Lecture 20: Point of Inflection & Riemann Integration

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Proposition 20.1 *Given $I \subseteq \mathbb{R}$ be an interval,$c \in I$ be an interior point and $f : I \to \mathbb{R}$, be any function. Then we have the following:*

1. [**Necessary Condition for a Point of Inflection**] *Let f be twice differentiable at c. If c is a point of inflection for f, then $f''(c) = 0$.*

2. [**Sufficient Condition for a Point of Inflection**] *Let f be thrice differentiable at c. If $f''(c) = 0$ and $f'''(c) \neq 0$, then c is a point of inflection for f.*

Example 20.2

1. *The condition in part 1 of the Proposition 20.1 is not sufficient. Consider, for example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^4$. Then 0 is not a point of inflection for f (because $f''(x) = 12x^2 \geq 0, \quad \forall\, x \in \mathbb{R}$ hence f is convex on entire real line) , but $f''(0) = 0$.*

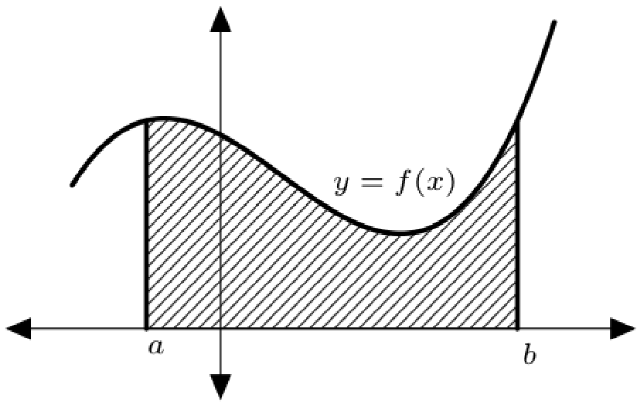
2. *The condition in part 2 of the Proposition 20.1 is not necessary. Consider, for example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^5$. Then 0 is a point of inflection for f (because $f''(x) = 20x^3 \geq 0$, if $x > 0$ hence f is convex on $(0, \infty)$ and $f''(x) < 0$ if $x < 0$ hence f is concave on $(-\infty, 0)$), but $f'''(0) = 0$.*

To conclude the topic of local extrema and points of inflection, let us consider the following example.

Example 20.3 *Let $f(x) := x^4 - 4x^3$ for $x \in \mathbb{R}$. Then $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \iff x \in \{0, 3\}$, $f''(x) = 12x^2 - 24x = 12x(x-2) = 0 \iff x \in \{0, 2\}$. Also, $f'''(x) = 24(x-1)$ for $x \in \mathbb{R}$. Since $f''(0) = 0 = f''(2)$ and $f'''(0) = -f'''(2) = -24 \neq 0$, we see that 0 and 2 are points of inflection for f. Since $f' < 0$ on $(-\infty, 3)$ hence f does not have a local extremum at 0. Further, since $f'(3) = 0$ and $f''(3) = 36 > 0$, the function f has a local minimum at 3.*

20.1 Riemann Integration

The notion of integration was developed much earlier than differentiation. Although the ideas involved in defining integrals date as far back as to Archimedes, it was Riemann who gave a systematic formulation to the theory. The main idea of integration is to assign a real number A, called the "area", to the region R bounded by the curves $x = a, x = b, y = 0$, and $y = f(x)$, where we assume that f is non-negative.



The number, A, the area of the region R, is called the integral of f over $[a, b]$ and denoted by the symbol $\int_a^b f(x)dx$. The most basic geometric region for which the area is known is a rectangle. The area of a rectangle whose sides have lengths l and b is lb. This suggests that our definition of an integral should be such that if $f(x) = c$, a constant, then $\int_a^b f(x)dx = c(b - a)$. We use this basic notion of area as the building block to assign an area to the regions under the graphs of bounded functions. To understand the concepts and results of this section, it is suggested that you may assume that f is non-negative and draw pictures whenever possible.

Definition 20.4 *By a partition of an interval $[a, b]$ (where $a, b \in \mathbb{R}$ and $a < b$) we mean a finite ordered set $\{x_0, x_1, \cdots, x_n\}$ of points in $[a, b]$ such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Example 20.5 1. *The simplest partition of $[a, b]$ is given by $P_1 := \{a, b\}$.*

2. *For $n \in \mathbb{N}$, the partition $P_n := \{x_0, x_1, \cdots, x_n\}$, where*

$$x_i = a + \frac{i(b - a)}{n}, \quad \text{for } i = 0, 1, \cdots, n$$

subdivides the interval $[a, b]$ into n subintervals, each of length $\frac{b-a}{n}$.

Definition 20.6 *Given two partitions P and Q of $[a,b]$, we say that Q is a refinement of P if $P \subset Q$.*

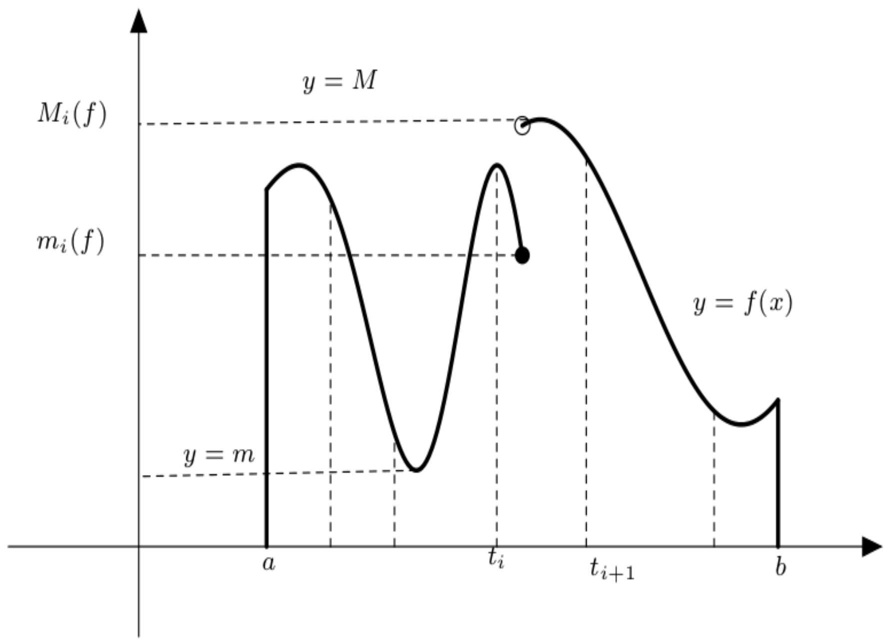
In Example 20.5 (2) if we take $[a,b]=[0,1]$, then $P_1 \subset P_2 \subset P_4$. But $P_2 \not\subseteq P_3$ as $\frac{1}{2} \in P_2$ but $\frac{1}{2} \notin P_3$.

Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Let us define

$$m(f) := \inf\{f(x) : x \in [a,b]\} \text{ and } M(f) := \sup\{f(x) : x \in [a,b]\}.$$

Given a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$, for $i = 1, \cdots, n$ let us define

$$m_i(f) := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$



Exercise 20.7 *Let A, B be nonempty and bounded subsets of $\mathbb{R}$ with $A \subset B$. Prove that*

$$\inf B \le \inf A \le \sup A \le \sup B.$$

Draw pictures and explain.

Since $[x_{i-1}, x_i] \subset [a,b]$ for each $i = 1, 2, \cdots, n$, so

$$\{f(x) : x \in [x_{i-1}, x_i]\} \subseteq \{f(x) : x \in [a,b]\}.$$

By Exercise 20.7 we have

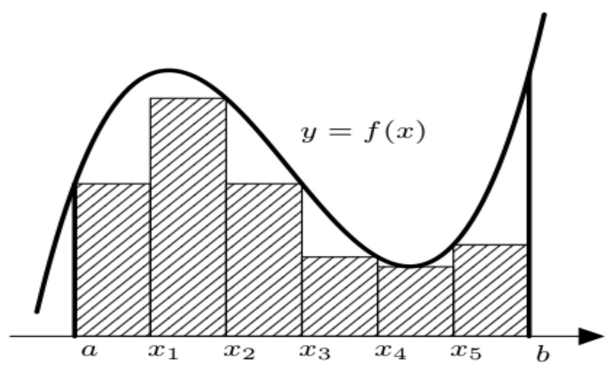
$$m(f) \le m_i(f) \le M_i(f) \le M(f) \quad \text{for all } i = 1, \cdots, n.$$

Definition 20.8 *Given $f : [a, b] \to \mathbb{R}$ be a bounded function and a partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$. Then the lower sum for the function f with respect to the partition P is defined as follows:*

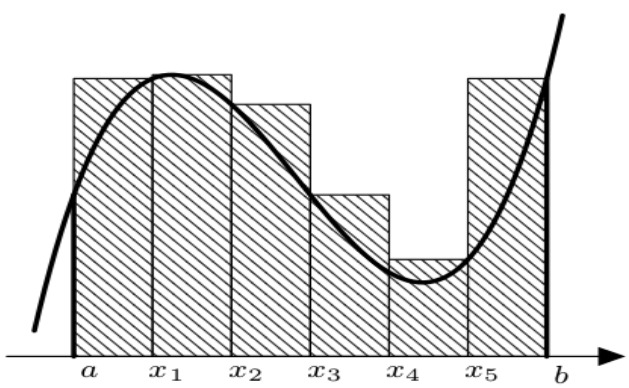
$$L(P, f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

Similarly, the upper sum for the function f with respect to the partition P is defined as follows:

$$U(P, f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$



Lower Sum



Upper Sum

Proposition 20.9 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then for any partition P of $[a, b]$, we have*

$$m(f)(b - a) \leq L(P, f) \leq U(P, f) \leq M(f)(b - a).$$

Proof: Let $P = \{x_0, \cdots, x_n\}$ be partition of $[a, b]$. Since $m(f) \leq m_i(f) \leq M_i(f) \leq M(f)$ for each $i = 1, \cdots, n$ and $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, the desired inequalities follow. ■

Definition 20.10 *Given partitions P_1 and P_2 of $[a, b]$, the partition $P^* = P_1 \cup P_2$ is called the common refinement of P_1 and P_2. (arrange the elements in the increasing order)*

Lemma 20.11 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

1. *If P is partition of $[a, b]$, and P^* is a refinement of P, then*

$$L(P, f) \leq L(P^*, f) \quad \text{and } U(P^*, f) \leq U(P, f),$$

and consequently,

$$U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f)$$

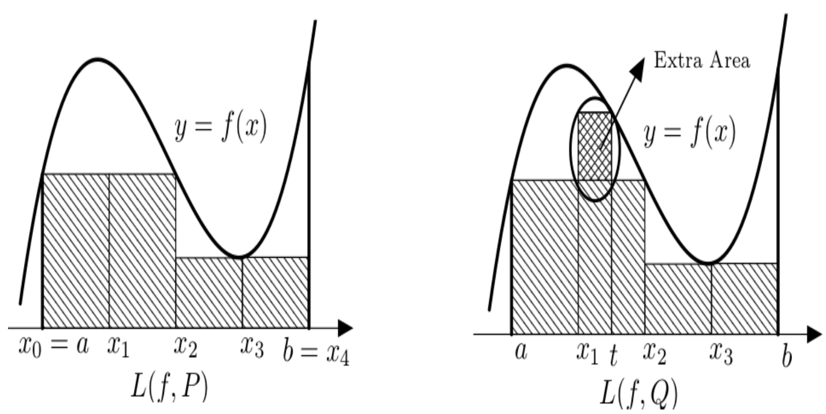
2. *If P_1 and P_2 are partitions of $[a, b]$, then $L(P_1, f) \leq U(P_2, f)$.*

3. *Define*

$L(f) := \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$, $U(f) := \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$

Then $L(f) \leq U(f)$.

Proof:



1. Figure 6.4: $L(f, P) \leq L(f, Q)$.

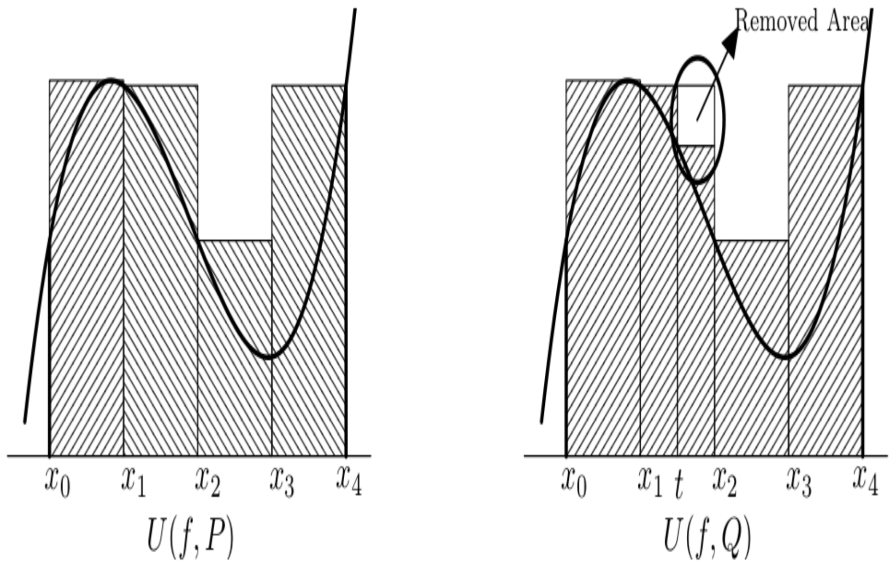


Figure 6.5: $U(f,P) \geq U(f,Q)$.

2. Let P^* denote the common refinement of partitions P_1 and P_2 . Then in view of 1 above,
$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$
3. Let us fix a partition P_0 of $[a, b]$. By 2 above, we have $L(P_0, f) \leq U(P, f)$ for any partition P of $[a, b]$. Hence $L(P_0, f)$ is a lower bound of the set $\{U(P, f) : P \text{ is a partition of } [a, b]\}$. Since $U(f)$ is the greatest lower bound of this set, we have $L(P_0, f) \leq U(f)$. Now, since P_0 is an arbitrary partition of $[a, b]$, we see that $U(f)$ is an upper bound of the set $\{L(P_0, f) : P_0 \text{ is a partition of } [a, b]\}$. Again, since $L(f)$ is the least upper bound of this set, we have $L(f) \leq U(f)$.

■

Definition 20.12 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is said to be Riemann integrable (on $[a, b]$) if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the Riemann integral of f (on $[a, b]$) and it is denoted by $\int_a^b f(x)dx$ or $\int_a^b f$. The number $U(f)$ is known as the upper Riemann integral of f and the number $L(f)$ as the lower Riemann integral of f. Thus, a bounded function on $[a, b]$ is integrable if its upper Riemann integral is equal to its lower Riemann integral.*

Admittedly, the definition of a Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$ is rather involved. This is because we need to consider lower sums for the function f with respect to all possible partitions of $[a, b]$ and calculate their supremum on the one hand, and also consider the corresponding upper sums and calculate their infimum on the other.