Lecture 07: Sub-Sequences

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Theorem 7.1 (Ratio test for sequences) *Let (a_n) be a sequence of real numbers such that $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$. Then*

1. *if $\lambda < 1$ then $\lim_{n\to\infty} a_n = 0$*
2. *if $\lambda > 1$ then $\lim_{n\to\infty} a_n = \infty$*
3. *if $\lambda = 1$ then test fails, i.e., we can not say anything about convergence or divergence of the sequence (a_n).*

Remark 7.2 *Using the fact that $|a_n| \to 0 \implies a_n \to 0$, the first part of the above theorem can be extended as follows:*

Suppose $(a_n)_{n\geq 1}$ is a real sequence such that $a_n \neq 0$ for all n and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lambda$. If $\lambda < 1$, then $a_n \to 0$.

Example 7.3 1. *Let $x_n = \frac{n}{2^n}$. Then*

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}(1+\frac{1}{n}) \implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{2} < 1 \implies x_n \to 0$$

2. *Let $x_n = (-1)^n n y^{n-1}$ for some $y \in (0,1)$. Then*

$$\frac{x_{n+1}}{x_n} = -\frac{(n+1)y^n}{ny^{n-1}} = -\frac{(n+1)y}{n} = -y(1+\frac{1}{n}) \implies \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = y < 1 \implies x_n \to 0$$

3. *Let $x_n = \frac{y^n}{n^2}$ for some $y > 1$. Then*

$$\frac{x_{n+1}}{x_n} = \frac{y^{n+1}}{(n+1)^2} \times \frac{n^2}{y^n} = \frac{n^2 y}{(n+1)^2} = \frac{y}{(1+\frac{1}{n})^2} \implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = y > 1 \implies x_n \to \infty$$

4. (a) *Let $x_n = n$. Then*

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

Test fails but we know that $x_n \to \infty$.

(b) *Let* $x_n = \frac{1}{n}$. *Then*

$$\frac{x_{n+1}}{x_n} = \frac{n}{n+1} = 1 + \frac{1}{n} \implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

Test fails but we know that $x_n \to 0$.

Subsequences

Let (a_n) be a sequence. If $n_1, n_2, \cdots$ are positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$, then the sequence (a_{n_k}), whose terms are

$$a_{n_1}, a_{n_2}, \cdots,$$

is called a subsequence of (a_n). Let us observe that $n_k \geq k$ for all $k \in \mathbb{N}$, which implies that $n_k \to \infty$ as $k \to \infty$.

Example 7.4 1. *Let* $a_n = \frac{1}{n}$ *and* $n_k = 2k$ *then the sequence* (a_{2n}) *is a subsequence of* (a_n). *Other subsequences are* $(a_{2n-1}), (a_{n!})$.

2. *Let* $a_n = \frac{1}{n}$, *then the following sequence*

$$\left(\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \cdots\right)$$

is not a subsequence of (a_n) *because* $n_1 = 2, n_2 = 1, n_3 = 4, n_4 = 3$ *(order is very important about sequences). Similarly the sequence*

$$\left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \cdots\right)$$

is not a subsequence of (a_n) *because there is a zero term which is not there in* (a_n).

3. *Every sequence is a subsequence of itself (take* $n_k = k$ *for* $k \in \mathbb{N}$*)*

Let (a_n) be a sequence and (a_{n_k}) be a subsequence. What does it mean to say that the subsequence (a_{n_k}) converges to a? Let us define a new sequence (x_k) where $x_k := a_{n_k}$. Then we say $a_{n_k} \to a$ if $x_k \to a$.

Definition 7.5 *Let* (a_n) *be a sequence and* (a_{n_k}) *be a subsequence. We say that the subsequence* (a_{n_k}) *converges to* a *if for every* $\epsilon > 0$, *there exists* $k_0 \in \mathbb{N}$ *such that for* $k \geq k_0$, *we have* $|a_{n_k} - a| < \epsilon$.

Theorem 7.6 *If a sequence (a_n) converges to a then show that every subsequence of (a_n) converges to a.*

Proof: Let $\epsilon > 0$ be given. Since $a_n \to a$ hence there is a $n_0 \in \mathbb{N}$ such that for all $k \geq n_0 \implies |a_k - a| < \epsilon$. Note that if $k \geq n_0$ then $n_k \geq k \geq n_0$. Hence, for $k \geq n_0$, we have $|a_{n_k} - a| < \epsilon$. ■

Remark 7.7 *The Theorem 7.6 can be used to conclude divergence of a sequence. If (a_n) has two convergent subsequences (a_{n_k}) and (a_{m_k}) whose limits are not equal then sequence (a_n) diverges.*

Example 7.8

1. *Let $a_n = (-1)^n$. Then the subsequence (a_{2n}) is constant sequence 1 and the subsequence (a_{2n-1}) is constant sequence -1. Hence sequence (a_n) diverges.*
2. *Let $(a_n) = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \cdots)$. Then $a_{2n} = \frac{1}{2n}$ which converges to 0 and $a_{2n-1} = 2n - 1$ which tends to ∞. Hence (a_n) diverges.*

Theorem 7.9 (Bolzano-Weierstrass Theorem for sequences) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

For example the sequence $(\sin n)$ is bounded and hence has convergent subsequence. Though it is difficult to give explicitly that subsequence.

Remark 7.10 *Converse of Theorem 7.9 is not true as in Example 7.8 (2), sequence has a convergent subsequence (a_{2n}) but the sequence (a_n) is unbounded.*