7 Oct '19

LECTURE 23

IMPROPER INTEGRAL

In Riemann $f \rightarrow$ bounded on $[a,b]$ closed and bounded

In integrand $(f(x))$ become infinite at some point in $[a,b]$ or limit either 'a' or 'b' or both infinite.

$\int_a^b f(t)dt$ is called Improper Integral

eg: $\int_1^\infty \frac{dt}{t}$, $\int_{-\infty}^{\infty} \frac{dt}{1+t^2}$, $\int_0^1 \frac{dt}{t(1-t)}$

Improper Integral Of First Kind

Let $a \in R$ and $f: [a,\infty) \rightarrow R$
take an interval $[a,x]$

Suppose $\int_a^x f(t)\,dt$ is Riemann integral for $x > a$.

or $\int_a^x f(t)dt$ exist for $x > a$ then $\int_a^\infty f(t)dt$ is called improper integral of first kind.

eg: $\int_0^\infty \sin(t)dt = \int_0^x \sin t\,dt = -\cos t\Big]_0^x = 1-\cos x$.

$\lim_{x\to\infty} \int_0^x \sin(t)dt = \lim_{x\to\infty}[1-\cos x] = 1-\cos\infty$
Not defined

$\therefore \int_0^\infty \sin(t)\,dt$ diverges.

COMPARISON TEST

Suppose $0 \le f(t) \le g(t)$ for $t \ge a$ where $f, g: [a,\infty) \rightarrow R$

(i) If $\int_a^\infty g(t)dt$ converges then $\int_a^\infty f(t)\,dt$ also converges.

(ii) If $\int_a^\infty f(t)\,dt$ diverges then $\int_a^\infty g(t)\,dt$ also diverges.

Note: The convergence of $\int_a^\infty f(t)\,dt$ is not affected by changing initial point [i.e. a] although value of Integral may change.

eg: Test the convergence of $\int_1^\infty \sin(1/t)\,dt$.

suppose $f(t) = \sin 1/t$ then take $g(t) = 1/t$.

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{\sin(1/t)}{1/t} = 1 \text{ (non-zero finite)}.$$

By LCT, $\int_1^\infty \sin(1/t)$ and $\int_1^\infty 1/t\,dt$ have same nature

$\because \int_1^\infty 1/t\,dt$ diverges, so $\int_1^\infty \sin(1/t)\,dt$ diverges.

eg: Test the convergence of $\int_1^\infty \frac{\cos^2 t}{t^2}\,dt$.

$$\frac{\cos^2 t}{t^2} \le \frac{1}{t^2} \quad \forall\ t > 1 \quad [\cos t \le 1]$$

By comparison test,
$\int_1^\infty 1/t^2\,dt$ converges

So, $\int_1^\infty \frac{\cos^2 t}{t^2}\,dt$ also converges.

eg: Test the convergence of $\int_1^\infty \frac{2 + \sin t}{t}\,dt$.

$$1/t \le \frac{2 + \sin t}{t} \le 3/t$$

By Comparison test :- $\int_1^\infty 1/t \, dt$ diverges.

So, $\int_1^\infty \frac{2+\sin t}{t}$ diverges.

Remark:-

The convergence of an improper integral $\int_a^\infty f(t)\,dt$ is not affected by changing the initial point 'a' of interval $[a,\infty)$. Although value of integral may be changed.

If $a' > a$, $\int_a^\infty f(t)\,dt$ converges $\Leftrightarrow$ $\int_{a'}^\infty f(t)\,dt$ converges.

$$\int_a^\infty f(t)\,dt = \int_a^{a'} f(t)\,dt + \int_{a'}^\infty f(t)\,dt.$$

~~23 Sept '19~~
9 Oct '19

LECTURE 23

PROPER INTEGRALS

$$\int_a^{\infty} f(t)\,dt \quad \text{where } a \in R$$

$$\lim_{x\to\infty} \int_0^x f(t)\,dt = \text{finite and unique} = M$$

$f(t), g(t) : [a,\infty) \to R$

$f(t) \geq 0,\ g(t) > 0 \quad , t > a.$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L = \text{non-zero finite}$$

$\int_0^{\infty} g(t)$; $\int_0^{\infty} f(t)$; have same nature.

eg: Test the convergence of $\int_0^{\infty} \sin(t)\,dt$

Suppose $f(t) = \sin(1/t)$

then $g(t) = 1/t$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{\sin(1/t)}{1/t} = 1, \text{ non zero finite.}$$

by LCT,

$\int_1^{\infty} \sin(1/t)$ and $\int_1^{\infty} \frac{1}{t}\,dt$ have same nature.

Result: $\int_1^{\infty} \frac{1}{t^p}\,dt$ converges ~~to~~ if $p > 1$, diverges $p \leq 1$.

$\because \int_1^{\infty} \frac{1}{t}\,dt$ diverges. So $\int_1^{\infty} \sin(1/t)\,dt$ diverges.

eg: Test the convergence of $\int_0^\infty e^{-t} \cdot t^p dt$, $p \in R$.

Suppose $f(t) = e^{-t} t^p$
$g(t) = 1/t^2$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{e^{-t} t^p}{1/t^2} = \lim_{t\to\infty} \frac{t^{p+2}}{e^t} = 0$$

$\because \int_1^\infty 1/t^2 \, dt$ converges, so, $\int_1^\infty e^{-t} t^p dt$ also converges.

DINCHLET'S TEST

Suppose $f, g : [a, \infty) \to R$ such that

1. f is monotonically decreasing on $[a, \infty)$ and $f(t) \to 0$ as $t \to \infty$

2. g is continuous and $\exists\, M \in R$ such that $\left|\int_a^x g(t)dt\right| \le M \;\forall$ $x \in R$ and $x > a$ then

$$\int_a^\infty f(t) g(t) \, dt \text{ converges.}$$

eg: $\int_\pi^\infty \frac{\sin t}{t} dt$

Let $f(t) = 1/t$ and $g(t) = \sin(t)$

(1) Clearly, 'f' is decreasing on $[0, \infty)$ and $\lim_{t\to\infty} 1/t = 0$,

(2) $g(t) = \sin t$ is continuous on $[\pi, \infty)$

$$\int_a^x \sin(t) dt = \left[-\cos x\right]_a^x = [\cos a - \cos x]$$

25
9

$$\int_a^x g(t)\,dt = \cancel{-2\sin\frac{a+x}{2}\sin\frac{a-x}{2}} = -2\sin\frac{a+x}{2}\sin\frac{a-x}{2}$$

$$\left|\int_a^x g(t)\,dt\right| \le 2$$

by Dirichlet's test,

$$\int_\pi^\infty \frac{\sin t}{t}\,dt \text{ converges.}$$

Improper Integral of Second Kind

Suppose $\int_x^b f(t)\,dt$ exist for all $a \le x \le b$ [Function may be unbounded on $[a,b]$]

then $\int_a^b f(t)\,dt$ is called improper integral, of second kind.

if $\lim_{x\to a^+} \int_x^b f(t) = M$ finite unique Real.

$\int_a^b f(t)\,dt$ converges and converges to M.

if $\lim_{x\to a^+} \int_x^b f(t)\,dt = \pm\infty$ then,

$\int_a^b f(t)\,dt$ converges.

eg: $\int_0^1 \frac{1}{t^p}\,dt$. Test its convergence.

if $p = 1$. $\int_x^1 \frac{1}{t}\,dt = \log 1 - \log x = -\log x = \log\left(\frac{1}{x}\right)$

$$\lim_{x\to 0^+}\int_x^1 \frac{1}{t}\,dt = \lim_{x\to 0^+}\log\left(\frac{1}{x}\right) = \log(\infty) = \infty$$

$\int_0^1 1/t\,dt$ diverges.

if $p>1$.

$$\int_x^1 1/t^p\,dt = \left[-\frac{1}{(p-1)t^{p-1}}\right]_x^1 = \frac{1}{(p-1)}\left[\frac{1}{x^{p-1}} - 1\right] \quad -(1)$$

if $p>1 \Rightarrow p-1>0$

$$\lim_{x\to 0^+}\int_x^1 f(t)\,dt = \lim_{x\to 0^+}\left[\frac{1}{(p-1)}\left(\frac{1}{x^{p-1}} - 1\right)\right]$$

$$= \lim_{x\to 0^+}\left[\frac{1}{p-1}\left[\frac{1}{(0+x)^{p-1}} - 1\right]\right] = \infty.$$

$\int_0^1 f(t)\,dt$ diverges.

if $p<1 \Rightarrow p-1<0$

$$\lim_{x\to 0^+}\int_0^1 f(t)\,dt = \lim_{x\to 0^+}\left[\frac{1}{(p-1)}\left[\frac{1}{x^{p-1}} - 1\right]\right] = \frac{-1}{p-1}$$

$\therefore \int_0^1 f(t)\,dt$ converges.

$$\int_a^{\infty} f(t)\,dt = \int_a^{a'} f(t)\,dt + \int_{a'} f(t)\,dt.$$

11th Oct'19

LECTURE 25

Comparison Test

Suppose $0 \le f(x) \le g(x) \quad \forall\, t > a.$

If $\int_a^b f(t)\,dt$ diverges then $\int_a^b g(t)\,dt$ also diverges.

Second Kind

if $\int_a^b g(t)\,dt$ converges then $\int_a^b f(t)\,dt$ also converges.

Limit Comparison Test

Suppose $f(t) \ge 0$, $g(t) > 0$ for $t > a$

If $\lim_{t \to a^+} \frac{f(t)}{g(t)} = c$ (non-zero finite number) then $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ both either converge or diverge.

If $c=0$ and $\int_a^b g(t)\,dt$ converges then $\int_a^b f(t)dt$ also converges.

If $c=\pm\infty$ and $\int_a^b g(t)dt$ diverges then $\int_a^b f(t)\,dt$ also diverges.

eg: $\int_0^\infty \frac{1-e^{-t}}{t^p}\,dt = \underbrace{\int_0^1 \frac{1-e^{-t}}{t^p}\,dt}_{2^{nd}\text{ Type}} + \underbrace{\int_1^\infty \frac{1-e^{-t}}{t^p}\,dt}_{1^{st}\text{ Type}}$

$\underline{I_1}$ $f(t) = \frac{1-e^{-t}}{t^p}$, $g(t) = \frac{1}{t^{p-1}}$

$$\lim_{t\to 0^+} \frac{f(t)}{g(t)} = \lim_{t\to 0^+} \frac{1-e^{-t}}{t} = \lim_{t\to 0^+} \frac{e^{-t}}{1} = 1$$

Since, $\int_0^1 \frac{1}{t^{p-1}}\,dt$ converges for $p-1<1 \Rightarrow p<2$

By LCT, $\int_0^1 \frac{1-e^{-t}}{t^p}\,dt$ converges for $p<2$ —— (1)

* If we choose $g(t) = \frac{1}{t^{p-2}}$ then answer will be wrong.!!

$\underline{\underline{I_2}}$ $f(t) = \frac{1-e^{-t}}{t^p}$, $g(t) = 1/t^p$

$\lim_{t\to\infty} 1-e^{-t} = 1$ (non-zero finite).

$\int_1^\infty \frac{1}{t^p}\,dt$ converges for $p>1$.

So, $\int_1^\infty \frac{1-e^{-t}}{t^p}\,dt$ converges for $p>1$ — (2)

$\int_0^\infty \frac{1-e^{-t}}{t^p}\,dt$ converges for $1<p<2$.

Absolute Convergence

Improper integral $\int_a^b f(t)\,dt$ is absolutely convergent if $\int_a^b f(t)\,dt$ and $\int_a^b |f(t)|\,dt$ both converges.

Conditionally Convergent

Improper integral $\int_a^b f(t)\,dt$ is called conditionally convergent if $\int_a^b f(t)\,dt$ converges but $\int_a^b |f(t)|\,dt$ diverges.

Theorem:
Every absolute convergent improper integral is also convergent.

eg 1: $\int_1^\infty \frac{\sin t}{t^p}\,dt$. $0<p<1$. Test the absolute convergence.

Solution:

1. Let $f(t) = \sin t$, $g(t) = 1/t^p$

$\because$ $t > 1$ and $0<p<1$. So, t^p is increasing.

$g(t) = \frac{1}{t^p}$ is decreasing on $[1,\infty)$ for $0<p<1$.

$\lim_{t\to\infty} \frac{1}{t^p} = 0$

2. $f(t)$ is continuous on $[1,\infty)$. since $\sin t$ is continuous.

$$\int_1^x \sin t = -\cos t\Big|_1^x = \cos 1 - \cos x.$$

$$\left|\int_1^x \sin t\,dt\right| \le |\cos 1| + |\cos x| \le 2.$$

By Dirchlet's Test,
$$\int_1^\infty \frac{\sin t}{t}\, dt \text{ converges.}$$

Absolute Convergence

$$\int_1^\infty \left|\frac{\sin t}{t^p}\right| dt$$

If $-1 \le a \le 1 \Rightarrow a^2 \le |a|$

$\therefore \sin^2 t \le |\sin t| \quad \{\because -1 \le \sin t \le 1\}$

$$\Rightarrow \frac{\sin^2 t}{t^p} \le \frac{|\sin t|}{t^p}$$

$$\Rightarrow \frac{1-\cos 2t}{2t^p} = \frac{\sin^2 t}{t^p} \Rightarrow 0 \le \frac{1-\cos 2t}{2t^p} \le \frac{|\sin t|}{t^p} \quad t>1 \quad \text{— (1)}$$

$$\int_1^\infty \frac{1-\cos 2t}{t^p}\, dt = \int_1^\infty \frac{1}{t^p}\, dt - \underbrace{\int \frac{\cos(2t)}{t^p}\, dt}_{\text{converges}} \quad \text{— (2)}$$

$$\Rightarrow \int_1^\infty \frac{1}{t^p}\, dt \text{ diverges}, \ 0<p<1$$

So, $\int_1^\infty \frac{1-\cos 2t}{t^p}\, dt$ diverges for $0<p<1$.

By equation (1) and Comparison Test,
$$\int_1^\infty \left|\frac{\sin t}{t^p}\right| dt \text{ diverges.}$$

LECTURE 26

TAYLOR'S THEOREM

Let $n \in \mathbb{Z}$ and $n \geq 0$. Let $f : [a, b] \to \mathbb{R}$ such that $f', f'', \ldots f^{n}(x)$ is continuous on $[a, b]$ and exist on $[a, b]$ and differentiable on (a, b). Then $\exists\, a < c < b$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{n}(a)}{n!}(b-a)^n + \frac{f^{n+1}(c)(b-a)^{n+1}}{(n+1)!} \quad \text{--- (1)}$$

(i) if $n = 0$, by Taylor's Theorem,

$$f(b) = f(a) + f'(c)(b-a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{which is MVT.}$$

(ii) if $n = 1$,

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)(b-a)^2}{2!}$$

This is known as Extended Mean Value Theorem.

\# if $x \in [a, b]$. $\exists\, a < c < x$, then by Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{n}(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \text{ --}$$

Equation (2) is called Taylor's Formula of $f(x)$ around 'a'. The polynomial is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{n}(a)}{n!}(x-a)^n$$

is known as n^{th} Taylor's polynomial of f around 'a'.

$$R_n = f - P_n = \frac{f^{n+1}(a)(x-a)^{n+1}}{(n+1)!} = \text{Remainder of order } n.$$

R_n is called remainder of order n in Taylor's formula.

eg: Write the n^{th} Taylor's polynomial if $f(x) = e^x$ about 0.

$$P_n(x) = f(a) + f'(0)\cdot x + \frac{f''(0)x^2}{2!} + \dots + \frac{f^n(0)x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

eg: Approximate the value of $\sqrt{2}$ by Taylor's first and second approximation.

First Approximation

Let $f: [m, m+1] \longrightarrow R$ such that $f(x) = \sqrt{x}$.

By Taylor's Theorem about 'm', $x \in [m, m+1]$.

$$f(x) = f(m) + f'(c)(m+1-m) \quad \left[\text{First approximation } x = m+1\right]$$

$$f(m+1) = f(m) + f'(c) \Rightarrow \sqrt{m+1} = \sqrt{m} + \frac{1}{2\sqrt{c}}$$

$c \in [m, m+1]$

$$\sqrt{m+1} - \sqrt{m} = 1/2\sqrt{c}$$

$$\frac{1}{2\sqrt{m+1}} < \sqrt{m+1} - \sqrt{m} < \frac{1}{2\sqrt{m}} \quad [\because m < c < m+1]$$

$$\sqrt{m} + \frac{1}{2\sqrt{m+1}} < \sqrt{m+1} < \sqrt{m} + \frac{1}{2\sqrt{m}}$$

Take $m = 1$,

$$1 + \frac{1}{2\sqrt{2}} < \sqrt{2} < 1 + \frac{1}{2}$$

$$1 + \frac{1}{3} < \sqrt{2} < 1 + \frac{1}{2}$$

$$1.33 < \sqrt{2} < 1.5 \quad \text{—— (2)}$$

Second Approximation

$$f(m+1) = f(m) + f'(m)(m+1-m) + \frac{f''(c)(m+1-m)^2}{2!} \quad m < c < m+1$$

$$\sqrt{m+1} = \sqrt{m} + \frac{1}{2\sqrt{m}} - \frac{1}{8c^{3/2}}$$

$$\sqrt{m} + \frac{1}{2\sqrt{m}} - \frac{1}{8m^{3/2}} < \sqrt{m+1} < \frac{1}{\sqrt{m}} + \frac{1}{2\sqrt{m}} - \frac{1}{8(m+1)^{3/2}}$$

Let $m = 1$

$$1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{8.2^{3/2}}$$

$(2\sqrt{2} \approx 3/2)$

$(\sqrt{2} < 3/2)$

$$\Rightarrow \frac{11}{8} < \sqrt{2} < \frac{3}{2} - \frac{1}{18 \cdot \sqrt{2}}$$

$$\Rightarrow \frac{11}{8} < \sqrt{2} < \frac{3}{2} - \frac{1}{16 \times 3/2}$$

$$\Rightarrow 1.375 < \sqrt{2} < 1.50 - 0.04017$$

$$\Rightarrow 1.375 < \sqrt{2} < 1.4583 \quad \text{—— (2)}$$

LECTURE 27

16th oct'19

\# $x \in R^n$

$x = (x_1, x_2, \ldots, x_n)$

where $x_i \in R, 1 \le i \le n$

x_i = ith co-ordinate of n.

$V(r, h) = \pi r^2 h$

Sequence: A sequence in R^2 is a mapping from N to R^2 denoted by ~~x_n~~ X_n.

$X_n = (x_n, y_n)$ is a sequence in R^2.

x_n, y_n are sequence in R.

Convergent: A sequence $X_n = (x_n, y_n)$ in R^2 is called convergent or converges to $(x_0, y_0) \in R^2$ if $x_n \to x_0$ and $y_n \to y_0$.

eg: $X_n = (1/n, 1/n)$.

We know that,

$1/n \to 0$

$X_n = (1/n, 1/n)$ converges to $(0, 0)$.

eg: $X_n = (1/n, (-1)^n)$

$1/n \to 0$

$(-1)^n$ does not converge

$\therefore$ $X_n = (1/n, (-1)^n)$ does not converge.

Bounded Sequence

A sequence $X_n = (x_n, y_n) \in R^2$ is ~~a~~ bounded if $\exists$ a +ve real number M such that

$$\|X_n\| \leq M \ \forall\ n \in N.$$

Theorem Every convergent sequence is bounded.

- Let $X_n = (x_n, y_n)$ be any sequence in R^2 and converges to (x_0, y_0).

$$\Rightarrow x_n \to x_0 \text{ and } y_n \to y_0.$$

In one variable, every convergent sequence is bounded.

$\exists$ ~~two~~ +ve real numbers M_1 and M_2 such that,

$$|x_n| < M_1 \text{ and } |y_n| < M_2 \ \forall\ n$$

$$\|X_n\| = \sqrt{x_n^2 + y_n^2} \leq |x_0| + |y_0|$$

$$\|X_n\| \leq M_1 + M_2 \ \forall\ n.$$

Theorem Every ~~conv~~ bounded sequence is not convergent.

LIMIT suppose $D \subset R^2$ and $(x_0, y_0) \in D$.
A real number L is called limit of function $f(x,y)$ defined on D i.e. $f: D \subset R^2 \to R$ as $(x,y) \to (x_0, y_0)$ if for each $\epsilon > 0$, however small, $\exists$ a $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \text{ whenever } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

denoted by

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{(finite and unique)}$$
(not path dependent)

if limit exist and finite, 'L' is called simultaneous limit as $(x, y) \to (x_0, y_0)$ simultaneously.

CONTINUITY A function $f(x,y): D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(x_0, y_0) \in D$.

1. (a) $f(x,y)$ must be defined at (x_0, y_0).

(b) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist.

(c) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

2. ϵ-δ Definition. Function 'f' is called continuous at $(x_0, y_0) \in D$ if for each $\epsilon > 0$. $\exists\ \delta > 0$ such that.

$$|f(x,y) - f(x_0,y_0)| < \epsilon \text{ whenever } \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta.$$

3. Sequential Definition.

A function f is called continuous at $(x_0, y_0) \in D$ if whenever sequence $(x_n, y_n) \to (x_0, y_0)$; we must have

$$f(x_n, y_n) \to f(x_0, y_0).$$

NOTE: If f is continuous at each point of D then f is continuous on D.

eg: $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

Check the continuity at $(0,0)$.

Let $\epsilon > 0$ be given.

$$|f(x,y) - f(0,0)| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$$

$$\leq \frac{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$$

$$\leq \sqrt{x^2+y^2}$$

~~Choose~~ Let $\delta = \epsilon$

Now,

$$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)|$$

$$\leq \sqrt{x^2+y^2} < \epsilon.$$

Hence, proved.

eg: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Check the continuity at $(0,0)$.

Let $y = mx$.

$\Rightarrow$ if $y \to 0$, $x \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

take $y = mx$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{x\to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{m}{1+m^2}$$ (path depends)

limit does not exist ; so not continuous at $(0,0)$.

18th Oct 19

eg: $f(x,y) =$

Check t

Alt

eg

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LECTURE 28

eg: $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$

Check the limit and continuity at (0,0).

Let $y = mx$

$\Rightarrow$ if $y \to 0$, $x \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$$

Along any axis $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Along the curve, $y = mx^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4 m}{x^4 + m^2x^4}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ path dependent.

limit does not exist at (0,0),

so not continuous at (0,0).

The function has a limit 0, if we approach a function along any straight line

eg: $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ ~~st~~ Limit does not exist.

Along $x = my^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{2m y^4}{m^2y^4 + y^4} = \lim_{y\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

Limit is path dependent $\therefore$ does not exist

Imp.* If we say limit exists, we have to prove it only by $\varepsilon-\delta$ method.

eg: Show that : $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy(x^2-y^2)}{x^2+y^2} = 0.$

Let $\varepsilon > 0$ be given.

$$|f(x,y)-0| = \left| \frac{xy(x^2-y^2)}{x^2+y^2} \right| \leq |x||y|$$
$$\leq \sqrt{x^2+y^2}\,.\,\sqrt{x^2+y^2}$$
$$\leq x^2+y^2 < \varepsilon$$

Take $\delta > 0$ such that, $\delta = \sqrt{\varepsilon}$

$$\Rightarrow \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y)-0| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Imp.* For finding limit, we do not consider the existence of func. but for continuity, we need to know if the func. exists at that particular point.

eg: LIMIT PROPERTIES

Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L \in R.$

$\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M \in R.$

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L \pm M$ Sum/Difference Rule

(ii) $\lim\limits_{(x,y)\to(x_0,y)} (f.g) = L.M.$ Product Rule.

(iii) $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x)}{g(x)} = L/M$, But $M \neq 0$, Quotient Rule

(iv) $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{k/l} = L^{k/l} \in R$.

eg: $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \times \left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$

$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = \underline{0}$

eg: $f(x,y) = \begin{cases} \frac{\sin^2(x-y)}{|x|+|y|}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0). \end{cases}$

Check the continuity at (0,0).

Let $\varepsilon > 0$ be given

$|f(x,y) - f(0,0)| = \left|\frac{\sin^2(x-y)}{|x|+|y|}\right|$

$\leq \left|\frac{(x-y)^2}{|x|+|y|}\right| \leq \frac{(|x|+|y|)^2}{|x|+|y|}$

$x-y \leq |x|+|y|$

$\Rightarrow |f(x,y) - f(0,0)| \leq |x|+|y|$.

$\leq 2\sqrt{x^2+y^2}$ —(1)

Let $\delta = \varepsilon/2$ then,

$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \varepsilon$.

$\therefore$ function is continuous at (0,0).

eg: $f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0) \end{cases}$

Check the continuity at (0,0).

Let $y = mx^2$.
$\Rightarrow$ if $x \to 0$, $y \to 0$.

$$f(x,y) = \frac{x^4 - m^2x^4}{x^4 + m^2x^4} = \frac{1-m^2}{1+m^2}$$

This is path dependent
$\therefore$ limit does not exist at (0,0).
$\therefore$ Not continuous at (0,0).

eg: $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0). \end{cases}$

Check the continuity at (0,0) all points in R^2 by using the sequential definition.

Solⁿ At (0,0)
Suppose sequence $(x_n, y_n) \to (0,0) \Rightarrow x_n \to 0, y_n \to 0$. 19th Oct

$\because y_n \to 0$. So, for $\varepsilon > 0$ $\exists$ $m \in N$,
such that
$|y_n - 0| < \varepsilon$ $\forall$ $n \geq m$

$\Rightarrow |y_n| < \varepsilon$ $\forall$ $n \geq m$ —— (1)

$$\left| f(x_n, y_n) - f(0,0) \right| = \left| \frac{x_n^2 y_n}{x_n^2 + y_n^2} \right|$$

$|f(x_n, y_n) - f(0,0)| \leq |y_n|$ — (2)

From (1) and (2),

$|f(x_n, y_n) - f(0,0)| < \varepsilon \;\; \forall \;\; n \geq m$

$\Rightarrow f(x_n, y_n) \longrightarrow f(0,0)$.

So, it is continuous at (0,0).

At non-zero points, (x_0, y_0) [Simultaneously x_0 and y_0 cannot be zero]

Suppose $(x_n, y_n) \longrightarrow (x_0, y_0)$

$\Rightarrow x_n \to x_0$ and $y_n \to y_0$

By Limit's theorem of sequence,

$x_n^2 y_n \to x_0^2 y_0$

$x_n^2 + y_n^2 \to x_0^2 + y_0^2 \neq (0)$

$$\frac{x_n^2 y_n}{x_n^2 + y_n^2} \longrightarrow \frac{x_0^2 y_0}{x_0^2 + y_0^2}$$; $f(x_n, y_n)$ converges to $f(x_0, y_0)$.

$\therefore$ function is continuous on all points of R^2.

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$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Check continuity at (0,0).

Let $\varepsilon > 0$ be given

Suppose $(x_n, y_n) \longrightarrow (0,0)$.

$\Rightarrow x_n \to 0$ and $y_n \to 0$.

$$|f(x_n, y_n) - f(0,0)| = \left|\frac{x_n^3 y_n}{x_n^4 + y_n^2}\right| \quad \text{--- (1)}$$

AM $\geq$ GM

$$\frac{x_n^4 + y_n^2}{2} \geq \sqrt{x_n^4 \cdot y_n^2}$$

$$\Rightarrow x_n^4 + y_n^2 \geq 2 x_n^2 |y_n|$$

$$\Rightarrow \left|\frac{1}{2x_n^2 y_n}\right| \geq \frac{1}{x_n^4 + y_n^2} \quad \text{--- (2)}$$

Using 2 in 1

$$|f(x_n, y_n) - f(0,0)| \leq \frac{1}{2}\left|\frac{x_n^3 \cdot y_n}{x_n^2 \cdot y_n}\right|$$

$$\leq \frac{|x_n|}{2} \quad \text{--- (3)}$$

Since $x_n \to 0$ $\therefore$ $\frac{x_n}{2} \longrightarrow 0$

So, for $\varepsilon > 0$ $\exists$ $m \in N$
such that

$$\left|\frac{x_n}{2} - 0\right| < \varepsilon \quad \forall\ n \geq m$$

$$\Rightarrow \frac{|x_n|}{2} < \varepsilon \quad \forall\ n \geq m \quad \text{--- (4)}$$

Using (4) in (3)

$$|f(x_n, y_n) - f(0,0)| < \varepsilon \quad \forall\ n \geq m$$

$$f(x_n, y_n) \longrightarrow f(0,0)$$

$\therefore$ It is continuous at $(0,0)$.

$B_\delta(x_0, y_0)$

(x_0, y_0)

REPEATED LIMIT

Suppose $f(x,y)$ is defined in some neighbourhood of (x_0, y_0) (denoted by $B_\delta(x_0, y_0)$).

The $\lim_{y \to y_0} f(x_0, y)$, if exists, is the expression of x say $\phi(x)$.

Now, $\lim_{x \to x_0} \phi(x)$, if exists, then

$$\lim_{x\to x_0}\left[\lim_{y\to y_0} f(x,y)\right] = \lim_{x\to x_0} \phi(x) = \lambda.$$

where, λ is called REPEATED LIMIT of $f(x,y)$ as $f(x,y) \to (x_0, y_0)$.

Suppose, change the order of limit,
We have,

$$\lim_{y\to y_0}\left[\lim_{x\to x_0} f(x,y)\right] = \lim_{y\to y_0}[\phi(x)] = \lambda'.$$

λ' is also called another repeated limit as $(x,y) \to (x_0, y_0)$.
λ and λ' may or may not be the same value.

NOTE: If simultaneous limit exists, the repeated limit if exists, then both must be equal, but converse is not true
$(\lambda = \lambda')$

Remark: Repeated limits are not equal then simultaneous limit does not exist.

eg: $f(x,y) = \left(\frac{x-y}{x+y}\right)\left(\frac{1-x}{1+y}\right)$ Check the existence of repeated and simultaneous limits.

$(x_0, y_0) = (0,0)$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{x-y}{x+y}\right)\left(\frac{1-x}{1+y}\right)\right) = \lim_{y\to 0}\left(\frac{-y}{y}\right)\left(\frac{1}{1+y}\right) = -1$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x-y}{x+y}\right)\left(\frac{1-x}{1+y}\right)\right) = \lim_{x\to 0}\left(\frac{x}{x}\right)\left(\frac{1-x}{1}\right) = 1$$

Simultaneous limit does not exist as repeated limits are not equal

OR

For simultaneous limit, along $y = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left(\frac{x-mx}{x+mx}\right)\left(\frac{1-x}{1+mx}\right)$$

$$= \frac{1-m}{1+m}$$

Limit does not exist

eg: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Check the existence of simultaneous and repeated limits.
~~exis~~

For repeated limit

$$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{xy}{x^2+y^2}\right] = \lim_{x\to 0} (0) = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} \frac{xy}{x^2+y^2}\right] = \lim_{y\to 0} (0) = 0$$

Repeated limit exists

For simultaneous limit, let $y = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x(mx)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

Limit does not exist

* If repeated limit exists, simultaneous limit may / may not exist, but simultaneous limit does not exist if repeated limit is not equal.

PARTIAL DIFFERENTIATION

Let $f : D \subset R^2 \to R$ and $X_0 = (x_0, y_0) \in D$.

Then the partial derivative of 'f' w.r.t x at $X_0 = (x_0, y_0)$ is defined as,

$$f_x(x_0, y_0) = \left(\frac{\delta f}{\partial x}\right)_{X_0 = (x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided limit exists.

Then partial derivative of 'f' w.r.t y at $X_0 = (x_0, y_0)$.

$$f_y(x_0, y_0) = \left(\frac{\delta f}{\delta x}\right)_{X_0 = (x_0, y_0)} = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

f_x, f_y are called first order partial derivative.

f_x gives the rate of change of f along x-axis.
f_y gives the rate of change of f along y-axis.

Remark: if partial derivatives f_x, f_y exists at (x_0, y_0), then pair $[f_x(x_0, y_0); f_y(x_0, y_0)]$ is equal to the gradient of f. ∇f at (x_0, y_0).

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eg: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Find $f_x(0,0)$, $f_y(0,0)$ and check continuity at $(0,0)$.

Soln: $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \dfrac{0}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \dfrac{f(0,k) - 0}{k} = 0,$

Along $y = mx$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \dfrac{2mx^2}{x^2+m^2x^2} = \dfrac{2m}{1+m^2}$ Depends upon path

Not continuous.

NOTE: Existence of partial derivatives, does not imply the existence of continuity.

eg: $f(x,y) = \sqrt{x^2+y^2}$. Check $f_x(0,0)$, $f_y(0,0)$ and continuity at $(0,0)$.

Let $\varepsilon > 0$ be given

$|f(x,y) - f(0,0)| = |\sqrt{x^2+y^2}| = \sqrt{x^2+y^2}$

Let $\delta = \varepsilon$

$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \varepsilon$

'f' is continuous at $(0,0)$.

* Continuity and partial diff. does not depend upon each other unless sufficient cond^n is fulfilled.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h+0,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{h^2+0}-0}{h} = \lim_{h\to 0}\frac{|h|}{h}$$ Does not exist

$f_x(0,0)$ does not exist.

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0}\frac{|k|}{k}$$ does not exist.

Existence of continuity does not imply existence of partial Derivatives.

* **Sufficient Condition :** A sufficient condition for function to be continuous at (x_0, y_0) is that both partial derivatives exist. and at least one is bounded in same open $d \in \mathbb{R}$ centered at (x_0, y_0).

HIGHER Order Partial Derivatives

$$f(x,y) \leq D < \mathbb{R}^2 \to \mathbb{R}$$

$$f_{x^2}(x_0,y_0) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x)_{X_0=(x,y)} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

$$f_{y^2}(x_0,y_0) = \left(\frac{\partial^2 f}{\partial y^2}\right)_{X_0=(x_0,y_0)} = \frac{\partial}{\partial y}(f_y)_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}$$

$$f_{xy}(x_0,y_0) = (f_x)_y = \frac{\partial^2 f}{\partial y . \partial x} = \frac{f_x(x_0,y_0+k)-f_x(x_0,y_0)}{k}$$

$$f_{yx}(x_0,y_0) = (f_y)_x = \frac{\partial^2 f}{\partial x . \partial y} = \frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h}$$

f_{xy} and f_{yx} are called mixed second order partial derivatives

MIXED PARTIAL THEOREM

Let $f : D \subset R^2 \to R$ where D is OPEN SET such that $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist where $(x_0, y_0) \in D$.

If $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ exist at (x_0, y_0) and continuous at (x_0, y_0), then,

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

otherwise

$$f_{xy}(x_0, y_0) \neq f_{yx}(x_0, y_0).$$

eg: $f(x,y) = \begin{cases} \dfrac{xy(2x^2-3y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(\cancel{x+}h, 0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \rightarrow 0 \quad \text{— (1)}$$

$$f_{yx}(0,y) = \lim_{h\to 0} \frac{f_y(h,y) - f_y(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{hy(2h^2-3y^2)}{h^2+y^2} - 0}{h}$$

$$= -3y \quad \text{— (2)}$$

Using (2) in (1)

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-3k - 0}{k} = -3$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{— (3)}$$

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$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{xk(2x^2-3k^2)-0}{x^2+k^2}}{k} = 2x$$

Using in (3)

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{2h-0}{h} = 2$$

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

eg: $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

$f_{xy}(0,0) \neq f_{yx}(0,0).$

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Lecture 31

DIFFERENTIABILITY

Let I be any interval (not a singleton set) and $c \in I$, $f : I \to R$ then 'f' is differentiable at 'c' if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ exist.}$$

$$f : D \subset R^2 \to R$$

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0)}{(h,k)}$$

\# formal def'n:

Let $f : D \subset R^2 \to R$ and $(x_0, y_0) \in D$

'f' is called differentiable at (x_0, y_0).

if $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist and finite.

Result:

Every differentiable function is continuous but converse is not true.

and

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - hf_x(x_0,y_0) - kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

Pair $(f_x(x_0,y_0), f_y(x_0,y_0))$ is derivative of f at (x_0,y_0)

eg. $f: R^2 \to R$ such that

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

At $(0,0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$$

To prove: $\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = 0$

$$LHS = \lim_{(h,k)\to(0,0)} \frac{h^2k}{\sqrt{h^2+k^2}\,(h^4+k^2)}$$

along $k = mh$

$$= \lim_{h\to 0} \frac{\cancel{h}\,h^2(mh)}{\cancel{h}\sqrt{1+m^2}\,(h^4+m^2h^2)}$$

$$= \lim_{h\to 0} \frac{m}{\sqrt{1+m^2}\,(m^2)}$$

$$= \frac{1}{m\sqrt{1+m^2}}$$

path dependent.

$\therefore$ limit does not exist

$$\lim_{(h,k)\to(0,0)} \frac{h^2k}{(\sqrt{h^2+k^2})^3} \neq 0$$

Not differentiable

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = |xy|$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}}$$

Let $\varepsilon > 0$ be given,

$$\left| \frac{|hk|}{\sqrt{h^2+k^2}} - 0 \right| = \frac{|hk|}{\sqrt{h^2+k^2}} \leq \frac{(\sqrt{h^2+k^2})(\sqrt{h^2+k^2})}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \frac{|hk|}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2}.$$

Chose $\varepsilon = \delta > 0$ then

$$\sqrt{h^2+k^2} < \delta \Rightarrow \left| \frac{hk}{\sqrt{h^2+k^2}} \right| < \varepsilon$$

$$\lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0.$$

$\therefore$ Partial derivative exists and function is also differentiable.

eg: $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Check the continuity and differentiability at (0,0).

Function is continuous at (0,0).

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = 0.$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}}$$

Along $k = mh$

$$= \lim_{h\to 0} \frac{h(mh)}{(\sqrt{h^2+m^2h^2})(\sqrt{h^2+m^2h^2})} = \frac{m}{\sqrt{1+m^2}}$$ path dependent

$$\therefore \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2} \neq 0$$

Continuous but Not differentiable

eg: $f(x,y) = \begin{cases} x^2 \sin 1/x + y^2 \sin 1/y, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Check the differentiability at (0,0).

$f_x(0,0) = 0$, $f_y(0,0)$

To prove:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = 0$$

$$LHS = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2 \sin(1/h) + k^2 \sin(1/k)}{\sqrt{h^2+k^2}}$$

By $\varepsilon-\delta$ defn:

$$\left| \frac{h^2 \sin 1/h + k^2 \sin 1/k}{\sqrt{h^2+k^2}} - 0 \right| \le \left| \frac{|h^2| + |k^2|}{\sqrt{h^2+k^2}} \right|$$

$$\le \frac{|h||h|}{\sqrt{h^2+k^2}} + \frac{|k||k|}{\sqrt{h^2+k^2}}$$

$$\le |h| + |k|$$

$$\le 2\sqrt{h^2+k^2}$$

Chose $\delta = \varepsilon/2$, Now,

$$\sqrt{h^2+k^2} < \delta \Rightarrow \left| \frac{h^2 \sin 1/h + k^2 \sin 1/k}{\sqrt{h^2+k^2}} \right| < \varepsilon$$

$\therefore$ function is differentiable.

eg: $f(x,y) = \sqrt{|xy|}$. Check the differentiability at $(0,0)$.

$f_x(0,0) = 0 = f_y(0,0)$

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$$

Let $k = mh$

$$\lim_{h\to 0} \frac{\sqrt{|m|}}{\sqrt{1+m^2}}$$ path dependent, $\therefore$ Not differentiable.

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Lecture 32

Necessary Condition for Differentiability

If $f: D \subset R^2 \rightarrow R$ is differentiable at $(x_0, y_0) \in D$ then partial derivatives f_x and f_y at (x_0, y_0) exist and are finite.

Sufficient Condition

Let $f: D \subset R^2 \rightarrow R$ if f_x and f_y exist in same open disk (nbhd) centered at (x_0, y_0) and either f_x or f_y is continuous at (x_0, y_0) then 'f' is differentiable at (x_0, y_0).

eg: $f(x,y) = \sqrt{x^2+y^2}$. Check the differentiability of f.

Soln: At $(0,0)$, f is continuous.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h^2}}{h}, \quad \therefore \text{ limit does not exist}$$

Similarly, $f_y(0,0) = \lim_{k \to 0} \frac{\sqrt{k^2}}{k}$; limit does not exist

$\therefore$ 'f' is not differentiable.

eg: $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & x^2+y^2 \neq 0 \text{ or } (x,y) \neq (0,0) \\ 0, & x^2+y^2 = 0 \text{ or } (x,y) = (0,0) \end{cases}$$

Check differentiability at $(0,0)$.

Soln $$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

Similarly, $f_y(0,0) = 0$.

$$f_x = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$

$$f_x = \frac{(y)(x^4 - y^4 + 4x^2y^2)}{(x^2+y^2)^2}$$

$(x^2-y^2)^2 \geq 0$

$x^4 + y^4 \geq 2x^2y^2$

$\frac{2x^2y^2}{x^4+y^4} \leq 1$

Let $\varepsilon > 0$ be given,

$$|f_x - f_x(0,0)| = \frac{|y|\,|x^4 - y^4 + 4x^2y^2|}{(x^2+y^2)^2}$$

$$\leq \frac{|y|\,|x^4 + y^4 + 4x^2y^2|}{(x^2+y^2)^2}$$

$$\leq |y| \left| 1 + \frac{2x^2y^2}{(x^2+y^2)^2} \right|$$

$$\leq \sqrt{x^2+y^2}\,(1+1)$$

$$\leq 2\sqrt{x^2+y^2}$$

Let δ such that $\delta = \varepsilon/2$.

$$\sqrt{x^2+y^2} < \delta \Rightarrow |f_x - f_x(0,0)| < \varepsilon$$

$\therefore$ fnc. is differentiable.

DIRECTIONAL DERIVATIVES

Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\qquad X_0 = (x_0, y_0) \in D$

& $U = (u_1, u_2) \in \mathbb{R}^2$ such that $\|U\| = 1$

i.e.

$$u_1^2 + u_2^2 = 1$$

Here, U is unit vector.

The directional derivatives of 'f' at $x_0 = (x_0, y_0)$ in the direction of v is given by

$$D_v f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$$

Provided limit exist.

Remark 1

if $v = -u = (-u_1, -u_2)$

$$D_v f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 - tu_1, y_0 - tu_2) - f(x_0, y_0)}{t}$$

Let $-t = h$ $\therefore$ As $h \to 0$, $t \to 0$.

$$= \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{-h}$$

$$= -D_u f(x_0, y_0).$$

Remark - II

Along x-axis, $v = (1, 0)$.

$$D_v f(x_0, y_0) = \lim_{t \to 0} \frac{(x_0 + t, y_0) - f(x_0, y_0)}{t}$$

$$= f_x(x_0, y_0).$$

Remark - III

if along y-axis

$v = (0, 1)$.

$$D_v f(x_0, y_0) = f_y(x_0, y_0)$$

given limits exist.

Differentiability and Directional Derivatives

Let $f : D \subset R^2 \to R$ and $(x_0, y_0) \in D$ if 'f' is differentiable at (x_0, y_0) then for any unit vector, $v \in R^2$, the directional derivative $D_u f(x_0, y_0)$ exist and moreover.

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot v = f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2$$

$$= (f_x, f_y) \cdot (u_1, u_2)$$

eg: $f(x,y) = x^2 + y^2$
$v = (u_1, u_2) \in R^2$ such that $\|v\| = 1$.

$$D_v f(x_0, y_0) = f_x (u_1) + f_y \cdot u_2$$
$$= 2x u_1 + 2y u_2$$

Differentiable fnc.

eg: $f : R^2 \to R$ such that

$$f(x,y) = \begin{cases} \dfrac{x^2 \cdot y}{x^4 + y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Check the differentiability of directional derivative existence.

Let $X_0 = (x_0, y_0) \in R^2$ and $v = (u_1, u_2) \in R^2$ such that $\|v\| = 1$.

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$= \cancel{\lim x}$$

$$= \lim_{t \to 0} \frac{t^2 u_1^2 \cdot t u_2}{t(t^4 u_1^4 + t^2 u_2^2)}$$

$$= \lim_{t \to 0} \frac{u_1^2 \cdot u_2}{t^2(u_1^4) + u_2^2}$$

$\Rightarrow$

$$D_u f(0,0) = \begin{cases} 0 & \text{if } u_2 = 0 \\ \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0 \end{cases}$$

$\therefore$ Directional derivative exists.

NOTE: For function to have directional derivative, it need not to be differential or continuous.
DD

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Lecture 33

eg: $f(x, y) = \begin{cases} x/y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$

Check the dir. derivative at (0,0) in all direction).

Soln: Let $U = (u_1, u_2)$ such that $\|U\| = 1$ i.e. $u_1^2 + u_2^2 = 1$

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

$$= \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{tu_1}{t^2 u_2}$$

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{u_1}{t u_2}$$

Both u_1 and u_2 cannot be zero simultaneously, since $||U|| = 1$.

$$D_U f(x_0, y_0) = \begin{cases} 0 & \text{if } u_2 = 0, \text{ since for } y = 0,\ f = 0 \\ 0 & \text{if } u_2 \neq 0, u_1 = 0 \\ \infty & \text{if } u_2 \neq 0, u_1 \neq 0 \end{cases}$$

$$D_U f(x_0, y_0) = \begin{cases} 0 & , \text{ if } u_1 = 0 \text{ or } u_2 = 0 \text{ or } u_1 . u_2 = 0 \\ \underbrace{\infty}_{\text{does not exist.}} & \text{if } u_2 \neq 0 \end{cases}$$

eg: $f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Check the continuity, differentiability, ~~diff~~ directional derivative at (0,0). Also check for partial derivatives

(i) Function is continuous (Check in continuity q's).

(ii) $f_x = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

$f_y = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0$

Let $U = (u_1, u_2) \in R^2$ such that $u_1^2 + u_2^2 = 1$.

$$D_U f(x_0, y_0) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{t^4 u_2 u_1^3}{t(t^4 u_1^4 + t^2 u_2^2)}$$

$$D_u f(x_0, y_0) = \frac{t\, u_2 \cdot u_1^3}{t^2 u_1^4 + u_2^2}$$

$= \quad 0 \qquad$ if $u_2 = 0$ or $u_1 = 0$
if $u_2 \neq 0,\ u_1 \neq 0$

$\therefore$ Directional derivative exist in all directions.

$D_u f(0,0) = 0$

(III) $\Rightarrow D_u f(0,0) = \nabla f(0,0) \cdot U = (f_x(0,0), f_y(0,0)) \cdot (U_1, U_2)$

$= (0,0) \cdot (U_1, U_2)$

$= 0$

$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot U$ is scalar.

(IV) Differentiability

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^3 k}{\sqrt{h^2+k^2}\,(h^4 + k^2)}$$

along $k = mh^2$

$$= \lim_{h\to 0} \frac{h^3 \cdot mh^2}{h\sqrt{1+m^2h^2}\,(1+m^2)} = \frac{m}{(1+m^2)^{3/2}}$$

function is path dep.
$\therefore$ not differentiable.

\# $D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot U$

$$= (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot (u_1, u_2)$$

$$= u_1 f_x(x_0, y_0) + u_2 f_y(x_0, y_0)$$

$$D_u f(x_0, y_0) = |\nabla f(x_0, y_0)| \cdot |U| \cos\theta$$

$$D_u f(x_0, y_0) = |\nabla f(x_0, y_0)| \cdot \cos\theta \quad \text{--- (1)}$$

(1) $D_u f(x_0, y_0)$ is maximum i.e. $|\nabla f(x_0, y_0)|$ at (x_0, y_0) if $\cos\theta = 1$ or $\theta = 2n\pi$ near point (x_0, y_0), function f increases most rapidly in the direction $\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$

or $\dfrac{f_x(x_0, y_0), f_y(x_0, y_0)}{\sqrt{f_x^2 + f_y^2}}$

(2) $D_u f(x_0, y_0)$ is maximum i.e. $|\nabla f(x_0, y_0)|$ at (x_0, y_0) if $\cos\theta = -1$ i.e. $\theta = \pi$.
function f decreases most rapidly in the direction $-\dfrac{(\nabla f(x_0, y_0))}{|\nabla f(x_0, y_0)|}$

(III) $D_u f(x_0, y_0) = 0$ if $\cos\theta = 0$ i.e. $\theta = \pi/2$,

f has no rate of change in the direction

$$\pm \frac{f_x(x_0, y_0), -f_y(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

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eg: $f: R^2 \longrightarrow R$

$f(x, y) = 4 - x^2 - y^2$. Find the dir'n in which the function increases most rapidly at $(1, 1)$.

$f_x = -2x, \; f_y = -2y$.

$\nabla f(x, y) = (-2, -2)$

$|\nabla f(x, y)| = 2\sqrt{2}$

(I) $\dfrac{\nabla f(x, y)}{|\nabla f(x, y)|} = \left(\dfrac{-2}{2\sqrt{2}}, \dfrac{-2}{2\sqrt{2}}\right) = \left(\dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\right)$. In this dir'n f increases rapidly.

(II) $\dfrac{-\nabla f(x_0, y_0)}{|\nabla f(x, y)|} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. In this dir'n, f decreases rapidly.

(III) f has no rate of change in dir'n $= \pm \dfrac{(f_x(x_0, y_0), -f_y(x_0, y_0))}{|\nabla f(x_0, y_0)|}$

$= \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$.