Lecture 23: Fundamental Theorem of Calculus & Improper Integral

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23.1 Fundamental Theorem of Calculus

Differentiation is a local process, that is, the value of the derivative at a point depends only on the values of the function in a small interval about that point. On the other hand, integration is a global process in the sense that the integral of a function depends on the values of the function on the entire interval. Further, these processes are defined in entirely different manners without any apparent connection between them. Indeed, from a geometric point of view, differentiation corresponds to finding (slopes of) tangents to curves, while integration corresponds to finding areas under curves. At first glance, there seems to be no reason for these two geometric processes to be intimately related.

The Fundamental Theorem of Calculus or, for short, the FTC, says that the differentiation and integration are inverse to each other, i.e., if one first integrates a function and then differentiates it, one gets back the original function. Also, if one differentiates a function on an interval and then integrates it, again one gets back the original function.

If $f : [a, b] \to \mathbb{R}$ is integrable, then we obtain a new function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for } x \in [a, b].$$

Indeed, f is integrable on $[a, x]$ for every $x \in [a, b]$, and $F(a) = 0$. Hence the function F is well defined on $[a, b]$.

Proposition 23.1 *Let $f : [a, b] \to \mathbb{R}$ be integrable and $F : [a, b] \to \mathbb{R}$ be defined by*

$$F(x) := \int_a^x f(t)dt \quad \textit{for } x \in [a, b].$$

Then (a) F is continuous on $[a, b]$. (b) If f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$. In particular, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $f(x) = F'(x)$ for all $x \in [a, b]$.

Remark 23.2 *The above proposition says that although an integrable function f may be discontinuous on $[a,b]$, the function $F : [a,b] \to \mathbb{R}$ obtained by integrating f from a to $x \in [a,b]$ is continuous on $[a,b]$. Thus, integration is a smoothing process, unlike the process of differentiation (derivative may not be even continuous).*

Example 23.3 *(i) Let $f(x) := [x]$ for $x \in [-1,1]$. Define $F(x) := \int_{-1}^{x}[t]dt$ for $x \in [-1,1]$. Check that $F(x) = -1 - x$ if $x \in [-1,0]$, and $F(x) = F(0) = -1$ if $x \in (0,1]$. Note f is not continuous on $[-1,1]$, but F is continuous on $[-1,1]$.*

(ii) Let $f(x) := |x|$ for $x \in [-1,1]$. Define $F(x) := \int_{-1}^{x}|t|dt$ for $x \in [-1,1]$. Check that $F(x) = \int_{-1}^{x}(-t)dt = (1-x^2)/2$ if $x \in [-1,0]$, and $F(x) = F(0) + \int_0^t t dt = (1+x^2)/2$ if $x \in (0,1]$. Note: f is not differentiable on $[-1,1]$, but F is differentiable; in fact, $F'(x) = |x| = f(x)$ for $x \in [-1,1]$

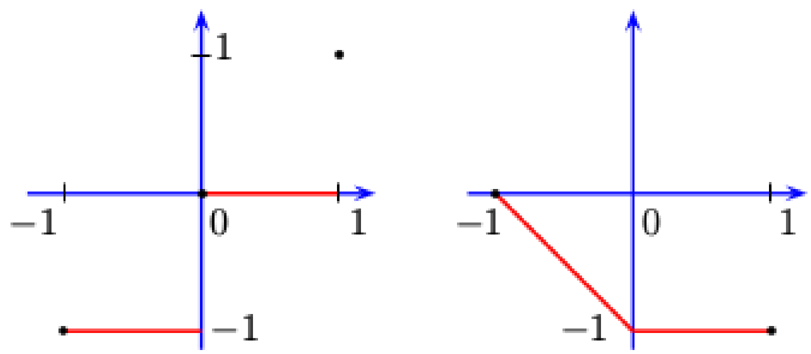

Figure: $y = [x]$ and $y = \int_{-1}^{x}[t]dt$

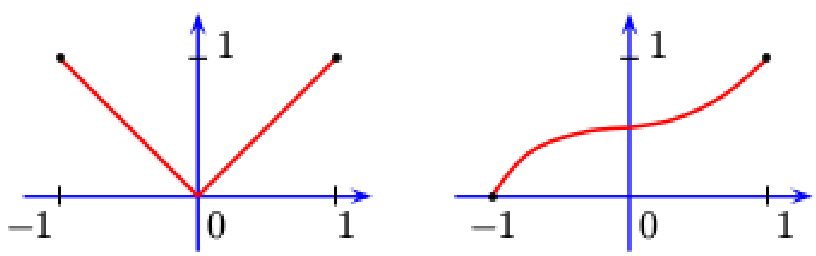

Figure: $y = |x|$ and $y = \int_{-1}^{x}|t|dt$

Definition 23.4 *Let I be an interval containing more than one point and $f : I \to \mathbb{R}$ be any function. We say that f has an antiderivative on I if there is a differentiable function*

$F : I \to \mathbb{R}$ such that $f = F'$. Such a function F is called an antiderivative of f. It is clear that if F is an antiderivative of f, then $F + c$ for any constant $c \in \mathbb{R}$ is also an antiderivative of f, that is antidervative is unique up to addition of a constant.

Example 23.5 *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then by part (b) of Proposition 23.1, the function $F : [a, b] \to \mathbb{R}$ defined as*

$$F(x) := \int_a^x f(t)dt \quad \text{for } x \in [a, b].$$

is an antiderivative of f.

Theorem 23.6 (Fundamental Theorem of Calculus) *Let $f : [a, b] \to \mathbb{R}$ be differentiable and f' be integrable on $[a, b]$, then*

$$\int_a^b f'(t)dt = f(b) - f(a)$$

If an integrable function $f : [a, b] \to \mathbb{R}$ has an antiderivative F , then F is called an indefinite integral of f , and it is denoted by $\int f(x)dx$. Note, however, that this notation is somewhat ambiguous, since an indefinite integral of f is unique only up to an additive constant. For this reason, one writes

$$\int f(x)dx = F(x) + C,$$

where C denotes an arbitrary constant.

23.2 Improper Integral

We have considered the Riemann integral of a bounded function defined on a closed and bounded interval. Now, we shall extend the process of integration to functions defined on a semi-infinite interval or a doubly infinite interval, and also to unbounded functions defined on bounded or unbounded intervals.

We begin by considering bounded functions defined on a semi-infinite interval of the form $[a, \infty)$, where $a \in \mathbb{R}$.

Definition 23.7 *Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, x]$ for every $x \geq a$. Then an integral of the form $\int_a^\infty f(x)dx$ is called an improper integral of the first kind.*

Definition 23.8 *We say that an improper integral* $\int_a^\infty f(x)dx$ *is convergent if the limit*

$$\lim_{x\to\infty}\int_a^x f(t)dt \quad \textit{exists.}$$

It is clear that if this limit exists, then it is unique, and we may denote it by I. When we write $\int_a^\infty f(x)dx = I$, we mean that I is a real number and the improper integral $\int_a^\infty f(x)dx$ is convergent with I as its value. An improper integral that is not convergent is said to be divergent. In particular, if $\int_a^x f(t)dt \to \infty$ or $\int_a^x f(t)dt \to -\infty$ as $x \to \infty$, then we say that the improper integral diverges to ∞ or to $-\infty$, as the case may be.

Example 23.9 *Let* $p \in \mathbb{R}$ *and* $f : [1,\infty) \to \mathbb{R}$ *be defined by* $f(x) := \frac{1}{x^p}$. *(Since* f *is continuous hence integrable on* $[1, x]$ *for every* $x \geq 1$.*) consider the improper integral* $\int_1^\infty \frac{1}{x^p}dx$. *Given any* $x \in [1,\infty)$, *we have*

$$\int_1^x \frac{1}{t^p}dt = \begin{cases} \dfrac{x^{1-p}-1}{1-p} & \text{if} \quad p \neq 1 \\ \ln x & \text{if} \quad p = 1 \end{cases}$$

It follows that if $p > 1$, *then* $\int_1^\infty \frac{1}{x^p}dx$ *converges to* $\frac{1}{1-p}$, *while if* $p \leq 1$, *then it diverges to* ∞.