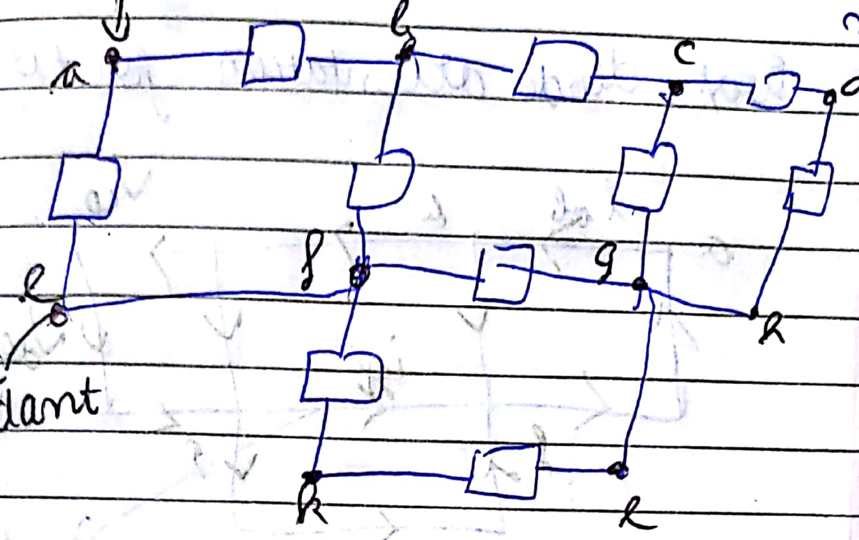
Pick up enough so that all devices = b/w nodes

Tellegen's Theorem

wire can also be a device, so nodes can be redundant no problem

Vertices / Nodes



Redundant

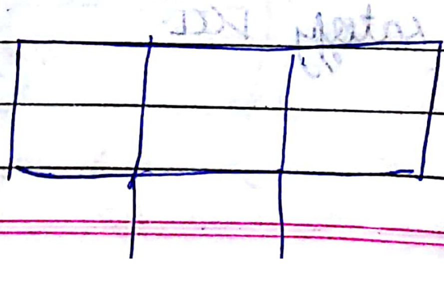
network with ◻ as elements/devices

can be anything like voltage source, current source, etc.

Could be even time dependent (like ac)

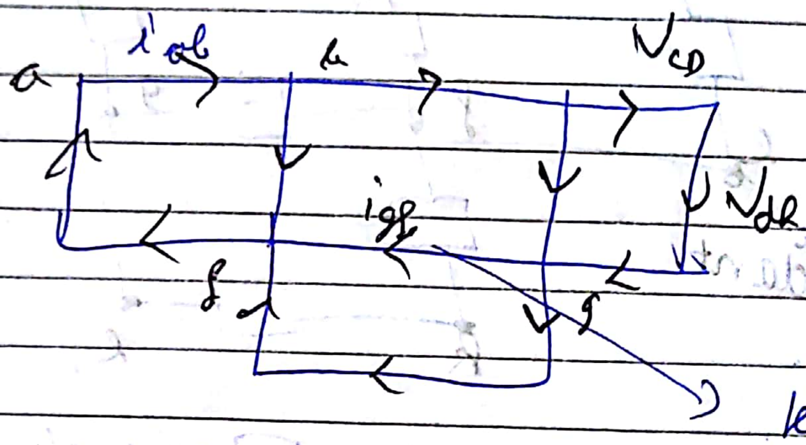
Draw this as a graph

Graph → omit devices just nodes.



on every edge I can define current & voltage.

First decide orientation for +ve current



Label all currents on edges.

We have 13 currents

$$\begin{bmatrix} i_{ab} \\ \vdots \\ i_{ek} \end{bmatrix} \quad I$$

These current should obey KCL (conservation law)

So these 13 numbers satisfy KCL

Similarly we can assign Voltages like

V_{cd}, V_{de}

Should obey KVL

V_{ab}

V_{ek}

B

$V_{gc} = -V_{cg}$ (obvi)

A

(Cons. of energy)

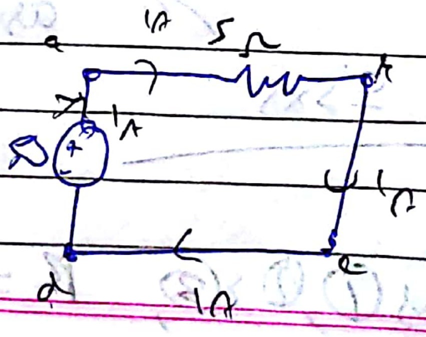
$\alpha \cdot \beta$ = ~~Power loss in the system~~

$V_{ab} i_{ab} + V_{bc} i_{bc} \ldots = 0$

would come every time out

(-ve effects of battery reverse)

eg.



$V_{ab} i_{ab} + V_{da} i_{da}$

$= (5)(1) + (-5)(1)$

$= 0$

Now keep current same but change voltages (replace elements)

only depends on structure of circuit relate with ratios

divergence of i following KCL = 0

vector space of i & V are always $\perp$r (in such structures)

Relation:
(R) rule of mapping set

Sets S & T

$S \times T = \{(s,t) \mid s \in S, t \in T\}$

$R \subseteq S \times T$

small these cases assume S as ϕ

Assume smallest domain

Consider $R \subseteq S \times S$

eg. for $R = \{(),()\}$ in $S \times S$

$\Rightarrow$ If R is ①, ② & ③, R = equivalent reflexive must contain set of all in S

(1) R is reflexive

iff $\Delta = \{(s,s) \mid s \in S\}$

& $\Delta \subseteq R$

(2) R is symmetric

iff $\forall (s,t) \in S$

if $(s,t) \in R$, then $(t,s) \in R$

$S = \mathbb{N} = \{0, 1, 2 \ldots\}$

$\{(i,i) \mid i \in \mathbb{N}\}$, $\{(i,j) \mid i,j \in \mathbb{N} \;\&\; i<j\}$

① or ② "equals" $<$

& ③ $=$ ③ $\{(i,j)$ i divides$\}$

$\mathbb{N} \times \mathbb{N}$, $0 \times \mathbb{N}$, ϕ ,

①,②,③ ③ ①,③ ①,②

(3) transitive

iff $(s,t) \in R$, $(t,u) \in R$

$\Rightarrow$ $(s,u) \in R$

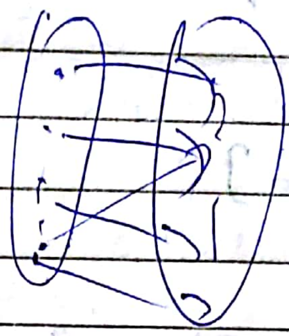
$\{ (i,j) \mid (i-j) \,\%\, 17 = 0 \}$

Transitive

<u>Function</u>

↓

Relation that is not one-to-many



<u>Graph</u>

<u>Directed multigraph</u>

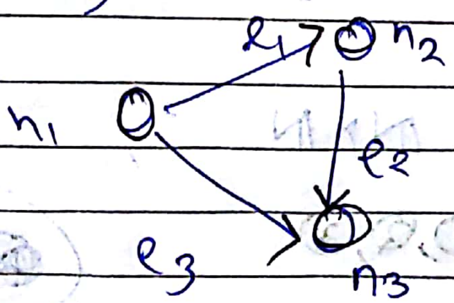
(source) $s: E \to N$

(target) $t: E \to N$

Nodes: N

Edges: E

eg.



then $N = \{n_1, n_2, n_3\}$

$E = \{e_1, e_2, e_3\}$

Scanned by CamScanner

Notice the arrows.
we defined s & t s.t. head of an edge = t & tail = s

$s(e_1) = n_1$ $\{t(e_1) = n_2\}$

$s(e_2) = n_2$ $t(e_2) = n_3$

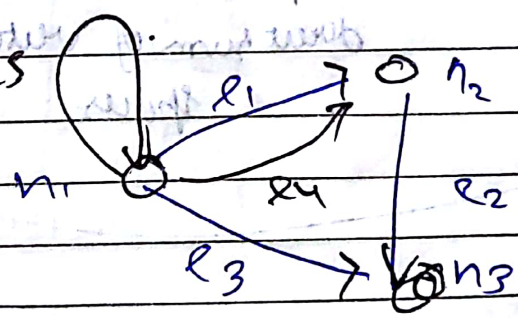
$s(e_3) = n_1$ $t(e_3) = n_3$

we can additionally define e_4 & e_5 acc. to ourselves.

$s(e_4) = n_1$ $t(e_4) = n_2$

$s(e_5) = t(e_5) = n_1$

Same nodes multiple edges



If S, T are sets, then $\{f : S \to T\}$ set of all possible functions

will be denoted by T^S → no. of all possible functions → cardinality of set

eg. $S = 3$, $T = 2$

total fn $= 2 \times 2 \times 2 = 8$

voltage fun. $v: E \to R$

current fun. $i: E \to R$

$$\{f: E \to R\}$$

$$\text{NOS} = R^E$$

vector space.

$$V_{KVL} \subset R^E$$

$$V_{KCL} \subset R^E$$

voltage & current that satisfies KCL & KVL

V_{KCL} & V_{KVL} turn out to be vector spaces

Tellegen's theorem

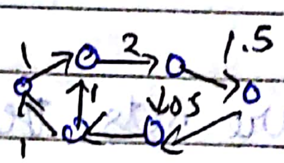
$$V_{KCL} \perp V_{KVL}$$

$$R^E = V_{KVL} \oplus V_{KCL}$$

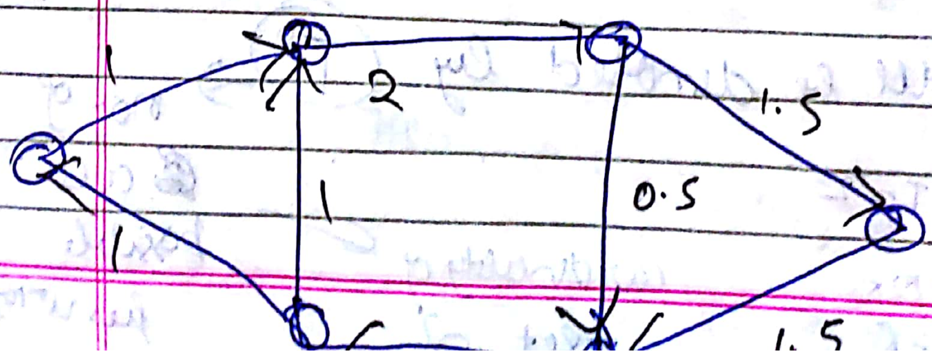
direct sum of vector spaces

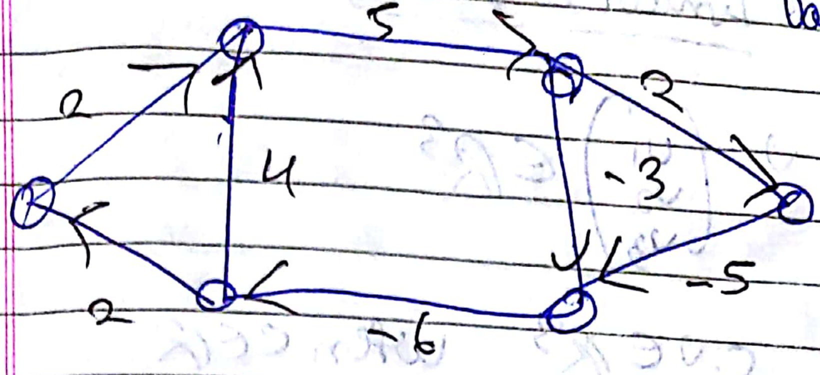
Take a random Example -

Take a graph



current distribution





voltage distribution

Taking dot product

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 0.5 \\ 0.5 \\ 1.5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 4 \\ 5 \\ -6 \\ -3 \\ 2 \\ -5 \end{bmatrix} = 0$$

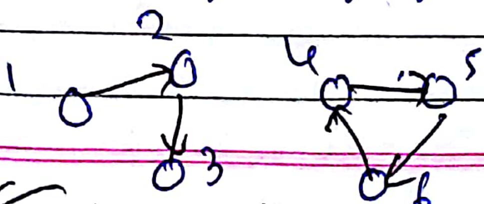
Claim this is zero

Claim is true

Graph needn't be nice

Valid for infact all types of graphs.

Reason for this



deg connected

no edge means current = 0, pot. = random.

current = same voltage adds up

Linear Algebra

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$$

$c \cdot v \in \mathbb{R}^3$ when $c \in \mathbb{R}$

$w + v = v + w \in \mathbb{R}^3$

$1 \cdot v = v$ (1 = scalar)

$0 + v = v + 0 = v$ (0 = vector)

$c \cdot (v + w) = c \cdot v + c \cdot w$

V vector space

$\left(V, \bar{0}, +_V, (F, 0, 1, +_F, \times_F), \cdot \right)$ — scalar set

contains vectors zerovector + $\cdot$ scalars

$\cdot$ takes a vector from a set of scalars & returns a vector

$\cdot : V \times F \to V$

Every scalar has inverse (both multiplicative & additive) except 0 which has no multiplicative

additive
0 is inverse of itself

Abelian group

A vector space forms an Abelian grp.

Set G

special Element 0

Only scalar multiplication missing for ~~group to form~~ V.S.

Operation $+_G : G \times G \to G$

$$g_1 +_G g_2 = g_2 +_G g_1$$ (Commutative)

In group every element has an inverse

$$g_1 +_G 0 = 0 +_G g_1 = g_1$$

(so 0 = additive identity)

Dropping subscript for now on $+_G$

Inverse -

Inverse function : $G \to G$

$g \to (-g)$
(element) (inverse) (additive)

$$g + (-g) = 0$$

Field

R is a set, has a 0 element, additn operation also has additive inverse

(R, 0, +, inverse) is also an example of abelian group.

R/{0} i.e (set of non zero ~~abstract~~ numbers)

Take another group

(R\{0}, 1, x, ( )$^{-1}$) ②

inverse.

Still an abelian grp.
Only difference addition replaced by multiplication

so new zero vector w.r.t. x is 1

See it satisfies everything for an abelian grp. !
(1) Is a set
(2) Has element 0 (1) that satisfies.

we should write x_G rather x_G

(2) behaves well with addition (scalar)

$a \times (b+c) = a \times b + a \times c$

Set F, elements $0, 1 \in F$, $0 \neq 1$ } not real numbers

(2): Set F, elements $0, 1 \in F$, $0 \neq 1$

operations - $+ : F \times F \to F$

$\times : F \times F \to F$

additive inverse.

(1) $(F, 0, +, \text{inv.})$ is an abelian grp.

(2) $(F \setminus \{0\}, 1, \times, \text{inv})$ is an abelian grp.

(3) $0 \cdot f = 0$ for $\forall f \in F$

(4) $a \times (b+c) = (a \times b) + (a \times c)$ $\forall a, b, c \in F$

here F is R

R ~~inverse~~, ~~contained~~ here.

Every field satisfies this.

(3) operation $\cdot$ (4) $g_1 \cdot g_2 = g_2 \cdot g_1$

$g_1 \cdot 1 = 1 \cdot g_2 = 1$

Say {0, 1} {9, 4}, anything

just a set of inverses

$\downarrow$

If field

$F = \{0, 1\}$

where 0 is additive identity
1 is multiplicative identity

& form an abelian grp.

we can define +

as

defining +

+	0	1
0	0	1
1	1	0

forced to be the only way

$0+0=0$ (zero additive identity)

$0+1=1$

$1+0=1$

$1+1=0$ $\because$ there needs to be an additive inverse of 1

$\therefore 1+1$ has to be 0

0 is identity so $0+0=0$

defining $\times$

$\times$	0	1
0	0	0
1	0	1

multiplicative inverse needs to exist for every element

Something $\times 0$ = always 0

$\therefore 0\times0=0$

$0\times1=0$

$1\times0=0$

1×1 = has to be 1

for mult. (-1) of 1 to exist

Field contains both.

$0, 1, +, \times$ & 0^{-1}

Abelian grp contains

$(0, +, \cancel{\times}, \text{Inverse})$

$\downarrow$ particular to special operation $+$

special operation.

mul. inverse of 0 needn't exist

while defining abelian group (S)

we exclude 0. so $F/\{0\}$

Say $F = \{0,1,2\}$

we can make both $2+2=0$

just remember additive (+) needs to exist for every element.

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$1+1=2$

$2+2=1$

so make 1 & 2 inverse of each other.

0 doesn't need inverse, 0 is the additive identity

$1+2=0$

can also be done by

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	2

→ but does mul. inverse needs to exist except for 0

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$0 \cdot \text{anything} = 0$
$1 \cdot \text{anything} = \text{anything}$

2 has mul. inverse
$2 \times 2 = 1$

1 has mul. inverse
$1 \times 1 = 1$

Confirmation

If
$a+b=0$
$a+c=0$
then $b=c$

Associativity

$a+b=0$
$(a+b)+c=c$
$a+(b+c)=c$
$a+(c+b)=c$
$(a+c)+b=c$
$a+b=c$
$b=c$

only unique additive identity.

①

Summary

To define fields,

(1) See components

(2) ~~(P)~~ Check points (1),(2),(3),(4)

(3) For + & × tables, make them abelian grps separately.

distribution must happen

point 4 that happens

For oper. ×, 1 is the 0 element of abelian grp.

Vector space

contains

(1) Abelian grp $(V, 0_V, +_V, \text{Inverse})$

(2) Field $(F, 0_F, 1_F, +_F, \times_F, (\text{Inv})_F,$ ↓ each ~~[illegible]~~ kinds

(3) $\cdot : F \times V \to V$

$\_F$ is additive in

(i) $0_F \cdot v = 0_V$

(ii) $1_F \cdot v = v$

(iii) $c \cdot (v+w) = c \cdot v + c \cdot w$

(iv) $(a \times_F b) \cdot v = a \cdot (b \cdot v)$

(v) $c \cdot 0_V = 0_V$

Examples of VS

Take two functions

$f \,\&\, g : [0,1] \to \mathbb{R}$

also exists $-f$ & $-g$ as additive inverse

$= \left(\{ f : [0,1] \to \mathbb{R} \}, 0, +, -_F \right)$ fun. abelian grp

0 Inverse

Basically V.S defines

Scalar (as field)
Vector (as Abelian group)

$\times$ & $+$ in scalars (operators in Field)
$\oplus$ of vector (abl. on group)

$\odot$ of vectors (separately defined in pt. ③)

Linear Map

$V, W := $ V.S over F scalar field.

①

$L : V \to W$

②

$$\boxed{L(a \cdot_F V_1 +_V b \cdot_F V_2) = a \cdot_F L(V_1) +_W b \cdot_F L(V_2)}$$

$\cdot_V$ $\cdot_W$ can be very diff
$+_V$ $+_W$ things

See page Ⓐ

for $\cdot$ & $+$

$\cdot_V$ means $F \times V \to V$ $\cdot_W$ means $F \times W \to W$

L (takes R^3 to R^3)

$M: R^3 \to R^2$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{pmatrix}_{2\times 3} \begin{pmatrix} - \\ - \\ - \end{pmatrix} \to \begin{pmatrix} - \\ - \end{pmatrix}$$

1. V & W over same ~~fields~~ defined fields because

RHS = LHS (should make sense)

$\therefore$ V & W if have diff. fields, $\therefore$ no sense b/w RHS & LHS doesn't exist

Q Is there $L: \mathbb{R}^3 \to \mathbb{R}^2$ linear map

~~s.t.~~ such that $(\tilde{\exists})$ matrix M

satisfying $Lv = Mv$

$\forall v \in \mathbb{R}^3$?

No ~~Yes~~

use both calculations to find M.

we can use isomorphism to say every vector space can be represented by $\mathbb{R}^d$

2 Hence for every $\mathbb{R}^m \to \mathbb{R}^n$ we can find a matrix by both calculation. ?

Last Time Revision

Abelian grp -

(1) Set G
(2) Element $0 \in G$
(3) Operation $+ : G \times G \to G$
(4) Inverse $G \to G$.

Satisfying $\forall\, g, h, k \in G$ by

(1) $g+(h+k) = (g+h)+k$ (Associative)
(2) $g+h = h+g$ (Commutative)

(3) $g+0 = 0+g = g$

(4) $g+$ inverse $(g) = 0$

Field -

(i) Abelian group $(F, 0, +,$ inverse$)$

(ii) Abelian group $(F)\{0\}, 1, \times, ()^{-1}$

Satisfying $\forall\, a, b, c \in F$

(1) $a \times (b+c) = (a \times b) + (a \times c)$ (distributi

(2) $0 \times a = 0$

Scanned by CamScanner

Vector Space

(i) Abelian grp $(V, 0_V, +_V, \text{Inverse})$

(ii) Field $(F, 0_F, 1_F, +, \times, \text{Inverse}, ()^{-1})$

(iii) Operation "Scalar multiplication" $(\cdot)$

$\therefore F \times V \to V$

Satisfying $\forall\, a, b, c \in F,\ u, v, w \in V$

(i) $a \cdot (u +_V v) = (a \cdot u) +_V (a \cdot v)$
(distributivity of scalar multiplication)

(ii) $0_F \cdot u = 0_V$

(iii) $1_F \cdot u = u$

Linear Map -

For a field F, let V, W be vector space over F. A function $L : V \to W$ is a linear map iff

$\forall\, v_1, v_2 \in V\ \forall\ c_1, c_2 \in F$:

$L(c_1 \cdot v_1 +_V c_2 \cdot v_2) = c_1 \cdot L(v_1) +_W c_2 \cdot L(v_2)$

$\exists$ ~~gathe~~ $X = \{$only set of those vectors whose c's need not be 0.$\}$

Linear dependence

Fix field F $(=\mathbb{R})$, vector space V over F

A subset $X \subseteq V$ is "linearly dependent" iff

$\exists\, n \in \mathbb{Z} > 0$, $\exists\, c_1, c_2 \ldots c_n \in F \setminus \{0\}$

$\exists\, v_1, \ldots v_n \in X$ distinct s.t.

$\sum_{i=1}^{n} c_i \cdot_F v_i = 0_V$

summation for n

If $\{0\} \subset X$, X = l.d. (obvi.)

Def

X is "linearly independent"

$:=$ X is not linearly dependent

i.e. $\sum_{i=1}^{n} c_i v_i = 0_V \implies c_i = 0 \;\forall i$

Empty set = lin. independent.

Spanning:

$X \subseteq V$ is spanning

iff $\forall v \in V$, $\exists n \in \mathbb{Z} > 0$,

$\exists c_1, c_2 \dots c_n \in F$,

$\exists v_1, v_2, \dots, v_n \in X$ s.t.

$$v = \sum_{i=1}^{n} c_i v_i$$

Note: If $X \subseteq V$ is l.i. & $Y \subseteq V$ is spanning, then

$$|X| \leq |Y|$$

Proof:

$x_1, x_2 \dots x_n$

$|X| > |Y|$ $\quad c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$

Then Y spans V $\quad x_1 = a_1y_1 + a_2y_2 + \dots + a_my_m$

Y also spans X $\quad x_2 = a_2 \cdot$

$|Y|$ $\quad c_1a_1 ~~c_2a_2~~ ~~c_3a_3~~ \dots = 0$

~~$c_1a_1y_1 + c_1a_2y_2 + \dots$~~

$X = \{x_1 — x_m\}$

$Y = \{y_1 — y_n\}$

~~To show~~ Assume $m > n$

$$x_1 = \sum_{i=1}^{n} c_{1i} y_i$$

$$x_n = \sum_{i=1}^{n} c_{ni} y_i$$

→ Matrix C

→ unique for every (Matrix = invertible) as $x_1 — x_4 = LI$

$$x_m = \sum_{i=1}^{n} c_{mi} y_i$$

$$\begin{pmatrix} y_1 \\ y_2 \\ | \\ y_n \end{pmatrix} = C^{-1} \begin{pmatrix} x_1 \\ | \\ x_n \end{pmatrix}$$

now $x_{n+1} = \sum_{i=1}^{n} d_i y_i$ & each y_i = in terms of x

$$x_{n+1} = \sum_{i=1}^{n} e_i x_i$$ → contradiction as x = L.I.

To prove:

$C = (c_{ji})_{n \times n}$ is invertible

X is linearly indep.

If $C^{-1} \neq$ exist

rank$^{(C)} < n$

Non zero lin. Comb. of rows of C which is 0.

Then

$\exists (a_1, \ldots a_n) \neq 0$

s.t. $(a_1 \ldots a_n)(C) = \bar{0}$ vector.

Then $\sum a_i x_i = \bar{a}\, Cy = 0$

$\Rightarrow \Leftarrow$ since X is lin indep.

Basis: subset B is a basis iff its L.I & spanning

If B = finite, then V is finite-dimensional

& $\dim(V) = |B|$

Claims:

(1) Every minimal spanning set = basis

(2) Every maximal L.I set = basis.

$\therefore m \leq n$

Claim -

* If $B_1, B_2 \subseteq V$ are basis, (finite dimensional)

$|B_1| = |B_2|$

just write elements in B_2 in terms of B_1
(spanning set.)

If diff. sizes $\neq$ LI
$\Rightarrow$ Same size.

New lecture

Last time

(1) linear dependence

$X \subseteq V$ is "lin. dep."

iff $\sum_{v_i \in X} c_i v_i = 0$, some $c_i \neq 0$

(2) Spanning

(3) If $X, Y \subseteq V$, X lin. indep., Y spanning,

then $|X| \leq |Y|$ (?)

(4) Basis = Both (1) & (2)

Claim:

I If $A = \{u_1, \_\_\_, u_m\}$

& $B = \{v_1 \_\_\_ v_n\}$

are bases of a vector space W, then $m = n$.

i.e. dim. (Basis) = constant

Pf:-

If $n > m$, $\mathcal{B}$ = lin. dependent.
using (v)

Since A is spanning,

$|A| \geq |\mathcal{B}|$ $\cancel{|\mathcal{B}| \geq |A|}$ $\because$ $\mathcal{B}$ = lin. indep.

or $m \geq n$ (contradiction)

without Loss of Gen

Same reverse argument of for $m > n$

Claim :- $V = V.S.$

II Let $A \subseteq V$ vector space
If A is a basis, then

(a) A is a maximal Lin. Indep. set.
(b) A is a minimal spanning set.

Proof-

A = spanning & L.I $\qquad$ v = any vector
(a) $\qquad$ $v \notin A$

Consider $A \cup \{v\} = A'$

A' = lin. dep.

$v = \sum c_i v_i \quad \because A$ = spanning
$v_i \in A$

$\therefore A'$ = lin. dep. $\therefore A$ = maximal L.I

(b) Let $A = A' + A''$ $\quad A'' \neq \phi$

where A' = Spanning (assumed)

Then $v = \sum c_i w_i \qquad w_i \in A'$

$v \in A''$

$\therefore A \neq$ lin. indep

(contradiction)

$\therefore A$ = minimal spanning

Claim:

(i) If A is max. lin. indep., then A = basis

(ii) If A is a min. spanning set, then A = basis.

Proof

(i) If A is not spanning, let $v \in V$ s.t. $v \notin \text{Span}(A)$

Let $A' = A \cup \{v\}$

$\Downarrow$

lin. indep. $\quad \because v$ can't be written as lin. comb. of vectors from A

$\therefore A \neq$ max lin. indep.

$\therefore$ Contradiction $\qquad \because v \notin \text{Span}(A)$

$\therefore$ ~~A must span~~, $\therefore A$ = basis.

(ii) A spans

let $A' = A \cup \{v\}$

s.t. A' = lin. dep.

but $v \in \text{Span}(A)$

$\therefore v = \sum c_i w_i \quad w_i \in A$

$\therefore A' \neq$ lin dep.

$\therefore A =$ L.I.

$\therefore A =$ Basis.

(ii) A spans.

Let A = lin. dep.

$\therefore \exists v \in A$

s.t. $v = \sum c_i v_i$

(Write it else (ii))

So $A \neq$ min. spanning set

$\therefore A$ = lin indep

$\therefore A$ = basis.

Maximal, minimal } extremal arguments.

Φ = lin. indep.

$$\begin{tikzcd}
 & S \cap T \arrow[dl] \arrow[dr] & \\
S \arrow[dr] & & T \arrow[dl] \\
 & S \cup T &
\end{tikzcd}$$

end of array = maximal elements

Collorary – Every vector space has a basis.

Linear Map

$L : V \to W$
over same field F

$$L(c_1 \cdot_F v_1 +_V c_2 \cdot_F v_2) = c_1 \cdot_F L(v_1) +_W c_2 \cdot_F L(v_2)$$

$? \cdot_F$

over same field so that it makes sense on LHS & RHS

* linear superposition

use

Isomorphism

V.S of Bvs = dim of $\cong \mathbb{R}^n$

Homomorphism -

$Hom(V, W) = \{ L : V \to W \}$

L: linear map

Search more?

set of all maps b/w V & W = V.S. linear

L = V.S. of the way its defined.

Homomorphism

Claim:

Hom(V, W) is a vector space.

Proof:- Matrices can be considered as $\mathbb{R}^n$ of n elements

& all vectors in $\mathbb{R}^n$ = vector space

i.e. $M = \mathbb{R}^n$

& $\mathbb{R}^n$ = Vec. Space

Laplace transform - Gives diagonal Basis, say for differentiation

Needs (0, +, ·, ·, inverse)

~~$M_{22} = d_2$~~ Buch calc. just use basis of

$= d_1 \cancel{e_1'} + d_2 e_2'$ both V & W.

$O: V \to W$

$w \to O_w$

$(L_1 +_{Hom} L_2) v = L_1(v) +_W L_2(v)$

$(c \cdot L)(v) = c \cdot_W (L(v))$

Verify associativity & commutativity

Trivial

$L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$

$L_1 + L_2 = L_2 + L_1$

$(c \cdot L)(v) = c \cdot (L(v))$

Isomorphism

Two sets V & W are isomorphic

$V \simeq W$ iff $V \underset{\hat{L}}{\overset{L}{\rightleftarrows}} W$

i.e. $\exists$ a map $L : V \to W$
& $\hat{L} : W \to V$

then $\hat{L} \circ L$ ($\hat{L}$ is composed with L)

i.e. L takes a element from V to W
& $\hat{L}$ takes the element from W back to V

or $\hat{L}(L(v)) = v \quad \forall v \in V$

& $L \circ \hat{L} = Id_W$

$\hat{L} \circ L$ means $\hat{L}(L(\ ))$

$\Leftrightarrow$ dim V = dim W
(Proof??)

but even if $V \cong W$, $V \neq W$ i.e. to be same v.s, sets must be same.

Dual vector Space

$\hookrightarrow V^* = \{ L : V \to \mathbb{R} \mid L \text{ linear map} \}$

Basically like 2 but its a V-S = of Homomorphism i.e. again it as V-S

if V = column vectors, V^* = row vector.

if $L \in V^*$, $v \in V$

then $L(v) \in \mathbb{R}$

similar to taking a column vector (v) multiplying to give $\mathbb{R}$

Don't need basis for V & V^*

$V \simeq V^*$

Prove:

~~Inner product~~

$(\mathbb{R}^3)$

define one coor of $\mathbb{R}^2$ to 1 rest all to 0 & is lin comb. of basis

L = row vector.

M is diagonalizable

$\Updownarrow$

$\forall N$

$MN = NM$

$L_M \circ L_N = L_N \circ L_M$

Inner Product

the dot product:

$$v \odot w = \sum v_i w_i$$

$\langle v_1, v_2 \rangle : V \times V \to \mathbb{R}$

$v_1, v_2 \in V$

IP1) $\langle v, w \rangle = \langle w, v \rangle$

IP2) $\langle u, av + bw \rangle$

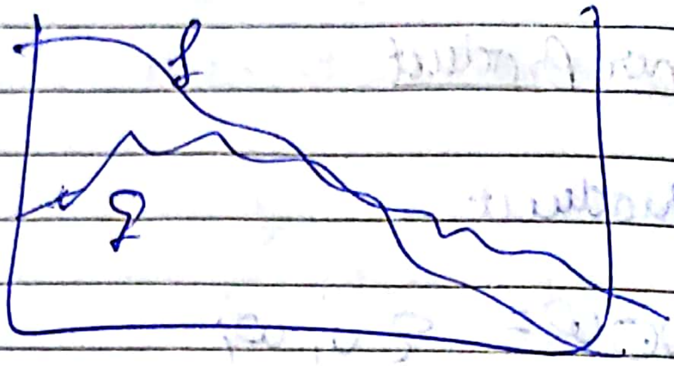
$= a\langle u, v \rangle + b\langle v, w \rangle$

IP3) $\langle u, u \rangle \geq 0$

with $\langle u, u \rangle = 0$ only if

$u = 0_V$

Fourier series Basis: sinusoids (because differential : diagonal) or sin & cos.



$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt$$

(satisfies all IPs)

New Lecture

Last time

Linear map $L : U \to V$

Hom(U, V) is a vector space

→ Dual space $V^* = \{ L : V \to \mathbb{R} \}$

→ Inner Product $\langle \, , \rangle : V \times V \to \mathbb{R}$

(1) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

(2) $\langle u, a v_1 + b v_2 \rangle = a \langle u, v_1 \rangle + b \langle u, v_2 \rangle$

(3) $\langle u, u \rangle \geq 0 \ \forall u \in V$ with equality if $u = 0_V$

Isomorphism

takes every vector to

Dual vector.

$\langle v, \_\_ \rangle : V \to \mathbb{R}$

$\langle v, \_ \rangle \in V^*$ (takes a vector & gives scalar)

Basis $B = \{v_1, \_\_ v_n\}$ for V

$B^* = \{\langle v_1, \_ \rangle, \langle v_2, \_ \rangle, \_\_ \langle v_n, \_ \rangle\}$

Claim: B^* = basis of V^*

$\phi_{\langle , \rangle} : V \to V^*$ (To create isomorphism b/w V & V')

$v_i \longmapsto v_i^* = \langle v_i, \_ \rangle$

$v = \sum c_i v_i \longmapsto \sum c_i v_i^*$

An inner product gives an isomorphism

$V \cong V^*$

Lang's

Suppose $f : V \to V^*$ is an isomorphism.

$\langle , \rangle_f : V \times V \to \mathbb{R}$

$(v_1, v_2) \longmapsto (f(v_1))(v_2)$

satisfies β

But α may not be satisfied

Why transpose depend on basis

$V^* = w^* \circ L$

$V \xrightarrow{L} W \to W^* \to \mathbb{R}$

$L(w^*)$: map of $V \to \mathbb{R}$

If transpose = Basis independent, it can be trusted.

$w^* \in W^* \xrightarrow{L^*(\text{adjoint})} V^*$

$w^* \circ L$

so if $L^*(w) = v^*$

v^* = adj of w^*

Isomorphism b/w W & W^*

Then apply L^*, then apply isomorphism b/w V^* V

Then we get $W \xrightarrow{LT} V$

Q. a matrix

$$\mathbb{R}^3 \to \mathbb{R}^2$$

$$(\mathbb{R}^2)^* \to (\mathbb{R}^3)^* \qquad (T^*)$$

$\mathbb{R}^2$ & $\mathbb{R}^{2*}$ are isom.

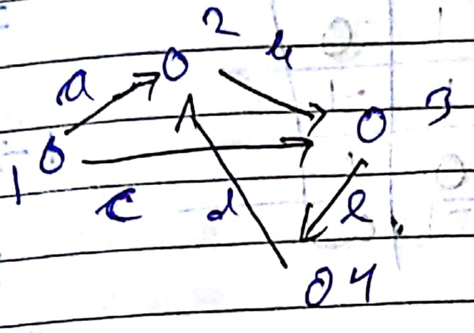
$\mathbb{R}^3$ & $\mathbb{R}^{3*}$ = isom

so we get a step map in T
that takes $\mathbb{R}^2 \to \mathbb{R}^3$
& is linear L^T

Killigen's

Graph-

$(V, E, s: E \to V, t: E \to V)$



$\mathbb{R}^E = \{ f : E \to \mathbb{R} \}$

a vector space usual + & · operation

edges are basis elements. (all of them)

a, b, c, d, e

1 0 0 0 0 (edge 1)

similarly 0 1 0 0 0 (edge 2) similarly

0 0 1 0 0 (edg 3) & so on

$\therefore$ all, a, b, c, d & e are indep.,

$\dim(\mathbb{R}^E) = |E|$

the v.s = current, then $\dim(\mathbb{R}^E) \leq |E|$ $\because b = e$ (dependency)

$v_{n \times v} \subset R^{E}$ ($v_{n \times v}$ = vector space)

Represent graph of matrix.
node × node.

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & 1 & 1 & 0 \\
2 & -1 & 0 & 1 & -1 \\
3 & -1 & -1 & 0 & 1 \\
4 & 0 & 1 & -1 & 0
\end{array}
$$

M

1→1
= no
= 0
1→2 yes
1,2 (y)
& 2→1 (y)

Subgraph

$G' \subseteq G = (N, E)$
s.t

$G' = (N' \subseteq N, E' \subseteq E,)$
s.t

Function
reserve
sam

Tree -
Subgraph not having a cycle

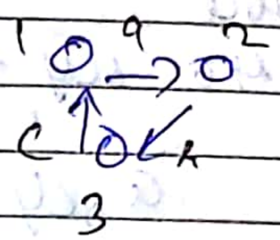
To define path

~~Cycle- Subgraph~~

Path- ~~Subgraph s.t.~~

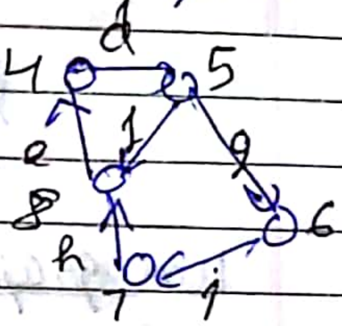
Graph Homomorphism

Map b/w graphs



another graph

We can define node to node function



e.g. $1 \to 4$

$2 \to 5$

$3 \to 8$

So a forced to d

b forced to f

c forced to e

* If G, H are graphs, a graph homomorphism $f: G \to H$ is a function $f: N_G \to N_H$

s.t. $\forall e \in E_G \ \exists e' \in E_H$

s.t. $f(s(e)) = s(e')$ & $f(t(e)) = t(e')$

Meaning of it, we can transport all edges from left to a node on right.

if there is a self node on G_2, there should be a self node on H also for homomorphism to exist.

Line graph - L_n

$n = 1$ (single node no edges)

Nodes: $1, 2, \ldots, n$

Edges: $1 \to 2 \to 3 - - n-1 \to n$

$n-1$ edges.

Cycle graph C_n

$n = 1$

Nodes $1, 2, \ldots, n$

Edges: $1 \to 2 \to 3 - - n-1 \to n$

n edges.

L_1 = edge ①

so no. of nodes in a graph G = no. of homomorphisms from ① to G.

$$|N| = |\{L_1 \to G\}|$$

No. of nodes = no. of homomorphisms from L_1 to G

$$|E| = |\{L_2 \to G\}|$$

Undirected graph

G = U.G. if
$\forall e \in E$
$\exists e' \in E'$
s.t.
$s(e) = t(e')$
& $t(e) = s(e')$

provided atmost one edge b/w a pair of nodes.

because if G can have such

one edge = directed & ~~gcle~~ G. Use graph ∄ multiple edges.

Directed cycle. of size n an injective (into) homomorphism

$C_n \to G$

undirected cycle if no homomorphism exist.

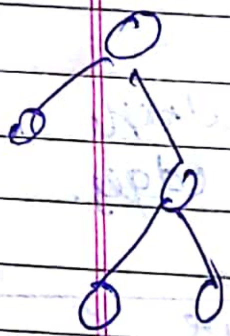
Injected so node don't collapse

Path - A directed path of length n from m_1 to m_2 in a graph is injective homomorphism $L_{n+1} \xrightarrow{f} G$

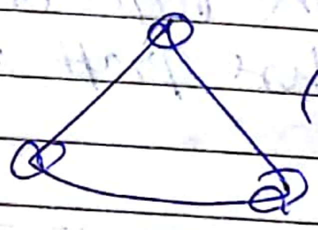
s.t. $f(1) = m_1$

& $f(n+1) = m_2$

Tree - undirected cycle free ($n \geq 3$) graph



Tree



not a tree

* A graph is strongly connected

if for every $m, n \in N_G$,

there is a directed path from m to n

Graph Homo - (Cross Rare Media)

New lecture

Last time -

Graph Homomorphism -

$f: N_G \to N_H$

s.t. $\forall e \in E_G, \exists e' \in E_H$ s.t.

$f(s(e)) = s(e')$

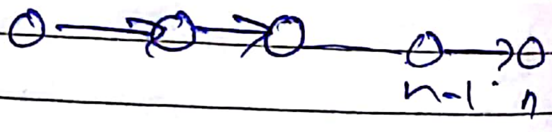
$f(t(e)) = t(e')$

Undirected

Line graph - L_n

(no arrows)

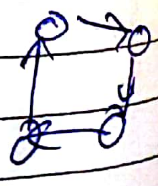
directed



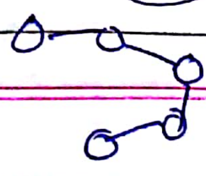
Cycle graph

Directed C_n

$n \geq 3$



undirected



undirected graph

$\forall e \in E \quad \exists e' \in E$

s.t.

$s(e) = t(e')$

$t(e) = s(e')$

same as

Give a graph G

Line-are graph

directed.

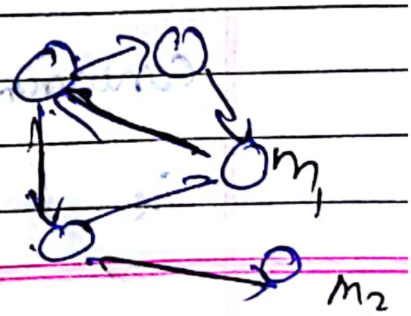
Let $m, n \in N_G$

(1) $m \rightsquigarrow n$:

means there is a directed path from m_1 to m_2 if you could reach n from m along arrow directions

Same as saying there is a homomorphism

From m to m_1 → undirected path but no directed path.



Same as saying
∃ injective
homomorphism

$$L_k \xrightarrow{f} G$$

s.t. $f(1) = m_1$
$f(k) = m_2$

i.e. after k consecutive nodes of line graph you may reach m_2 from m_1

(2) $m \sim n$:

If ∃ injective
homomorphism

$$\bar{L}_k \xrightarrow{f} G$$

s.t. $f(1) = m_1$
$f(k) = m_2$

means ~~[illegible]~~ undirected

If G is a graph, let $\bar{G}$ be
smallest undirected graph
containing all edges of G

i.e. if $e \in E_G$, then $e \in E_{\bar{G}}$
$e' \in E_{\bar{G}}$

looks like

actually

where

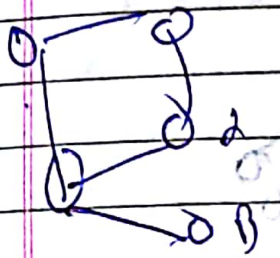
we care abt reachability

$s(e') = t(e)$

$t(e') = s(e)$

i.e. If $\exists$ edge in G $u(u+e')$, only one in $\bar{G}$

i.e. $\bar{G} = G$ without arrows.



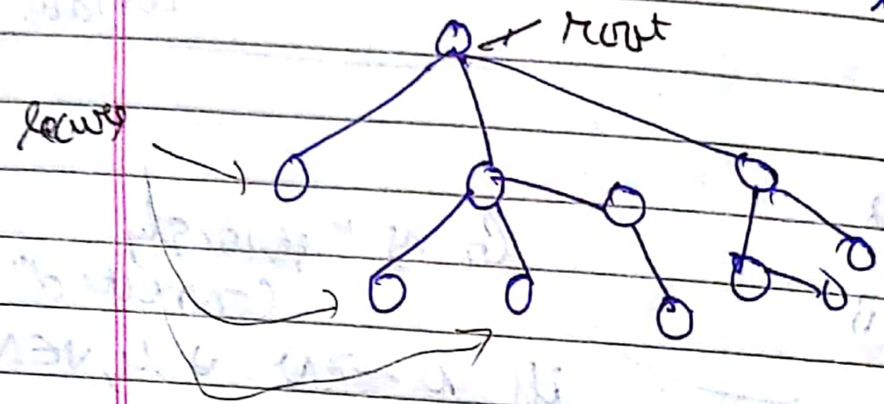
G is "strongly connected" iff $u \rightarrow v \;\forall\, u, v \in N_G$

G is "connected" iff $\bar{G}$ is strongly connected.

i.e. you can go every node from any node

Undirected tree

An undirected graph G is a tree iff $\forall u, v \in N_G$, $\exists$ unique path $u \rightsquigarrow v$



Claim:

Let G be an undirected graph, ~~then~~ without self loop

i.e. $s(e) \neq t(e)$ $\forall e \in E_G$

& without multi-edges.

i.e. If $s(e) = s(f)$, $t(e) = t(f)$

then $e = f$, $\forall e, f \in E_G$

injective so that walking on same path
many times is avoided
one-one, it would then become
many - one

Then
the following are
equivalent:

No of nodes

(1) G is connected & has $(n-1)$ edges
(counting each one).

(2) G is a tree

(3) G is connected & "cycle-free"

i.e. $C_k \not\to G \;\forall\; k \geq 3$

undirected cycle graph.

Connected graph with no cycles = tree.

Proof:-

$(1 \Rightarrow 2) \to 2$

$\therefore 1 \Leftrightarrow 2$ (equivalent)

$2 \Rightarrow 1$

Induction on n (Trivial case $n=2$)

Suppose
In nodes with $n-1$ edges
add a new node, new no. of edges
$= n-1+1$
$= n$

$\therefore n+1$ nodes $\to n$ edges $\therefore$ have a.

2 ⇒ 3

∵ tree ⇒ unique path

∴ connected

∴ G = cycle free. & connected

∵ cycle ~~path~~

∵ path = unique

3 ⇒ 2 & cycle ≠ unique path

path exists ∴ G is connected

no cycle ∴ all paths = unique

∴ G is a tree ?

∴ 2 ⇔ 3

(2) Base case

o—o connected

$|V| - 1 = 1$ edge

tree.

Inductive hypothesis

Extra node

o (B)

4. Tree of n nodes (n-1) edges

So ~~only~~ we have to add atleast one edge to have a connection with β.

we have to add atmost one edge to have a unique path.

∴ only one edge is added.
Resulting H_{n+1} = tree (connected + unique)

~~nodes~~ Edges in H_{n+1} $= n-1+1$
$= n$

∴ Proved

If ≥ 2 path, say $\beta \to h_1 \to v_1$
$\beta \to h_2 \to v_2$

$h_1 \to h_2$ already exists (v')
∵ H = tree (unique path already exists)

So unique path now doesn't exist.
as ~~edge~~ $h_1 \to h_2$
has two path. $v' + v_1$
& $v' + v$

∴ not a tree.

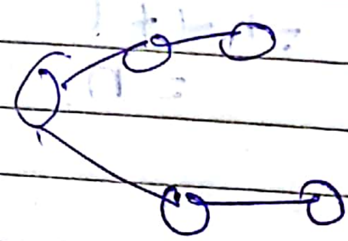
∴ $2 \Rightarrow 1$

$1 \Leftrightarrow 2$

For 2 ⇒ 1

Since G is a tree, we claim
∃ v ∈ N_G s.t. v is a leaf.

Suppose not - then every node has at least two edges incident on it.



∴ have to also prove every tree has a tree incident

No leaf ⇒ every ○ has 2 edges.

part is the next node ≠ same
then it will be a cycle ∵
finite graph ∴ there must be a final node not having further leaf in tree.

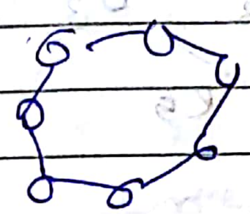
2 ⇔ 1 ∴ 1 ⇔ 3
2 ⇔ 3 ∴ all equivalent.

Pull out a leaf, still tree = tree.

2⇒3

G is a tree, suppose G has a cycle.

$\tilde{C_n}$



But ∃ 2 distinct paths $1 \sim k$ in $\tilde{C_n}$

$$\begin{tikzcd} C_n \arrow[r, "Id"] & \tilde{C_n} \arrow[r, "h"] & G \\ L_2 \arrow[ur, "f"] & & \end{tikzcd}$$

$f(1) = 1$

$f(2) = n$

3⇒2

To show ∃! $u \sim v$. Since G is connected ∃ $u \sim v$. Suppose not true.

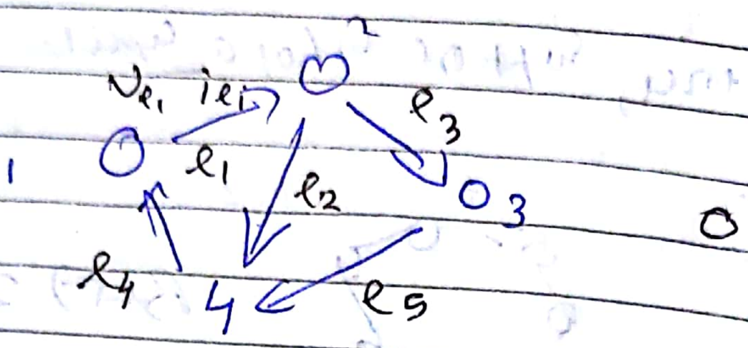
paths $(u, u_1, u_2, \ldots, u_k, v)$

& $(v, v_1, v_2, \ldots, v_k, v)$ are two paths,

let x be the first vertex common to both paths after u.

we get $(u, v_1, v_2, \ldots \Rightarrow \ldots u_2, u_1, u)$ a cycle ⇒⇐

New Lecture



$$\sum_{e\in E} v_e \cdot i_e = 0$$

$v_1 = 10$
$v_2 = 20$
$v_3 = 30$
$v_4 = 40$
$v_5 = 50$

$$\delta : \mathbb{R}^N \to \mathbb{R}^E$$

$$(v_n)_{n\in N} \longmapsto \left(v_{s(e)} - v_{t(e)}\right)_{e\in E}$$

$$(\delta(v))_{e_5} = 30 - 40 = -10$$

taking a function returning one

N, E

$s: E \to N \longrightarrow R$

$t: E \to N \longrightarrow$

$R^N = \{ f: N \to R \}$

$R^E = \{ g: E \to R \}$

defines function i.e. potential at each node

defines potential for every edge.

gives potential at node n.

so v_n defines ~~for~~ R^N

$\therefore v_{s(e)} - v_{t(e)}$ defines R^E

δ sends 0 to 0.

$R^N = 0$ means all nodes $= 0$

& all $R^E = 0$

$\delta(av + bw) \overset{?}{=} a\delta(v) + b\delta(w)$

check it

$\therefore \delta$ - linear map.

i.e. if V & W are different potential configurations of nodes they can be linearly superposed.

& the final pot. difference b/w the edges will be same.

$V \in R^{n}$

$\delta(w)$
$\delta(\tilde{w})$ $\in R^{E}$

Basis of R^{n} = I_5

~~the~~ $\forall$ [a b c d e]

represent voltage at node 1, 2, 3, 4 & 5

then basis vectors = $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Basis of R^{E} –

(5 edges)

each edge = independent

~~so Basis~~ = I^{5}

matrix for S (S: linear map ∴ have a matrix)

$$\underbrace{\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}}_{|E| \times |N|} \underbrace{\begin{bmatrix} \ \\ \ \end{bmatrix}}_{|N| \times 1} \rightarrow \underbrace{\begin{bmatrix} \ \\ \ \end{bmatrix}}_{|E| \times 1}$$

we know the elements of $|E| \times 1$

they are $v_1 - v_2$, $v_2 - v_4$, $v_2 - v_3$, $v_4 - v_1$, $v_3 - v_4$

(see edges)

$\therefore v_{\text{source}} - v_{\text{target}}$

Elements of $|N| \times 1$

= simply $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$

∴ $|E| \times |N|$ can be found

$$S = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

all info about graph in S.

δ^T, M_s^T

Accumulation

$\delta^T : \mathbb{R}^E \to \mathbb{R}^N$

$\delta = |E| \times |N|$

$\Rightarrow \delta^T = |N| \times |E|$

give a current in $\mathbb{R}^E$, get accumulation in $\mathbb{R}^N$

how much current is going in & how much going out

now define current

$$(i_e)_{e \in E} \longmapsto \left(+ \sum_{(l,m) \in E} i_{lm} - \sum_{(m,n)} i_{mn} \right)_{n \in N}$$

Define the vector that has dimension $|E|$

(l,m) = one node

(m,n) = successive of (l,m)

so at node m, we calculate all current going in ($\overset{l}{\to} m \overset{l}{\leftarrow}$) & subtracting current going out ($\leftarrow m \to n$, $\downarrow n$)

For a fixed graph $\exists$ fixed δ

Defn-

image of δ

(1) $W_{KVL} = \text{im}\,\delta = \{ v \in \mathbb{R}^E \mid \exists\, V \in \mathbb{R}^N$ with $v = \delta(V) \}$

(columns in the eq.) subset of $\mathbb{R}^E$

null space of δ^T (such that $\delta^T(M) = 0$)

(2) $W_{KCL} = \ker \delta^T$
$= \{ i \in \mathbb{R}^E \mid \delta^T(i) = 0 \}$

Claim:

(1) W_{KVL} is a vector space.

(2) W_{KCL} is a vector space.

(3) $v \in \mathbb{R}^E$ satisfies KVL iff $v \in W_{KVL}$

(1) all column satisfy KVL, all lin. comb. of columns satisfy KVL
nothing else satisfy KVL

0 is there $\delta(0) = 0$

$av_1 + bv_2$ in image. $= \delta(aV_1 + bV_2)$

$= av_1 + bv_2$ = entry fn cross. $= a\delta(V_1) + b\,\delta(V_2)$

(2) $0 \in W_{KCL}$ = $i = 0$ = in nullspace of S^T (obviously) trivial soln.

$S^T(i_1) = 0$ & $S^T(i_2) = 0$

$S^T(a i_1 + b i_2)$
$= a S^T(i_1) + b S^T(i_2)$
$= a(0) + b(0) = 0$

Same way verify for (not also)

$\Rightarrow a i_1 + b i_2 \in W_{KCL}$

(3) Have a cycle $\bar{C}_n$ (undirected)

$\bar{C}_n \to \bar{G}$

look at homomorphism

Suppose $1 \to 2 \to 3 \to \ldots \to n \to 1$ is a cycle in $\bar{G}$

Then $v_{(1,2)} + v_{(2,3)} + \ldots + v_{(n,1)}$

$= v_1 - v_2 + v_2 - v_3 + \ldots + v_n - v_1$

$= 0$

(a) Suppose $v \in \mathbb{R}^E$ satisfies KVL. Then $v \in W_{KVL}$

i.e. $\exists V \in \mathbb{R}^N$ s.t. $v = \delta(V)$

means potential function has unique soln

??

KCL no need for defn, W_{KCL} defn. takes care of that.

<u>Tellegen's theorem</u>

$$(W_{KVL})^{\perp} = W_{KCL}$$

(3)

2. $\circledast$ of W_{KVL}, $W \in W_{KCL}$

any vector

Figure out

Proof-

Suppose $v \in W_{KVL}$, $i \in W_{KCL}$

then $v \cdot i = \langle \delta V, i \rangle = \langle V, \delta^T i \rangle$

$= (M_\delta V) \cdot i$ & true for dot product

$= (i^T M_\delta) V = (M_\delta^T i) \cdot V$

$\langle v, \delta^T i \rangle = 0$

$v = \delta^T i = 0$

$= \langle v, i \rangle = 0$

<u>Thing left to prove</u> - (B)

New lecture

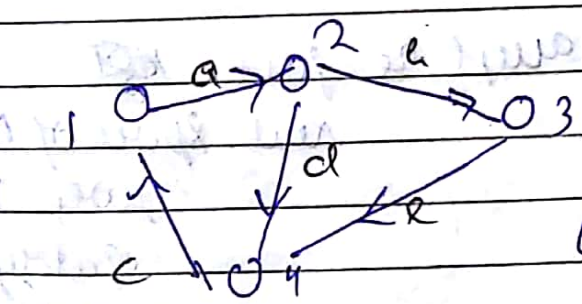
Last time

W_{KVL}, W_{KCL}

Tellegen's theorem - $(W_{KVL})^{\perp} = W_{KCL}$

Proof: (1) We will show that if $v \in W_{KVL}$, $i \in W_{KCL}$, then $v \cdot i = 0$

(2) To show If $i \in W_{KVL}^{\perp}$, $i \in W_{KCL}^{\perp}$



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & M_g = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

$|E| \times |N|$

$$M_g^T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

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$M_s^T i = 0$ gives 4 eqns.
& its same of Kirchoff current law

Outgoing current = B
+ Ingoing

∴ its now an eqn along ~~nodes~~ edges

M_s = eqn along nodes.

& the 4 eqns satisfying KCL.

~~@~~ The Each node has some no of i's as there are edges connected to it.

So just accept the fact ~~M_s^T~~ null space of M_s^T gives KCL satisfying value of edges

↓

Crosscheck.

Real thing is that putting KCL eqns in matrix A
& KVL values in B,
we somehow always find that

$A^T = B$

$M_S = \begin{bmatrix} m_1 & m_2 & - & m_e^T & - \\ - & - & & & \\ - & - & & & \end{bmatrix}$

Pf (i) :-

Since $v \in \ker L$, $v = \sum_{e \in N} c_e m_e$

$\therefore v \cdot i = \sum_{e \in N} c_e \cdot (m_e \cdot i) \to 0$

$= 0$

$v \cdot i = \langle \delta v, i \rangle = \langle v, \delta^T i \rangle = \langle v, 0 \rangle$
$= 0$

$\sum m_e \cdot i = (M_S^T) \cdot i$

$\because (A \cdot B)_{\ell m} = A_{\ell *} \cdot B_{* m}$

Dotting the columns of ~~some vector~~ dot product defined that way

so for dotting columns of M, = some of $M_S^T \cdot i$

& $i \in$ Nullspace of $M_S T$ $\therefore 0$.

$v = \delta(V)$

so prove if $i \in \omega_{KVL}^{\perp}$,
then $i \in \omega_{KCL}$

(ii) Proof:

Suppose $i \in \omega_{KVL}^{\perp}$, then since $m_1, m_2 — m_n \in \omega_{KVL}$,

$i \cdot m_e = 0$

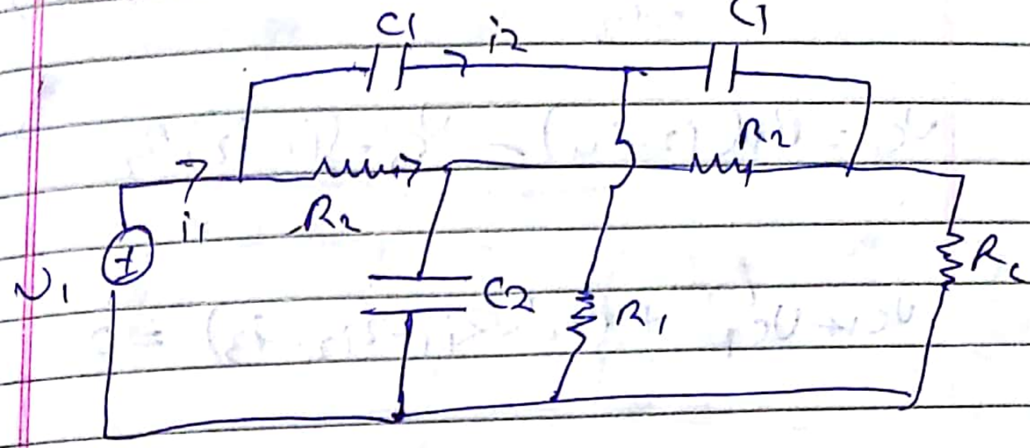
2) $i \in \ker(M_g^T) \Rightarrow i \in \omega_{KCL}$

Von Valkenberg

Ch-3 Q.4

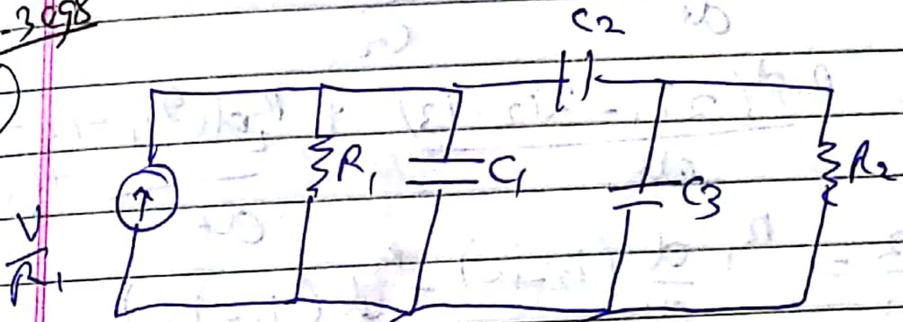
Network eqns-

I

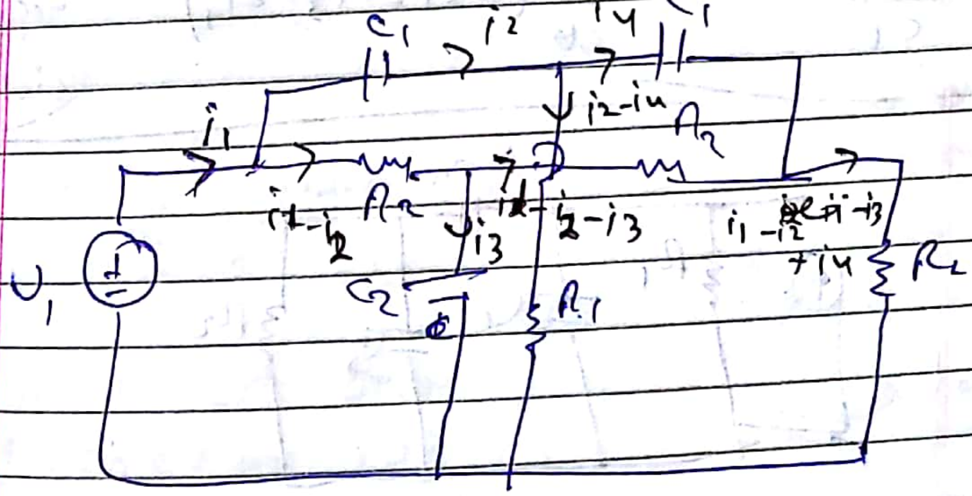


ch-3 Q.8

II



I



$\frac{Q}{C}$

$v_1 = (i_1 - i_2) R_2 + V_{C_2}$

$\cancel{i_3 = C_2 \frac{dV_{C_2}}{dt}} \quad \cancel{\frac{d(i_2 - i_1) R_2}{dt}} = C_2$

$v_1 = R_2(2i_1 - 2i_2 - i_3) + R_L(i_1 - i_2 - i_3 + i_4)$

$v_{C_1} = R_1(i_2 - i_4) - v_{C_2} - (i_1 - i_2)R_2$

$v_{C_1} + v_{C_2}' + R_2(2i_1 - 2i_2 - i_3) = 0$

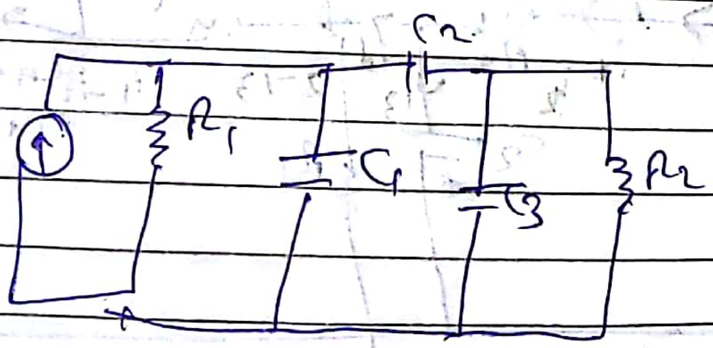
$0 = \frac{d(i_1 - i_2)}{dt}R_2 + \frac{i_3}{C_2}$

$0 = R_2\frac{d(2i_1 - 2i_2 - i_3)}{dt} + R_L\frac{d(i_1 - i_2 - i_3 + i_4)}{dt}$

$\frac{i_3}{C_2} + \frac{i_2}{C_1} = R_1\frac{d}{dt}(i_2 - i_4) - R_2\frac{d}{dt}(i_1 - i_2)$

$\frac{i_2 + i_4}{C_1} = R_2\frac{d}{dt}(2i_2 + i_3 - 2i_1)$

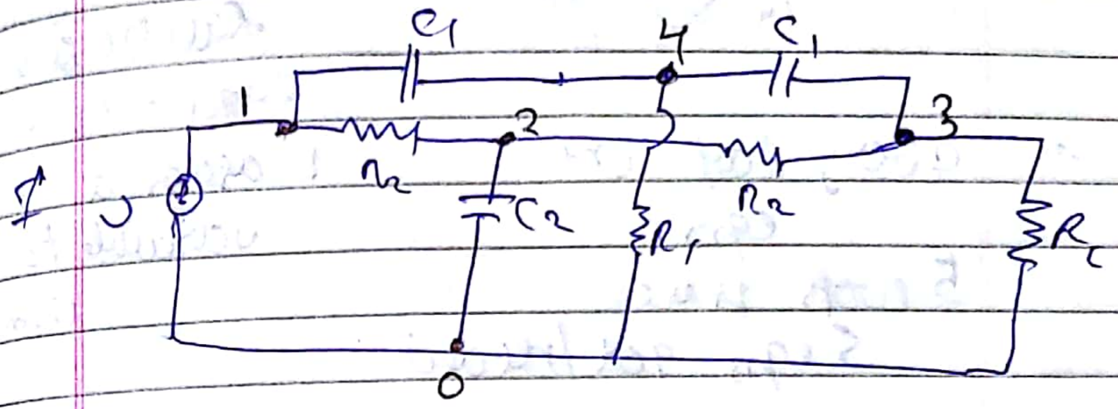
4 eqns
4 unknown

I 

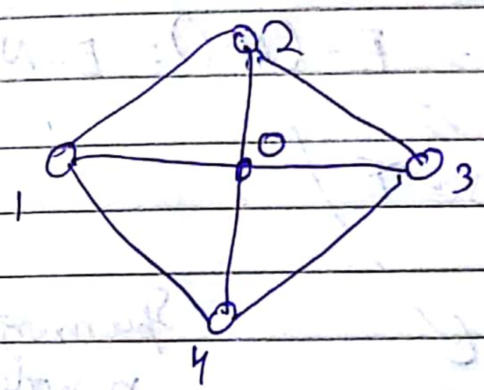
Spanning Tree = largest tree possible, touching all nodes.

Actual soln -

Draw a graph first, label every node.



5 nodes
0, 1, 2, 3, 4

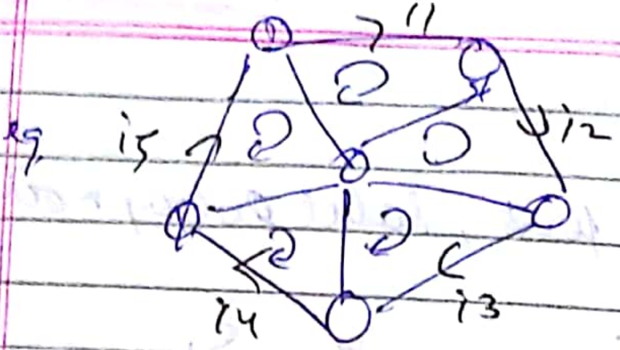


Few nodes
more cycles
↓
node variable analysis

Lets do loop variable analysis first

Many nodes
few cycles
↓
loop variable analysis

Define currents on the outermost edge & one in inner loops run



graph if not complete cycle type, loops ... all the edges of branches & assign a variable to them.

→ every loop = one eqns.
5 loops there
5 eqns. got/needed.

→ E edges, N nodes, connected

Then Branches = $E-(N-1) = E-N+1$

Loop Eqns = E

rest eqn come from KCL

Spanning tree for n nodes, n-1 edges.

Branch edges = $E-(N-1)$

Choose a spanning tree

Every connected graph has a spanning tree

Take the loop 0, ~~1~~ 0

$$\frac{\int i_1 \, dt}{C_1} + R_1(i_1 - i_4) = -v_0$$

$\dim(W_{KVL}) = N - 1 = 3$

$\dim(W_{KCL}) = |E| - |N| + 1$

$= 5 - 4 + 1 = 2$

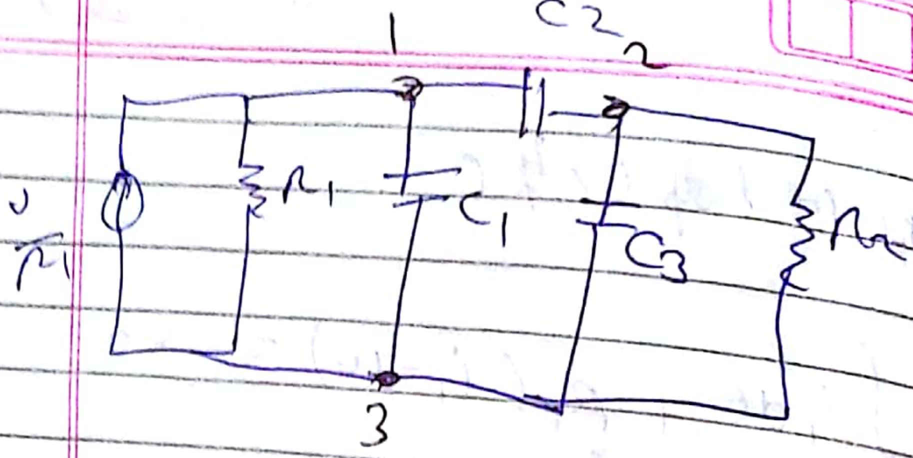
$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

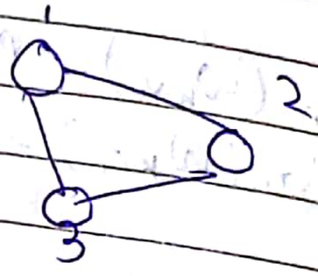
II 6 edges

3 nodes, need 3 eqn.

loop currents → 4 eqn. (4 loops.

Node analysis more suitable (less eqns)





Analysis help
to write less
eqns.

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New Lecture

Q1) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad 2x+3y-z=0, \; x,y,z \in \mathbb{R} \right\}$

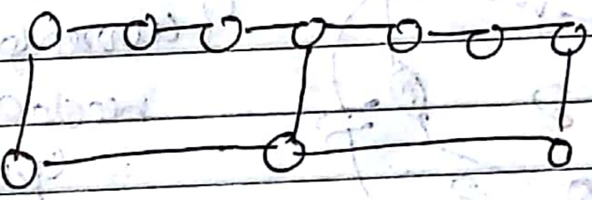
$\downarrow$

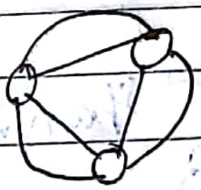
vector space over $\mathbb{R}$

a) $\dim W = 2$

b) $\dim W^* = 2$ (Isomorphism b/w W & W^*)

c) $\dim W^\perp = n-r = 3-2 = 1$

Q2) G =  (10 nodes)

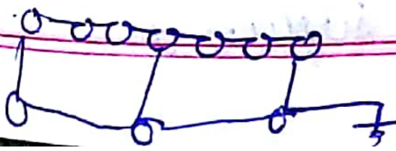
H =  (3 nodes)

a) For each graph, how many node variable eqns?

b) How many loop variable eqns?

Ans) G:

$|N| = 10$, comp $= 2$ — node variable

 $(10-1)$ (Kirchhoff's current law = like node analysis in EE114)

$= 9$

For node variables (9), cost of redundant eqs.

Branch 2 currents

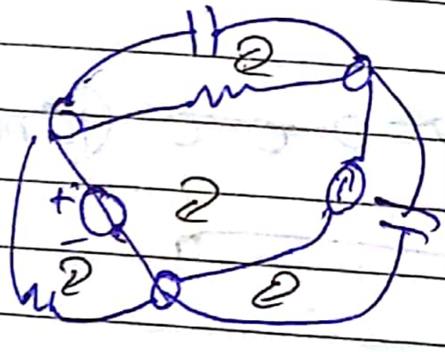
= 2 loop variable eqs.

4:

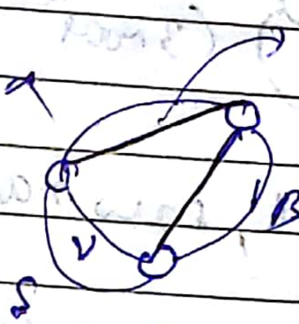
|N|=3

Cycles = 4 (minimum of v.s. needed to cover all currents)

Make a spanning tree & see the no. of edges needed.



4 currents needed = cycles 4.



Spanning tree

4 edges not on spanning tree.

So each edge gives a loop.

$\alpha, \beta, \gamma, \delta$ + edge of S.T

= they form a loop

= they are sufficient

Current
$\therefore$ KCL have true currents

S.T ≠ unique, but no. of edges = same in each S.T = n-1
& no. of edges left out = same.

You don't get every loop from S.T.

Mutual Inductance & Dot Convention

Ex-3.5 & 3.6 from book

→ device characterized by v_e, i_e, $\frac{dv_e}{dt}$, $\frac{di_e}{dt}$

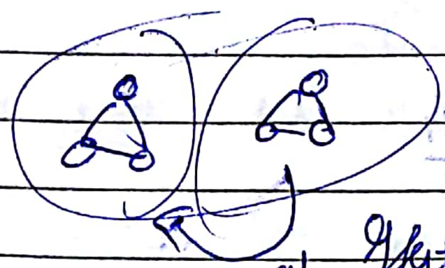
Relations b/w these 4 terms = depend on device

b/w A & B

eg. in inductor, $\frac{di_e}{dt}$ = linear with v_e or 0.

2 different isolated circuits influence each other. (mutual effects)

eg.

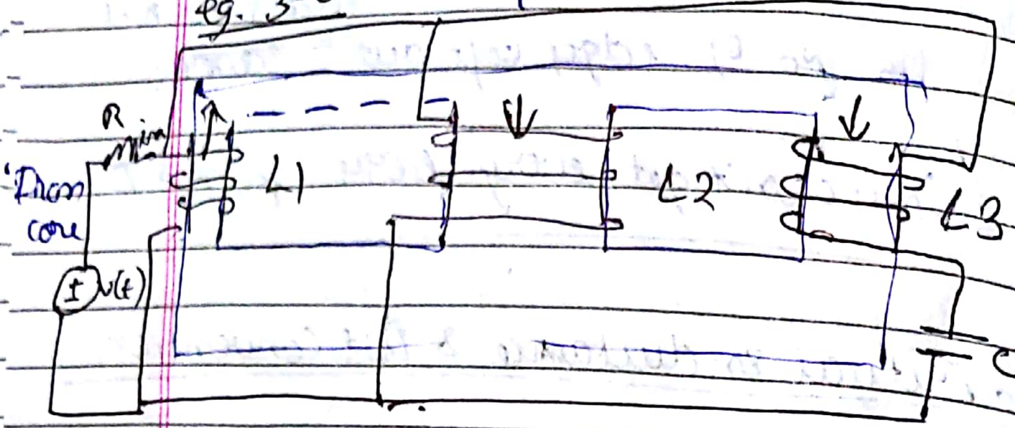


mutual effect

dot convention says choose current such that flux = same direction.

Notice polarity of flux

eg. 3-6

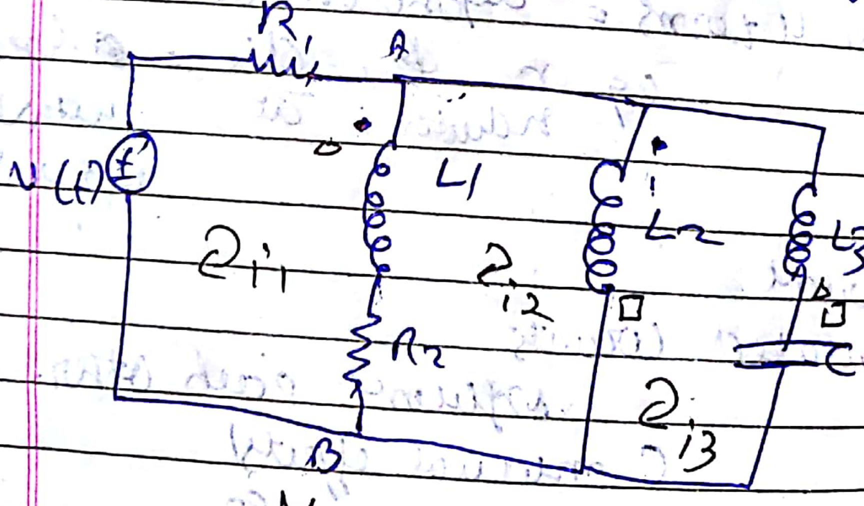


There is a circuit consisting of time varying voltage source

same circuit of.

Flux direction = fixed

⇒ current in coils ⇒ depends even hence isolated



$$v_2 = \cancel{X}^{M} \frac{di_1}{dt}$$

• only where flux enters = current direction

i_1 & i_2 push φ in same direction.

" represent where current enters from.
Φ_1, i.t. same cycle of ϕ induced
$\phi_2 \phi$

for 1 & 2 •
2 & 3 D
1 & 3 D

Now write eqns

$$v(t) = i_1 R_1 + L_1 \frac{d(i_1 - i_2)}{dt}$$

$$+ (i_1 - i_2) R_2$$

voltage on L_1 of current on L_2

$$+ M_{12} \frac{d(i_2 - i_3)}{dt}$$

$$- M_{13} \frac{d i_3}{dt}$$

See how i_3 ↑ inducing V_+ & V_- across AB

current flows out of dot
∴ potential = -ve
across L_1

Before writing i_1, i_2 & i_3 direction of
See how current increase in i_3,
aid or harm the flux through L_1
(desired direction always)

In our case i_3 ↑ ⇒ flux (L_1) ↓
∴ $-M_{13}$

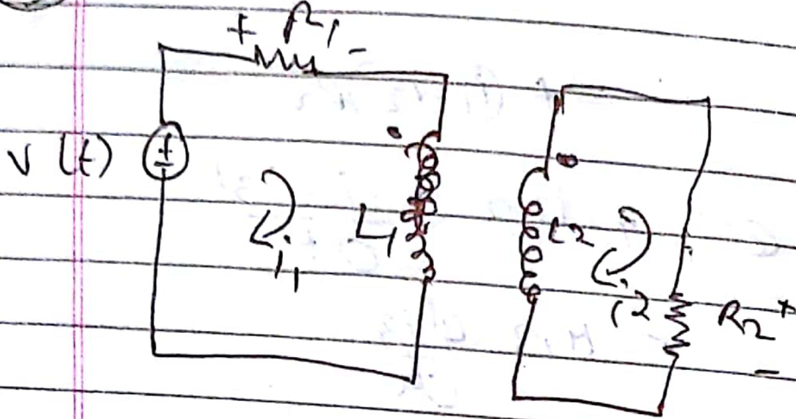
or you can see dot, if current goes ~~throu~~ (or ↗) from Φ_1 → $+M_{13}$

else $-M_{13}$. Same for i_2, i_2 entry through dot ∴ $+M_{12}$ (aid)

Just see dots, if i enter dot, M
ardy (+ve)
else not (-ve M

(2)

i1 enters dot
i2 enters opp. dot
so -ve in both cases.



$$v(t) = i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$ (see dots)

$$0 = i_2R_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt}$$

If both enter dot or both enter opp. dot, +ve, else -ve M

• decided by winding & flux always.

+ve or

-ve

New lecture

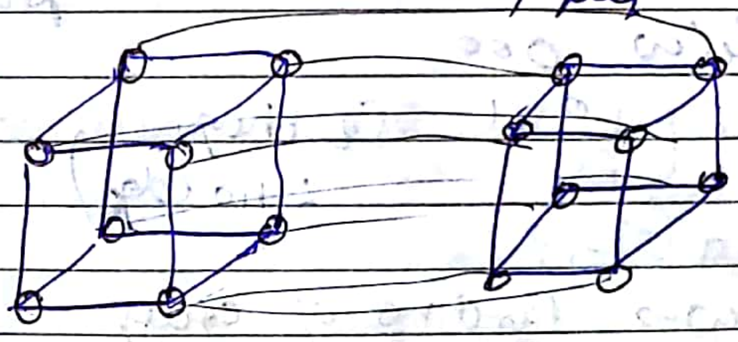
Last time

Dot notation for mutual inductance
controlled sources
quiz

Loop variable
↓
KVL
Node variable
↓
KCL

Spanning tree = Always a subgraph.
ie. ↓ it contains edges from graph only.

To find -
No. of cycles: draw spanning tree
find how many edges = left
= no. of cycles



$|N| = 16, \quad |E| = 12 + 12 + 8 = 32$

Spanning tree has 15 edges $(= N - 1)$ & $\Rightarrow$ cycles $= 32 - 15$
$= 17$

Node variable → 15 eqns (N-1)
Loop variable → 17 eqns (cycles)

∴ Node variable preferred.

H_n

$N = \{0,1\}^n$ → each position unique code

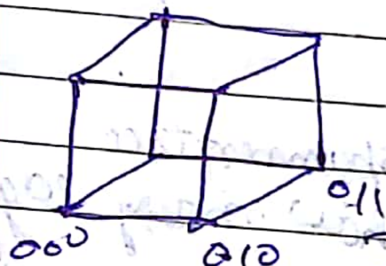
2^n nodes

$E = \{(n_1, n_2)\}$ $n_1, n_2 \in N$

& ∃ $e_j = (0\,0\,1\,0\,0)$

s.t. either $n_1 = n_2 + e_j$

or $n_2 = n_1 + e_j$



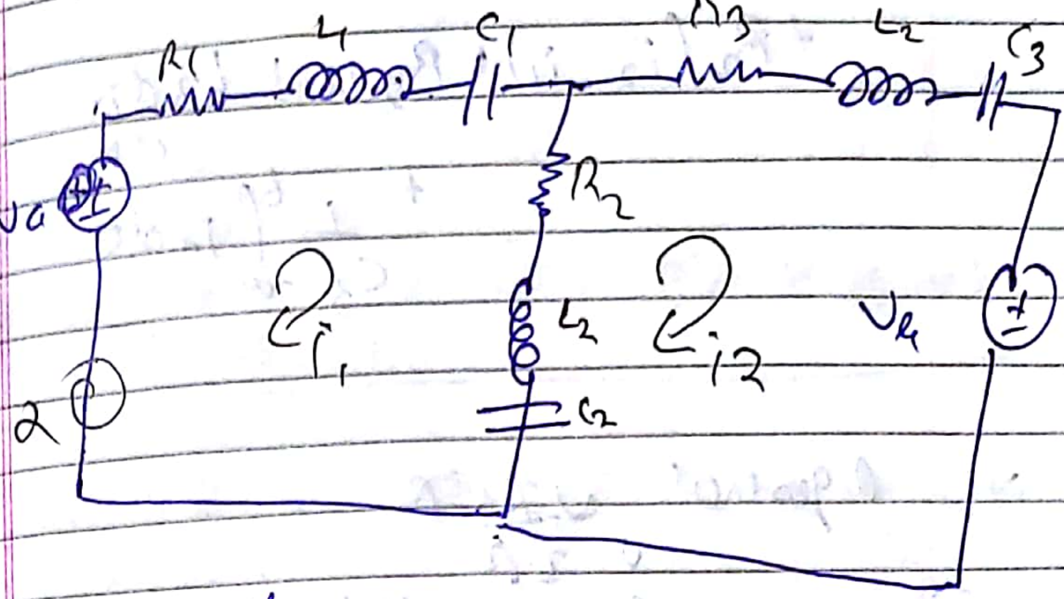
edge exists → x, y differ by only one place

C/W 000

011 (2 changes ∴ no edge)

H.W → Find no. of cycles

Ch 3 eg. 10



loop variable.

∴ cycle = less. node = too much.

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

V R I

① $v_a = i_1 R_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int_{-\infty}^{t} i_1\,dt$

$+ (i_1 - i_2) R_2$

$+ L_2 \frac{d(i_1 - i_2)}{dt}$

$+ \frac{1}{C_2}\int_{-\infty}^{t} (i_1 - i_2)\,dt$

(2) $-V_a = \frac{1}{C_2}\int_{-\infty}^{t} i_2 - i_1 \, dt + L_2 \frac{d(i_2 - i_1)}{dt}$

$+ R_2(i_2 - i_1) + R_3 i_2 + L_3 \frac{di_2}{dt}$

$+ \frac{1}{C_3}\int_{-\infty}^{t} i_2 \, dt$

* A general ~~$V = IR$~~
$V = IR$
form needed.
to solve all similar graphs

Current through all sources = V_a
& oriented in direction of current.
= + Va

can be written in matrix form:

$$\begin{pmatrix} V_a \\ V_a \end{pmatrix} = \begin{pmatrix} R_2 + L_2\frac{d}{dt} + R_1 + L_1\frac{d}{dt} + \frac{1}{C_1}\int + \frac{1}{C_2}\int \\ R_3 + \cancel{\frac{d}{dt}} L_3\frac{d}{dt} + \frac{1}{C_3}\int \end{pmatrix} i$$

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{bmatrix} \quad \end{bmatrix}_{2\times 2} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

$$\begin{bmatrix} \text{Components of } i_1 \text{ in eqn 1} & \text{Comp. of } i_2 \text{ in eqn 1} \\ \text{Components of } i_1 \text{ in eqn 2} & \text{Comp. of } i_2 \text{ in eqn 2} \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ \vdots \end{pmatrix}$$

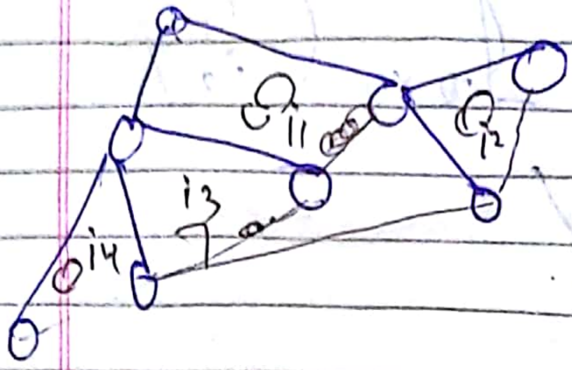
→ dimension is number of cycles → Same dimension fn.

$|E| - |V| + 1$

if connected

disconnected

Σ connected cycles.



4 cycles, 4 voltage sources

Just write voltage source of each loop of
$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

Z_{11} = net resistance of current 1,

Z_{12} = ~~ip~~ Impedance involved b/w i_1 & i_2 (i_2 dependance)

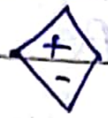
Z_{13} 0 ...

+ve sign if same direc.
-ve ~~sig~~ if opp. direc.

diagonal terms = ~~ab~~ impedance of whole loop corresponding to loop of i_k

Z_{kk}

* Add a current controlled voltage source ki_2

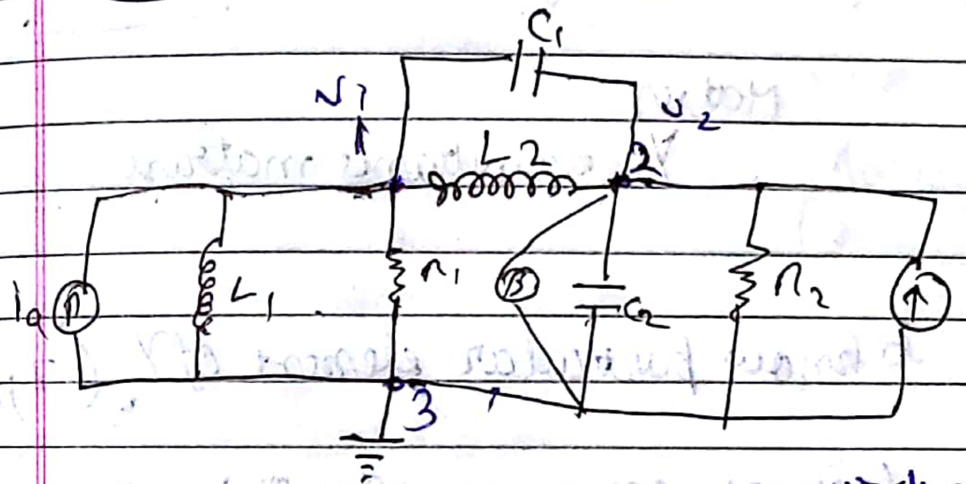


at α

Then
$$\begin{pmatrix} v_1 \\ - \end{pmatrix} = \left(\begin{array}{c|c} -\ - & -\ - \\ & -k \\ -\ - & -\ - \end{array}\right)\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

· Purpose to shift contents to V column
so put kI_2 on right (shifted by a constant)

Ex-12



node analysis = lower case eqs for currents.

$$i_a = \left(\frac{1}{R_1} + \frac{1}{L_1}\int + \frac{1}{L_2}\int + C_1\frac{d}{dt}\right)v_1 - \left(\frac{1}{L_2}\int + C_1\frac{d}{dt}\right)v_2$$

Sign decided by seeing if current enters node or leaves.

$$i_2 = \left(\frac{v_2}{R_2} + C_2\frac{d}{dt} + \frac{1}{L_2}\int + C_1\frac{d}{dt}\right)v_2 - \left(\frac{1}{L_2}\int + C_1\frac{d}{dt}\right)v_1$$

$$\begin{pmatrix} i_a \\ i_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{R_1} + \left(\frac{1}{L_1}+\frac{1}{L_2}\right)\int + C_1\frac{d}{dt} & -\frac{1}{L_2}\int - C_1\frac{d}{dt} \\ -\frac{1}{L_2}\int - C_1\frac{d}{dt} & \frac{1}{R_2} + (C_1+C_2)\frac{d}{dt} + \frac{1}{L_2}\int \end{pmatrix}$$

Matrix

Y = admittance matrix

To know particular element of Y (z_{jk})

look at common edge of j & k loops

2 adjust sign of currents.

Add control source.

$\boxed{\uparrow}$ kv_1 at B

Add $-k$ in y_{21}

↓

current going out from node 2 w.r.t. i_L

V_1

V_2

$$Z_{jk} = \frac{V_j}{i_k}$$

$i_k = 0$

$j \neq k$

send current through jth loop & keep all other loops = open

voltage around source not across (open circuit)

Q.

Duality

Change V to I & vice versa

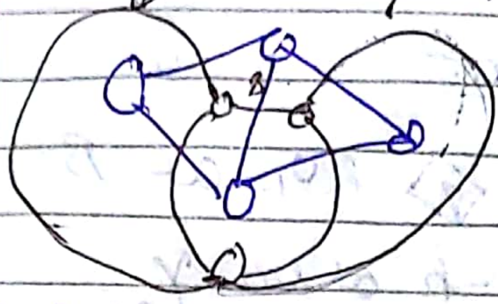
↓

equivalent circuit obtained.

$R \to 1/R$

$L \to C$

$C \to L$



Every loop = node

Put one node outside

Ensure each edge is cut once

When 2 regions share an edge give an edge (eg β)

$S : \mathbb{R}^N \to \mathbb{R}^E$

nodes

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & & n_1 & n_2 & n_3 & \cdots & \end{array} \\
M_S = \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_m \end{array} & \left[\begin{array}{ccccccc} 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array}\right]_{|E| \times |N|}
\end{array}
$$

edges

+1 for source
-1 for target
0 no connection

(Same as long ago)

means e_1 starts at n_2, ends at n_3, rest nodes = 0

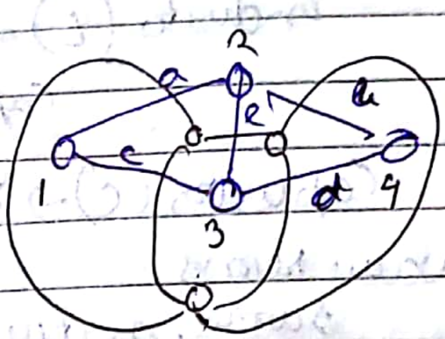
M_S^T (nodes of edges & edges of node)

corresponds to dual graph.

Duality → b/w nodes & cycles / loops, not nodes & edges.

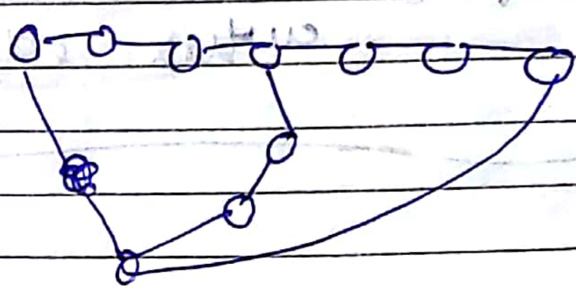
Def: Dual (G) is the graph with incidence matrix $M_S^T{}_G$ where M_{SG} is the incidence matrix of G.

So we get a dual graph:

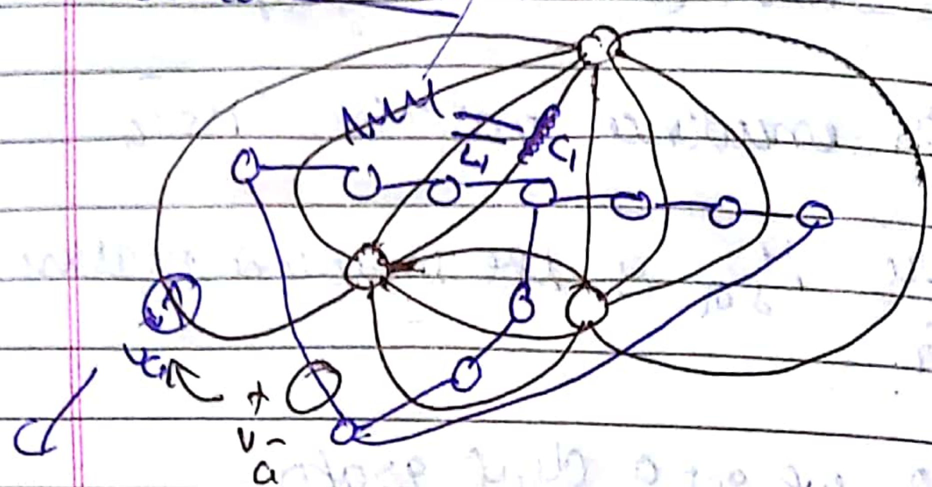


Dual circuit of Ex-10 :-

Graph →



Study carefully



To draw dual:

Va voltage source → current source short

Inductor (L) → Capacitor (C L)

Capacitor (C) → Inductor (C)

value remains same, nature changes

To see where to put

L1 || → C1 and 1/R1 ...

If Dual cuts ~~the~~ edge b/w α, β,

~~the~~ device b/w α, β = on cutting edge of dual

New lecture

Last time

write network equations in matrix form.

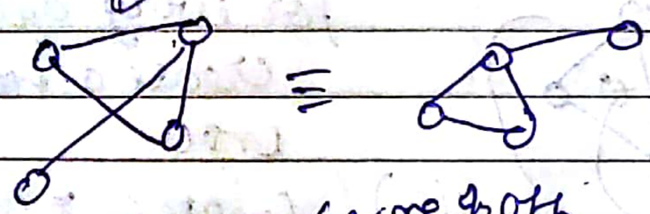
$I = YV$

$V = ZI$

-> "Duality" to be continued.

Planar graph

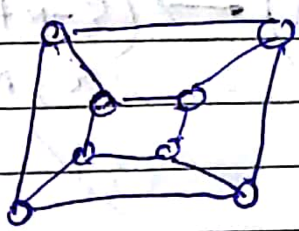
Can be drawn on 2d surface, edges shouldn't cross.

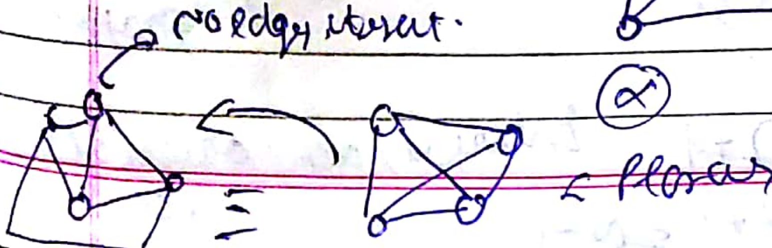


= Planar

(same graph)

Cube also planar.





= Planar.

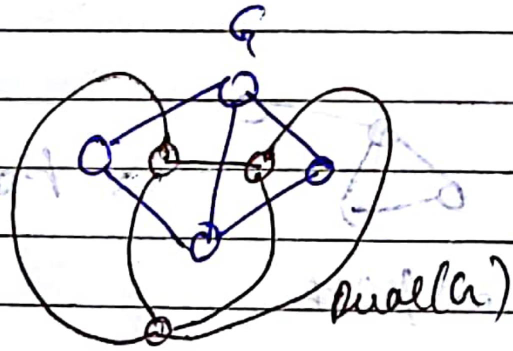
4 nodes ~~C~~ planar always (~~C~~ = new edge is still planar)

Pentagon: connect all nodes, can't be drawn planar.
(5 nodes)

In 3d no such concept exists, ∵ in need of lines, more space exists.

<u>Planar drawing of a planar graph</u>

drawing in which no lines cut.



for every cycle create a node inside.
Create a node outside.

$(N, \mathcal{E}, \mathcal{E}-N+1) \to (\mathcal{E}-N+2, \mathcal{E}, N-1)$

(you will get same number of edges of original graph)

Intersect each original edge

cycles = ~~$(\mathcal{E}-N+2)-(\mathcal{E})+1$~~

$\mathcal{E}-(\mathcal{E}-N+2)+1 = N-1$

now again form a dual,

$(\mathcal{E}-N+2,\ \mathcal{E},\ N-1) \longrightarrow (\mathcal{E}-(\mathcal{E}-N+2)+2,\ E,\ \mathcal{E}-(N-1))$

$(\mathcal{E}-N+2,\ \mathcal{E},\ N-1) \rightarrow (N,\ \mathcal{E},\ \mathcal{E}-N+1)$

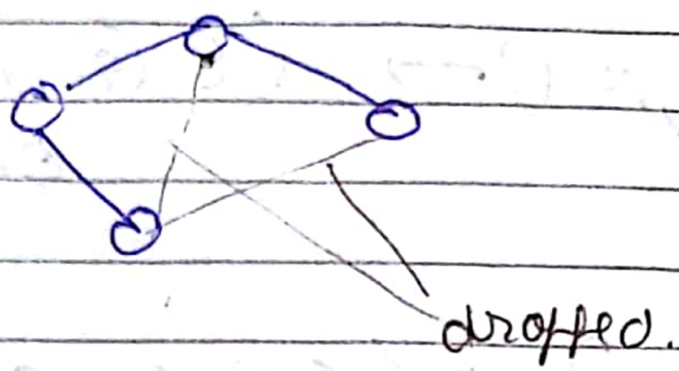
Dual is unique. Original graph obtained.

Planar networks → whose underlying graph = planar (network can be drawn planar)

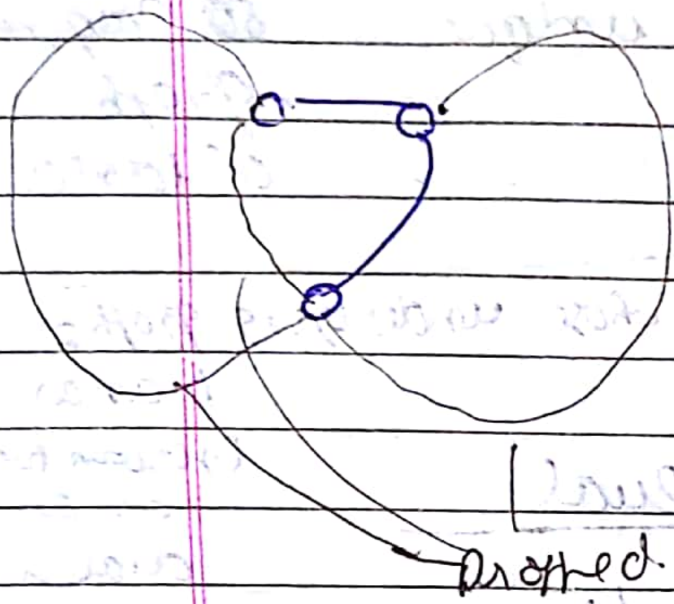
Original	Dual
$R\ \Omega$	$1/R\ \Omega$
L Henry (H)	L Farads (F)
C F	C H
$v(t)$ V	$v(t)$ A
$i(t)$ A	$i(t)$ V
NVA	LVA
Loop variable analysis (LVA)	Node variable analysis (NVA)

In dual, every cycle has become a node.
So

Spanning Tree of G - (ST(G))

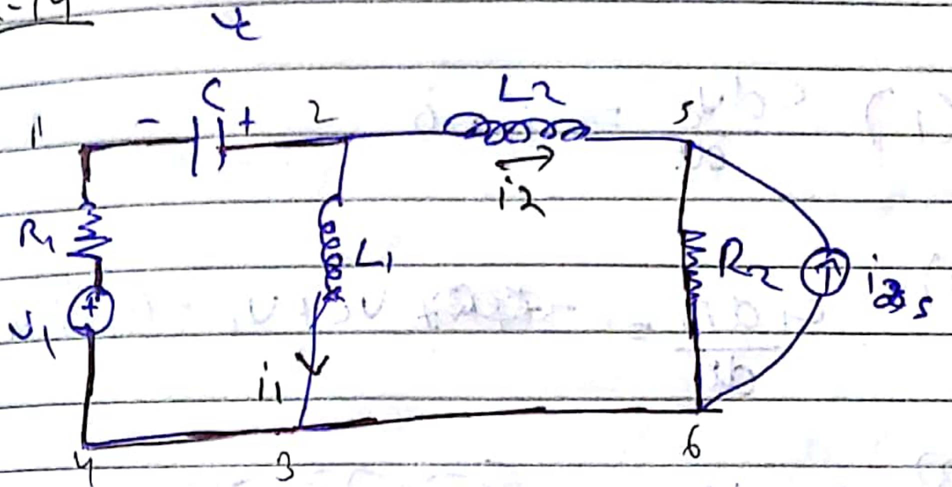


ST(Dual G) = corresponding edges in dual that were dropped in G



State space equations

Ex-14



$i = C\frac{dv}{dt}$ (cap.)

$v = L\frac{di}{dt}$ (Inductor)

Node variable: better to have a cap.

Loop variable: better to have an inductor.

Take a spanning tree; that includes all caps & avoids all inductors in NVA

vice versa in LVA

would avoid integration.

S.T(o), eg 14:
Q. (shown in class)

(1) $C\frac{dv_c}{dt} = -i_1 - i_2$

(2) $L_1\frac{di_1}{dt} = \cancel{}\ v_c + v_1 - R_1(i_1 + i_2)$

(3) $L_2\frac{di_2}{dt} = \cancel{-(R_1+R_2)i_2 + v_c - R_2 i_s}$
$v_1 + v_c - R_1(i_1 + i_2) - i_s R_2 - R_2 i_2$

1234 12~~3~~5634

(you have branch without L or C)

$$\frac{d}{dt}\begin{bmatrix} v_c \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{3\times3} \begin{bmatrix} v_c \\ i_1 \\ i_2 \end{bmatrix}$$

State variables

functions of L_1, L_2, C_1, R_1, R_2

$$+ \begin{bmatrix} 0 \\ v_1/L_1 \\ \frac{v_1}{L_2} - \frac{i_s R_2}{L_2} \end{bmatrix} \rightarrow u$$

State of capacitor = charge/voltage across it

Similarly State (inductor) = current/magnetic flux

$\rightarrow \vec{i}$

$$\begin{bmatrix} 0 & 0 \\ 1/L_1 & 0 \\ 1/L_2 & -\frac{R_2}{L_2} \end{bmatrix}_{3\times 2} \begin{bmatrix} v_1 \\ i_s \end{bmatrix}_{2\times 1} \rightarrow \text{source.}$$

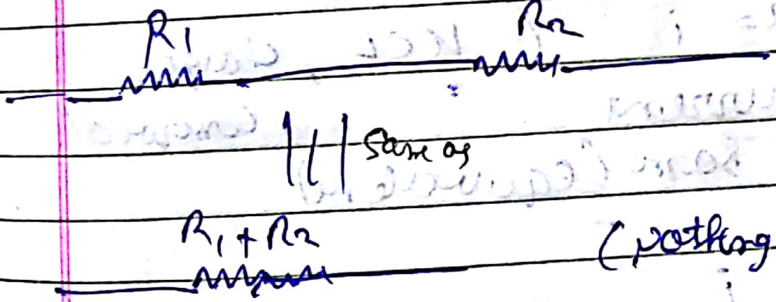
$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{u}$$

(Can always write it this way)

constant w.r.t. variables in $\vec{x}$ (the i_1, i_2, v_c)

could be a fn. of time.

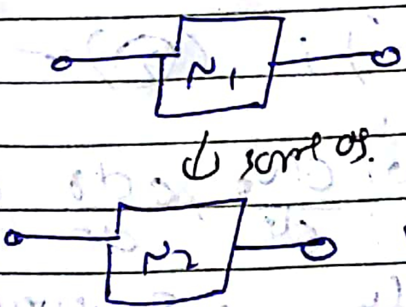
Equivalence



same as

(nothing changes)

Same for networks also



$v_1, i_1, \frac{dv_1}{dt}, \frac{di_1}{dt}$

same as

$v_2, i_2, \frac{dv_2}{dt}, \frac{di_2}{dt}$

To prove: $\overset{R_1}{-\!\!\!\wedge\!\!\wedge\!\!\wedge\!\!-}\overset{R_2}{\wedge\!\!\wedge\!\!\wedge}$

is equivalent to

$\underset{R_1+R_2}{-\!\!\wedge\!\!\wedge\!\!\wedge\!\!-}$

If we say characteristics always = same.

$$v_1, i_1, \frac{dv_1}{dt}, \frac{di_1}{dt}$$

$$v_2, i_2, \frac{dv_2}{dt}, \frac{di_2}{dt} \quad (1)$$

$$v, i, \frac{dv}{dt}, \frac{di}{dt} \quad (2)$$

$i_1 = i_2 = i$ (KCL, charge = conserved)

$\therefore$ Current = same (equivalence)

$v_1 = R_1 i$

$v_2 = R_2 i$

$v = v_1 + v_2 = (R_1 + R_2) i \quad (1)$

$v = (R_1 + R_2) i \quad (2)$ Equivalent

Similarly show $\frac{dv}{dt}$, & $\frac{di}{dt}$ also

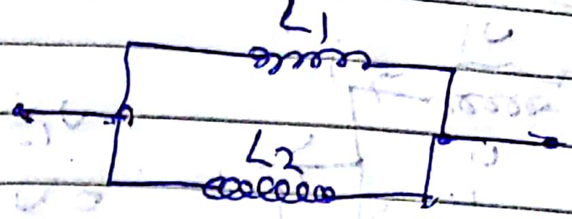
we have equivalently. $\therefore$ Equivalence proved.

Network's equivalence proved by same logic

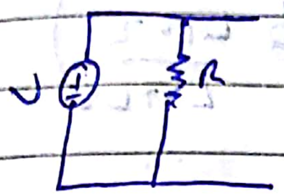
Figure 3.6

Not open circuits

(1)

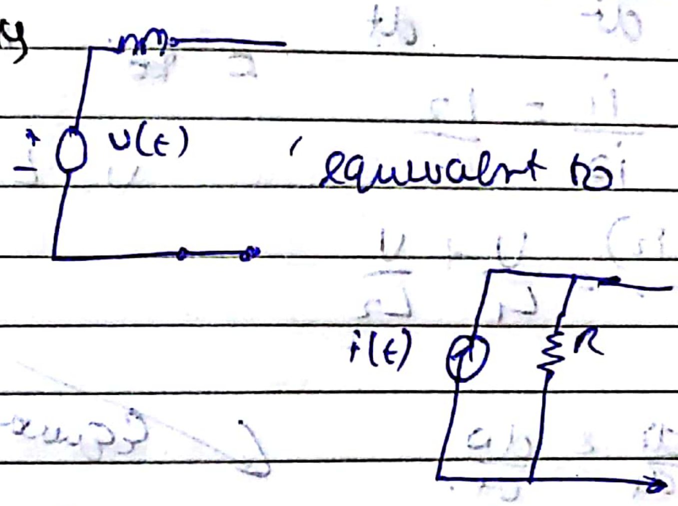


(2)



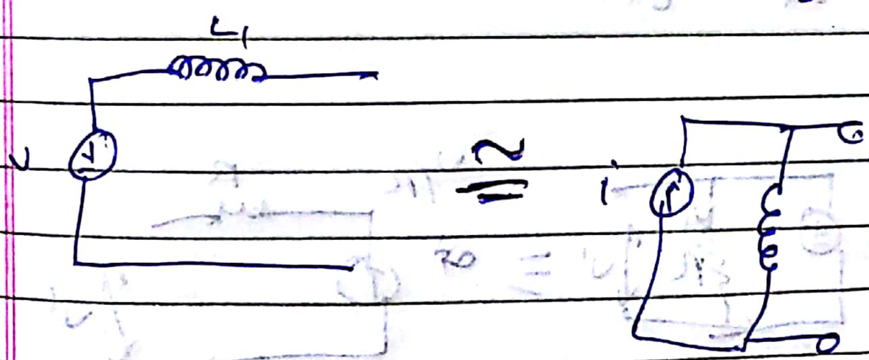
(3)

(4) when Rs

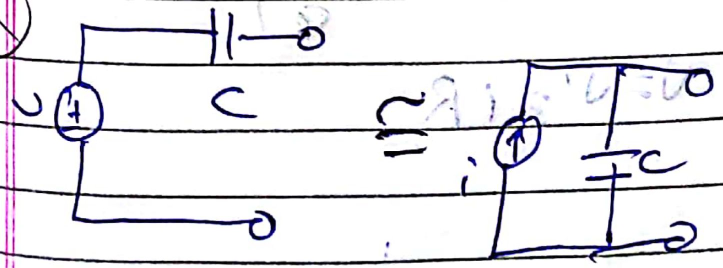


(5)

Rovls



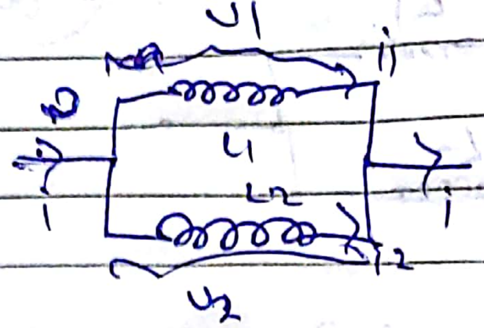
(6)



Answer

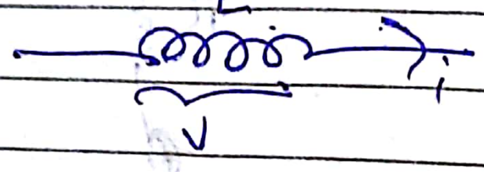
(1)

(a)



$v_1 = v_2 = v$ connected ||

(b)



$\left(\frac{L_1 L_2}{L_1+L_2}\right)$

(a): $L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} = v$

$\therefore \frac{i_1}{i_2} = \frac{L_2}{L_1}$

$\frac{d(i_1+i_2)}{dt} = \frac{v}{L_1} + \frac{v}{L_2}$

(b) $\frac{L_1L_2}{L_1+L_2}$

$v = L\frac{di}{dt} = \frac{d(i_1+i_2)}{dt}$

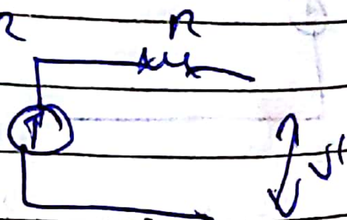
Equivalent property

Similarly $\frac{di}{dt}$ & $\frac{dv}{dt}$.

(2)

(a)

$= v/R$



(b)

(a) $v = v' = iR$