Lecture 06: Monotonic Sequences

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Applications of Sandwich Theorem:

1. Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, using density of rationals choose a rational number $r_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. Then by Sandwich Theorem $r_n \to x$. So for any real number x we can construct a sequence of rational numbers converging to it.

2. Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, using density of irrationals choose a rational number $t_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. Then by Sandwich Theorem $t_n \to x$. So for any real number x we can construct a sequence of irrational numbers converging to it.

3. Let $\alpha := \sup A$ where A is nonempty subset of $\mathbb{R}$. Then there exists a sequence (a_n) in A such that $a_n \to \alpha$. For each $n \in \mathbb{N}$, note that $\alpha - 1/n < \alpha$. Hence by necessary condition for supremum, there exists $a_n \in A$ be such that $\alpha - 1/n < a_n \leq \alpha$. Now by sandwhich theorem we are done.

4. Let $a_n := \frac{1}{n} \sin \frac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n \to 0$, since $|a_n| \leq \frac{1}{n} \to 0$

Definition 6.1 *A sequence (a_n) is called increasing, if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.*

Following result gives a sufficient condition for convergence of a sequence.

Proposition 6.2 *Let (a_n) be increasing and bounded above. Then (a_n) is convergent. Moreover,*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

Proof: Since (a_n) is bounded above, by completeness property of $\mathbb{R}$ the set $\{a_n : n \in \mathbb{N}\}$ has a supremum. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. Given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $a - \epsilon < a_{n_0}$. Since (a_n) is increasing, we have $a_{n_0} \leq a_n$ for all $n \geq n_0$. Hence

$$a - \epsilon < a_{n_0} \leq a_n \leq a < a + \epsilon, \quad \forall\, n \geq n_0 \implies |a_n - a| < \epsilon, \ \forall\, n \geq n_0.$$

Thus $a_n \to a$. ∎

Sequences whose n-th term is defined in terms of previous terms are called recursive sequences or sequences defined recursively. This type of sequence occurs naturally. We shall look at an example of sequences defined recursively and find their limits if they are convergent.

Example 6.3 *Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. We show that (x_n) is increasing by induction. $2 + \sqrt{2} > 2 \implies \sqrt{2+\sqrt{2}} > \sqrt{2}$. That is $x_2 > x_1$. Suppose $x_{k+1} > x_k$. Then $2 + x_{k+1} > 2 + x_k \implies \sqrt{2 + x_{k+1}} > \sqrt{2 + x_k}$. That is $x_{k+2} > x_{k+1}$. We show that $x_n \leq 2$ by induction. clearly $x_1 \leq 2$. Now suppose $x_k \leq 2$. Then $x_k + 2 \leq 4 \implies \sqrt{x_k + 2} \leq 2$. That is $x_{k+1} \leq 2$. Hence, by Theorem 6.2 (x_n) converges. Suppose $x_n \to \lambda$. Then $\lambda = \sqrt{2+\lambda} \implies \lambda = 2$.*

Proposition 6.4 *Let (a_n) be decreasing and bounded below. Then (a_n) is convergent. Moreover,*

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Example 6.5 *Let $x_1 = 2$. Define*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), n \geq 1.$$

Claim 6.6 *$(x_n)_{n\geq 1}$ is bounded below by $\sqrt{2}$.*

Note that $2x_{n+1}x_n = x_n^2 + 2 \implies x_n^2 - (2x_{n+1})x_n + 2 = 0$. Considering the quadratic equation $t^2 - 2x_{n+1}t + 2 = 0$. Since $x_n \in \mathbb{R}$ hence this quadratic equation must have real roots,i.e., $4x_{n+1}^2 - 4 \cdot 2 \geq 0$, i.e., $x_{n+1}^2 \geq 2$ for all $n \geq 1$.

Claim 6.7 *$(x_n)_{n\geq 1}$ is decreasing.*

We have

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n = -\frac{x_n}{2} + \frac{1}{x_n} = \frac{2 - x_n^2}{2x_n} \leq 0.$$

Hence by Theorem 6.4, (a_n) converges. Suppose its limit is l. Then

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) \implies l^2 = 2 \implies l = \sqrt{2}$$