Identities related to Gradient

$$\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g, \qquad \boldsymbol{\nabla}\cdot(\mathbf{A}+\mathbf{B}) = (\boldsymbol{\nabla}\cdot\mathbf{A}) + (\boldsymbol{\nabla}\cdot\mathbf{B}),$$

$$\boldsymbol{\nabla}\times(\mathbf{A}+\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A}) + (\boldsymbol{\nabla}\times\mathbf{B}),$$

$$\boldsymbol{\nabla}(kf) = k\boldsymbol{\nabla} f, \qquad \boldsymbol{\nabla}\cdot(k\mathbf{A}) = k(\boldsymbol{\nabla}\cdot\mathbf{A}), \qquad \boldsymbol{\nabla}\times(k\mathbf{A}) = k(\boldsymbol{\nabla}\times\mathbf{A}),$$

fg (product of two scalar functions),
$\mathbf{A}\cdot\mathbf{B}$ (dot product of two vector functions).

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f.$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}.$$

Identities related to Divergence & Curl

$f\mathbf{A}$ (scalar times vector),

$\mathbf{A} \times \mathbf{B}$ (cross product of two vectors)

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

∇T is a *vector* (1) Divergence of gradient: $\nabla \cdot (\nabla T)$

(2) Curl of gradient: $\nabla \times (\nabla T)$.

$\nabla \cdot \mathbf{v}$ is a *scalar*: Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$

$\nabla \times \mathbf{v}$ is a *vector*, Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$

Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

$$\begin{aligned}
\nabla \cdot (\nabla T) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \\
&= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \\
&= \nabla^2 T
\end{aligned}$$

$\nabla^2 \longleftarrow$ Laplacian operator

Laplacian of a *vector*,

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

The curl of a gradient is always *zero*.

$$\boxed{\nabla \times (\nabla T) = 0}$$

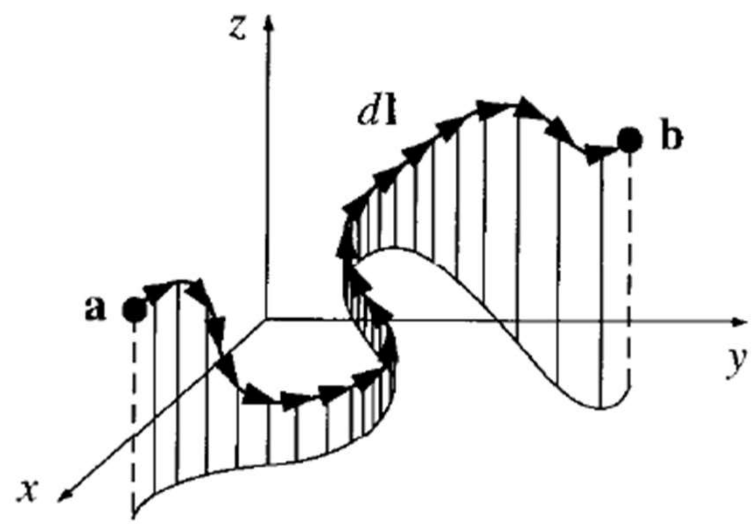
$$\nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

The divergence of a curl, like the curl of a gradient, is *always zero*.

$$\boxed{\nabla \cdot (\nabla \times \mathbf{v}) = 0}$$

Vector Line Integration

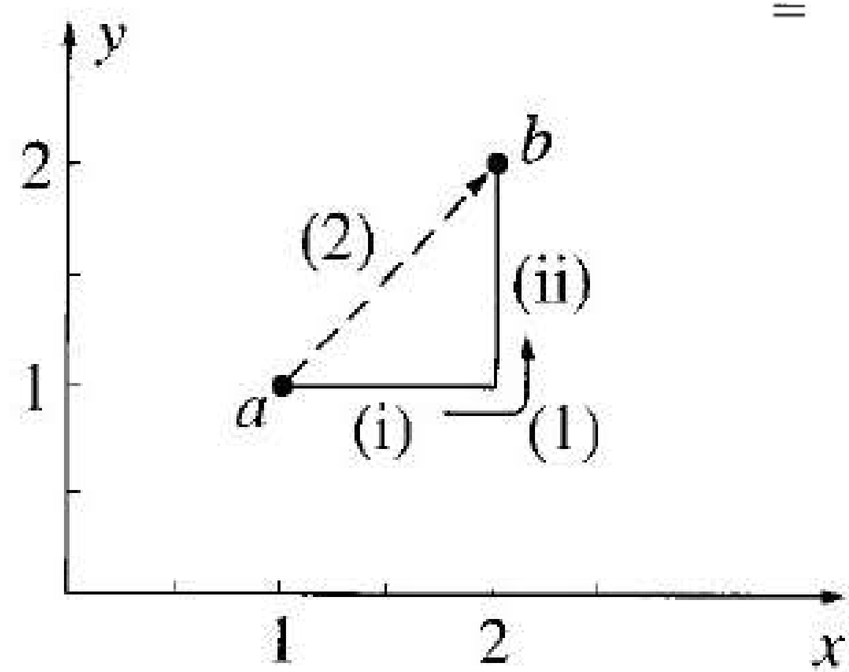


$(i)\quad \vec{V} = x\hat{i} + y\hat{j}$

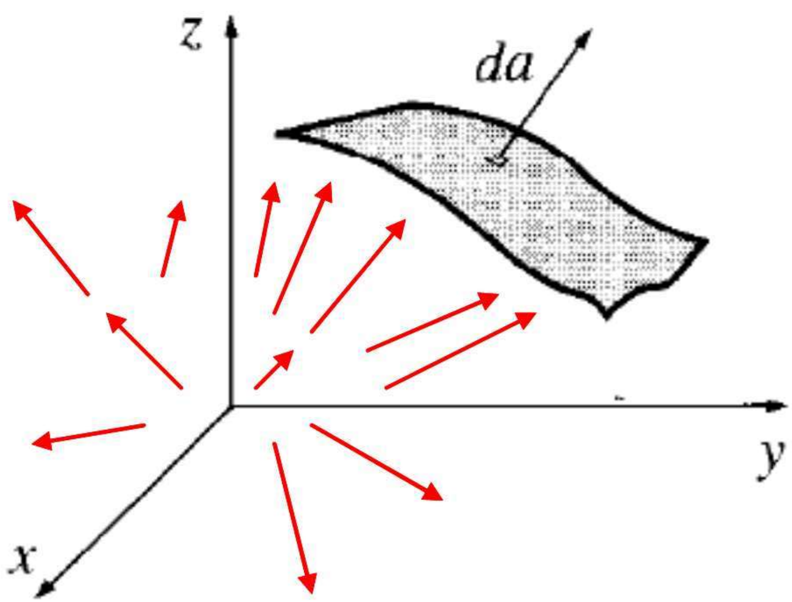
$(ii)\quad \vec{V} = xy^2\hat{i} + yx^2\hat{j}$

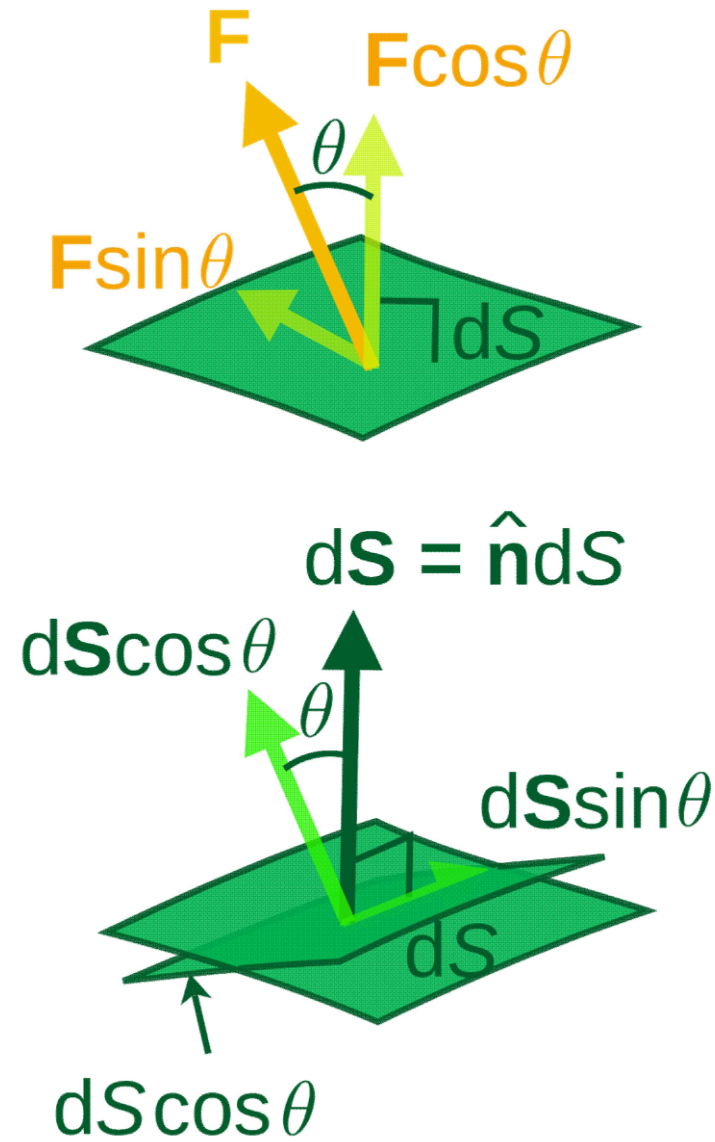
$(iii)\quad \vec{V} = -y\hat{i} + x\hat{j}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

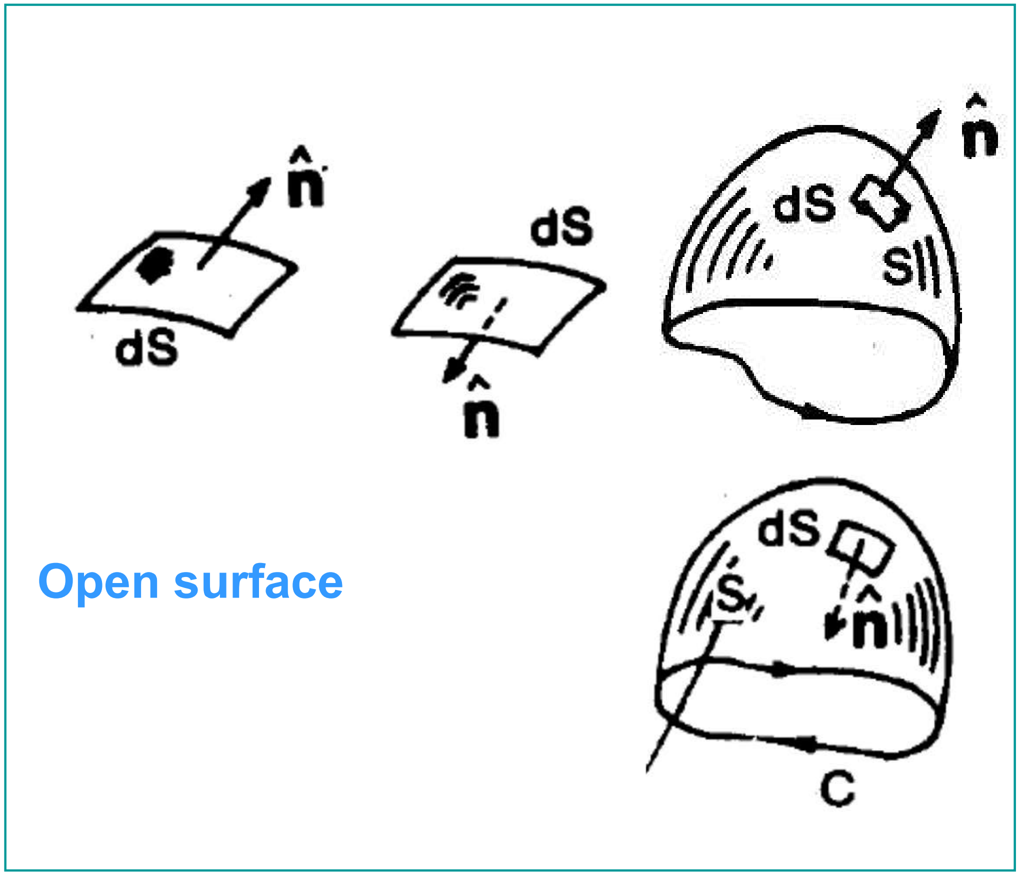


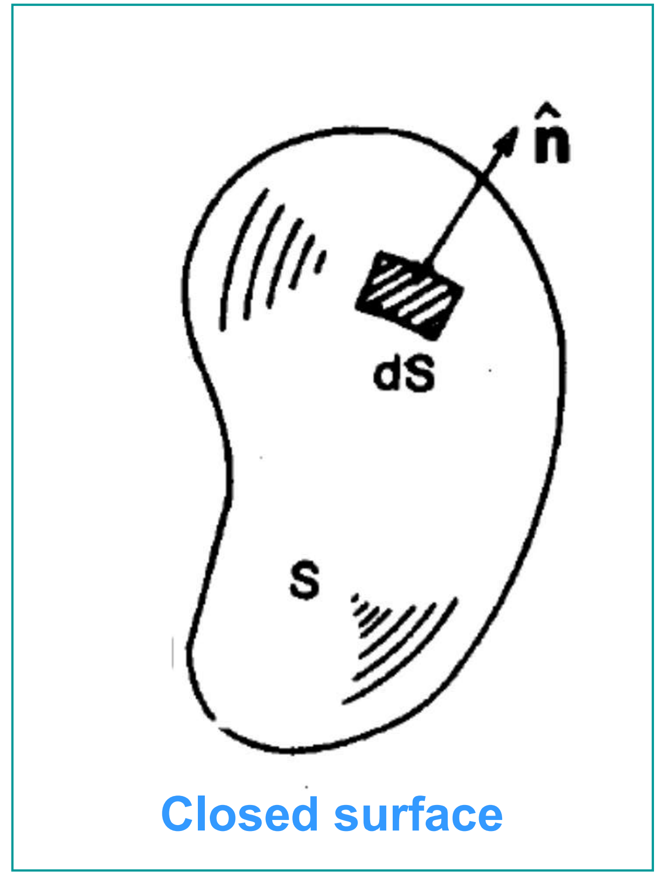
Vector Surface Integration


z
da
y
x


F
Fcosθ
θ
Fsinθ
dS
dS = n̂dS
dScosθ
θ
dSsinθ
dS
dScosθ

Vector surface Integral

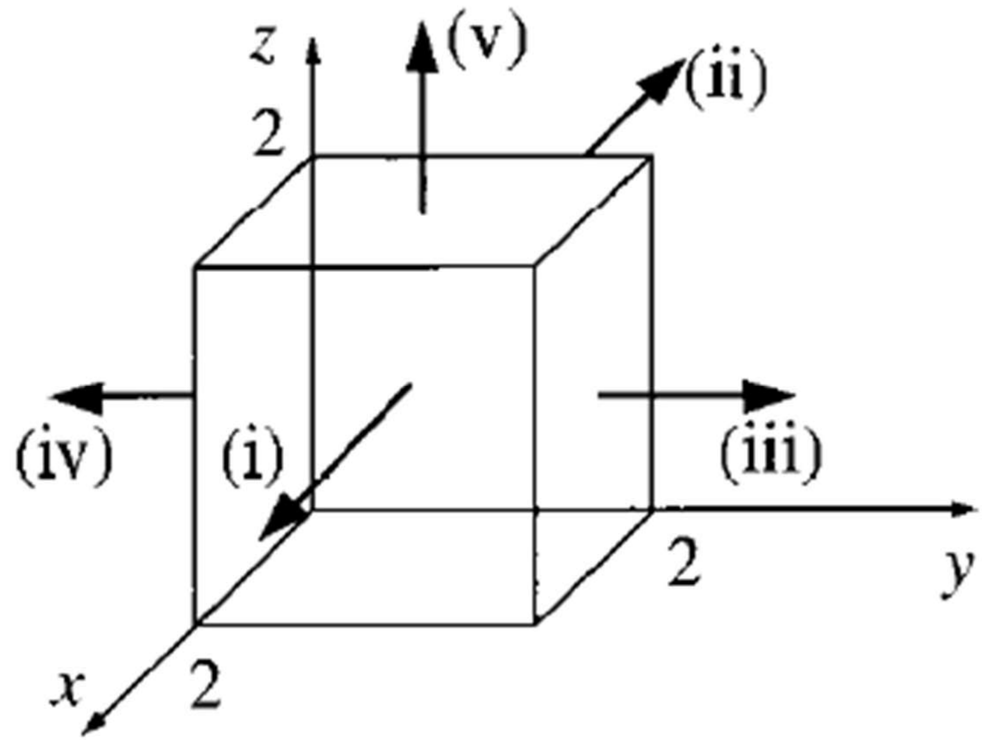




(*i*) $\vec{V}_1 = -y\hat{i} + x\hat{j}$

(*ii*) $\overline{V}_2 = -y\hat{i} + xy\hat{j}$

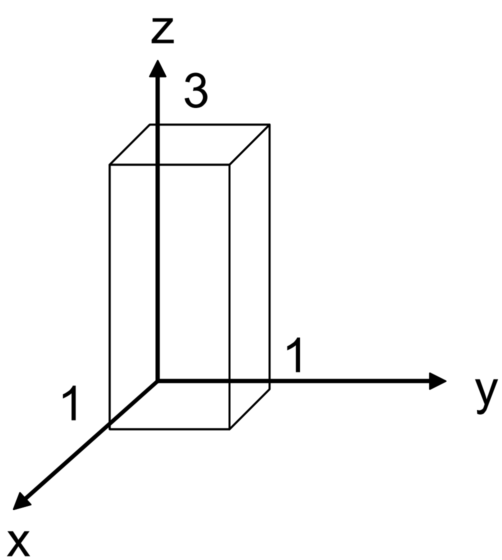
(*iii*) $\vec{V}_3 = xy\hat{i} + yz\hat{j} + zx\hat{k}$

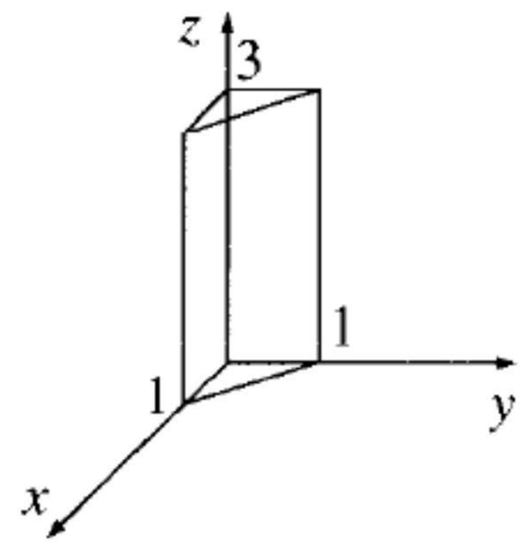


Volume Integral

$$\int_{\mathcal{V}} T \, d\tau \qquad d\tau = dx\,dy\,dz$$

Calculate the volume integral of $T = xyz^2$ over the prism





$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz =$$

$$\frac{1}{2} \int_0^3 z^2\,dz \int_0^1 (1-y)^2 y\,dy = \tfrac{1}{2}(9)(\tfrac{1}{12}) = \tfrac{3}{8}.$$