Lecture 14: Properties of Continuous Function

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Now we shall study some properties of continuous functions.

14.1 Closed Subset of $\mathbb{R}$

Definition 14.1 *Let $c \in \mathbb{R}$ and $r > 0$. Then open interval $(c-r, c+r)$ is called a symmetric neighborhood of point c with radius r.*

Definition 14.2 *Let $D \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a boundary point of D if every symmetric neighborhood of c contains points that lie outside of D as well as points that lie in D.*

For example $D = (0,1) \cup (2,3)$ has four boundary points $0, 1, 2, 3$.

Remark 14.3 *A boundary point of D itself need not belong to D.*

Definition 14.4 *A subset D of $\mathbb{R}$ is said to be closed if it contains all its boundary points.*

For example $D = (0,1) \cup (2,3)$ is not closed, $D = [0,1] \cup [2,3]$ is closed and $D = [0,1] \cup (2,3)$ is not closed. Any finite set is closed. $\mathbb{R}$ is closed.

14.2 Continuity and Boundedness

Definition 14.5 *Let $D \subseteq \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is bounded on D if there exist a real number $K \geq 0$ such that*

$$|f(x)| \leq K, \quad \forall\, x \in D.$$

Question: Is $f(x) = x^2$ is bounded on $[-10, 100]$?
Question: Is $f(x) = x^2$ is bounded on $\mathbb{R}$?

Definition 14.6 *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. We say that*

1. *f attains absolute maximum value (absolute maxima or global maxima) on D if there is $c \in D$ such that*
$$f(c) = \sup\{f(x) : x \in D\},$$

2. *f attains absolute minimum value (absolute minima or global minima) on D if there is $d \in D$ such that*
$$f(d) = \inf\{f(x) : x \in D\}.$$

Theorem 14.7 *Let D be closed and bounded subset of $\mathbb{R}$. If $f : D \to \mathbb{R}$ is continuous on D then f is bounded. Moreover, f attains its absolute maxima and absolute minima on D.*

Example 14.8

1. *Let $f(x) = x^2$ for $x \in [-1, 2]$, then f attains its absolute minima at 0 and its absolute maxima at 2.*

2. *Let $f(x) = \frac{1}{x}$ for $x \in (0, 1)$. Then f is continuous but not bounded, because for each $n \in \mathbb{N}$, we can find $x \in (0, 1)$ such that $\frac{1}{x} > n$. Hence hypothesis in Theorem 14.7 that domain of function is a closed interval can not be dropped.*

3. *Let $f(x) = x$ for $x \in (0, 1)$. Then f is continuous and bounded. But f does not attain its absolute maxima and absolute minima on $(0, 1)$.*

4. *Dirichlet function on interval $[0, 1]$ is bounded but not continuous.*

14.3 Intermediate value property

Definition 14.9 *A subset $I \subseteq \mathbb{R}$ is said to be an interval if $a, b \in I$ and if $a < x < b$, we then have $x \in I$.*

If I is an interval then I is one of the following sets

$$(a, b), [a, b], (a, b], [a, b), (-\infty, a), (-\infty, a], (a, \infty), [a, \infty), (-\infty, \infty)$$

where $a \leq b$ be real numbers.

Definition 14.10 *Let I be an interval and $f : I \to \mathbb{R}$ be a function. We say that f has the IVP on I if for any $a, b \in I$ with $a < b$ and $r \in \mathbb{R}$ such that r lies between $f(a)$ and $f(b)$ (means either $f(a) \leq r \leq f(b)$ or $f(b) \leq r \leq f(a)$) $\implies$ $r = f(x)$ for some $x \in [a, b]$.*

Remark 14.11 1. *Note that if $f : I \to \mathbb{R}$ has the IVP on I, and J is a subinterval of I, then f has the IVP on J.*

2. *If $r \in (f(a), f(b))$ or $r \in (f(b), f(a))$ then $r \in (a, b)$. Since by definition of function a and b can not have two images.*

Example 14.12 1. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := 2x + 1$. It is continuous. Take any $a, b \in \mathbb{R}$ such that $a < b$. If r lies between $f(a)$ and $f(b)$ Then straight line $y = r$ will intersect the graph of f at some point lying in interval $[a, b]$. Because graph of f is a continuous curve or has no breaks.*

2. *Let $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} 1 & \text{if } \ x \geq 0 \\ -1 & \text{if } \ x < 0 \end{cases}$$

Then f does not have the IVP on $\mathbb{R}$. Since $\frac{1}{2}$ lies between $-1 = f(-1)$ and $1 = f(0)$ but there is no $x \in [-1, 0]$ such that $f(x) = \frac{1}{2}$.

Theorem 14.13 (IVP) *Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function. Then f has the IVP on I.*

Remark 14.14 *The converse of the Theorem 14.13 does not hold in general, that is, a discontinuous function may have the IVP on an interval I. For example:*

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } \ 0 < x < 1 \\ 0 & \text{if } \ x = 0 \end{cases}$$

One can show that f is discontinuous at $x = 0$. Graphically it is very easy to observe that f has IVP on $[0, 1]$.

14.4 Application of IVP

Example 14.15 *Let $n \in \mathbb{N}$ be odd, and let $p(x) := a_0 + a_1 x + \cdots + a_n x^n$ for $x \in \mathbb{R}$, where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Then p has at least one real root.*

Solution: Suppose $a_n > 0$ without loss of generality. Write

$$p(x) = x^n \left(a_n + \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x}\right)$$

$p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$ (this is the place at which we need n to be odd)

If we take $x := b > 0$ large enough, then $p(b) > 0$. If we take $x := a < 0$ small enough, then $p(a) < 0$. Since p is continuous, it has the IVP on $\mathbb{R}$. The number 0 lies between $p(a)$ and $p(b)$. There is $c \in \mathbb{R}$ such that $a < c < b$ and $p(c) = 0$. ∎