Lecture 13: Continuity

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13.1 $\epsilon - \delta$ Definition of Continuity

Definition 13.1 *Let $D \subseteq \mathbb{R}$. Consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that f is continuous at c if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Theorem 13.2 *Sequential definition of continuity, i.e., Definition 12.1 is equivalent to $\epsilon-\delta$ definition of continuity, i.e., Definition 13.1.*

Example 13.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$$

Show that f is continuous at $x = 0$ using $\epsilon - \delta$ definition.

Solution: Let $\epsilon > 0$ be given. Take $\delta = \epsilon$. If $|x - 0| = |x| < \delta$ then

$$|f(x) - f(0)| = |f(x)| \leq |x| < \epsilon.$$

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Example 13.4 *Let $f(x) = x^2$ for all $x \in [a, b]$, where $a, b, \in \mathbb{R}, a < b$. Show that f is continuous on $[a, b]$ using $\epsilon - \delta$ definition.*

Solution: Let $c \in [a, b]$ and $\epsilon > 0$ be given. Define $R = \max\{|a|, |b|\}$. Choose $\delta < \frac{\epsilon}{2R}$. Let $x \in [a, b]$ be such that $|x - c| < \delta$. Then $|x + c| \leq |x| + |c| < R + R$. Consider

$$\begin{aligned} |f(x) - f(c)| &= |x^2 - c^2| \\ &= |x + c||x - c| \\ &\leq (|x| + |c|)|x - c| \\ &< 2R|x - c| \\ &< 2R\delta \\ &< \epsilon. \end{aligned}$$

Thus, f is continuous at $c \in [a, b]$ and hence on $\mathbb{R}$, since c was an arbitrary element of $\mathbb{R}$. ■

Example 13.5 *Let $f(x) = x^2$ for all $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$ using $\epsilon - \delta$ definition.*

Solution: Let $c \in \mathbb{R}$ and $\epsilon > 0$ be given. Choose $\delta < \min\{1, \frac{\epsilon}{2+|c|}\}$. Let $x \in \mathbb{R}$ be such that $|x - c| < \delta$. Then $|x - c| < 1$ so that $|x| = |x - c + c| \leq |x - c| + |c| < 1 + |c|$. Consider

$$\begin{aligned} |f(x) - f(c)| &= |x^2 - c^2| \\ &= |x + c||x - c| \\ &\leq (|x| + |c|)|x - c| \\ &< (2 + |c|)|x - c| \\ &< (2 + |c|)\delta \\ &< \epsilon. \end{aligned}$$

Thus, f is continuous at $c \in \mathbb{R}$ and hence on $\mathbb{R}$, since c was an arbitrary element of $\mathbb{R}$.

Remark 13.6 *Let us see that how easy is working with sequences. We prove the continuity of the function $f(x) = x^2$ using Definition 12.1. Let $c \in \mathbb{R}$ and (x_n) be a sequence in $\mathbb{R}$ such that $x_n \to c$. Then by part 3 of Limit Theorem for sequences we have $x_n^2 \to c^2$ which is same as saying $f(x_n) \to f(c)$.*