Lecture 11: Cauchy condensation Test, Ratio test, Root Test, Liebnitz Test

August 28, 2019

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Theorem 11.1 (Cauchy's Condensation Test) *Let (a_k) be a monotonically decreasing sequence of nonnegative real numbers. Then the series $\sum_k a_k$ is convergent if and only if the series $\sum_k 2^k a_{2^k}$ is convergent.*

Example 11.2 *Deduce the convergence and divergence of $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where $p > 0$.*

Solution: Note that if $p > 0$, $k(\ln k)^p$ increases as k increasing and hence $a_k = \frac{1}{k(\ln k)^p}$ is a decreasing sequence. Note that

$$2^k a_{2^k} = \frac{2^k}{2^k(\ln 2^k)^p} = \frac{1}{k^p \ln 2}.$$

We know that $\sum_{k=2} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$. Hence by Cauchy's condensation test, $\sum_{k=2} 2^k a_{2^k} = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$. ■

Theorem 11.3 (Ratio Test) *if $a_k \neq 0$ for all k and*

$$\frac{|a_{k+1}|}{|a_k|} \to l, \quad \text{as } k \to \infty \quad \text{where } 0 \leq l \leq \infty.$$

Then we have

1. *If $l < 1$, $\sum_k a_k$ is absolutely convergent*
2. *If $l > 1$, $\sum_k a_k$ is divergent.*

Test fails if $l = 1$.

Example 11.4 *Let $a_k := \dfrac{k^2}{2^k}$ for all $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, we have*

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)^2}{2^{k+1}} \times \frac{2^k}{k^2} = \frac{1}{2}\left(1+\frac{1}{k}\right)^2 \implies \frac{|a_{k+1}|}{|a_k|} \to \frac{1}{2} \quad \text{as } k \to \infty$$

So Ratio Test $\sum_k a_k$ is absolutely convergent.

Example 11.5 *For the series $\sum_{n=1}^{\infty} \frac{1}{n}$,*

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{n+1} \times \frac{n}{1} = \frac{1}{\left(1+\frac{1}{n}\right)} \implies \frac{|a_{n+1}|}{|a_n|} \to 1 \quad \text{as } n \to \infty$$

So ratio test fails but we know that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{(n+1)^2} \times \frac{n^2}{1} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \implies \frac{|a_{n+1}|}{|a_n|} \to 1 \quad \text{as } n \to \infty$$

So ratio test fails but we know that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Example 11.6 *Let (a_k) be a real sequence such that (a_{2k}) and (a_{2k+1}) converges to the same limit, say $l \in \mathbb{R}$. Then Show that (a_k) converges to l.*

Solution: Let $\epsilon > 0$ be given. Then there exist positive integers k_1 and k_2 such that

$$\begin{aligned} k \geq k_1 &\implies |a_{2k} - l| < \epsilon \\ k \geq k_2 &\implies |a_{2k+1} - l| < \epsilon. \end{aligned}$$

Set $n_0 := \max\{2k_1, 2k_2+1\}$. Then

$$n \geq n_0 \implies |a_n - l| < \epsilon.$$

(If $n \geq n_0$ is odd integer, then $n = 2k+1$ for some k. Also $n \geq n_0 \implies 2k+1 \geq 2k_2+1 \implies k \geq k_2$) ∎

Theorem 11.7 (Root Test) *If $|a_k|^{\frac{1}{k}} \to l$ as $k \to \infty$, where $0 \le l \le \infty$. Then*

1. *If $l < 1$, then $\sum_k a_k$ is absolutely convergent*
2. *If $l > 1$, then $\sum_k a_k$ is divergent.*

If $l = 1$ test fails.

Example 11.8 *Discuss the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$ where*

$$a_n := \begin{cases} \dfrac{n}{2^n}, & n \text{ is odd} \\ \dfrac{1}{2^n}, & n \text{ is even} \end{cases}$$

Solution: Note that

$$\frac{a_{n+1}}{a_n} := \begin{cases} \dfrac{1}{2n}, & n \text{ is odd} \\ \dfrac{n+1}{2}, & n \text{ is even} \end{cases}$$

Hence $\frac{a_{n+1}}{a_n}$ does not converge. Therefore ratio test is not applicable. However,

$$a_n^{\frac{1}{n}} := \begin{cases} \dfrac{n^{\frac{1}{n}}}{2}, & n \text{ is odd} \\ \dfrac{1}{2}, & n \text{ is even} \end{cases}$$

Now $a_{2n+1}^{\frac{1}{2n+1}} = \frac{(2n+1)^{\frac{1}{2n+1}}}{2} \to \frac{1}{2}$ and $a_{2n}^{\frac{1}{2n}} = \frac{1}{2} \to \frac{1}{2}$, hence by Example 11.6, we get $a_n^{\frac{1}{n}} \to \frac{1}{2} < 1$. Hence by root test the given series converges. ∎

Remark 11.9 *Both the Root Test and the Ratio Test deduce absolute convergence of a series by comparing it with the geometric series. The Ratio Test is often simpler to use than the Root Test because it is easier to calculate ratios than roots. But the Root Test has a wider applicability than the Ratio Test in the following sense. Whenever the Ratio Test gives (absolute) convergence of a series, so does the Root Test, and moreover, the Root Test can yield (absolute) convergence of a series for which the Ratio Test is inconclusive.*

Definition 11.10 *A convergent series that is not absolutely convergent is said to be conditionally convergent.*

Following is the tests for Conditional Convergence.

Theorem 11.11 (Leibniz Test) *Let (a_k) be a monotonic sequence (either decreasing or increasing) such that $a_k \to 0$ as $k \to \infty$. Then $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ is convergent.*

Example 11.12 *Consider series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^p}$, $p > 0$. Since $(\frac{1}{k^p})$ is decreasing and converge to 0 so by Leibniz Test series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^p}$ converges. Note that series does not converges absolutely for $0 < p \leq 1$.*