The LNM Institute of Information Technology
Jaipur, Rajasthan

MATH-I ■ Assignment #5

(Limits, Continuity, Differentiability, Rolle's & Mean Value Theorems, L'Hospital Rule)

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1. Let $\alpha \in \mathbb{R}$, $D \subseteq \mathbb{R}$ and $f, g : D \to \mathbb{R}$ be continuous at $c \in D$. Then following are also continuous at c: $f + g$, αf, fg, $|f|$, $h(x) := \max\{f(x), g(x)\}$, $k(x) := \min\{f(x), g(x)\}$. (Statement only)

2. Using the **sequential** definition of continuity show that the function $f : [0, 1] \to [0, 1]$ defined by
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$
is discontinuous everywhere. Moreover, show that $\lim_{x\to 0} \sin \dfrac{1}{2x}$ does not exist.
[Hint for f: Rational and Irrational numbers are dense in real line, so for every real number x there exists a sequence of rational numbers and also there exists a sequence of irrational numbers which converges to x. Hint for limit: Take a sequence $(\frac{1}{(2n+1)\pi}) \to 0$]

3. Determine the points of continuity for the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 2x, & \text{if } x \text{ is rational} \\ x + 3, & \text{if } x \text{ is irrational} \end{cases}$$

4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy *Lipschitz condition*, if there exists $M > 0$, such that $|f(x) - f(y)| \leq M|x - y|$ for each $x, y \in \mathbb{R}$. Prove that such function f is always continuous.

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{wherep, q} \in \mathbb{N} \text{ and p, q have no common factors} \\ & 0, \text{ if } x \text{ is irrational} \end{cases}$$

Show that f is continuous at every irrational, but discontinuous at every rational.

5. Check the function $f(x) = \begin{cases} x^n \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ for the continuity, differentiability. Moreover, check the continuity and differentiability of $f'(x)$, where $n \in \mathbb{N}$.

6. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that f has a fixed point, that is, there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.
[Hint: Apply IVP on the function $g(x) := f(x) - x$ and use $0 \leq f(x) \leq 1$.]

7. Show that the polynomial $x^4 + 2x^3 - 9$ has at least two real roots.

8. Prove that the polynomial $f(x) = x^7 + 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.

   Hint: Apply IVP + Rolle's theorem.

9. Let $f : [a, b] \to \mathbb{R}$ be differentiable, then f is constant if and only if $f'(x) = 0$ for every $x \in [a, b]$.
   Hint: Apply MVT

10. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is differentiable, and the derivative is zero.

11. Using Mean Value Theorem, show that $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

12. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct c_1, $c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

13. Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be any function. If $f'(x)$ never vanishes on I then show that f is one-one.
    Hint: Use MVT

14. Find $\lim\limits_{x \to 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \to 0^+} \left(1 + \dfrac{1}{x}\right)^x$.

15. Can we apply L'Hospital Rule on the following:

    (a) $\lim\limits_{x \to \infty} \dfrac{x - \sin x}{2x + \sin x}$

    (b) $\lim\limits_{x \to 0} \dfrac{2x + x \sin \frac{1}{x}}{3x - x \sin \frac{1}{x}}$