Lecture 05: Sequences

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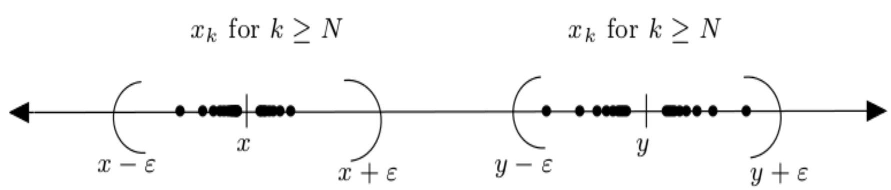
Let us recall the definition of a convergent sequence. We say that a real sequence (a_n) is convergent if

$$\exists a \in \mathbb{R}(\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|a_n - a| < \epsilon))))$$

As far as the definition is considered, we are not demanding the uniqueness of a but at least one such a must exists. That's all our expectation.

An observant reader must have noticed that in the cases when (a_n) is convergent, the moment we guessed a possible limit say a, we stopped looking for other real numbers b such that $a_n \to b$. Why did we do so? Is it possible for a sequence a_n to converge to two distinct real numbers a and b? The following picture must convince you that it is not possible.



In fact, unless we prove the uniqueness of the limit of sequence it is not legitimate to write $\lim_{n\to\infty} a_n = a$, because which a we mean here. So let prove the following proposition.

Proposition 5.1 *A convergent sequence has a unique limit.*

Proof: Suppose $a_n \to a$ as well as $a_n \to b$. Suppose $b \neq a$, Then $\epsilon = \dfrac{|a-b|}{2} > 0$. Since $a_n \to a$, there is $n_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon \ \forall n \geq n_1$, and since $a_n \to b$, there is $n_2 \in \mathbb{N}$ such that $|a_n - b| < \epsilon \ \forall n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a - b| \leq |a - a_{n_0}| + |a_{n_0} - b| < \epsilon + \epsilon = |a - b| \implies |a - b| < |a - b|.$$

which is a contradiction. ■

Definition 5.2 1. *A sequence (x_n) is said to be bounded above if there is $\alpha \in \mathbb{R}$ such that $x_n \leq \alpha$ for all $n \in \mathbb{N}$.*

2. *A sequence (x_n) is said to be bounded below if there is $\beta \in \mathbb{R}$ such that $x_n \geq \beta$ for all $n \in \mathbb{N}$.*

3. *The sequence (x_n) is said to be bounded if it is bounded above as well as bounded below.*

Some examples of sequence are

1. $(-n)$ is bounded above by -1.

2. $a_n = n$ for $n \in \mathbb{N} : 1, 2, 3, \cdots$. Bounded below by 1

3. $a_n = \dfrac{1}{n}$ for $n \in \mathbb{N} : 1, \frac{1}{2}, \frac{1}{3}, \cdots$ bounded by zero from below and by 1 from above.

4. $a_n = (-1)^n$ for $n \in \mathbb{N} : -1, 1, -1, 1, \cdots$, Bounded by -1 from below and by 1 from above.

5. $a_n = 1$ for $n \in \mathbb{N} : 1, 1, \cdots$. This is an example of a constant sequence. Bounded

6. $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3 : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$. This sequence is known as the Fibonacci sequence. Bounded below.

Example 5.3 *Show that a sequence (x_n) is bounded if and only if there is $\gamma \in \mathbb{R}$ such that $|x_n| \leq \gamma$ for all $n \in \mathbb{N}$.*

Solution: Assume sequence (x_n) is bounded, then there exists $\alpha, \beta \in \mathbb{R}$ such that $\beta \leq x_n \leq \alpha$ for all $n \in \mathbb{N}$. Take $\gamma = \max\{|\beta|, |\alpha|\}$. Then $x_n \leq \alpha \leq |\alpha| \leq \gamma$. Using the fact that for any real number x, $x \geq -|x|$, we have $-\gamma \leq -|\beta| \leq \beta \leq x_n$. converse is trivial. ■

The following result gives a necessary condition for the convergence of a sequence.

Theorem 5.4 *If $a_n \to a$ then (a_n) is bounded.*

Proof: Since $a_n \to a$ so for $\epsilon = 1$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq n_0$. Note $|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a| < 1$. This implies $|a_n| < 1 + |a|$ for all $n \geq n_0$. Now take $\gamma = \max\{|a_1|, |a_2|, \cdots, |a_{n_0-1}|, |a| + 1\}$ Then $|a_n| \leq \gamma$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. ■

Remark 5.5

1. *The above condition is necessary for convergence of a sequence, but it is not sufficient. For example $((-1)^n)$ is bounded but not convergent.*

2. *Theorem 5.4 implies that an unbounded sequence is divergent.*

In general, proving the convergence directly from the definition is a difficult task. Now we state some results that are useful in proving convergence or divergence of a variety of sequences.

Theorem 5.6 (Limit Theorem for Sequences) *Let $a_n \to a$ and $b_n \to b$. Then*

1. $a_n \pm b_n \to a \pm b$,
2. $ra_n \to ra$ *for any* $r \in \mathbb{R}$,
3. $a_n b_n \to ab$,
4. *If* $b_n \neq 0, \ \forall\, n$ *and* $b \neq 0$ *then* $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$
5. *If* $a_n \leq b_n$ *for all* $n \geq n_0$ *(where* n_0 *is some fixed positive integer), then* $a \leq b$.
6. **Sandwich-Theorem** *Let* $(a_n), (b_n), (c_n)$ *be sequences and* $c \in \mathbb{R}$ *be such that* $a_n \leq c_n \leq b_n$ *for all* $n \geq n_0$ *(where* n_0 *is some fixed positive integer) and* $a_n \to c$ *as well as* $b_n \to c$. *Then* $c_n \to c$.

Example 5.7 *Let* $a \in \mathbb{R}$, *and* $a_n := a^n$ *for* $n \in \mathbb{N}$. *Then show that* (a_n) *is convergent* $\iff -1 < a \leq 1$.

Solution: Clearly, if $a := 0$, then $a_n \to 0$, and if $a := 1$, then $a_n \to 1$. Also, if $a := -1$, then we have seen that (a_n) is divergent.

Let $0 < |a| < 1$, and $r := 1/|a|$. Then $r > 1$, and so $r = 1 + h$ with $h > 0$. By the binomial theorem, $r^n = (1+h)^n = 1 + nh + \cdots + h^n > nh$ for all $n \in \mathbb{N}$. Hence $0 \leq |a_n| = |a|^n = (1/r^n) \leq (1/nh) \to 0$. Thus $a_n \to 0$.

Let $s := |a| > 1$. Then $s = 1 + h$ with $h > 0$, and for all $n \in \mathbb{N}$, $|a_n| = s^n = 1 + nh + \cdots + h^n > nh$. Hence (a_n) is unbounded, and so it is divergent. ■