The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #3

(Cauchy Sequence & Infinite Series)

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1. Suppose that $0 < \alpha < 1$ and that $(x_n)_{n\geq 1}$ is a sequence which satisfies the following condition:
$$|x_{n+1} - x_n| \leq \alpha^n \quad n = 1, 2, 3, \ldots$$
Then prove that (x_n) is a Cauchy sequence.

2. For a real sequence (x_n), show that condition $|x_{n+1} - x_n| < |x_n - x_{n-1}|$ does not guarantee the convergence of (x_n). Give examples.

3. For the sequence $a_n = \dfrac{1}{n}; n \in \mathbb{N}$, show that there does not exist $\alpha \in [0, 1)$ such that
$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \quad \text{for all } n \geq 2.$$

4. Show that there exists a sequence (a_n) of rational numbers which is a Cauchy sequence but it does not converge in $\mathbb{Q}$.

5. Find all Cauchy sequences of integers.

6. Since the convergence of a series is defined in terms of the convergence of the sequence of its partial sums, many results about the convergence of series follow from the corresponding results given for the convergence of sequences. Use your knowledge of sequences to prove the following results.

   (a) Show that the sequence of partial sums of a convergent series is bounded.

   (b) Let $\sum_k a_k = A$ and $\sum_k b_k = B$. Then show that

      i.
      $$\sum_k (a_k + b_k) = A + B$$

      ii.
      $$\sum_k (ra_k) = rA, \quad \text{for any } r \in \mathbb{R}.$$

      iii. Further, if $a_k \leq b_k$ for all k, then show that $A \leq B$.

   (c) **(Sandwich Theorem for series)** If $(a_k), (b_k)$, and (c_k) are sequences of real numbers such that $a_k \leq c_k \leq b_k$ for each k, and further, $\sum_k a_k = A$ and $\sum_k b_k = A$, then show that $\sum_k c_k = A$.

7. Give examples to show that if $\sum_k a_k$ and $\sum_k b_k$ are convergent series of real numbers, then the series $\sum_k a_k b_k$ may not be convergent. Also show that if $\sum_k a_k = A$ and $\sum_k b_k = B$, then $\sum_k a_k b_k$ may be convergent, but its sum may not be equal to AB.

8. Let $a_n \geq 0$. Then show that both the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} \dfrac{a_n}{a_n + 1}$ converge or diverge together.

9. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$, where a_n equals:
(a) $1 - n \sin \frac{1}{n}$, (b) $\frac{1}{n} \log(1 + \frac{1}{n})$, (c) $1 - \cos \frac{1}{n}$,

10. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Show that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges. Test the series $\sum_{n\geq 1} \dfrac{n^{n-2}}{e^n n!}$ for convergence.