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# Complex Networks 2

Assignment 2

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## Problem 1

**What does it mean for a network to be typical in a model?**

For a network to be *typical* in a model, we assume it is meant to be $(f, \epsilon)$-typical for some chosen graph-distance metric, $d : G^2 \to \mathbb{R}^2$, with threshold, $\bar{d} < \epsilon$, and sufficient statistics in the model, $\{f\}$, which are fixed and concentrated.

**Describe typical networks in:**

1. Canonical $G(n, p)$

   The canonical $G(n, p)$ model is that which fixes an average degree $\bar{k} = np$. Thus, all graphs with average degree $\bar{k}$ are typical in the canonical $G_{np}$ model.

2. Microcanonical $G(n, p)$

   All graphs with $m = n\bar{k} = p\binom{n}{2}$ edges are typical in the microcanonical $G_{np}$ model.

3. Barabási-Albert Model $BA(n, m)$

   There are no known rigorous notions of typicality in the BA Model `\(°O°)/`

4. The "simplest" model of scale-free networks

   Because sufficient statistics for this model are unclear, there are no rigorous notions of typicality in this model either.

5. microcanonical configuration model $CM(\{k_1, k_2, ..., k_n\})$

   All labeled graphs with degree sequence $\{k_1, k_2, ..., k_n\}$ are typical in the model.

6. soft (canonical) configuration model $SCM(\{\bar{k}_1, \bar{k}_2, ..., \bar{k}_n\})$

   All labeled graphs with expected degrees $\{\bar{k}_1, \bar{k}_2, ..., \bar{k}_n\}$ are typical in the model.

## Problem 2

**(a)** To prove that $P_f(G) = P^*(G)$, where $P^*(G)$ is the ensemble which maximizes ensemble entropy $S[P] = \sum_{G\in\mathbb{G}} P(G) \log P(G)$, we treat this as an optimization problem. The function to optimized being the entropy $S[P]$, with constraints coming from the normalization and definition of $\langle f_s(G)\rangle$ with Lagrange multipliers $\beta$ and $\lambda$, respectively. To do this we maximize the functional

$$\mathcal{L} = -\sum_{\mathbb{G}} P(G) \log P(G) + \beta[1 - \sum_{\mathbb{G}} P(G)] + \sum_{s} \lambda_s[\sum_{\mathbb{G}} f_s(G)P(G) - \bar{f}_G] \tag{1}$$

We find the optima along $\frac{\partial \mathcal{L}}{\partial P(G_i)} = 0$ where $G \in \mathbb{G}$ is a single graph instance to be

$$P^*(G) = e^{-(1-\beta) - \sum_s \lambda_s f_s(G)} \tag{2}$$

The optima along the level set $\frac{\partial \mathcal{L}}{\partial \beta} = 0$ are then found defined as

$$\sum_{\mathbb{G}} P^*(G) = 1 \tag{3}$$

Inserting Eq. 2 into Eq. 3, we find arrive at the condition that

$$\sum_{\mathbb{G}} e^{-(1-\beta)-\sum_s \lambda_s f_s(G)} = 1 \Longrightarrow e^{-(1-\beta)} = \frac{1}{e^{-\sum_s \lambda_s f_s(G)}} = Z_{\bar{f}}^{-1} \tag{4}$$

Finally, we can take the leftmost equivalence in 4 and plug it into 2 to get

$$P^*(G) = \frac{1}{e^{-\sum_s \lambda_s f_s(G)}} \cdot e^{-\sum_s \lambda_s f_s(G)} \tag{5}$$

□

**(b)** Using the definition of $Z_{\bar{f}}$ in the problem, and as concurrently derived in 4, we show that $\langle f_s(G)\rangle = -\frac{\partial \ln Z}{\partial \lambda_s}$ where $\langle \cdot \rangle$ is an average.

To begin, we use the chain rule as

$$\frac{\partial \ln Z}{\partial \lambda_s} = \frac{1}{Z}\frac{\partial Z}{\partial \lambda_s} \tag{6}$$

Opening up the summation over $\mathbb{G}$, and using the chain rule again we see that

$$\frac{\partial Z}{\partial \lambda_s} = e^{-\sum_s \lambda_s f_s(G_1)} \cdot \frac{\partial}{\partial \lambda_s}(-\sum_s \lambda_s f_s(G_1)) + ... + e^{-\sum_s \lambda_s f_s(G_k)} \cdot \frac{\partial}{\partial \lambda_s}(-\sum_s \lambda_s f_s(G_k)) \tag{7}$$

where $k = ||\mathbb{G}||$. The derivative which remains simply evaluates to $\frac{\partial}{\partial \lambda_s}(-\sum_s \lambda_s f_s(G)) = -\sum_s f_s(G)$ and thus we're left with

$$\langle f_s(G)\rangle = \frac{\partial \ln Z}{\partial \lambda_s} = -\frac{1}{Z}(-\sum_{\mathbb{G}} f_s(G) e^{-\sum_s \lambda_s f_s(G)}) \tag{8}$$

The sum over $s$ is dropped because the $\partial/\partial\lambda_s$ drops all terms, $i$ in the sum where $i \neq s$. Finally, bringing $1/Z$ into the sum, we get

$$-\sum_{\mathbb{G}} \frac{1}{Z} e^{-\sum_s \lambda_s f_s(G)} f_s(G) = -\sum_{\mathbb{G}} P(G) f_s(G) \tag{9}$$

and thus,

$$\langle f_s(G)\rangle = -\frac{\partial \ln Z}{\partial \lambda_s} = \sum_{\mathbb{G}} P(G) f_s(G) \tag{10}$$

□

## Problem 3

We begin by sampling from the Pareto Distribution described on the Homework. Using the Numpy python package, a bit of algebra shows that the shape parameter in their documentation, $a$, is equal to $\gamma - 1$. From there, it follows that what they call $m$ is what we call $x_-$.

**(a)** See code attached.

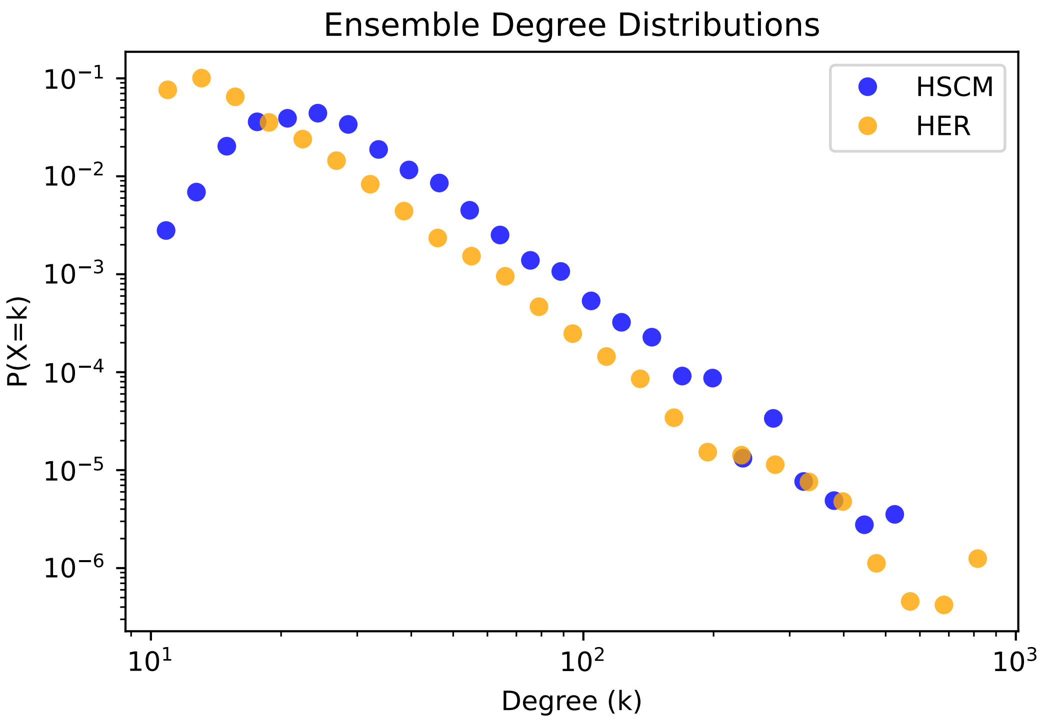


Figure 1: Degree distributions from the Hypersoft Configuration Model (HSCM), sampling all nodes from all $N_{HSCM} = 10$ graphs sampled from the model; and the Hypercanonical Erdoš-RenyiM̀odel (HER), sampling all nodes from all $N_{HER} = 10^4$ graphs sampled from the model. Both distributions were log-binned to normalize to 1.

**(b, c)** As shown in 1, the log-binned distributions of the two *ensembles'* degree distributions look very similar.

Both appear to show a clear power-law degree distribution with similar tail exponent. This is to be expected because by considering the degree distribution of all graphs in the ensemble, we are effectively sampling the same Pareto Distribution which defines $p_{\bar{k},\gamma}$. Thus, because the $p$-values for the *ensemble* of HER graphs came from the same Pareto Density as the $p$-values for *each edge in each graph* for the HSCM, it makes sense that the distributions align very well.

However, although $p$ in a single HER model comes from the same Pareto Density, it is fixed. Thus, as seen in 2, the degree distribution of any *single* ER graph in the HER model is very different from that of any single graph in the HSCM model because in the HSCM graph, *every edge probability* is pulled from this Pareto Distribution, while in the ER graph, although $p$ may be pulled from the Pareto, each edge is still coming from a Bernoulli Distribution with bias $p$. As such, instead of a power law distribution, we see a Binomial degree distribution for the sampled ER graph.

## 0.1 Appendix: Code

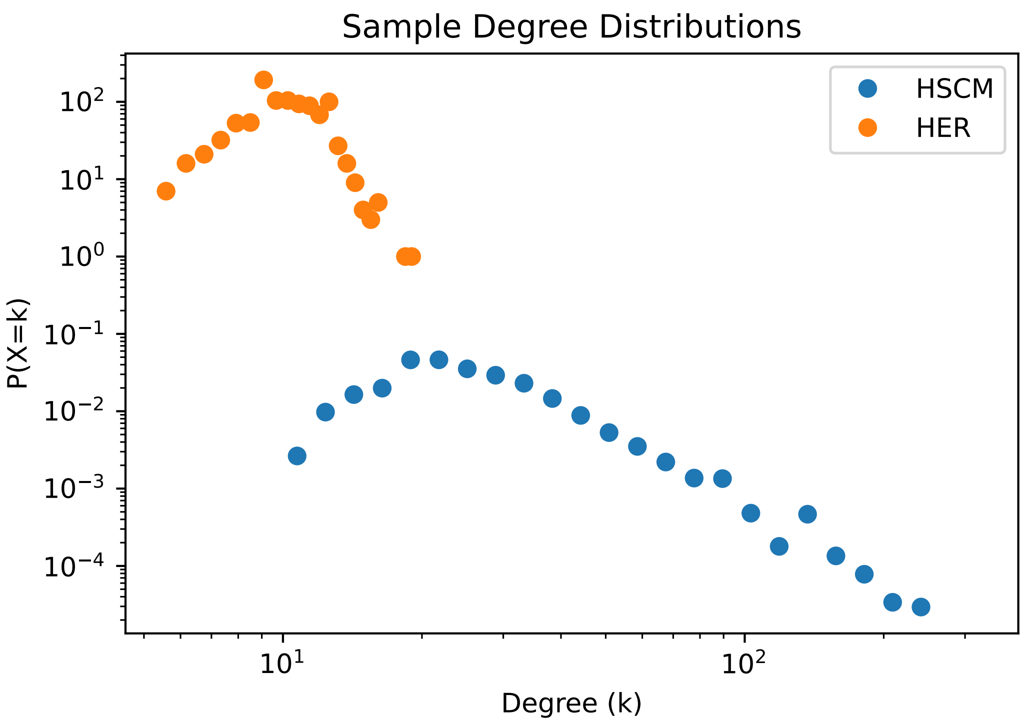


Figure 2: Degree distribution of a single, random sampled graph from both the HSCM and HER models. The HSCM distribution was log-binned, while the HER distribution was linearly binned. In an attempt to maintain the same number of points for both, the bin width of the HER graph renders probabilities greater than 1.