Lecture 04: Sequences

August 7, 2019

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

The word sequence is almost self-explanatory. It refers to a succession of certain objects.

Definition 4.1 *Let X be any set. A sequence in X is a function from the set $\mathbb{N}$ of natural numbers to the set X.*

The value of this function at $n \in \mathbb{N}$ is denoted by $a_n \in X$, and a_n is called the nth term of the sequence. We shall use the notation (a_n) to denote a sequence.

Initially, we let $X := \mathbb{R}$, that is, we consider sequences in $\mathbb{R}$. Later, we shall consider sequences in $\mathbb{R}^2$ and in $\mathbb{R}^3$.

Some examples of sequence are

1. $a_n = n$ for $n \in \mathbb{N} : 1, 2, 3, \cdots$.
2. $a_n = \dfrac{1}{n}$ for $n \in \mathbb{N} : 1, \frac{1}{2}, \frac{1}{3}, \cdots$
3. $a_n = (-1)^n$ for $n \in \mathbb{N} : -1, 1, -1, 1, \cdots$,
4. $a_n = 1$ for $n \in \mathbb{N} : 1, 1, \cdots$. This is an example of a constant sequence.
5. $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3 : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$. This sequence is known as the Fibonacci sequence.

If we see a sequence, it is natural to ask where it leads. Observe that all the sequences above, do not behave in same way. The terms of the sequence (n) “tends to $+\infty$”. Terms of the sequence $\left(\dfrac{1}{n}\right)$ “approaches 0” as n increases. Terms of the sequence $((-1)^n)$ keeps on “oscillating” between 1 and -1. Terms of the sequence (1) have the constant value 1 regardless of n.

From your knowledge about limits, you may say that $\lim\limits_{n\to\infty} n = +\infty, \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} (-1)^n$ does not exists.

We want to understand the “precise” meaning of the statement $\lim\limits_{n\to\infty} a_n = a$. You may say, what a big deal! The meaning of $\lim\limits_{n\to\infty} a_n = a$ is “a_n tends to a, as n tends to infinity”. or

" a_n approaches to a, as n approaches to infinity". If someone further ask you, what is the meaning of a_n approaches to a as n approaches to infinity ? Many students thinks the meaning of "a_n tends to a, as n tends to infinity" is "as n becomes equal to infinity a_n is equal to a". Actually this is not the meaning. We do manipulation like $\frac{1}{\infty} = 0$ in order to calculate the limit of the sequence $\left(\frac{1}{n}\right)$, but this is not the meaning of a limit of sequence. Because there are situations when blind manipulations does not work. For example if we want to calculate limit of the sequence $\frac{n}{n+1}$ then just substituting $n = \infty$, you get an indeterminate form $\frac{\infty}{\infty}$. But you know that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. You must realize that what you have learnt in your 11^{th} or 12^{th} standard, are techniques to find the limit of a sequence, not the "precise meaning of a limit of sequence". Of course you have some intuitive understanding of the concept of a limit. We are trying to transform that intuition into a rigorous mathematical definition. Following are steps towards this goal.

1. Let us add some mathematical concepts to clarify the meaning of "a_n approaches to a, as n approaches to infinity". We say that meaning of statement "a_n approaches to a, as n approaches to infinity" is that "distance between a_n and a becomes small as n becomes large." Or mathematically $|a_n - a|$ is small , whenever n is large".

2. As a second step let us paraphrase step 1, so as to bring out the fact that smallness of $|a_n - a|$ depends on largeness of n. We say that meaning of statement "a_n approaches to a, as n approaches to infinity" is that " $|a_n - a|$ can be made small by taking n large".

3. The statement in step 2 is still vague, since it does not specify how small is small. The standard of smallness can, in fact very. The next statement makes allowance for such variation. "$|a_n - a|$ can be made arbitrarily small by taking n sufficiently large"

Let us try to understand this with the help of the following example.

Example 4.2 *We know $1/n \to 0$. To obtain more detailed information about how $1/n$ approaches 0 when n is large, we ask the following question: How large n we should have to be so that $1/n$ differs from 0 by less than 0.1? So our problem is find a natural number n_0*

$$|1/n - 0| < 0.1, foralln \geq n_0$$

$$1/n < 0.1 \iff n > 1/0.1 = 10$$

So we may choose $n_0 = 11$. If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $1/n$ will differ from 0 by less than 0.01 provided that $n \geq 101$, i.e.

$$n \geq 101 \implies |1/n - 0| < 0.01$$

Similarly

$$n \geq 1001 \implies |1/n - 0| < 0.001$$

The numbers $0.1, 0.01,$ *and* 0.001 *that we have considered are error tolerances that we might allow. For* 0 *to be the precise limit of sequence* $1/n$ *as* n *approaches* ∞*, we must not only be able to bring the difference between* $1/m$ *and 0 below each of these three numbers; we must be able to bring it below any positive number. And, by the same reasoning, we can for* ϵ *an arbitrary positive number, then we find as before that*

$$n \geq \left[\frac{1}{\epsilon}\right], |1/n - 0| < \epsilon$$

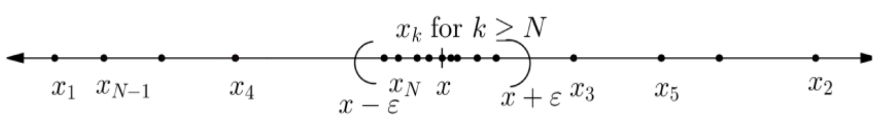
The catch here is that the standard of smallness of $|a_n - a|$ has to be specified first. Only after knowing this standard, can we decide what degree of largeness of n will enable us to meet this standard. The standard method of specifying the standard of smallness is, of course, to specify an upper bound on the size. This leads us into the following definition.

Definition 4.3 *We say that a real sequence* (a_n) *is convergent if there is* $a \in \mathbb{R}$ *such that for every* $\epsilon > 0$*, there is* $n_0 \in \mathbb{N}$ *such that*

$$|a_n - a| < \epsilon, \quad \textit{for all } n \geq n_0.$$

In this case, we say that (a_n) *converges to* a *or that* a *is a limit of* (a_n)*, and write* $a_n \to a$ *(as* $n \to \infty$*) or* $\lim\limits_{n\to\infty} a_n = a$*.*

Geometrically, if we represent the term of the sequence (x_k) on the number line. Then $x_n \to x$ means, for a given ϵ-neighborhood of x, i.e., open interval $(x - \epsilon, x + \epsilon)$, after certain stage onwards all the terms of the sequence (x_k) will lie within in the ϵ-neighborhood of x.



Remark 4.4

1. *Once again we emphasize that* ϵ *has to be specified first, only then can an appropriate* n_0 *be found.*
2. *If* n_0 *works for a particular value of* ϵ *will obviously work for any larger value of* ϵ *but in general not for a smaller value.*
3. *Unless the sequence is constant (or eventually constant) there is no single* n_0 *which will work for every* ϵ*. Thus* n_0 *very much depends on* ϵ*. There is however, no restriction on* ϵ *other than that it be positive. It could be as small as we like.*

4. *When we are asked to prove that a_n converges to a, our proof must begin with a statement like "Let $\epsilon > 0$ be given". Our task is to use the given data (sequence a_n, a, ϵ) find n_0 which works for this $\epsilon > 0$. Typically n_0 will be something like $\epsilon, \frac{\epsilon}{2}, \epsilon^2, \min\left\{\frac{\epsilon}{2}, M\right\}$ where M is positive constant obtained from given data. It is important to know that we are not generally required to find the best (i.e. the minimum) n_0 that will work for a given $\epsilon > 0$. All we need to find some n_0 which will work. Strictly speaking, you are not even required to define n_0 by an explicit formula. It is enough to prove merely that it exists.*

Example 4.5 *Let $a_n = 1$ for all n, then show that $a_n \to 1$.*

Solution: Let $\epsilon > 0$ be given. Then

$$0 = |a_n - 1| < \epsilon, \ \forall n \in \mathbb{N}$$

Hence we can take $n_0 = 1$. ■

Example 4.6 *Show that the sequence $\left(\dfrac{2}{n^2+1}\right)$ is convergent.*

Solution: We claim that $\dfrac{2}{n^2+1} \to 0$. Let $\epsilon > 0$ be given. We choose $n_0 \in \mathbb{N}$ (by Archimedian property) such that $n_0 > \dfrac{2}{\epsilon}$. Then for all $n \geq n_0$,

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \leq \frac{2}{n} < \epsilon,$$

■

One must realize that the purpose of definition of a convergent sequence is not to find a limit of the sequence. In fact, the last exercise tells us a crucial point regarding convergence of a sequence is that, first we have make a guess for a limit and then proceed further. The precise meaning to a mathematical concept is needed if we want to prove rigorously theorems using that concept.

Example 4.7 *Is sequence $((-1)^n)$ convergent ? Justify your answer.*

Solution: The sequence is not convergent. We show this by contradiction. Suppose the sequence converges to a. Then if we choose $\epsilon = \frac{1}{2}$, there exist n_1 such that we have

$$|(-1)^n - a| < \frac{1}{2}, \forall n \geq n_1$$

That is

$$|-1-a| = |a+1| < \frac{1}{2} \text{ and } |1-a| = |a-1| < \frac{1}{2}$$

That is $a \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$ and $a \in \left(\dfrac{1}{2}, \dfrac{3}{2}\right)$, which is absurd. ■