Lecture 21: Examples: Riemann Integration

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Example 21.1 *Consider the constant function on $[a, b]$ defined by $f(x) := c$ for all $x \in [a, b]$, for some $c \in \mathbb{R}$. Then for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, we have $m_i(f) = c = M_i(f)$ for all $i = 1, \cdots, n$ and so*

$$L(P, f) = U(P, f) = \sum_{i=1}^{n} r(x_i - x_{i-1}) = r(b - a).$$

Hence $L(f) = r(b - a) = U(f)$. Thus f is integrable and $\int_a^b r dx = r(b - a)$.

Example 21.2 *Consider the Dirichlet function on $[a, b]$ defined by*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \textit{ is rational and } x \in [a, b] \\ 0 & \text{if} \quad x \textit{ is irrational and } x \in [a, b] \end{cases}$$

Let $P = \{x_0, x_1, \cdot, x_n\}$ be a partition of $[a, b]$. Since each $[x_{i-1}, x_i]$ contains a rational number as well as an irrational number, we see that $m_i(f) = 0$ and $M_i(f) = 1$ for all $i = 1, \cdots, n$, and so

$$L(P, f) = \sum_{i=0}^{n} 0(x_i - x_{i-1}) = 0, \quad \textit{but } U(P, f) = \sum_{i=0}^{n} 1(x_i - x_{i-1}) = b - a.$$

Hence $L(f) = 0$ and $U(f) = b - a$. Since $a < b$, we have $L(f) \neq U(f)$, that is, f is not integrable.

Example 21.3 *Consider $f : [0, 2] \to \mathbb{R}$ defined as*

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1. \end{cases}$$

Let $P = \{x_0 = 0, x_1, \cdots, x_n = 2\}$ be a partition of $[0, 2]$. There are two possibilities:

If $1 \in P$: *Then $x_i = 1$ for some $i \in \{1, 2, \cdots, n - 1\}$. In this case $L(P, f) = 0$ and $U(P, f) = (x_i - x_{i-1}) + (x_{i+1} - x_i) = x_{i+1} - x_{i-1}$.*

If $1 \notin P$**:** *Then* $x_{i-1} < 1 < x_i$ *for some* $i \in \{1, 2, \cdots, n\}$*. In this case* $L(P, f) = 0$ *and* $U(P, f) = (x_i - x_{i-1})$*.*

Since $L(P, f) = 0$ *for every partition* P*, therefore* $L(f) = 0$*.*

Claim 21.4 $U(f) = 0$*.*

To see this, for each $n \in \mathbb{N}$*, let* P_n *be the partition, which divides the interval* $[0, 2]$ *into* n *equal sub-intervals. Then*

$$U(P_n, f) = \frac{4}{n} \text{ or } U(P_n, f) = \frac{2}{n}. \implies U(P_n, f) \leq \frac{4}{n}.$$

Since $L(f) \leq U(f) \leq U(P_n, f)$*, hence we get*

$$0 \leq U(f) \leq \frac{4}{n}, \text{ for each } n \in \mathbb{N}$$

This means $U(f) = 0$*. This complete the proof of the claim.*

The above example illustrates the difficulty in proving the integrability of a bounded function f on $[a, b]$ by showing $U(f) = L(f)$. We now give a necessary and sufficient condition for the integrability of such a function, which is much easier to verify.

Theorem 21.5 (Riemann's Criterion for Integrability) *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function. Then* f *is integrable if and only if for every* $\epsilon > 0$*, there is a partition* P *(depending on* ϵ*) of* $[a, b]$ *such that*

$$U(P, f) - L(P, f) < \epsilon.$$

Example 21.6 *Let* $f : [0, 1] \to \mathbb{R}$ *such that* $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$ *Show that* f *is integrable on* $[0, 1]$ *and* $\int_0^1 f(x)dx = 0$*.*

Solution: We will use the Riemann criterion to show that f is integrable on $[0, 1]$. Let $\epsilon > 0$ be given. We need to find a partition P such that $U(P, f) - L(P, f) < \epsilon$. By the density of irrationals, in any subinterval $[x_{i-1}, x_i]$ of a given partition P of $[0, 1]$, $m_i(f)$'s are zero and hence $L(f, P) = 0$. This further implies that $L(f) = 0$. Hence we need only show $U(f, P) < \epsilon$.

If $\epsilon \geq 1$**:** In this case, take $P = \{0, 1\}$. Then $\sup f(x) = 1$ and $U(P, f) = 1 \leq \epsilon$.

If $0 < \epsilon < 1$. Note that f is non-zero only at points $\frac{1}{n}$, so we choose $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \epsilon$. Hence $\frac{1}{n} \in [0, \epsilon]$ for all $n \geq N$.

So only $1, \dfrac{1}{2}, \dfrac{1}{3}, \cdots, \dfrac{1}{k}$ (where $k \leq N$) lie in the interval $[\epsilon, 1]$, which will give non-zero contribution to $U(P, f)$. Note that there are at most N points in $[\epsilon, 1]$ where f is not zero. Also, other than endpoint 1 each of $1, \dfrac{1}{2}, \dfrac{1}{3}, \cdots, \dfrac{1}{k}$ can belong to two subintervals in a given partition P (this happens if each point is a node of the partition). Now take a partition $P = \{0 = x_0, x_1 = \epsilon, x_2, x_3, \cdots, x_n = 1\}$ such that $x_i - x_{i-1} \leq \dfrac{\epsilon}{2N}$ for all $i = 2, \cdots, n$. (One can do it by dividing the interval $[\epsilon, 1]$ into n equal subintervals, where $\dfrac{1-\epsilon}{n} \leq \dfrac{\epsilon}{2N}$)

Let $A := \{2 \leq i \leq n : \frac{1}{j} \in [x_{i-1}, x_i],\ \text{for } j = 1, 2, \cdots, k\}$. Set A contains all indices of those subintervals which contains $1, \dfrac{1}{2}, \dfrac{1}{3}, \cdots, \dfrac{1}{k}$. Therefore number of elements in A will be at most $2N - 1$. Note that $M_i(f) \leq 1$ for all $i \in A$

Define $B := \{2, \cdots, n\} \setminus A$. So B is the collection of those indices of subintervals in which f is identically zero. Therefore $M_i(f) = 0$ for all $i \in B$.

Also $M_1(f) \leq 1$.

$$\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \\
&= M_1(f)(\epsilon - 0) + \sum_{i=2}^{n} M_i(f)(x_i - x_{i-1}) \\
&\leq \epsilon + \sum_{i \in A} M_i(f)(x_i - x_{i-1}) + \sum_{i \in B} M_i(f)(x_i - x_{i-1}) \\
&\leq \epsilon + \sum_{i=2}^{2N-1} (x_i - x_{i-1}) \\
&\leq \epsilon + (2N - 2)\frac{\epsilon}{2N} \\
&< 2\epsilon
\end{aligned}$$

Hence by the Reimann criterion the function is integrable.

Since the lower integral is 0 and the function is integrable, $\displaystyle\int_0^1 f(x)dx = 0$. ■