Lecture 22: Riemann Integration

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In defining $\int_a^b f(x)dx$, we have assumed $a < b$ so far. In order to obtain uniformity of presentation and simplicity of notation, we adopt the following definitions.

Conventions:

1. Let $b = a$. Then every $f : [a, a] \to R$ is integrable, and $\int_a^a f(x)dx := 0$.

2. Let $b < a$ and $f : [b, a] \to \mathbb{R}$ be integrable. Then

$$\int_b^a f(x)dx := -\int_a^b f(x)dx.$$

   We emphasize that the Riemann integral of $f : [a, b] \to \mathbb{R}$ is defined over the subset $\{x : a \leq x \leq b\}$ of $\mathbb{R}$ and that we have not associated any direction or orientation with this subset. What we have mentioned above are mere conventions; they are not results that follow from our definition.

Interpretation of Riemaan Integral If a function $f : [a, b] \to \mathbb{R}$ is integrable and nonnegative, then the area of the region R_f under the curve given by $y = f(x), x \in [a, b]$, is defined to be

$$\text{Area}(R_f) = \int_a^b f(x)dx, \text{ where } R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, 0 \leq y \leq f(x)\}.$$

Suppose f is equal to a negative constant c throughout the interval $[a, b]$. Then $\int_a^b f < 0$. In this case we regard $\int_a^b f$ as negative area.

Let us consider $f : [0, 2\pi] \to \mathbb{R}$ be $f(x) = \sin x$, then from your previous classes you know $\int_0^{2\pi} \sin x dx = 0$.

This suggests that the Riemann integral of f on $[a, b]$ can be interpreted as the "signed area" of the planar region delineated by the curve $y = f(x), x \in [a, b]$.

22.1 Sufficient conditions for Riemaan Integrability

Using the Riemann's criterion one can prove the integrability of a wide variety of functions.

Theorem 22.1 *Let $f : [a,b] \to \mathbb{R}$ be a function.*

(a) If f is monotonic, then it is integrable.

(b) If f is continuous, then it is integrable.

(c) If f is bounded and has finitely many jump discontinuities, then it is integrable.

Remark 22.2 *1. It may be noted that the above theorem gives no clue about the evaluation of the Riemann integral of f.*

2. In part (a) and (b) we did not assume that f is bounded on $[a,b]$, which is a necessary condition in order to define Riemaan integral. Actually it is implicit in the hypothesis.

If f is monotonically increasing on $[a,b]$ then $f(a) \le f(x) \le f(b)$, hence f is bounded. Similarly, If f is monotonically decreasing on $[a,b]$ then $f(b) \le f(x) \le f(a)$, hence f is bounded.

Any continuous function on a closed and bounded interval is bounded.

Example 22.3 *Define $f : [0,1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := 1/n$ if $1/(n+1) < x \le 1/n$ for $n \in \mathbb{N}$. Hence*

$$f(x) = \begin{cases} 1, & \frac{1}{2} < x \le 1 \\ \frac{1}{2}, & \frac{1}{3} < x \le \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{4} < x \le \frac{1}{3} \end{cases}$$

Note that f is discontinuous at infinitely many points $x = \frac{1}{n}$ for $n \ge 2$, but f is increasing on $[0,1]$, f is integrable on $[0,1]$.

Example 22.4 *A polynomial function is continuous everywhere hence integrable on any interval $[a,b]$.*

The function f in Example 21.3, is discontinuous only at $x = 1$, and f is bounded by 1 hence it is integrable by part (b) of above theorem.

22.2 Riemann Sum

Although Riemann's criterion for integrability is very useful to prove several interesting results regarding integration, there are a number of difficulties in employing it to test the integrability of an arbitrary bounded function.

1. First, the calculation of $U(P, f)$ and $L(P, f)$, for a given partition P , involves finding the absolute maxima and minima of f over several subintervals of $[a, b]$. This task is rarely easy.
2. Second, it is not clear how to go about choosing a partition P, so as to obtain $U(P, f) - L(P, f) < \epsilon$.

If a function is found to be integrable, Riemann's criterion does not give any clue how to compute its Riemann integral. How does one actually find at least an approximate value of its integral?

To overcome the first difficulty mentioned above, namely, of having to calculate several maxima and minima of f , we give an alternative approach. While calculating maxima and minima of f over several subintervals of $[a, b]$ may be difficult, evaluating f at several points of $[a, b]$ is relatively easy. With this in mind, we introduce the following concept.

Definition 22.5 *Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$, and let s_i be a point in the ith subinterval $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$. Then*

$$S(P, f) := \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

is called a Riemann sum for f corresponding to P.

Note that the upper sum $U(P, f)$ and the lower sum $L(P, f)$ are determined by P and f, whereas a Riemann sum $S(P, f)$ depends on P and f, and also on the choice of the points $s_i \in [x_{i-1}, x_i]$ for $i = 1, \cdots, n$.

Let us now take up the second question regarding the choice of a partition P so as to make $U(P, f) - L(P, f)$ small.

Definition 22.6 *For a partition P of $[a, b]$, we define the mesh of P to be the length of the largest subinterval of P. Thus, if $P = \{x_0, x_1, \cdots, x_n\}$, then*

$$\mu(P) := \max\{x_i - x_{i-1} : i = 1, \cdots, n\}.$$

Theorem 22.7 (Approximation of a Riemann Integral) *Let $f : [a,b] \to \mathbb{R}$ be integrable and let (P_n) be a sequence of partition of $[a,b]$ with $\mu(P_n) \to 0$. Then,*

$$S(P_n, f) \to \int_a^b f(x)dx,$$

where $S(P_n, f)$ is Riemaan sum of f corresponding to P_n (with any choices of s_i)

The above theorem is useful in two ways.

1. If $\int_a^b f(x)dx$ is known, then we can find $\lim_{n\to\infty} a_n$, if $a_n := S(P_n, f)$ for $n \in \mathbb{N}$ and $\mu(P_n) \to 0$.
2. (ii) If $\int_a^b f(x)dx$ is not known, then $S(P_n, f)$ gives us an approximation of $\int_a^b f(x)dx$ if $\mu(P_n) \to 0$.

Example 22.8 *Consider $a_n := \sum_{i=1}^{n} \frac{1}{\sqrt{n^2 + in}}$ for $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$.*

Solution:

$$a_n = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{1+\frac{i}{n}}} = \sum_{i=1}^{n} \frac{1}{\sqrt{1+\frac{i}{n}}}\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

We observe that if we consider $f : [0,1] \to \mathbb{R}$ defined by $f(x) := \frac{1}{\sqrt{1+x}}$ and let $P_n := \left\{0, \frac{1}{n}, \cdots, \frac{n}{n}\right\}$ and $s_i := \frac{i}{n}$ for $n \in \mathbb{N}$ and $i = 1, \cdots, n$, then $a_n = S(P_n, f)$. Note that $1 + x$ is increasing on $[0,1]$ hence $\sqrt{1+x}$ is increasing, therefore f is a decreasing function on $[0,1]$, hence integrable on $[0,1]$. In this case, f clearly has an antiderivative, namely $2\sqrt{1+x}$. Since $\mu(P_n) = \frac{1}{n} \to 0$, so we have

$$a_n = S(P_n, f) \to \int_0^1 \frac{1}{\sqrt{1+x}}dx = 2(\sqrt{2} - 1) \text{ as } n \to \infty$$

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Example 22.9 *Let r be a nonnegative rational number and consider $a_n := \sum_{k=1}^{n} \frac{k^r}{n^{r+1}}$ for $n \in \mathbb{N}$. Determine the limit of the sequence (a_n).*

Solution: $a_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\frac{i}{n}\right)^r = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^r\left(\frac{i}{n}-\frac{i-1}{n}\right)$. Consider $f:[0,1]\to\mathbb{R}$ defined by $f(x) := x^r$ and let $P_n := \left\{0, \dfrac{1}{n}, \cdots, \dfrac{n}{n}\right\}$ and $s_i := \dfrac{i}{n}$ for $n \in \mathbb{N}$ and $i = 1, \cdots, n$, then $a_n = S(P_n, f)$. Since $\mu(P_n) = \dfrac{1}{n} \to 0$, so we have $a_n = S(P_n, f) \to \displaystyle\int_0^1 x^r dx = \frac{1}{1+r}$ as $n \to \infty$ ■