Lecture 03: Completeness of Real Numbers

06 August 2019

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT, Jaipur

3.1 Completeness Property

The most important, property of $\mathbb{R}$ that we shall assume is the following.

Completeness Property: Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Exercise 3.1 *State True/False.*

1. *Every nonempty subset of $\mathbb{N}$ that is bounded above has a supremum in $\mathbb{N}$. Ans. True*
2. *Every nonempty subset of $\mathbb{Z}$ that is bounded above has a supremum in $\mathbb{Z}$. Ans. True*
3. *Every nonempty subset of $\mathbb{N}$ that is bounded above has a supremum in $\mathbb{Q}$. Ans. True*
4. *Every nonempty subset of $\mathbb{Z}$ that is bounded above has a supremum in $\mathbb{Q}$. Ans. True*

Completeness property implies the following.

Proposition 3.2 *Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Then S has an infimum.*

Remark 3.3 *Proposition 3.2 is equivalent to the completeness property.*

The set of natural numbers $\mathbb{N}$ is not bounded above in $\mathbb{R}$. This is evident if we take any real number on number line, then we may take integers bigger than it.

Proposition 3.4 (Archimedean property) *Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

Alternate form of Archimedean Property: If $a, b \in \mathbb{R}$ and $a > 0$, then there is a positive integer n such that $na > b$.

Suppose $a, b \in \mathbb{R}$ with $a > 0$ is given. Then set $x = b/a$. By Proposition 3.4, there exists n such that $n > b/a$, i.e., $na > b$.

To go other way around take $a = 1$ and $b = x$.

The next couple of results are easy consequences of the Archimedean property.

Example 3.5 *1. Given $x > 0$, show that there exists $n_0 \in \mathbb{N}$ such that*

$$n \geq n_0 \implies 0 < 1/n < x.$$

2. Given $x > 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies 2^{-n} < x.$$

3. Let $x \geq 0$. Then show that $x = 0$ iff for each $n \in \mathbb{N}$, we have $x \leq 1/n$.

Solution:

1. Take $a = x$ and $b = 1$ then AP says that there exist n_0 such that $n_0 x > 1$. Now $n_0 x > 1 \iff x > 1/n_0$. If $n \geq n_0$ then $1/n \leq 1/n_0$. Hence we are done.
2. Note that $2^n > n$ for all $n \geq 1$. One can prove it using induction. For $n = 1$, result is true. For $n > 1$, adding n on both sides on inequality we get $n + n > 1 + n$, i.e., $2n > n+1$. Now suppose result is true for $n = m > 1$. Now $2^{m+1} = 2 \cdot 2^m > 2m > m+1$. Now by part 1, it follows that there exists n_0 such that
$$\frac{1}{n_0} < x$$
For $n \geq n_0$, $1/n \leq 1/n_0$ also $2^{-n} < 1/n$ hence we get the result.
3. If $x = 0$ then evidently $x < 1/n$ for each n. Suppose conversely, that $x \leq 1/n$ for each n. Suppose $x > 0$, then by part 1, it follows there exists n_0 such that $1/n_0 < x$. This is a contradiction to the assumption that $x \leq 1/n$ for each n.

Example 3.6 *Let $J_n := (1/n, 1)$. Show that $\displaystyle\bigcup_{n=1}^{\infty} J_n = (0, 1)$.*

Solution: Note that $J_1 = (1, 1) = \emptyset$. Also note that $J_1 \subset J_2 \subset J_3 \subset \cdots$. Each $J_n \subset (0, 1)$ hence their union is also a subset of $(0, 1)$. We need to show that $(0, 1) \subset \bigcup_{n=1}^{\infty} J_n$. Let $x \in (0, 1)$ be given. Then by AP, there exist n_0 such that $1/n_0 < x < 1$. hence $x \in J_{n_0}$.

Example 3.7 *Let $J_n := [n, \infty)$. Show that $\bigcap_n J_n = \emptyset$.*

Solution: Note that $J_1 \supset J_2 \supset J_3 \supset \cdots$. We prove $\bigcap_n J_n = \emptyset$ by contradiction. Suppose $\bigcap_n J_n \neq \emptyset$. There there must exists at least one $x \in \mathbb{R}$ such that $x \in \bigcap_n J_n$, i.e., $x \in J_n$ for all n, i.e. $x \geq n$ for all n. That is x is an upper bound for set of natural number, which is absurd.

A very important consequence of Archimedean property is the existence of greatest integer function stated in the following theorem.

Proposition 3.8 *Let $x \in \mathbb{R}$. Then there exists a unique $m \in \mathbb{Z}$ such that $m \leq x < m + 1$.*

Definition 3.9 *The unique integer m such that $m \leq x < m + 1$ is called greatest integer less than or equal to x. It is denoted by $[x]$. It is also called the floor of x and is denoted by $\lfloor x \rfloor$ in computer science. The number $x - [x]$ is called the fractional part of x. We observe that $0 \leq x - [x] < 1$.*

The Archimedean property leads to the "density of rationals in $\mathbb{R}$" and "density of irrationals in $\mathbb{R}$.

Proposition 3.10 *Given any $a, b \in \mathbb{R}$ with $a < b$, there exists a rational number as well as an irrational number between a and b.*

Proof: Using Archimedean-Property with $x = 1, y = \dfrac{1}{b-a}$, we can find $n \in \mathbb{N}$ such that $n > \dfrac{1}{b-a}$. Let $m = [na] + 1$. Then $m - 1 \leq na < m$, and hence

$$a < \frac{m}{n} \leq \frac{na + 1}{n} = a + \frac{1}{n} < a + (b - a) = b.$$

Thus we have found a rational number (namely, $\dfrac{m}{n}$) between a and b. Now we show that given any $a, b \in \mathbb{R}$ with $a < b$, there exists an irrational number , $a < b \implies a + \sqrt{2} < b + \sqrt{2}$, and we can find r a rational number between $a + \sqrt{2}$ and $b + \sqrt{2}$. Then we claim that $r - \sqrt{2}$ is an irrational number between a and b. Clearly $a + \sqrt{2} < r < b + \sqrt{2} \implies a < r - \sqrt{2} < b$. Let us assume that $r - \sqrt{2}$ is rational, i.e.

$$\begin{aligned} r - \sqrt{2} &= \frac{m}{n}, \text{ for some } m \in \mathbb{Z}, n \in \mathbb{N} \\ \implies r - \frac{m}{n} &= \sqrt{2} \end{aligned}$$

which is contradiction as left hand side is a rational number. ∎

Example 3.11 *Show that $\mathbb{Q}$ does not have completeness property,i.e. show that there is a non-empty subset of $\mathbb{Q}$ that is bounded above but does not have supremum in $\mathbb{Q}$.*

Solution: Let $A = \{r \in \mathbb{Q} : r^2 < 2\}$; this is a non-empty and bounded subset of $\mathbb{Q}$. It is clear that $\sqrt{2}$ is an upper bound of set A (If $b_1, b_2 \in \mathbb{R}$ with $0 \leq b_1 < b_2$, then $\sqrt{b_1} < \sqrt{b_2}$). We claim that it the supremum. Let us assume contrary, that is there exist $\alpha < \sqrt{2}$ such that α is an upper bound for A. As α is an upper bound and $1 \in A$ so we can say that $1 \leq \alpha$. Now $1 \leq \alpha < \sqrt{2} \implies \exists\, r \in \mathbb{Q}$ such that $\alpha < r < \sqrt{2}$. This implies $r^2 < 2 \implies r \in A$. This contradicts that α is an upper bound.