Lecture 09: Infinite Series

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9.1 Motivation

If $a_1, \cdots, a_n$ are any real numbers, then we can add them together and form their sum $a_1 + \cdots + a_n$. Also, we can add them in any order. For example if we want to numbers $1, 2, 3, 4, 46, 47, 48, 49$, we would add the pairs $(1, 49), (2, 48), (3, 47)$ and $(4, 46)$ since each pair add up to 50 and hence the total sum of these pairs is 200.

Now we shall investigate whether we can 'add' infinitely many real numbers. In other words, if (a_k) is a sequence of real numbers, then we ask whether we can give a meaning to a symbol such as '$a_1 + a_2 + \cdots$' or '$\sum_{k=1}^{\infty} a_k$'. Before we go into formal aspect of it, let us recall something you might have seen in earlier classes.

You have probably seen the following geometric series

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \tag{9.1}$$

You have probably seen the following trick to sum this series: if we call the above sum S, then if we multiply both sides by 2, we obtain

$$2S = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + S \implies S = 1. \tag{9.2}$$

However, if we apply the same trick to the infinite series $\sum_{n=1}^{\infty} 2^n$ Then we get

$$2S = \sum_{n=1}^{\infty} 2^{n+1} = S - 2 \implies S = -2. \tag{9.3}$$

Why we don't trust (9.3) but (9.2), though we have applied the same reasoning?

Let us consider the following example dealing with infinite sums.

$$\begin{aligned} 1-1+1-1+\cdots &= (1-1)+(1-1)+\cdots \\ &= 0+0+\cdots+0+\cdots \\ &= 0, \end{aligned}$$

$$\begin{aligned} 1-1+1-1+\cdots &= 1+(-1+1)+(-1+1)+\cdots \\ &= 1+0+0+\cdots \\ &= 1. \end{aligned}$$

This absurdity shows that we should define the "the sum" of infinitely many real numbers in a rigorous way so as to avoid this.

Recall that we have been writing $1/3 = 0.3333\cdots$. It is interesting to note that the right side is the infinite sum $3/10 + 3/10^2 + 3/10^3 + \cdots$. We would like to arrive at a definition of the sum of infinite real numbers, using which we should be able to prove rigorously

$$\frac{1}{3} = 3/10 + 3/10^2 + 3/10^3 + \cdots \tag{9.4}$$

9.2 Infinite Series

Definition 9.1 *Give a sequence $(a_k)_{k\geq 1}$ of real numbers, an expression of the form $\sum\limits_{k=1}^{\infty} a_k$ is called an infinite series.*

The notion of convergence of a series gives a precise meaning to the idea of forming the sum of infinitely many real numbers.

Definition 9.2 *For a series of real numbers $\sum\limits_{k=1}^{\infty} a_k$, $S_n := \sum\limits_{k=1}^{n} a_k$ is called the n^{th} partial sum of the series.*

It is clear that for each n, S_n is some real number, hence $(S_n)_{n\geq 1}$ is a real sequence.

Definition 9.3 *We say that a series $\sum\limits_{k=1}^{\infty} a_k$ is convergent, if the sequence of partial sums $(S_n)_{n\geq 1}$ is convergent. If $(S_n)_{n\geq 1}$ diverges we say series $\sum\limits_{k=1}^{\infty} a_k$ diverges.*

If $(S_n)_{n\geq 1}$ converges to S, then by uniqueness of limit of sequence, real number S is unique, and it is called the sum of the series $\sum_{k=1}^{\infty} a_k$. Thus, when we write

$$S = \sum_{k=1}^{\infty} a_k.$$

we mean that S is a real number, the series $\sum_{k=1}^{\infty} a_k$ is convergent, and its sum is equal to S.

In this case we also say that $\sum_{k=1}^{\infty} a_k$ converges to S.

Example 9.4 *Let $a \in \mathbb{R}$. Define $a_0 = 1$ and $a_k := a^k$ for $k \in \mathbb{N}$. If $a \neq 1$, then for $n = 0, 1, 2, \cdots$, we have sum of n terms of a geometric progression*

$$S_n = a_0 + a_1 + a_2 + \cdots + a_n = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

We have proved that if $|a| < 1$, then $a^n \to 0$. Hence using limit theorem of sequences we conclude

$$S_n \to \frac{1}{1-a}, \quad \text{if } |a| < 1$$

Thus

$$1 + \sum_{k=1}^{\infty} a^k = \frac{1}{1-a}, \quad \text{if } |a| < 1.$$

This is perhaps the most important example of a convergent series. Its special feature is that we are able to give a simple closed-form formula for each of its partial sums as well as its sum.

As promised, we must now verify (9.4) as per the Definition 9.3. In fact from Example 9.4,

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = 3\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k = 3\left[\frac{1}{1-\frac{1}{10}} - 1\right] = 3\left(\frac{10}{9} - 1\right) = 1/3.$$

Proposition 9.5 (Necessary condition for convergence) *If $\sum_k a_k$ is convergent, then $a_k \to 0$ as $k \to \infty$.*

Proof: Let $\sum_k a_k$ be a convergent series. If (S_n) is its sequence of partial sums and S is its sum, then we have $a_k = S_k - S_{k-1}$. Now $S_k \to S \implies S_{k-1} \to S$, hence by limit theorems for sequences implies $a_k \to 0$. ■

Proposition 9.5 can be used to conclude divergence of a series, that is, if $a_k \nrightarrow 0$ then series $\sum_k a_k$ is divergent.

Example 9.6 *Show that if* $|a| \geq 1$ *then the series* $\sum_{k=0}^{\infty} a^k$ *is divergent.*

Solution: By Example 5.7, If $|a| > 1$ then (a^k) is divergent. Therefore by Proposition 9.5, series $\sum_{k=0}^{\infty} a^k$ is divergent.

If $a = 1$ then $(a_k) = (1)$ is a constant sequence which does not converge to zero. Therefore by Proposition 9.5, series $\sum_{k=0}^{\infty} a^k$ is divergent.

If $a = -1$ then $a_k = (-1)^k$ which diverges. Therefore by Proposition 9.5, series $\sum_{k=0}^{\infty} a^k$ is divergent. ■

9.3 Sequences Diverging to $\pm\infty$

Recall that both the sequences $((-1)^n), (n)$ are divergent. But divergence of sequence (n) has a different reason. As n increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$\lim_{n\to\infty} n = \infty.$$

In writing infinity as the limit of a sequence, we are not saying that the differences between the terms a_n and ∞ become small as n increases. Nor are we asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that a_n eventually gets and stays larger than any fixed number as n gets large.

Definition 9.7 *The sequence* (a_n) *diverges to* $+\infty$ *if for every number* $M \in \mathbb{R}$ *there is a* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $a_n \geq M$. *If this condition holds we write*

$$\lim_{n\to\infty} a_n = \infty \quad \textit{or} \quad a_n \to \infty.$$

Similarly one can formulate an analogous notion of a sequence diverging to $-\infty$.

Definition 9.8 *We say sequence (a_n) diverges to $-\infty$, if for every number $m \in \mathbb{R}$ there is a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $a_n \leq m$. If this condition holds we write*

$$\lim_{n\to\infty} a_n = -\infty \quad \textit{or} \quad a_n \to -\infty.$$

Remark 9.9 *Note that any sequence diverging to $+\infty$ or $-\infty$ is unbounded. But converse is not true, for example $(1, 2, 1, 4, 1, 6, \cdots)$.*

Example 9.10 *Let $(a_n)_{n\geq 1}$ be an increasing sequence which is not bounded above. Then show that $a_n \to \infty$.*

Solution: Let $M \in \mathbb{R}$ be given. Since (a_n) is not bounded above, so there exists $n_0 \in \mathbb{N}$ such that

$$a_{n_0} \geq M.$$

Since (a_n) is increasing, $n \geq n_0 \implies a_n \geq a_{n_0} \geq M$. Which shows that $a_n \to \infty$.