Lecture 08: Cauchy Sequences

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In general, proving the convergence of a sequence (a_n) is difficult since we must correctly guess the limit of (a_n) beforehand. There is a way of avoiding this guesswork, which we now describe.

Definition 8.1 *A sequence (a_n) in $\mathbb{R}$ is called a Cauchy sequence if for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$|a_n - a_m| < \epsilon, \forall\, n, m \geq n_0.$$

Example 8.2 *Show that every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.*

Solution: Let (x_n) be convergent sequence in $\mathbb{R}$ and $x \in \mathbb{R}$ be it's limit. Then for given $\epsilon > 0$, there is n_0 such that

$$\begin{aligned} n \geq n_0 &\implies |x_n - x| < \frac{\epsilon}{2} \\ m \geq n_0 &\implies |x_m - x| < \frac{\epsilon}{2} \end{aligned}$$

Hence if $m, n \geq n_0$ then

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

Theorem 8.3 (Cauchy Criterion) *A sequence (a_n) in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

Above theorem says that the only examples of Cauchy sequences in $\mathbb{R}$ are convergent sequences.

Checking whether a given sequence is Cauchy or not directly from the definition is very difficult. The following result gives a useful sufficient condition for a sequence to be Cauchy.

Proposition 8.4 *Let (a_n) be a sequence of real numbers and α be a real number such that $0 \leq \alpha < 1$ and*

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \quad \forall n \in \mathbb{N} \text{ with } n \geq 2.$$

Then (a_n) is a Cauchy sequence.

Proof: For $n \in \mathbb{N}$, we have

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \leq \alpha^2 |a_{n-1} - a_{n-2}| \leq \cdots \leq \alpha^{n-1} |a_2 - a_1|.$$

Hence for all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq \alpha^{m-2}|a_2 - a_1| + \alpha^{m-3}|a_2 - a_1| + \cdots + \alpha^{n-1}|a_2 - a_1| \\
&= |a_2 - a_1|(\alpha^{m-2} + \alpha^{m-3} + \cdots + \alpha^{n-1}) \\
&= |a_2 - a_1|\alpha^{n-1}(1 + \alpha + \alpha^2 + \cdots \alpha^{m-n-1}) \\
&= |a_2 - a_1|\alpha^{n-1}\frac{(1 - \alpha^{m-n})}{1 - \alpha} \\
&< |a_2 - a_1|\alpha^{n-1}\frac{1}{1 - \alpha}
\end{aligned}$$

If we interchange the role of m and n in above argument, we have for all $m, n \in \mathbb{N}$ with $n > m$

$$|a_m - a_n| < |a_2 - a_1|\alpha^{m-1}\frac{1}{1 - \alpha}$$

If $a_2 = a_1$, then it is clear that $a_n = a_1$ for all $n \in \mathbb{N}$, and (a_n) is a Cauchy sequence. Suppose $a_2 \neq a_1$ and let $\epsilon > 0$ be given. Since $\alpha < 1$, hence $\alpha^n \to 0$. Consequently, there is $n_0 \in \mathbb{N}$ such that

$$\alpha^{n-1} \leq \frac{\epsilon(1 - \alpha)}{|a_2 - a_1|}, \ \forall n \geq n_0$$

It follows that $|a_m - a_n| < \epsilon$ for all $m, n \in \mathbb{N}$ with $m, n \geq n_0$. Thus (a_n) is a Cauchy sequence. ■

Example 8.5 *Consider the sequence (a_n) defined by*

$$a_1 := 1 \quad \text{and } a_{n+1} := 1 + \frac{1}{a_n} \text{ for } n \in \mathbb{N}.$$

First we show that (a_n) is a Cauchy sequence. By induction it is easy to see that $a_n \geq 1$ for all $n \in \mathbb{N}$ and hence

$$a_n a_{n-1} = \left(1 + \frac{1}{a_{n-1}}\right) a_{n-1} = a_{n-1} + 1 \geq 2 \ \ \forall\, n \in \mathbb{N} \text{ with } n \geq 2.$$

Since

$$a_{n+1} - a_n = \left(1 + \frac{1}{a_n}\right) - \left(1 + \frac{1}{a_{n-1}}\right) = \frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{a_{n-1} - a_n}{a_{n-1}a_n}$$

we see that

$$|a_{n+1} - a_n| = \frac{|a_n - a_{n-1}|}{a_{n-1}a_n} \leq \frac{1}{2}|a_n - a_{n-1}| \quad \forall n \in \mathbb{N} \text{ with } n \geq 2.$$

Hence by Proposition 8.4, (a_n) is a Cauchy sequence and by Cauchy criterion, it is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$, and since $a_{n+1} = 1 + \frac{1}{a_n}$, we have $a = 1 + \frac{1}{a} \implies a = \frac{1 \pm \sqrt{5}}{2}$. Also, $a_n \geq 1$ for all $n \in \mathbb{N}$ implies that $a \geq 1$. Hence $a = \frac{1+\sqrt{5}}{2}$.

It may be noted that (a_n) is not a monotonic sequence. In fact, from the relation

$$a_{n+1} - a_n = \frac{a_{n-1} - a_n}{a_{n-1}a_n}, \quad n \geq 2.$$

we have

$$\begin{aligned} a_{n+1} - a_n \leq 0 &\iff a_{n-1} - a_n \leq 0, \quad n \geq 2 \\ \text{and } a_{n+1} - a_n \geq 0 &\iff a_{n-1} - a_n \geq 0, \quad n \geq 2. \end{aligned}$$