cancel{(z - z_0)}} = \lim_{z \to z_0} \left(\sum_{i=0}^{n-1} z^i z_0^{n-1-i} \right)$$

$$\boxed{n z_0^{n-1}}$$

(vii) $f = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$\lim_{z \to 0} \frac{\frac{z}{\bar{z}} - 0}{z} = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$$

$$\lim_{z \to 0}$$

(iv) $f(z) = Re(z)$ $\qquad z = x + iy$

$z = z_0 = x_0 + iy_0$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad h \in \mathbb{R} \qquad \lim_{h \to 0} \frac{f(z+h) - f(z_0)}{h}$$

$$\frac{x+h-x}{h} = 1$$

$$\lim_{h \to 0} \frac{f(z+ih) - f(z_0)}{ih} = \frac{x-x}{ih} = 0.$$

Holomorphic func^n are also called conformal mapping.

$z \quad \bar{z}$

$$x = \frac{z+\bar{z}}{2}$$

$x \quad y$

$$y = \frac{z-\bar{z}}{2i}$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow \text{Cauchy Riemann}$$

$$\frac{\partial f}{\partial z} = 0 \Rightarrow \text{Antiholomorphic.}$$

(vii) $f(z) = \bar{z}$.

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{\bar{z}+h - \bar{z}}{h} = 1$$

$$\lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih}$$

$$= \lim_{h\to 0} \frac{\bar{z} - ih - \bar{z}}{ih} = -1.$$

If f is C^1 fun^n the converse of complex and real differentiation is true.

Page
Date

(4) If in an open connected set $\Omega \subseteq \mathbb{R}^n$, $Df = 0$ then f is constant on Ω.

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f : \overset{\Omega \subseteq \mathbb{C}}{\mathbb{C}} \rightarrow (\mathbb{R} \subseteq \mathbb{C})$$

then f' doesn't exist at a point. or $f' = 0$ at that point.

Every real valued holomorphic fun^n is locally const

$h \in \mathbb{R}$, z.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{R}$$

$$\lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} \in i\mathbb{R}$$

If this limit exists $\lim_{h \to 0} f = 0$

as real and purely imaginary numbers meet at zero —

Scanned by CamScanner

MA-205 (Tut)-2

(1) $f(z) = e^x(\cos y + i\sin y)$ is holomorphic one.

$= (e^x\cos y) + i(e^x\sin y)$

$u_x = e^x\cos y \qquad u_y = -e^x\sin y$

$v_x = e^x\sin y \qquad v_y = e^x\cos y.$

$$\left.\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}\right\} \checkmark$$

$$u_x(x_0,y_0) \lim_{h\to 0} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}$$

(2) $f = u + iv$

$f = u(r,\theta) + i\, v(r,\theta)$

$z =$

$x = r\cos\theta \quad y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$u_r = u_x\cos\theta + u_y\sin\theta$

$u_\theta = -r u_x\sin\theta + r u_y\cos\theta$

$v_r = v_x\cos\theta + v_y\sin\theta$

$v_\theta = -v_x r\sin\theta + v_y r\cos\theta.$

$u_r = \frac{1}{r} v_\theta. \qquad v_r = \frac{-u_\theta}{r}$

(3) u and v are Harmonic Conjugates.

If f is holomorphic and $f = u + iv$

$v \xrightarrow{HC} u \Rightarrow u_x = v_y$ and $v_x = -u_y$,

$u \xrightarrow{HC} v \Rightarrow v_x = u_y$ and $u_x = -v_y$,

$u_x = 0 \qquad v_x = 0$

$v_y = 0 \qquad u_y = 0$

④ $u(x,y) = xy + 3x^2y - y^3.$

$u_x = y + 6xy = v_y$

$\Rightarrow v = \frac{y^2}{2} + 3xy^2 + \phi(x)$

$u_{xx} = 6y$

$v_x = 3y^2 + \phi'(x)$

$u_y = x + 3x^2 - 3y^2 = -v_x$

$u_{yy} = -6y$

~~$3xy^2 - \frac{x^3}{} - \frac{x^2}{2} + k(y)$~~

$\phi'(x) = -x - 3x^2$

$\boxed{u_{xx} + u_{yy} = 0}$

$\phi(x) = -\frac{x^2}{2} - x^3 + C$

$v = \frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3 + C$

$u(x,y) = 3x^2 + 2x - 3y^2 - 1$

$u_x = 6x + 2 = v_y$

$\Rightarrow v = 6xy + 2y + \phi(x)$

$u_{xx} = 6$

$v_x = 6y + \phi'(x)$

$u_y = -6y = -v_x$

$\phi'(x) = 0$

$u_{yy} = -6$

$\phi(x) = C$

$u_{xx} + u_{yy} = 0.$

① $\sum_{k=1}^{\infty} kz^k$ ② $\sum_{p=\text{prime}} z^p$ ③ $\sum_{k=1}^{\infty} \frac{k!\, z^k}{k^k}$

Root Test - $R = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{|a_i|}}$

Ratio Test - $R = \limsup_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

Date

$$\limsup_{i\to\infty} \sqrt[i]{|a_i|} \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\epsilon > 0$ then $\exists\ N_0$ s.t.

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon \quad \forall\, n \ge N_0.$$

$$L - \epsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon \quad \forall\, n \ge N_0$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon \quad \forall\, n \ge N_0$$

$$\left|\frac{a_{N_0+1}}{a_{N_0}}\right| < L + \epsilon \quad , \quad \left|\frac{a_{N_0+2}}{a_{N_0+1}}\right| < L + \epsilon$$

$$\cdots \quad \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon$$

$$\left|\frac{a_{n+1}}{a_{N_0}}\right| < (L+\epsilon)^{n-N_0}$$

$$\Rightarrow |a_{n+1}| < (L+\epsilon)^{n-N_0} \cdot |a_{N_0}|$$

$$\Rightarrow \cancel{\frac{|a_{n+1}|}{|a_n|} < (L+\epsilon)^{n-N_0} \left|\frac{a_{N_0}}{a_n}\right|}$$

$$\Rightarrow |a_{n+1}|^{\frac{1}{(n+1)}} < |a_{N_0}|^{\frac{1}{n+1}} (L+\epsilon)^{\frac{n-N_0}{n+1}}$$

$\cancel{(L+\epsilon)^{\frac{N_0}{n+1}}}$

As $n \to \infty$ $\to 1$. $\quad \to L+\epsilon$

$$R = \limsup_{k \to \infty} \left| \frac{(k+1) z^{k+1}}{k z^k} \right|$$

$$\limsup_{k \to \infty} \left| \left(1 + \frac{1}{k}\right) z \right|$$

(2) $$a_i = \begin{cases} 1 & i = \text{prime} \\ 0 & \text{o.w.} \end{cases}$$

$$y_n = \sup\{ \sqrt[i]{|a_i|} \mid i \geq n \}$$

$$= 1$$

$$\limsup \sqrt[i]{|a_i|} = \lim_{n \to \infty} y_n = 1$$

$$R = 1.$$

$a_n = \dfrac{n!}{n^n}$ Ratio test.

$$\cancel{\limsup_{n \to \infty} \left| \frac{(n+1)!\, n^n}{(n+1)^{(n+1)}\, n!} \right|}$$

$$\cancel{\limsup_{n \to \infty} \left| \left(\frac{n}{n+1}\right)^n \right|}$$

$$R = \limsup_{n \to \infty} \left| \left(\frac{n+1}{n}\right)^n \right|$$

$n \cancel{\ln}\left(1 + \frac{1}{n}\right)$

e

$e^{\cancel{n+1}}$

Page
Date

(c) $1 + \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{2^2} + \ldots$

$$a_n = \begin{cases} 2^{-n}, & n : \text{even} \\ 2^{-n+3}, & n : \text{odd} \end{cases}$$

$\left| \cancel{\frac{2^{-n+1}}{2^{-n+3}}} \right|$ $\cancel{2^{-4}}$

even odd

$n \to$ even $\frac{2^{-n}}{2^{-n+2}}$ $\cancel{2^{2}}$

$\left| \frac{2^{-n+3}}{2^{-n-1}} \right|$

$\cancel{\frac{2^{-n}}{2^{-n+2}}}$ $\cancel{2^{2}}$

$n \to$ odd 2^{5}

2^{-n+3}

$a_{2n} = \frac{1}{3^n}$ $a_{2n+1} = \frac{1}{2^n}$

$\left| \frac{a_{2n+1}}{a_{2n}} \right|$

$\limsup \sqrt[i]{|a_i|} = \frac{1}{\sqrt{2}}$

$\left| \frac{3^n}{2^{n+1}} \right|$

$\frac{1}{2} \left| \frac{3}{2} \right|^n > 1$

Tut - 3

Cauchy's Th.

$\gamma : [0, 1] \longrightarrow \mathbb{C}$ Continuous

$I = \int_\gamma f(z)\, dz$, $f = F'$ then

$I = F(\gamma(1)) - F(\gamma(0))$

For a closed curve $I = 0$.

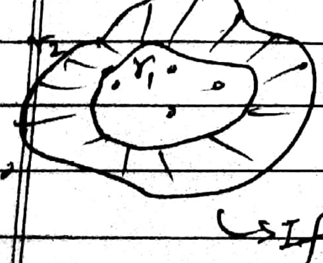
① $\int_{S^1} \frac{1}{z}\, dz = 2\pi i$

$\frac{1}{2\pi i} \int \frac{1}{z-0}\, dz = 1$ / ev at 0.

Cauchy's Integral Formula -

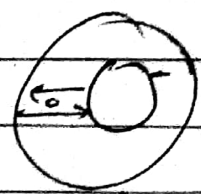
II $\int_{\partial D} f(z)\, dz$

int of D is simply connected then anti derivative exists.



$\int_{\gamma_1} f = \int_{\gamma_2} f$

↳ If in joining the curves one doesn't pass through any hole



Keyhole contour.

$\sin(x+iy) = \sin x \cos(iy) + \cos(x)\sin(iy)$

$\sin x \cosh y + i \cos x \sinh y$

$\frac{e^{-y}-e^{y}}{2i} = i\,\frac{e^{y}-e^{-y}}{2}$

Page
Date

$\frac{e^{-y}+e^{y}}{2}$

② (i) $\sin z =$ ~~$\sin x \cos$~~

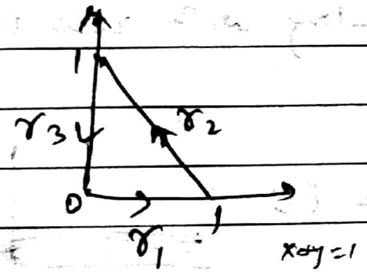
$x \in \mathbb{R}$ $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$y \in \mathbb{R}$ $\cosh y = \frac{e^{y}+e^{-y}}{2}$ $\sinh y = \frac{e^{y}-e^{-y}}{2}$

$$\frac{e^{ix+y}-e^{-ix+y}+e^{ix-y}-e^{-ix-y}}{4i} + i\,\frac{e^{ix+y}+e^{-ix+y}-e^{ix-y}-e^{-ix-y}}{4}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$=$



$\gamma_1 : [0,1] \to \mathbb{C}$
$\gamma_1(t) = (t, 0)$

$\gamma_2 : [0,1] \to \mathbb{C}$
$\gamma_2(t) = (t, i(1-t))$

$\gamma_3 : [0,1] \to \mathbb{C}$
$\gamma_3(t) = (0, it)$

$$\int_\gamma Re(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} Re(z)\,dz$$

$$= \int_0^1 t\cdot(1)\,dt + \int_1^0 t\cdot(1-i)\,dt$$

$\int_\gamma f(z)\,dz =$
$\underbrace{\quad}_{u(z)+iv(z)}$

$\gamma : [0,1] \to \mathbb{C}$

$\int_0^1 (u(\gamma(t)) + i\,v(\gamma(t)))\,\gamma'(t)\,dt$

$\frac{1}{2} + \left[\frac{-1}{2} + i\,\frac{1}{2}\right].$

$= i/2$

Scanned by CamScanner

(3) $\bar{z}^2$ $\qquad t^2 - i t^2$

$$\int_0^1 t^2 \cdot 1\,dt + \int_1^0 \left[t^2-(1-t)^2+2i\,t(1-t)\right]\left[1-i\right]dt$$

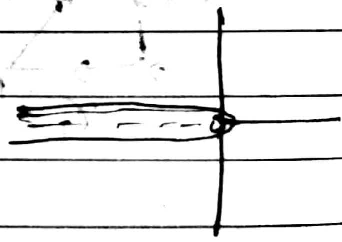
$$+ \int_1^0 -t^2\,(i)\,dt$$

$$\frac{1}{3} + \frac{i}{3} - \frac{1}{3}$$

NOTE $\quad F'(z) = \frac{1}{z}$ $\qquad \int_\gamma \frac{1}{z}\,dz$

Excluding (-ve) real axis along with zero we can define log.

$$\log re^{i\theta} = \log r + i\theta$$



Simpy connected → Put a rubber band on domain, if it can be shrink then domain is simply connected.

$\sqrt{z}$ is also defined only on simply connected domains.

$w \in \mathbb{C}$

Page
Date

(4) $W = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$e^{iz} - e^{-iz} = 2iw$$

$$e^{2iz} - 1 = 2iwe^{iz}$$

$$e^{2iz} - 2iwe^{iz} - 1 = 0.$$

$$u^2 - 2iwu - 1 = 0.$$

$$u = \frac{2iw \pm \sqrt{-4w^2+4}}{2}$$

$$z = -i\ln\left(iw \pm \sqrt{1-w^2}\right)$$

FUNFACT
$a, b, c \in \{0, 1\}$ ~~a,b~~ ax^2+bx+c

Quadratic formula doesn't hold.

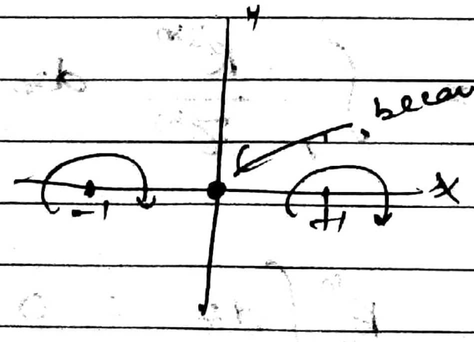
$0+0 = 0$ $\quad 1\cdot 0 = 0$

$0+1 = 1$ $\quad 1\cdot 1 = 1$

$1+0 = 1$ $\quad 0\cdot 0 = 0$

$1+1 = 0$

For $\log$ and $\sqrt{z}$, branch points are 0.



because ~~of taking log~~ we have to consider zero as a branch point

$$z = -i\log\left(iw + \sqrt{1-w^2}\right) + 2n\pi \quad n \in \mathbb{Z}.$$

$\sin(z + 2n\pi) = \sin z$. (Proved)

Page
Date

$\exp: \mathbb{C} \to \mathbb{C}$

$\text{Range}(\exp) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Little Picard theorem - An analytic fn can miss at most one point.

(5) $\gamma : [0,1] \to C$

$\gamma(t) = Re^{2\pi i t}$

$$\int_\gamma z^m \, dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m = -1 \end{cases}$$

$m \in \mathbb{Z}$

$z \in \gamma \qquad z\bar{z} = R^2$

$$\bar{z} = \frac{R^2}{z}$$

$$\int_\gamma \bar{z}^m \, dz = \int_\gamma \frac{R^{2m}}{z^m} \, dz$$

$$\int_\gamma |z|^m \, dz = R^m \int_\gamma dz = 0$$

$$\int_{\partial M} \omega = \int_M d\omega$$

$$\int_{\partial D \subseteq R^2} M dx + N dy = \int_D \frac{\partial M}{\partial x} \underbrace{dx \times dx}_{=0}$$

$$+ \int_D \frac{\partial M}{\partial y} dy \times dx$$

$$+ \int_D \frac{\partial N}{\partial x} dx \times dy$$

$$+ \int_D \frac{\partial N}{\partial y} \underbrace{dy \times dy}_{=0}$$

$$= \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

Div.

$$\int_{\partial V} A\, dx \times dy + B\, dy \times dz + C\, dz \times dz$$

$$= \int_V \underbrace{\left(\frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} + \frac{\partial C}{\partial z} \right)}_{\vec{\nabla} \cdot \vec{F}} dV$$

$\vec{F} = (A, B, C)$

D domain ∂D C^1

$f(z) = \bar{z} = x - iy.$

$$\int_{\partial D} \underbrace{f(z)}_{u+iv} dz = dx + i dy$$

$$\int_{\partial D} (u dx - v dy) + i \int_{\partial D} (u dy + v dx)$$

$$\int_{\partial D} (x dy + y dx) + i \int_{\partial D} x dy + y dx.$$

$$= \int_D (1-1) dA + i \int_D (1+1) dA = 2i \,\text{Area}(D)$$

Scanned by CamScanner