Lecture 25: Taylor's Theorem & Taylor Series

October 11, 2019

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25.1 Taylor's Theorem

Taylors Theorem paves approximation of functions by polynomials.

Theorem 25.1 (Taylor's Theorem) *Let $n \in \mathbb{Z}, n \geq 0$, and $f : [a,b] \to \mathbb{R}$ be such that $f, f', \cdots, f^{(n)}$ exist on $[a,b]$ and further, $f^{(n)}$ is continuous on $[a,b]$ and differentiable on (a,b). Fix $x_0 \in [a,b]$. Then for each $x \in [a,b]$ with $x \neq x_0$, there exists c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The polynomial given by

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

is called the nth Taylor polynomial of f around x_0. The difference $R_n(x) = f(x) - P_n(x)$ is called the remainder of order n. Note that the Taylor formula for f around x_0 shows that the remainder R_n is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \text{ for some } c \text{ between } x_0 \text{ and } x.$$

The above expression for $R_n(x)$ is sometimes called the Lagrange form of remainder in Taylor formula. Notice that if the degree of Taylors polynomial is higher, the approximation is better.

Example 25.2 *Let $f(x) = e^x$. Then the nth degree Taylor's polynomial of f around 0 is*

$$P_n(x) = 1 + x + \cdots + \frac{x^n}{n!}.$$

Example 25.3 *Suppose we are stuck on an island without calculator and a demon ask us to find an approximate value of* $2^{1/3}$. *Since we can easily compute the addition and multiplication of rational numbers hence let us take* $f : [1, 2] \to \mathbb{R}$ *defined as* $f(x) = x^{1/3}$.

Note that

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, \quad f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}, \quad f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$$

The Taylor's polynomial involves evalaution of derivatives hence we should choose Taylor's ployonomial around such a point for which derivattives are easy to compute. So we take $x_0 = 1$ *and consider second degree polynomial*

$$P_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{2!}\frac{2}{9}(x-1)^2$$

Now $P_2(2) = 1 + \frac{1}{3}(2-1) - \frac{1}{9}(2-1)^2 = \frac{4}{3} - \frac{1}{9} = \frac{11}{9} = 1.2222\cdots$.

Note that $2^{1/3} = 1.25992105$, *hence we do get a good estimate by second degree polynomial.*

Suppose we use third degree polynomial Then The third Taylor's polynomial around $x = 1$ *for* f *is*

$$P_3(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{10}{27}\frac{1}{3!}(x-1)^3$$

Now $P_3(2) = 1 + \frac{1}{3}(2-1) - \frac{1}{9}(2-1)^2 + \frac{10}{27}\frac{1}{3!} = \frac{11}{9} + \frac{5}{81} = \frac{99+5}{81} = 1.283950617$.

If we use higher degree polynomial then we should expect better approximation.

You may feel that P_2 *and* P_3 *both gives correct value of* $2^{1/3}$ *upto one decimal place but a close look at the error shows that in* P_3 *we have reduced the error.* $|1.283950617 - 1.25992105| = 0.024029567$ *and* $|1.25992105 - 1.2222| = 0.0377$.

So there is indeed some improvement in the approximation.

Suppose we choose the Taylor's polynomial around $x = 8$ *then*

$$P_2(x) = 1 + \frac{1}{3}2^{-2}(x-8) - \frac{2}{9}2^{-5}(x-8)^2$$

Now $P_2(2) = 1 + \frac{1}{3}2^{-2}(2-8) - \frac{2}{9}2^{-5}(2-8)^2 = 1 - \frac{1}{2} - \frac{2}{9}\frac{6^2}{2^5} = 1/2 - \frac{1}{4} = \frac{1}{4}$. *This is very bad estimate, because Taylor's polynomial approximate the function in the neighborhood of its center. Point 2 is quite far from 8, hence we get this bad estimate.*

Example 25.4 *Suppose* f *is a three times differentiable function on* $[-1, 1]$ *such that* $f(-1) = 0, f(1) = 1$ *and* $f'(0) = 0$. *Using Taylor's theorem prove that* $f'''(c) \geq 3$ *for some* $c \in (-1, 1)$.

Solution: By Taylor's theorem,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2!} + \frac{f'''(c_1)}{3!} \quad \text{for some} \quad c_1 \in (0,1).$$

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2!} - \frac{f'''(c_2)}{3!} \quad \text{for some} \quad c_1 \in (-1,0).$$

Therefore, $\frac{f'''(c_1)+f'''(c_2)}{6} = 1$. Hence either $f'''(c_1)$ or $f'''(c_2) \geq 3$. ■

Example 25.5 *For $x > -1$, $x \neq 0$ prove that*
(a) $(1+x)^\alpha > 1 + \alpha x$ whenever $\alpha < 0$, or $\alpha > 1$
(b) $(1+x)^\alpha < 1 + \alpha x$ whenever $0 < \alpha < 1$.

Solution: $f(x) = (1+x)^\alpha$. For any $x > -1$ and we consider interval $[-1, y]$ where $y > \max\{x, 0\}$. Then by Taylor's Theorem

$$f(x) = f(0) + f'(0)x + f''(c)\frac{x^2}{2} = 1 + \alpha x + \alpha(\alpha-1)(1+c)^{\alpha-2}\frac{x^2}{2}$$

for some c between 0 and x. Observe that, $x \neq 0$ implies that $\frac{x^2}{2}$ greater than 0 and since $x > -1$ so $c > -1$ hence $(1+c)^{\alpha-2}$ is also greater than 0.

1. Now $\alpha(\alpha-1) > 0$ whenever $\alpha < 0$ and $\alpha > 1$. Hence $f(x) > 1 + \alpha x$, whenever $\alpha < 0$, or $\alpha > 1$.

2. Now $\alpha(\alpha-1) < 0$ whenever $0 < \alpha < 1$. Hence $f(x) < 1 + \alpha x$.

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Example 25.6 *Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that*

$$x - \frac{1}{2}x^2 + \ldots - \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$

Solution: Let $f(x) = \log(1+x)$ which is defined for $x > -1$. We have

$$f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2}, f'''(x) = \frac{2}{(1+x)^3}.$$

By induction it is easy to see that

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

By Taylor's theorem, $\exists\, c \in (0, x)$ s.t.

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}.$$

Note that, for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} > 0$ if $n = 2k$ Hence

$$\log(1+x) > x - \frac{x^2}{2} + \ldots + \frac{(-1)^{2k-1}}{2k}x^{2k}$$

if $n = 2k+1$ then for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} < 0$ Hence

$$\log(1+x) < x - \frac{x^2}{2} + \ldots + \frac{(-1)^{2k}}{2k+1}x^{2k+1}$$

■

25.2 Taylor Series

Let $a \in \mathbb{R}, I$ be an interval containing a, and $f : I \to \mathbb{R}$ be an infinitely differentiable function at a. The infinite series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

is called the Taylor series of f around a. It is clear that $P_n(x)$ (nth Taylor polynomial of f around a) is the nth partial sum of the Taylor series. If f is infinitely differentiable at a then the corresponding Taylor series is defined. Also the Taylor series of f around a converges to $f(a)$ at $x = a$. However, at some other point $x \in I$, this series may be divergent.

Example 25.7 *Consider $f : (-1, \infty) \to \mathbb{R}$ defined by $f(x) := \log(1+x)$. Then*

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} \\
f''(x) &= -\frac{1}{(1+x)^2} \\
f^{(3)}(x) &= \frac{(-1)(-2)}{(1+x)^3} \\
f^{(k)}(x) &= \frac{(-1)^{k-1}(k-1)!}{(1+x)^n}
\end{aligned}$$

So f is infinitely times differentiable on $(-1, \infty)$ and its Maclaurin series is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k.$$

Now we apply ratio test to determine the values of x for which Taylor series converges convergence.

$$\lim_{k \to \infty} \frac{|x|^{k+1}}{k+1} \frac{k}{|x|^k} = |x|$$

Hence series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Note that f is infinitely times differentiable on $(-1, \infty)$. However, the Taylor series does not converge for $x > 1$.

The following result gives a sufficient condition for the convergence of a Taylor series of a function.

Theorem 25.8 *Let $a \in \mathbb{R}, I$ be an interval containing a, and $f : I \to \mathbb{R}$ be an infinitely differentiable function. If there is $\alpha > 0$ such that*

$$\left| f^{(n)}(x) \right| \leq \alpha^n, \quad \text{for all } n \in \mathbb{N} \text{ and } x \in I,$$

then the Taylor series of f converges to $f(x)$ for each $x \in I$.