Tutorial - 5

Liouvilles Theorem.

A bounded entire function is a constant. → holomorphic over all

$|f(z)| < c \qquad z \in \mathbb{C}$

Indentity theorem

$S_n \xrightarrow[n\to\infty]{\lim} z_0$

Converging sequence — Sequence must be converse.

$S_n \quad \frac{1}{n} \to 0$

$f(S_n) = g(S_n)$, for all S_n.

$f(z) = g(z)$ if f and g are holomorphic.

~~$h(\frac{1}{n}) = 0 \quad g(\frac{1}{n}) = 0$~~

$f(z) = g(z) = h(z) \to$ holomorphic

$z - z_0$

If a holomorphic function has arbitrarily close zeros the function $\Rightarrow$

$f(z_0) \Rightarrow f(z_1) = 0$

$|z_1 - z_0| < \varepsilon.$

1) $f \to$ entire $\qquad g(z) = \frac{1}{f(z)}$

$|f(z)| > c > 0$

$g(z)$ is bounded $\quad$ as $|g(z)| \leq \frac{1}{c}$

$g(z)$ is entire as $f(z) \neq 0$ $\quad$ by liouvil theorem it is constant.

(ii) $Re(f(z)) \geq 0$

Let $g(z) = (f(z)+1)$ Apply real Part $Re f(z) > 0$

$|g(z)| > c > 0$

as $Re(g(z)) > 1$

So using the result of (i)

$g(z) = $ constant

or $f(z) = g(z) - 1 = $ constant.

(iii) $\overline{f(C)} \neq C$

here $f(C) \longrightarrow$ range / image of

$\overline{f(C)}$ – closer of f defined as $f(C) \cup$ Boundary of $f(C)$

~~there is~~

There exist a disk of radius δ around a z

which has only Point

$\overline{f(C)} \neq C$

there exist a z_0 which is an enterior Point w.r.t $f(C)$

and z_0

$|f(z) - z_0| > \varepsilon$ for Some ε

$g(z) = \dfrac{1}{f(z) - z_0}$

$|g(z)| < C$

using liouvilles theorem $g(z)$ and hence $f(z)$ is a

Constant.

Q2) $f(\frac{1}{n}) = \frac{1}{n} \to n$ Even

$= \frac{-1}{n} \to$ if n odd

1) $f(\frac{1}{2n}) = \frac{1}{2n}$

2) $f(\frac{1}{2n+1}) = \frac{-1}{2n+1}$ $\forall n \in N$

from (1) $f(z) = z$ by identity theorem

as taking $S_n = \frac{1}{2n}$ and $z, \Rightarrow$ $g(z) = z$

from (2) $f(z) = -z$ by indentity theorem.

$f(z) = z$

$f(z) = -z$

So such a f holomorphic doesn't exist is there is a contradiction

f(z) is come out two different f^n

take g : $g(z^2) = z$

3) ~~$f(\frac{1}{n^2}) = \frac{1}{n}$~~

~~$g(z) = \frac{1}{\sqrt{z}} \sqrt{z}$~~ $S_n = \frac{1}{n^2}$ f = entire

$f(\frac{1}{n^2}) = \frac{1}{n}$

~~$g\ f(S_n) = g(S_n)$~~

taking the sequence $\frac{1}{n^2}$ (converging to 0)

~~$f(z) = g(z)$~~

$g(z) = \frac{1}{\sqrt{z}} \sqrt{z}$ we get $f(z^2) = z$ for all z

using identity theorem

we can say $\sqrt{z}$ because it is not holomorphic so we not used Identity theorem.

taking derivative w.r.t.

$2z f'(z') = 1$

$z = 0$ $\quad 0 = 1 \quad$ a con

4.)

$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots \frac{z^n}{n!} f^n(0)$

$+ \frac{z^{n+1}}{(n+1)!} \int_0^1 (1-t)^n f^{(n+1)}(tz)\, dt$

$f^{n+1}(z) =$

$\left| \frac{z^{n+1}}{(n+1)!} \int_0^1 (1-t)^n \cancel{f^{(n+1)}} e^{(z)} \right| = e^z$ here

$\int_0^1 (1-t)^n e^z\, dt \Big| < \int e^z\, dt$

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$$\int_{|z|=1} \left(\frac{z^2+1}{z}\right)^n dz$$

$$\int \frac{(z^2+1)^{2n}}{z^{2n+1}} dz$$

$$f^n(z_0) = \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$\cancel{g(z) = (z^2+1)^{2n}}$

$f(z) = (z^2+1)^{2n}$

$$\frac{f^{2n}(0)}{} = \frac{2n!}{2\pi i} \int_{|z|} z$$

$f(z) = (z^2+1)^{2n}$

$f^{2n}(0) = C$ only the constant term of $f^{2n}(z) \neq 0$

∴ for getting the constant ter

the z^{2n} term of $f(z)$ diff $2n$ terms

givens of the constan

coefficient of z^{2n} in $f(z) = {}^{2n}C_n$ using binomial exp.

after differentiating $2n$ time is ${}^{2n}C_n \cdot 2n!$

$$\int_0^{2\pi} (\cos\theta)^{2n} d\theta$$

$\bar{z} = \frac{1}{z}$ on $z = 1$

So $z + \frac{1}{z} = 2Re(z) = 2\cos\theta$ on $|z| = 1$

taking $z = e^{i\theta}$

$$\int_{|z|=1} (2\cos\theta)^{2n} \frac{dz}{z} \qquad dz = iz\, d\theta$$

$$\int_0^{2\pi} 4^n (\cos\theta)^{2n} d\theta =$$

$$\int \qquad \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \frac{2n!}{n!\, n!} \times \frac{2n!}{2n!}\ 2\pi i$$

$= 2\pi i\ {}^{2n}C_n.$

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Tutorial- 6

Singularility :- A Point on which the function is not holomorphic.

It have two type.

1. Isolated Singularity : disc around Point z_0 where function is holomorphic

1.1 Removable.

z_0 is Removable

$\lim_{z \to z_0} f(z)$ exist.

1.2 Pole.

$\lim_{z \to z_0} f(z) \longrightarrow \infty$

1.3 Essential - Not

2. Non Isolated Singularility :-

1) $f(z) = \dfrac{1}{\sin(\frac{1}{z})}$

$\sin(\frac{1}{z})$ is not singular at $z=0$

$f(z)$ is not singular when $\sin(\frac{1}{z}) = 0$

$\sin(\frac{1}{z}) = 0$ $\qquad$ $z = \frac{1}{n\pi}$

$\frac{1}{z} = n\pi$

To check at $z = \frac{1}{n\pi}$

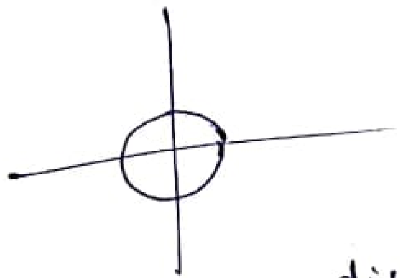
$\lim_{z \to \frac{1}{n\pi}} \frac{\left(z - \frac{1}{n\pi}\right)}{\sin\left(\frac{1}{z}\right)}$ — if it exist then $z = \frac{1}{n\pi}$ is a Pole of order 1

$\lim_{z \to \frac{1}{n\pi}} \frac{z - \frac{1}{n\pi}}{\sin\left(z - \frac{1}{n\pi}\right)}$ for n even $= 1$

and $\lim_{z \to \frac{1}{n\pi}} = -\frac{\left(z - \frac{1}{n\pi}\right)}{\sin\left(\frac{1}{z} - n\pi\right)}$ for n odd

So at $z = \frac{1}{n\pi}$ we have a pole of order

for $z = 0$



$\frac{1}{f(z)} = z \sin\left(\frac{1}{z}\right) = 0$ taking a disc of radius ε →

$\sin \frac{1}{z} = 0$ has many soln $= \frac{1}{n\pi}$ with $n = \frac{1}{\varepsilon}$

Q2.) (i) $\frac{1}{(z^4+1)^2}$

$\frac{1}{f(z)} = 0$ $\quad (z^4+1)^2 = 0$

$z^4 = -1$

$z^4 = e^{i\pi + 2n\pi}$

$z = e^{\frac{i\pi}{4} + \frac{n\pi}{2}}$

Pole is $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$

$\lim_{z \to z_0} (z-z_0)^m f(z)$ is exist.

the function is called order of m and

order of Pole – is given by order of zero in $\frac{1}{f(z)}$

All the Pole are of order 2

~~or~~ alternative way to Prove the the function is m order.

order of a zero = The number of repeated root.

if order of a zero = m

$f(z_0) = 0$

$f'(z_0) = 0$

$\vdots$

$f^{m-1}(z_0) = 0$

Q5) $f(z) = \frac{p(z)}{q(z)}$ p(z) and q(z) are differentiable

$p(z_0) \neq 0$

$q(z_0) = 0$

Pole of order 1 at z_0 $q'(z_0) \neq 0$

laurent

Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=-1}^{-\infty} b_m (z-z_0)^m$$

If $f(z)$ has singularity at z_0

If z_0 is removable all $b_m = 0$

pole of order m_0 then all $b_m = 0$ for $m < m_0$

* we can written Laurent series for non-isolated singularity

$$\int f(z)\,dz = \int \frac{b_{-1}}{z}$$

PJ)

$$f(z) = \left(\sum a_n (z-z_0)^n + \frac{b_{-1}}{z-z_0} \right)$$

$$\lim_{z \to z_0} (z-z_0) f(z) = \left(\sum a_n (z-z_0)^n + b_{-1} \right)$$

$$\lim_{z \to z_0} = \frac{p(z)}{q(z)} \times (z-z_0)$$

$q(z_0) = 0$

$$= \frac{p(z)(z-z_0)}{q(z)-q(z_0)}$$

$$\lim_{z \to z_0} = \frac{p(z)}{q'(z_0)}$$

6)

(ii) $\frac{1}{z^2 \sin z}$

a) It have pole has order 3.

$\lim_{z \to 0} \frac{z^3}{z^2 \sin z} = 1$

So It has pole of order 3 at 0.

$f(z) = \frac{1}{(z^2 \sin z)} = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_{-1}}{z} + \frac{b(-2)}{z^2} + \frac{b(-3)}{z^3}$

for getting a residue of at a pole of order m

take $g(z) = (z - z_0)^m f(z)$

the residue is $b_{-1} = \frac{g^{m-1}(z_0)}{(m-1)!}$

$b_{-1} = \frac{g^{m-1}(z_0)}{(m-1)}$:

$g(z) = \frac{z}{\sin(z)}$

b_{-1} is residue of any Laurent series.

$1 + \frac{1}{8} r + r^2 + \ldots = \frac{1}{1-r}$

$r < 1$

$1 + 2 + 2^2 \ldots = \frac{1}{1-2}$