Lecture 16: Differentiation

September 09, 2019

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Let us try to understand what is meaning of the limit $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$. Define new function $g(x) := \frac{f(x)-f(c)}{x-c}$. This is not defined at $x = c$, Otherwise it is define at every point of the domain of f, which is the interval I.

Proposition 16.1 *Let I be an interval (which is not a singleton), $c \in I$ and $f : I \to \mathbb{R}$ be function.*

1. $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ *exists* $\iff$ $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ *exists.*
2. *If f is differentiable at $c \in I$, then f is continuous at c.*

Example 16.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$$

Show that f is differentiable only at $x = 0$. Find $f'(0)$.

Solution: For $h \neq 0$,

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} h & \text{if} \quad h \text{ is rational} \\ 0 & \text{if} \quad h \text{ is irrational} \end{cases}$$

Hence for any sequence $(h_n) \in \mathbb{R} \setminus \{0\}$ which converges to 0, we have

$$\left| \frac{f(h_n)}{h_n} \right| \leq |h_n| \implies \frac{f(h_n)}{h_n} \to 0.$$

Hence $f'(0) = 0$. Now we show that f is not continuous at any non-zero point, which in turn proves that at any non-zero point derivative does not exists. If $c \neq 0$ is a rational number then by density of irrationals in $\mathbb{R}$ and Sandwich Theorem we can find a sequence (x_n) in $\mathbb{R}$ of irrationals such that $x_n \to c$. Then $f(x_n) = 0$ for all $n \in \mathbb{N}$, while $f(c) = c^2 \neq 0$. On the other hand, if c is an irrational, then by density of rationals in $\mathbb{R}$ and Sandwich Theorem we can find a sequence (x_n) in $\mathbb{R}$ of rationals such that $x_n \to c$. Then $f(x_n) = x_n^2$ for all $n \in \mathbb{N}$, while $f(c) = 0$. Thus in both cases, $x_n \to c$, but $f(x_n) \nrightarrow f(c)$.

Example 16.3 *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim_{n\to\infty} (n\{f(c+1/n) - f(c)\})$. However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.*

Solution: Since $f'(c)$ exists, by taking $h_n = 1/n$ which converges to zero, we must have $f'(c) = \lim_{n\to\infty}(n\{f(c+1/n) - f(c)\})$.

However, in Example 16.2, take $c = \sqrt{2}$. Then $\sqrt{2} + \frac{1}{n}$ is also irrational for each n (prove by contradiction). Now

$$f(\sqrt{2} + \frac{1}{n}) - f(0) = 0 - 0 \implies n\left(f(\sqrt{2} + \frac{1}{n}) - f(0)\right) = 0$$

Hence desired limit exists, but we have already proved that f is not continuous at every non-zero real number hence $f'(2)$ does not exists. ■

Example 16.4 *Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that*

$$\lim_{x\to 0} \frac{1}{x}\left\{f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \cdots + f(\frac{x}{k})\right\} = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

Solution: Write

$$\begin{aligned} \frac{f(x)}{x} &= \frac{f(x) - f(0)}{x - 0} \\ \frac{f(x/2)}{x} &= \frac{1}{2}\frac{f(x/2) - f(0)}{x/2 - 0} \\ \frac{f(x/k)}{x} &= \frac{1}{k}\frac{f(x/k) - f(0)}{x/k - 0} \end{aligned}$$

Also note that $x \to 0 \iff \frac{x}{2} \to 0 \iff \cdots \iff \frac{x}{k} \to 0$. ■

Differentiability and local extrema

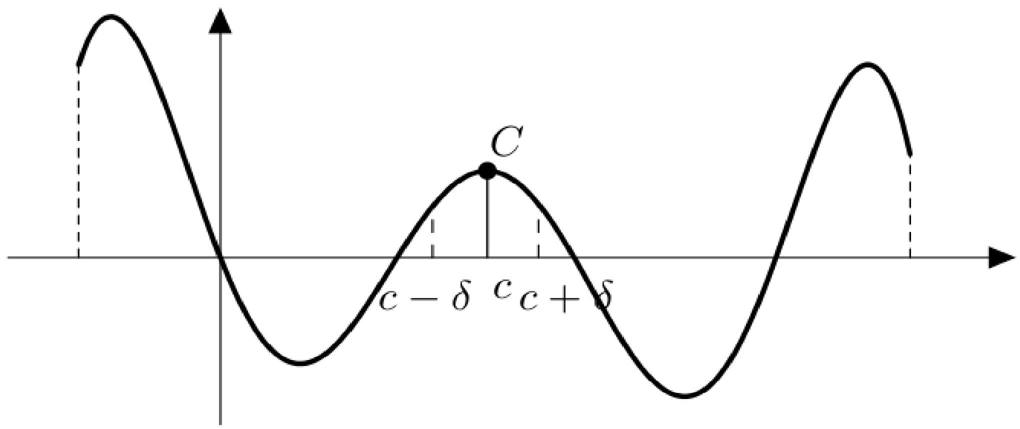
Definition 16.5 *Let $D \subset \mathbb{R}$ and $c \in D$. We say c is an interior point of the set D if there exists $r > 0$ such that $(c - r, c + r) \subset D$.*

For example if $D = [1, 2]$ then all the points in the interval $(1, 2)$ are interior points of D but 1 and 2 are not interior points of D.

Definition 16.6 *Let $D \subset \mathbb{R}$ and let c be an interior point of D. We say that a function $f : D \to \mathbb{R}$ has*

(a) *local maximum at c if there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset D$ and $f(x) \le f(c)$ for all $x \in (c-\delta, c+\delta)$.*

(b) *local minimum at c if there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset D$ and $f(x) \ge f(c)$ for all $x \in (c-\delta, c+\delta)$.*



A point c is said to be a local extremum (local extrema) if it is either a local maximum or a local minimum.

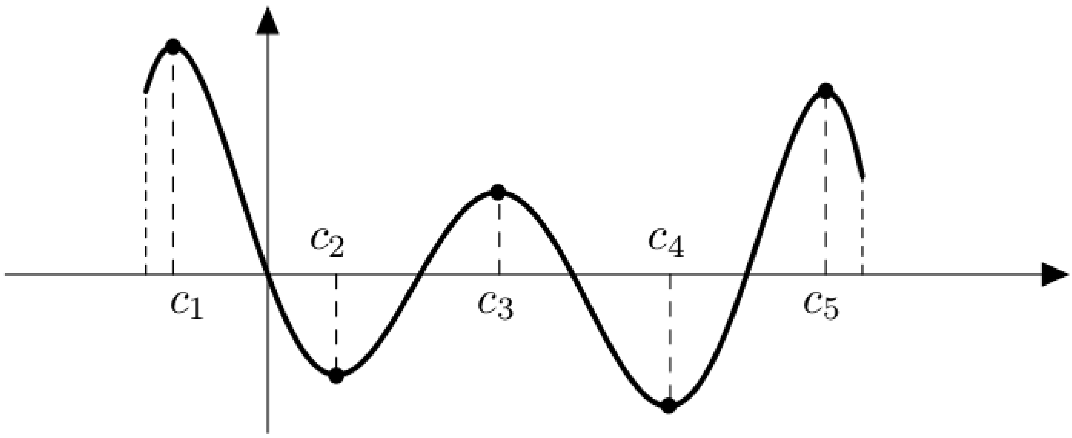
Definition 16.7 *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. We say that*

1. *A point $c \in D$ is said to be a point of absolute maximum value (absolute maxima or global maxima) on D if*
$$f(c) = \sup\{f(x) : x \in D\},$$

2. *A point $c \in D$ is said to be a point of absolute minimum value (absolute minima or global minima) on D if*
$$f(c) = \inf\{f(x) : x \in D\}.$$

A point c is said to be a global extremum (global extrema) if it is either a global maximum or a global minimum.

Remark 16.8 *Note that a local extremum need not be a global extremum. Similarly, a global extremum need not be a local extremum. The points of local minimum and local maximum should be "interior points" in the domain.*



The points, c_1, c_3 , and c_5 are local maximum whereas c_2 and c_4 are points of local minimum. The point c_1 is the global maximum and c_4 is the global minimum.

Theorem 16.9 *Let $D \subseteq \mathbb{R}$ and c be an interior point of D. If $f : D \to \mathbb{R}$ be differentiable at c and has a local extremum at c, then $f'(c) = 0$.*