SET Theory

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \setminus B = \{x \mid x \in A, \text{but } x \notin B\}$

difference

$A_1 \cup A_2 \cup \ldots\ldots = \bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n \in N\}$

$= \bigcup_{n \in N} A_n = \{x \mid x \in A_n, \text{ for every } n \in N\}$

$N = \{1, 2, 3 \ldots\}$

Eg:

$A_n = (-n, n), n \in N$

Ans: $\bigcup_{n=1}^{\infty} A_n = R$

$\bigcap_{n=1}^{\infty} A_n = (-1, 1)$

Q) Proof $\sqrt{2}$ is irr.

$\sqrt{2} = \frac{r}{s}$, assume r, s have no common divison, i.e. GCD =

$2s^2 = r^2$

~~...~~ 2 divides r^2, so 2 divides r

~~...~~

$\therefore$ 2 divides r^2

$r^2 = 2^2 K^2$

$2s^2 = r^2 = 4k^2 \Rightarrow s^2 = 2k^2$

$\Rightarrow$ 2 divides s^2

$\Rightarrow$ 2 divides s

i.e. is contradiction.

$\therefore \sqrt{2}$ is irrational

Q (i) $\sqrt{6}$, (ii) $\sqrt{2} + \sqrt{3}$

Eg: Find $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$. $= \phi$

Eg: Prove $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

* QUANTIFIER :

$\forall$: Universal Quantifier $\leftarrow$ for all

$\exists$: Existential quantities $\leftarrow$ there exisits

Eg: There is atleast one student who has 5 apple.

$\Rightarrow$ " $\exists x$ (x has 5 apples) " ation "All student does not have 5 apples"

$\Rightarrow \forall x$ (x does not have 5 apples)

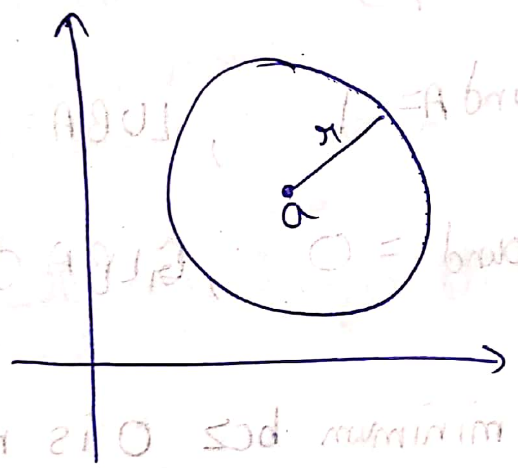
g: $(\forall \epsilon > 0 \, (\exists n_0 \in N \, (\forall_{n > n_0} \, (|x_n - x| < \epsilon))))$

property $x < x_n < x + \epsilon$

Negation:

$(\exists \epsilon > 0 (\forall n_0 \in N (\exists n > n_0$ ("$|x_n - x| < \epsilon$" fails))))

i.e. $(\exists \epsilon > 0 (\forall n_0 \in N (\exists n > n_0 (|x_n - x| \geq \epsilon))))$



$A \subseteq \{x \mid |x - a| < r\}$

$a - r < x < a + r$

$x \in (a - r, a + r)$

★ Definition:

① A non-empty subset A of R is <u>bounded below</u> if $\exists\, m \in R$ such that $m \leq x \;\forall x \in A$

② A non-empty subset A of R is bounded above if $\exists M \in R$ such that $x \leq M, \forall x \in A$

→ 6 & 10
1 & 2 are common divisor
gcd = 2

→ 30, 60, 120, ... are cm
LCM = 30

Eg:

$A = \{ \frac{1}{n} \mid n \in N \}$

$= \{ 1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n} \}$

Least uppor bound A = 1 (LUB) , LUB A = 1

Greatest lower bound = 0 (GLB) , GLB A = 0

0 is not minimum bcz 0 is not in the set

Definition: Bounded above

① Let S be a non-empty subset of R.
The set S is called bounded above if
$\exists M \in R$, such than $x \leq M, \forall x \in S$

In this case we call m is an upper bound of S.

2) The real no. α is called least upper bound (LUB) if (i) α is an upper bound of S (That is $x \le \alpha, \forall x \in S$)
(ii) If M is an upper bound of S, $\alpha \le M$

Bounded Below

1) Let S be a non-empty subset of R.
The set S is called bounded below if
$\exists\, m \in R\ \forall x \in S$

2) β is GLB of S if
(i) $\beta \le x, \forall x \in S$
(ii) $m \le \beta$ if m is a l.b. of S

(GLB, LUB are always Unique)

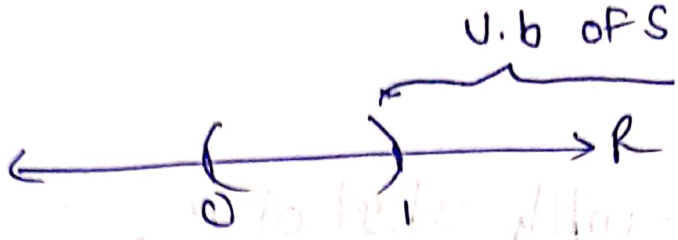
Definition:

A real no. M is called <u>maximum element</u> of the set $S \subseteq R$ if

(i) $M \in S$
(ii) $x \le M, \forall x \in S$

Eg: $S = (0,1)$

Clearly 1 is not maximum & 1 is an Upper Bound ~~of the set~~ of S.



Q) Prove 1 is LUB : (i.e. any no. less than 1 is not LUB)

Sol Let $0 < \alpha < 1$

want to show that there is an element of the set b/w α & 1.

Clearly,

$$0 < \frac{1+\alpha}{2} < 1 \quad \text{and if} \quad \frac{1+\alpha}{2} \geq 1$$

$$1+\alpha \geq 2$$

$$\alpha \geq 1$$

which is a contradiction.

α is lub of S

$\Leftrightarrow$ $\forall \epsilon > 0, \exists x \in S$ such that

$$\alpha - \epsilon < x < \alpha$$

β is glb of S

$\Leftrightarrow \forall \varepsilon > 0, \exists x \in S, \quad \beta \leq x < \beta + \varepsilon$

(2) $S = [0,1] \cup \{2\}$

LUB $S = 2$

GLB $S = 0$

(3) $S = \{x \mid x^2 < 2\}$

LUB $S = \sqrt{2}$

GLB $S = -\sqrt{2}$

ARCHIMEDEAN PROPERTY

Theorem : Let $x, y \in R$, let $x > 0$

Then $\exists n \in N$, such that $nx > y$

* LUB Property (Completeness of R) :-

Let S be a non-empty subset of R.

If S has an upper bound then there exists a ~~subset~~ lub S in R

Proof of Arch. Prop. :-

Suppose the conclusion is false

$\forall n \in N \ (nx \le y)$

Consider $S = \{nx \mid n \in N\} = \{x, 2x, 3x, \dots\}$

$\therefore y$ is an upper bound of S

$\therefore$ By LUB property there exists a real no. α such that α is $=$ LUB S ($nx \le \alpha \ \forall n \in N$)

$\therefore (n+1)x \le \alpha \ , \ \forall n \in N$

$nx \le \alpha - x \ , \ \forall n \in N$

$\therefore \alpha - x$ is an upper bound of S

This is a contradiction to the fact that α is LUB S ($\because \alpha - x < \alpha$)

Eg: $S = \{\frac{1}{n} : n \in N\}$

1 is maximum element of S

0 is a lower bound of S

Prove, GLB S $= 0$

Ans: Let 0 is glb S

$\Leftrightarrow \forall \ \varepsilon > 0 \ , \exists x \in S \Rightarrow 0 < x < \varepsilon$

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$\Leftrightarrow \exists n \in \mathbb{N} \longrightarrow \boxed{n\varepsilon > 1}$

This follows from A.P. when $x = \varepsilon$, $y = 1$

<u>mple</u>

Find lub & glb of :

$S = \{ 1 - \frac{1}{n} : n \in \mathbb{N} \}$

LUB $S = 1$

GLB $S = 0$

(i) To show glb = 0 we need to show that

$\forall \varepsilon > 0$, $\exists x \in S$

$\ni \quad 0 \leq x < \varepsilon$

$\Rightarrow \quad 0 \leq \boxed{1 - \frac{1}{n} < \varepsilon}$

$\Rightarrow \exists n \in \mathbb{N},\ 1 - \frac{1}{n} < \varepsilon$

$\Rightarrow \quad n\varepsilon > n - 1$

This follows from A.P, where $x = \varepsilon$, $y = n-1$

Alt

To show 0 is glb we need to show $\forall \varepsilon > 0$ $\exists x \in S \ni \varepsilon > x$

clearly $x = 0$ satisfies this

(ii) For LUB we want to show that

$\forall \varepsilon > 0$, $\exists x \in S$

$\ni \quad 1 - \varepsilon < x \leq 1$

$\Rightarrow 1 - \varepsilon < 1 - \frac{1}{n} < 1$

$\Rightarrow \cancel{1} - \varepsilon < \cancel{1} - \frac{1}{n}$

$\Rightarrow -\varepsilon n < -1 \quad \Rightarrow n\varepsilon > 1$

This follows from A.P, where $x = \varepsilon$, $y = 1$

Theory

α is upper bound of S.

α is lub if :

$\forall \varepsilon > 0, \exists x \in S$

$\ni \alpha - \varepsilon < x \leq \alpha$

Archimedean Property :-

Let $x, y \in R$

Let $x > 0$, then $\exists\, n \in N$ such that $nx > y$

① **Result**: Between any two distinct real no. there exists a rational number.

Note :

$3 < \pi < 4$

$[\pi] = 3$

Every real no. c, there exists an integer m such that $m-1 \leq c < m$

Proof: Let a & b are two distinct no.

Without lose of generality assume

$\boxed{a < b}$

choose $x = b - a > 0$

choose $y = 1$

By Archimedean property $nx > y$,

i.e. $\boxed{n(b-a) > 1} \Rightarrow nb > 1 + na$

We can find an integer such that

$\boxed{m-1 \leq na < m}$

$\therefore a < \frac{m}{n} \leq \frac{1+na}{n} < \frac{nb}{n} = b$,

$\boxed{a < \frac{m}{n} < b}$

H.P

② **Result**:

Proof:

Example:

2) **Result:** B/w any two distinct real no. there exist an irrational no.

Proof: Let a and b are distinct real no.

Assume $a<b$

$$\Rightarrow \frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$$

By the previous result, there exists a rational no. $\frac{r}{s}$, such that

$$\frac{a}{\sqrt{2}} < \frac{r}{s} < \frac{b}{\sqrt{2}} \quad ; \; r \in Z, \; s \in N$$

$$\Rightarrow a < \left(\frac{r}{s}\right)\sqrt{2} < b$$

Prove it irr.

& hence this result is proved

Example:

$f: R \to R$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Reiman Integration

Example: find lub & glb of

$$S = \left\{ \frac{m}{m+n},\ m, n \in N \right\}$$

Ans: To show 0 is glb, we need to show $\forall \varepsilon > 0, \exists x \in S,$

$\ni\ 0 < x < \varepsilon$ $\qquad \frac{1}{1+n} < \varepsilon$

$\Rightarrow\ 0 < \frac{m}{m+n} < \varepsilon$ $\quad \Rightarrow 1 < \varepsilon(1+n)$

$\Rightarrow (1+n)\varepsilon > 1$

$m = 1$ $\quad \Rightarrow\ m < \varepsilon(m+n)$ By AP glbS $= 0$

$\varepsilon > \frac{1}{n}$ $\Rightarrow$ ~~$\varepsilon > \frac{1}{n+1}$ for some~~

~~$n \in N$~~

To show 1 is lub,

$\forall\ \varepsilon > 0,\ \exists\ x \in S \ni 1 - \varepsilon < x$

$\left[\cancel{1} - \varepsilon < \frac{m}{m+1} = \cancel{1} - \frac{1}{1+m},\right.$ for some $\left. m \in N \right]$

$\Rightarrow\ \varepsilon > \frac{1}{1+m}$ $\Rightarrow (1+m)\varepsilon > 1$

$\Rightarrow K\varepsilon > 1$, $K = (1+m)$

$\therefore$ By A.P. LUBS $= 1$

Q) Given a real no. a, such that $1+a \geq 0$.
Prove that $(1+a)^n \geq 1+na$, $\forall\, n \in N$

Result : S is any non-empty subset of R and α is an upper bound of S.

(if & only if)

Then $\alpha = \sup S \Leftrightarrow \forall \epsilon > 0\ (\exists x \in S\ (\alpha - \varepsilon < x))$

Proof : $(\Rightarrow)$ Suppose $\alpha = \sup S$

Assume that "$\forall \epsilon > 0\ (\exists x \in S\ (\alpha - \varepsilon < x))$" is false

i.e. $\exists \epsilon > 0\ (\forall x \in S\ (\alpha - \varepsilon \geq x))$

$\Rightarrow$ $\alpha - \varepsilon$ is an upper bound

$\therefore$ We have a contradiction ($\alpha - \varepsilon < \alpha$ & $\alpha = \sup S$)

$\therefore$ $\forall \epsilon > 0\ (\exists x \in S\ (\alpha - \varepsilon < x))$ is true

$(\Leftarrow)$ Suppose $\forall \epsilon > 0\ (\exists x \in S\ (\alpha - \varepsilon < x))$

Assume that α is not supremum of S

$\therefore$ There exist some upper bound m such that $M \not< \alpha$

chose $\varepsilon = \alpha - M > 0$ $(\because M < \alpha)$

but ~~$\alpha - (\alpha - M)$~~

$$\alpha - \varepsilon = (\alpha - (\alpha - M)) = M \geq x \; \forall \; x \in S$$

$\therefore$ $x \leq \alpha - \varepsilon, \; \forall x \in S$

$\therefore$ We get a contradiction

$\therefore$ $\boxed{\alpha \text{ is } \sup S}$

OTE:

$$\left\{\frac{m}{m+n} \,\middle|\, m,n \in N\right\}$$

$$1-\varepsilon < \frac{m}{m+n}$$

$$\Leftrightarrow \quad m+n-m\varepsilon-n\varepsilon < m$$

$$\Leftrightarrow \quad n < m\varepsilon + n\varepsilon \Leftrightarrow \frac{n}{m+n} < \varepsilon$$

{We can use $n < n\varepsilon$ only when $\varepsilon > 1$}

~~$(1+a)^n \geq 1+na, \forall n \in N$~~

Eg: Which of the following sets are bounded:—

(i) $\left\{\frac{1}{n^2} : n \in N\right\} = \left\{1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2} \cdots \frac{1}{n^2} \cdots\right\}$
$\{x_1, x_2, x_3, x_4 \cdots x_n \cdots\}$

(no need to tell sup. & infr.)

(ii) $\left\{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) : n \in N\right\}$
$= \{\sqrt{2}-1, \sqrt{2}(\sqrt{3}-\sqrt{2}), \sqrt{3}(\sqrt{4}-\sqrt{3}) \cdots\}$

(iii) $\{(-1)^n : n \in N\}$ $(-1, 1, -1, 1, \cdots)$

(iv) $\left\{\left(1+\frac{1}{n}\right)^n : n \in N\right\}$

$$x_n = \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$ (but it is bounded)

(iv) $(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \dots + b^n$ — (i)

also $(1+a)^n \geq 1+na$, $\forall n \in N$ — (ii)

$\therefore \left(1+\frac{1}{n}\right)^n \geq 1 + \left(\frac{1}{n}\right)n = 2$

$\left(\because (1+a)^n \geq na, \text{ where } a = \frac{1}{n}\right)$

Using (i),

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \dots + \left(\frac{1}{n}\right)^n$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\frac{1}{n!}$$

$\left(\because \frac{n-1}{n} < 1\right)$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}}$$

$$= 1 + \left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)$$

$$= 1+2-\frac{1}{2^{n-1}} < 3$$

$\therefore$ (iv) is also bounded b/w 2 & 3

SEQUENCE

sequence : $x_1, x_2, \dots x_n \dots$

series : $x_1 + x_2 + x_3 + \dots + x_n + \dots = \sum_{n=1}^{\infty} x_n$

Convergence of series : When you add ∞ no. of terms, you get a real no.

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

(We need to prove)

Eg.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

Eg!

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Take till k, then replace k by ∞

Eg: $\sum_{n=1}^{\infty} (-1)^n$

This series is not convergent

Definition:

A SEQUENCE of real no. is a fn
$f: N \to R$

We use notation x_n for $f(n)$, where $n \in N$

Notation for this sequence is:
$\{x_n\}$ or (x_n)

Example: $x_n = \sqrt{n}$

$y_n = \frac{1}{n}$

$z_n = 2y_n = \frac{2}{n}$ (any no. times sequence is also a sequence)

Definition:

Let (x_n) be a sequence of real no.
The sequence (x_n) converges to some [redacted] $l \in R$
[redacted]
if
" $\forall \epsilon > 0 \left(\exists K \in N \left(\forall n \geq K \underbrace{\left(|x_n - l| < \epsilon \right)}_{} \right) \right)$ "

$l - \epsilon < x_n < l + \epsilon$
if $n \geq K$, $\exists K \in N$

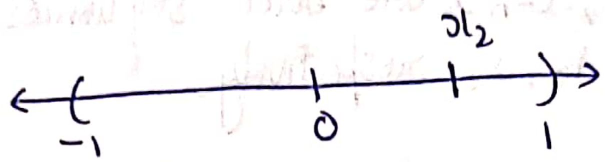
$\underbrace{x_K, x_{K+1}, x_{K+2}, \ldots}_{K\text{-tail}}$

Eg: $(x_n) = \left(\frac{1}{n}\right)$

$l = 0$

$\varepsilon = 1$

$K = 2$



Now let's prove that it actually converges to 0.

$l = 0$

$|x_n - l| < \varepsilon$

$\Leftrightarrow \left|\frac{1}{n} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon$

$\Leftrightarrow n\varepsilon > 1$

By the A.P, $\exists\, K \in \mathbb{N}$ such that $K\varepsilon > 1$

($\because x = \varepsilon, y = 1$)

$\therefore \forall\, n \geq K$, $\boxed{n\varepsilon \geq K\varepsilon > 1}$

<u>Theorem:</u> Let (x_n) & (y_n) be sequences of real no. such that $(x_n) \to l_1$ & $(y_n) \to l_2$

Then
1. $(x_n + y_n) \to l_1 + l_2$
2. $(\alpha \cdot x_n) \to \alpha l_1$, where $\alpha \in \mathbb{R}$
3. $(x_n \cdot y_n) \to l_1 l_2$
4. $\left(\frac{x_n}{y_n}\right) \to \frac{l_1}{l_2}$, where $y_n \neq 0$ & $l_2 \neq 0$
5. $(x_n^p) \to l_1^p$, $p \in \mathbb{N}$

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(l_1 is limit of sequence)

$$x_n^{1/p} \longrightarrow l_1^{1/p} \text{ if } p \in \mathbb{N}$$

Theorem: (Sandwich or squeeze)

If (x_n), (y_n), (z_n) are three sequences converges to l_1, l_2, l_3 respectively.

and $x_n \leq y_n \leq z_n \quad \forall\, n \in \mathbb{N}$

and $l_1 = l_3$

then $l_1 = l_2 = l_3$

Example:

$$x_n = 0 \longrightarrow 0$$
$$y_n = \frac{1}{n} \longrightarrow 0$$
$$z_n = \frac{1}{n^2}$$

$$\underset{\substack{\downarrow \\ 0}}{x_n} \leq z_n \leq \underset{\substack{\downarrow \\ 0}}{y_n}$$

By Sandwich theorem,

$$z_n \longrightarrow 0$$

Example: Is the sequence $\left(\frac{\sin n}{n}\right)$ convergent?

Solution: $x_n = \dfrac{\sin(n)}{n}$ $\qquad \left\{ -1 \leq \sin(n) \leq 1 \right\}$

$$\underset{\substack{\downarrow \\ 0}}{0} \leq |x_n| = \left| \frac{\sin(n)}{n} \right| \leq \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

By sandwich thm

$|x_n| \to 0$

$\therefore x_n \to 0$ (when $|x_n| \to 0$, $x_n \to 0$)

(only for 0)

<u>Example:</u>

$$x_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$$

$$x_1 = \frac{1}{2}$$ (as n^{th} term is $\frac{1}{2}$)

$$x_2 = \frac{2}{5} + \frac{2}{6}$$

$$\frac{n}{n^2+n} \le \frac{n}{n^2+j} \le \frac{n}{n^2+1}, \quad 1 \le j \le n$$

$$\Rightarrow n\left(\frac{n}{n^2+n}\right) \le x_n \le n\left(\frac{n}{n^2+1}\right)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$1 \qquad\qquad\qquad 1$$

$\therefore x_n \to 1$ $\quad \left\{ \frac{n^2}{n^2+n} = \frac{1}{1+\frac{1}{n}} \to 1 \right\}$

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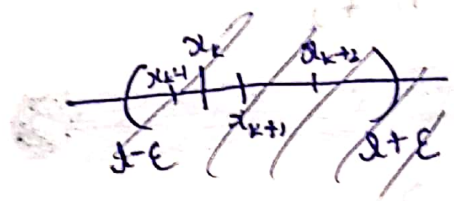
Definition : let (x_n) be a sequence of real no.
(x_n) converges to a real no. l, if
$\forall\ \varepsilon > 0,\ \exists\ K \in \mathbb{N}$, such that
$x_n \in (l-\varepsilon,\ l+\varepsilon),\ \forall\ n \geq K$
$(x_K, x_{K+1}, x_{K+2}, \ldots)$

In other words,



Result: Limit of a sequence is unique

Proof: Let l_1 & l_2 are two real no. s.t.
$(x_n) \to l_1$ & $(x_n) \to l_2$

Let $\varepsilon > 0$ be given
$\to$ By the definition of convergence of a sequence,
$\exists\ K_1, K_2 \in \mathbb{N}$

$\to \quad x_n \in \left(l_1 - \frac{\varepsilon}{2},\ l_1 + \frac{\varepsilon}{2}\right)\ \forall\ n \geq K_1$

$x_n \in \left(l_2 - \frac{\varepsilon}{2},\ l_2 + \frac{\varepsilon}{2}\right)\ \forall\ n \geq K_2$

Choose $K = \max\{K_1, K_2\}$ $(\because n \geq K_1,\ n \geq K_2)$ & so that both range will get covered

$\forall\ n \geq K,\ |l_1 - l_2|$

$$|l_1 - l_2| = |l_1 - x_n + x_n - l_2| \le |l_1 - x_n| + |l_2 - x_n|$$

$\{\because |a+b| \le |a| + |b|\}$ $\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ $\{\because \frac{\varepsilon}{2} > |x_n - l|$

$< \varepsilon$

$\therefore \boxed{l_1 = l_2}$

RESULT: Every convergent sequence is bounded

Prove not convergent.

Example: $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ (This seq. as it have finite term)

($1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} \cdots$ This is sereis as it have ∞ terms)

Solution: $x_1 = 1$

$x_2 = 3/2$

$x_3 = \frac{11}{6}$

$\vdots$

$\therefore$ Inc. seq.

$n = 2^k$

$x_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}$ $\{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2}\}$

$\{\frac{1}{3} + \frac{1}{4} > \frac{1}{2}\}$

$> 1 + \underbrace{\frac{1}{2} + \frac{\cancel{2}^{1}}{\cancel{4}_{2}} + \frac{\cancel{4}^{1}}{\cancel{8}_{2}} + \cdots \frac{1}{2}}_{k\text{ times}}$

$= 1 + \frac{k}{2}$

x_{2^k} goes to ∞ as k approaches to ∞

$\therefore \{x_n\}$ is not bounded, and hence it is not convergent

Result: Every convergent sequence is bounded.

Proof:

let $\varepsilon > 0$, since $(x_n) \to l$,

$\{x_k, x_{k+1}, \ldots\}$ is bounded below by $l-\varepsilon$ and above by $l+\varepsilon$

~~let m = min $\{x_1, \ldots, x_n, \ldots\}$ is bounded~~

let $m = \min\{x_1, \ldots, x_{k-1}\}$

$M = \max\{x_1, \ldots, x_{k-1}\}$

$\{x_1, \ldots, x_n, \ldots\}$ is bounded above by $\max\{M, l+\varepsilon\}$ and bounded below by $\min\{m, l-\varepsilon\}$

Example: $x_n = (-1)^n$

Sol.

- $y_k = x_{2k} = (-1)^{2k} = 1 \to$ sub-seq
- $z_k = x_{2k+1} = (-1)^{2k+1} = -1 \to$ sub-seq's

It is bounded but not convergent.

Definition:
monotonic

Eg: 1, 1,

To show
show,

Result: A
bo

Example:

We

(Be
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Definition: A sequence (x_n) of real no. is called <u>monotonically increasing</u> if

$$x_n \le x_{n+1}, \quad \forall\, n \in N$$

Eg: 1, 1, 1, 1.9, 1.99, 1.999, -----

To show, $x_n \le x_{n+1}$, $\forall\, n \in N$

show, either $\boxed{x_{n+1} - x_n \ge 0}$ or $\boxed{\dfrac{x_n}{x_{n+1}} \le 1}$

Result: A sequence (x_n) is monotonically inc. and bounded above, then it converges to the $\sup\{x_n : n \in N\}$

Example: $x_n = \left(1+\frac{1}{n}\right)^n$

We know that this sequence is bounded ($\because$ $2 \le x_n \le 3$, $\forall\, n \in N$)

$$x_n = \left(1+\frac{1}{n}\right)^n$$

(Bernwolli inequality
$(1+a)^n \ge 1+na$)

Bin. exp. :-

$$x_n = \left(1+\frac{1}{n}\right)^n$$

$$= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots + \frac{n(n-1)\dots 1}{n!}\,\frac{1}{n^n}$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \dots + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)}{n!}$$

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1 + (n+1)\left(\frac{1}{n+1}\right) + \frac{(n+1)n}{2!}\frac{1}{(n+1)^2} + \cdots + \frac{(n+1)n\ldots 2.1}{(n+1)!}\frac{1}{(n+1)^{n+1}}$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n+1}\right)}{2!} + \cdots + \frac{\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)}{n!} + \frac{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}{(n+1)!}$$

$$x_{n+1} \geq x_n, \quad \forall\, n \in \mathbb{N}$$

Definition: The sequence (x_n) is increasing (monotonically) if $x_n \leq x_{n+1}$, $\forall\, n \in \mathbb{N}$

i.e. $x_1 \leq x_2 \leq x_3 \leq \ldots$

Two ways to show this:

(i) $x_{n+1} - x_n \geq 0 \quad \forall n \in \mathbb{N}$

(ii) If $x_n \neq 0$, $\forall\, n \in \mathbb{N}$

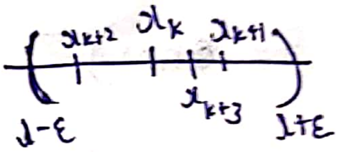
then $\boxed{\frac{x_{n+1}}{x_n} \geq 1}$, $\forall\, n \in \mathbb{N}$

Theorem (Monotone convergence theorem):

If (x_n) is monotonically increasing sequence of real no. which is bounded above, then (x_n) converges to the $\alpha = \sup\{x_n \mid n \in \mathbb{N}\}$

Proof: $(x_n) \to l \iff \forall\, \varepsilon > 0,\ \exists\, K \in \mathbb{N}$

s.t. $|x_n - l| < \varepsilon,\ \forall\, n \geq K$



$\alpha = \sup\{x_n : n \in \mathbb{N}\}$

that means,

$\forall\, \varepsilon > 0,\ \alpha - \varepsilon < x_n$, for some $n \in \mathbb{N}$

using this we will prove

$\forall\, \varepsilon > 0,\ \exists\, K \in \mathbb{N}$,

s.t. $\alpha - \varepsilon < x_K$

$\forall\, n \geq K$,

$$\alpha - \varepsilon < x_K \leq x_n \leq \alpha < \alpha + \varepsilon$$

$\therefore\ \forall\, n \geq K,\ \alpha - \varepsilon < x_n < \alpha + \varepsilon$

$\therefore\ l = \alpha$

Example: Take $x_1 = 2$ & $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), n \geq 1$

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n$$

$$= \frac{2 - x_n^2}{2x_n}$$

$x_{n+1} - x_n \geq 0 \iff 2 - x_n^2 \geq 0 \quad (\forall n \in N)$
$(\forall n \in N)$

(False for $n=1$)

$x_{n+1} - x_n \leq 0 \iff 2 - x_n^2 \leq 0 \quad (\forall n \in N)$
(x_n is dec.) $(\forall n \in N)$

$\iff x_n \geq \sqrt{2}, \forall n \in N$

To show $x_n \geq \sqrt{2}, \forall n \in N$

$x_1 = 2 \geq \sqrt{2}$

Suppose, $x_n \geq \sqrt{2}$ ~~then x_{n+1}~~

then $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{2}$

$\iff x_n^2 + 2 \geq 2\sqrt{2}x_n$

$\iff x_n^2 - 2\sqrt{2}x_n + 2 \geq 0$

$\iff (x_n - \sqrt{2})^2 \geq 0 \quad \forall n \in N$

This is always true

$\therefore$ seq. is mon. dec. seq. & bounded below ~~so it is convergent to~~ its infimum

(x_n) is d

the mono

x_n

If (x_n) -

$\therefore$ x_n

Theorem: (

Let x_n

Let λ

(i) If

(ii) I

(x_n) is decreasing and it is bounded below by the monotone convergence theorem,

x_n converges to $\inf \{x_n \mid n \in \mathbb{N}\}$

If $(x_n) \to l$, then $(x_{n+1}) \to l$

$\therefore$ $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ gives

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) \Rightarrow 2l = l + \frac{2}{l}$$

$$\Rightarrow l^2 + 2 - 2l^2 = 0$$

$$\Rightarrow l^2 = 2$$

$$\Rightarrow l = \pm\sqrt{2}$$

but terms of the seq. is +ve.

$\therefore$ $l = -\sqrt{2}$ is rej.

$\therefore$ $\boxed{l = \sqrt{2}}$

Theorem: (Ratio) (For +ve terms)

Let $x_n > 0$, $\forall\, n \in \mathbb{N}$

Let $\lambda = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$

(i) If $\lambda < 1$, then $(x_n) \to 0$ (i.e. x_n converges to 0)

(ii) If $\lambda > 1$, then $(x_n) \to \infty$

(i.e. x_n diverges to ∞)

Example :

(1) $x_n = \dfrac{n}{2^n}$

$\therefore x_{n+1} = \dfrac{n+1}{2^{n+1}}$

$$\lambda = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \times \frac{2^n}{n}$$

$$= \frac{1}{2} < 1$$

$\because \lambda < 1$

$\therefore x_n$ converges to 0

Example:

Let $b > 1$

$y_n = \dfrac{b^n}{n^2}$

$y_{n+1} = \dfrac{b^{n+1}}{(n+1)^2}$

$$\lambda = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} > 1$$

$\therefore \lambda > 1$

$\therefore x_n$ diverges to ∞

Note: $\frac{x_{n+1}}{x_n} < 1$, means the seq. is dec., it does not mean it conv. to 0

Example: let $0 < r < 1$ then $x_n = r^n$ converges to 0 as $n \to \infty$

i.e. (r^n) conv. to 0 as $n \to \infty$

Soln.

$r < 1$

$\frac{1}{r} > 1$, let $\underbrace{h}_{\text{i am defining}} := \frac{1}{r} - 1 > 0$

$\therefore r = \frac{1}{1+h}$

$\therefore 0 \le r^n = \frac{1}{(1+h)^n} \le \frac{1}{1+nh} \le \frac{1}{nh} = \left(\frac{1}{n}\right)\frac{1}{h}$

$\downarrow \ \downarrow \qquad\qquad \downarrow \qquad \downarrow$

$0 \ \ 0 \qquad\qquad 0 \qquad$ fixed no.

Using $\boxed{(1+h)^n \ge 1+nh}$

$\therefore (r^n)$ conv. to 0 (Using sandwich thm)

Cauchy Sequence

Let $(x_n) \to l$

$\forall \varepsilon > 0, \exists K \in N \Rightarrow |x_n - l| < \varepsilon \ \forall n \geq k$

that is

$x_n \to l$ as $n \to \infty$

Suppose $n, m \geq k$,

$$|x_n - x_m| = |x_n - l + l - x_m|$$
$$< |x_n - l| + |x_m - l| < \varepsilon + \varepsilon = 2\varepsilon$$

Definition: The sequence (x_n) of real no. is called <u>cauchy</u> if.

$\forall \varepsilon > 0, \exists K \in N$ such that

$$|x_n - x_m| < \varepsilon, \forall n, m \geq k$$

Result:

$$(x_n) \to l \Leftrightarrow \boxed{(x_n) \text{ is Cauchy}}$$

Definition: A sequence (x_n) of real no. is called <u>contractive</u> if $\exists \alpha \in R$, such that $0 < \alpha < 1$ satisfying

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, \text{ for } n = 1, 2, 3 \ldots$$

$\therefore \ |x_3 - x_2| \leq \alpha |x_2 - x_1| \quad (n=1)$

$\therefore \ |x_4 - x_3| \leq \alpha |x_3 - x_2| \quad (n=2)$

$\leq \alpha^2 |x_2 - x_1|$

$$\boxed{|x_{n+2} - x_{n+1}| \leq \alpha^n |x_2 - x_1|}$$

Result : Every contractive sequence is Cauchy.

Proof : If $n = m$, the cauchy condn. is trivial

$\therefore$ We may assume, $m > n$, assume

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m|$$

$$\leq |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \cdots + |x_{m-1} - x_m|$$

$$\leq \alpha^{n-1} |x_2 - x_1| + \alpha^{n} |x_2 - x_1| + \cdots + \alpha^{m-2} |x_2 - x_1|$$

$$= \alpha^{n-1} \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}\right) |x_2 - x_1|$$

$$= \alpha^{n-1} \left(\frac{1 - \alpha^{m-n}}{1 - \alpha}\right) |x_2 - x_1|$$

$$= \left(\alpha^{n-1} - \alpha^{m-1}\right) \frac{|x_2 - x_1|}{(1 - \alpha)}$$

$\downarrow$

converges to 0 as $n, m \to \infty$ $\quad$ ($\because 0 < \alpha < 1$) ($\alpha^n \to 0$)

$\therefore$ (x_n) is Cauchy sequence.

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Example: Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$, $\forall n \geq 1$

Is this sequence convergent. Also, if (x_n) is convergent, find the limit l.

Solution: $|x_{n+2} - x_{n+1}|$

$$= \left| \frac{1}{2+x_{n+1}} - \frac{1}{2+x_n} \right|$$

$$= \frac{|x_{n+1} - x_n|}{(2+x_{n+1})(2+x_n)}$$

$$\leq \frac{1}{4} |x_{n+1} - x_n|$$

$\therefore (x_n)$ is contractive ($\because$ We can choose $\alpha = \frac{1}{4}$)

$\therefore$ It is Cauchy

$\therefore$ It is convergent

$$l = \frac{1}{2+l}$$

$$\Rightarrow 2l + l^2 = 1$$

$$\Rightarrow l^2 + 2l - 1 = 0$$

$$l = \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= -1 \pm \sqrt{2}$$

$\therefore l = \sqrt{2} - 1$ (−ve case rej. as all terms are +ve)

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Definition: Let (n_k) be an increasing sequence of natural nos.

That is $n < n_2 < \dots < n_k < \dots$

The sequence (x_{n_k}) is called a <u>subsequence</u> of a given sequence (x_n)

Example: ① Let $(x_n) = \left(\frac{1}{n}\right)$

Let $n_k = 2K$

$$x_{n_k} = \frac{1}{n_k} = \frac{1}{2K} \longleftarrow \text{sub-sequence}$$

$K=1, \ \frac{1}{2}$

$K=2, \ \frac{1}{4}$

$\vdots$

Result: The sequence of real no.

$(x_n) \to l$. Then every subsequence (x_{n_k}) of (x_n) converges to l.

Example: $(x_n) = (-1)^n$

$n_k = 2K$

$\therefore (x_{n_k}) = (-1)^{2k} = 1 \longrightarrow 1$

$n_k = 2K+1$

$\therefore (x_{n_k}) = (-1)^{2k+1} = -1 \longrightarrow -1$

$(x_{n_k}) \to 1$, $(y_{n_k}) \to -1$, and (x_{n_k}) and (y_{n_k}) are subsequence of (x_n)

$\therefore$ From the previous result, (x_n) is divergent

$\left(\frac{1}{n}\right) \to 0$

$0 < r < 1$

$(r^n) \to 0$

$\exists\, \alpha$ s.t $0 < \alpha < 1 \Rightarrow$ $\boxed{|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, \forall\, n \in \mathbb{N}}$

this is condition of contractivity.

$$1 \xleftarrow{\text{converge}} \left| \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \right| \leq \alpha$$

$\boxed{|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|}$

Q) Give a sequence which is divergent but satisfy

Theorem (Bolzano - Weierstrass)

Every bounded sequence has a convergent subsequence

(Bounded above and bounded below both)

Example: $(x_n)=((-1)^n)$, x^n is bounded

$n_k = 2k$

$\therefore (x_{n_k}) \longrightarrow 1$ which is convergent

SERIES

Series is sum of infinitely many real numbers,
$x_1, x_2, x_3, \ldots$

NOTATION :

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots + x_n + \ldots$$

n^{th} partial sum of the series $\sum_{n=1}^{\infty} x_n$ is defined by S_n, where

$$S_n = x_1 + x_2 + \ldots + x_n$$

→ The series $\sum_{n=1}^{\infty} x_n$ is convergent if the sequence of n^{th} partial sum (S_n) converges.

let $0<r<1$

Example : $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + \ldots + r^n + \ldots$

$$S_n = x_1 + x_2 + \ldots + x_n$$

$$S_n = 1 + r + r^2 + \ldots + r^{n-1}$$

$$= \frac{1-r^n}{1-r}$$

$$\therefore \lim_{n\to\infty} S_n = \frac{1-0}{1-r} \quad (\because (r^n) \to 0 \text{ if } 0<r<1)$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r}$$

$$\therefore \sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + \dots + r^n + \dots = \frac{1}{1-r}$$

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S_n = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$(S_n) \to 1$ as $n \to \infty$

NOTE: (x_n) converges means (x_n) approaches to l

NOTE: If (x_n) approaches to ∞, here we say (x_n) diverges.

$(x_n) = (-1)^n$ diverges.

Definition:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \text{ is called series.}$$

The series $\sum_{n=1}^{\infty} a_n$ converges if

→ n^{th} partial sum of this series denoted by:
$S_n = a_1 + \cdots + a_n$ converges as a sequence.

Eg : $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Soln. : $a_n = \frac{1}{n(n+1)}$

$$\therefore S_n = a_1 + a_2 + \cdots + a_n$$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \longrightarrow 1 \text{ as } n \longrightarrow \infty$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \cdots + \frac{1}{n(n+1)} + \cdots = 1$$

Example: Let $0 < r < 1$, $\sum_{n=1}^{\infty} r^{n-1}$

Check this series is conv./diverg.

Soln. $S_n = 1 + r + r^2 + \dots + r^{n-1}$

$$= \frac{1 - r^n}{1 - r} \longrightarrow \frac{1}{1-r} \quad \text{as } n \to \infty$$

$$\therefore \sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$$

Remark: The series $\sum_{n=1}^{\infty} a_n$ is convergent means:

$a_1 + a_2 + \dots + a_n + \dots$ is a real no.

The series $\sum_{n=1}^{\infty} a_n$ diverges means either $a_1 + a_2 + \dots$ is either $\infty, -\infty$ or we cannot find that:

Eg = $1 + (-1) + 1 + (-1)$

Q: $x_n \geq 0$, $y_n \geq 0$

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \dots + \frac{1}{2^n}$$

$$\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{n \text{ times}}$$

$$y_n = 1 + \frac{n}{2}$$

$$\therefore x_n \geq y_n \quad \forall n \in \mathbb{N}$$

Since $y_n = 1 + \frac{n}{2} \to \infty$ as $n \to \infty$

$\therefore x_n \to \infty$ as $n \to \infty$ ($\because x_n \geq y_n, \forall n \in \mathbb{N}$)

Remark:

Let (a_n) and (b_n) be two sequences of non-negative numbers which are monotonically inc., and $a_n \leq b_n, \forall n \in \mathbb{N}$

Then,

① If (a_n) diverges to ∞, then (b_n) also diverges to ∞

② If (b_n) is convergent, then (a_n) is convergent.

Theorem:

If $a_n \geq 0, \forall n \in \mathbb{N}$ and (S_n) is bounded above, then $\sum_{n=1}^{\infty} a_n$ converges.

$$S_n = a_1 + \cdots + a_n$$

$$S_{n+1} = a_1 + \cdots + a_n + a_{n+1}$$

$$\boxed{a_{n+1} = S_{n+1} - S_n}$$

Solution:

By the definition of S_n,

$a_{n+1} = S_{n+1} - S_n$. Also $a_{n+1} \geq 0$

$\therefore S_{n+1} \geq S_n \; \forall n \in \mathbb{N}$

$\therefore$ By monotone convergence theorem sequence (S_n) converges, therefore $\sum_{n=1}^{\infty} a_n$ converges.

Theorem: (n^{th} term test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$ as $n \to \infty$

Points to be noted for this thm:
- The condition given is necessary but not sufficient i.e. It is possible that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.
- If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges bcz $a_n \not\to 0$

Proof:

Since $\sum_{n=1}^{\infty} a_n$ converges,

$(S_n) \to S$ as $n \to \infty$

Then, $(S_{n+1}) \to S$ as $n \to \infty$

$\therefore a_{n+1} = S_{n+1} - S_n \to 0$ as $n \to \infty$

(Never use this result in reverse direction)

Theorem: (comparison test)

Let $0 \leq a_n \leq b_n \quad \forall\, n \geq k$

If ① If $\sum_{n=1}^{\infty} b_n$ converges, then

$\sum_{n=1}^{\infty} a_n$ converges

② If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges

<u>Example:</u> Check the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

<u>Solution:</u> $a_n = \frac{1}{n^2} \le \frac{1}{n(n-1)}$

Let $b_n = \frac{1}{n(n-1)}$

$\therefore 0 \le a_n \le b_n \;,\; \forall\, n \geq K$ (as $n \neq 1$) (for the K tail only)

Since, $\sum_{n=2}^{\infty} b_n$ is converges

By comparison test,

$\sum_{n=2}^{\infty} a_n$ is convergent

$\therefore \sum_{n=1}^{\infty} a_n$ is convergent ($\because \sum_{n=2}^{\infty}$ is convergent)

<u>Theorem</u> (Limit Comparison test) for only +ve term series

Let $a_n \geq 0$ and $b_n > 0 \;\forall\; n \geq K$

consider $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

Then ① If L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges or diverges together.

Note: $\sum \frac{1}{n(n+1)}$ convergent

$\sum r^n$ convergent

$\sum \frac{1}{n}$ is divergent

$\sum \frac{1}{n^2}$ is convergent

(2) If $L=0$, and $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

(3) If $L=\infty$, and $\sum_{n=1}^{\infty} b_n$ diverges to ∞, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Note: $\sum_{n=1}^{\infty} r^n = 1 + r + r^2 + \dots + r^n + \dots$

If $0<r<1$, then this series is convergent

If $r=1$, this series is divergent

If $r>1$, this series is divergent.

(nth term test) $\sum_{n=1}^{\infty} a_n$ converges, if $(a_n) \to 0$

Ex: $\sum_{n=1}^{\infty} 1 + n \sin \frac{1}{n}$ $\qquad (\because |x| \geq 1)$

$a_n = 1 + n \sin \frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right) = 2$

$\therefore$ Series is divergent

Ex. $\sum_{n=1}^{\infty} 1-n\sin\frac{1}{n}$

$a_n = 1-n\sin\frac{1}{n}$

$b_n = \frac{1}{n}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-n\sin\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n\left(\frac{1}{n}-\sin\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}-\sin\frac{1}{n}}{\frac{1}{n^2}}$$

$$= \lim_{x\to 0^+} \frac{x-\sin x}{x^2}$$

$$= \lim_{x\to 0^+} \frac{1-\cos x}{2x}$$

$$= \lim_{x\to 0} \frac{\sin x}{2} = 0$$

But we have no conclusion. So it's just wastage of time, take $b_n = \frac{1}{n^2}$ ($\because \frac{1}{n}$ is not conv.)

$$L = \lim_{n\to\infty} \frac{1-n\sin\frac{1}{n}}{\frac{1}{n^2}} = \frac{n\left(\frac{1}{n}-\sin\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}-\sin\frac{1}{n}}{\frac{1}{n^3}}$$

$$= \lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

$L \in R^+$

$\therefore \Sigma a_n$ & Σb_n converges or diverges together

but $\Sigma \frac{1}{n^2}$ conv. $\therefore$ Σa_n conv.

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Theorem (Ratio Test) :

Suppose $a_n \neq 0$, $\forall n = k, k+1, \dots$

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Then ① If $L < 1$, then $\sum_{n=1}^{\infty} |a_n|$ is convergent.

② If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

③ If $L = 1$, we cannot concalude anything.

(if $a_n > 0$
$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$)

Example: Is the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ convergent ?

Solution: $a_n = \frac{(2n)!}{n!\,n!}$

$a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$

$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$

$= 4$

$\therefore \sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ is divergent by Ratio test.

Theorem (Root test):

Suppose $\forall\ n = k,\ k+1, \ldots$

Let $L = \lim_{n\to\infty} (a_n)^{1/n}$

Then,

(i) If $L < 1$, then $\sum_{n=1}^{\infty} |a_n|$ is convergent

(ii) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $L = 1$, we cannot calculate anything

Example: Is the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} \text{ convergent}$$

Soln.

$$a_n = \left(\frac{n+1}{n}\right)^{n^2}$$

$$L = (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n} = e > 1$$

This series is divergent, by root test.

1^{∞} form: $e^{\lim_{x\to\infty} x\left(\frac{x+1}{x} - x\right)}$

$e^{1} = 1$

Theorem (Cauchy condensation test)

If $a_n \geq 0$, $\forall\, n \in \mathbb{N}$ and $(a_n) \to 0$, then

" $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent ".

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$

$a_n = \frac{1}{n}$

$a_{2^k} = \frac{1}{2^k}$

$\therefore \sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \left(\frac{1}{2^k}\right) = \sum_{k=0}^{\infty} 1$

is divergent.

Example: for which values of p, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Soln. $a_n = \frac{1}{n^p}$

$a_{2^k} = \frac{1}{(2^k)^p}$

$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{(2^k)^{p-1}}$

For $p=1$, this series is divergent, therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the above test

For $p<1$, therefore $1-p>0$. Hence, $\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} (2^k)^{1-p}$

$$= \sum_{k=0}^{\infty} (2^k)^{1-p} = \sum_{k=0}^{\infty} (2^{1-p})^k$$

$r = 2^{1-p} > 1$

$\therefore$ Divergent

If $p > 1$,

then $\sum_{k=0}^{\infty} 2^k . a_{2^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k$

hence $1 > r = \frac{1}{2^{p-1}} > 0$

$\therefore$ The series convergent if $p > 1$

Directly use

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$
is convergent if $p > 1$

Q) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$p < 1$

$\therefore$ series is divergent.

Theorem (Leibniz Test):

Let (a_n) be a decreasing sequence which converges to 0. (i.e. $a_n > 0$)

Then the ~~test~~ alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$

$\left(\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + \dots\right)$ is convergent.

Theorem :

If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then

$\sum_{n=1}^{\infty} a_n$ is convergent.

(converse not true)
(we can't say anything if $\sum |a_n|$ is div.)

Example : $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\therefore a_n = \frac{(-1)^n}{n^2}$ $\therefore |a_n| = \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,

$\sum \frac{(-1)^n}{n^2}$ is conv. by prev. thm.

Alt

By Lebniz test,

$a_n = \frac{1}{n^2}$

$\therefore$ It is conv.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

We can use only Leibniz test not the other result.

Example: $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 3^n}$ for what x it is convergent.

Here $a_n = \frac{(x-1)^{2n}}{n^2 3^n}$

$$a_{n+1} = \frac{(x-1)^{2n+2}}{(n+1)^2\, 3^{n+1}}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{2n+2}}{(n+1)^2\, 3^{n+1}} \times \frac{n^2 3^n}{(x-1)^{2n}}\right|$$

$$= \lim_{n\to\infty} \frac{(x-1)^2}{3} \times \frac{n^2}{(n+1)^2}$$

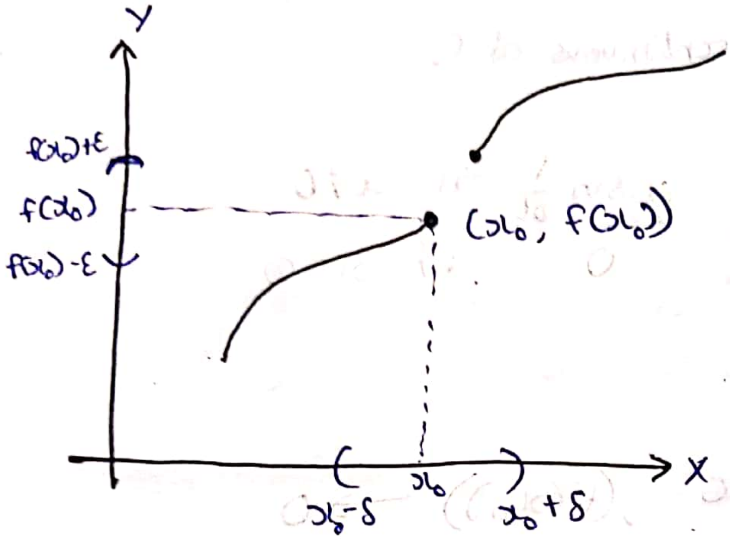
$$= \frac{(x-1)^2}{3} < 1 \iff (x-1)^2 < 3$$

$$\iff |x-1| < \sqrt{3}$$

$$1-\sqrt{3} < x < 1+\sqrt{3}$$

CONTINUITY OF FUNCTION

Function is discontinous [redacted] if $\exists \varepsilon > 0, \forall \delta > 0$,
" $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ "
fails.



$\therefore$ It is continuous if
$\forall \varepsilon > 0, \exists \delta > 0$, " $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ "

Definition:

The function $f: R \to R$ is continuous at point x_0 if
$(x_n) \to x_0 \Rightarrow (f(x_n)) \to f(x_0)$

Example: $f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

$= \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Check continuity at $x = 0$.

Proof:

We want to find a sequence $(x_n) \to 0$, such that $f(x_n) = (-1)^n$.

Take $x_n = \frac{(-1)^n}{n} \to 0$, $f(x_n) = (-1)^n$ which is divergent

$\therefore$ fn is discontinuous at 0

Example: $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

To Prove:

If $(x_n) \to 0$, $(f(x_n)) \to 0$

Proof:

(Note: $x_n \to 0 \Leftrightarrow |x_n| \to 0$, only if conv. to 0)

$$0 \le |f(x_n)| = \left|x_n \sin \frac{1}{x_n}\right| \le |x_n| \to 0$$

$\downarrow$

0

$\therefore |f(x_n)| \to 0$

$\therefore f(x_n) \to 0$

Q) ~~Find one seq. $x_n \to 0$ f.~~ Using seq. prove $\sin \frac{1}{x}$ is dist. at $x = 0$

Definition :

$f: \mathbb{R} \to \mathbb{R}$ is continuous at $x = x_0$, if $\forall\ \varepsilon > 0$,
$\exists\ \delta > 0$, s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$x \in (x_0 - \delta, x_0 + \delta)$

$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

or.

$\lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow$ whenever $(x_n) \to x_0 \Rightarrow (f(x_n)) \to f(x_0)$

Example: $f(x) = \begin{cases} 2x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Check if this function is cont. at $x = 0$.

In:

Take $x_0 = 0$

$\therefore f(x_0) = 0$

To prove : $\forall\ \varepsilon > 0, \exists\ \delta > 0$,
s.t. $|x| < \delta \Rightarrow |f(x)| < \varepsilon$

$\Rightarrow |2x \sin \frac{1}{x}| < \varepsilon$

Rough work

$|\sin \frac{1}{x}| < 1$

$|2x \sin \frac{1}{x}| < 2|x| < \varepsilon$ → The condition I propose

→ $|x| < \frac{\varepsilon}{2}$

$\therefore \delta = \frac{\varepsilon}{2}$

$\therefore \forall\ \varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$ s.t.

$|x| < \delta \Rightarrow |f(x)| = |2x \sin \frac{1}{x}| \leq 2|x| < \epsilon$

Hence proved

Example : Show that the function :

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational no.} \\ x & \text{if } x \text{ is a irrational no.} \end{cases}$$

is continuous only at $x = 0$. soln →

Result : Let c be any real number, then there exist a sequence of rational numbers converges to c.

Proof : For every ~~[illegible]~~ $n \in \mathbb{N}$

$\exists$ a rational no. r_n

s.t.

$$c < r_n < c + \frac{1}{n} \quad \text{(rational density)}$$

As $n \to \infty$,

$$c \leq r_n \leq c + \frac{1}{n} \to c$$

c

$$\boxed{\lim_{n \to \infty} r_n = c} \quad \text{(squeeze thm)}$$

Note

$\{1, 1.4, 1.41, 1.414, \ldots\}$ conv. to $\sqrt{2}$

ution :

Let c be any real no. where f is continuous
By the previous result, $\exists$ a sequence of rational number r_n s.t. $(r_n) \to c$

$\therefore$ By continuity definition,
$$(f(r_n)) \to f(c)$$

$\therefore$ $f(c) = 0$

$\therefore$ c is a rational no. (from definition of fn given)
($\because$ Using contrapositive)
$\therefore$ f is not contin. at irrationals.

Assume f is continuous at a rational no. c.

$\therefore$ There exists a sequence (x_n) of irrational no. which converges to c.
$\therefore$ $(f(x_n)) \to f(c)$
$\therefore$ $(x_n) \to f(c)$

$\therefore$ $f(c) = c$
$\therefore$ $\boxed{c=0}$
$\therefore$ f is not continous at non-zero rationals.

To show f is continuous at $x_0 = 0$,
we need to prove that $(x_n) \to 0 \Rightarrow f(x_n) \to 0$
$\underset{0}{\parallel}$ or $\underset{x_n}{=}$
both conv. to 0

This is true from the definition of the function.
H.P.

Eg: Show that the fn.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational no.} \\ 0 & \text{if } x \text{ is an irrational no.} \end{cases}$$

Note:

$f : R \to R$ is cont.
if $\varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \varepsilon \Rightarrow |f(x) - l| < \varepsilon$
$\hookrightarrow x \in (x_0 - \delta, x_0 + \delta)$

$x \neq x_0$

R

$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

or

$\lim_{x \to x_0} f(x) = l$

P_3

$\Leftrightarrow$ whenever $(x_n) \to x_0 \Rightarrow (f(x_n)) \to l$
$x_n \neq x_0$

Example: Show that the function

$f(x) = \frac{1}{x}$

$f : R - \{0\} \to R$

Solution: **Goal:** Find $(x_n) \to 0$, $(x_n \neq 0) \Rightarrow (f(x_n))$ diverges

Choose $x_n = \frac{1}{n} \to 0$

$f(x_n) = \frac{1}{x_n} = n$ diverges to ∞.

R

Q. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Check continuity of $f'(x)$.

$f'(0) = \lim_{x \to 0} x \sin\frac{1}{x} = 0$

Soln. We know that,

$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

We want to find $\lim_{x \to 0} f'(x) = \lim_{x \to 0} 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$\boxed{(x_n) \longrightarrow 0 \Rightarrow \cos\frac{1}{x_n} = (-1)^n}$

$\frac{1}{x_n} = n\pi$

$\cos(n\pi) = (-1)^n$

$x_n = \frac{1}{n\pi} \longrightarrow 0$

$f'(x_n) = 2x_n \sin\frac{1}{x_n} - \cos\frac{1}{x_n} \longrightarrow -(-1)^n$

$= (-1)^{n+1}$ which is divergent

$\therefore$ $f'(x_0)$ is not continuous.

~~Introduction~~

Definition: Let $f: S \to R$ be a function.

f has maximum value at $x = x_0 \in S$, if

$$f(x) \leq f(x_0) \quad \forall x \in S$$

Example: $f: (0,1] \to R$

$$f(x) = \frac{1}{x}$$

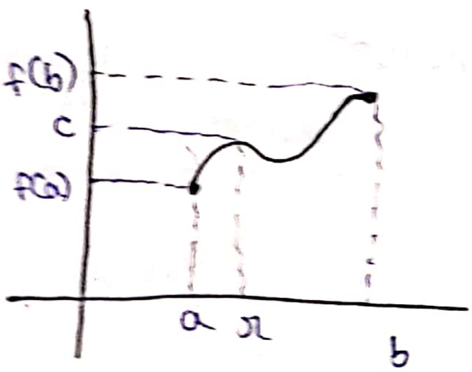
This fn. is continuous but $f(x)$ does not assume maximum.

Theorem: If $f: [a,b] \to R$ is a continuous function, f has maximum and minimum.

In other words....

If other words,

$$\underbrace{m}_{\text{min}} \leq f(x) \leq \underbrace{M}_{\text{Max}}, \quad \forall x \in [a,b]$$



Theorem: (Intermediate Value Property)

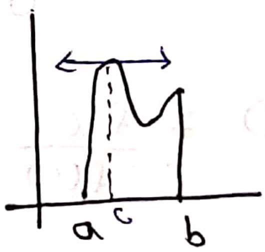
If $f:[a,b] \rightarrow R$ is continous and c is any point between $f(a)$ and $f(b)$, then $\exists\, x \in [a,b] \rightarrow f(x) = c$

Example: Show that there exists a point:
$x \in (0,1)$, s.t. $(1-x)\cos x = \sin x$

Sol.
Let $f : [0,1] \rightarrow R$
s.t. $f(x) = (1-x)\cos x - \sin x$

f is continous on $[0,1]$,
$f(0) = 1$ and $f(1) = -\sin 1$
$\therefore \exists\, c \in [0,1]$, s.t. $f(c) = 0$
$\Rightarrow (1-c)\cos(c) - \sin(c) = 0$

Theorem (Rolle's theorem):
Let $f:[a,b] \rightarrow R$ be a continous function which is differentiable on (a,b) and $f(a) = f(b)$, then $\exists\, c \in (a,b)$
s.t. $f'(c) = 0$



Q) $f(x) = x^{13} + 7x^3 - 5$ has exactly one real root.

Ans. $f(0) < 0$, $f(1) > 0$
$\therefore$ By I.V.P. $\exists\, x \in (0,1)$ s.t. $f(x) = 0$
$\therefore$ f has atleast one real root.

It is clear that f cannot have -ve roots.

Suppose a & b are two +ve roots of f.

$\therefore f(a) = f(b) = 0$

$\therefore$ By rolle's theorem,

$\exists c \in (a,b)$

s.t $f'(c) = 0$

$\therefore \exists c > 0$

s.t $f'(c) = 13c^{12} + 21c^2 = 0$

This is a contradiction

$\therefore$ This polynomial has atmost 1 real root

$\therefore$ It has exactly one real root.

Q) Let f and g be two continuous functions on $[a,b]$ and they are differentiable on (a,b) s.t. $f(a) = f(b) = 0$, then $\exists\ c \in (a,b)$, s.t.

$f'(c) + f(c)g'(c) = 0$

~~Soln~~

Note : $g'(c) = \dfrac{f'(c)}{f(c)}$

$\Rightarrow g(c) = -\ln(f(c))$

$\Rightarrow f(c) = e^{-g(c)}$

$\Rightarrow \boxed{f(c)e^{g(c)} = 1}$

Soln: Define $h: [a,b] \to R$
by
$h(x) = f(x) e^{g(x)}$

$h(a) = f(a) e^{g(a)} = 0$

Similarly
$h(b) = 0$

$\therefore$ Since f & g are cont. on $[a,b]$ and diff. (a,b), h is cont. on $[a,b]$ and diff. on (a,b).

By rolle's thm:

$h'(c) = 0$ for $c \in (a,b)$

$\therefore h'(c) = f'(c) e^{g(c)} + f(c) g'(c) e^{g(c)} = 0$

Mult. by $e^{-g(c)}$

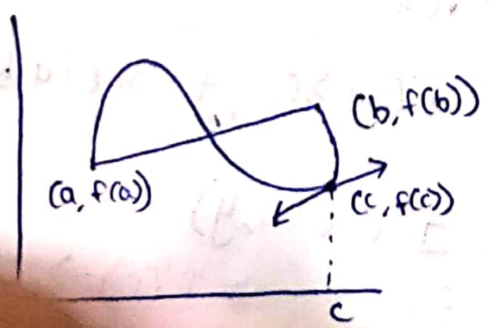
$\therefore \boxed{f'(c) + f(c) g'(c) = 0}$

H.P

Theorem: (MVT)

Let $f: [a,b] \to R$ be a continuous function and f is differentiable on (a,b).

Then $c \in (a,b)$ ~~s.t~~ $\frac{f(b) - f(a)}{b-a} = f'(c)$



Definition :

$f : [a,b] \to R$ is increasing whenever $f(x) \le f(y)$ if $x \le y$

Result : $f[a,b] \to R$, Let f be differentiable on (a,b)

Then, (1) $f'(x) \geq 0 \ \forall x \in (a,b) \Leftrightarrow f$ is increasing

(2) $f'(x) \le 0, \forall x \in (a,b) \Leftrightarrow f$ is decreasing on (a,b)

(3) $f'(x) > 0, \forall x \in (a,b) \Rightarrow f$ is strictly increasing on (a,b)

(4) $f'(x) < 0, \forall x \in (a,b) \Rightarrow f$ is strictly dec.

Note : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

PROOF :

(i) Let f be an increasing function on (a,b)

$\therefore \frac{f(x+h) - f(x)}{h} \geq 0$

$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$

Note : $h > 0 \Rightarrow f(x+h) \geq f(x)$

$h < 0 \Rightarrow f(x+h) \le f(x)$

Conversely, suppose $f'(x) \geq 0, \forall x \in (a,b)$

Choose $y \in (a,b)$

Then by MVT, $\exists \ c \in (a,y)$

s.t. $\frac{f(y) - f(a)}{\underbrace{y-a}_{>0}} = f'(c) \geq 0$

$\Rightarrow f(y) \geq f(a)$

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$\therefore f$ is increasing on (a,b)

Proof of ③:

Example: $f(x)=x^3$

$f(x)=x^3$ is strictly increasing, but $f'(x)=3x^2$

$\Rightarrow 3x^2=0 \Leftrightarrow x=0$

To prove ③ is not if & only if

Example: $f(x) = x - \sin x$ on $[0, \frac{\pi}{2}]$

$\therefore f'(x) = 1-\cos x \geq 0$

$f(x) \geq f(0) \quad \forall x \in (0, \frac{\pi}{2})$

$\therefore \sin x \leq x$

Theorem: (Cauchy MVT)

Let $f,g:[a,b] \longrightarrow R$ which are continuous.

Let f,g be differentiable on (a,b), s.t. $g'(x) \neq 0$, $\forall x \in (a,b)$,

then $\exists\, c \in (a,b)$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Example: $f(x) = 1-\cos x$, $g(x) = \frac{x^2}{2}$, $\frac{\pi}{2} > x > 0$

Consider the interval $[0,x]$

Soln: $\therefore \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(c)}{g'(c)}$, for some $0<c<x$

$$\therefore \frac{1-\cos x}{\frac{x^2}{2}} = \frac{\sin(c)}{c} \leq 1$$

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$$\therefore \; 1-\cos x \leq \frac{x^2}{2}$$

$$\therefore \; \boxed{1-\frac{x^2}{2} \leq \cos x}$$

<u>Example:</u> Prove that $\exists c \in (a,b)$

s.t. $$\frac{bf(a)-af(b)}{b-a} = f(c) - cf'(c)$$

<u>Soln.</u>

$$\frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{b-a}{ab}} = f(c) - cf'(c)$$

$$\Rightarrow \frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = f(c) - cf'(c)$$

$$F(x) = \frac{f(x)}{x}$$

$$G(x) = \frac{1}{x}$$

$$\Rightarrow \frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} = \frac{\frac{cf'(c)-f(c)}{c^2}}{\frac{-1}{c^2}}$$

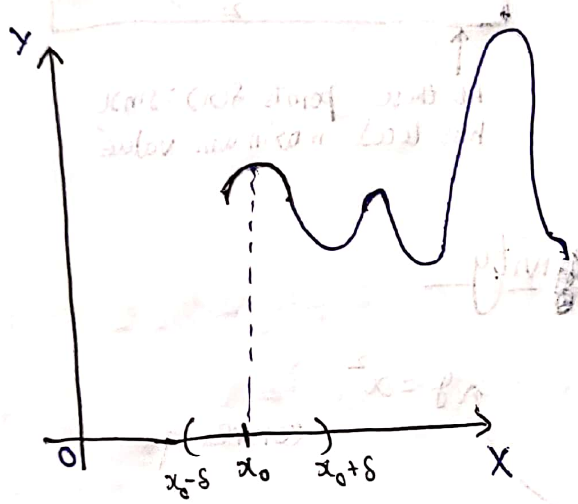
Prove that:

$|\sin x - \sin y| \le |x-y| \quad \forall\, x, y \in \mathbb{R}$

Definition: Let $f : S \to \mathbb{R}$ be a function. Here $S \subseteq \mathbb{R}$ which is not empty.

For example S is any interval,

This function has local maximum at $x_0 \in S$, if $\exists\, \delta > 0$ s.t. $f(x) \le f(x_0)$, $\forall\, x \in (x_0 - \delta, x_0 + \delta)$.



(Let f be a continuous function).

Theorem: When $f(x)$ is increasing in $(x_0 - \delta, x_0)$ and decreasing on $(x_0, x_0+\delta)$, then f has local maximum at $x = x_0$

Theorem: If f is differentiable on S and $\exists\, \delta > 0$, s.t. $f'(x) \ge 0$ on $(x_0 - \delta, x_0)$ and $f'(x) \le 0$ on $(x_0, x_0+\delta)$, then f has local maximum at $x = x_0$.

Theorem: If f is differentiable on S and f has local maximum at $x_0 \in S$, then $f'(x_0) = 0$.

Critical points: where the derivative is 0.

Theorem: (Second derivative test)

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has local maximum at $x = x_0$.

Example: $f(x) = \sin x$

$\therefore f'(x) = \cos x = 0$

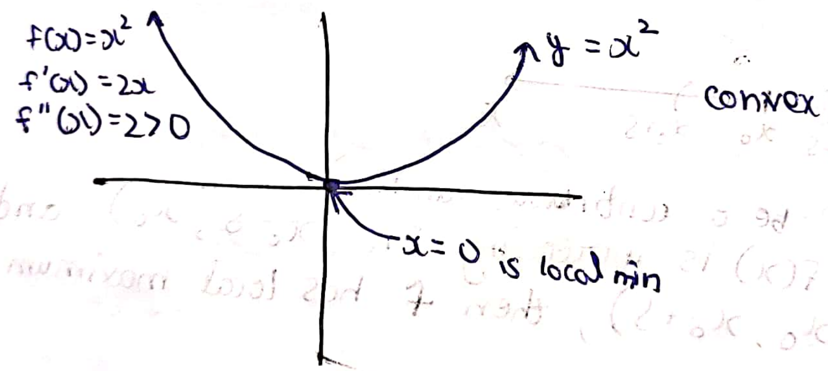
$\Leftrightarrow x = (2k+1)\frac{\pi}{2}, \ k \in \mathbb{Z}$ $\leftarrow$ critical pts.

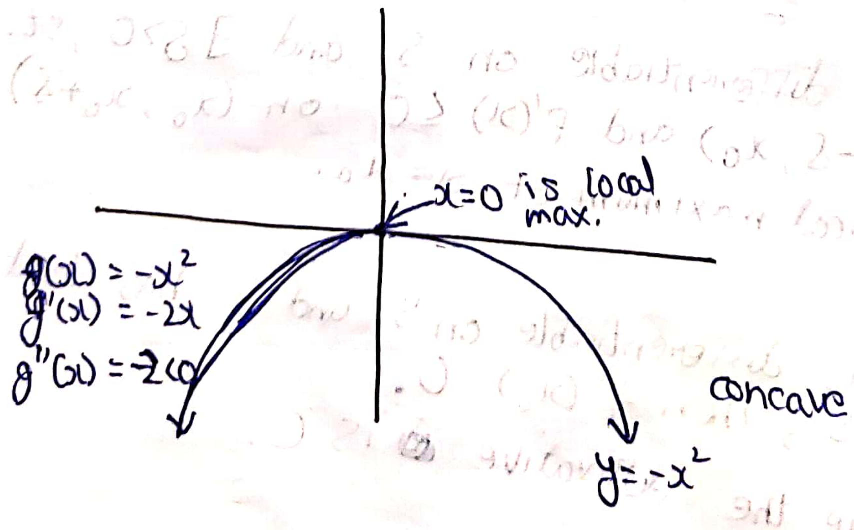
Here, $f''(x) = -\sin x$

$\therefore f''(x) < 0 \Leftrightarrow \boxed{x = 2n\pi + \frac{\pi}{2}, \ n \in \mathbb{Z}}$

At these points $f(x) = \sin x$ has local maximum value

Concavity & Convexivity





Definition:
Let $f: S \to R$ be a differentiable function. Also, f' is strictly increasing on an Interval I in S, then f is convex on I.

Theorem: If f is twice differentiable on (a,b) and $f''(x) > 0$, then f is convex in (a,b).

Definition: Let f be a function from S to R.
f has point of inflection at $x = x_0$, if f is either
• "concave on $(x_0 - \delta, x_0)$ and convex on $(x_0, x_0 + \delta)$"

or

• "convex on $(x_0 - \delta, x_0)$ and concave on $(x_0, x_0 + \delta)$"

Theorem: If f has point of inflection at $x = x_0$, then $f''(x_0) = 0$

Theorem: If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then $x = x_0$ is a point of inflection for f.

Ex: $f(x) = x^3$
$f'(x) = 3x^2$
$\left.\begin{aligned} f''(x) &= 6x = 0 \\ f'''(x) &= 6 \neq 0 \end{aligned}\right\}$ at $x = 0$
$\therefore$ $x = 0$ is pt of inflection

$f(x) = x^5$
$f'(x) = 5x^4$
$\left.\begin{aligned} f''(x) &= 20x^3 = 0 \\ f'''(x) &= 60x^2 = 0 \end{aligned}\right\}$ at $x = 0$
$\therefore$ We can't use previous thm.
$x < 0$, $x > 0$
$f'' < 0$, $f'' > 0$
$\therefore$ concavity to convexity
$\therefore$ $x = 0$ is pt. of inflection

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Q) Find pt where the fn is greater than 0 & less than zero. Find interval where it is inc. & dec.

Soln. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$\Rightarrow 4(x-1)(x-2)(x-3)$

$\therefore f'(x) \geq 0$

$x \in (1,2) \cup (3, \infty)$

$f'(x) \leq 0$

$x \in (-\infty, 1) \cup (2,3)$

Q) $\begin{cases} \sin x \leq x & \text{if } x \in [0, \pi/2] \\ \sin x \leq x & x \geq 1 \end{cases}$

$\rightarrow \sin x \leq x$ for all $x \geq 0$

Example: Using Cauchy Mean Value theorem:
Prove that : $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$

Solution: Assume $x > 0$

Consider $[0, x]$

Rough

$f(x) = \cos x$

$\frac{\cos x - \cos 0}{x - 0} = f'(c) = -\sin(c)$

$\therefore \frac{\cos x - 1}{x} = -\sin(c)$

$$g(x) = \frac{x^2}{2}$$

$$\boxed{\frac{\frac{x^2}{2} - 0}{x - 0} = g'(c) = c}$$

Apply Cauchy MVT with $f(x) = \cos x$, $g(x) = \frac{x^2}{2}$

$\therefore \exists\, c \in (0, x)$, s.t. $\dfrac{\cos x - 1}{\frac{x^2}{2} - 0} = \dfrac{-\sin(c)}{c}$

since $c > 0$, $\dfrac{\sin(c)}{c} < 1$

$$\therefore \frac{\cos x - 1}{\frac{x^2}{2}} > -1$$

$\therefore \boxed{\cos x > 1 - \frac{x^2}{2}}$ for $x > 0$ ——— (1)

Suppose, $x < 0$,

$\therefore -x > 0$

From (1),

we get,

$$\cos(-x) > 1 - \frac{(-x)^2}{2}$$

$\therefore \boxed{\cos x > 1 - \frac{x^2}{2}}$ for $x < 0$

$f(x) = \cos x - 1$

Example: Let $f:[a,b] \longrightarrow R$ be a continuous function. Then f is a constant function of $[a,b]$ $\Leftrightarrow f'(x)=0, \ \forall x \in (a,b)$.

Solution: $[\Rightarrow]$ is clear.

$[\Leftarrow]$ Assume $f'(x)=0, \ \forall x \in (a,b)$

Apply mean value theorem, on $[a,x]$, we get

$$\frac{f(x)-f(a)}{x-a} = f'(c) \text{ for some } a<c<x$$

$\therefore f(x)=f(a) \quad (\because f'(c)=0)$

$\therefore f(x)=f(a), \ \forall x \in (a,b]$.

$\therefore f$ is a constant function

Example: If $f'(x)>0$, $x \in (a,b)$ then f is one to one on (a,b).

Solution: If $f'(x)>0$,

$x<y \Rightarrow f(x)<f(y)$

$\therefore x \neq y \Rightarrow f(x) \neq f(y)$

$\therefore f$ is one-to-one function.

Curve Tracing

Example : $f(x) = x^3 - 6x^2 + 9x + 1$

Draw graph of $f(x)$.

Solution: $f'(x) = 3x^2 - 12x + 9 = 0$

$\Leftrightarrow \quad 3(x^2 - 4x + 3) = 0$

$\Leftrightarrow \quad 3(x-1)(x-3) = 0$

Interval	$(-\infty, 1)$	$(1,3)$	$(3,\infty)$
f'	+ve	-ve	+ve

$f''(x) = 6x - 12 = 0$

$\Rightarrow x = 2$

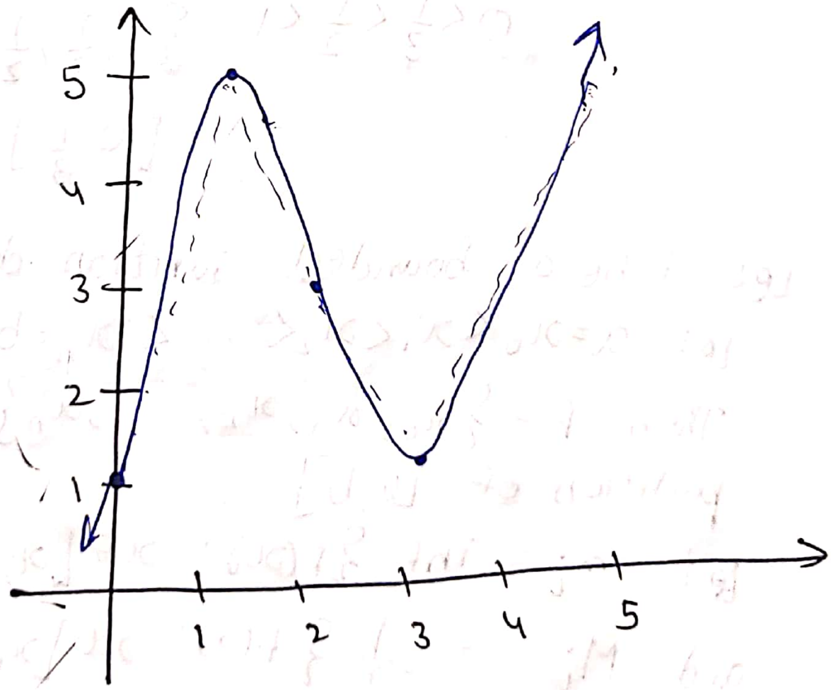
Interval	$(-\infty, 2)$ concave	$(2,\infty)$ convex
f''	-ve	+ve

$f(0) = 1$

$f(1) = 5$

$f(2) = 3$

$f(3) = 1$



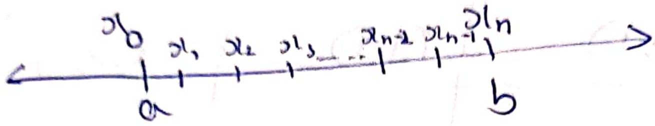
Reimann Integration

Q) $f : [0,1] \to R$

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

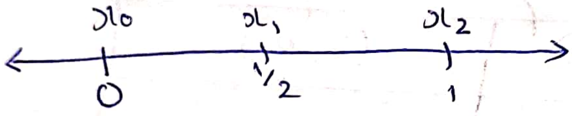
Soln. Assuming the fn is bounded.

Let f be a bounded function defined on $[a,b]$



$x_0 < x_1 < x_2 < \cdots < x_n$

Similarly:



$0 < \frac{1}{2} < 1$ $\{0, 1/2, 1\}$.

$0 < \frac{1}{3} < \frac{1}{2} < 1$ $\{\overset{x_0}{0}, \overset{x_1}{\frac{1}{3}}, \overset{x_2}{\frac{1}{2}}, \overset{x_3}{1}\}$

$[0,\frac{1}{3}], [\frac{1}{3},\frac{1}{2}], [\frac{1}{2},1]$

→ Let f be a bounded function defined on $[a,b]$
Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ are points.
Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partition of $[a,b]$
Let $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$
and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, $1 \le i \le n$

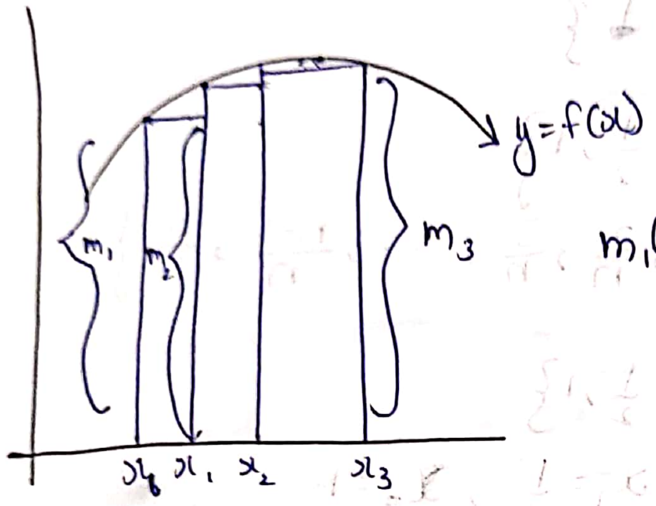
Lower Riemann Sum, $L(P,f) := \sum_{i=1}^{n} m_i \Delta x_i$ $(\Delta x = x_i - x_{i-1})$

Upper sum, $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ (where $\Delta x_i = x_i - x_{i-1}$)

Definition: Let $f [a,b] \to R$ be a bounded function

The set $P = \{a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b\}$ is a partition of $[a,b]$ if :

$x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n$



$m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2)$

$m_1 = \inf \{f(x) : x \in [x_0, x_1]\}$

The Upper Riemann Sum $= U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$

The lower Reimann Sum $= L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$

let $m_i := \inf \{f(x) ; x \in [x_{i-1}, x_i]\}$, $i = 1, 2, \dots n$

$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$, $i = 1, 2, \dots n$

$\Delta x_i = x_i - x_{i-1}$

Example:

① $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n}\}$ is a partition of $[0,1]$

② $P = \{0, \frac{1}{3}, 1\}$ is a partition of $[0,1]$

Example: find upper and lower Riemann Sum with respect to the function $f(x) = x^2$ on $[0,1]$ and the partition.

① $P_2 = \{0, \frac{1}{2}, 1\}$

② $P_3 = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$

③ $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n}\}$

Solution: ① $P_2 = \{0, \frac{1}{2}, 1\}$

$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$

$m_1 = \inf \{f(x) : x \in [x_0, x_1]\} = 0$

$m_2 = \inf \{f(x) : x \in [x_1, x_2]\} = \frac{1}{4}$

$$L(P_2, f) = \sum_{i=1}^{2} m_i \Delta x_i$$

$$= 0\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8} \approx 0.125$$

$M_1 = 1/4$

$M_2 = 1$

$$U(P_2, f) = \frac{1}{4}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8} \approx 0.625$$

$L(P_3, f) = 31/216 \approx 0.14$

(2) $U(P_3, f) = \frac{125}{216} \approx 0.578$

(3) $L(P_n, f) = \frac{1}{n}(m_1 + \dots + m_n)$

$= \frac{1}{n}\left(0 + \frac{1}{n^2} + \dots + \left(\frac{(n-1)}{n}\right)^2\right)$

$= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + (n-1)^2\right)$

$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} \frac{n(2n+1)(n+1)}{6n^3} = \frac{1}{3}$

$\lim_{n\to\infty} U(P_n, f) = \frac{1}{3}$

$\therefore \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$

$\therefore$ fn is integrable

Example: $f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$

$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\right\}$

$L(P_n, f) = 0$ $\quad \cancel{\exists\ n\ st.\ \frac{1}{n} < \varepsilon}$

$U(P_n, f) = 1\left(\frac{1}{n}\right) = \frac{1}{n}$

$\lim_{n\to\infty} \left(U(P_n, f) - L(P_n, f)\right) = 0$

$\boxed{\begin{array}{l} f \text{ is Riemann Integrable if} \\ \text{If } \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a,b] \\ \text{st. } U(P,f) - L(P,f) < \varepsilon \end{array}}$

Definition : Let $f : [a,b] \to R$ be a bounded function. Consider the lower Riemann integral.

$$\underline{\int_a^b} f(x)\, dx := \sup \{ L(P,f) : P \text{ is any partition of } [a,b] \}$$

Upper Riemann integral and

$$\overline{\int_a^b} f(x)\, dx := \inf \{ U(P,f) : P \text{ is any partition of } [a,b] \}$$

$\Rightarrow$ When $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)\, dx$, we say that f is RIEMANN INTEGRAL.

$\Rightarrow$ In this case, we write

$$\int_a^b f(x)\, dx = \underline{\int_a^b} f(x) dx = \overline{\int_a^b} f(x) dx$$

Example: $f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \le 1 \end{cases}$ on $[0,1]$

Solution: Let $P = \{ 0 = x_0, x_1, \dots, x_{n-1}, x_n = 1 \}$ be a partition of $[0,1]$

$\therefore 0 < x_1 < x_2 < \dots < x_{n-1} < 1$

$\therefore L(P,f) = 0$ and

$U(P,f) = 1(x_1 - x_0) = x_1$

$$\therefore \underline{\int_0^1} f(x) dx = 0$$

$\therefore \overline{\int_0^1} f(x)dx = \inf \{U(P,f) \mid P$ is a partition of $[0,1]\}$

$= \inf \{x_1 \mid 0 < x_1 \leq 1\}$

$= 0$ (We have to prove it)

Since $\underline{\int_0^1} f(x)dx = \overline{\int_0^1} f(x)dx = 0$

$\therefore f$ is integrable and $\int_0^1 f(x)dx = 0$

Definition: Let f be a bounded function.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$, be a partition of $[a,b]$

$L(P,f) = \sum m_i \Delta x_i$, where $m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$

$\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$

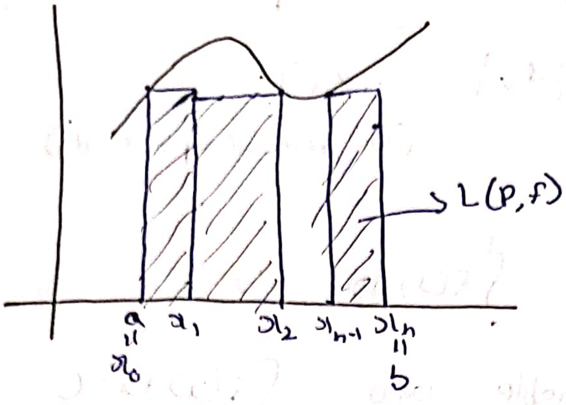
$\underline{\int_a^b} f(x)dx = \sup \{L(P,f) \mid P$ is any partition of $[a,b]\}$

Example: Let $f : [a,b] \to \mathbb{R}$

defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Solution: Let P be a partition of $[a,b]$, s.t. $P = \{a = x_0, x_1, \ldots, x_n = b\}$

For every subinterval $[x_{i-1}, x_i]$ by rational density and irrational density there are atleast one rational no. and one irrational no. belongs to $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$



$m_i = 0$

$M_i = 1 \quad \forall\, i = 1, 2, \dots n$

$L(P,f) = 0$

$U(P,f) = \sum_{i=1}^{n} \Delta x_i = b-a$

$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \text{ is any partion of } [a,b]\}$

$= \sup\{0 : P \text{ is any partition of } [a,b]\}$

$= 0$

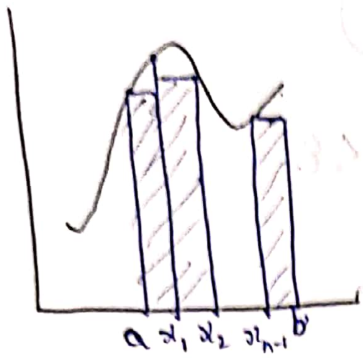
$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \text{ is any partion of } [a,b]\}$

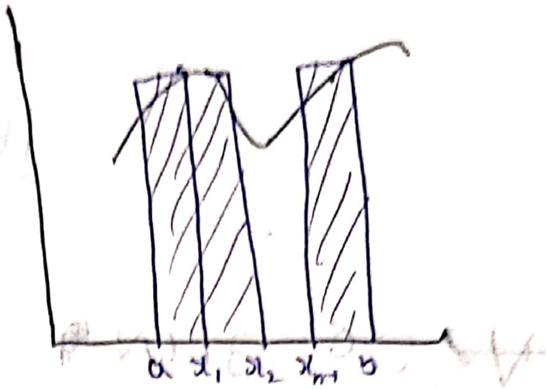
$= b-a$

Since $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$

$\therefore$ f is not Riemann integral

Theorem: Let $f: [a,b] \longrightarrow R$ be a bounded function, f is Riemann integral (if & only if) $\forall\ \varepsilon > 0$, $\exists$ a partition P such that $U(P,f) - L(P,f) < \varepsilon$





$(M_i - m_i)\Delta x_i$

$U(P,f) - L(P,f) < \varepsilon$

Example: Let $f(x) = x$. Show that f is Riemann integrable on $[a,b]$ $a < b$.

Sol. Let $P = \{x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a,b]$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= \sum_{i=1}^{n} x_i (x_i - x_{i-1})$$

$$\therefore U(P,f) - L(P,f) = \sum_{i=1}^{n} (x_i - x_{i-1})^2$$

This is we are choosing partition $\{a = x_0, a + \frac{b-a}{n}, \dots\}$

Assume, $x_i - x_{i-1} = \dfrac{b-a}{n}$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)^2$$

$$= n\left(\frac{b-a}{n}\right)^2$$

$$= \frac{(b-a)^2}{n} < \varepsilon$$

$\forall \varepsilon > 0, \exists n \in \mathbb{N}$

s.t. $\dfrac{(b-a)^2}{n} < \varepsilon$

__Example:__ $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x \neq \frac{1}{2} \end{cases}$$

__Solution:__ Consider $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$

$L(P_n, f) = 0$

$0 \leq U(P_n, f) \leq \frac{2}{n}$

$\therefore 0 \leq U(P_n, f) - L(P_n, f) \leq \frac{2}{n}$

There exists n s.t. $\frac{2}{n} < \varepsilon$

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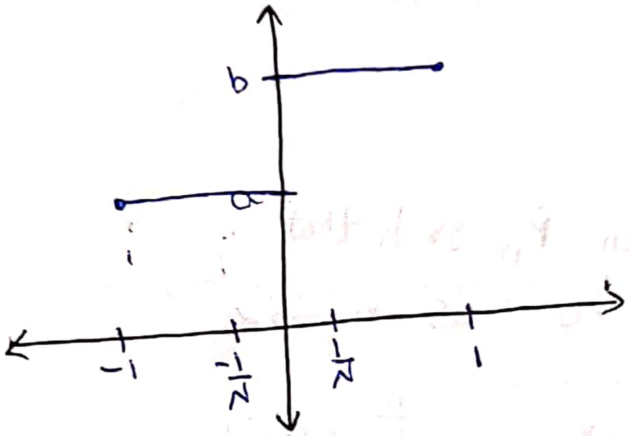
Example: Let $f:[-1,1]\to\mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} a & \text{if } -1\le x<0 \\ 0 & \text{if } x=0 \\ b & \text{if } 0<x\le 1. \end{cases} \quad (\text{where } a,b>0)$$

Prove that f is Riemann integrable.

Solution: Without loss of generality assume that $a<b$.



$$P = \left\{-1, -\frac{1}{N}, \frac{1}{N}, 1\right\}$$

$$L(P,f) = a\left(1-\frac{1}{N}\right) + b\left(1-\frac{1}{N}\right) + 0$$

$$U(P,f) = a\left(1-\frac{1}{N}\right) + b\left(1+\frac{1}{N}\right)$$

$$U(P,f) - L(P,f) = \frac{2b}{N}$$

This follows from archaemedian property that $\exists N\in\mathbb{N}$ s.t. $\frac{2b}{N}<\varepsilon$

Similarly, when $b>a$

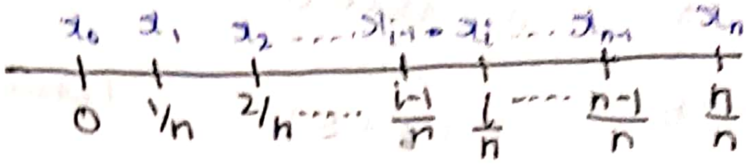
& Similarly, when $a=b$

$\therefore$ f is Riemann integrable

H.W. Do on interval [1,2]

Example: $f(x) = x^2$ on $[0,1]$

Solution:



$$U(P_n, f) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \longrightarrow \frac{1}{3} = \int_0^1 x^2\,dx$$

$$= \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1}{n}$$

Definition: Choose partition P_n such that
$\max\limits_{1\le i\le n} \Delta x_i \longrightarrow 0$ as $n \longrightarrow \infty$

Then $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\, \frac{b-a}{n}$$

where c_i is either left end point of $[x_{i-1}, x_i]$ or right end-point of $[x_{i-1}, x_i]$

Example: find $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{n}{k^2+n^2}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^2}{k^2+n^2} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1} \times \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \times \frac{1}{n}, \quad \text{where } f(x) = \frac{1}{x^2+1}$$

$\therefore$ Partition is $\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n}\}$

$$= \int_0^1 f(x)\,dx = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$$

Result: All continuous functions are Riemann Integrable. & We can use Riemann sum on Remann Sum method.

Theorem: (First Fundamental theorem of Calculus)

If $f:[a,b] \longrightarrow R$ be an integrable function, define

$$F(x) := \int_a^x f(t)\,dt \quad ; \quad a \le x \le b$$

then:

$F(x)$ is continuous on $[a,b]$ and if f is continuous at $x = x_0$, then $F'(x_0) = f(x_0)$

Example: $f(x) = x$ on $[0,1]$

$$F(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \frac{x^2}{2}$$

$$F'(x) = x = f(x), \quad \forall x \in [0,1]$$

Example: Let f be a continuous function on $[0, \frac{\pi}{2}]$ and $\int_0^{\pi/2} f(t)\,dt = 0$. Then $\exists\, c \in (0, \pi/2)$ s.t. $f(c) = 2\cos 2c$.

Solution: we want $g'(c) = f(c) - 2\cos 2c = 0$

Consider $g(x) = \int_0^x f(t)\,dt - \sin 2x$

$\therefore \quad g(0) = g(\frac{\pi}{2}) = 0$,

$\therefore$ By Rolle's theorem

$\exists\, c \in (0, \pi/2)$ s.t. $g'(c) = 0$ $\quad \therefore$ we shown what we wanted

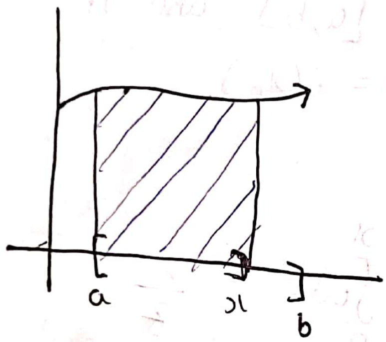
Theorem (Second fundamental theorem) : If $f'(x)$ is integrable on $[a, b]$, then $\int_a^b f'(x)\,dx = f(b) - f(a)$

Example: Show that $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4} = \frac{1}{3}$

Solution

Rough

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$



$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4}\,dt$$

$$= \lim_{x \to 0} \frac{\int_0^x \frac{t^2}{1+t^4}\,dt}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{1+x^4}}{3x^2} = \frac{1}{3}$$

E: $\int_1^\infty f(x)\,dx = \lim_{x\to\infty} \int_1^x f(t)\,dt$

$$\sum_{k=1}^{\infty} a_n = \lim_{n\to\infty} \underbrace{\left(\sum_{k=1}^{n} a_n\right)}_{S_n}$$

IMPROPER INTEGRAL OF FIRST KIND

inition:

- Let f be a function which is integrable on $[a, b]$ where $x > a$
- Then the improper integral $\int_a^\infty f(x)\,dx$ converges if $\lim_{x\to\infty}\left(\int_a^x f(t)\,dt\right)$ exist
- whenever this limit does not exist (either limit does not exist or limit is ∞ or $-\infty$) we say that this improper integral diverges.

mple: For which value of $p \in \mathbb{R}$ the integral $\int_1^\infty \frac{1}{t^p}\,dt$ is convergent?

tion: $\int_1^\infty \frac{1}{t^p}\,dt := \lim_{x\to\infty} \int_1^x \frac{1}{t^p}\,dt$

Case 1: $(p=1)$

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{x\to\infty}\int_1^x \frac{1}{t}\,dt = \lim_{x\to\infty}(\log x) = \infty$$

In this case the integral is divergent.

case 2: $p>1$

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{x\to\infty}\left(\frac{t^{-p+1}}{-p+1}\right)_{t=1}^{t=x}$$

$$= \lim_{x\to\infty}\left(\frac{x^{1-p}-1}{1-p}\right) \quad 1-p<0$$

$$= \frac{1}{p-1}$$

In this case, the improper integral is convergent and value of the integration is $\frac{1}{p-1}$

case 3: $p<1$

In this case, the improper integral is divergent

Example: (i) $\int_0^\infty te^{-t^2}\,dt$

$$= \lim_{x\to\infty}\int_0^x te^{-t^2}\,dt$$

$$= \lim_{x\to\infty}\left[-\frac{1}{2}e^{-t^2}\right]_0^x$$

$$= \lim_{x\to\infty}\left(-\frac{1}{2}e^{-x^2}+\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

(ii) $\int_0^\infty \sin t \, dt$

$= \lim_{x\to\infty} \int_0^x \sin t \, dt$

$= \lim_{x\to\infty} (1 - \cos x)$

This limit does not exist, therefore the improper integral is divergent.

Theorem: (comparison test)

let $0 \leq f(t) \leq g(t)$, $\forall\ t \geq a$

And $\int_a^\infty g(t)$ is convergent, then $\int_a^\infty f(t)\,dt$ is convergent

Example: Check the convergence of

(i) $\int_1^\infty \frac{\cos^2 t}{t^2} \, dt$ (ii) $\int_1^\infty \frac{2+\sin t}{t} \, dt$

(i) $\frac{\cos^2 t}{t^2} \leq \frac{1}{t^2}\ \forall\ t \geq 1$

$\therefore \int_1^x \frac{\cos^2 t}{t^2} \, dt \leq \int_1^x \frac{1}{t^2} \, dt$

Since $\int_1^\infty \frac{1}{t^2} \, dt$ is convergent,

by comparison test, $\int_1^\infty \frac{\cos^2 t}{t^2} \, dt$ is convergent.

(ii) $\int_1^\infty \frac{1}{t} \leq \int_1^\infty \frac{2+\sin t}{t}$

$\therefore$ By comparision test, this integral is divergent.

Theorem: (Limit Comparison test)

Suppose $f(x) \geq 0$ and $g(x) > 0$, $\forall\ x \geq a$

Let $c = \lim_{x\to\infty} \frac{f(x)}{g(x)}$

(i) if $c > 0$, then $\int_a^\infty f(t)\,dt$ and $\int_a^\infty g(t)\,dt$ converges or diverges together.

(ii) if $c = 0$, and $\int_a^\infty g(t)\,dt$ is convergent, then $\int_a^\infty f(t)\,dt$ is convergent

Example: (i) $\sin\frac{1}{x}$

Here $f(t) = \sin\frac{1}{t}$

Take $g(t) = \frac{1}{t}$

$c := \lim_{x\to\infty} \frac{f(x)}{g(x)}$

$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$= 1 > 0$

$\therefore$ By limit comparision test,

$\int_{1}^{\infty} \sin\frac{1}{x}$ is divergent ($\because \int_1^\infty \frac{1}{t}\,dt$ is divergent)

(ii) $\int_1^\infty e^{-t} t^p \, dt$, let $p \in \mathbb{R}$

$f(t) = e^{-t} t^p$

$g(t) = t^{-p}$

$c := \lim_{t \to \infty} \frac{f(t)}{g(t)}$

$= \lim_{t \to \infty} e^{-t} t^{2p}$

$= \lim_{t \to \infty} \frac{t^{2p}}{e^t}$

$p = 0 \Rightarrow c = 0$

$p < 0 \Rightarrow c = 0$

$p > 0 \Rightarrow c = 0$

$\therefore$ This integral is convergent for $p > 1$

if $f(t) = e^{-t} t^{-p}$

$g(t) = t^{-p}$

then integral is also conv. for $p \leq 1$

IMPROPER INTEGRAL OF 2ND TYPE

Definition:

Suppose the function f is discontinuous at $x = a$ and if it is integrable on $(x, b]$ $\forall$ $a < x \leq b$.

Then the improper integral:

$$\int_a^b f(x)\,dx = \lim_{x \to a^+} \int_x^b f(x)\,dx$$

Example: Find $p \to \int_0^1 \frac{1}{t^p}\,dt$ is convergent.

Solution:

$$\int_0^1 \frac{1}{t^p}\,dt = \lim_{x \to 0^+} \int_x^1 \frac{1}{t^p}\,dt = \lim_{x \to 0^+} \left[\frac{t^{1-p}}{1-p}\right]_x^1$$

$$= \lim_{x \to 0^+} \frac{1}{1-p}\left[1 - x^{1-p}\right]$$

$$= \frac{1}{1-p} \quad \text{if } 1-p > 0 \Leftrightarrow \boxed{p < 1}$$

For $p \geq 1$,

it is divergent.

Example: Determine the value of p for which $\int_0^\infty \frac{1-e^{-x}}{x^p}\,dx$ is convergent.

Solution: We will first find values of p s.t.

$$\int_0^1 \frac{1-e^{-x}}{x^p}\,dx \text{ is convergent}$$

Here $f(x) = \dfrac{1-e^{-x}}{x^p}$ $\qquad \left[\lim_{x \to 0} \dfrac{1-e^{-x}}{x} = 1\right]$

Take $g(x) = \dfrac{1}{x^{p-1}}$

$\therefore \lim_{x \to 0^+} \dfrac{1-e^{-x}}{x^p} \Big/ \dfrac{1}{x^{p-1}} = 1$

$\therefore$ By limit comparison test $\int_0^1 \dfrac{1-e^{-x}}{x^p} \, dx$ converges

$\Leftrightarrow \int_0^1 \dfrac{1}{x^{p-1}} \, dx$ converges

→ is convergent if $p-1 < 1$
$\Rightarrow p < 2$

Now, Consider $\int_1^{\infty} \dfrac{1-e^{-x}}{x^p} \, dx$

$\therefore f(x) = \dfrac{1-e^{-x}}{x^p}$. Take $g(x) = \dfrac{1}{x^p}$

$\therefore \lim_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim_{x \to \infty} 1-e^{-x} = 1$

By limit comparison test $\int_1^{\infty} \dfrac{1-e^{-x}}{x^p} dx$ converges if $\boxed{p > 1}$

$\therefore \int_0^{\infty} \dfrac{1-e^{-x}}{x^p} \, dx$ converges if $1 < p < 2$ or $p \in (1,2)$

Theorem:
If $\int_a^{\infty} |f(x)| \, dx$ is convergent then $\int_a^{\infty} f(x) \, dx$ is converg.

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Example: Show that $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is absolutely convergent.

Solution: $\int_1^\infty \left|\frac{\sin^2 x}{x^2}\right| dx \leq \int_1^\infty \frac{1}{x^2}\,dx$

By comparision test.

since $\int_1^\infty \frac{1}{x^2}dx$ is conv.

$\therefore \int_1^\infty \left|\frac{\sin^2 x}{x^2}\right| dx$ is convergent.

Dirichlet Test for improper integral:

Let $f, g : [a, \infty) \longrightarrow R$ be functions such that:

(i) f is monotonically decreasing on $[a, \infty)$ and $\lim_{x\to\infty} f(x) = 0$

(ii) g is a continuous function such that $G(x) := \int_a^x g(t)\,dt$ is bounded for all $x \geq a$

$\boxed{\text{i.e. } |G(x)| \leq M, \forall\, x \geq a}$

Then $\int_a^\infty f(x)\, g(x)\, dx$ is convergent.

Example: Show that $\int^\infty \frac{\sin x}{x}\,dx$ is convergent but it is not absolutely convergent.

Solution: Let $f(x) = \frac{1}{x}$ and $g(x) = \sin x$

(i) clearly, $f(x)$ is decreasing on $[1, \infty)$ and $\lim_{x\to\infty} \frac{1}{x} = 0$

(ii) $|G(x)| = \left|\int_1^x \sin x\,dx\right|$

$= |\cos 1 - \cos x| \leq 1 + |\cos 1|$

Using Dirichlet theorem,

$\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent.

Now $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$

$|\sin x| \leq 1 \Rightarrow \sin^2 x \leq |\sin x|$

$\Rightarrow \int_1^t \left|\frac{\sin x}{x}\right| dx \geq \int_1^t \frac{\sin^2 x}{x}\,dx = \int_1^t \frac{1-\cos 2x}{2x}\,dx$

$= \underbrace{\int_1^t \frac{1}{2x}\,dx}_{\text{divergent}} - \underbrace{\int_1^t \frac{\cos 2x}{2x}\,dx}_{\text{(it is conv. by dirichlet test \& prove it.)}}$

$\underbrace{\int_1^t \left|\frac{\sin x}{x}\right| dx + \int_1^t \frac{\cos 2x}{2x}\,dx}_{\text{conv.}} \geq \underbrace{\int_1^t \frac{1}{2x}\,dx}_{\text{div.}}$

By comp. test.

not abs. conv.

3 3 3
10 11

9: (8 1/2)

TAYLOR'S ~~[illegible]~~ THEOREM

Let f be a fn on $[a,b]$. Let $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ exists on $[a,b]$.

Suppose $f^{(n)}$ is continuous on $[a,b]$ and differentiable on (a,b). Let $x_0 \in [a,b]$. Then for any $x \neq x_0$, $\exists$ c between x & x_0 s.t.

$$f(x) = \overbrace{f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{f^{n}(x_0)(x-x_0)^n}{n!}}^{\text{Taylor's Polynomial}}$$
$$+ \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Remainder}}$$

$n=0 \Rightarrow f(x) = f^{0}(x_0) + f'(c)(x-x_0)$

$n=1 \Rightarrow f(x) = f(x_0) + f^{1}(x_0)(x-x_0) + \frac{f''(c)(x-x_0)^2}{2!}$

<u>Taylor's Polynomial</u>:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

Example: $f(x) = e^x$

$x_0 = 0$.

$$P_n(x) = f(0) + f'(0)x + \ldots + \frac{f^{n}(0)x^n}{n!}$$
$$= 1 + x + \ldots + \frac{x^n}{n!}$$

$P_n(1) = 1+1+\frac{1}{2!} + \ldots + \frac{1}{n!}$

$P_1(1) = 1+1 = 2$

$P_2(1) = 1+1+\frac{1}{2} = 2.5$

Example: Suppose f is 3 times differentiable function on $[-1,1]$ such that $f(-1)=0$, $f(1)=1$, $f'(0)=0$. Using taylor's theorem prove that $\exists c \in (-1,1)$, s.t. $f'''(c) \geq 3$.

Solution: By taylor's theorem, for $x_0=0$, we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(c)}{6}x^3$$

$\boxed{\exists c_1 \in (0,1)}$

for $x=1$, $f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(c_1)}{6}$

for $x=-1$, $f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f'''(c_2)}{6}$

$\boxed{\exists c \in (-1,0)}$

$$\Rightarrow f(1) - f(-1) = 2f'(0) + \frac{f'''(c_1)+f'''(c_2)}{6}$$

$$\Rightarrow 6 = f'''(c_1) + f'''(c_2)$$

$\exists\, c \in (-1,1)$, s.t $\boxed{f'''(c) \geq 3}$

$\boxed{1+\alpha x < (1+x)^\alpha}$

Example: For $x>-1$, $x \neq 0$, prove that

(a) $(1+x)^\alpha > 1+\alpha x$, when $\alpha<0$ or $\alpha > 1 \Rightarrow \alpha(\alpha-1)>0$

(b) $(1+x)^\alpha < 1+\alpha x$, when $0<\alpha<1 \Rightarrow \alpha(\alpha-1)<0 \rightarrow 1+\alpha x > (1+x)^\alpha$

Solution: $f(x) = (1+x)^\alpha \Rightarrow f'(x) = \alpha(1+x)^{\alpha-1}$

$$\Rightarrow f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

$\therefore$ By taylor's thm, with $x_0=0$ & $n=1$

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2!}x^2$$, where c is b/w 0 & x

$$\therefore (1+x)^\alpha = 1+\alpha x + \alpha(\alpha-1)(1+c)^{\alpha-2}\frac{x^2}{2!}$$

Top

Example: Using Taylor's theorem, for any $k \in \mathbb{N}$ and $x > 0$, show that

$$x - \frac{x^2}{2} + \dots - \frac{1}{2k} x^{2k} < \log(1+x) < x - \frac{x^2}{2} + \dots + \frac{1}{2k+1} x^{2k+1}$$

NOTE:

Solution: $f(x) = \log(1+x)$

$\Rightarrow f'(x) = \frac{1}{1+x}$

$\Rightarrow f''(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow f'''(x) = \frac{2}{(1+x)^3}$, ...

$\Rightarrow f^{n}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

REMARK

Defin

$f(0) = 0 \Rightarrow f'(0) = 1, \Rightarrow f''(0) = -1,$

$\Rightarrow f'''(0) = 2 \Rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$\therefore \log(1+x) = x - \frac{x^2}{2!} + \dots + \frac{(-1)^{n-1}(n-1)!}{n!} x^n$

$+ \frac{(-1)^n n!}{(n+1)!} f'''(c)$

REMA

$n = 2k$

& $n = 2k+1$

$$x - \frac{x^2}{2} + \dots - \frac{1}{2k} x^{2k} < \log(1+x) < x - \frac{x^2}{2} + \dots + \frac{1}{2k+1} x^{2k+1}$$

NOTE: $f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

$$\lim_{x \to a} f(x) = L$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. when $0 < |x-a| < \delta$, we have $|f(x) - L| < \varepsilon$

REMARK:

When $(x_n) \longrightarrow a \quad (x_n \neq a)$

$\Rightarrow f(x_n) \longrightarrow L$

Definition:

Let $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

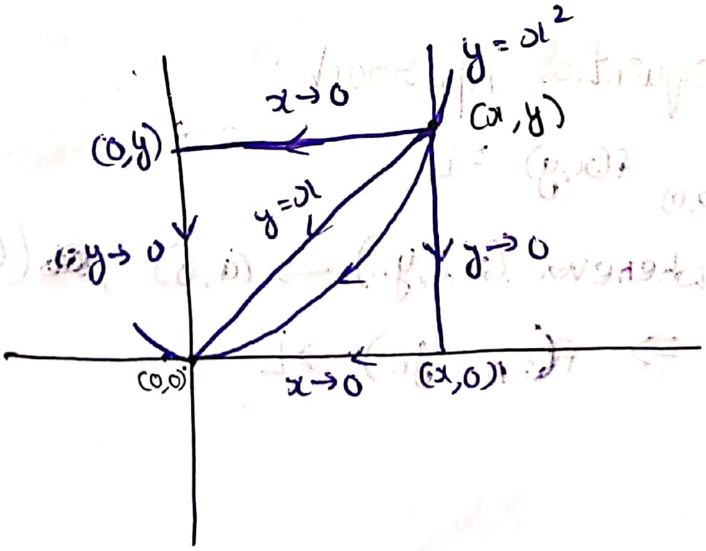
$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if $\forall \varepsilon > 0, \exists \delta > 0$, s.t. whenever $0 < |(x,y) - (a,b)| < \delta$

$\Rightarrow |f(x,y) - L| < \varepsilon$

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

REMARK:



$(x,y) \longrightarrow (0,0)$

Example: Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ where

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2-y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^2-y^2}$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2+y^2}{x^2-y^2}\right) = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2+y^2}{x^2-y^2}\right) = -1$$

Along these two paths, we are getting different ~~values~~ Limits.

∴ This Limit does not exist

REMARK: Sequential Approach:

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$

$\Leftrightarrow$ whenever $(x_n, y_n) \to (a,b)$, $((x_n,y_n) \neq (a,b))$

$\Rightarrow f(x_n, y_n) \to L$

Remark:

$$(x_n, y_n) \longrightarrow (a,b)$$

$$\Leftrightarrow \left| (x_n, y_n) - (a,b) \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Leftrightarrow \sqrt{(x_n - a)^2 + (y_n - b)^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Leftrightarrow |x_n - a| \longrightarrow 0 \text{ and } |y_n - b| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Leftrightarrow (x_n) \longrightarrow a \text{ and } (y_n) \longrightarrow b$$

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find $\lim\limits_{(x,y) \to (0,0)} f(x,y)$, if it exist.

Solution:

Rough

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y=mx}} \frac{x^2 y}{x^2+y^2} = \lim_{x \to 0} \frac{mx^3}{x^2 + m^2 x^2} = 0$$

& other paths limit $= 0$

$\therefore$ Limit may exist

We want to show that,

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

if $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$

s.t. whenever $0 < |(x,y) - (0,0)| < \delta$

$$\Rightarrow \left| \frac{x^2 y}{x^2+y^2} - 0 \right| < \varepsilon$$

We want to show that $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. whenever $0 < \sqrt{x^2+y} < \delta$

we have, $\left|\dfrac{x^2 y}{x^2+y^2}\right| < \varepsilon$

Take $\varepsilon = \dfrac{\delta}{2}$

$\therefore$ $\left|\dfrac{x^2 y}{x^2+y^2}\right| \leq \dfrac{|x|}{2}$ $\left(\because \left|\dfrac{xy}{x^2+y^2}\right| \leq \dfrac{1}{2},\ (x-y)^2 \geq 0\right)$

$\Rightarrow \left|\dfrac{x^2 y}{x^2+y^2}\right| \leq \dfrac{|x|}{2} < \dfrac{\delta}{2} = \varepsilon$ $\left(\because |x| < \sqrt{x^2+y^2} < \delta\right)$

Alt. Want to show $(x_n, y_n) \to (0,0)$ $((x_n, y_n) \neq (0,0))$

$\Rightarrow f(x_n, y_n) \to 0$ as $n \to \infty$.

$\therefore$ We want to show

$(x_n) \to 0$ and $(y_n) \to 0$

$\Rightarrow f(x_n, y_n) = \dfrac{x_n^2 y_n}{x_n^2+y_n^2} \to 0$

as $n \to \infty$

Consider,

$0 \leq \left|\dfrac{x_n^2 y_n}{x_n^2+y_n^2}\right| \leq |x_n| \left|\dfrac{x_n y_n}{x_n^2+y_n^2}\right|$

$\leq \dfrac{|x|}{2}$ $\left(\because \left|\dfrac{x_n y_n}{x_n^2+y_n^2}\right| \leq \dfrac{1}{2}\right)$

$$\therefore \quad 0 \leq \left| \frac{x_n^2 y_n}{x_n^2 + y_n^2} \right| \leq \frac{|x_n|}{2} \to 0 \text{ as } n \to \infty$$

$\therefore$ By Squeeze theorem, the limit exists and it is 0.

Q) (i) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$

(ii) $\lim_{(x,y)\to(0,0)} (x^2+y^2) \sin\left(\dfrac{1}{x^2+y^2}\right)$

Solution (i) $y = mx$; $my = x$

$$\lim_{x\to 0} \frac{m^2x^3}{x^2 + m^4x^4} = 0$$

$x = y^2$

(ii) $\lim_{y\to 0} \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

$\therefore$ Limit does not exist

Solution (ii)

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) \sin\left(\frac{1}{x^2+y^2}\right)$$

Here limit exist (by chking various paths) & then use squeeze thm.

NOTE:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Limit exist

ϵ-δ

$(x_n, y_n) \to (a,b)$

$\Rightarrow f(x_n, y_n) \to f(a,b)$

Limit does not exist

Find two paths along that you get two diff. limit.

Q. $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{xy^3}{x^2+y^6}$$

$$= \lim_{x\to 0} \frac{m^3 x^4}{x^2+m^6x^6}$$

$$= \lim_{x\to 0} \frac{m^3 x^2}{1+m^6x^4} = 0$$

AM-GM Inequality :-

$$xy^3 \leq \frac{x^2+y^6}{2}$$

$$x^2 = y^6$$

$$x = y^3$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y^3}} \frac{xy^3}{x^2+y^6}$$

$$= \lim_{y\to 0} \frac{y^6}{x^2+y^6} = \frac{1}{2}$$

Q $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$$g(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

NOTE: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Definition:

Let $f : \mathbb{R} \to \mathbb{R}$,

→ The partial derivative wrt x is defined by:

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Another notation ~~for~~ for f_x is $\dfrac{\partial f}{\partial x}$

→ The partial derivative wrt y is defined by

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} := \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k}$$

Another notation ~~for~~ is $f_y(x_0,y_0)$

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Example: Let $f(x,y) = x^2 + 2xy - y^2$. find $f_x(1,1)$ & $f_y(1,1)$.

Solution: $f_x(x,y) = 2x + 2y$ (Alt: Do by definition)

$f_y(x,y) = 2x - 2y$

$\therefore f_x(1,1) = 4$

& $f_y(1,1) = 0$

Example: Let $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Note: $f_{xy}(x,y) = \lim_{k \to 0} \dfrac{f_x(x,y+k) - f_x(x,y)}{k}$

Solution:

$f_x(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{0-0}{h}$

$= 0$

$f_y(0,0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0$

Let $(x,y) \neq (0,0)$

$f_x(x,y) = \dfrac{\partial}{\partial x}\left(xy\dfrac{x^2-y^2}{x^2+y^2}\right)$

$= y\left(\dfrac{x^2-y^2}{x^2+y^2}\right) + xy\dfrac{\partial}{\partial x}\left(\dfrac{x^2-y^2}{x^2+y^2}\right)$

$= y\left(\dfrac{x^2-y^2}{x^2+y^2}\right) + xy\left(\dfrac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2}\right.$

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$$= y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{4xy^2}{(x^2+y^2)^2}\right)$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\right)$$

$$= \left(\frac{(x^3-3xy^2)(x^2+y^2)-2y(yx^3-xy^3)}{(x^2+y^2)^2}\right)$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$\left[f_x(0,k) = k\left(\frac{-k^2}{k^2}\right) = -k\right]$$

$$\therefore\ f_{xy}(0,0) = \lim_{k\to 0}\frac{-k - 0}{k} = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

$$\left[f_y[h,0] = \frac{(h^3)(h^2)}{h^4} = h\right]$$

$$\therefore\ f_{xy}(0,0) \neq f_{yx}(0,0)$$

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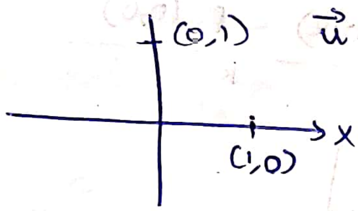
$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{f((a,b)+h(1,0)) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

$$= \frac{f((a,b)+k(0,1)) - f(a,b)}{k}$$

$\vec{u} = (u_1, u_2)$, $|\vec{u}| = 1$



$$D_{\vec{u}}(a,b) = \lim_{h\to 0} \frac{f((a,b)+h(u_1,u_2)) - f(a,b)}{h}$$

Directional derivative

Example: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

$\therefore$ $f_x(0,0) = 0$

$f_y(0,0) = 0$

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition: f is differentiable at (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{h^2+k^2}} = 0$$

Q) $f(x,y) = x^2 + y^2$. Check f is continuous at point (a,b)

$$\lim_{(h,k)\to(a,b)} = \frac{f(2a,2b) - f(a,b) - 2a^2 - 2b^2}{\sqrt{a^2+b^2}}$$

$$= \frac{4a^2 + 4b^2 - 2a^2 - 2b^2 - 2a^2 - 2b^2}{\sqrt{a^2+b^2}}$$

$$= 0$$

~~$\therefore$ f is continuous at pt (a,b)~~

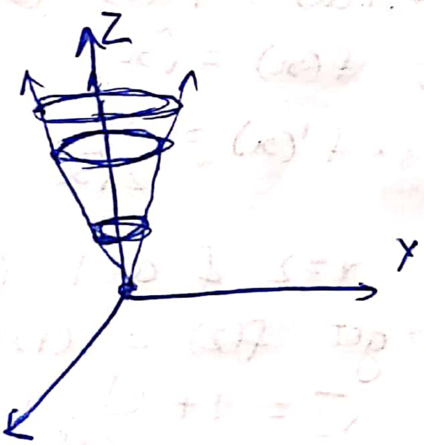
Alt

$$\lim_{(h,k)\to(0,0)} \frac{(a+h)^2 + (b+k)^2 - (a^2+b^2) - 2ha - 2kb}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2} = 0$$

Graph of $z = x^2 + y^2$

$z = f(x,y)$



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Example: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$, find pt s.t. $f_x(a,b)=0$ and $f_y(a,b)=0$

Solution: $f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

$f_x(a,b) = 0 \Rightarrow 3a^2 - 3 = 0$

$\Rightarrow a = \pm 1$

$f_y(a,b) = 0 \Rightarrow 3b^2 - 12 = 0 \Rightarrow b = \pm 2$

$(a,b) = (1,2), (1,-2), (-1,2), (-1,-2)$

Result:

$$\frac{f(b)-f(a)}{b-a} = f'(c), \quad c \text{ is b/w } a \text{ \& } b.$$

MVT (i) $f(b) = f(a) + f'(c)(b-a)$

Taylor's (ii) $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(c)$,

where c is b/w a & x

Example: Find $\sqrt{2}$ using M.V.T

Solution: $f(x) = f(a) + f'(c)(x-a)$, c is b/w x & a.

Here, $f(x) = \sqrt{x}$

$\therefore f'(x) = \frac{1}{2\sqrt{x}}$

Take $x=2$ & $a=1$ (Interval $(1,2)$)

we get $f(2) = f(1) + \frac{1}{2\sqrt{c}}$

$\sqrt{2} = 1 + \frac{1}{2\sqrt{c}}$ and $c \in (1,2)$

$1 < \sqrt{c} < \sqrt{2}$ and we get

$$\frac{1}{2\sqrt{2}} < \sqrt{2} - 1 < \frac{1}{2}$$

$$1 + \frac{1}{2\sqrt{2}} < \sqrt{2} < \frac{3}{2}$$

$$\frac{4}{3} < \sqrt{2} < \frac{3}{2}$$

Q) Find $\sqrt{2}$ using Taylor's Expansion :-

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$, $f''(x) = \frac{-1}{4x\sqrt{x}}$

Take $x = 2$ & $a = 1$ (Interval $(1, 2)$)

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{-1}{4c\sqrt{c}}\right) \quad \& \quad 1 < c < 2$$

$$-8\left(\sqrt{2} - \frac{3}{2}\right) = \frac{1}{c\sqrt{c}}$$

$$1 < c < 2$$

$$1 < \sqrt{c} < \sqrt{2}$$

$$1 < c\sqrt{c} < 2\sqrt{2}$$

$$\frac{1}{2\sqrt{2}} < -8\left(\sqrt{2} - \frac{3}{2}\right) = \frac{1}{c\sqrt{c}} < 1$$

$\frac{-1}{16\sqrt{2}} + \frac{3}{2} > \sqrt{2} >$

$\left(\quad - \sqrt{2} > -\frac{3}{2} \right.$

$\frac{3}{2} - \frac{1}{16\sqrt{2}} < \frac{-2}{3\times 16} + \frac{3}{2}$

$= \frac{36-1}{24}$

$\left. = \frac{35}{24} \right)$

$\therefore \boxed{\frac{35}{24} > \sqrt{2} > \frac{11}{8}}$

Example: $f(x,y) = \sin(x^2 y)$

$f_x = 2xy \cos(x^2 y)$

$f_y = x^2 \cos(x^2 y)$

Example: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Show that $f_{xx} f_{yy} - f_{xy^2}$ is +ve at $(1,2)$

$f_x = 3x^2 - 3$

$f_{xx} = 6x$

$f_{xx}(1,2) = 6$

$f_y = 3y^2 - 12$

$f_{yy} = 6y$

$f_{yy}(1,2) = 12$

$f_{xy} = 0$

$f_{xy^2} = 0$

$(6x)(6y) - 0 = 6(12) = 72$

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Differentiability

Definition:

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

f is differentiable at $(a,b) \in D$ if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}} = 0$$

Gradient: $\nabla f(a,b) := (f_x(a,b), f_y(a,b))$

Example: $f(x,y) = \sqrt{|x y|}$

Check diffentiability of f at $(0,0)$

Solution: $f_x(0,0)$ & $f_y(0,0) = 0$

For differentiability at $(0,0)$,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0$$

(Using) $\left(|h| \leq \sqrt{h^2+k^2}\right)$

$$\left(0 \leq \frac{|hk|}{\sqrt{h^2+k^2}} \leq |k| \quad \text{\& squeeze thm}\right)$$

$\therefore$ Fn is differentiable.

Q. $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Check for differentiability.

Solution:

$$\cancel{f_x(0,0) = \lim \frac{3x^2(x^2+y^2) - 2x(x^3-y^3)}{(x^2+y^2)^2}}$$

$$\cancel{= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \frac{-k}{k} = -1$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h + k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\dfrac{h^3-k^3}{h^2+k^2} - h + k}{\sqrt{h^2+k^2}}$$

Iterated limits :

$$= \frac{h^3 - k^3 - h^3 + k^3 + kh^2 - hk^2}{\sqrt{h^2+k^2}}$$

$$= \frac{kh^2 - hk^2}{(h^2+k^2)^{3/2}}$$

$$= \frac{kh(h-k)}{(h^2+k^2)^{3/2}}$$

For $K = 2h$

limits are non zero

& for other path limit is 0.

∴ It is not differentiable

$(u_1,u_2) = (0,1) = j$ $\quad f_y(x_0,y_0) = \lim_{k\to 0} \dfrac{f(x_0, y_0+k) - f(x_0,y_0)}{h}$

Definition :

$f : D \subseteq R^2 \longrightarrow R$

Let $(x_0,y_0) \in D$ be an interior point

let $u = (u_1,u_2)$ be a unit vector, that is $u_1^2+u_2^2=1$

$$D_u f(x_0,y_0) = \lim_{t\to 0} \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t}$$

f is differentiable at (x_0,y_0) if

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

Theorem: Let $f : D \subseteq R^2 \longrightarrow R$ be a fn and (x_0,y_0) be an interior point of D. If f is differentiable at (x_0,y_0), then $D_u f(x_0,y_0)$ exists for every $u = (u_1,u_2)$ and $D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot u$

$$D_u f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0)) \cdot (u_1,u_2)$$
$$= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

· It is path dependent, ∴ It is not continuous.
∴ It is not differentiable.

Find the directional derivative of f at $(0,0)$ in the direction of $(1,1)$.

Solution: Here $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$D_u f(0,0) := \lim_{t \to 0} \frac{f\left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)}{t}$$

$$\cancel{= \lim_{t \to 0} \frac{\frac{t^3}{2\sqrt{2}}}{t^3}} \cancel{= \frac{1}{2\sqrt{2}}}$$

$$= \frac{\left(\frac{t}{\sqrt{2}}\right)^3}{\left(\frac{t}{\sqrt{2}}\right)^4 + \left(\frac{t}{\sqrt{2}}\right)^2} \times \frac{1}{t}$$

$$= \lim_{t \to 0} \frac{\frac{1}{2\sqrt{2}}}{\frac{t^2}{4} + \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$f_x(0,0) = 0$

$f_y(0,0) = 0$

$\therefore \nabla f(0,0).u = 0$

$(f_x(0,0), f_y(0,0)).u = (0,0).u = 0$

$\therefore D_u f(0,0) \neq \nabla f(0,0).u$.

$\therefore f$ is not differentiable

Q) $f(x,y) := \begin{cases} \dfrac{x^3 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Find the directional derivative of f at $(0,0)$ in the direction of unit vector (u_1, u_2).

$$D_{u} f(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2)}{t}$$

$$= \frac{(tu_1)^3 \cdot tu_2}{(tu_1)^4 + (tu_2)^2}$$

If $u_2 = 0$, then $u_1 = 1$

$D_{(0,1)} f(0,0) = 0$

If $u_2 \neq 0$

$$D_u f(0,0) := \lim_{t \to 0} \frac{t^2 u_1^3 u_2}{t^2 u_1^4 + u_2^2} = 0$$

$$\therefore f_x(0,0) = 0 \text{ and } f_y(0,0) = 0$$

$$\nabla_f (0,0) = 0$$

$$D_u f(0,0) = \nabla f(x_0, y_0) \cdot u = 0$$

Now, checking another way:

$$\lim_{(h,k) \to (0,0)} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k) \to (0,0)} \frac{h^3 k}{h^4 + k^2} \times \frac{1}{\sqrt{h^2 + k^2}}$$

for $h = k$

$$\lim_{h \to 0} \frac{h^4}{h^4 + h^2} \times \frac{1}{\sqrt{2h^2}} = 0$$

for $h = k^2$

$$\lim_{h \to 0} \frac{h^5}{h^4 + h^4} \times \frac{1}{\sqrt{h^2 + h^4}}$$

$$\lim_{h \to 0} \frac{h^4}{2h^4} \times \frac{1}{\sqrt{1 + h^2}}$$

$$= \frac{1}{2}$$

Theorem: (Sufficient condition for differentiability)

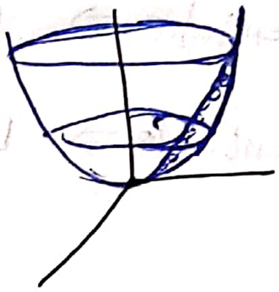
→ Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

→ Let (x_0, y_0) is contained in a disk $U \subseteq D$

→ If f_x, f_y exists in U, and if f_x is continuous or f_y is continuous at (x_0, y_0), then f is differentiable.

For parabolic surface

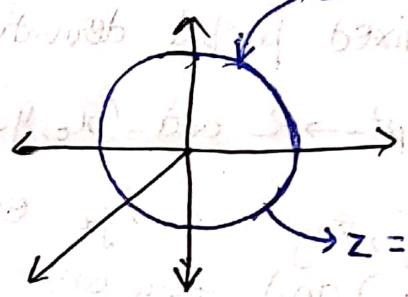
$z = x^2 + y^2$



For spherical surface

$x^2 + y^2 + z^2 = 1$

$\Rightarrow z = \pm\sqrt{1 - x^2 - y^2}$



Note: $D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u = |\nabla f(x_0, y_0)| \cos\theta$

$|u| = |(u_1, u_2)| = 1$

That is $u_1^2 + u_2^2 = 1$

$a = (a_1, a_2)$

$b = (b_1, b_2)$

$a \cdot b = |a||b| \cos\theta$, $\quad 0 \leq \theta \leq \pi$

$\dfrac{a \cdot b}{|a||b|} = \cos\theta$

$\cos(0) = 1$

$\cos(\pi) = -1$

$\cos\left(\frac{\pi}{2}\right) = 0$

For rapidly inc.

$a \cdot b = |a|$, if $b = \dfrac{a}{|a|}$

For rapidly dec.

$a \cdot b = -|a|$, if $b = -\dfrac{a}{|a|}$

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$\therefore \nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$

At $\theta = 0$,

$D_u f(x_0, y_0) = |\nabla f(x_0, y_0)| \longrightarrow f$ is incr. rapidly

if $u = \dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$

At $\theta = \pi/2$,

$D_u f(x_0, y_0) = 0 \longrightarrow f$ does not change

At $\theta = \pi$

$D_u f(x_0, y_0) = -|\nabla f(x_0, y_0)| \longrightarrow f$ is decr. rapidly

if $u = -\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$

$\nabla f(x_0, y_0) . u = 0$

$(f_x(x_0, y_0), f_y(x_0, y_0)) . (u_1, u_2) = 0$

$\Rightarrow f_x . u_1 + f_y . u_2 = 0$

if $u = \pm \dfrac{(f_y(x_0, y_0) - f_x(x_0, y_0))}{|\nabla f(x_0, y_0)|}$

Q) $f(x, y) = 4 - x^2 - y^2 \quad (x_0, y_0) = (1, 1)$

Find unit vector, along which $D_u f$ is rapidly inc, dec. & does not change