Lecture 01: The Real Number System

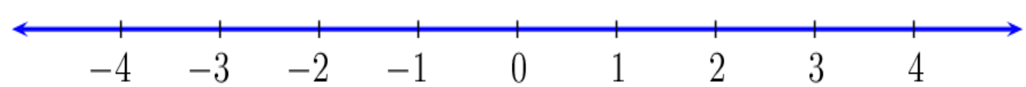
02 August 2019

Sunil Kumar Gauttam

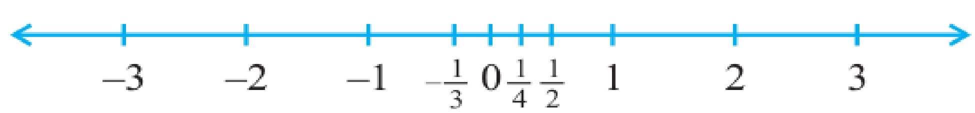
Department of Mathematics, LNMIIT, Jaipur

When we learn the script of a language, such as the English language, we begin with the letters of the alphabet , A, B, C, D; when we learn the sounds of music, such as those of Indian classical music, we begin with the notes Sa, Re, Ga, Ma Likewise, in mathematics, one begins with $1, 2, 3, \cdots$ these are the positive integers or the natural numbers. We shall denote the set of positive integers by $\mathbb{N} = \{1, 2, 3, \cdots\}$. Multiplication and addition are operations in $\mathbb{N}$, meaning that the sum or product of two natural numbers is a natural number. Subtraction, however, may not make sense if we have only the natural numbers at our disposal. For example, $3 - 5$ has no meaning in $\mathbb{N}$. Therefore we must consider the larger number system.

We denote the set of integers by $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Although addition, multiplication, and subtraction make sense in the integers, division may not. For instance, the expression $3/5$ does not represent an integer. Quotients of integers are called rational numbers. We shall denote the set of all rational numbers by $\mathbb{Q} = \left\{\frac{m}{n} \,\middle|\, m, n \in \mathbb{Z}, n \neq 0\right\}$. Geometrically, the integers can be represented by points on a line by fixing a base point (signifying the number 0) and a unit distance. Such a line is called the number line and it may be drawn as in the following figure



By suitably subdividing the segment between 0 and 1, we can also represent rational numbers such as $\frac{1}{n}$, where $n \in \mathbb{N}$, and this can, in turn, be used to represent any rational number by a unique point on the number line.



In general, the numbers that we encounter in everyday life: prices, speed limits, weights, temperatures, interest rates, and so on-are rational numbers. Also addition, multiplication, subtraction and division (by non-zero number) are operations in $\mathbb{Q}$.

However, there are also numbers that are not rational. As a simple example, the hypotenuse of a right triangle with base and height 1 , is not a rational number. As another example, the number π, which is defined as the circumference of a circle of diameter 1, is not a rational number.

The rational numbers and the irrational numbers together constitute the set $\mathbb{R}$, called the set of real numbers. Once again, addition, multiplication, subtraction and division (by non-zero number) are operations in $\mathbb{R}$.

Order relation on $\mathbb{R}$: Given any $a, b \in \mathbb{R}$, exactly one of the following statements is true.

$$a < b;\ a = b;\ b < a.$$

Definition 1.1 *Let S be a non-empty subset of $\mathbb{R}$. We say that S is bounded above if there exists $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in S$. Any such α is called an upper bound of S.*

For example, $S = \{1, 2, 3\}, \{x : 0 \leq x \leq 3\}, (-\infty, 0)$ are bounded above.

Remark 1.2 *It's time to focus on language. In definition we are saying that α is an upper bound. This indefinite articles 'an' before upper bound tell us that it is not unique. In fact if α is an upper bound of set S and so is $\alpha + 1$. In fact any number in the interval (α, ∞) is an upper bound. Definite article 'The' means unique in mathematical statement.*

Definition 1.3 *Let S be a non-empty subset of $\mathbb{R}$. An element $M \in \mathbb{R}$ is called a supremum or a least upper bound of the set S if*

1. *M is an upper bound of S, and*
2. *If α an upper bound of S then $M \leq \alpha$.*

For example, if $S_1 = \{x \in \mathbb{R} : 0 < x \leq 1\}, S_2 = (0, 1)$, then $\sup S_1 = 1$ and $\sup S_2 = 1$. But $\sup S_1$ is an element of S_1, but $\sup S_2$ is not an element of S. So we remark that The least upper bound of a set S may not belong to the set S.