Lecture 19: Global Extrema, Local Extrema and Point of Inflection

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Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

19.1 Critical Points & Absolute/Global extrema

Recall if D is a closed and bounded subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ continuous is continuous, then f attains its absolute maximum and absolute minimum on D.

Question: How does one find the absolute extreme values, and the points where they are attained?

It turns out that we can considerably narrow down the search for the points where the absolute extrema are attained if we look at the derivative of f . To make this precise, let us first formulate a couple of definitions.

Definition 19.1 *Given $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, a point $c \in D$ is called a critical point of f if c is an interior point of D (i.e., there exist $r > 0$ such that $(c - r, c + r) \subseteq D$) such that either f is not differentiable at c, or f is differentiable at c and $f'(c) = 0$.*

Proposition 19.2 *Let D be a closed and bounded subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ be a continuous function. Then the absolute minimum as well as the absolute maximum of f is attained either at a critical point of f or at boundary points of D.*

19.2 Finding Global Extrema

In practice, the critical points of a function of its domain are few in number. Thus, in view of the above theorem, we have a simple recipe to determine the absolute extrema and the points where they are attained. Namely, determine the critical points of a function and the boundary points of its domain; then calculate the values at these points, and compare these values. The greatest value among them is the absolute maximum, while the least value is the absolute minimum. This recipe is illustrated by the following example.

Example 19.3 *Let $f : [-1, 2] \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} -x & \text{if} \quad -1 \leq x \leq 0 \\ 2x^3 - 4x^2 + 2x & \text{if} \quad 0 < x \leq 2 \end{cases}$$

Find the absolute extrema of f.

solution: First, note that f is continuous on $[0, 2]$. On the other hand,

$$f'(x) = \begin{cases} -1 & \text{if} \quad -1 < x < 0 \\ 6x^2 - 8x + 2 & \text{if} \quad 0 < x < 2 \end{cases}$$

Next, f is not differentiable at 0 since $f'_-(0) = -1$ and $f'_+(0) = 2$. So, $f'(x) = 0$ only at $x = \frac{1}{3}$ and $x = 1$. It follows that $x = 0, x = \frac{1}{3}$ and $x = 1$ are the only critical points of f. The end points of our domain $[-1, 2]$ are -1 and 2. Thus we make the following table.

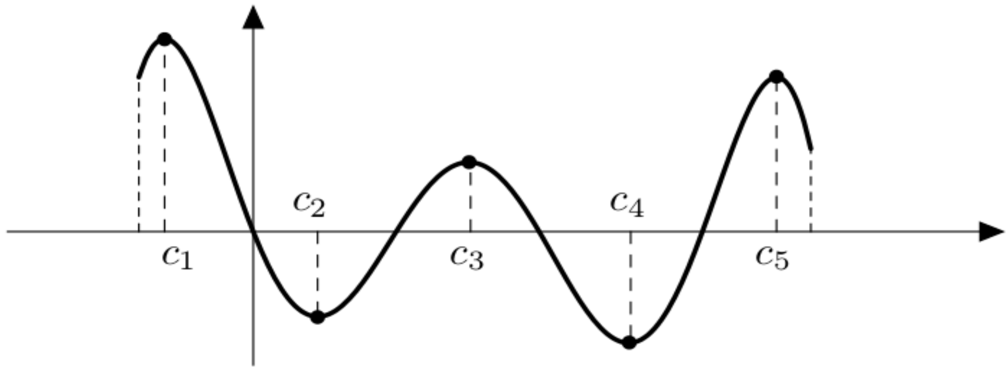
x	$f(x)$
-1	1
0	0
$\frac{1}{3}$	$\frac{8}{27}$
1	0
2	4

From this we conclude that the absolute minimum of f is 0, which is attained at $x = 0$ as well as at $x = 1$, whereas the absolute maximum of f is 4, which is attained at $x = 2$.

Remark 19.4 *While finding global extrema of a continuous function f defined on a closed and bounded subset of $\mathbb{R}$, it is of no use to know whether a local extremum of f is in fact a local maximum or a local minimum. Hence do not find the second derivative of f.*

19.3 Sufficient conditions for a local extremum

To get an idea of the relation between derivatives and the notions of local minimum/maximum, we may look at the behavior of the graph of the function around its peaks or dips (or even a plateau). We see that as we approach a dip (local minimum) from the left, the graph is decreasing and the tangents have negative slopes, whereas as we approach it from the right, the graph is increasing and the tangents have positive slopes. Similarly, as we approach a peak (local maximum) from the left, the graph is increasing and the tangents have positive slopes, whereas as we approach it from the right, the graph is decreasing and the tangents have negative slopes. In case the tangent is defined at a peak or a dip, then it is necessarily horizontal, that is, it has slope zero.



The above observations about the behavior of the graph lead to the following sufficient condition for a local extremum.

Theorem 19.5 (First Derivative Test) *Let $D \subseteq \mathbb{R}$, c be an interior point of D, and $f : D \to \mathbb{R}$ be any function. Suppose f is continuous at c.*

1. *If there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \leq 0$ for all $x \in (c-\delta, c)$, and $f'(x) \geq 0$ for all $x \in (c, c+\delta)$, then f has a local minimum at c.*
2. *If there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \geq 0$ for all $x \in (c-\delta, c)$, and $f'(x) \leq 0$ for all $x \in (c, c+\delta)$, then f has a local maximum at c.*

Remark 19.6 *An informal, but easy, way to remember the First Derivative Test is as follows: Suppose f is continuous at c.*
f' changes from $-$ to $+$ at $c \implies f$ has a local minimum at c;
f' changes from $+$ to $-$ at $c \implies f$ has a local maximum at c;

Note, however, that apart from differentiability in a neighborhood about c, except possibly at c, the continuity at the point c is essential in order to apply first derivative test.

Example 19.7 *Let $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} -x-1 & \text{if } \; x < 0 \\ 0 & \text{if } \; x = 0 \\ x+1 & \text{if } \; x > 0 \end{cases}$$

Note that for all $x \in (-\infty, 0)$, $f'(x) = -1$ and for all $x \in (0, \infty)$, $f'(x) = 1$. But $f(0) > f(x)$ for all $x \in (-\infty, 0)$ and $f(0) < f(x)$ for all $x \in (0, \infty)$. Hence there does not exist $\delta > 0$ such that either $f(x) \geq f(0)$ for all $x \in (-\delta, \delta)$ or $f(x) \leq f(0)$ for all $x \in (-\delta, \delta)$. The function $f(x)$ does not have local extremum at $x = 0$.

f is not continuous at $x = 0$.

First Derivative Test provides sufficient conditions for a local extremum. The following examples shows that conditions are not necessary.

Example 19.8 *Consider $f : (-1, 1) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x^2 & \text{if } 0 < |x| < 1 \\ -1 & \text{if } x = 0 \end{cases}$$

Then it is clear that $f(0) < f(x)$ for all nonzero $x \in (-1, 1)$, and thus f has a local minimum at $x = 0$. Indeed, f is differentiable on $(-1, 0)$ as well as on $(0, 1)$ and $f'(x) = 2x$ hence it changes sign from $-$ to $+$ at $x = 0$ but f is not continuous at 0 $(\lim_{x\to 0} f(x) = 0 \neq f(0))$.

Example 19.9 *Let $f(x) := |x|$ for $x \in \mathbb{R}$. Then f is continuous at 0, $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. Thus by first derivative test, f has a local minimum at 0.*

Example 19.10 *Let $f(x) := x + 2\sin x$ for $x \in (0, 2\pi)$. Then $f'(x) = 1 + 2\cos x = 0 \iff x \in \{2\pi/3, 4\pi/3\}$. f is continuous at $2\pi/3$ and at $4\pi/3$. We note the following:*

$$f' > 0 \text{ on } (0, 2\pi/3), f' < 0 \text{ on } (2\pi/3, 4\pi/3), f' > 0 \text{ on } (4\pi/3, 2\pi/3)$$

Hence by first derivative test, f has a local maximum at $2\pi/3$ and a local minimum at $4\pi/3$.

Theorem 19.11 (Second Derivative Test) *Let $I \subseteq \mathbb{R}$ be an interval, c be an interior point of I, and $f : I \to \mathbb{R}$ be any function. If f is is twice differentiable at c and satisfies $f'(c) = 0$ at c. Then we have the following:*

1. *If $f''(c) > 0$ then f has a local minimum at c.*
2. *If $f''(c) < 0$ then f has a local maximum at c.*

Remark 19.12 *The Second Derivative Test is valid under a restrictive hypothesis, namely, twice differentiability, and usually needs more checking (values of both the derivatives). But it has the advantage of being short and easy to remember.*

Example 19.13 *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^4$. Then $f(0) = 0 < f(x)$ for all nonzero $x \in \mathbb{R}$, and thus f has a local minimum at $x = 0$. However, the conditions of the Second Derivative Test are not satisfied. Indeed, f is twice differentiable and $f'(0) = 0$, but $f''(0)$ is not positive. So conditions in second derivative test are sufficient but not necessary.*

This example shows that the first derivative test is more general than the second. Since $f' < 0$ on $(-1, 0)$ and $f' > 0$ on $(0, 1)$, the first derivative test shows that f has a local minimum at 0.

Example 19.14 *Let $f(x) := x^4 - 2x^2$ for $x \in \mathbb{R}$. Then $f'(x) = 4x^3 - 4x = 4(x+1)x(x-1) = 0 \iff x \in \{-1, 0, 1\}$. f is twice differentiable, and $f''(x) = 4(3x^2 - 1)$ for $x \in \mathbb{R}$. Since $f''(-1) = 8 > 0, f''(0) = -4 < 0$ and $f''(1) = 8 > 0$, f has a local maximum at 0, and has a local minimum at ± 1.*

19.4 Point of Inflection

Points where the graph of a function changes from the convexity to concavity (or vice versa), are of great interest in calculus and its applications.

Definition 19.15 *Given $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, a point $c \in D$ is called a point of inflection for f, if c is an interior point and if there is $\delta > 0$ with $\delta \leq r$ such that f is convex in $(c-\delta, c)$, while f is concave in $(c, c+\delta)$, or vice versa, that is, f is concave in $(c-\delta, c)$, while f is convex in $(c, c+\delta)$.*

Let $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3, g(x) = x^{1/3}$, for both functions 0 is a point of inflection.

Theorem 19.16 (Derivative tests for a point of inflection) *Given $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, a point $c \in D$ be an interior point.*

1. **(First derivative test)** *Suppose there is $\delta > 0$ such that f is differentiable on $(c-\delta, c) \cup (c, c+\delta)$. Then c is point of inflection for $f \iff f'$ is increasing on $(c-\delta, c)$ and f' is decreasing on $(c, c+\delta)$, or the other way round.*

2. **(Second derivative test)** *Suppose there is $\delta > 0$ such that f is twice differentiable on $(c-\delta, c) \cup (c, c+\delta)$. Then c is point of inflection for $f \iff f'' \geq 0$ on $(c-\delta, c)$ and $f'' \leq 0$ is on $(c, c+\delta)$, or the other way round.*

Thumb Rule: f'' changes sign at $c \iff c$ is a point of inflection for f.