4 MOMENTUM

4.1 Introduction

So far we have viewed nature as if it were composed of ideal particles rather than real bodies. Sometimes such a simplification is justified—for instance in the study of planetary motion, where the size of the planets is of little consequence compared with the vast distances of our solar system, or in the case of elementary particles moving through an accelerator, where the size of the particles, about 10^{-15} m, is minute compared with the size of the machine. However, most of the time we deal with large bodies that may have elaborate structure. For example, consider the landing of an explorer vehicle on Mars. Even if we could calculate the gravitational field of such an irregular and inhomogeneous body as Mars, the explorer itself hardly resembles a particle—it has wheels, gawky antennas, extended solar panels, and a lumpy body.

Furthermore, the methods of the last chapter fail when we try to analyze systems such as rockets in which there is a flow of mass. Rockets accelerate forward by ejecting mass backward; it is not obvious how we can apply $\mathbf{F} = M\mathbf{a}$ to such a system.

In this chapter we shall generalize the laws of motion to overcome these difficulties. We begin by restating Newton's second law in a slightly modified form. In Chapter 2 we wrote the law in the familiar form

$$\mathbf{F} = M\mathbf{a}.$$

Newton, however, wrote it in the form

$$\mathbf{F} = \frac{d}{dt}(M\mathbf{v}).$$

In Newtonian mechanics, the mass M of a particle is a constant and $(d/dt)(M\mathbf{v}) = M(d\mathbf{v}/dt) = M\mathbf{a}$, as before. The quantity $M\mathbf{v}$ plays a prominent role in mechanics and is called *momentum*, or sometimes *linear momentum*, to distinguish it from angular momentum. Momentum is a vector because it is the product of a vector $\mathbf{v}$ and a scalar M. Denoting momentum by $\mathbf{P}$, Newton's second law becomes

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}.$$

This form is preferable to $\mathbf{F} = M\mathbf{a}$ because it is readily generalized to complex systems, as we shall soon see, and because momentum turns out to be more fundamental than mass or velocity separately. The units of momentum are kg · m/s in the SI system, and g · cm/s in CGS. There are no special names for these units.

4.2 Dynamics of a System of Particles

To generalize the laws of motion to extended bodies consider a system of interacting particles, for instance the Sun and the planets. The bodies of our solar system are so far apart compared with their diameters that they can be treated as particles to an excellent approximation. All particles in the solar system interact gravitationally. Planets interact primarily with the Sun, although their interactions with one another also influence their motion. In addition, the entire solar system is attracted by distant matter.

On a much smaller scale, the system could be a billiard ball resting on a table. Here the particles are atoms (disregarding for now that atoms are not particles but are themselves composed of smaller particles) and the interactions are primarily interatomic electric forces. The external forces on the billiard ball include the gravitational force of the Earth and the contact force of the tabletop.

We shall now prove some simple properties of physical systems. It is important to be clear about what we mean by "system." We are free to choose the boundaries of a system as we please, but once the choice is made, we must be consistent about which particles are included in the system and which are not.

We suppose that the particles in the system interact with particles outside the system as well as with each other. To make the argument general, consider a system of N interacting particles with masses $m_1, m_2, m_3, \cdots, m_N$. The position of the jth particle is $\mathbf{r}_j$, the force on it is $\mathbf{f}_j$, and its momentum is $\mathbf{p}_j = m_j\dot{\mathbf{r}}_j$. The equation of motion for the jth particle is

$$\mathbf{f}_j = \frac{d\mathbf{p}_j}{dt}.$$

The force on particle j can be split into two terms:

$$\mathbf{f}_j = \mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}}.$$

Here $\mathbf{f}_j^{\text{int}}$, the *internal* force on particle j, is the force due to all other particles in the system, and $\mathbf{f}_j^{\text{ext}}$, the *external* force on particle j, is the force due to sources outside the system. The equation of motion of particle j can therefore be written

$$\mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}} = \frac{d\mathbf{p}_j}{dt}.$$

Now let us focus on the system as a whole by the following stratagem: add the equations of motion of all the particles in the system.

$$\begin{aligned}
\mathbf{f}_1^{\text{int}} + \mathbf{f}_1^{\text{ext}} &= \frac{d\mathbf{p}_1}{dt}\\
\mathbf{f}_2^{\text{int}} + \mathbf{f}_2^{\text{ext}} &= \frac{d\mathbf{p}_2}{dt}\\
&\cdots\cdots\cdots\\
\mathbf{f}_N^{\text{int}} + \mathbf{f}_N^{\text{ext}} &= \frac{d\mathbf{p}_N}{dt}.
\end{aligned}$$

The result of adding these equations can be written

$$\sum_{j=1}^{N} \mathbf{f}_j^{\text{int}} + \sum_{j=1}^{N} \mathbf{f}_j^{\text{ext}} = \sum_{j=1}^{N} \frac{d\mathbf{p}_j}{dt}. \tag{4.1}$$

The summations extend over all particles, $j = 1, \ldots, N$.

The first term in Eq. (4.1), $\Sigma\mathbf{f}_j^{\text{int}}$, is the sum of all internal forces acting on all the particles. According to Newton's third law, the forces between any two particles are equal and opposite so that their sum is zero. It follows that the internal forces all cancel in pairs so that the sum of all the forces between all the particles is also zero. Hence

$$\sum_{j=1}^{N} \mathbf{f}_j^{\text{int}} = 0.$$

The second term, $\sum_{j=1}^{N} \mathbf{f}_j^{\text{ext}}$, is the sum of all external forces acting on all the particles. It is the *total external force* $\mathbf{F}_{\text{ext}}$ acting on the system:

$$\sum_{j=1}^{N} \mathbf{f}_j^{\text{ext}} \equiv \mathbf{F}_{\text{ext}}.$$

Equation (4.1) then simplifies to

$$\mathbf{F}_{\text{ext}} = \sum_{j=1}^{N} \frac{d\mathbf{p}_j}{dt}.$$

The right-hand side, $\Sigma(d\mathbf{p}_j/dt)$, can be written $(d/dt)\Sigma\mathbf{p}_j$, because the derivative of a sum is the sum of the derivatives. $\Sigma\mathbf{p}_j$ is the *total momentum* of the system, which we designate by $\mathbf{P}$:

$$\mathbf{P} \equiv \sum_{j=1}^{N} \mathbf{p}_j.$$

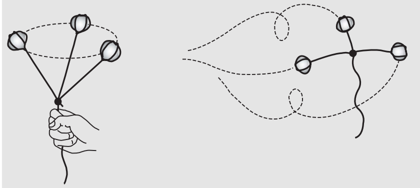
With this substitution, Eq. (4.1) becomes

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt}.$$

In words, the total force applied to a system equals the rate of change of the system's momentum. This is true regardless of the details of the interaction; $\mathbf{F}_{\text{ext}}$ could be a single force acting on a single particle, or it could be the resultant of many tiny interactions involving each particle of the system.

Example 4.1 The Bola

The bola is used by gauchos for entangling their cattle. It consists of three balls of stone or iron connected by thongs. The gaucho whirls the bola in the air and hurls it at the animal. What can we say about its motion?



Consider a bola with masses m_1, m_2, and m_3. Each ball is pulled by its binding thong and by gravity. (We neglect air resistance.) Since the constraining forces depend on the instantaneous positions of all three balls, it is a real problem even to write the equation of motion of one ball. However, the total momentum obeys the simple equation

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} = \mathbf{f}_1{}^{\text{ext}} + \mathbf{f}_2{}^{\text{ext}} + \mathbf{f}_3{}^{\text{ext}}$$
$$= m_1\mathbf{g} + m_2\mathbf{g} + m_3\mathbf{g}$$

or

$$\frac{d\mathbf{P}}{dt} = M\mathbf{g},$$

where M is the total mass. This equation represents an important first step in finding the detailed motion. The equation is identical to that of a single particle of mass M with momentum $\mathbf{P}$. This is instinctively clear to the gaucho when he hurls the bola; although it is a complicated system, he need only aim it like a single mass.

4.3 Center of Mass

According to Eq. (4.1),

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}, \tag{4.2}$$

where we have dropped the subscript "ext" with the understanding that $\mathbf{F}$ stands for the external force. This result is identical to the equation of motion of a single particle, although it may in fact refer to a system of several particles. It is tempting to push the analogy between Eq. (4.2) and single-particle motion even further by writing

$$\mathbf{F} = M\ddot{\mathbf{R}}, \tag{4.3}$$

where M is the total mass of the system and $\mathbf{R}$ is a vector yet to be defined. Because $\mathbf{P} = \sum_{j=1}^{N} m_j\dot{\mathbf{r}}_j$, Eqs. (4.2) and (4.3) give

$$M\ddot{\mathbf{R}} = \frac{d\mathbf{P}}{dt} = \sum_{j=1}^{N} m_j\ddot{\mathbf{r}}_j,$$

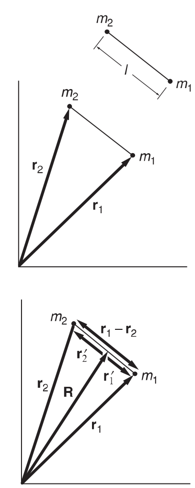
which is true if

$$\mathbf{R} = \frac{1}{M}\sum_{j=1}^{N} m_j\mathbf{r}_j. \tag{4.4}$$

$\mathbf{R}$ is a vector from the origin to a point called *the center of mass*. The motion of a system's center of mass behaves as if all the mass were concentrated there and all the external forces act at that point.

We are often interested in the motion of comparatively rigid bodies like baseballs or automobiles. Such a body is merely a system of particles that are fixed relative to each other by strong internal forces. Equation (4.4) shows that with respect to external forces, the body behaves as if it were a single particle. In Chapters 2 and 3, we casually treated every body as if it were a particle; we see now that this is justified provided that we focus attention on the center of mass.

You may wonder whether this description of center of mass motion isn't also an oversimplification—experience tells us that an extended body like a plank behaves differently from a compact body like a rock, even if the masses are the same and we apply the same force. Indeed, center of mass motion is only part of the story. The relation $\mathbf{F} = M\ddot{\mathbf{R}}$ describes only the translation of the body (the motion of its center of mass); it does not describe the body's orientation in space. In Chapters 7 and 8 we shall investigate the rotation of extended bodies. It turns out that, as we expect, the rotational motion of a body depends on its shape and where the forces are applied. Nevertheless, as far as translation of the center of mass is concerned, $\mathbf{F} = M\ddot{\mathbf{R}}$ is true for any system of particles, not just for those fixed in rigid objects, as long as the forces between the particles obey Newton's third law. It is immaterial whether or not the particles move relative to each other and whether or not there happens to be any matter at the center of mass.



Example 4.2 Drum Major's Baton

A drum major's baton consists of two masses m_1 and m_2 separated by a thin rod of length l. The baton is thrown into the air. Find the baton's center of mass and the equation of motion for the center of mass.

Let the position vectors of m_1 and m_2 be $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The position vector of the center of mass, measured from the same origin, is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{1}$$

where we have neglected the mass of the thin rod. The center of mass lies on the line joining m_1 and m_2; the proof is left as a problem.

Assuming that air resistance is negligible, the external force on the baton is

$$\mathbf{F} = m_1\mathbf{g} + m_2\mathbf{g}.$$

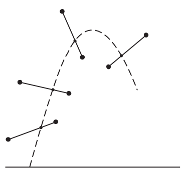
The equation of motion of the center of mass is

$$(m_1 + m_2)\ddot{\mathbf{R}} = (m_1 + m_2)\mathbf{g}$$

or

$$\ddot{\mathbf{R}} = \mathbf{g}.$$

The center of mass follows the parabolic trajectory of a single mass in a uniform gravitational field. With the methods to be developed in Chapter 8, we shall be able to find the motion of m_1 and m_2 about the center of mass, completing the solution.



Although it is a simple matter of algebra to find the center of mass of a system of particles, finding the center of mass of an extended body normally requires integration. We proceed by dividing the body of mass M into N mass elements. If $\mathbf{r}_j$ is the position of the jth element, and m_j is the element's mass, then

$$\mathbf{R} = \frac{1}{M}\sum_{j=1}^{N} m_j\mathbf{r}_j. \tag{4.5}$$

In the limit where N approaches infinity, the size of each element approaches zero and the approximation becomes exact:

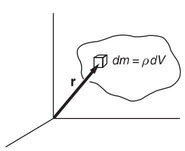
$$\mathbf{R} = \lim_{N\to\infty}\frac{1}{M}\sum_{j=1}^{N} m_j\mathbf{r}_j.$$

This limiting process defines an integral. Formally

$$\lim_{N\to\infty}\sum_{j=1}^{\infty} m_j\mathbf{r}_j = \int \mathbf{r}\,dm,$$

where dm is a differential mass element at position $\mathbf{r}$. Then

$$\mathbf{R} = \frac{1}{M}\int_V \mathbf{r}\,dm.$$



To visualize this integral, think of dm as the mass in an element of volume dV located at position $\mathbf{r}$. If the mass density at the element is ρ, then $dm = \rho\,dV$ and

$$\mathbf{R} = \frac{1}{M}\int_V \mathbf{r}\rho\,dV.$$

This integral is called a volume integral. It is sometimes written with three integral signs (a triple integral) to emphasize that the integration proceeds over all three space coordinates:

$$\mathbf{R} = \frac{1}{M}\iiint_V \mathbf{r}\rho\,dV.$$

The integral of a sum of terms equals the sum of the integrals over each term. We can use this fundamental property of integrals to express the center of mass of several extended bodies in terms of the centers of mass of the individual bodies. Consider extended body 1 with mass M_1 and body 2 with mass M_2. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the position vector of each center of mass. From Eq. (4.5)

$$\mathbf{R}_1 = \frac{1}{M_1}\int_{V_1}\mathbf{r}_1\,dm$$

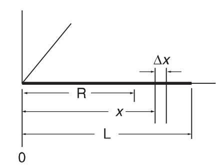
$$\mathbf{R}_2 = \frac{1}{M_2}\int_{V_2}\mathbf{r}_2\,dm.$$

The position vector $\mathbf{R}$ of the system's center of mass can then be written

$$(M_1 + M_2)\mathbf{R} = \int_{V_1}\mathbf{r}_1\,dm + \int_{V_2}\mathbf{r}_2\,dm$$
$$= M_1\,\mathbf{R}_1 + M_2\,\mathbf{R}_2.$$

In other words, to find the center of mass of a system of several extended bodies, treat each body as if its mass were concentrated at its center of mass.

We shall only be concerned with a few simple cases of calculating the center of mass of extended bodies, as illustrated by the following examples. Further examples are given in Note 4.1 at the end of the chapter.



Example 4.3 Center of Mass of a Non-uniform Rod

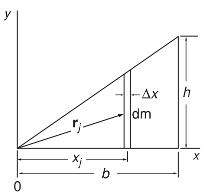
A rod of length L has a non-uniform density. The mass per unit length of the rod, λ, varies as $\lambda = \lambda_0(x/L)$, where λ_0 is a constant and x is the distance from the end marked 0. Find the center of mass.

It is apparent that $\mathbf{R}$ lies on the rod. Let the x axis lie along the rod, with the origin $x = 0$ at the end of the rod. The mass in an element of length dx is $dm = \lambda dx = \lambda_0 x\,dx/L$. The rod extends from $x = 0$ to $x = L$ and the total mass is

$$M = \int dm$$
$$= \int_0^L \lambda\,dx$$
$$= \int_0^L \frac{\lambda_0 x\,dx}{L}$$
$$= \tfrac{1}{2}\lambda_0 L.$$

The center of mass is at

$$\begin{aligned}\mathbf{R} &= \frac{1}{M}\int \lambda\,\mathbf{r}\,dx \\ &= \frac{2}{\lambda_0 L}\int_0^L (x\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})\frac{\lambda_0 x\,dx}{L} \\ &= \frac{2}{L^2}\frac{x^3}{3}\bigg|_0^L \hat{\mathbf{i}} \\ &= \frac{2}{3}L\hat{\mathbf{i}}.\end{aligned}$$



Example 4.4 Center of Mass of a Triangular Plate

Calculating the center of mass is straightforward if the object can be subdivided into parts with known centers of mass. Consider the two-dimensional case of a uniform right triangular plate of mass M, base b, height h, and small thickness t.

Divide the triangle into strips of width Δx parallel to the y axis, as shown.

The jth strip at x_j has its center of mass halfway up, because the plate is uniform, and the total height of the jth strip is $x_j h/b$ by similar triangles. The position vector to the strip's center of mass is therefore

$$\mathbf{r}_j = x_j\hat{\mathbf{i}} + \frac{x_j h}{2b}\hat{\mathbf{j}}.$$

The center of mass of the plate is located at $\mathbf{R}$, and in the limit of very narrow strips,

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm \tag{1}$$

where

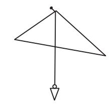
$$M = \rho A t = \rho t b h/2$$

$$dm = \rho t y dx = \rho t\frac{xh}{b}\,dx.$$

Then Eq. (1) can be written

$$\begin{aligned}\mathbf{R} &= \left(\frac{2}{\rho t b h}\right)\int \mathbf{r}\rho t\frac{xh}{b}\,dx \\ &= \frac{2}{b^2}\int_0^b x\,\mathbf{r}\,dx \\ &= \frac{2}{b^2}\int_0^b \left(x^2\hat{\mathbf{i}} + \frac{x^2 h}{2b}\hat{\mathbf{j}}\right)dx \\ &= \tfrac{2}{3}b\hat{\mathbf{i}} + \tfrac{1}{3}h\hat{\mathbf{j}}.\end{aligned}$$

To find the center of mass if the plate is not uniform, we would need to use multiple integrals, as discussed in Note 4.1.

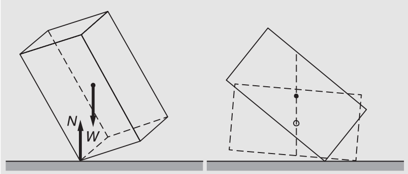


Physical arguments are sometimes able to take the place of complicated calculations. Suppose we want to find the center of mass of a thin irregular non-uniform plate. Let it hang from a pivot and draw a plumb line from the pivot. The center of mass will hang directly below the pivot (this may be intuitively obvious, and can easily be proved with the methods of Chapter 7), so the center of mass is somewhere on the plumb line. Repeat the procedure with a different pivot point. The two lines intersect at the center of mass.

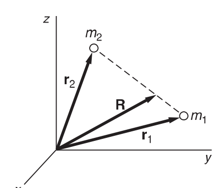
Example 4.5 Center of Mass Motion

A rectangular crate is held with one corner resting on a frictionless table. The crate is gently released and falls in a complex tumbling motion. We are not yet prepared to predict the full motion because it involves rotation, but there is no difficulty in finding the trajectory of the center of mass.

The external forces acting on the box are gravity and the normal force of the table. Both of these are vertical, so the center of mass must accelerate vertically. If the box is released from rest its center falls straight down.



4.4 Center of Mass Coordinates

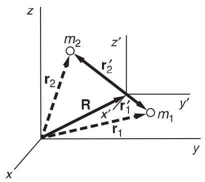


Often a problem can be simplified by a clever choice of coordinates. The center of mass coordinate system, in which the origin lies at the center of mass, is particularly useful. Consider the case of a two-particle system with masses m_1 and m_2.

In the initial coordinate system x, y, z, the particles are located at $\mathbf{r}_1$ and $\mathbf{r}_2$ and their center of mass is at

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$

We now set up the center of mass coordinate system, x', y', z', with its origin at the center of mass. The origins of the old and new system are



displaced by $\mathbf{R}$. The center of mass coordinates of the two particles are

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R}$$
$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R}.$$

Center of mass coordinates are the natural coordinates for an isolated two-body system. Such a system has no external forces, so the motion of the center of mass is trivial—it moves uniformly. Furthermore, $m_1\mathbf{r}_1' + m_2\mathbf{r}_2' = 0$ by the definition of center of mass, so that if the motion of one particle is known, the motion of the other particle follows directly. Here are two examples.

Example 4.6 Exoplanets

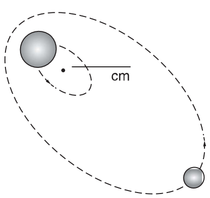
For many centuries people have wondered if there might be life on other planets. Searching for life on the other planets and moons of our solar system is an active field of inquiry, and has been extended to other stars to discover orbiting planets that might be able to sustain life. Planets not members of our own solar system are called exoplanets (Greek $\epsilon\chi o$, "outside of"). In a few favorable cases, telescopes have seen planets orbiting a nearby star, but for distant stars, a small dark planet is undetectable in the star's bright glare. This example shows how exoplanets can be detected using the concept of center of mass.

Newton was the first to calculate the motion of two gravitating bodies. As we shall show in Chapter 10, two bodies bound by gravity move so that the vector joining them traces out an ellipse with its focus at the center of mass. Consider a single planet of mass m orbiting a star of mass M. Let $\mathbf{r}_\mathrm{p}$ and $\mathbf{r}_\mathrm{s}$ be the position vectors of the planet and star, respectively. Taking the origin at the center of mass,

$$m\mathbf{r}_\mathrm{p} + M\mathbf{r}_\mathrm{s} = 0$$

which gives

$$\mathbf{r}_\mathrm{s} = -\left(\frac{m}{M}\right)\mathbf{r}_\mathrm{p}. \qquad (1)$$



As the planet swings around the star, Eq. (1) shows that the center of the star also moves in an orbit, but a much smaller one, because $m \ll M$, as shown schematically in the sketch. The line joining the star and the planet always passes through the center of mass as they orbit. As seen edge on to the orbits, the star is advancing toward the observer as the planet completes half its revolution, and the star is receding the other half.

Another relation comes from considering the dynamics. The Earth's orbit is very nearly circular, only modestly elliptical, probably a good situation for life because the temperature on such a planet would not

vary greatly. A highly elliptical elongated orbit could lead to large temperature swings, with water going between freezing and boiling.

Assuming a circular orbit, the angular velocity $\dot\theta$ is constant, and

$$mr_\mathrm{p}\dot\theta^2 = \frac{GmM}{(r_\mathrm{p}+r_\mathrm{s})^2}$$
$$r_\mathrm{p}\dot\theta^2 \approx \frac{GM}{r_\mathrm{p}^2}$$
$$\dot\theta = \sqrt{\frac{GM}{r_\mathrm{p}^3}}. \qquad (2)$$

Integrating,

$$\theta = \sqrt{\frac{GM}{r_\mathrm{p}^3}}\,t.$$

The planet makes a complete orbit as θ goes from 0 to 2π, so the time T for a complete orbit (the planet's "year") is

$$T = 2\pi\sqrt{\frac{r_\mathrm{p}^3}{GM}}.$$

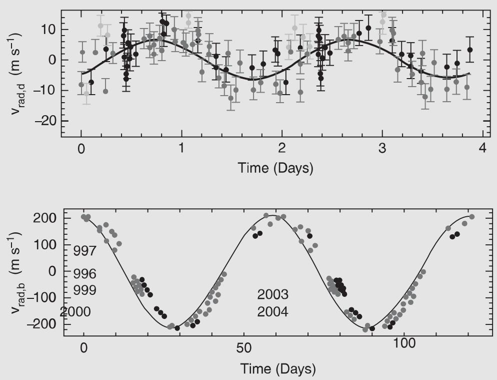
Incidentally, this result is a special case of one of the seventeenth-century astronomer Johannes Kepler's laws of planetary motion, that the square of the "year" is proportional to the cube of the orbital radius. We shall derive the general case in Chapter 10. One consequence of this law is that a planet orbiting close to a star has a much shorter period than a planet farther away.

Ever since large telescopes became available in the late eighteenth century, astronomers have attempted to observe the "wobble" of stars as a means of detecting exoplanets. Although the wobble is typically too small to detect directly, many exoplanets have been discovered from the effect of the wobble on a star's spectrum.

The Kepler space satellite telescope has detected hundreds of exoplanets using a different, but related, technique. If the planet's orbit is nearly edge-on as seen from the Earth, the wobbling star will move periodically toward and away from the Earth. The velocity of this motion can be detected by the *Doppler shift* of the star's light, using the same principle underlying police or baseball radar guns. (We shall discuss the Doppler shift in Chapter 12.) For the edge-on case, Eq. (2) gives the variation in velocity of the star's center as

$$r_s\dot{\theta} = \pm\sqrt{\frac{GMr_s^{2}}{r_p^{3}}} = \pm\sqrt{\frac{Gm^2}{Mr_p}}$$

where the last step follows from Eq. (1). For the Earth–Sun system, this is ± 0.09 m/s. With the method's sensitivity of a few m/s, our Earth would not be readily detectable from a distant solar system, but a planet as massive as Jupiter or Saturn could be. A planet of moderate mass could be detected if it orbits close to the star, but then it might be too hot to support life as we know it.



The exoplanet Gliese 876d (upper figure) has a period of 1.9 Earth days, and Gliese 876b (lower figure) has a period of 61 days. The figures are from NSF press release 05-097 (2005), based on data published by Eugenio Rivera *et al.*, *The Astrophysical J.* **634**(1):625–640 (2005).

The figures show measured radial speeds inferred from the Doppler shift for two different exoplanets orbiting the same star "Gliese 876" located 15.3 light years = 1.4×10^{17} m from Earth. The upper figure is for exoplanet "d" close to the star; we see that the "year" for this planet is only a few days, and the "wobble" it induces in the star gives a radial speed variation of only a few m/s. The lower figure is for exoplanet "b" farther from the star. It has a much longer "year" than exoplanet "d" but it induces a much larger "wobble" because of its much greater mass.

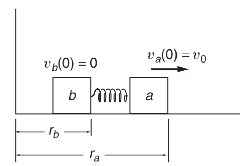
We would like to know the exoplanet's mass and orbital radius to see if it might be suitable for sustaining life. We have three unknowns, m, M, and r_p (or r_s) but only two measured values, the Doppler shift velocity, and the period T from the time variation of the Doppler shift.

The third quantity we need comes from estimating the star's mass M using stellar models based on color and brightness.

One weakness of this approach is its lack of sensitivity to planets of smaller mass, such as the Earth. Another weakness is our incomplete understanding of the different forms life might take. Even on the Earth life takes on unexpected forms. For instance, tube worms live without light near thermal vents in the deep ocean, sustained by chemicals from the vent, with the help of bacteria. The field of astrobiology is concerned with expanding our conception of life that might exist on exoplanets.

Example 4.7 The Push Me–Pull You

Two identical blocks a and b each of mass m slide without friction on a straight track. They are attached by a spring with unstretched length l and spring constant k; the mass of the spring is negligible compared to the mass of the blocks. Initially the system is at rest. At $t = 0$, block a is hit sharply, giving it an instantaneous velocity v_0 to the right. Find the velocity of each block at later times. (Try this yourself if there is a linear air track available—the motion is unexpected.)



Since the system slides freely after the collision, the center of mass moves uniformly and therefore defines an inertial frame.

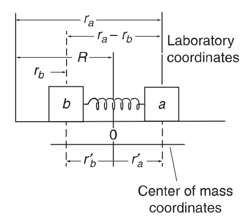
Let us transform to center of mass coordinates. The center of mass lies at

$$R = \frac{mr_a + mr_b}{m + m} = \frac{1}{2}(r_a + r_b).$$

R always lies halfway between a and b, which is hardly surprising. The center of mass coordinates of a and b are

$$r'_a = r_a - R = \tfrac{1}{2}(r_a - r_b)$$
$$r'_b = r_b - R = -\tfrac{1}{2}(r_a - r_b) = -r'_a.$$

The sketch shows these coordinates.



The instantaneous length of the spring is $r_a - r_b = r'_a - r'_b$. The instantaneous departure of the spring from its equilibrium length l is $r_a - r_b - l = r'_a - r'_b - l$. The equations of motion in the center of mass system are

$$m\ddot{r}'_a = -k(r'_a - r'_b - l)$$
$$m\ddot{r}'_b = +k(r'_a - r'_b - l).$$

The form of these equations suggests that we subtract them, obtaining

$$m(\ddot{r}'_a - \ddot{r}'_b) = -2k(r'_a - r'_b - l).$$

It is natural to introduce the departure of the spring from its equilibrium length as a variable. Letting $u = r'_a - r'_b - l$, we have

$$m\ddot{u} + 2ku = 0.$$

This is the equation for simple harmonic motion that we discussed in Chapter 3. The general solution is

$$u = A\sin\omega t + B\cos\omega t,$$

where $\omega = \sqrt{2k/m}$. Since the spring is unstretched at $t = 0$, $u(0) = 0$, which requires $B = 0$. Then $u = A\sin\omega t$ so that $\dot{u} = A\,\omega\cos\omega t$. At $t = 0$

$$\dot{u}(0) = A\omega\,\cos(0)$$

and since $u = r'_a - r'_b - l = r_a - r_b - l$

$$\dot{u}(0) = v_a(0) - v_b(0)$$
$$= v_0,$$

so that

$$A = v_0/\omega.$$

Therefore

$$u = (v_0/\omega)\sin\omega t$$

and

$$\dot{u} = v_0\cos\omega t.$$

Since $v'_a - v'_b = \dot{u}$, and $v'_a = -v'_b$, we have

$$v'_a = -v'_b = \tfrac{1}{2}v_0\cos\,\omega t.$$

The laboratory velocities are

$$v_a = \dot{R} + v'_a$$
$$v_b = \dot{R} + v'_b.$$

Since $\dot{R}$ is constant, it is always equal to its initial value

$$\dot{R} = \tfrac{1}{2}(v_a(0) + v_b(0))$$
$$= \tfrac{1}{2}v_0.$$

Putting these results together gives

$$v_a = \frac{v_0}{2}(1 + \cos\omega t)$$
$$v_b = \frac{v_0}{2}(1 - \cos\omega t).$$

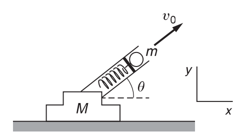
The masses move to the right on the average, but they alternately come to rest in a push me–pull you fashion.

4.5 Conservation of Momentum

In Section 4.2, we found that the total external force **F** acting on a system is related to the total momentum **P** of the system by

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}.$$

Consider the implications of this for an isolated system. In this case $\mathbf{F} = 0$, and $d\mathbf{P}/dt = 0$. The total momentum of an isolated system is constant, no matter how strong the interactions among its constituents, and no matter how complicated the motions. This is the *law of conservation of momentum*. As we shall show, this apparently simple law can provide powerful insights into complex systems.



Example 4.8 Spring Gun Recoil

A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble (the speed with which the marble is ejected, relative to the muzzle) is v_0. What is the final motion of the gun?

Take the physical system to be the gun and marble. Gravity and the normal force of the table act on the system. These external forces are both vertical. Because there are no horizontal external forces, the x component of the vector equation $\mathbf{F} = d\mathbf{P}/dt$ is

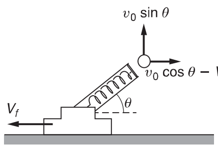
$$0 = \frac{dP_x}{dt}. \tag{1}$$

According to Eq. (1), P_x is conserved:

$$P_{x,\text{initial}} = P_{x,\text{final}}. \tag{2}$$

Let the initial time be prior to firing the gun. Because the system is initially at rest, $P_{x,\text{initial}} = 0$, After the marble has left the muzzle, the gun recoils to the left with some speed V_f, and its final horizontal momentum is $-MV_f$.

Finding the final velocity of the marble involves a subtle point, however. Physically, the marble's acceleration is due to the force of the gun,



and the gun's recoil is due to the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so at the instant the marble and the gun part company, the gun has its final speed $-V_f$. At that same instant the speed of the marble *relative to the gun* is v_0. Hence, the final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$. By conservation of horizontal momentum, we therefore have

$$0 = m(v_0 \cos\theta - V_f) - MV_f$$

or

$$V_f = \frac{mv_0 \cos\theta}{M + m}.$$

The law of conservation of momentum follows directly from Newton's third law, so that the conservation of momentum appears to be a natural consequence of Newtonian mechanics. However, conservation of momentum turns out to hold true even in the realms of quantum mechanics and relativity where Newtonian mechanics proves inadequate. Conservation of momentum can also be generalized to apply to light, because light can be thought of as a stream of particles called *photons* that are massless but nevertheless possess momentum. For these reasons, the law of conservation of momentum is generally regarded as being more fundamental than the laws of Newtonian mechanics. In this view, Newton's third law is a simple consequence of the conservation of momentum for interacting particles. For our present purposes it is purely a matter of taste whether we wish to regard Newton's third law or conservation of momentum as more fundamental.

4.6 Impulse and a Restatement of the Momentum Relation

The relation between force and momentum is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}. \tag{4.6}$$

As a general rule, any law of physics that can be expressed in terms of derivatives can also be written in an integral form. The integral form of the force–momentum relationship is

$$\int_0^t \mathbf{F}\, dt = \mathbf{P}(t) - \mathbf{P}(0). \tag{4.7}$$

The change in momentum of a system is given by the integral of force with respect to time. Equation (4.7) contains essentially the same physical information as Eq. (4.6), but it gives a new way of looking at the effect of a force: the change in momentum is the time integral of the force. To produce a given change in the momentum in time interval t

requires only that $\int_0^t \mathbf{F}\, dt$ have the appropriate value; we can use a small force acting for much of the time or a large force acting for only part of the interval.

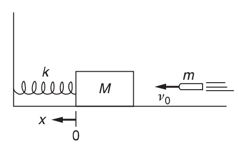
The integral $\int_0^t \mathbf{F}\, dt$ is called the *impulse*. The word impulse calls to mind a short, sharp shock, as in Example 4.7, where a blow to a mass at rest gave it a velocity v_0. However, the physical definition of impulse can just as well apply to a weak force acting for a long time. Change of momentum depends only on $\int \mathbf{F} dt$, independent of the detailed time dependence of the force.

Here are three examples involving impulse and momentum.

Example 4.9 Measuring the Speed of a Bullet

Faced with the problem of measuring the speed of a bullet, our first thought might be to turn to a raft of high-tech equipment—fast photodetectors, fancy electronics, whatever. In this example we show that a simple mechanical system can make the measurement, with the aid of conservation of momentum.

We take a simplified model to emphasize the fundamental principles. Consider a block of soft wood on a horizontal frictionless surface. A compression spring with spring constant k and uncompressed length l connects the block to a wall. The block has mass M, and the spring has negligible mass.



At $t = 0$ a gun fires a bullet of mass m and speed v_0 into the block, which moves back at initial speed V_i due to the impulse. Our system is the bullet and the block, and by conservation of momentum applied at very short times after the collision

$$mv_0 = (M + m)V_i. \tag{1}$$

Conservation of momentum is accurate here, because during the very short time of the collision the horizontal force of the spring has very little time to act. During the ensuing time, however, the spring force has plenty of time to act—it brings the system momentarily to rest. Measuring how far the system moves can tell us the speed of the bullet.

After the initial impulse, the equation of motion of the system is

$$(M + m)\ddot{x} = -kx.$$

The spring length does not appear in the equation of motion because we have taken a coordinate system with $x = 0$ when the spring is uncompressed. We recognize the equation for simple harmonic motion, which has the general solution

$$x = A \sin\omega t + B \cos\omega t \tag{2}$$

where

$$\omega = \sqrt{\frac{k}{M+m}}.$$

Using the initial conditions $x(0) = 0$, $\dot{x}(0) = V_i$ in Eq. (2) we find $A = V_i/\omega$ and $B = 0$. The position and speed of the system are then

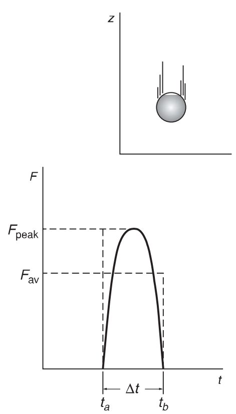
$$x = \frac{V_i}{\omega}\sin\omega t \qquad (3)$$
$$\dot{x} = V_i \cos\omega t.$$

The system first comes to rest at $t = t_f$ when $\omega t_f = \pi/2$. Using Eqs. (1), (2), and (3),

$$x(t_f) = \frac{V_i}{\omega} = \frac{mv_0}{\sqrt{k(M+m)}}$$

so that

$$v_0 = \frac{\sqrt{k(M+m)}}{m}\, x(t_f).$$



Example 4.10 Rubber Ball Rebound

A rubber ball of mass 0.2 kg falls to the floor. The ball hits with a speed of 8 m/s and rebounds with approximately the same speed. High speed photographs show that the ball is in contact with the floor for $\Delta t = 10^{-3}$ s. What can we say about the force exerted on the ball by the floor?

The momentum of the ball just before it hits the floor is $\mathbf{P}_a = -1.6\,\hat{\mathbf{k}}$ kg · m/s and its momentum 10^{-3} s later is $\mathbf{P}_b = +1.6\,\hat{\mathbf{k}}$ kg · m/s. Using $\int_{t_a}^{t_b} \mathbf{F}\,dt = \mathbf{P}_b - \mathbf{P}_a$ gives $\int_{t_a}^{t_b} \mathbf{F}\,dt = 1.6\,\hat{\mathbf{k}} - (-1.6\,\hat{\mathbf{k}}) = 3.2\,\hat{\mathbf{k}}$ kg · m/s.

Although the exact variation of $\mathbf{F}$ with time is not known, it is easy to find the average force. If the collision time is $\Delta t = t_b - t_a$, the average force $\mathbf{F}_{\rm av}$ acting during the collision is

$$\mathbf{F}_{\rm av}\Delta t = \int_{t_a}^{t_a+\Delta t} \mathbf{F}\,dt.$$

Since $\Delta t = 10^{-3}$ s,

$$\mathbf{F}_{\rm av} = \frac{3.2\,\hat{\mathbf{k}}\ \text{kg}\cdot\text{m/s}}{10^{-3}\ \text{s}} = 3200\,\hat{\mathbf{k}}\ \text{N}.$$

The average force is directed upward, as we expect. In English units, 3200 N ≈ 720 lb—a sizable force. The instantaneous force on the ball is even larger at the peak, as the sketch implies.

If the ball hits a softer surface the collision time is longer, and the peak and average forces are less.

Actually, there is a weakness in our treatment of the rubber ball rebound. In calculating the impulse $\int \mathbf{F}\,dt$, $\mathbf{F}$ is the total force. This includes the gravitational force, which we have neglected. Proceeding more carefully, we write

$$\mathbf{F} = \mathbf{F}_{\rm floor} + \mathbf{F}_{\rm grav} = \mathbf{F}_{\rm floor} - Mg\hat{\mathbf{k}}.$$

The impulse equation then becomes

$$\int_0^{10^{-3}} \mathbf{F}_{\rm floor}\,dt - \int_0^{10^{-3}} Mg\,\hat{\mathbf{k}}\,dt = 3.2\,\hat{\mathbf{k}}\ \text{kg}\cdot\text{m/s}.$$

The impulse due to the gravitational force is

$$-\int_0^{10^{-3}} Mg\,\hat{\mathbf{k}}\,dt = -Mg\,\hat{\mathbf{k}}\int_0^{10^{-3}} dt = -(0.2)(9.8)(10^{-3})\,\hat{\mathbf{k}}$$
$$= -1.96\times 10^{-3}\,\hat{\mathbf{k}}\ \text{kg}\cdot\text{m/s}.$$

This is less than one-thousandth of the total impulse, and we can neglect it with little error. Over a long period of time, gravity can produce a large change in the ball's momentum (the ball gains speed as it falls, for example). In the short time of contact, however, gravity contributes little momentum change compared with the tremendous force exerted by the floor. Contact forces during a short collision are generally so huge that we can neglect the impulse due to other forces of moderate strength, such as gravity or friction.

The rubber ball rebound example shows why a quick collision is more violent than a slow collision, even when the initial and final velocities are identical. This is the reason that a hammer can produce a force far greater than the carpenter could produce on his own; the hard steel hammerhead rebounds in a time short compared to the time of the hammer swing, and the force driving the hammer is correspondingly amplified. Contrariwise, pounding a nail into a tall fence picket can be difficult, because the thin picket can spring back under the blow, increasing the collision time and therefore decreasing the force of the hammer.

Many devices to prevent bodily injury in accidents are based on prolonging the time of the collision, which is the design basis for bicycle helmets and automobile airbags. The following example shows what can happen in even a relatively mild collision, as when you jump to the ground.

Example 4.11 How to Avoid Broken Ankles
Animals, including humans, instinctively reduce the force of impact with the ground by flexing while running or jumping. Consider what happens to someone who hits the ground with legs rigid.

Suppose a person of mass M jumps to the ground from height h, and that their center of mass moves downward a distance s during the time of collision with the ground. The average force during the collision is

$$F = \frac{Mv_0}{\Delta t}, \tag{1}$$

where Δt is the collision time and v_0 is the velocity with which they hit the ground. As a reasonable approximation, we can take the acceleration due to the force of impact to be constant, so that the person comes uniformly to rest. In this case the collision time is given by $v_0 = 2s/\Delta t$, or

$$\Delta t = \frac{2s}{v_0}.$$

Inserting this in Eq. (1) gives

$$F = \frac{Mv_0{}^2}{2s}. \tag{2}$$

For a body in free fall under gravity through height h, $v_0{}^2 = 2gh$. Inserting this in Eq. (2) gives

$$F = Mg\frac{h}{s}.$$

If the person hits the ground rigidly in a vertical position, their center of mass will not move far during the collision. Suppose that the center of mass moves downward by only 1 cm. If they jump from a height of 2 m, the force is 200 times their weight! If this person has mass 90 kg ($\approx$ 200 lb), the force on them is

$$\begin{aligned} F &= 90\ \text{kg} \times 9.8\ \text{m/s}^2 \times 200 \\ &= 1.8 \times 10^5\ \text{N}. \end{aligned}$$

Where is a bone fracture most likely to occur? Because the mass above a horizontal plane through the body decreases with height, the force is maximum at the feet. Thus the ankles will break, not the neck. If the area of contact of bone at each ankle is 5 cm^2, then the force per unit area is

$$\begin{aligned} \frac{F}{A} &= \frac{1.8 \times 10^5\ \text{N}}{10\ \text{cm}^2} \\ &= 1.8 \times 10^4\ \text{N/cm}^2. \end{aligned}$$

This is approximately the compressive strength of human bone, and so there is a good probability that the ankles will snap.

Of course, no one would be so rash as to jump rigidly. We instinctively cushion the impact when jumping by flexing as we hit the ground, in the extreme case collapsing to the ground. If the center of mass drops 50 cm instead of 1 cm during the collision, the force is only four times their weight, and there is no danger of compressive fracture.

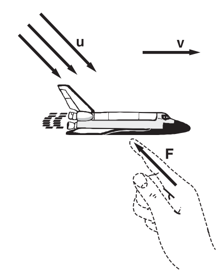
4.7 Momentum and the Flow of Mass

Analyzing the forces on a system in which there is a flow of mass can be totally confusing if you try to apply Newton's laws blindly. A rocket provides the most dramatic example of such a system, although there are many other everyday problems where the same considerations apply—for instance, the problem of calculating the reaction force on a fire hose.

There is no fundamental difficulty in handling any of these problems provided that we keep clearly in mind exactly what is included in the system. Recall that $\mathbf{F} = d\mathbf{P}/dt$ was established for a system composed of a certain set of particles. When we apply this equation in integral form,

$$\int_{t_a}^{t_b} \mathbf{F}\, dt = \mathbf{P}(t_b) - \mathbf{P}(t_a),$$

it is essential to deal with the same set of particles throughout the time interval t_a to t_b; we must keep track of all the particles that were originally in the system. Consequently, the integral form applies correctly only to systems defined so that the system's mass does not change during the time of interest.

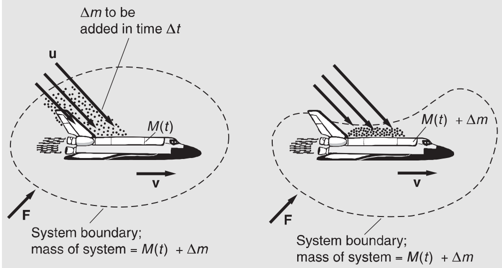


Example 4.12 Mass Flow and Momentum
A spacecraft moves through space with constant velocity $\mathbf{v}$. The spacecraft encounters a stream of dust particles that embed themselves in the hull at rate dm/dt. The dust has velocity $\mathbf{u}$ just before it hits.

At time t the total mass of the spacecraft is $M(t)$. The problem is to find the external force $\mathbf{F}$ necessary to keep the spacecraft moving uniformly. (In practice, $\mathbf{F}$ would most likely come from the spacecraft's own rocket engines. Until we discuss rocket motion in Section 4.8, we can for simplicity visualize the source $\mathbf{F}$ to be completely external—an "invisible hand".)

Let us focus on the short time interval between t and $t + \Delta t$. The drawings show the system at the beginning and end of the interval. The system consists of $M(t)$ and the mass increment Δm added to the craft during Δt. The initial momentum is

$$\mathbf{P}(t) = M(t)\mathbf{v} + (\Delta m)\mathbf{u}.$$



Time t Time $t + \Delta t$

The final momentum is

$$\mathbf{P}(t + \Delta t) = M(t)\mathbf{v} + (\Delta m)\mathbf{v}.$$

The change in momentum is

$$\begin{aligned}\Delta\mathbf{P} &= \mathbf{P}(t + \Delta t) - \mathbf{P}(t) \\ &= (\mathbf{v} - \mathbf{u})\Delta m.\end{aligned}$$

The rate of change of momentum is approximately

$$\frac{\Delta\mathbf{P}}{\Delta t} = (\mathbf{v} - \mathbf{u})\frac{\Delta m}{\Delta t}.$$

In the limit $\Delta t \to 0$, the result is exact:

$$\frac{d\mathbf{P}}{dt} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt}.$$

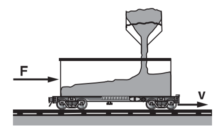
Since $\mathbf{F} = d\mathbf{P}/dt$, the required external force is

$$\mathbf{F} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt}.$$

Note that $\mathbf{F}$ can be either positive or negative, depending on the direction of the stream of mass. If $\mathbf{u} = \mathbf{v}$, the momentum of the system is constant, and $\mathbf{F} = 0$.

The procedure of isolating the system, focusing on differentials, and taking the limit may appear a trifle formal but it helps to avoid errors in a subject where it is easy to become confused. For instance, a common mistake is to argue that $\mathbf{F} = (d/dt)(m\mathbf{v}) = m(d\mathbf{v}/dt) + \mathbf{v}(dm/dt)$. In the last example this would have led to the incorrect result that $\mathbf{F} = \mathbf{v}(dm/dt)$ rather than $(\mathbf{v} - \mathbf{u})(dm/dt)$. The origin of the error is that the expression for the momentum of a single particle $\mathbf{p} = m\mathbf{v}$ cannot be

applied blindly to a system of many particles. The limiting procedure used in Example 4.12 expresses the physical situation correctly.



Example 4.13 Freight Car and Hopper

Sand falls from a stationary hopper onto a freight car that moves with uniform velocity v. The sand falls at the rate dm/dt. What force is needed to keep the freight car moving at the speed v?

The system is the loaded freight car of mass M and the incoming mass increment Δm added in time Δt. The initial horizontal speed of the sand is $v = 0$, so taking horizontal components of momentum we have

$$\begin{aligned}P(t) &= Mv + 0 \\ P(t + \Delta t) &= (M + \Delta m)v \\ P(t + \Delta t) - P(t) &= v\Delta m.\end{aligned}$$

Dividing by Δt and taking the limit $\Delta t \to 0$, the required force is $F = dP/dt = v\, dm/dt$.

Example 4.14 Leaky Freight Car

Now consider a case related to Example 4.13. A freight car leaks sand at the rate dm/dt. What force is needed to keep the freight car moving uniformly with speed $\mathbf{v}$?

Here the mass is decreasing. However, the velocity of the sand just after leaving the freight car is identical to its initial velocity, and its momentum does not change

$$\begin{aligned}P(t) &= (M + \Delta m)v \\ P(t + \Delta t) &= Mv + v\Delta m.\end{aligned}$$

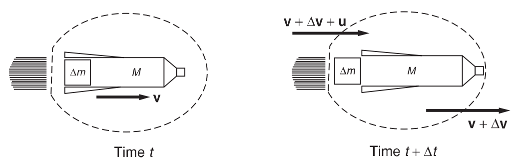
Since $dP/dt = 0$, no force is required.

4.8 Rocket Motion

We can readily explain the principle of rocket motion by focusing on momentum. During a time interval Δt the engine exerts a force that accelerates some of the fuel Δm, expelling it from the rocket with exhaust velocity $\mathbf{u}$. By Newton's third law, there is an equal and opposite force on the rocket, propelling the rocket in the opposite direction. Another way to look at this is that the center of mass of the expelled mass and the rocket moves at constant velocity. Hence, if Δm is accelerated backward, the rocket must be accelerated forward.

Suppose that a rocket coasts in deep space with its engines turned off and that external forces are negligible. Let the mass of the rocket be $M + \Delta m$, and let the rocket coast at velocity $\mathbf{v}$ with respect to our coordinate system. At time t a thruster engine fires and expels mass Δm

in the time interval Δt. What is the velocity $\mathbf{v} + \Delta\mathbf{v}$ of the rocket body at time $t + \Delta t$?



Comparing the momentum at the initial time t and at a slightly later time $t + \Delta t$, when mass Δm has been expelled with velocity $\mathbf{u}$ with respect to the rocket, we have

$$\begin{aligned}\mathbf{P}(t) &= (M + \Delta m)\mathbf{v}\\ \mathbf{P}(t + \Delta t) &= M(\mathbf{v} + \Delta\mathbf{v}) + \Delta m(\mathbf{v} + \Delta\mathbf{v} + \mathbf{u})\\ \Delta\mathbf{P} &= M\Delta\mathbf{v} + \Delta m\,(\Delta\mathbf{v} + \mathbf{u}).\end{aligned}$$

Hence the rate of change of the system's momentum is

$$\begin{aligned}\frac{\mathbf{d}P}{dt} &= \lim_{\Delta t\to 0}\left(M\frac{\Delta\mathbf{v}}{\Delta t} + \frac{\Delta m}{\Delta t}(\Delta\mathbf{v} + \mathbf{u})\right)\\ &= M\frac{d\mathbf{v}}{dt} + \mathbf{u}\frac{dm}{dt}\\ &= M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}.\end{aligned}$$

In this last equation we used the identity $dm/dt = -dM/dt$, because the expelled mass decreases the total mass of the rocket. The equation of rocket motion in free space is therefore

$$M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt} = 0.$$

If an external force $\mathbf{F}$ such as gravity acts on the system, the general equation for rocket motion becomes

$$\mathbf{F} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}.$$

Example 4.15 Center of Mass and the Rocket Equation

In this example we shall derive the rocket equation using center of mass considerations. The general expression for the velocity $\dot{\mathbf{R}}$ of the center of mass of a system of two masses m_1 and m_2 is

$$\dot{\mathbf{R}} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{(m_1 + m_2)}.$$

If no external forces act on the system, $\ddot{\mathbf{R}} = 0$, so the center of mass moves with constant velocity. In the inertial frame moving along with the rocket, $\dot{\mathbf{R}} = 0$.

Using the same notation as in our previous derivation (Example 4.14),

$$\dot{\mathbf{R}} = \frac{M\Delta v + \Delta m(\mathbf{u} + \Delta v)}{M + \Delta m} = 0.$$

Hence

$$M\Delta v + \Delta m(\mathbf{u} + \Delta v) = 0.$$

Dividing by Δt and taking the limit $\Delta t \to 0$ gives

$$M\frac{d\mathbf{v}}{dt} + \mathbf{u}\frac{dm}{dt} = 0.$$

The second-order term $\Delta m\Delta\mathbf{v}$ does not contribute in the limit. Then, with the identity $dm/dt = -dM/dt$,

$$M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt} = 0$$

as before.

Our approach to rocket motion illustrates a powerful method for analyzing physical problems. It is easy to become confused trying to take into account the detailed acceleration of Δm and the rocket body while they separate. But these details vanish when taking the limit; their effect is actually included in the final equation of motion. The correct equation of motion results from taking the limit and including only the nonvanishing "first-order" terms. Terms beyond first order, such as $\Delta m\Delta\mathbf{v}$, vanish in the limit $\Delta t \to 0$.

Here are three examples on rocket motion.

Example 4.16 Rocket in Free Space

If there is no external force on a rocket, $\mathbf{F} = 0$, and the rocket's motion is given by

$$M\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dM}{dt}$$

or

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt}.$$

Checking signs—always useful—we expect the rocket to accelerate $(d\mathbf{v}/dt > 0)$ while its mass decreases $(dM/dt < 0)$. To make both sides of the last equation positive, $\mathbf{u} < 0$, which means that the mass is expelled in the backward direction, as expected.

The exhaust velocity $\mathbf{u}$ is usually constant, in which case it is easy to integrate the equation of motion:

$$\int_{t_0}^{t_f} \frac{d\mathbf{v}}{dt}\,dt = \mathbf{u}\int_{t_0}^{t_f} \frac{1}{M}\frac{dM}{dt}\,dt$$

$$\int_{\mathbf{v}_0}^{\mathbf{v}_f} d\mathbf{v} = \mathbf{u}\int_{M_0}^{M_f} \frac{dM}{M}$$

or

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\frac{M_f}{M_0} = -\mathbf{u}\ln\frac{M_0}{M_f}.$$

If $\mathbf{v}_0 = 0$, then

$$\mathbf{v}_f = -\mathbf{u}\ln\frac{M_0}{M_f}.$$

The final velocity is independent of how the mass is released—the fuel can be expended rapidly or slowly without affecting $\mathbf{v}_f$. The only important quantities are the exhaust velocity and the ratio of initial to final mass.

The situation is quite different if a gravitational field is present, as shown by the next example.

Example 4.17 Rocket in a Constant Gravitational Field

If a rocket takes off in a constant gravitational field $\mathbf{F} = M\mathbf{g}$, the equation for rocket motion becomes

$$M\mathbf{g} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt},$$

where $\mathbf{u}$ and $\mathbf{g}$ are directed down and are assumed to be constant

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt} + \mathbf{g}.$$

Integrating with respect to time we obtain

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\left(\frac{M_f}{M_0}\right) + \mathbf{g}(t_f - t_0).$$

Let $\mathbf{v}_0 = 0$ and $t_0 = 0$, with velocity $\mathbf{v}$ positive upward:

$$v_f = -u\ln\left(\frac{M_0}{M_f}\right) - gt_f.$$

Now there is a premium attached to burning the fuel rapidly. The shorter the burn time, the greater the final velocity. This is why the takeoff of a large rocket is so spectacular—it is essential to burn the fuel as quickly as possible.

Example 4.18 Saturn V

The Saturn V ("Five") three-stage rocket, one of the most powerful expendable launch vehicles ever constructed, fulfilled its purpose by sending Apollo astronauts to land on the Moon on six different missions. The first stage was powered by five enormous F-1 rocket engines (each nearly 6 m tall and 4 m diameter at the outlet). The F-1 engines burned a hydrocarbon similar to kerosene, with liquid oxygen as the oxidizer. All of these materials had to be carried by the rocket; a fully fueled Saturn V had a total mass of 3.0×10^6 kg, of which 2.1×10^6 kg was the fuel for the first stage. All the first-stage fuel was expended in 168 seconds.

The rocket equation with constant gravity is

$$M\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dM}{dt} + M\mathbf{g}. \tag{1}$$

The first term on the right-hand side of Eq. (1) is called the "thrust." The second term is the weight of the rocket, directed vertically downward. At launch, the weight was 2.9×10^7 N, but this decreased rapidly as the first stage burned its fuel. The five F-1 engines in the first stage produced a total thrust of 3.4×10^7 N, somewhat greater than the initial weight. The initial upward acceleration, as you can easily verify, was only about 0.17 g.

Where does the thrust come from? Because $\mathbf{u}\Delta M$ is the momentum carried off by the expelled gases in time Δt, the thrust is the rate at which momentum is carried off by the burning fuel. Because both $\mathbf{u}$ and dM/dt are negative, the thrust is positive, opposite to $\mathbf{g}$.

Fuel is a precious commodity on rockets. To minimize the fuel mass required for a given thrust, the exhaust velocity must be as large as possible. The exhaust velocity for the first stage F-1 engines was 2600 m/s, but the second and third stages used liquid hydrogen and liquid oxygen, giving an exhaust velocity of 4100 m/s.

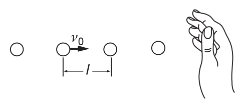
Evaluating the right-hand side of Eq. (1) for the first stage gives $(2600\ \text{m/s})(2.1 \times 10^6\ \text{kg})/168\ \text{s} = 3.4 \times 10^7$ N, in good agreement with the thrust.

Rocket data tables often do not list the exhaust velocity but instead a quantity called the "specific impulse," which is the exhaust velocity divided by g. Specific impulse has units of seconds, and is therefore independent of whether we use SI, CGS, or English units.

4.9 Momentum Flow and Force

When catching a ball one expects to feel a recoil force or, more precisely, to experience an impulse. The concepts of momentum and impulse are reasonably intuitive: the recoil that we experience is merely the reaction to the impulse we must deliver to the ball to bring it to rest. Closely related to these concepts, though perhaps less intuitive, is the concept of momentum flow. Anyone who has been on the receiving end of a stream of water from a hose knows that a stream can exert a force. If the stream is intense, as in the case of a fire hose, the push can be dramatic—a jet of high pressure water can break through the wall of a burning building.

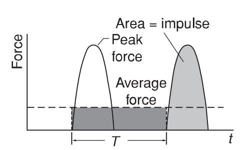
How can a column of water flying through the air exert a force that is every bit as real as a force transmitted by a rigid steel rod? The origin of the force can be visualized by picturing the stream as a series of small uniform droplets each of mass m traveling with velocity v. Let the droplets be distance l apart. Assume that the drops collide with your hand without rebounding, with final velocity $v_f = 0$, and then simply fall to the ground. Consider the force exerted by your hand on the stream. As each drop hits your hand there is a large force for a short time. Although we do not know the instantaneous force, we can find the impulse I_{droplet} given *to* each drop *by* your hand:



$$\begin{aligned} I_{\text{droplet}} &= \int_{1\text{ collision}} F\,dt \\ &= \Delta p \\ &= m(v_f - v) \\ &= -mv. \end{aligned}$$

By Newton's third law, the impulse delivered to your hand by the droplet is equal and opposite to the impulse delivered to the droplet by your hand:

$$I_{\text{hand}} = mv.$$



The positive sign means that the impulse to your hand is in the same direction as the velocity of the droplet. The impulse equals the area under one of the peaks of the instantaneous force shown in the drawing.

If there are many collisions per second, you feel the average force F_{av} (indicated by the dashed line in the drawing) rather than the shock of individual drops. If the average time between collisions is T, then the area under F_{av} during the time T is identical to the impulse due to one droplet

$$F_{\text{av}}T = \int_{1\text{ collision}} F\,dt = mv.$$

The average distance between droplets is $l = vT$ and so the average force exerted by the stream can be written

$$F_{\text{av}} = \frac{mv}{T} = \frac{mv^2}{l}. \tag{4.8}$$

Momentum transfer by a stream is the physics underlying the force on a wind turbine blade and the lift on an airplane wing.

This description of successive collisions generating an average force employed an idealized model of a stream of water, but the model is pretty accurate for a related scenario: laser slowing of atoms. Just as a stream of water can exert a force on a hand, a stream of light can exert a force on an atom. The force can be so large that atoms are brought to near rest almost instantaneously. This process is the first step in the creation of ultracold atomic gases, in which atoms are cooled into the sub-microkelvin temperature regime using laser light.

Example 4.19 Slowing Atoms with Laser Light

Understanding how laser light can slow atoms requires a few facts from quantum physics that we will simply state without stopping to explain them.

The starting point is the classical description of light. According to the electromagnetic theory of James Clerk Maxwell, light is a wave of oscillating electric and magnetic fields that carries energy. The speed of a light wave, c, its wavelength λ, and its frequency ν are related by the familiar wave condition $c = \lambda\nu$.

Einstein put forth an alternate picture of light that seems at first sight to be totally incompatible with Maxwell's: light energy is received in discrete bundles or *quanta*, now called *photons*, with particle-like properties. Einstein argued that the energy of a photon associated with light of frequency ν is $h\nu$, where the constant h, known as Planck's constant, has the numerical value of 6.63×10^{-34} kg · m^2/s. Einstein also argued that each photon carries momentum $h\nu/c$ or, equivalently, h/λ.

If a gas of atoms is heated or excited by an electrical discharge, the atoms radiate light at characteristic wavelengths. They can also absorb light at those wavelengths. Niels Bohr proposed that the energy of atoms cannot vary arbitrarily, as we expect in classical physics, but that atoms exist only in certain states that he called *stationary states*. If the lowest-lying state, called the *ground state*, has energy E_a and an excited state has energy E_b, then an excited atom can get rid of its energy by creating a photon. Conservation of energy requires that $h\nu = E_b - E_a$. Thus, the different colors radiated by atoms reflect their particular stationary states.

An excited atom rapidly jumps back to its ground state by emitting a photon, a process called *spontaneous emission*. The process is similar to radioactive decay with a characteristic decay time τ that is typically tens of nanoseconds. To complete the description, we need one further concept, also proposed by Einstein, *stimulated emission*. An atom in an excited state will in time spontaneously emit a photon, but if it is illuminated by that frequency, the emission will occur sooner. Stimulated emission is the fundamental process in the generation of laser light. (The term "LASER" is an acronym for Light Amplification by Stimulated Emission of Radiation.)

In a laser-cooling apparatus, a stream of atoms initially in their ground state flows into a high vacuum through a small aperture. Laser light tuned to one of the atom's spectral lines is directed toward the aperture. Lasers are so intense that their light causes absorption and stimulated emission in times much shorter than the decay time τ. In such a situation, the atom can be viewed as being in the ground state half the time, and in its excited state the other half.

Because the laser light is directed against the motion of the atoms, every time an atom absorbs a photon it recoils with a momentum kick, or impulse, $\Delta p = h/\lambda$. The atom also experiences a momentum kick when it emits a photon, causing it to recoil in the opposite direction from the emission. However, spontaneous emission occurs in random directions and its momentum kicks average out. Consequently, the atom experiences a series of kicks that retard its motion. The time-average force is given by

$$F_{\text{av}} = \frac{1}{2}\frac{\Delta p}{\tau} = \frac{1}{2}\frac{h}{\lambda\tau},$$

where the factor of 1/2 takes into account that the atom is in the excited state only half the time. If the mass of the atom is M, then the average acceleration is

$$a_{\text{av}} = \frac{F_{\text{av}}}{M} = \frac{1}{2M}\frac{h}{\lambda\tau}.$$

The first experiments in laser slowing were to a stream of sodium atoms. For sodium, the wavelength of the excited state transition is $\lambda = 589 \times 10^{-9}$ m, $M = 3.85 \times 10^{-26}$ kg, and $\tau = 15 \times 10^{-9}$ s. Using $h = 6.6 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$,

$$a_{\text{av}} = 9.7 \times 10^5\ \text{m/s}^2.$$

This acceleration is about 10^5 times larger than the acceleration of gravity! The average speed of sodium atoms at room temperature is about 560 m/s, and laser slowing can bring them essentially to rest in less than a meter.

4.10 Momentum Flux

In Section 4.9 we found that the average force on a surface due to a perpendicular stream of droplets of mass m moving with velocity v and separated by distance l is given by Eq. (4.8):

$$F_{\text{av}} = \frac{m}{l}v^2.$$

This expression has a natural interpretation. The quantity mv/l is the average momentum per unit length in the stream of particles. If we multiply this by v, the distance per second that the droplets travel, we obtain the momentum per second carried by the stream past any point, that is, the *rate* of momentum transport. Thus, the average force exerted on a surface is the rate at which the stream transports momentum to the surface.

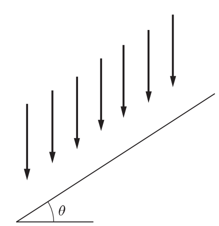
More realistic than a hypothetical stream of particles is a real stream of matter, for instance a stream of water in a hose with cross-section A, flowing with speed v in the direction given by the unit vector $\hat{\mathbf{v}}$. If the water has mass density $\rho_m(\text{kg/m}^3)$, then the mass per unit length in the stream is $\rho_m A$ and the momentum per unit length is $\rho_m vA$. The rate at which momentum flows through a hypothetical surface across the stream is

$$\dot{\mathbf{P}} = \rho_m v^2 A\,\hat{\mathbf{v}}.$$

If the stream is brought to a halt by striking a solid surface, the force exerted by the surface must cause the stream to lose momentum at the same rate the stream transports momentum to the surface. The force of the surface on the stream is therefore $-\dot{\mathbf{P}}$. The reaction force of the stream on the surface is

$$\mathbf{F}_{\text{on surface}} = +\dot{\mathbf{P}} = \rho_m v^2 A\,\hat{\mathbf{v}}.$$

As expected, the direction of the force on the surface is in the direction of flow $\hat{\mathbf{v}}$. If the stream does not come to rest at the surface but is reflected straight back, then the surface must exert the force needed not only to cancel the incoming momentum but also to generate the outgoing momentum, doubling the force on the surface. On the other hand, if the surface is transparent so that the matter simply passes through, then momentum is carried to and away from the surface at the same rate; the net rate of momentum transfer to the surface is zero and hence there is no force.



If the surface is not perpendicular to the flow, but tilted at angle θ as shown in the drawing, then the momentum flow to the surface is

$$\dot{P} = \rho_m v^2 A\cos\theta$$

if the momentum is cancelled at the surface.

It is useful to introduce the vector $\mathbf{J}$ with magnitude $\rho_m v^2$, and directed along the flow $\hat{\mathbf{v}}$:

$$\mathbf{J} = \rho_m v^2\hat{\mathbf{v}}.$$

The vector $\mathbf{J}$ is called the *flux density* of the stream.

It is also useful to describe the area by a vector $\mathbf{A}$. The magnitude of $\mathbf{A}$ is numerically equal to the area and its direction is perpendicular to the surface, as described by a unit vector $\hat{\mathbf{n}}$ normal to the surface. The normal vector can lie in either of two directions. We choose the following convention: in evaluating the momentum transfer through a surface into a system, $\hat{\mathbf{n}}$ is positive if it points inward.

Because the momentum transfer by a stream is in the direction of flow $\hat{\mathbf{v}}$, the rate of momentum transfer to a surface—consequently, the force on the system—is

$$\dot{\mathbf{P}} = (\mathbf{J} \cdot \mathbf{A})\,\hat{\mathbf{v}}.$$

The quantity $\dot{\mathbf{P}} = (\mathbf{J} \cdot \mathbf{A})\,\hat{\mathbf{v}}$ is called the *flux* (or *flow*) of momentum to the surface.

Momentum is a vector, so momentum flux is a vector. Flux as a vector occurs often in physics, particularly in fluid dynamics and electromagnetic theory.

In situations such as a stream of water rebounding from a surface, momentum can be transported both to the surface and away from it. According to our convention with $\hat{\mathbf{n}}$ pointing inward through a surface surrounding a system, flux is positive $\mathbf{J} \cdot \mathbf{A} > 0$ if the momentum flows in and negative $\mathbf{J} \cdot \mathbf{A} < 0$ if it flows out. The total force on a system due to a number of sources of momentum flow can be written

$$\mathbf{F}_{\text{tot}} = \sum_k \dot{\mathbf{P}}_k = \sum (\mathbf{J}_k \cdot \mathbf{A}_k)\hat{\mathbf{v}}_k$$

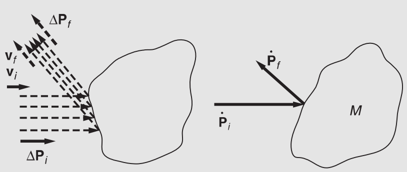
where the sum is over all the surface elements through which momentum flows. As the number of elements is increased, the sum becomes what is called a *surface integral*. However, there is no need for us to take that limit now because we shall be concerned here only with surfaces of simple geometry.

A helpful way to calculate the force on a system in which there are several sources of momentum flow is to sum the inward flow terms into a total inward flow $\dot{\mathbf{P}}_{\text{in}}$, and all the outgoing flow terms into a total outward flow $\dot{\mathbf{P}}_{\text{out}}$. The total force on the system can then be written

$$\mathbf{F}_{\text{tot}} = \dot{\mathbf{P}}_{\text{in}} - \dot{\mathbf{P}}_{\text{out}}.$$

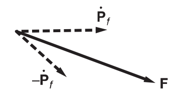
Example 4.20 Reflection from an Irregular Object

A stream is reflected from an object, as shown in the drawing.



The incident momentum flux is $\dot{\mathbf{P}}_{\text{i}}$ and the reflected (outgoing) flux is $\dot{\mathbf{P}}_{\text{o}}$. Hence, the total force on the system is

$$\mathbf{F} = \dot{\mathbf{P}}_{\text{i}} - \dot{\mathbf{P}}_{\text{o}}.$$



Example 4.21 Solar Sail Spacecraft

Interplanetary exploration has many applications for a lightweight fuel-saving spacecraft capable of carrying a small instrument package across the solar system. One attractive design is the solar sail craft, which sports a large sail made of thin plastic sheet; propulsion is the force exerted by sunlight—the momentum carried by photons.

Light from the Sun arrives at the Earth with an energy flux density called the *solar constant*, which has the value $S_{\text{sun}} = 1370\,\text{watts/m}^2$. The solar constant can be viewed as a flux of photons with energies $h\nu$ that vary over a wide frequency range. Details of the Sun's spectrum are unimportant because every photon carries momentum $h\nu/c$. Consequently, the momentum flux density of sunlight at the Earth is simply $S_{\text{sun}}/c = 1370/(3\times10^8) = 4.6\times10^{-6}\ \text{kg/(m\,s}^2)$, where we have used 1 watt = 1 joule/s = 1 kg m^2/s^3. This can also be written $S_{\text{sun}}/c = 4.6\times10^{-6}$ (kg m/s)/(m^{2}s), showing that it has the units of momentum per unit area per unit time.

In 2010, Japan launched a solar sail craft called IKAROS ("Interplanetary Kite-craft Accelerated by Radiation Of the Sun"). The sail was very thin polyimide (Kapton®), only 7.5×10^{-6} m thick. The sail's area was $A = 150\,\text{m}^2$, and the craft's total mass was $M = 1.6\,\text{kg}$. The craft was lifted by a chemical rocket into space, where the sail unfurled. Rotation at 25 revolutions/minute about the axis kept the sail flat, eliminating the need for struts that would have added to the mass. IKAROS traveled to the orbit of Venus, the first craft to demonstrate solar sail technology in deep space.

We shall calculate the initial acceleration of IKAROS when it started out near the Earth. Suppose that the solar sail is a perfect reflector, and that all the sunlight is reflected back. The total force on the sail is

$$\mathbf{F} = \dot{\mathbf{P}}_{\text{in}} - \dot{\mathbf{P}}_{\text{out}} = \frac{2S_{\text{sun}}\mathbf{A}}{c}.$$

Near Earth orbit, the magnitude of the acceleration a_{photon} due to photons is therefore

$$\begin{aligned} a_{\text{photon}} &= \frac{F}{M} = \left(\frac{2S_{\text{sun}}}{c}\right)\frac{A}{M} \\ &= 2\left(4.6\times10^{-6}\ \text{kg/(m s}^2)\right)\left(\frac{150\ \text{m}^2}{1.6\,\text{kg}}\right) \\ &= 8.6\times10^{-4}\ \text{m/s}^2. \end{aligned}$$

The Sun exerts an inward gravitational acceleration g_{sun}. Near Earth orbit,

$$g_{\text{sun}} = \frac{GM_{\text{sun}}}{R_E{}^2}$$

where $R_E = 1.50 \times 10^{11}$ m is the mean distance of the Earth from the Sun, known as the *astronomical unit (AU)*

$$g_{\text{sun}} = \frac{(6.7 \times 10^{-11}\text{m}^3/(\text{kg s}^2))(2.0 \times 10^{30}\text{ kg})}{(1.5 \times 10^{11}\text{ m})^2} = 5.9 \times 10^{-3}\text{ m/s}^2.$$

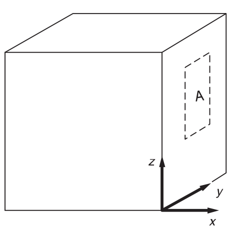
The net acceleration is

$$\begin{aligned} a_{\text{net}} &= a_{\text{photon}} - g_{\text{sun}} \\ &= (0.86 - 5.9) \times 10^{-3}\text{ m/s}^2 \\ &= -5.0 \times 10^{-3}\text{ m/s}^2. \end{aligned}$$

The craft falls inward toward the Sun. Because the solar intensity and the solar gravity both vary as the inverse square, the acceleration increases as the craft moves toward the Sun, but is always directed inward. However, during IKAROS' flight the radiation force slowed the craft enough to allow it to come near Venus. A craft with a much larger sail would be needed to travel outward toward Jupiter.

Example 4.22 Pressure of a Gas

The pressure of a gas arises from momentum flow to and from the enclosing surfaces due to the random motion of the particles in the gas. The tiny gas molecules exert a real force, because of their momentum. Consider a thin aluminum can containing a carbonated beverage. Before opening, the can feels strong and rigid because of the outward pressure of the gas on the walls. When the tab is popped and the excess pressure is released, the can is weak and easily crushed.



Consider a gas of n particles per unit volume, each having mass m. The mass density is $\rho = nm$ kg/m^3. Let us find the momentum transport to a surface of the container having area A, oriented in the y–z plane, as shown.

Although particles move in all directions, we shall be concerned only with motion in the x direction. We suppose for the moment that the particles have only a single x velocity, v_x, but they are just as likely to move in the $-x$ direction as the $+x$ direction. Hence, at any instant the density of particles moving toward the wall is $\rho/2$. They transport momentum in the $+x$ direction at the rate

$$\dot{P}_x = \frac{\rho}{2} v_x^2 A.$$

Particles must leave the wall at the same rate as they approach; otherwise, they would accumulate at the surface. Hence, the momentum flow away from the wall is

$$\dot{P}_{-x} = \frac{\rho}{2} v_x^2 A$$

and the total force on the wall is

$$F_x = \dot{P}_x - \dot{P}_{-x} = \rho v_x^2 A.$$

Let us now drop the simplistic assumption that atoms move only in the positive or negative x directions, with a single speed v_x. We expect atoms to move in random directions with speeds that change as they collide. If we calculate the contribution to the pressure for particles moving in a small range of velocities, we would obtain the above result, with the understanding that it is for the *average* force. Hence

$$\overline{F_x} = \overline{\dot{P}_x} - \overline{\dot{P}_{-x}} = \rho \overline{v_x^2} A$$

where the bar indicates an average over all the particles.

The pressure of a gas is the force per unit area on the surface. Consequently, the pressure on the area normal to the x–y plane is

$$\mathcal{P}_x = F_x/A = \rho \overline{v_x^2}.$$

(Here we use the symbol $\mathcal{P}$ for pressure, to distinguish it from P, the symbol for total momentum.) Because the pressure of a gas is the same in all directions, we expect a similar result for pressure on surfaces that are normal to the y and z axes. This would be the case if

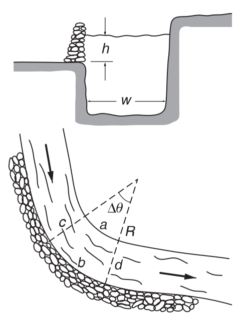
$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2},$$

where $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$. Consequently,

$$\mathcal{P} = (1/3)\rho\overline{v^2}.$$

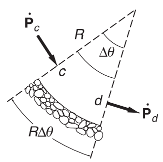
This result provides a crucial link connecting the concepts of heat, energy, and microscopic motion, topics we shall pursue in Chapter 5.

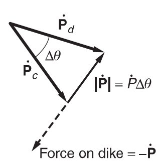
Example 4.23 Dike at the Bend of a River



The problem is to build a dike at the bend of a river to prevent flooding when the river rises. The dike must be strong enough to withstand the static pressure of the river ρgh, where ρ is the density of the water and h is the height from the base of the dike to the surface of the water. We can understand from Newton's laws that the outer bank and the dike must exert a sideways force on the stream to deflect it from straight-line flow. The dike must therefore withstand the dynamic pressure due to the deflection of the flow in addition to the static pressure. How do the dynamic and static pressures compare?

We approximate the bend by a circular curve with radius R, and focus our attention on a short length of the curve subtending angle $\Delta\theta$. We need concern ourselves only with the height h of the river above the





base of the dike. Let us calculate the momentum flux to the volume bounded by the river bank a, by the dike b, and by the fictitious surfaces across the river c and d. The river flows with velocity $\mathbf{v}$; because the cross-sectional area is constant, the magnitude of $\mathbf{v}$ is constant.

Momentum flows through surfaces c and d at rates $\dot{\mathbf{P}}_{\text{in,c}} = \rho v^2 A\hat{\mathbf{v}}_{\text{c}}$ and $\dot{\mathbf{P}}_{\text{out,d}} = \rho v^2 A\hat{\mathbf{v}}_{\text{d}}$, respectively. Here $A = hw$ is the cross-sectional area of the river lying above the base of the dike. The total rate of momentum transfer to the bounded volume is

$$\dot{\mathbf{P}} = \dot{\mathbf{P}}_{\text{in,c}} - \dot{\mathbf{P}}_{\text{out,d}} = \rho v^2 A(\hat{\mathbf{v}}_{\text{c}} - \hat{\mathbf{v}}_{\text{d}}).$$

From the drawing, the magnitude of the momentum transfer is

$$\dot{P} = \rho v^2 A(2\sin\Delta\theta/2).$$

The momentum transfer points to the center of the bending circle. The dike must provide a force to account for this momentum transfer, and the reaction to that force gives rise to the dynamic pressure on the dike.

To calculate the pressure we can take the small angle limit $\sin\Delta\theta/2 \approx \Delta\theta/2$, and consider the force arising from a section of the dike subtending angle $\Delta\theta$, with area $(R\Delta\theta)h$. The dynamic force on the dike is radially outward, and has magnitude $\dot{P} \approx \rho v^2 A\Delta\theta$. The force is exerted over the area $(R\,\Delta\theta)h$, and the dynamic pressure is therefore

$$\begin{aligned}\text{dynamic pressure} &= \frac{\dot{P}}{R\,\Delta\theta\, h}\\ &= \frac{\rho v^2 A\Delta\theta}{R\,\Delta\theta h}\\ &= \frac{\rho v^2 A}{Rh}\\ &= \frac{\rho v^2 w}{R}.\end{aligned}$$

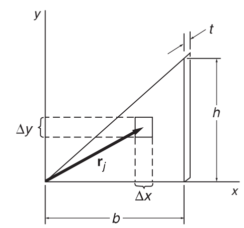
The ratio of dynamic to static pressure is

$$\begin{aligned}\frac{\text{dynamic pressure}}{\text{static pressure}} &= \frac{\rho v^2 w}{R}\frac{1}{\rho g h} = \frac{w}{h}\frac{v^2}{Rg}\\ &= \frac{\text{width}}{\text{height}} \times \frac{\text{centripetal acceleration}}{g}.\end{aligned}$$

For a river in flood with a speed of 10 mph (approximately 15 ft/s), a radius of 2000 ft, a flood height of 3 ft, and a width of 200 ft, the ratio is 0.22, so that the dynamic pressure is by no means negligible.

Note 4.1 Center of Mass of Two- and Three-dimensional Objects

In this note we shall find the center of mass of some multidimensional objects. These examples are straightforward if you have had experience evaluating two- or three-dimensional integrals. Otherwise, read on.



1. Uniform Triangular Plate

Consider the two-dimensional case of a uniform right triangular plate of mass M, base b, height h, and small thickness t. We treated this problem in Example 4.4, reducing it to a single integral. Here we shall treat it in a more general way using a double integral, an approach that would also apply to cases where the density is not uniform.

Divide the plate into small rectangular areas of sides Δx and Δy, as shown. The volume of each element is $\Delta V = t\,\Delta x\,\Delta y$, and

$$\begin{aligned}\mathbf{R} &\approx \frac{\Sigma\, m_j \mathbf{r}_j}{M}\\ &= \frac{\Sigma\,\rho_j\, t\,\Delta x\,\Delta y\,\mathbf{r}_j}{M},\end{aligned} \tag{1}$$

where j is the label of a volume element and ρ_j is its density. Because the plate is uniform,

$$\rho_j = \text{constant} = \frac{M}{V} = \frac{M}{At},$$

where $A = bh/2$ is the area of the plate.

We can evaluate the sum in Eq. (1) by summing first over the Δx's and then over the Δy's, instead of over the single index j. This gives a double sum that can be converted to a double integral by taking the limit, as follows:

$$\begin{aligned}\mathbf{R} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left(\frac{M}{At}\right)\left(\frac{t}{M}\right)\sum\sum \mathbf{r}_j\,\Delta x\,\Delta y\\ &= \frac{1}{A}\iint \mathbf{r}\,dx\,dy.\end{aligned}$$

Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ be the position vector of the element $dx\,dy$. Then, writing $\mathbf{R} = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}$, we have

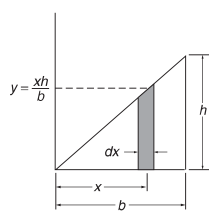
$$\begin{aligned}\mathbf{R} &= X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}\\ &= \frac{1}{A}\iint (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\,dx\,dy\\ &= \frac{1}{A}\left(\iint x\,dx\,dy\right)\hat{\mathbf{i}} + \frac{1}{A}\left(\iint y\,dx\,dy\right)\hat{\mathbf{j}}.\end{aligned}$$

Hence the coordinates of the center of mass are

$$\begin{aligned}X &= \frac{1}{A}\iint x\,dx\,dy\\ Y &= \frac{1}{A}\iint y\,dx\,dy.\end{aligned}$$

The double integrals may look strange, but they are easily evaluated. Consider first the double integral for X:

$$X = \frac{1}{A}\iint x\,dx\,dy.$$



This integral instructs us to take each element, multiply its area by its x coordinate, and sum the results. We can do this in stages by first considering the elements in a strip parallel to the y axis. The strip runs from $y = 0$ to $y = xh/b$. (By similar triangles, any point on the slanted boundary obeys the relation $y/x = h/b$.)

Each element in the strip has the same x coordinate, and the contribution of the strip to the double integral is

$$\frac{1}{A} x\,dx \int_0^{xh/b} dy = \frac{h}{bA} x^2\,dx.$$

Finally, we sum the contributions of all such strips $x = 0$ to $x = b$ to find

$$\begin{aligned} X &= \frac{h}{bA}\int_0^b x^2\,dx = \frac{h}{bA}\frac{b^3}{3} \\ &= \frac{hb^2}{3A}. \end{aligned}$$

Since $A = \frac{1}{2}bh$,

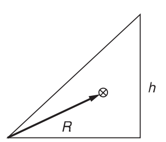
$$X = \tfrac{2}{3}b.$$

Similarly for Y,

$$\begin{aligned} Y &= \frac{1}{A}\int_0^b \left(\int_0^{xh/b} y\,dy\right)dx \\ &= \frac{h^2}{2Ab^2}\int_0^b x^2\,dx = \frac{h^2 b}{6A} \\ &= \tfrac{1}{3}h. \end{aligned}$$

Hence

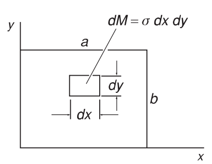
$$\mathbf{R} = \tfrac{2}{3}b\hat{\mathbf{i}} + \tfrac{1}{3}h\hat{\mathbf{j}}.$$



Although the coordinates of $\mathbf{R}$ depend on the particular coordinate system we choose, the position of the center of mass with respect to the triangular plate is, of course, independent of the coordinate system.

2. Non-uniform Rectangular Plate

Find the center of mass of a thin non-uniform rectangular plate with sides of length a and b, whose mass per unit area σ varies as $\sigma = \sigma_0(xy/ab)$, where σ_0 is a constant.



$$\mathbf{R} = \frac{1}{M}\iint (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\sigma\,dx\,dy.$$

We first find M, the mass of the plate:

$$\begin{aligned} M &= \int_0^b\int_0^a \sigma\,dx\,dy \\ &= \int_0^b\int_0^a \sigma_0\frac{x}{a}\frac{y}{b}\,dx\,dy. \end{aligned}$$

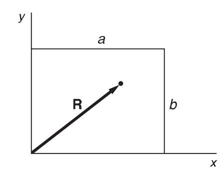
Integrate over x, treating y as a constant

$$\begin{aligned} M &= \int_0^b \sigma_0\frac{y}{b}\left(\int_0^a \frac{x}{a}\,dx\right)dy \\ &= \int_0^b \sigma_0\frac{y}{b}\left(\left.\frac{x^2}{2a}\right|_{x=0}^{x=a}\right)dy \\ &= \int_0^b \sigma_0\frac{y}{b}\frac{a}{2}\,dy. \end{aligned}$$

Then integrate over y

$$= \left.\frac{\sigma_0 a}{2}\frac{y^2}{2b}\right|_{y=0}^{y=b} = \frac{1}{4}\sigma_0 ab.$$

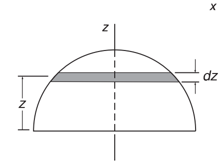
Using the same approach, the x component of $\mathbf{R}$ is



$$\begin{aligned} X &= \frac{1}{M}\iint x\sigma\,dx\,dy \\ &= \frac{1}{M}\int_0^b \frac{\sigma_0}{ab}y\left(\int_0^a x^2\,dx\right)dy \\ &= \frac{1}{M}\int_0^b \frac{\sigma_0}{ab}y\left(\left.\frac{x^3}{3}\right|_0^a\right)dy \\ &= \frac{1}{M}\frac{\sigma_0}{ab}\int_0^b \frac{ya^3}{3}\,dy \\ &= \frac{1}{M}\frac{\sigma_0}{ab}\frac{a^3}{3}\frac{b^2}{2} \\ &= \frac{4}{\sigma_0 ab}\frac{\sigma_0 a^2 b}{6} \\ &= \frac{2}{3}a. \end{aligned}$$

Similarly, $Y = \frac{2}{3}b$.

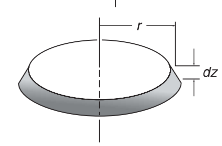
3. Uniform Solid Hemisphere



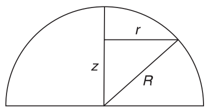
Find the center of mass of a uniform solid hemisphere of radius R and mass M.

From symmetry it is apparent that the center of mass lies on the z axis. Its height above the equatorial plane is

$$Z = \frac{1}{M}\int z\,dM.$$



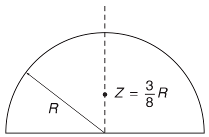
The integral is over three dimensions, but the symmetry of the situation lets us treat it as a one-dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider a circular disk of radius r and thickness dz. Its volume is $dV = \pi r^2 dz$, and its mass is $dM = \rho\,dV = (M/V)(dV)$, where $V = \frac{2}{3}\pi R^3$.



Hence

$$Z = \frac{1}{M}\int \frac{M}{V} z\, dV$$
$$= \frac{1}{V}\int_{z=0}^{R} \pi r^2 z\, dz.$$

To evaluate the integral we need to find r in terms of z.
Since $r^2 = R^2 - z^2$, we have



$$Z = \frac{\pi}{V}\int_0^R z(R^2 - z^2)\, dz$$
$$= \frac{\pi}{V}\left(\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\right)\Big|_0^R$$
$$= \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right)$$
$$= \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3}$$
$$= \frac{3}{8}R.$$

Problems

*For problems marked *, refer to page 521 for a hint, clue, or answer.*

4.1 *Center of mass of a non-uniform rod**
The mass per unit length of a non-uniform rod of length l is given by $\lambda = A\cos(\pi x/2l)$, where x is position along the rod, $0 \le x \le l$.
(*a*) What is the mass M of the rod?
(*b*) What is the coordinate X of the center of mass?

4.2 *Center of mass of an equilateral triangle*
Find the center of mass of a thin uniform plate in the shape of an equilateral triangle with sides a.

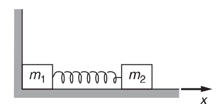
4.3 *Center of mass of a water molecule*
A water molecule H_2O consists of a central oxygen atom bound to two hydrogen atoms. The two hydrogen–oxygen bonds subtend an angle of 104.5°, and each bond has a length of 0.097 nm.
Find the center of mass of the water molecule.

4.4 *Failed rocket*
An instrument-carrying rocket accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The rocket breaks into two pieces that fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to Earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the Earth's curvature.

4.5 *Acrobat and monkey*
A circus acrobat of mass M leaps straight up with initial velocity v_0 from a trampoline. As he rises up, he takes a trained monkey of mass m off a perch at a height h above the trampoline.
What is the maximum height attained by the pair?

4.6 *Emergency landing*
A light plane weighing 2500 lb makes an emergency landing on a short runway. With its engine off, it lands on the runway at 120 ft/s. A hook on the plane snags a cable attached to a 250-lb sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is 0.4, and if the plane's brakes give an additional retarding force of 300 lb, how far does the plane go before it comes to a stop?

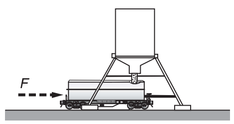
4.7 *Blocks and compressed spring*
A system is composed of two blocks of mass m_1 and m_2 connected by a massless spring with spring constant k. The blocks slide on a frictionless plane. The unstretched length of the spring is l. Initially m_2 is held so that the spring is compressed to $l/2$ and m_1 is forced against a stop, as shown. m_2 is released at $t = 0$.
Find the motion of the center of mass of the system as a function of time.



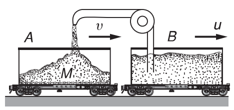
4.8 *Jumper*
A 50 kg woman jumps straight into the air, rising 0.8 m from the ground. What impulse does she receive from the ground to attain this height?

4.9 *Rocket sled*
A rocket sled moves along a horizontal plane, and is retarded by a friction force $f_{\text{friction}} = \mu W$, where μ is constant and W is the weight of the sled.
The sled's initial mass is M, and its rocket engine expels mass at constant rate $dM/dt \equiv \gamma$; the expelled mass has constant speed v_0 relative to the rocket.
The rocket sled starts from rest and the engine stops when half the sled's total mass is gone. Find an expression for the maximum speed.

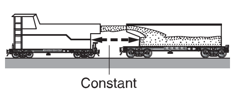
4.10 *Rolling freight car with sand*
A freight car of mass M contains a mass of sand m. At $t = 0$ a constant horizontal force F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at constant rate dm/dt. Find the speed of the freight car when all the sand is gone. Assume the freight car is at rest at $t = 0$.



4.11 *Freight car and hopper**
An empty freight car of mass M starts from rest under an applied force F. At the same time, sand begins to run into the car at steady rate b from a hopper at rest along the track.
Find the speed when a mass of sand m has been transferred.



4.12 *Two carts and sand*
Material is blown into cart A from cart B at a rate b kilograms per second, as shown. The material leaves the chute vertically downward, so that it has the same horizontal velocity u as cart B. At the moment of interest, cart A has mass M and velocity v. Find dv/dt, the instantaneous acceleration of A.



4.13 *Sand sprayer*
A sand-spraying locomotive sprays sand horizontally into a freight car as shown in the sketch. The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate $dm/dt = 10$ kg/s with a velocity 5 m/s relative to the locomotive. The freight car starts from rest with an initial mass of 2000 kg. Find its speed after 100 s.

4.14 *Ski tow*
A ski tow consists of a long belt of rope around two pulleys, one at the bottom of a slope and the other at the top. The pulleys are driven by a husky electric motor so that the rope moves at a steady speed of 1.5 m/s. The pulleys are separated by a distance of 100 m, and the angle of the slope is 20°.
Skiers take hold of the rope and are pulled up to the top, where they release the rope and glide off. If a skier of mass 70 kg takes the tow every 5 s on the average, what is the average force required to pull the rope? Neglect friction between the skis and the snow.

4.15 *Men and flatcar*
N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.
(*a*) What is the final velocity of the flatcar if all the men jump off at the same time?
(*b*) What is the final velocity of the flatcar if they jump off one at a time? (The answer can be left in the form of a sum of terms.)
(*c*) Does case (*a*) or case (*b*) yield the larger final velocity of the flatcar? Can you give a simple physical explanation for your answer?

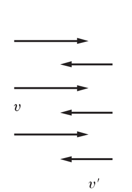
4.16 *Rope on table**
A rope of mass M and length l lies on a frictionless table, with a short portion, l_0, hanging through a hole. Initially the rope is at rest.

(*a*) Find a general equation for $x(t)$, the length of rope through the hole.
(*b*) Find the particular solution so that the initial conditions are satisfied.

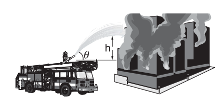
4.17 *Solar sail 1*
With reference to Example 4.21, what is the maximum film thickness for a space sail like IKAROS to be accelerated outward away from the Sun? Take the density of Kapton® to be 1.4 g/cm^3.

4.18 *Solar sail 2*
With reference to Example 4.21, consider the design of a solar sail intended to reach escape velocity from the Earth $\sqrt{2gR_e} = 11.2$ km/s using only the pressure due to sunlight. The sail is made of a Kapton® film 0.0025 cm thick with a density 1.4 g/cm^3. Take the solar constant to be 1370 watts/m^2, assumed to be constant during the acceleration.
(*a*) What is the acceleration near the Earth due to sunlight pressure alone?
(*b*) How far from the Earth, as measured in units of the Earth's radius, R_e, would the sail have to be launched so that it could escape from the Earth?
(*c*) What area of sail would be needed to accelerate a 1 kg payload at half the rate of the sail alone?

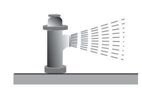
4.19 *Tilted mirror*
On the Earth, a mirror of area 1 m^2 is held perpendicular to the Sun's rays.
(*a*) What is the force on the mirror due to photons from the Sun, assuming that the mirror is a perfect reflector? The momentum flux density from the Sun's photons is $J_{\text{sun}} = 4.6 \times 10^{-6}$ kg/(m · s^2).
(*b*) Find how the force varies with angle if the mirror is tilted at angle α from the perpendicular.



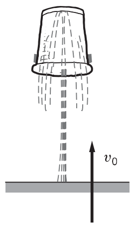
4.20 *Reflected particle stream**
A one-dimensional stream of particles of mass m with density λ particles per unit length, moving with speed v, reflects back from a surface, leaving with a different speed v', as shown. Find the force on the surface.



4.21 *Force on a fire truck*
A fire truck pumps a stream of water on a burning building at a rate K kg/s. The stream leaves the truck at angle θ with respect to the horizontal and strikes the building horizontally at height h above the nozzle, as shown. What is the magnitude and direction of the force on the truck due to the ejection of the water stream?



4.22 *Fire hydrant*
Water shoots out of a fire hydrant having nozzle diameter D, with nozzle speed V_0. What is the reaction force on the hydrant?



4.23 *Suspended garbage can**
An inverted garbage can of weight W is suspended in air by water from a geyser. The water shoots up from the ground with a speed v_0, at a constant rate K kg/s. The problem is to find the maximum height at which the garbage can rides. Neglect the effect of the water falling away from the garbage can.

4.24 *Growing raindrop*
A raindrop of initial mass M_0 starts falling from rest under the influence of gravity. Assume that the drop gains mass from the cloud at a rate proportional to the product of its instantaneous mass and its instantaneous velocity:

$$\frac{dM}{dt} = kMV,$$

where k is a constant.

Show that the speed of the drop eventually becomes effectively constant, and give an expression for the terminal speed. Neglect air resistance.

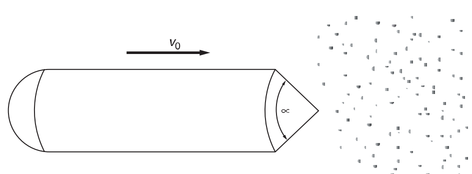
4.25 *Bowl of water*
A bowl full of water is sitting out in a pouring rainstorm. Its surface area is 500 cm^2. The rain is coming straight down at 5 m/s at a rate of 10^{-3} g/cm^{2}s. If the excess water drips out of the bowl with negligible velocity, find the force on the bowl due to the falling rain.

What is the force if the bowl is moving uniformly upward at 2 m/s?

4.26 *Rocket in interstellar cloud*
A cylindrical rocket of diameter $2R$ and mass M is coasting through empty space with speed v_0 when it encounters an interstellar cloud. The number density of particles in the cloud is N particles/m^3. Each particle has mass $m \ll M$, and they are initially at rest.

(*a*) Assume that each cloud particle bounces off the rocket elastically, and that the collisions are so frequent they can be treated as continuous. Prove that the retarding force has the form bv^2, and determine b. Assume that the front cone of the rocket subtends angle $\alpha = \pi/2$, as shown.

(*b*) Find the speed of the rocket in the cloud.



4.27 *Exoplanet detection*
The data plots in Example 4.6 show that with the methods then in use, a shift of 1 m/s in radial velocity of a star is just barely detectable. Could an astronomer on a far-off planet using these same methods detect that our Sun has a planet? The biggest effect would be due to Jupiter.

Use only the following data:
mass of the Sun = 1.99×10^{30} kg
mass of Jupiter = 1.90×10^{27} kg
mean radius of Jupiter's orbit = 7.8×10^8 km
period of Jupiter's orbit = 4330 days.