**The LNM Institute of Information Technology**
**Jaipur, Rajsthan**

**MATH-I** ■ Assignment #1 (Real Number System)

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Notation: $\mathbb{Q}$ : set of rational numbers, $\mathbb{R}$ : set of real numbers, $\mathbb{N}$: set of natural numbers, $a|b$ : $a$ divides $b$.

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1. Show that
   (a) $\sqrt{p}$ is not a rational for any prime $p$.
   (b) Prove that there is no rational number whose square is $n$, where $n$ is not a perfect square.
   (c) $\sqrt{15}, \sqrt[3]{2}, \sqrt[5]{16}$ are not rational.

   (Hint: Let $p$ be a prime, and $m$ and $n$ be natural numbers. If $p|m^n$, then $p|m$. )

2. Prove that there is no rational number whose square is 12.

3. Show that $\log_{10} 2$ is not rational.
   (Hint: Take $\log_{10} 2 = \frac{a}{b} \in \mathbb{Q}$ with $gcd(a,b) = 1$. Take $2 = 10^{\frac{a}{b}} \Rightarrow 2^b = 10^a \Rightarrow 2^b = 2^a 5^a \Rightarrow \frac{2^b}{5^a} = 2^a$)

4. If $r$ is rational ($r \neq 0$) and $x$ is irrational, prove that $r + x$ and $rx$ are irrational.

5. Prove that $\sqrt{2} + \sqrt{3}$ is irrational. (Hint: The square of a rational number is always a rational number.)

6. Find the infimum and supremum (if exists) of the following sets:
   (a) $S_1 = \left\{\frac{2}{n} : n \in \mathbb{N}\right\}$.
   (b) $S_2 = \left\{\frac{(-1)^n}{5n} : n \in \mathbb{N}\right\}$.
   (c) $S_3 = \left\{\frac{-3}{n} : n \in \mathbb{N}\right\}$.
   (d) $S_4 = \left\{\frac{2m}{m+n} : m, n \in \mathbb{N}\right\}$.
   (e) $S_5 = \left\{(1 + \frac{1}{n})^n : n \in \mathbb{N}\right\}$.

7. Let $S$ be a non-empty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. If $\alpha = Sup\ S$, then show that for any $\epsilon > 0$, there is some $x \in S$ such that $\alpha - \epsilon < x$.

8. Let $S$ be a non-empty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$. If $\beta = Inf\ S$, then show that for any $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$.

9. Use the Archimedean property of real numbers to show that $\displaystyle\bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right]$ is an empty set.

10. Show that $S = \{x : x \in \mathbb{Q}, x > 0\ and\ x^2 < 2\}$ has no supremum in $\mathbb{Q}$.

    **Hint:** Take $S = \{x : x \in \mathbb{Q}, x > 0\ and\ x^2 < 2$ and show that $S$ has no supremum in $\mathbb{Q}$, i.e. Sumpremum of $S$ does not belongs to $\mathbb{Q}$. Take $\alpha = SupS \in \mathbb{Q}$ such that $\alpha^2 < 2$ and $y = \frac{4+3\alpha}{3+2\alpha}$. Then show $\alpha - y < 0$ and $y^2 2 < 0$. Which indicates that $\alpha < y < \sqrt{2}$.