The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #7

(Improper Integral, Taylor's Theorem)

---

1. Test the convergence/divergence of the following improper integrals:
$(a)\ \int_0^1 \frac{dx}{\log(1+\sqrt{x})}$ $\quad (b)\ \int_0^1 \frac{dx}{x-\log(1+x)}$ $\quad (c)\ \int_0^1 \frac{\log x}{\sqrt{x}}dx$ $\quad (d)\ \int_0^1 \sin\left(\frac{1}{x}\right)dx$
$(e)\ \int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{x}dx$ $\quad (f)\ \int_0^\infty e^{-x^2}dx$ $\quad (g)\ \int_0^{\pi/2} \frac{dx}{x-\sin x}$ $\quad (h)\ \int_0^{\pi/2} \mathrm{cosec} x dx.$

2. In each case, determine the values of p for which the following improper integrals converge
$(a)\ \int_0^\infty \frac{1-e^{-x}}{x^p}$ $\quad (b)\ \int_0^\infty \frac{t^{p-1}}{1+t}dt.$

3. Show that the integrals $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ and $\int_0^\infty \frac{\sin x}{x}dx$ converge. Further, prove that
$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

4. Show that $\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = 0.$

5. Prove that $\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$.

6. Show that $\int_0^s \frac{1+x}{1+x^2}dx$ and $\int_{-s}^0 \frac{1+x}{1+x^2}dx$ do not approach a limit as $s \to \infty$. However $\lim_{s\to\infty}\int_{-s}^s \frac{1+x}{1+x^2}dx$ exists.

7. For $x > -1$, $x \neq 0$ prove that
(a) $(1+x)^\alpha > 1+\alpha x$ whenever $\alpha < 0$, or $\alpha > 1$
(b) $(1+x)^\alpha < 1+\alpha x$ whenever $0 < \alpha < 1$.

8. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that
$$x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$