Lecture 15: Limits of Real-valued Functions of a Real Variable

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We have seen what is meant by the limit of a sequence. As we know, a sequence is a function whose domain is the set $\mathbb{N}$ of all natural numbers. We shall now define the concept of a limit of a function at a point in $\mathbb{R}$ around which there are infinitely many points of the domain. We begin with the notion of a limit point of a set, which can be viewed as a legitimate point at which limits of functions defined on that set can be considered.

15.1 Limit Point

Definition 15.1 *Let $D \subset \mathbb{R}$ and $c \in \mathbb{R}$. Then c is a limit point of D if for every $r > 0$, there is $x \in D$ such that $0 < |x - c| < r$.*

In other words we are saying that c is a limit point of set D if every deleted neighborhood of point c (i.e., set of the form $(c - r, c) \cup (c, c + r)$) contains at least one point of set D.

Example 15.2 *Let $D = (0, 1)$.*

1. *Then $c = 0$ is a limit point since for every $r > 0$, $(0, r) \cap D$ contains infinitely many points of D.*
2. *Similarly $c = 1$ is also a limit point of D.*
3. *Also, Every point of D is a limit point of D.*
4. *$c = 1.001$ is not a limit point, since if we take $r = 0.001$ then deleted neighborhood of c with radius r is $(1, 1.001) \cup (1, 1.002)$, which does not contain any point of D.*

Example 15.3 *If $D = \{\frac{1}{n} : n \in \mathbb{N}\}$ then 0 is the only limit point of D.*

Example 15.4 *If $D = \mathbb{N}$ then D has no limit points.*

15.2 Limit

Definition 15.5 *Let $D \subset \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. We say that a limit of f as x tends to c exists if there is a real number l such that*

$$\forall \epsilon > 0 (\exists \delta > 0 (\forall x \in D, 0 < |x - c| < \delta (|f(x) - l| < \epsilon)))$$

Note that c need not be in the domain of f. Even if c lies in the domain of f, l need not be $f(c)$.

Example 15.6 *Consider function $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x) = 0$ using $\epsilon - \delta$ definition.*

Solution: Let $\epsilon > 0$ be given. Then choose $\delta = \epsilon$, If

$$x \in \mathbb{R},\ 0 < |x - 0| < \delta$$

Then

$$|f(x) - 0| = |x \sin \frac{1}{x} - 0| \leq |x| < \delta = \epsilon.$$

■

Definition 15.7 *Let $D \subset \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. We say that $\lim_{x \to c} f(x)$ exists if there exists $l \in \mathbb{R}$ such that for every sequence (x_n) in $D \setminus \{c\}$ with the property that $x_n \to c$, we have $f(x_n) \to l$.*

Example 15.8 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \text{ is rational} \\ 1 & \text{if} \quad x \text{ is irrational} \end{cases}$$

Show that $\lim_{x \to -1} f(x) = 1$, using sequential definition of limit.

Solution: Let (x_n) be a sequence in $\mathbb{R} \setminus \{-1\}$ such that $x_n \to -1$. hence $x_n^2 \to (-1)^2 = 1$. This implies $x_n^2 - 1 \to 0$. Now

$$|f(x_n) - 1| \leq |x_n^2 - 1| \quad \forall n \in \mathbb{N}$$

Hence by Sandwich theorem, we get $f(x_n) \to 1$. ■

15.3 Why we need a limit point to define limits?

Suppose we make the following definition of limit.

Definition 15.9 *Let $D \subset \mathbb{R}$ and let $c \in \mathbb{R}$ Let $f : D \to \mathbb{R}$ be a function. We say that a limit of f as x tends to c exists if there is a real number l such that for every $\epsilon > 0$, $\exists \delta > 0$ such that*

$$x \in D, 0 < |x - c| < \delta \implies |f(x) - l| < \epsilon$$

Now let $f : \mathbb{N} \to \mathbb{R}$ be any function. we are interested in determining limit of this function say at $c = 2$. Let $\epsilon > 0$ be given. Choose any $\delta < 1$. Then there is no point of domain $\mathbb{N}$ such that $0 < |x - 2| < 1$. hence condition $|f(x) - l| < \epsilon$ is vacuously true for every real number l. Therefore any real number is a limit of this function at $c = 2$. So we loose the uniqueness of the limit.

In order have uniqueness of the limit, we need to define limits only at limit point.

15.4 Why Continuity before Limits?

We now relate the concepts of continuity and limit.

Theorem 15.10 *Let $D \subset \mathbb{R}$ and let $c \in D$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. Then f is continuous at c if and only if $\lim\limits_{x \to c} f(x)$ exists and is equal to $f(c)$.*

In your earlier classes you first study the limits and then define continuity in term of limits.

The drawback of using limits to define continuity is easily understood, while we can define continuity on an arbitrary subset of $\mathbb{R}$, to define limit, we need to put a restriction on D.

15.5 Differentiation

Differentiation is a process that associates to a real-valued function f another function f', called the derivative of f. This process is local in the sense that the value of f' at a point c depends only on the values of f in a small interval around c.

Definition 15.11 *Let I be an interval (which is not a singleton), and $c \in I$. A function $f : I \to \mathbb{R}$ is said to be differentiable at point c if the limit*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In this case, the value of the limit is denoted by $f'(c)$ and is called the derivative of f at c.

Note that c is a point of interval ensure that it is a limit of the interval I (where is $I \neq [a, a]$).