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$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot U$

$$= (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot (u_1, u_2)$$

$$= u_1 f_x(x_0, y_0) + u_2 f_y(x_0, y_0).$$

$$D_u f(x_0, y_0) = |\nabla f(x_0, y_0)| \cdot |U| \cos\theta$$

$$D_u f(x_0, y_0) = |\nabla f(x_0, y_0)| \cdot \cos\theta \quad \text{—} \quad (1)$$

(1) $D_u f(x_0, y_0)$ is maximum i.e. $|\nabla f(x_0, y_0)|$ at (x_0, y_0) if $\cos\theta = 1$ or $\theta = 2n\pi$ near point (x_0, y_0), function f increases most rapidly in the direction $\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$

or $\dfrac{f_x(x_0, y_0), f_y(x_0, y_0)}{\sqrt{f_x^2 + f_y^2}}$.

(2) $D_u f(x_0, y_0)$ is minimum i.e. $|\nabla f(x_0, y_0)|$ at (x_0, y_0) if $\cos\theta = -1$ i.e. $\theta = \pi$.
function f decreases most rapidly in the direction $-\dfrac{(\nabla f(x_0, y_0))}{|\nabla f(x_0, y_0)|}$

(III) $D_u f(x_0, y_0) = 0$ if $\cos\theta = 0$ i.e. $\theta = \pi/2$,

f has no rate of change in the direction

$$\pm \frac{f_x(x_0, y_0), -f_y(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

eg: $f: R^2 \longrightarrow R$
$f(x, y) = 4 - x^2 - y^2$. Find the dir'n in which the function increases most rapidly at $(1,1)$.

$f_x = -2x$, $f_y = -2y$.
$\nabla f(x, y) = (-2, -2)$
$|\nabla f(x, y)| = 2\sqrt{2}$

(I) $\dfrac{\nabla f(x, y)}{|\nabla f(x, y)|} = \left(\dfrac{-2}{2\sqrt{2}}, \dfrac{-2}{2\sqrt{2}}\right) = \left(\dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\right)$. In this dir'n f increases rapidly.

(II) $-\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x, y)|} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. In this dir'n, f decreases rapidly.

(III) f has no rate of change in dir'n $= \pm \dfrac{(f_x(x_0, y_0), -f_y(x_0, y_0))}{|\nabla f(x_0, y_0)|}$

$= (-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$.

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Lecture 34

CHAIN RULE

$W = f(x)$ is diff. w.r.t x and $x = g(t)$ is differentiable w.r.t t.

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

FOR TWO VARIABLES (1) $w = g(z)$ and $z = f(x, y)$
$\Rightarrow$ w is the function of x and y.

$$\frac{dw}{dx} = \frac{dw}{dz} \cdot \frac{dz}{dx}, \quad \frac{dw}{dy} = \frac{dw}{dz} \cdot \frac{dz}{dy}$$

(II) if $z = f(x,y)$ and $x = f(t)$ and $y = g(t)$

$\Rightarrow$ z is a function of t.

$$\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}$$

(III) if $z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Mean Value Theorem

Let $D \subseteq R^2$ be an open disk and $f : D \to R$ is differentiable. Given any two distinct points $A = (x_0, y_0)$ and $B = (x_1, y_1)$ in D, then $\exists$ $C = (c, d)$ lying on the line segment joining A and B with $C \neq A$ and $C \neq B$ such that

$$f(B) - f(A) = f'(C)\cdot(B - A)$$

OR

$\exists$ $s \in (0,1)$ such that

$$f(x_1, y_1) - f(x_0, y_0) = \nabla f((x_0 + (x_1 - x_0)s), (y_0 + (y_1 - y_0)s)) \cdot [(x - x_0), (y - y_0)]$$

$$= (f_x(x_0 + (x_1 - x_0)s, y_0 + (y_1 - y_0)s), f_y(\cancel{y_0 + y}\ x_0 + (x_1 - x_0)s, y_0 + (y_1 - y_0)s)) \cdot ((x_1 - x_0), (y_1 - y_0))$$

$$\Rightarrow f(x_1, y_1) - f(x_0, y_0) = (x_1 - x_0) f_x(x_0 + sx_1 - sx_0, y_0 + sy_1 - sy_0) + (y_1 - y_0) f_y(x_0 + sx_1 - sx_0, y_0 + sy_1 - sy_0)$$

Proof: Let $\phi : [0,1] \to R$ such that

$$\phi(t) = f(x_0 + (x_1 - x_0)t \,;\, y_0 + (y_1 - y_0)t)$$

Given f is differentiable,

$x(t) = x_0 + (x_1 - x_0)t$ and $y(t) = (y_1 - y_0)t + y_0$ is also differentiable w.r.t. t.

So, Q is differentiable w.r.t. t

$$Q(t) = \frac{dQ}{dt} = f_x(x_0+(x_1-x_0)t,\ y_0+(y_1-y_0)t)(x_1-x_0) + f_y(x_0+(x_1-x_0)t,\ y_0+(y_1-y_0)t)(y_1-y_0)$$

Applying M.V.T for one variable on Q, $\exists$ $s \in (0,1)$ such that

$$Q(1)-Q(0) = Q'(s).(1-0).$$

$$f(x_1,y_1)-f(x_0,y_0) = f_x(x_0+(x_1-x_0)s,\ y_0+(y_1-y_0)s).(x_1-x_0) + f_y(x_0+(x_1-x_0)s,\ y_0+(y_1-y_0)s).(y_1-y_0).$$

Theorem: Let D be an open disk in R^2 and $f: D \to R^1$. 'f' is constant if and only if, 'f' is differentiable and both f_x and f_y vanish identically on D.

Proof: Suppose 'f' is constant

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = 0$$

$$f_y(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k} = 0$$

- By definition, 'f' is differentiable.

Conversely, Given 'f' is differentiable and $f_x = 0 = f_y$ in D.

Let (x_0,y_0) be any fixed pt. in D.

Consider another point (x_1,y_1) in D such that $(x_1,y_1) \neq (x_0,y_0)$.

Given 'f' is differentiable on D, By mean value theorem for the variables

$$f(x_1,y_1)-f(x_0,y_0) = 0.$$

$B_\delta(x_0, y_0)$ – open disk with radius (x_0, y_0) and radius δ.

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MAXIMA and MINIMA

LOCAL MAXIMA and MINIMA

Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function. A point (x_0, y_0) in D is called local maxima if (x_0, y_0) is an interior point of D and if $\exists$ $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and

$$f(x, y) \leq f(x_0, y_0) \quad \forall \quad (x, y) \in B_\delta(x_0, y_0)$$

~~then~~

~~point (x_0, y_0)~~ is called local minima if (x_0, y_0) is interior point of D and if $\exists$ $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and

$$f(x, y) \geq f(x_0, y_0) \quad \forall \quad (x, y) \in B_\delta(x_0, y_0).$$

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Lecture 35

Absolute Maxima or Minima

Critical Point

Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ & $(x_0, y_0) \in D$ is called critical point if either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ i.e. $\nabla f(x_0, y_0)$ does not exist or if exist $\nabla f(x_0, y_0) = 0$.

Theorem

Let $D \subset \mathbb{R}^2$ be closed and bounded and $f : D \to \mathbb{R}$ is continuous on D. Then absolute maxima or minima of f are attained either at a critical point or boundary point.

Necessary Condition for Local Extremum

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ and (x_0, y_0) be an interior point of D. Suppose 'f' has local extremum at (x_0, y_0). Then $\nabla f(x_0, y_0)$ (if exist) is equal to $(0, 0)$.

eg: $f(x,y) = x^2+y^2$
$(f_x, f_y) = \nabla f(x,y) = (2x, 2y)$

$f_x = 2x,\ f_y = 2y$

At $(0,0)$, $f_x = 0$, $f_y = 0$
Clearly, $(0,0)$ is local minima.

eg: $f(x,y) = -(x^2+y^2)$
$(f_x, f_y) = (-2x, -2y)$
At $x=0, y=0$, $f_x = 0, f_y = 0$
Clearly, $(0,0)$ is local maxima.

eg: $f(x,y) = xy$
$f_x = y = 0$ $\quad\quad \nabla f(x,y) = 0$
$f_y = x = 0$
$(0,0)$.

\# If for $\delta > 0$, there exists points in $B_\delta(0,0)$ where
$f(x,y) > f(0,0)$.

& $f(x,y) < f(0,0)$.

$0 < t < \delta$.

$f(t,t) = t^2 > 0 = f(0,0)$
$f(t,-t) = -t^2 < 0 = f(0,0)$
$(0,0)$ is neither local maxima, nor local minima.
Such point is called SADDLE POINT.

P.T.O.

Second Derivative Test

$f : D \subset R^2 \to R$

For stationary points (critical point)

f_x, f_y and $f_x = 0$, $f_y = 0$

We have a point (x_0, y_0)

For each (x_0, y_0)

$$r = \frac{\partial^2 f(x_0, y_0)}{\partial x^2}, \; s = \frac{\partial^2 f(x_0, y_0)}{\partial x y}, \; t = \frac{\partial^2 f(x_0, y_0)}{\partial y^2}$$

(i) if $rt - s^2 > 0$

(a) and if $r > 0$, f has local minima on (x_0, y_0).

(b) and if $r < 0$, f has local maxima on (x_0, y_0)

(ii) if $rt - s^2 < 0$

f has neither local maxima nor local minima

$\to (x_0, y_0)$ is saddle point.

(iii) if $rt - s^2 = 0$ TEST FAILS.

eg: $f(x, y) = x^3 + y^3 - 3axy$. Find saddle point.

For stationary point.

$f_x = 0$, $f_y = 0$

$f_x = 3x^2 - 3ay = 0$, $f_y = 3y^2 - 3ax = 0$

$\Rightarrow x^2 = ay$; $y^2 = ax$

Checking for points $(0,0)$ and (a,a).

$$\frac{x^4}{a^2} = ax \Rightarrow x(x^3 - y^3) = 0$$

$f_{xx} = 6x$; $f_{xy} = -3a$; $f_{yy} = 6y$.

At $(0,0)$,

$r = f_{xx}(0,0) = 0$

$t = f_{yy}(0,0) = 0$

$s = f_{xy}(0,0) = -3a$

$rt - s^2 = -9a^2 < 0$

Hence, $(0,0)$ is a saddle point

At (a,a),

$r = 6a,\ t = 6a;\ s = -3a$

$rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

if $a > 0 \Rightarrow r > 0$; f has local maxime at (a,a).

$f_{min} = a^3 + a^3 - 3a^3 = -a^3$

if $a < 0 \Rightarrow r < 0$; f has local maxima at $(a,0)$.

Here take

$a = -k,\ k > 0$.

$f_{max} = -k^3 + (-k^3) + 3k^3 = k^3$

eg: $f(x,y) = x^4 + y^4$

$f_x = 4x^3$; $f_y = 4y^3$

$f_{xx} = 12x^2$; $f_{yy} = 12y^2$

$f_{xy} = 0$.

At point $(0,0)$

$r = 0;\ t = 0;\ s = 0$

$rt - s^2 = 0$ (Test fails)

but $(0,0)$ is local maxima.

eg: $f(x,y) = (x^2+y)^2 - x^4$.

$f_x = 2x + 2y - 4x^3$ $\qquad f_y = 2x + 2y$

$f_{xx} = 2 - 12x^2$ $\qquad f_{yy} = 2$.

$f_{xy} = 2$

For critical points, $f_x = 0$, $f_y = 0$

At point $(0,0)$

$r = 2$, $t = 2$, $s = 2$

$rt - s^2 = 0$ $\therefore$ Test fails.

For $0 < \delta < 1$

$0 < t \leq \delta$ $B_\delta(0,0)$

$f(t,t) = 4t^2 - t^4$
$= t^2(4 - t^2)$

$f(t,t) > 0 = f(0,0)$

$f(t,-t) < 0 = f(0,0)$.

$\therefore$ $(0,0)$ is a saddle point.

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Lecture 36

Absolute Extrema

Consider a rectangular region with perimeter 16.

$f(x,y) = xy$

such that $2(x+y) = 16$

$\Rightarrow x + y = 8$.

eg: $f : [-2,2] \times [-2,2] \to \mathbb{R}$ such that

$f(x,y) = 4xy - 2x^2 - y^4$ 16–24

Find absolute extrema.

For Critical Point

$f_x = 0$, $f_y = 0$

$f_x = 4y - 4x = 0 \Rightarrow x = y$

$f_y = 4x - 4y^3 = 0 \Rightarrow x = y^3$

(x,y)	$(0,0)$	$(1,1)$	$(-1,-1)$	$(-2,2)$	$(2,2)$	$(2,-2)$	$(2, 2^{1/3})$	$(-2,-2^{1/3})$	$(-2,-2)$
$f(x,y)$	0	1	1	-40	-8	-40	$6\cdot 2^{1/3}-8$	$+6\cdot 2^{1/3}-8$	-8

$\Rightarrow y^3 - y = 0$

$\Rightarrow y(y^2-1) = 0$

$\Rightarrow y = 0, 1, -1$

Similarly, $x = 0, 1, -1$

Critical points are $(0,0), (1,1), (-1,-1)$

Boundary Point

$(x,y) \in [-2,2] \times [-2,2]$ is a boundary point if
$x = \pm 2$ and $y = \pm 2$.

Now restricted function for $x = 2, -2$ and $y = 2, -2$ are

$$\left.\begin{aligned} f &= -8y - 8 - y^4 = f(-2, y) ; \\ f &= 8y - 8 - y^4 = f(2, y) ; \end{aligned}\right\} -2 \le y \le 2$$

$$\left.\begin{aligned} f &= -8x - 2x^2 - 16 = f(x,-2) ; \\ f &= 8x - 2x^2 - 16 = f(x, 2) ; \end{aligned}\right\} -2 \le x \le 2$$

We need to solve only 2 fns. out of these 4 for abs. extrema since $f(-x,-y) = f(x,y)$ (there is symmetry in opp. quads).

Solving $f(x,2) = 8x - 2x^2 - 16$

$$\frac{\partial f}{\partial x} = 8 - 4x = 0 \Rightarrow x = 2$$

$f(x,2)$ has absolute extrema at $x = \pm 2$.

$f(x,y)$ has absolute extrema at $(-2,2), (2,2)$

For $f(2,y) = 8y - 8 - y^4$; $-2 \le y \le 2$

$$\frac{\partial f}{\partial y} = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$$

For critical point, $y = 2^{1/3}$

$f(2,y)$ have absolute extrema at $y = -2, 2$ and at $y = 2^{1/3}$

Moreover, $\frac{\partial f}{\partial y} > 0$ for $y = (-2, 2^{1/3})$ $\therefore$ f is increasing.

$\partial f / \partial y < 0$ for $y = (2^{1/3}, 2)$ $\therefore$ f is decreasing.

Lagrange's Method (Constrained extrema)

$f(x,y) = x.y$ such that $2(x+y) = 16$

or $(x+y) = 8$

$f(x) = x(8-x)$

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Lecture 37

Zero set of $g = \{(x,y) \in \mathbb{R}^2 ; g(x,y) = 0\}$

Lagrange's Method – for solving constrained extrema i.e. with given conditions

Lagrange's Multiplier Method

eg: Find the absolute extrema of a function $f(x,y)$ subject to constraint $g(x,y) = 0$.

(i) solve the following equation for λx and y.

$$\nabla f(x,y) = \lambda \nabla g(x,y).$$

$$g(x,y) = 0 \quad (\nabla g(x,y) \neq 0).$$

(ii) If it can be ensured that 'f' has an absolute extrema on the zero set of g (which will certainly be the case of zero set of $g(x,y)$ is closed and bounded and f is cont.) Then absolute extremum of 'f' is also a local extremum of 'f' and g is necessarily attained either at a simultaneous solution (x_0, y_0) of above equations for which $\nabla g(x_0, y_0) \neq 0$ or at a point of the zero set of g (x,y) at which ∇g or ∇f does not exist or at which ∇g vanishes. $(\nabla g = 0)$.

eg: Find the absolute extremum of $f(x,y) = xy$ on the unit circle $x^2 + y^2 = 1$.

Soln: $g(x,y) = x^2+y^2-1$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\Rightarrow (y, x) = \lambda(2x, 2y)$

$\Rightarrow \quad y = 2\lambda x \text{ --- } (1) \quad ; \quad x = 2\lambda y \text{ --- } (2)$

$x^2+y^2-1=0 \text{ --- } (3)$

The zero set of $g(x,y)$ is all points on the circumference of unit circle which is closed and bounded and f is continuous in zero set of $g(x,y)$. so, 'f' has absolute extremum on zero set of $g(x,y)$.

$(0,0)$ is not solution of (1), (2) and (3).

Using (2) in (1),

$y = 4\lambda^2 y$

$\Rightarrow y(1-4\lambda^2) = 0$

$\Rightarrow \lambda = \pm 1/2$

$\Rightarrow y = \pm x$ (from (1))

Using in (3),

$2x^2-1 = 0$

$\Rightarrow x = \pm 1/\sqrt{2} \quad \Rightarrow y = \pm \frac{1}{\sqrt{2}}$

Thus, points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Absolute maxima of $f = \frac{1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

Absolute minima of $f = \frac{-1}{\sqrt{2}}$ at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

eg: Find absolute maxima of $f(x,y) = x+y$ as $xy = 16$.

Soln: $g(x,y) = xy - 16$

$\Rightarrow \nabla f(x,y) = \lambda \nabla g(x,y)$ → This is necessary condition and not sufficient condition i.e. it does not guarantee the existence of extrema.

$\Rightarrow (1,1) = \lambda(y,x)$

$\Rightarrow \quad 1 = \lambda y$ — (1) ; $1 = \lambda x$ — (2)

$xy - 16 = 0$ — (3)

Using (1), (2) and (3).

$\frac{1}{\lambda^2} - 16 = 0 \Rightarrow \lambda = \pm 1/4$

$\Rightarrow x = \pm 4 \quad , y = \pm 4$

$\therefore$ Points are $(4,4)$ and $(-4,-4)$.

But, as we move away from the origin of $xy = 16$ (hyperbola) i.e. (5,5), the function becomes larger and larger i.e. will not attain absolute maxima i.e. it is continuous and OPEN $\therefore$ Langrange's method fail $\Rightarrow$ funtion has no absolute maxima.

eg: Find the point on the curve $(x-1)^3 = y^2$ which is closed to origin (0,0).

Soln: Let $A = (x,y)$ be the point on the curve which is close to origin O.

Distance b/n AO $= \sqrt{x^2+y^2} = h(x,y)$.

Minimizing $h(x,y)$ is same as minimizing $h^2(x,y)$.

So, now problem becomes : Find minima of $f(x,y)$.

$f = x^2+y^2$ such that $(x-1)^3 = y^2$,
$g(x,y) = (x-1)^3 - y^2 = 0$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\Rightarrow (2x, 2y) = \lambda(3(x-1)^2, -2y)$

$\Rightarrow 2x = 3\lambda(x-1)^2$ —— (1)
$2y = -2\lambda y$ —— (2)
$(x-1)^3 - y^2 = 0$ —— (3)

(I) if $y \neq 0$; (2) $\Rightarrow \lambda = -1$ using in (1)
$2x = -3(x^2+1-2x)$
$\Rightarrow 3x^2 - 4x + 3 = 0$
$b^2 - 4ac = 16 - 36 < 0$
$\therefore$ No solution on R.

(II) if $y = 0$; (3) $\Rightarrow x = 1$ (using in (1))
$2 = 0$ which is absurd.

* Geometrically, it has absolute minima but no maxima.

Now, $\nabla g(x,y) = [3(x-1)^2, -2y]$

$\Rightarrow \nabla g(1,0) = (0,0) \rightarrow$ Lagrange's 3rd Condition.

$\Rightarrow$ Minima will be at $(1,0)$

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Lecture 38

Q Find the volume of largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln: Let $2x, 2y, 2z$ be the length width and height of the rectangular parallelopiped.

$$\text{Volume} = 8xyz$$

Problem: Find the absolute maxima $f(x,y,z) = 8xyz$ subject to $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

by Langrange,

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

(1) $\times x$ + (2) $\times y$ + (3) $\times z$

$$\Rightarrow \cancel{32}\,24xyz = 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right]$$

$$\Rightarrow 12xyz = \lambda \quad \text{— (5)}$$

Using (5) in (1), $8yz = 12xyz \cdot \frac{2x}{a^2} \Rightarrow 3x^2 = a^2$

$\Rightarrow x = a/\sqrt{3}$

similarly, $y = b/\sqrt{3}$

$z = c/\sqrt{3}$

$$\boxed{V = 8abc/3\sqrt{3}}$$

Since, the zero set of $g(x,y,z)$ is closed & bounded (bcoz $g(x,y,z)$ is an ellipsoid), & f is continuous so by Langrange's method f attains absolute maxima, $V_{max} = \frac{8abc}{3\sqrt{3}}$.

Find absolute extrema $f(x,y,z)$ such that $Q_1(x,y,z) = 0$ and $Q_2(x,y,z) = 0$

$$\nabla f(x,y,z) = \lambda \nabla Q_1(x,y,z) + \mu \nabla Q_2(x,y,z)$$

λ and μ are undetermined coefficients.

$$Q_1(x,y,z) = 0$$
$$Q_2(x,y,z) = 0$$

eg: Find the absolute extrema of $U = x^2 + y^2 + z^2$ such that $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.

Soln: $Q_1 = ax^2 + by^2 + cz^2 - 1 = 0$

$Q_2 = lx + my + nz = 0$

The intersection of ellipsoid Q_1 and plane $Q_2 = 0$ is a ellipse which is closed and bounded. Also U is continuous on ellipse, so U attains absolute extreme. $\nabla U = \lambda \nabla Q_1 + \mu \nabla Q_2$

$$(2x, 2y, 2z) = \lambda(2ax, 2by, 2cz) + \mu(l, m, n)$$

$$2x = 2\lambda ax + \mu l \quad -(1)$$
$$2y = 2\lambda by + \mu m \quad -(2)$$
$$2z = 2\lambda cz + \mu n \quad -(3)$$

$$ax^2 + by^2 + cz^2 - 1 = 0 \quad -(4)$$
$$lx + my + nz = 0 \quad -(5)$$

$(1) \times x + (2) \times y + (3) \times z$

$\Rightarrow 2(x^2 + y^2 + z^2) = 2\lambda(ax^2 + by^2 + cz^2) + \mu(lx + my + nz)$

$\Rightarrow 2U = 2\lambda + 0$

$$\Rightarrow \quad \boxed{\lambda = u}$$

Using in (1), (2) and (3)

$$2x = 2uax + \mu l$$

$$\Rightarrow \quad x = \frac{\mu l}{2(1-au)}$$

Similarly,

$$y = \frac{\mu l}{2(1-bu)}$$

$$z = \frac{\mu l}{2(1-cu)}$$

Using value of x, y, z in (5)

$$\frac{\mu}{2}\left[\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu}\right] = 0$$

$$\Rightarrow \frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu} = 0.$$

which gives required extremum.

eg: Find the absolute extrema for $u = ax^2 + by^2 + cz^2$ such that $x^2 + y^2 + z^2 = 1$, and $lx + my + nz = 0$.

Soln: $Q_1 = x^2 + y^2 + z^2 - 1$

$Q_2 = lx + my + nz$

Ans: $\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$

eg Find the point on the intersection of two planes given by $x + y + z = 1$ and $3x + 2y + z = 6$ that is closest to the origin.

Soln: Let $P(x, y, z)$ be any point on the intersection of planes.

Ans:

Distance from origin, $PO = \sqrt{x^2+y^2+z^2}$.

Minimizing PO is same as minimizing $(PO)^2$.

Now problem is: Minimize $f = (PO)^2 = x^2+y^2+z^2$

$$Q_1 = x+y+z=1 \quad — (4)$$
$$Q_2 = 3x+2y+z-6 \quad — (5)$$

The intersection of two planes is a line. So, geometrically there exists a pt. on the line that is closest to the origin. So by Langrange's method,

Now,

$$\nabla f = \lambda \nabla Q_1 + \mu \nabla Q_2$$
$$(2x, 2y, 2z) = \lambda(1,1,1) + \mu(3,2,1)$$
$$\Rightarrow \quad 2x = \lambda + 3\mu \quad — (1)$$
$$2y = \lambda + 2\mu \quad — (2)$$
$$2z = \lambda + \mu \quad — (3)$$

Putting x, y, z in (4) and (5),

$$3\lambda + 6\mu = 2$$
$$3\lambda + 7\mu = 6$$
$$\Rightarrow \mu = 4, \ \lambda = -22/3$$

Substituting in (1), (2), (3)

$\cancel{x = -15/3}$, $\cancel{y = -16/3}$

$$x = 7/3, \ y = 1/3, \ -5/3$$

$$\therefore \quad P = \left(7/3, 1/3, -5/3\right).$$

Q: Find the extreme value of $f = 2x+3y+z$ such that $x^2+y^2 = 5$ and $x+z=1$.

$\Rightarrow \boxed{\lambda = u}$

Using in (1), (2) and (3)

$$2x = 2uax + \mu l$$

$$\Rightarrow x = \frac{\mu l}{2(1-au)}$$

Similarly,

$$y = \frac{\mu l}{2(1-bu)}$$

$$z = \frac{\mu l}{2(1-cu)}$$

Using value of x, y, z in (5)

$$\frac{\mu}{2}\left[\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu}\right] = 0$$

$$\Rightarrow \frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu} = 0.$$

which gives required extremum.

eg: Find the absolute extrema for $u = ax^2 + by^2 + cz^2$ such that $x^2 + y^2 + z^2 = 1$, and $lx + my + nz = 0$.

Soln: $Q_1 = x^2 + y^2 + z^2 - 1$

$Q_2 = lx + my + nz$

Ans: $$\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$$

eg Find the point on the intersection of two planes given by $x + y + z = 1$ and $3x + 2y + z = 6$ that is closest to the origin.

Soln: Let $P(x, y, z)$ be any point on the intersection of planes.

Ans:

Distance from origin, $PO = \sqrt{x^2+y^2+z^2}$.

Minimizing PO is same as minimizing $(PO)^2$.

Now problem is: Minimize $f = (PO)^2 = x^2+y^2+z^2$

$$Q_1 = x+y+z = 1 \quad \text{— (4)}$$

$$Q_2 = 3x+2y+z = 6 \quad \text{— (5)}$$

The intersection of two planes is a line. So, geometrically, there exists a pt. on the line that is closest to the origin. So by Langrange's method,

Now,

$$\nabla f = \lambda \nabla Q_1 + \mu \nabla Q_2$$

$$(2x, 2y, 2z) = \lambda(1,1,1) + \mu(3,2,1)$$

$$\Rightarrow \quad 2x = \lambda + 3\mu \quad \text{— (1)}$$

$$2y = \lambda + 2\mu \quad \text{— (2)}$$

$$2z = \lambda + \mu \quad \text{— (3)}$$

Putting x, y, z in (4) and (5),

$$3\lambda + 6\mu = 2$$

$$3\lambda + 7\mu = 6$$

$$\Rightarrow \mu = 4, \ \lambda = -22/3$$

Substituting in (1), (2), (3)

$\cancel{x = -15/3}$, $\cancel{y = -16/3}$

$$x = 7/3, \ y = 1/3, \ -5/3$$

$$\therefore \ P = \left(7/3, 1/3, -5/3\right).$$

eg: Find the extreme value of $f = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

\# Whenever nothing is specified, we take that the intersection happened horizontally.

The intersection of cylinder and plane is a circle. So, by Lagrange's method.......

Soln: $\nabla f = 2 + \lambda \nabla Q_1 + \mu \nabla Q_2$

$$Q_1 = 0 \Rightarrow x^2 + y^2 - 5 = 0 \quad —(1)$$

$$Q_2 = 0 \Rightarrow x + z - 1 = 0 \quad —(2)$$

Ans:

$$P = \left[-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right] = \left[-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right]$$

$$\therefore f = 1 - 5\sqrt{2}$$

$$P' = \left[\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\right] = \left[+\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right]$$

$$\therefore f = 1 + 5\sqrt{2}$$

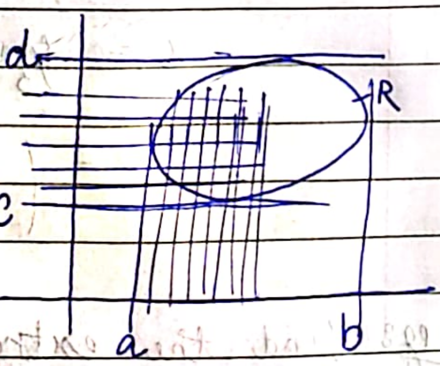
18th Nov '19

Lecture 39

DOUBLE INTEGRAL

Double integral on a non-negative function $f(x,y)$ on a region R is defined as the volume of a 3-dimensional solid on the x-y plane bounded below by region R and bounded above by surface $z = f(x,y)$.

$D = [a,b] \times [c,d]$



$f : Q \to R$

Suppose $P_1 = [a = x_0, x_1, \ldots, x_n = b]$

& $P_2 = [c = y_0, y_1, \ldots, y_n = d]$

be any partition of $[a,b]$ and $[c,d]$.

$P = P_1 \times P_2$ decompose Q into mn subinterval

$$m_{ij} = \inf[f(x,y) : x \in [x_{i-1}, x_i],\ y \in [y_{j-1}, y_j]]$$

$$M_{ij} = \sup[f(x,y) : x \in [x_{i-1}, x_i],\ y \in [y_{j-1}, y_j]]$$

$$L(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij} \cdot \Delta x_i \cdot \Delta y_i$$

$$U(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij} \cdot \Delta x_i \cdot \Delta y_j$$

$$\underline{\iint} f(x,y)\, dx.dy = \sup_P L(P,f)$$

$$\overline{\iint} f(x,y)\, dx\, dy = \inf_P U(P,f)$$

if

$$\boxed{\underline{\iint} f\, dx\, dy = \overline{\iint} f\, dx\, dy}$$

then f is double integral.

So, $\iint f\, dx\, dy = \iint f\, dA$

<u>Theorem</u>: If $f : Q[a,b] \times [c,d] \to R$ is ~~connected~~ continuous on Q, then f is integrable.

FUBINI'S THEOREM (First form)

Let $f : Q = [a,b] \times [c,d] \to R$ be continuous then,

$$\iint_Q f(x,y)\, dx\, dy = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\, dy\right) dx.$$

$$= \int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\, dx\right) dy.$$

FUBINI'S THEOREM (Second/Stronger Form).

Let $f(x,y)$ be a bounded function in region D.

(i) If $D = [(x,y) : a \le x \le b]$ & $f_1(x) \le y \le f_2(x)$, for some continuous function $f_1, f_2 : [c,d] \to R$. then,

$$\iint_D f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx.$$

(ii) If $D = [(x,y) : c \le y \le d]$ and $g_1(y) \le x \le g_2(y)$ for some continuous function $g_1, g_2 : [a,b] \to R$ then,

$$\iint_D f(x,y)\,dx\,dy = \int_{y=c}^{d}\left(\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\right)dy$$

eg: Find the volume of the region bounded above by $z = 10 + x^2 + 3y^2$ and below by rectangle $D: 0 \le x \le 1, 0 \le y \le 2$.

$$\iint_D f(x,y)\,dx\,dy = \int_0^1\int_0^2 \left((10 + x^2 + 3y^2)\,dy\right)dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\,dx$$

$$= 28 + 2/3$$

$$= \underline{\underline{86/3}}$$

eg: Evaluate $\iint_D (x+y)^2 \, dx\, dy$ where D is the region bounded by lines joining the points (0,0), (0,1) and (2,2).

Solⁿ: $x=0$ to $x=2$

Eqⁿ of line OB, $y=x$

Eqⁿ of line AB, $y-1 = \frac{2-1}{2-0}(x-0) \Rightarrow y = \frac{x}{2}+1$

$\therefore\ y = x$ to $y = \frac{x}{2}+1$

$$I = \iint_D f(x,y)\,dx\,dy = \int_0^2 \left(\int_x^{x/2+1} (x+y)^2 \, dy \right) dx$$

$$= \int_0^2 \left(x^2 y + 2xy^2 + \frac{y^3}{3} \right)_x^{x/2+1} dx$$

$$\cancel{= \int_0^2 x\left(1-\frac{x}{2}\right)\left[x+1-\frac{x}{2}\right] + \frac{1}{3}\left[\left(\frac{x}{2}+1\right)^3 - x^3\right] dx}$$

$$= \int_0^2 x^2\left(\frac{x}{2}+1\right) + x\left(\frac{x}{2}+1\right)^2 + \frac{1}{3}\left(\frac{x}{2}+1\right)^3 - \left(x^3+x^3+\frac{x^3}{3}\right) dx$$

$$= \int_0^2 \left(\frac{x}{2}+1\right)x^2 + \left(\frac{x}{2}+1\right)^2\left(x+\frac{x}{6}+\frac{1}{3}\right) - \left(\frac{7x^3}{3}\right) dx$$

$$= \int_0^2 \left(\frac{x}{2}+1\right)\left[x^2 + \left(\frac{x}{2}+1\right)\left(\frac{7x}{6}+\frac{1}{3}\right)\right] - \frac{7x^3}{3}\, dx$$

$$= \int_0^2 \left(1+\frac{x}{2}\right)\left[\frac{19x^2}{12} + \frac{4x}{3} + \frac{1}{3}\right] - \frac{7x^3}{3}\, dx = \int_0^2 -\frac{37x^3}{24} + \frac{27x^2}{12} + \frac{3x}{2} + \frac{1}{3}$$

$$= -\frac{37\times 16}{24\times 4} + \frac{3\times 8}{4} + \frac{3\times 4}{4} + \frac{2}{3} = \frac{21}{6}$$

Evaluate,

eg: $\iint_R \frac{\sin x}{x}\, dx\, dy$, where R is triangle in x-y plane bounded by x-axis, lines $y=x$ and $x=1$.

Soln: $\iint_R \frac{\sin x}{x}\, dx\, dy = \int_{y=0}^{1}\int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{x} \frac{\sin x}{x}\, dy\right) dx$$

$$= \int_{x=0}^{1} \frac{\sin x}{x} [y]_0^x\, dx = \int_0^1 \frac{\sin x}{x} \times x\, dx$$

$$= (-\cos x)_0^1$$

$$= -\cos 1 + 1.$$

Properties:

Let f, g be continuous function on the bounded region R.

(i) $\iint_R (f(x,y) \pm g(x,y))\, dx\, dy = \iint_R f(x,y)\, dx\, dy \pm \iint_R g(x,y)\, dx . dy$

(ii) if $f(x,y) \geq 0 \;\forall (x,y) \in$ Region, then

$$\iint_R f(x,y)\, dx\, dy \geq 0$$

(iii) if $f(x,y) \geq g(x,y)$ over R.

$$\iint_R f\, dx\, dy \geq \iint_R g (dx\, dy)$$

(4) $\iint_R cf(x,y)\,dx\,dy = c\iint_R f(x,y)\,dx\,dy$ for $c \in R$.

(5) $\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy.$