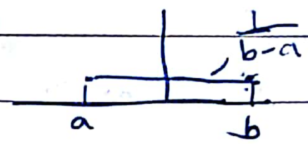
Recap:

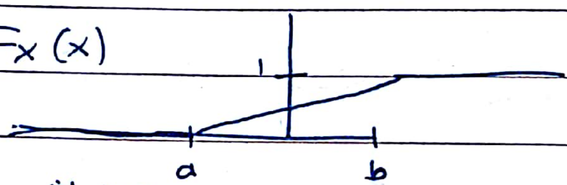
Continuous distribution

Uniform $[a,b]$

$$f_X(x) = \frac{1}{b-a} \quad \text{if } x \in [a,b]$$
$$= 0 \quad \text{otherwise}$$



$F_X(x)$

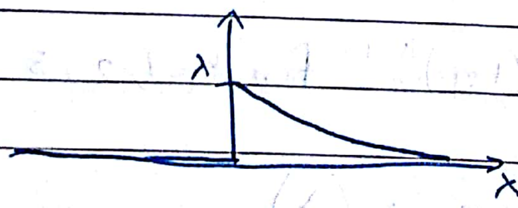


$$F_X(x) = 0 \quad \text{if } x \le a$$
$$= \frac{x-a}{b-a} \quad \text{if } x \in (a,b]$$
$$= 1 \quad \text{if } x > b$$

Exponential (λ); $\lambda > 0$

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{if } x = 0$$
$$= 0 \quad \text{otherwise}$$

f_X

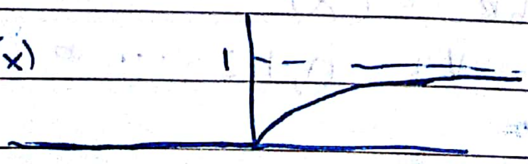


$\lambda e^{-\lambda x} dx$

$1 - e^{-\lambda x}$

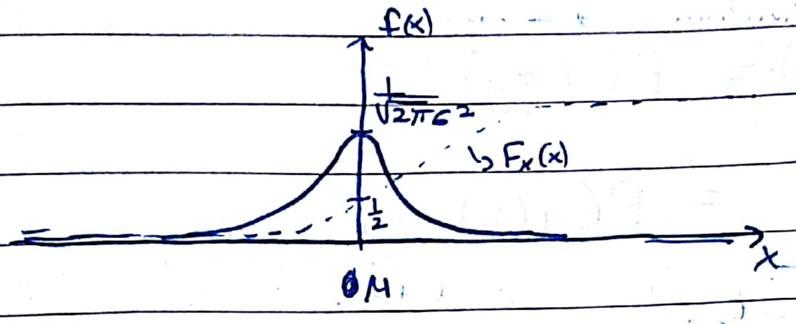
$1 - e^{-\lambda x}$ if $x > 0$ $= F_X(x)$

0 otherwise $=$



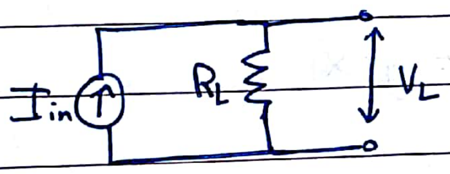
Gaussian Distribution (μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \quad \forall x \in \mathbb{R}_e$$



$F_X(0) = \frac{1}{2}$

Function of Random Variables:



ϕ_0

$$Y = aX + b$$

$X \sim F_X$ (x has distr. F_X)

Find F_Y

$$F_Y(y) = P(Y \leq y)$$

$$= P(aX + b \leq y)$$

$$= P(aX \leq y - b)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & \text{if } a > 0 \to F_X\left(\frac{y-b}{a}\right) \\ P\left(X \geq \frac{y-b}{a}\right) & \text{if } a < 0 \to 1 - P\left(X < \frac{y-b}{a}\right) \end{cases}$$

$$\| $$

$$1 - \lim_{u \uparrow y} F_X\left(\frac{u-b}{a}\right)$$

~~Find $F_y(\cdot)$~~

~~$F_y : \mathbb{R} \to [0,1]$~~

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$Y = g(X)$

g is a borel measurable fn from $\mathbb{R} \to \mathbb{R}$

$F_Y(y) = P(Y \leq y)$

$= P(g(X) \leq y$

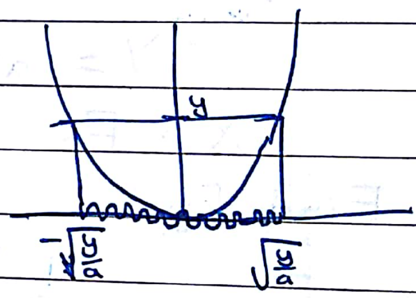
find ~~root~~ ~~of~~ x s.t. $g(x) - y \leq 0$ $\quad = \{x : g(x) \leq y\}$

$= \tilde{g}^{-1}(y)$

$Y = aX^2, \quad Y = g(X)$

$g(x) = ax^2$

$a > 0$



$\tilde{g}^{-1}(y) = \{x : g(x) \leq y\}$

$x \in [-\sqrt{\frac{y}{a}}, \sqrt{\frac{y}{a}}]$

$F_Y(y) = P\left(X \in \left[-\sqrt{\frac{y}{a}}, \sqrt{\frac{y}{a}}\right]\right) = \left(F_X(\sqrt{y/a}) - F_X(-\sqrt{y/a})\right) + P\left(X = -\sqrt{\frac{y}{a}}\right)$

Now to find density

$P(X=x) = P(X \leq x) - P(X < x)$ (if $A \subseteq B$, then $P(B \setminus A) = P(B) - P(A)$

$F_Y(y) = F_X(\sqrt{y/a}) - F_X(-\sqrt{y/a})$

$f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y/a}) - F_X(-\sqrt{y/a})\right]$

$$\frac{d}{dy} F_X\left(\sqrt{\tfrac{y}{a}}\right) = \frac{d}{dy}\int_{-\infty}^{\sqrt{y/a}} F_X(u)\,du = f_X\left(\sqrt{\tfrac{y}{a}}\right)\frac{d}{dy}\sqrt{\tfrac{y}{a}}$$ leibniz

$$= f_X\left(\sqrt{\tfrac{y}{a}}\right)\frac{1}{\sqrt{a}}\,\frac{1}{\sqrt{y}}\times\frac{1}{2}$$

$$\frac{d}{dy} F_X\left(-\sqrt{\tfrac{y}{a}}\right) = f_X\left(-\sqrt{\tfrac{y}{a}}\right)\times\frac{(-1)}{2\sqrt{ay}}$$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{2\sqrt{ya}} + \frac{f_X\left(-\sqrt{\frac{y}{a}}\right)}{2\sqrt{ya}} = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)+f_X\left(-\sqrt{\frac{y}{a}}\right)}{2\sqrt{ya}}$$

Theorem: Let X be a continuous r.v. & let $g()$ be a differentiable f^n

Then

$$f_Y(y) = \sum_{K=1}^{\infty}\frac{f_X(x_K)}{|g'(x_K)|}$$

Where x_1, x_2 are at most countably many roots of $g(x) = y_1$ i.e. $g(x) = y$ $\forall i$

$Y = aX^2$ $\quad g(x) = ax^2$ $\quad g'(x) = 2ax$

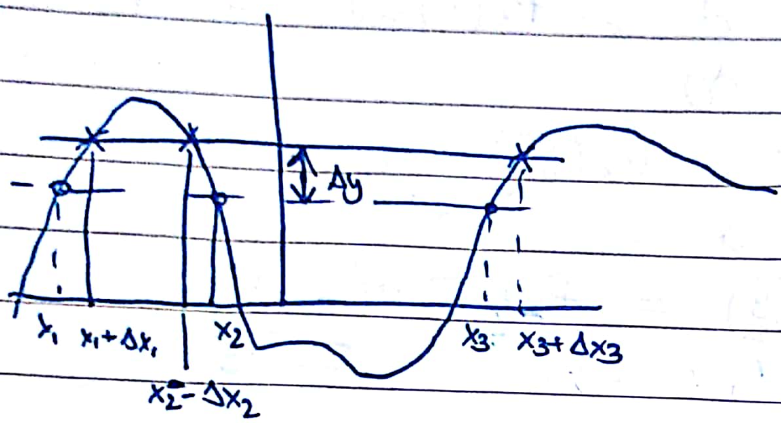
$f_Y(y)$ to find $\qquad$ Roots of $y = ax^2$ for $y>0$

$$x = -\sqrt{\tfrac{y}{a}},\ \sqrt{\tfrac{y}{a}}$$

$$g'\left(-\sqrt{\tfrac{y}{a}}\right) = \left|-2a\sqrt{\tfrac{y}{a}}\right| = 2\sqrt{ya} \qquad g'\left(\sqrt{\tfrac{y}{a}}\right) = 2\sqrt{ya}$$

$f_Y(y) =$ (same as above)

Now we can integrate to get F



$$f_Y(y) = \lim_{\Delta y \downarrow 0} \frac{F_Y(y+\Delta y) - F_Y(y)}{\Delta y}$$

$$= \lim_{\Delta y \downarrow 0} P(y < Y \leq y + \Delta y) / \Delta y$$

$$= \lim_{\Delta y \downarrow 0} \frac{P(x_1 < X < x_1 + \Delta x) + P(x_2 - \Delta x_2 < X < x_2) + P(x_3 < X < x_3 + \Delta x)}{\Delta y}$$

because g is cont as $\Delta y \downarrow 0$ Δx also $\downarrow 0$

Expectation

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$E[x] = \sum_{K=1}^{\infty} a_K P\{x = a_K\}$$

Discreet r.v.

Bernoulli r.v.

$x = 1$ w prob p

0 o.w

$$E[x] = \int 1(1\times p) + 0(1-p) = p$$

Binomial

$x = K$ w prob $\binom{n}{K}(p^K)(1-p)^{n-K}$

$$E[x] = \sum_{K=0}^{\infty} K \cdot \binom{n}{K} p^K (1-p)^{n-K}$$

$$= n \sum_{K=1}^{n} \binom{n-1}{K-1} p^K (1-p)^{n-K}$$

$$= n \sum_{v} \binom{n-1}{v} p^{v+1} (1-p)^{n-v-1}$$

$$= np \sum \binom{n-1}{v} p^v (1-p)^{(n-1)-v}$$

$$= np\, [p + (1-p)]^{n-1}$$

$$= np$$

Poisson r.v. $(\lambda > 0)$

$X = K$ wp $\frac{e^{-\lambda}\lambda^K}{K!}$ $\forall K = 0, 1 \ldots$

$$E[X] = \sum_{K=0}^{\infty} K e^{-\lambda} \frac{\lambda^K}{K!}$$

$$= \sum_{K=1}^{\infty} \frac{e^{-\lambda}\lambda^K}{(K-1)!}$$

$K-1 = U$

$$= \lambda \sum_{U=0}^{\infty} \frac{e^{-\lambda}\lambda^U}{U!}$$

$$= \lambda (e^{-\lambda} \times e^{\lambda}) = \lambda$$

- Uniform $[a, b]$

$$f_X(x) = \frac{1}{b-a} \quad \forall x \in [a, b]$$

$$= 0 \quad \text{otherwise}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= \int_a^b x f_X(x) dx = \frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right) = \frac{a+b}{2}$$

$\frac{d}{dx}\left(-x\left(\frac{e^{-\lambda x}}{\lambda}\right)\right) = \lambda x e^{-\lambda x} - e^{-\lambda x} + \frac{e^{\lambda x}}{\lambda}$

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Exponential r.v. $(\lambda > 0)$

$$f_X(x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$$
$$= 0 \quad \text{otherwise}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

Properties of Expectation

1) If $X \geq 0$, then $E[X] \geq 0$

↑
Pointwise

$$X(\omega) \geq 0 \quad \forall \omega \in \Omega$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad F_X(x) = 0 \; \forall x < 0$$
$$\therefore f_X(x) = 0 \; \forall x < 0$$

$$= \int_0^{\infty} x f_X(x)\,dx \geq 0$$

Note $F_X(x)$ is inc so $f_X(x)$ is +ve

2) If $f_X()$ is symmetric around $a \in \mathbb{R}$, i.e. $f_X(a-x) = f_X(a+x)$, then $E[X] = a$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$
$$= \int_{-\infty}^{a} x f_X(x)\,dx + \int_a^{\infty} x f_X(x)\,dx$$

$$a - x = y$$

$$= \cancel{\int_{-\infty}^{\infty} -u f_X(-u)\,du}$$
$$\int_{\infty}^{0} (a-y) f_X(a-y)(-dy) + \int_0^{-\infty} (a-y) f_X(a-y)(-dy)$$

$$= \int_0^{\infty} (a-y) f_X(a-y)\,dy + \int_{-\infty}^{0} (a-y) f_X(a-y)\,dy$$

$$\cancel{= \left(a \int_{-\infty}^{\infty} f_X(a-y)\,dy \; (= a) \right.} +$$

$$a \int_{-\infty}^{\infty} f_X(a-y)\,dy = a$$

$$- \left[\int_{-\infty}^{0} y\, f_X(a-y)\,dy + \int_{0}^{\infty} y\, f_X(a-y)\,dy \right]$$

$y = -u$

$$\left[\int_{\infty}^{0} (-u) f_X(a+u)(-du) \right.$$

$$= \left[-\int_{0}^{\infty} u f_X(a-u)\,du + \int_{0}^{\infty} y\, f_X(a-y)\,dy \right]$$

$$= \int_{0}^{\infty} y\, f_X(a-y)\,dy\,(-1+1)$$

$$= 0$$

$$\therefore \quad E[X] = a$$

If $X \geq 0$ then

$$E[X] = \int_{0}^{\infty} (1 - F_X(u))\,du$$

$$\cancel{E[X] = \int_{-\infty}^{\infty} x f_X(}$$

Discreet version

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k)$$

assuming $X \in \mathbb{N}$

$$E[X] = \sum_{k=1}^{\infty} k\, P(X = k)$$

$K=1$ $\quad P(X=1)$

$K=2$ $\quad P(X=2) + P(X=2)$

$K=3$ $\quad P(X=3) + P(X=3) + P(X=3)$

$$E[X] = P(X\geq 1) + P(X\geq 2) + P(X\geq 3) \;\ldots$$

In a differ notation

$$\sum_{k=1}^{\infty} k\,P(X=k) = \sum_{k=1}^{\infty}\left(\sum_{u=1}^{k} 1\right) P(X=k)$$

$$= \sum_{u=1}^{\infty} \underbrace{\sum_{k=u}^{\infty} P(X=k)}_{P(X\geq u)}$$

$$E[X] = \int_0^{\infty} x f_X(x)\,dx = \int_0^{\infty}\int_0^{x} du\, f_X(x)\,dx$$

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

$$F_X(x) = 1 - e^{-\lambda x} =$$

$$1 - F_X(x) = e^{-\lambda x}$$

$$E[X] = \int_0^{\infty} e^{-x\lambda}\,dx = -\frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}$$

Law of unconscious Statistician

nontrivial result

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\,dx$$

$$Var(x) = E[(x - Ex)^2]$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$$

To prove → $Y = x^2$ find f_y

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

<u>Moments of r.v.</u>

k^{th} moment $m_k = E[x^k]$ (not necessary n^{th} moment exists

<u>Central moment</u>

$$C_k = E[(x - Ex)^k]$$

C_2 is called variance of r.v.

$$\sigma^2 = E[(x - Ex)^2]$$

<u>Moment generating function</u>

$$\psi_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f_x(x)\,dx$$

$\psi_x(t)$ exists if $\exists\, t_0$ s.t. $\forall\, t \in (0, t_0)$
$\psi_x(t)$ is defined & finite

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$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \cdots$$

(Assume expectation is linear)

$$E[e^{tx}] = 1 + t\,E[x] + \frac{t^2 E[x^2]}{2!} + \frac{t^3 E[x^3]}{3!} \cdots$$

$|E[e^{tx}]| < \infty$ only if $|E[x^k]| < \infty$ for every k

Finding moments using m.g.f. (at $t=0$ mgf $=1$) $\int_{-\infty} f_X(x)dx$

$$\frac{d}{dt} E[e^{tx}] = E[x] + tE[x^2] + \frac{t^2 E[x^3]}{2!} \cdots$$

$$\lim_{t \to 0} \left(\frac{d}{dt} E[e^{tx}] \right) = E[x^2]$$

$$\lim_{t \downarrow 0} \frac{d}{dt} E(e^{tx}) = E[x]$$

- If $E[e^{tx}] = E[e^{ty}] \ \forall\ t > 0$
  then $F_X(x) = F_Y(x)$ except for ~~finitely~~ countably many points (like at point of discontinuity)

Characteristics fn

$$\Phi_X(t) = E[e^{jtx}] = \int_{-\infty}^{\infty} e^{jtx} f_X(x)\,dx$$

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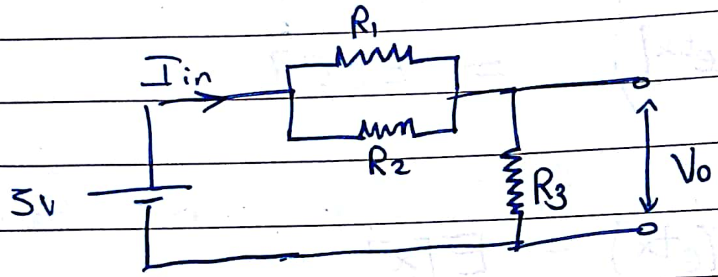
Two random variables:

$X \& Y$

$X + Y$

$\forall \omega \in \Omega, \quad X(\omega) + Y(\omega)$

Both must be defined on some probability space.



$\Omega = \{ (r_1, r_2, r_3) : r_1, r_2, r_3 \in \mathbb{R}^+ \}$

$$I_{in}(r_1, r_2, r_3) = \frac{5}{(r_1 // r_2) + r_3}$$

$$V_0(r_1, r_2, r_3) = r_3 \cdot I_{in}(r_1, r_2, r_3)$$

For a fixed I_{in}, we may still see many values of V_0

∴ ~~V_0 & I_{in} are not functions of each other~~

Like knowing I_{in} I only have partial info about V_0

Even when we have all these similarities, we can't find V_0 if given I_{in}

Two Random Variables

$X : \Omega \to \mathbb{R}_e$

$Y : \Omega \to \mathbb{R}e$

Joint distribution fn (, means AND)

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$P(\underbrace{\{\omega : X(\omega) \le x\}}_{\in F} \cap \underbrace{\{\omega : Y(\omega) \le y\}}_{\in F})$$

$$F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0 \quad \forall x, y \in Re$$

$$F_{XY}(+\infty, y) = F_Y(y)$$

↑ marginal distribution fn

Partial Order

$(x_1, y_1) \preceq (x_2, y_2)$ whenever $x_1 \le x_2$, $y_1 \le y_2$

Only for partial order we can talk about monotonic

Joint Density function, $f_{XY}(x,y)$
any non-negative f^n satisfying

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\,\partial y}$$

Conditional distribution

$P(X \le x \mid Y \le y)$

$$P(X \in B_1 \mid Y \in B_2) = \frac{P(X \in B_1, Y \in B_2)}{P(Y \in B_2)}$$

$$F_Y(y \mid X \in B) = \frac{P(Y \le y, X \in B)}{P(X \in B)}$$

$$F_Y(\cdot \mid X \in B) : \mathbb{R} \to [0,1]$$

Conditional Density

$$f_Y(y \mid X \in B) = \frac{d\, F_Y(y \mid X \in B)}{dy}$$

$$F_Y(y \mid X \le x) = \frac{F_{XY}(x,y)}{F_X(x)}$$

$$f_Y(y \mid X \le x) = \frac{1}{F_X(x)} \frac{\partial}{\partial y} F_{XY}(x,y)$$

———× ———× ——

Just a notation — $F_Y(y \mid X = x) = \frac{P(Y \le y, X = x)}{P(X = x)}$

means $\lim_{\Delta x \downarrow 0} F_Y(y \mid X \in (x, x+\Delta x))$

whenever the limit exists

$$\lim_{\Delta x \downarrow 0} \frac{P(Y \le y, X \in [x, x+\Delta x]) / \Delta x}{P(X \in [x, x+\Delta x]) / \Delta x}$$

$$\left(\frac{\partial}{\partial x} F_{XY}(x,y)\right) \frac{1}{f_X(x)}$$

$$f_Y(y \mid X = x) = \frac{\partial}{\partial y} F_Y(y \mid X = x)$$

$$= \frac{f_{XY}(x,y)}{f_X(x)}$$

$$P(Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_Y(y \mid X = x) f_X(x)\, dx\, dy$$

$$P(X \in [a,b]) = \int_a^b f_X(x)\, dx$$

Approx $\rightarrow P(X \in [a, a+\Delta]) \approx f_X(x)\Delta$ → Approximation for small Δ.

Law of total prob:

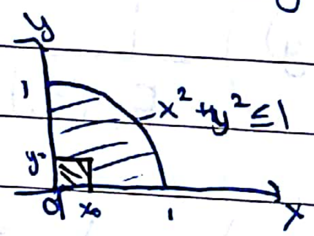
$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

Where $\{B_i\}_{i \in 1 \dots n}$ is partition of Ω

$$P(Y \le y) = \int_{-\infty}^{\infty} \underbrace{P(Y \le y \mid X = x)} f_X(x)\,dx$$

$$\int_{-\infty}^{y} f_{Y|X}(u \mid X = x)\,du$$

Let (X, Y) denote the coordinates of a point chosen uniformly at random from the shaded area



$P((X,Y) \in$ Black area$)$

$f_{XY}(x,y) = c, \quad c > 0$

only in the shaded area

$= 0$ outside

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\cancel{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c \, dx \, dy}$$

$$\iint f_{xy}(x,y) = \int_0^1 \int_0^{\sqrt{1-y^2}} dx\,dy = \frac{c\pi}{4} = 1$$

$$\Rightarrow c = \frac{4}{\pi}$$

$$P(X \le x_0, Y \le y_0)$$

$$= \int_0^{x_0} \int_0^{y_0} f_{XY}(x,y)\, dy\, dx$$

$$= x_0 y_0 \frac{4}{\pi}$$

$$P(\cancel{Y \le y}\, X \le x) = \int_0^1 \int_0^x f_{XY}(u,v)\, du\, dv$$

To show $f_Y(y \mid X=x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$$F_Y(y \mid x < X < x+\Delta x) = \frac{P(Y \le y,\ x < X \le x+\Delta x)}{P(x < X \le x+\Delta x)}$$

$$f_Y(y \mid x < X \le x+\Delta x) = \frac{1}{P(x < X < x+\Delta x)} \int_x^{x+\Delta x} f_{XY}(u,y)\, du$$

$$f_Y(y \mid X=x) = \lim_{\Delta x \to 0} \frac{1}{P(x<X<x+\Delta x)/\Delta x} \int_x^{x+\Delta x} f_{XY}(u,y)\, du/\Delta x$$

$$= \frac{f_{XY}(x,y)}{f_X(x)}$$

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Law of Total Probability

$$f_Y(y) = \int_{-\infty}^{\infty} f_Y(y|X=x)\, f_X(x)\, dx$$

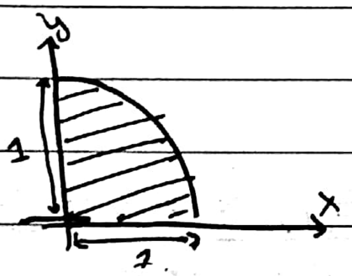
Bayes Formula

$$f_Y(y|X=x) = \frac{f_X(x|Y=y) \cdot f_Y(y)}{f_X(x)}$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Conditional Expectation

$$E[Y|A] = \int_{-\infty}^{\infty} y\, f_Y(y|A)\, dy$$



(X, Y) denotes co-ordinates of a point picked uniformly at random in the shaded area

$$f_{XY}(x,y) = \frac{4}{\pi} \quad \text{if } (x,y) \in \text{Shaded area}$$
$$0 \quad \text{otherwise}$$

$$f_Y(y|X=x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{0}^{+\sqrt{1-x^2}} \frac{4}{\pi}\,dy = \frac{4}{\pi}\sqrt{1-x^2}$$

$$f_Y(y|X=x) = \frac{1}{\sqrt{1-x^2}} \rightarrow \text{if } x,y \in \text{Shaded area}$$

here we know y is a uniform random variable between $(0, \sqrt{1-x^2})$

$$E[Y, X=x] = \frac{1}{2}\sqrt{1-x^2}$$

—x—x—

$$E[g(Y)|A] = \int_{-\infty}^{\infty} g(y)\, f_Y(y|A)\,dy$$

Function of two random variables

$$Z = X+Y$$

We know $F_{XY}(x,y)$ To find $F_Z(z)$

$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(u,v)\,du\,dv$$

Diff approach

$$= P(X+Y \le z \,|\, X=x)\, f_X(x)\,dx$$
$$= \int_{-\infty}^{\infty} P(X+Y \le z \,|\, X=x)\, f_X(x)\,dx$$
$$= \int_{-\infty}^{\infty} P(Y \le z-x \,|\, X=x)\, f_X(x)\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_Y(u|X=x)\,du\, f_X(x)\,dx$$

classmate

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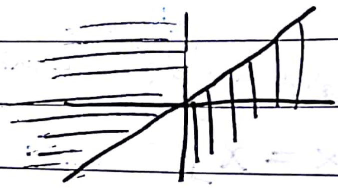
If $z = \frac{y}{x}$

$F_z(z) = P(z \le z)$

$= P(Y/x \le z)$

$\frac{y}{x} \le z \Rightarrow y \le zx$ when $x > 0$

$y \ge zx$ when $x < 0$



Expectation

1. $E[g(X,Y)]$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

2. Monotonicity of Expectation

if $\underbrace{X \leq Y}_{\text{means pointwise greater}}$, then $E[X] \leq E[Y]$

$$Z = Y - X \geq 0$$

When $Z > 0$

$$E[Z] = \int_{-\infty}^{\infty} z\, f_Z(z)\, dz$$

because $Z > 0$

$$= \int_{0}^{\infty} z\, f_Z(z)\, dz \geq 0$$

$\therefore E[Y-X] \geq 0$

$$E[Y-X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-x)\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\, dx\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, dy - \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$E[Y] - E[X]$$

Since $E[Y-X] > 0$

$$E[Y] > E[X]$$

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Markov's Inequality

Let $X \geq 0$, then $\forall\ K \geq 1$

Called Tail Probability $\rightarrow$ $P(X > \varepsilon) \leq \frac{E X^K}{\varepsilon^K}$

To convert Prob. to Expectation

$I_{\{X>\varepsilon\}} = 1$ if $X > \varepsilon$
$= 0$ otherwise

$I\{X > \varepsilon\}(\omega) = 1$ if $X(\omega) > \varepsilon$
$= 0$ otherwise

$E[I_{\{X>\varepsilon\}}] = \int 1 \times P(I\{X>\varepsilon\} = 1)$
$+ \int 0 \times P(I\{X>\varepsilon\} = 0)$

$= 1 \cdot P(X>\varepsilon) + 0 \times P(X \leq \varepsilon)$
$= P(X > \varepsilon)$

$\therefore$ to show $E[I\{X>\varepsilon\}] \leq E\left[\frac{X^K}{\varepsilon^K}\right]$

Enough to Show

$I\{X>\varepsilon\} \leq \frac{X^K}{\varepsilon^K} \quad \forall\ K \geq 1$

Consider $\omega \in \Omega$

Case 1 : $X(\omega) > \varepsilon$

$I\{X>\varepsilon\}(\omega) = 1$

$\frac{X^K(\omega)}{\varepsilon^K} > 1$ holds Since $\frac{X(\omega)}{\varepsilon} > 1$

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cause X is positive r.v.

Case 2: $0 \le X(\omega) \le \varepsilon$

$$I_{\{X>\varepsilon\}}(\omega) = 0$$

$$\frac{X(\omega)}{\varepsilon} \ge 0 \quad \forall \omega$$

Chebyshev's Inequality

$$P(\underbrace{|X - E[X]|^2}_{Y} > \varepsilon) \le \frac{Var(X)}{\varepsilon^2}$$

$Y>0$ $k=2$ Apply Markov's Ineq

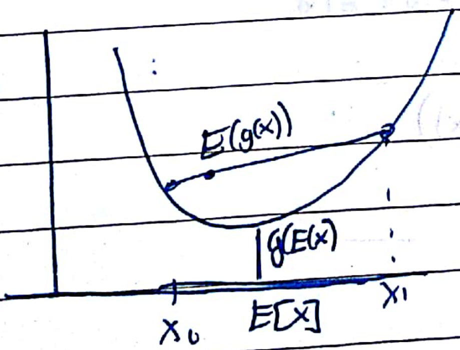
$$P(|X-E[X]|^2 > \varepsilon) = P(Y > \varepsilon) \le \frac{E[Y^2]}{\varepsilon^2} = \frac{E(|X-E[X]|^2)}{\varepsilon^2} = \frac{Var(X)}{\varepsilon^2}$$

$$\cancel{E[g(X)] \le g(E[X])}$$

Jensen's Inequality

If $g(x)$ is a convex fn, then

$$E[g(x)] \le g(E[x])$$



$x = x_0$ wp p

$x = x_1$ wp $1-p$

$$E[X] = p x_0 + (1-p) x_1$$

$g(E(x))'$

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$$E[g(x)] = g(x_0)p + g(x_1)(1-p)$$

$$\frac{y - g(x_0)}{x - x_0} = g'(x_0)$$

$$y = g'(x_0)(x - x_0) + g(x_0)$$

$$g(x) \geq g'(x_0)(x - x_0) + g(x_0) \qquad \forall x \neq x_0$$

$$g(X) \geq g'(x_0)(X - x_0) + g(x_0)$$

Choose $x_0 = E[x]$

$$g(x) \geq g'(\cancel{x}E[x]) \cdot (x - E[x]) + g(E[x])$$

$$E[g(x)] \geq E[g'[Ex) \cdot (x - E(x))] + g[E(x))]$$

$$E[g(x)] \geq g'(Ex)\,E[\underset{E(x)}{x} - \underset{E[E(x)] = E(x)}{Ex}] + g(E(x))$$

$$E(g(x)) \geq g(E(x))$$

— x — x —

E.G. $E[x^2] \geq E^2[x]$

$g(x) = x^2$

$$E(Y|X) = \int_{-\infty}^{\infty} E[Y|X=x]\, \delta(x-x)\, dx$$

$$E[Y|X=x] = g(x)$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y|\, X=x)\, dy \rightarrow \cancel{f_{X,Y}(x,y)}$$

$$\frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \int_{-\infty}^{\infty} g(x)\, \delta(x-x)\, dx$$

$$= g(x)$$

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Tutorial - 2

1. $k(1+2+4) + k(2+4+5) = 1$

$k(7+11) =$

(a) $k = \frac{1}{18}$

(b) $E[x] = \int_{-\infty}^{\infty} x f_x[x]\, dx$ $\qquad f_x(x) = \frac{1}{18} P(x=x)$

$= 1\times\frac{1}{18} + 2\times\frac{2}{18} + 4\times\frac{4}{18} + \frac{2}{18}\times 3 + \frac{4}{18}\times 5 + \frac{5}{18}\times 6$

$= \frac{\cancel{1+4+16+6} + 20 + 30}{18}$

$= \frac{77}{18}$

(c) $E[x^2] = 1\times\frac{1}{18} + 4\times\frac{2}{18} + 16\times\frac{4}{18} + \frac{2}{18}\times 9 + \frac{4}{18}\times 25 + \frac{5\times 36}{18}$

$1 + \cancel{8} + 64 + \cancel{18} + 100 + \cancel{180}$

$\frac{371}{18}$

2. mean $= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}\, x\,dx$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, x\,dx$

$\frac{(x-\mu)^2}{2\sigma^2} = t$

$dt = \frac{2(x-\mu)\,dx}{2\sigma^2}$

$\int_{-\infty}^{\infty} \frac{e^{-t}(\sigma^2 dt + \mu dx)}{\sqrt{2\pi\sigma^2}}$ $\qquad \mu dx + \sigma^2 dt = x dx - \mu dx$

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3. $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$$\sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \frac{\lambda^k e^{-\lambda}}{(k-1)!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = \lambda$$

Variance $= \cancel{E}\ E(x^2) - (E(x))^2$

$$E(x^2) = \sum k^2 \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{(k-1)!}$$

$$e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} = e^{-\lambda} \lambda \sum_{k=1} k \frac{\lambda^{k-1}}{(k-1)!} = \sum_{0}^{\infty} (k+1) \frac{\cancel{e}\lambda^k}{k!}$$

$$\frac{d}{dx} \sum \frac{\lambda^{k+1}}{(k+1)!}$$

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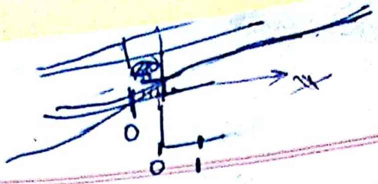
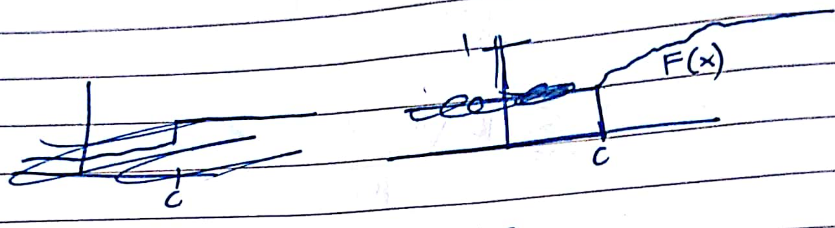
$\omega \to \mathbb{R}$



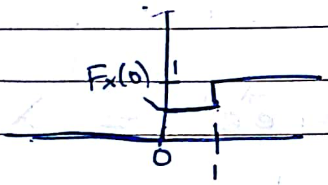
4.



$= 0 \quad X(\omega) \leq C$

$= F_X(x) \quad X(\omega) \geq C$

5.



$F_X(0)$

2.

1.2 $Y = -\log X$

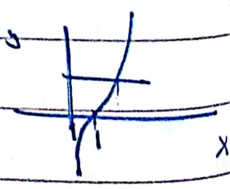
$Y \leq y$

$-\log X \leq y$

as log

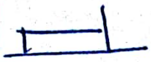
$\log X \geq -y$

$X \geq e^{-y}$ as ~~log X is~~



$[e^{-y}, \infty)$ is a Borel σ field So $Y \leq y$ is

Borel Set Dikhaga is suff.



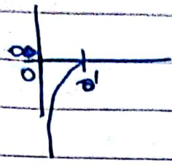
$f_X(x) = 1 \quad x \in (0,1)$

~~$f_Y(y)$~~

$f_X(e^{-y}) = 1$

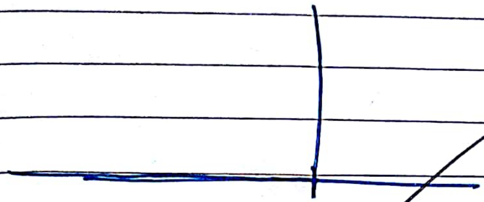
~~$F_Y =$~~

$Y =$



$\int f_Y(y)\,dy = 1$

F_Y



$F_Y(y) = \int_x^\infty f_X(x)\,dx$

$= \int_{e^{-y}}^\infty f_X(x)\,dx$ $\quad e^{-y} > 0$

$= \int_{e^{-y}}^\infty f_X(x)\,dx$

$= \int_{e^{-y}}^1 1\,dx$

$= 1 - e^{-y}$ for $y > 0$

3. $P(X > n^{\frac{1}{2}}) \le E\left(\frac{X}{n^{\frac{1}{2}}}\right) = \frac{n}{n^{\frac{1}{2}}} = n^{\frac{1}{2}}$

4.

$\perp\!\!\!\perp = \pi 2$ ulter

Independence

$X \perp\!\!\!\perp Y$

x is independent of Y

σ-field generated by a r.v.

$X^{-1}(\mathcal{B}) = \{A \cdot \cancel{A \in \mathcal{F}}$ $A \in \Omega$ & $\exists\, B \in \mathcal{B}$ s.t. $X(A) = B\}$

$\subseteq \mathcal{F}$

X^{-1} of Collection of sets

Subsets of $\mathbb{R}$

$A = X^{-1}(B)$ $\subseteq \Omega$

We also have $Y^{-1}(\mathcal{B})$

X & Y are said to be independent if $X^{-1}(\mathcal{B})$ & $Y^{-1}(\mathcal{B})$ are independent collection

$\forall A_X \in X^{-1}(\mathcal{B})$ & $A_Y \in Y^{-1}(\mathcal{B})$

$$P(A_X \cap A_Y) = P(A_X) \cdot P(B_Y)$$

$X \perp\!\!\!\perp Y$ if

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \qquad \forall x,y \in \mathbb{R}e$$

$$P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y)$$

$(-\infty, x]$ $(-\infty, y]$

Needs a proof.

It is enough to check for $(-\infty, x]$ type ke liee hold ho, then it'll hold for all subsets of $\mathcal{B}$. To be proved in 3rd year

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$