

The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #8

(Real-valued Functions of Several Variables: Continuity, limits and Iterated Limits)

1. Examine the following functions for continuity at the point $(0, 0)$ where $f(0, 0) = 0$ and $f(x, y)$ for $(x, y) \neq (0, 0)$ is given by

(a) $|x| + |y|$, (b) $\frac{-x}{\sqrt{x^2 + y^2}}$, (c) $\frac{2x}{x^2 + x + y^2}$, (d) $\frac{x^4 - y^2}{x^4 + y^2}$, (e) $\frac{x^4}{x^4 + y^2}$.

2. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or if } y = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that the function satisfy the following:

- (a) The iterated limits $\lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} f(x, y) \right]$ and $\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow 0} f(x, y) \right]$ exist and equals 0,
(b) $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist,
(c) $f(x, y)$ is not continuous at $(0, 0)$.

3. Consider the function

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Find iterated limits $\lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} f(x, y) \right]$ and $\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow 0} f(x, y) \right]$ (if exists). Also find $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ (if exists).

4. Consider the function

$$f(x, y) = \begin{cases} \sin(xy) & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

Find $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ if it exists.

5. Using $\epsilon - \delta$ definition show that

$$\lim_{(x, y) \rightarrow (0, 0)} y \sin \frac{1}{x^2 + y^2} = 0$$

6. Suppose $f(x, y) := \frac{x^3 y}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that f is continuous using $\epsilon - \delta$ definition.