Lecture 18: Consequence of Mean Value Theorem

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18.1 Derivatives and Monotonicity

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be any function. We say that f is said to be (monotonically) increasing on I if $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$. Also, f is said to be (monotonically) decreasing on I if $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$. One says that f is monotonic on I if it is monotonically increasing on I or monotonically decreasing on I. The function f is said to be strictly increasing [resp. strictly decreasing] on I if $x_1, x_2 \in I$, $x_1 < x_2$, implies $f(x_1) < f(x_2)$ [resp. $f(x_1) > f(x_2)$]. Also, one says that f is strictly monotonic on I if it is strictly increasing on I or strictly decreasing on I.

Proposition 18.1 *Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be a differentiable function. Then we have the following:*

1. *f' nonnegative throughout I $\iff$ f is monotonically increasing on I.*
2. *f' nonpositive throughout I $\iff$ f is monotonically decreasing on I.*
3. *f' positive throughout I $\implies$ f is strictly increasing on I.*
4. *f' negative throughout I $\implies$ f is strictly decreasing on I.*

Remark 18.2 *The converse of the implication in part 3 and 4 of Proposition 18.1 is not true. In fact, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is strictly increasing but $f'(0) = 0$. Similarly, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^3$ is strictly decreasing but $g'(0) = 0$.*

Example 18.3 *Consider the polynomial function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$. Then f is differentiable and one can easily check that $f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x - 1)(x - 2)(x - 3)$. Therefore, $f'(x) \geq 0$ if $x \geq 3$ or $1 \leq x \leq 2$, whereas $f'(x) \leq 0$ if $x \leq 1$ or $2 \leq x \leq 3$. Thus, f is monotonically increasing on $[1, 2]$ and on $[3, \infty)$, whereas f is monotonically decreasing on $[2, 3]$ and on $(-\infty, 1]$. In fact, since f vanishes only at $x = 1, 2$, and 3, we see that f is strictly increasing on $(1, 2)$ and on $(3, \infty)$, whereas f is strictly decreasing on $(2, 3)$ and on $(-\infty, 1)$.*

18.2 Cauchy's Mean Value Theorem

Theorem 18.4 (Chauchy's Mean Value Theorem) *If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that*

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]. \tag{18.1}$$

Remark 18.5 *Cauchy's mean value theorem is sometimes called generalized mean value theorem. Because, if we take $g(x) = x$ in CMVT we obtain the MVT.*

The Cauchy's mean value theorem is quite useful in proving certain inequalities. Here are some samples.

Example 18.6 *Using Cauchy's mean value theorem show that*

$$1 - \frac{x^2}{2!} \leq \cos x \textit{ for } x \in \mathbb{R}$$

Solution: For $x = 0$, the equality holds. Let $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$. Let $x > 0$, then f, g are continuous on $[0, x]$ and differentiable on $(0, x)$. So by CMVT there exists $c \in (0, x)$ such that

$$c(1 - \cos x - 0) = \sin c(x^2/2 - 0) \implies 1 - \cos x = \left(\frac{\sin c}{c}\right)\frac{x^2}{2} \leq \frac{x^2}{2}.$$

where last inequality follows from Example 17.7.

If $x < 0$, then $-x > 0$ hence there exists $c \in (0, -x)$ such that

$$c(1 - \cos(-x) - 0) = \sin c((-x)^2/2 - 0) \implies 1 - \cos x = \left(\frac{\sin c}{c}\right)\frac{x^2}{2} \leq \frac{x^2}{2}.$$

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Example 18.7 *Let f be continuous on $[a, b], a > 0$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that*

$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

Solution: Note that $\frac{f(x)}{x}$ and $\frac{1}{x}$ are continuous on $[a, b]$ (since $a > 0$) and differentiable on (a, b). So by CMVT, there exists $c \in (a, b)$ such that

$$-\frac{1}{c^2}\left(\frac{f(b)}{b} - \frac{f(a)}{a}\right) = \frac{cf'(c) - f(c)}{c^2}\left(\frac{1}{b} - \frac{1}{a}\right) \implies \frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

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18.3 Convexity & Concavity

Next, we discuss more subtle properties of a function, known as convexity and concavity. Geometrically, these notions are easily described. A function is convex if the line segment joining any two points on its graph lies on or above the graph. A function is concave if any such line segment lies on or below the graph.

Definition 18.8 *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. We say that*

1. *f is convex on I (or concave upward on I) if for any $x_1, x_2 \in I$ and any $t \in (0,1)$ we have*
$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2). \tag{18.2}$$
2. *f is concave on I (or concave downward on I) if for any $x_1, x_2 \in I$ and any $t \in (0,1)$ we have*
$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2). \tag{18.3}$$

A function $f : I \to \mathbb{R}$ is called strictly convex on I if inequality $\leq$ in 18.2 is strict inequality $<$. Similarly A function $f : I \to \mathbb{R}$ is called strictly concave on I if inequality $\geq$ in 18.2 is strict inequality $>$.

Example 18.9 *Let $a, b \in \mathbb{R}$, and let $f(x) := a + bx$ for $x \in \mathbb{R}$. Then f is convex as well as concave on $\mathbb{R}$ (but not strictly).*

Example 18.10 *Let $f(x) := x^2$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$. $f(x) = -x^2$ is strictly concave on $\mathbb{R}$.*

Note that convexity and concavity are purely geometric notions and a priori they have no relation with derivatives. However, in the case of differentiable functions, there is an intimate relation between derivatives and the notions of convexity and concavity. The key idea can once again be gleaned by looking at the graphs. Namely, if we draw tangents at each point, then we see that as we move from left to right, the slopes increase if the function is convex, whereas the slopes decrease if the function is concave. A more precise analytic formulation of this is given in the proposition below. In practice, this greatly simplifies checking whether a differentiable function is convex or concave.

Proposition 18.11 *Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be a differentiable function. Then we have the following:*

1. *f' is increasing on I $\iff$ f is convex on I.*
2. *f' is decreasing on I $\iff$ f is concave on I.*
3. *f' is strictly increasing on I $\iff$ f is strictly convex on I.*
4. *f' is strictly decreasing on I $\iff$ f is strictly concave on I .*

For twice differentiable functions, testing convexity or concavity can sometimes be simpler using the following result.

Proposition 18.12 *Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be a differentiable function. Then we have the following:*

1. *f'' is nonnegative throughout I $\iff$ f is convex on I.*
2. *f'' is nonpositive throughout I $\iff$ f is concave on I.*
3. *$f'' > 0$ throughout I $\implies$ f is strictly convex on I.*
4. *$f'' < 0$ throughout I $\implies$ f is strictly concave on I.*

The converse of (iii) in the above proposition is false: Let $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$, but $f''(0) = 0$. Similarly, the converse of (iv) is false for $f(x) = -x^4$.