# Network Theory Homework - 2

1.

ⓐ Graph Homomorphism

Suppose $G = (V, E)$ and $H = (V', E')$ are graphs.
A homomorphism from G to H is a mapping f from V to V', s.t. for each edge $\underset{e_1}{x,y}$ in G, $\underset{e_2}{f(x), f(y)}$ is an edge of H.

~~$v(e_1) = v(e_2)$~~ $s(e_1) = s(e_2)$, $t(e_1) = t(e_2)$

ⓑ A directed Line graph $(L_n)$
is a graph with n nodes, ordered as $n_1, n_2 \dots n_n$
with $n-1$ edges ordered as $e_1, e_2 \dots e_{n-1}$
correspondingly
s.t. $v(e_1) = n_1$, $t(e_{n-1}) = n_n$

$v(e_i) = n_i$ $t(e_i) = n_{i+1}$

Undirected line graph → ~~for every edge~~ $(\bar{L_n})$
Graph with n nodes, ordered as $n_1, n_2 \dots n_n$
with $2(n-1)$ edges $e_1, e_2 \dots e_{n-1}$ and $e_1', e_2', \dots e_{n-1}'$
s.t. $v(e_i) = n_i$, $t(e_i) = n_{i+1}$
$t(e_i') = n_i$, $v(e_i') = n_{i+1}$

ⓒ Directed Cycle graph $(C_n)$
Graph with n nodes, ordered as $n_1, n_2 \dots n_n$
with n edges ordered $e_1, e_2 \dots e_n$
s.t. $v(e_i) = n_i$, $t(e_i) = n_{i+1}$ $(i < n)$
$v(e_n) = n_n$, $t(e_n) = n_1$

Undirected Cycle graph $(\bar{C_n})$
" with $2n$ edges, $e_1, e_2 \dots e_n$ & $e_1', e_2' \dots e_n'$
$v(e_i) = n_i$, $t(e_i) = n_{i+1}$ $v(e_i') = n_{i+1}$, $t(e_i') = n_i$ $(i<n)$
$v(e_n) = n_n$, $t(e_n) = n_1$ $v(e_n') = n_n$, $t(e_n') = n_i$

$m_1 \rightsquigarrow m_2$

(d) Directed Path of length $n$ from node $m_1$ to $m_2$ in a graph $(G)$ is an injective homomorphism $L_{n+1} \to G$
s.t. $f(1) = m_1$, $f(n+1) = m_2$ (1, 2 ... n+1 are nodes of $L_{n+1}$)

$m_1 \sim m_2$

(e) Undirected Path of length $n$ from node $m_1$ to $m_2$ in a graph $(G)$ is ~~two~~ an injective homomorphism~~s~~ $L_{n+1} \xrightarrow{f} G$, ~~$L'_{n+1} \xrightarrow{f'} G$~~
s.t. $f(1) = m_1$ and $f(n+1) = m_2$
~~$f'(1) = m_2$ and $f'(n+1) = m_1$ and $f(i) = f'($~~
~~and $f(i) = f(n+2-i)$ must hold~~

(f) $G$ is an undirected graph if for every $e \in E$ there exists $e' \in E$ s.t. $s(e) = t(e')$ and $t(e) = s(e')$

(h), (g) If $G$ is a graph, let $\bar{G}$ be the smallest undirected graph containing (all edges of) $G$
i.e. if $e \in E_G$, then $e \in E_{\bar{G}}$, $e' \in E_{\bar{G}}$
where $s(e') = t(e)$, $t(e') = s(e)$

(h) → $G$ is strongly connected iff $\exists\ u \rightsquigarrow v$ ~~an injective homomo~~ $\forall u, v \in N_G$
(i.e. if there exists a directed path from node $u$ to node $v$ for all $(u, v)$ pairs ~~belonging~~ where $u, v$ belong to $N_G$)

(g) $G$ is connected iff $\bar{G}$ is strongly connected

(i) ~~A tree graph $G$ is a tree if $G$ is connected and "cycle free" i.e. $C_K \to G$ DNE $\forall K$~~

(i) An undirected graph $G$ is a tree iff $\forall u, v \in N_G$ $\exists$ unique path $u \sim v$

(j) An undirected cycle of size $n$ in a graph is an injective homomorphism $\overline{C_n} \to G$

(i) A directed cycle of size $n$ in a graph is an injective homomorphism $C_n \to G$

(l) For a connected graph $G$ with $n$ nodes, a spanning Tree is a homomorphism from a Tree with $n$ nodes to $G$.

(For a connected graph such a Tree exists necessarily as every connected graph has a spanning tree)

2. $(N, E)$ is a graph

$\delta : \mathbb{R}^N \to \mathbb{R}^E$

where $\mathbb{R}^N$ is ~~a~~ the set of all functions ~~from~~ $f : N_G \to \mathbb{R}$

and $\mathbb{R}^E$ is ~~a~~ the set of all functions $g : E_G \to \mathbb{R}$

$\therefore \delta : f(n) \to g(e)$

~~$\delta$ is a map~~

Domain is $\mathbb{R}^N$

Range is $\mathbb{R}^E$

~~Property~~ $(V_n)_{n \in N} \mapsto (V_{s(e)} - V_{t(e)})_{e \in E}$

$\delta(aV) \overset{\text{where } a \in \mathbb{R}}{=} (aV_n)_{n \in N} \mapsto (aV_{s(e)} - aV_{t(e)})_{e \in E}$

$= a(V_n)_{n \in N} \mapsto a(V_{s(e)} - V_{t(e)})_{e \in E} = a\delta(V)$

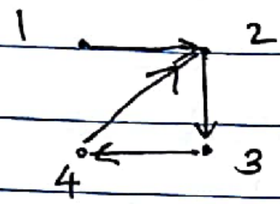
$\therefore$ ~~$\delta a(V)$~~ $\delta(aV) = a\,\delta(V)$

$\delta(aV + bW) = (aV_n + bW_n)_{n \in N} \mapsto \big( (aV_{s(e)} + bW_{s(e)}) - (aV_{t(e)} + bW_{t(e)}) \big)$

$= (aV_n + bW_n)_{n \in N} \to (a(V_{s(e)} - V_{t(e)})) + (b(W_{s(e)} - W_{t(e)}))$

$= a\,\delta(V) + b\,\delta(W)$

Hence $\delta$ is a linear map




$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \rightarrow \begin{bmatrix} v_1 - v_2 \\ v_2 - v_3 \\ v_3 - v_4 \\ v_4 - v_2 \end{bmatrix}$$

Map corresponding to $\delta$

$$\delta^T : \mathbb{R}^E \rightarrow \mathbb{R}^N$$

$$(i_e)_{e \in E} \mapsto \Big( \overset{\text{Inflow}}{+\sum_{(l,m) \in E} i_{lm}} - \overset{\text{outflow}}{\sum_{(m,n) \in E} i_{mn}} \Big)$$

Domain $\rightarrow \mathbb{R}^E$

Codomain $\rightarrow \mathbb{R}^N$

3. $W_{KVL} = im(\delta)$

$\cancel{= \{ f \in \mathbb{R}^E \mid \exists}$

$:= \{ g \in \mathbb{R}^E \mid \exists f \in \mathbb{R}^N \text{ with } g = \delta(f) \}$

$W_{KCL} = Ker(\delta^T)$

$:= \{ g \in \mathbb{R}^E \mid \delta^T(g) \equiv 0 \}$

1) Assume $V \in W_{KVL}$

Consider any loop of the given graph and label its nodes as $1, 2, 3 \ldots n$ such that there is an edge between adjacent nodes. Call these edges $e_{1-2}, e_{2-3} \ldots e_{(n-1)-n}, e_{n-1}$

Kirchoff's Voltage law states that sum of potential drop across all edges ~~on~~ along a loop equals zero.

Fix the orientation of the loop along which you want to calculate the potential drop.

$\therefore V \in W_{KVL}$

$\therefore$ ~~$\forall e_{i-(i+1)}$~~ $\therefore V(e_{ij}) = f(s(e_{ij})) - f(t(e_{ij}))$

where $f \in \mathbb{R}^N$

$\therefore V(e_{ij})$ will be the potential drop ~~on~~ across edge $e_{ij}$ if its direction ~~on~~ is same as ~~on~~ the orientation of our loop.

$-V(e_{ij})$ will be the potential drop across edge $e_{ij}$ if its direction is opposite to that of orientation of our loop.

$$\therefore \sum_{\substack{\text{all} \\ \text{edges} \\ \text{of loop}}} \text{Potential Drop} = \sum_{\text{(i)} \begin{cases} e_{ij} \in \text{loop} \\ \text{and has} \\ \text{same} \\ \text{direction} \\ \text{as loop} \end{cases}} V(e_{ij}) + \sum_{\text{(ii)} \begin{cases} e_{ij} \in \text{loop} \\ \text{and has} \\ \text{opposite} \\ \text{direction} \\ \text{as loop} \end{cases}} -V(e_{ij})$$

Lets fix the orientation of loop as going from (1 to 2 to 3 ... n to 1)

$$= \sum_{\text{(i)}} f(s(e_{ij})) - f(t(e_{ij})) + \sum_{\text{(ii)}} f(t(e_{ij})) - f(s(e_{ij}))$$

$$= \sum_{\text{(i)}} f(i) - f(j) + \sum_{\text{(ii)}} f(i) - f(j)$$

$$= \sum_{\substack{i,j \in N \text{ and } j = i+1 \\ \text{except when } i = n \text{ then } j = 1}} f(i) - f(j) = \boxed{0}$$

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Hence proved $v \in W_{KVL} \Rightarrow v$ satisfies KVL

2) Assume $v$ satisfies KVL

Again, consider any loop of the given graph. label nodes as $1, 2, \ldots n$ such that there is an edge between adjacent nodes. Call them $e_{1-2}, e_{2-3}, \ldots e_{(n-1)-n}, e_{n-1}$

Fix an orientation of loop, say 1 to 2 .. to n to 1.

KVL tells us
$$\sum_{\substack{e_{i-j} \in \text{loop}, \\ \text{has same direction} \\ \text{as loop}}} V(e_{i-j}) + \sum_{\substack{e_{i-j} \in \text{loop}, \\ \text{has opposite direction} \\ \text{as loop}}} -V(e_{i-j}) = 0$$

Now we will try to construct a function $f \in \mathbb{R}^N$ such that $\delta(f) = v$

Define $f(1) = 0$

$$f(2) = \begin{cases} -V(e_{1-2}) & \text{if } s(e_{1-2}) = 1 \\ V(e_{1-2}) & \text{if } s(e_{1-2}) = 2 \end{cases}$$

~~In general~~ $$f(3) = \begin{cases} f(2) - V(e_{2-3}) & \text{if } s(e_{2-3}) = 2 \\ f(2) + V(e_{2-3}) & \text{if } s(e_{2-3}) = 3 \end{cases}$$

In general,
$$f(K) = \begin{cases} f(K-1) - V(e_{(K-1)-K}) & \text{if } s(e_{(K-1)-K}) = K-1 \\ f(K-1) + V(e_{(K-1)-K}) & \text{if } s(e_{(K-1)-K}) = K \end{cases}$$

$\forall K = 1, 2, \ldots n$

observe that $V(e_{(K-1)-K}) = f(s(e_{(K-1)-K})) - f(t(e_{(K-1)-K}))$

$\forall K = 1, 2, \ldots n$

$\therefore$ If $V(e_{n-1}) = f(s(e_{n-1})) - f(t(e_{n-1}))$ then $S(f) = V$ is true.

$$\text{LHS of KVL eqn} = f(n) - f(1) + \begin{cases} -V(e_{n-1}) & \text{if } s(e_{n-1}) = n \\ V(e_{n-1}) & \text{if } s(e_{n-1}) = 1 \end{cases}$$

$= 0$ (given)

$\therefore V(e_{n-1}) = f(s(e_{n-1})) - f(t(e_{n-1}))$ | Hence proved $v$ satisfies KVL $\Rightarrow v \in W_{KVL}$

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4. Given: $v \in W_{KVL}$

$\therefore \exists f \in \mathbb{R}^N$ s.t. $\delta(f) = v$

Given: $i \in W_{KCL}$

$\therefore \delta^T(i) = 0$

$$\sum_{e \in E} i_e v_e = \langle v, i \rangle$$

$$= \langle \delta(f), i \rangle$$

Using:

$$\langle AB, C \rangle = (AB)^T C = (B^T A^T) C$$

$$= B^T (A^T C)$$

$$\cancel{= \langle B^T} = \langle B, A^T C \rangle$$

Where $A, B, C$ are matrices.

$$\langle \delta(f), i \rangle = \langle f, \delta^T(i) \rangle = \langle f, 0 \rangle = 0$$

(Here think of $\delta_{m \times n}$, $f_{n \times 1}$ and $i_{m \times 1}$ as matrices)

$m$ = number of edges

$n$ = number of nodes

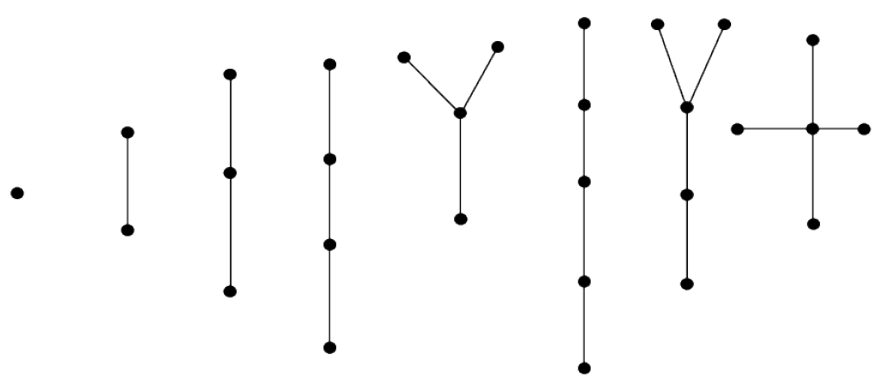
# 4. Trees

One of the important classes of graphs is the trees. The importance of trees is evident from their applications in various areas, especially theoretical computer science and molecular evolution.

## 4.1 Basics

**Definition:** A graph having no cycles is said to be *acyclic*. A *forest* is an acyclic graph.

**Definition:** A *tree* is a connected graph without any cycles, or a tree is a connected acyclic graph. The edges of a tree are called *branches*. It follows immediately from the definition that a tree has to be a simple graph (because self-loops and parallel edges both form cycles). Figure 4.1(a) displays all trees with fewer than six vertices.



(a)

**Fig. 4.1(a)**

The following result characterises trees.

**Theorem 4.1** A graph is a tree if and only if there is exactly one path between every pair of its vertices.

**Proof** Let $G$ be a graph and let there be exactly one path between every pair of vertices in $G$. So $G$ is connected. Now $G$ has no cycles, because if $G$ contains a cycle, say between vertices $u$ and $v$, then there are two distinct paths between $u$ and $v$, which is a contradiction. Thus $G$ is connected and is without cycles, therefore it is a tree.

Conversely, let $G$ be a tree. Since $G$ is connected, there is at least one path between every pair of vertices in $G$. Let there be two distinct paths between two vertices $u$ and $v$ of $G$. The union of these two paths contains a cycle which contradicts the fact that $G$ is a tree. Hence there is exactly one path between every pair of vertices of a tree. ❑

The next two results give alternative methods for defining trees.

**Theorem 4.2** A tree with n vertices has $n-1$ edges.

**Proof** We prove the result by using induction on $n$, the number of vertices. The result is obviously true for $n = 1$, 2 and 3. Let the result be true for all trees with fewer than $n$ vertices. Let $T$ be a tree with $n$ vertices and let $e$ be an edge with end vertices $u$ and $v$. So the only path between $u$ and $v$ is $e$. Therefore deletion of $e$ from $T$ disconnects $T$. Now, $T-e$ consists of exactly two components $T_1$ and $T_2$ say, and as there were no cycles to begin with, each component is a tree. Let $n_1$ and $n_2$ be the number of vertices in $T_1$ and $T_2$ respectively, so that $n_1+n_2 = n$. Also, $n_1 < n$ and $n_2 < n$. Thus, by induction hypothesis, number of edges in $T_1$ and $T_2$ are respectively $n_1-1$ and $n_2-1$. Hence the number of edges in $T = n_1-1+n_2-1+1 = n_1+n_2-1 = n-1$. ❑

**Theorem 4.3** Any connected graph with $n$ vertices and $n-1$ edges is a tree.

**Proof** Let $G$ be a connected graph with $n$ vertices and $n-1$ edges. We show that $G$ contains no cycles. Assume to the contrary that $G$ contains cycles.

Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph $H$ is a tree. As $H$ has $n$ vertices, so number of edges in $H$ is $n-1$. Now, the number of edges in $G$ is greater than the number of edges in $H$. So $n-1 > n-1$, which is not possible. Hence, $G$ has no cycles and therefore is a tree. ❑

**Definition:** A graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

Here is the next characterisation of trees.

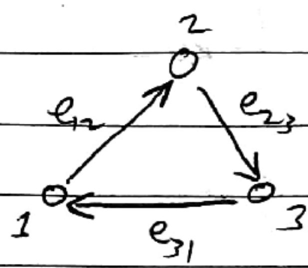
**Theorem 4.4** A graph is a tree if and only if it is minimally connected.

**Proof** Let the graph $G$ be minimally connected. Then $G$ has no cycles and therefore is a tree.

Conversely, let $G$ be a tree. Then $G$ contains no cycles and deletion of any edge from $G$ disconnects the graph. Hence $G$ is minimally connected. ❑

6. False.

Counter example :




Let $f(1) = 1$

$f(2) = 2$

$f(3) = 3$

Then $v(e_{12}) = f(1) - f(2) = -1$ $\quad$ ($\because v \in W_{KVL}$)

$v(e_{23}) = f(2) - f(3) = -1$

$v(e_{31}) = f(3) - f(1) = 2$

Let $i(e_{12}) = 0$

$i(e_{23}) = 2$

$i(e_{31}) = 1$

Observe that $\sum_{e \in E} i_e V_e = 0 + 2 \times (-1) + 1 \times (2)$

$= 0$

But obviously $i \notin W_{KCL}$

7. To Prove that: $W_{KCL} \cap W_{KVL} = \{0\}$

Assume there exists $P \in \mathbb{R}^E$ such that $P \in W_{KCL}$ and $P \in W_{KVL}$

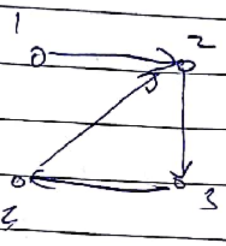
We have Proved that $\langle v_1, v_2 \rangle = 0$ if $v_1 \in W_{KCL}$ and $v_2 \in W_{KVL}$

$\therefore \langle P, P \rangle = 0$

This is possible iff $P \equiv 0$

This is the only element which belongs to both $W_{KCL}$ and $W_{KVL}$

$\Rightarrow W_{KCL} \cap W_{KVL} = \{0\}$

8. 


Matrix of $\delta = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$W_{KVL} = im(\delta)$

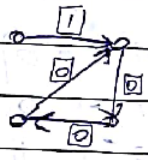
$=$ column space (matrix of $\delta$)

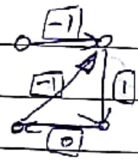
$\dim(W_{KVL}) = \dim(\text{col}(\text{matrix of } \delta))$

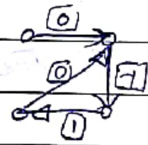
$= \text{rank}(\text{matrix of } \delta)$

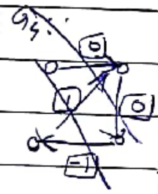
$= \boxed{3}$

Basis for $W_{KVL}$ : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ ~~$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$~~

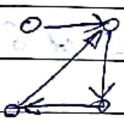
$G_1$ : 

$G_2$ : 

$G_3$ : 




~~which has any $V(e)$ satisfying KVL can be~~

$G$ :  which has any $V(e)$ satisfying KVL can be constructed by linear superposition of $G_1, G_2, G_3$ ~~and $G_4$~~

$W_{KCL} = \text{Ker}(\delta^T)$

$=$ Null space (transpose of matrix of $\delta$)

$\dim(W_{KCL}) = \dim(\text{Ker}(\delta^T))$

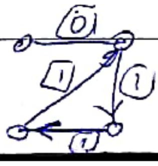
~~$= \dim(\mathbb{R}^E) - \dim(\text{rank}(\delta))$~~

~~(By rank-nullity Thm) $= \dim(\mathbb{R}^N) - \dim(\text{rank}(\delta))$~~

$= \dim(\mathbb{R}^E) - \text{rank}(\delta^T)$ (By Rank-Nullity theorem and using $\delta : \mathbb{R}^N \to \mathbb{R}^E$)

$= |E| - \text{rank}(\delta^T)$

$= 4 - \text{rank}(\delta) = 4 - 3 = \boxed{1}$

$G_1$ : 

Basis : $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Any $i(e)$ is scalar multiple of that in $G_1$.

Q 9. and 10 ~~If we look at a connected graph with~~ n n

To show connected graph $G$ with $n$ nodes and $|E|$ edges has $\dim W_{KVL} = n-1$

Let us ~~look at $\mathbb{R}^N$~~ construct a basis for $W_{KVL}$, by :-

Labeling nodes $n_1$ to $n_n$

1) ~~Let us assign~~ Let us fix Potential at node 1 = 0 ~~(without loss~~ (without loss of generality)

2) Define ~~vectors~~ functions $f_2, \ldots f_n$

~~$f_2 = (0, 1, 0, 0, 0, 0 \ldots)$~~

$$f_i(n_j) \begin{cases} = 1 & \text{if } i=j \\ = 0 & \text{if } i \neq j \end{cases}$$

given that $V_1 = 0$

(This is a basis for Potential) (Prove it yourself)

3) ~~This~~ The Potential differences corresponding to this form a basis of $W_{KVL}$

Proof: Claim 1: Spanning

Consider $\vartheta \in W_{KVL}$

Corresponding to this we have ~~$V \in \mathbb{R}^N$~~ $f \in \mathbb{R}^N$ ~~V can be~~ s.t. $\delta(f) = \vartheta$

$f = \sum_{i=2}^{n} c_i f_i$ (because $f_i$ is a basis)

$\delta f = \delta(\Sigma c_i f_i) = \Sigma c_i \, \delta(f_i)$

$\therefore \delta f_i$ is spanning on $W_{KVL}$

Claim 2: It is linearly independent

$\sum_{i=2} c_i \delta f_i = 0$ means $c_i = 0 \; \forall i \in [2, n]$

$\Rightarrow \delta(\sum_{i=2}^{n} c_i f_i) = 0 \Rightarrow \sum_{i=2}^{n} c_i f_i = k$, $(\delta(0) = 0)$

$k \in \mathbb{R}$ (constant)

~~$\sum_{i=2}^{n} c_i f_i(n_j) = c_j = 0$~~

$\sum_{i=2}^{n} c_i f_i(n_1) = 0$ identically $\therefore k = 0$

$\Sigma c_i f_i(n_j) = c_j = 0 \quad \forall j \in [2, n]$ $\therefore$ It is linearly independent

This ~~basis~~ basis has ~~dimension~~ $n-1$ elements, $W_{KVL}$ has dimension $(n-1)$

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For a connected graph with n nodes

$\dim W_{KVL} = n-1$

If we look at non-connected ~~with~~ graph $G$ with K connected components.

~~SG~~ ~~Let us denote~~

Let us denote each connected component by a Subgraph $SG_i$, $1 \le i \le K$

Let each subgraph contain $a_i$ nodes

~~Dim of basis~~ Number of elements of basis $B_i$ of $W_{KVL}$ of each subgraph $= a_i - 1$

Basis of graph $G = \{\bigcup_{i=1}^{K} B_i\}$

$\therefore$ Number of elements of basis of $G$

$= \sum_{i=1}^{K}$ no. of elements of $B_i$

$= \sum_{i=1}^{K}(a_i - 1) = n - K \quad \left(\sum_{i=1}^{K} a_i = n\right)$

$\therefore \dim(W_{KVL}) = n - K$ (on Graph $G$)

We know that for a graph $G$ with K connected components, number of equations from node variable analysis $= n - K$.

Hence Q10/9 proved.

~~Similarly we can show for WKCL using rank nullity theorem~~

Once we know $\dim W_{KVL}$, using rank nullity theorem we can show $\dim W_{KCL} = |E| - n + K$