The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #1

(Real Number System, Sequences)

Q1. Let x be a real number such that x^2 is irrational. Show that x is also irrational. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.

Q2. Using the result "Let $m \in \mathbb{Z}, n \in \mathbb{N}$ and p be a prime. If $p|m^n$, then $p|m$." show that

1. $\sqrt{p}$ is irrational for any prime p.
2. $\sqrt{15}, \sqrt[3]{2}, \sqrt[5]{16}$ are irrational.

Q3. Find the infimum and supremum (if exists) of the sets $S_1 = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}, S_2 = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Q4. Using Archimedean property of real numbers show that for any $a \in \mathbb{R}$, there is some $m, n \in \mathbb{N}$ such that $-m < a < n$.

Q5. Use the Archimedean property of real numbers to show that $\displaystyle\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi$.

Q6. Let $a_n \to a$ and $a \neq 0$. Then show that there is $m \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq m$.

Q7. Investigate the convergence/divergence of the following sequences:
(a) $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}$
(b) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$
(c) $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0,1)$
(d) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$
(e) $x_n = \frac{n!}{(2n+1)!!}$.

Q8. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.

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MATH-I ■ Solutions Assignment #1

(Real Number System, Sequences)

Q1. Let x be a real number such that x^2 is irrational. Show that x is also irrational. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.

Ans. Let if possible x is rational then $x = \frac{m}{n}$ for some $m, n \in \mathbb{Z}, n \neq 0 \implies x^2 = \frac{m^2}{n^2}$ is also rational which is a contradiction.
$(\sqrt{2}+\sqrt{3})^2 = 5+\sqrt{6}$. Now if $5+\sqrt{6} = \frac{m}{n} \in \mathbb{Q}$ then $\sqrt{6} = \frac{m-5n}{n} \in \mathbb{Q}$, which is not true.

Q2. Find the infimum and supremum (if exists) of the sets $S_1 = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}, S_2 = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Ans. Let us first consider the set $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$. Note that $0 < \frac{m}{m+n} < 1$. We guess that $\inf = 0$ because $\frac{1}{1+n}$ is in the set and it approaches 0 when n is very large. It is clear that 0 is a lower bound. So in order to show that 0 is the infimum, it remians to show that 0 is the least among all the lower bounds of the set. For this it is enough to show that a number $\alpha > 0$ cannot be a lower bound of the given set. This is true because we can find an n such that $\frac{1}{1+n} < \alpha$ using the Archimedean property and $\frac{1}{1+n} \in S_1$ and so α can not be a lower bound.
Similarly, we can show that $\sup = 1$.

Clearly $S_2 \neq \Phi$ and 0 is a lower bound for S_2. Therefore, S has an infimum (by completeness property). Let $\alpha = \inf S_2$ so $\alpha \geq 0$. Let $\epsilon > 0$ be arbitrary then by the Archimedean property $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Therefore, $0 \leq \alpha \leq \frac{1}{n} < \epsilon \implies \alpha = 0$ (since ϵ is arbitrary).

Q3. Use the Archimedean property of real numbers to show that $\displaystyle\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi$.

Ans. Suppose if possible $x \in \displaystyle\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right]$. Then for all $n \in \mathbb{N}$, $x \in \left(0, \frac{1}{n}\right]$, that is, for all $n \in \mathbb{N}$, $0 < x \leq \frac{1}{n}$. But by the Archimedean Property, if $x > 0$, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < x$, this is a contradiction. Therefore, $\displaystyle\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi$.

Q4. Investigate the convergence/divergence of the following sequences:
(a) $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}$
(b) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$

(c) $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0,1)$
(d) $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$
(e) $x_n = \frac{2^n}{n!}$
(f) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$
(g) $x_n = \frac{n!}{(2n+1)!!}$.

Ans. (a) Note that
$$\frac{1+2+\ldots+n}{n+n^2} \leq x_n \leq \frac{1+2+\ldots+n}{1+n^2}.$$
By Sandwich theorem $x_n \to \frac{1}{2}$.
(b) Note that
$$\frac{n.n^2}{n^3+2n} \leq x_n \leq \frac{n.n^2}{n^3+n+1}.$$
By Sandwich theorem $x_n \to 1$.
(c) $x_n = n^\alpha\left[\left(1+\frac{1}{n}\right)^\alpha - 1\right]$. As $0 < \alpha < 1$, $\left(1+\frac{1}{n}\right)^\alpha < \left(1+\frac{1}{n}\right)$. Thus $x_n < n^\alpha\left[1+\frac{1}{n}-1\right] = n^{\alpha-1} = \frac{1}{n^{1-\alpha}} \to 0$.
(d) Note that $x_n > 0$ for all n and $\frac{x_{n+1}}{x_n} = \frac{(n+1)^s(1+p)^n}{n^s(1+p)^{n+1}} = \frac{1}{1+p}\left(1+\frac{1}{n}\right)^s \to \frac{1}{1+p} < 1$. Hence, $\lim\limits_{n\to\infty} x_n = 0$.
(e) Note that $x_n > 0$ for all n and $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n} = 0 < 1$. Hence $\lim\limits_{n\to\infty} x_n = 0$.
(f) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$. Note that $0 \leq x_n \leq (\sqrt{2}-1)^n$. Therefore, $x_n \to 0$.
(g) Here $\frac{x_{n+1}}{x_n} < 1\ \forall n$. Hence (x_n) is a decreasing sequence. Since $0 \leq x_n\ \forall n$, therefore the sequence converges.

Q5. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.

Ans. Note that $x_n > 0$ and $x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{a - x_n^2}{x_n}\right)$. Also $2x_{n+1}x_n = x_n^2 + a \geq 2\sqrt{x_n^2 a}$ (by the A.M - G.M. inequality). Thus $x_{n+1} \geq \sqrt{a}$. This implies that $x_{n+1} - x_n \leq 0$. Therefore the sequence is decreasing and bounded below. Hence it converges. Let $\lim\limits_{n\to\infty} x_n = l$. Then $l = \frac{1}{2}\left(\frac{l^2+a}{l}\right) \Rightarrow l = \sqrt{a}$.

Q6. Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence which satisfies one of the following conditions:
(a) $|x_{n+1} - x_n| \leq \alpha^n, \quad n = 1, 2, 3, \ldots$
(b) $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|, \quad n = 1, 2, 3, \ldots$
Then prove that (x_n) satisfies the Cauchy criterion. *Whenever you use this result,*

you have to show that the number α that you get, satisfies $0 < \alpha < 1$. The condition $|x_{n+2}-x_{n+1}| \leq |x_{n+1}-x_n|$ does not guarantee the convergence of (x_n). Give examples.

Ans. (a) Let $n > m$. Then

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ldots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m| \\ &\leq \alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^m = \alpha^m[1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1+m}] \\ &\leq \alpha^m[1 + \alpha + \alpha^2 + \ldots] \\ &= \frac{\alpha^m}{1-\alpha} \quad \text{as} \quad m \to \infty. \end{aligned}$$

Thus (x_n) satisfies the Cauchy criterion.
(b) Note that

$$|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n| \leq \alpha^2|x_n - x_{n-1}| \leq \ldots \leq \alpha^n|x_2 - x_1|.$$

For $n > m$,

$$\begin{aligned} |x_n - x_m| &\leq \alpha^{n-2} + \alpha^{n-3} + \ldots + \alpha^{m-1}|x_2 - x_1| \\ &\leq \frac{\alpha^m}{1-\alpha}|x_2 - x_1| \to 0 \quad \text{ax} \quad m \to \infty. \end{aligned}$$

Thus (x_n) satisfies the Cauchy criterion.
Examples:
(i) $x_n = n$. Here, $|x_{n+2} - x_{n+1}| = 1 = |x_{n+1} - x_n|$.
(ii) $x_n = \sqrt{n}$. Here
$|x_{n+2} - x_{n+1}| = |\sqrt{n+2} - \sqrt{n+1}| = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} = |x_{n+1} - x_n|$.

Q7. Let $x_1 \in \mathbb{R}$ and let $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ for $n \in \mathbb{N}$. Show that (x_n) converges for $0 < x_1 < 1$. Also conclude that it converges to a root of $x^3 - 7x + 2$ lying between 0 and 1. Does the sequence converge for any starting value of $x_1 > 1$.

Ans. Note that
$|x_{n+2} - x_{n+1}| = \frac{1}{7}|x_{n+1}^3 - x_n^3| = \frac{1}{7}|x_{n+1}^2 + x_{n+1}x_n + x_n^2||x_{n+1} - x_n| \leq \frac{3}{7}|x_{n+1} - x_n|$.
By problem 6 (b), (x_n) satisfies the Cauchy criterion, hence it converges.
It is clear that for $x_1 = 7$, $x_n \to \infty$.

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MATH-I ■ Assignment #2

(Sequences Cont.)

Q1. Investigate the convergence/divergence of the following sequences:
(a) $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$
(b) $x_n = \frac{2^n}{n!}$

Q.2 Is the sequence $a_n = 1 + (-1)^n$ a cauchy sequence ?

Q.3 Is the sequence $a_n = \frac{1}{n}$ a cauchy sequence ?

Q4. Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence which satisfies one of the following conditions:
(a) $|x_{n+1} - x_n| \leq \alpha^n, \quad n = 1, 2, 3, \ldots$
(b) $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, \quad n = 1, 2, 3, \ldots$
Then prove that (x_n) satisfies the Cauchy criterion. *Whenever you use this result, you have to show that the number* α *that you get, satisfies* $0 < \alpha < 1$. *The condition* $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ *does not guarantee the convergence of* (x_n). *Give examples.*

Q5. Let $x_1 \in \mathbb{R}$ and let $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ for $n \in \mathbb{N}$. Show that (x_n) converges for $0 < x_1 < 1$. Also conclude that it converges to a root of $x^3 - 7x + 2$ lying between 0 and 1. Does the sequence converge for any starting value of $x_1 > 1$.

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MATH-I ■ Solutions Assignment #2

(Limits, Continuity, Intermediate Value Property)

Q1. Suppose that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to zero. Find $\lim\limits_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x}$.

Ans. Apply sandwich theorem for limits to get $\lim\limits_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x} = 1$.

Q2. Given $f(x) = mx$, $m > 0$ and numbers $l = 2m$, $x_0 = 2$, $\epsilon = 0.003$. Find an open interval around x_0 and a $\delta > 0$ such that the inequality $|f(x) - l| < \epsilon$ holds for all x satisfying $|x - x_0| < \delta$.

Ans. Open interval= $\left(2 - \frac{0.003}{m}, 2 + \frac{0.003}{m}\right)$, $\delta = \frac{0.003}{m}$.

Q3. Determine the points of continuity for the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Ans. f is discontinuous everywhere.
For, if x is a rational point, then we can find a sequence of irrationals (x_n) converging to x. However, $(f(x_n)) \to 1 \neq f(x) = 0$. Similarly, f is not continuous at any irrational point.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Ans. Since $f(x_0) - c > 0$, choose ϵ such that $0 < \epsilon < f(x_0) - c$. Since, f is continuous at x_0, for this choice of ϵ, there exists a $\delta > 0$, such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Hence, for all $x \in (x_0 - \delta, x_0 + \delta)$ $f(x) > f(x_0) - \epsilon > c$.

Q5. Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ is also one-one then either f is a strictly increasing function or f is a strictly decreasing function.

Ans. Since f is $1 - 1$, if $x \neq y$, $f(x) \neq f(y)$. Assume $x < y$ and $f(x) < f(y)$. Let $c \in (x, y)$. We claim that $f(c) \in (f(x), f(y))$.
Clearly, $f(c) \neq f(x), f(y)$. If possible, assume that $f(x) > f(c)$. Then, $\frac{f(x)+f(c)}{2}$ lies in both the intervals $(f(c), f(x))$ and $(f(c), f(y))$. By the intermediate value theorem, we can find, $x_1 \in (x, c)$ and $x_2 \in (c, y)$ such that $f(x_1) = \frac{f(x)+f(c)}{2}$ and $f(x_2) = \frac{f(x)+f(c)}{2}$. Here, $x_1 \neq x_2$, but $f(x_1) = f(x_2)$, a contradiction. Thus $f(x) < f(c)$.

Q6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which takes only rational values. Show that f is a constant function.

Ans. Suppose $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}$. Let us choose an irrational number α between $f(x)$ and $f(y)$. Since f is continuous by IVP, there exists $z \in (x, y)$ such that $f(z) = \alpha$ which is a contradiction (as f takes only rational values).

Q7. Show that the polynomial $x^4 + 2x^3 - 9$ has at least two real roots.

Ans. Let $p(x) = x^4 + 2x^3 - 9$. Then $p(0) = -9$ and $p(x) \to \infty$ as $x \to \pm\infty$. Therefore, by intermediate value theorem there exist two real roots say $a > 0, b < 0$ of $p(x)$. Note that $p'(x) = 4x^3 + 6x^2 = 2x^2(2x+3)$ and $p\left(-\frac{3}{2}\right) =\neq 0$. a, b are simple roots of p. Since complex roots occur in pair, if p has three real roots, it will have all four as real roots. Also none of them is a repeated root. Therefore, p' must vanish at three distinct points, which is not true. Hence p has exactly two real roots.

Q8. Let $f : [1, 3] \to \mathbb{R}$ be a continuous function. Prove that there exist real numbers $x_1, x_2 \in [1, 3]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}(f(3) - f(1)).$$

Ans. Define $g(x) = f(x+1) - f(x) - \frac{1}{2}(f(3) - f(1))$. Then $g(1) = -g(2)$. By the intermediate value property, there exists $c \in (1, 2)$ such that $g(c) = 0$ i.e. $f(c+1) - f(c) = \frac{1}{2}(f(3) - f(1))$. Here $x_1 = c$ and $x_2 = c + 1$.

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MATH-I ■ Assignment #3

(Infinite Series)

Q1. Let $a_n \geq 0$. Then show that both the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

Q2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$, where a_n equals:
(a) $1 - n\sin\frac{1}{n}$, (b) $\frac{1}{n}\log(1+\frac{1}{n})$, (c) $1-\cos\frac{1}{n}$,
(d) $2^{-n-(-1)^n}$, (e) $\left(1+\frac{1}{n}\right)^{n(n+1)}$, (f) $\frac{n\log n}{2^n}$.

Q3. Test the series $\sum_{n\geq 1} \tan^{-1}(e^{-n})$ and the series $\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^{n^2}$ for convergence.

Q4. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\ldots+a_n}{n}$. Show that $\sum_{n\geq 1}(-1)^n b_n$ converges.

Q5. Show that if $\sum_{n=1}^{\infty}|a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges. Give an example to show that the converse need not be true.

Q6. Determine the values of x for which the following series converges:
(a) $\sum_{n\geq 1}\frac{(x-1)^{2n}}{n^2 3^n}$, (b) $\sum_{n\geq 1}\frac{n^3}{3^n}x^n$, (c) $\sum_{n\geq 1}\frac{(2n)!}{(2^n n!)^2}\frac{x^{2n+1}}{2n+1}$.

Q7. Let (a_n) be a constant sequence. If $\sum_{n} a_n$ converges then show that $a_n = 0$ for all n.

Q8. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Show that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges. Test the series $\sum_{n\geq 1}\frac{n^{n-2}}{e^n n!}$ for convergence.

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MATH-I ■ Solutions Assignment #3

(Infinite Series)

Q1. Let $a_n \geq 0$. Then show that both the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

Ans. Suppose $\sum_{n\geq 1} a_n$ converge. Since $0 \leq \frac{a_n}{a_n+1} \leq a_n$ by comparison test $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. Suppose $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. By the necessary condition $\frac{a_n}{a_n+1} \to 0$. Hence $a_n \to 0$ and therefore $1 \leq 1 + a_n < 2$ eventually. Hence $0 \leq \frac{1}{2}a_n \leq \frac{a_n}{a_n+1}$. Apply the comparison test.

Q2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$, where a_n equals:

(a) $1 - n\sin\frac{1}{n}$, (b) $\frac{1}{n}\log(1+\frac{1}{n})$, (c) $1-\cos\frac{1}{n}$,
(d) $2^{-n-(-1)^n}$, (e) $\left(1+\frac{1}{n}\right)^{n(n+1)}$, (f) $\frac{n\log n}{2^n}$.

Ans. (a) Use Limit Comparison Test (LCT) with $\frac{1}{n^2}$. Since $1 - n\sin\frac{1}{n} \leq \frac{1}{3!n^2} < \frac{1}{n^2}$, one can also use comparison test.
(b) Use LCT or comparison test with $\frac{1}{n^2}$.
(c) Use LCT with $\frac{1}{n^2}$ or comparison test because $1-\cos\frac{1}{n} \leq \frac{1}{2!n^2} < \frac{1}{n^2}$ or $1-\cos\frac{1}{n} = 2\sin^2\frac{1}{2n} < \frac{1}{2n^2}$.
(d) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $\frac{1}{2}$ and therefore the series converges.
(e) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $e > 1$ and hence the series is divergent.
(f) By ratio test, we get $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ and therefore the series converges.

Q3. Test the series $\sum_{n\geq 1} \tan^{-1}(e^{-n})$ and the series $\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^{n^2}$ for convergence.

Ans. Applying ratio test we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\tan^{-1}(e^{-(n+1)})}{\tan^{-1}(e^{-n})} = \lim_{n\to\infty}\frac{\frac{-e^{-(n+1)}}{1+e^{-2(n+1)}}}{\frac{-e^{-n}}{1+e^{-2n}}}$$
$$= \frac{1}{e}\lim_{n\to\infty}\frac{1+e^{-2n}}{1+e^{-2(n+1)}} = \frac{1}{e}.$$

Since $\frac{1}{e} < 1$, therefore by Ratio test the series converges.

Applying root test, we get $|a_n|^{\frac{1}{n}} = \left(1 - \frac{1}{n}\right)^n$. Also,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} < 1$$

$\Big($To see this, Let $y = \left(1 - \frac{1}{n}\right)^n$. Then $\ln y = n \ln\left(1 - \frac{1}{n}\right) = \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}}$. Therefore, using L'Hopital rule we get $\lim_{n\to\infty} \ln y = -1 \implies \lim_{n\to\infty} y = \dfrac{1}{e}.\Big)$

Hence, the series converges by root test.

Q4. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\ldots+a_n}{n}$. Show that $\sum_{n\geq 1}(-1)^n b_n$ converges.

Ans. $b_{n+1} - b_n = \frac{a_1+a_2+\ldots+a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n} = \frac{a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n(n+1)}$. Since $\{a_n\}$ is decreasing, $a_1 + a_2 + \ldots + a_n \geq n a_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1}-a_n}{n+1} \leq 0$. Therefore, $\{b_n\}$ is decreasing.

We now need to show that $b_n \to 0$. For a given $\epsilon > 0$, since $a_n \to 0$, there exists n_0 such that $a_n < \epsilon/2$, $\forall n \geq n_0$.

Therefore, $\left|\frac{a_1+a_2+\ldots+a_n}{n}\right| = \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n} + \frac{a_{n_0+1}+a_2+\ldots+a_n}{n}\right| \leq \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n}\right| + \frac{n-n_0}{n}\frac{\epsilon}{2}$.

Choose $N \geq n_0$ large enough so that $\frac{a_1+a_2+\ldots+a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1+a_2+\ldots+a_n}{n} < \epsilon$. Hence, $b_n \to 0$. Use the Leibnitz test for convergence.

Q5. Show that if $\sum_{n=1}^{\infty}|a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges. Give an example to show that the converse need not be true.

Ans. Let $\sum_{n=1}^{\infty}|a_n|$ converges. Then the sequence of partial sums of $\sum_{n=1}^{\infty}|a_n|$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ also satisfies the Cauchy criterion (why?). This shows that the series $\sum_{n=1}^{\infty} a_n$ converges.

For the converse part, consider the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n}$. This series converges by Leibnitz test, however the series $\sum_{n=1}^{\infty}\dfrac{1}{n}$ obtained on taking the absolute values of the terms of the original series diverges.

Q6. Determine the values of x for which the following series converges:

$$(a) \sum_{n\geq 1} \frac{(x-1)^{2n}}{n^2 3^n}, \qquad (b) \sum_{n\geq 1} \frac{n^3}{3^n} x^n, \qquad (c) \sum_{n\geq 1} \frac{(2n)!}{(2^n n!)^2} \frac{x^{2n+1}}{2n+1}.$$

Ans. (a) By root test the series converges for $|x-1| < \sqrt{3}$. If $x-1 = \pm\sqrt{3}$ the series converges. Therefore the series converges for $|x-1| \leq \sqrt{3}$.
(b) Use the ratio test to see that the series converges for $|x| < 3$. For $x = \pm 3$, the series diverges.
(c) Use ratio test to see that the series converges for $|x| < 1$. For $x = \pm 1$, the corresponding series will converge.

Q7. Let (a_n) be a constant sequence. If $\sum_n a_n$ converges then show that $a_n = 0$ for all n.

Ans. Let $a_n = c$ for some $c \in \mathbb{R}$ such that $c \neq 0$. Then $S_n = nc$. Given that S_n converges, $\frac{1}{c} S_n = n$ converges, which is a contradiction.

Q8. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Show that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges. Test the series $\sum_{n\geq 1} \frac{n^{n-2}}{e^n n!}$ for convergence.

Ans. Clearly, $c_{n+1} = \frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} = c_n \; \forall n \geq \mathbb{N}$. Thus $0 < c_n = \frac{a_n}{b_n} < \frac{a_N}{b_N} \; \forall n > \mathbb{N}$. Use the comparison test.
For the other part, note that $\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n}\right)^{n-2}}{e} = \frac{\left(1+\frac{1}{n}\right)^{n}}{\left(1+\frac{1}{n}\right)^2 e} < \frac{e}{\left(1+\frac{1}{n}\right)^2 e} = \frac{\frac{1}{(1+n)^2}}{\frac{1}{n^2}}$.

The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #4

(Continuity, Intermediate Value Property, Derivatives)

Q1. Determine the points of continuity for the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Q2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Q3. Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ is also one-one then either f is a strictly increasing function or f is a strictly decreasing function.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which takes only rational values. Show that f is a constant function.

Q5. Show that the polynomial $x^4 + 2x^3 - 9$ has at exactly two real roots.

Q6. Let $f : [1, 3] \to \mathbb{R}$ be a continuous function. Prove that there exist real numbers $x_1, x_2 \in [1, 3]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}(f(3) - f(1)).$$

Q7. Show that the function $f(x) = x|x|$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

Q8. Check the function $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ for differentiability.

Q9. Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not bounded on the interval $[-1, 1]$. From this deduce that $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

Q10. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function [$(f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an odd function [$(f(-x) = -f(x)$ for all $x \in \mathbb{R}$].

The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Solutions Assignment #4

(Continuity, Intermediate Value Property, Derivatives)

Q1. Determine the points of continuity for the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Ans. f is discontinuous everywhere.
For, if x is a rational point, then we can find a sequence of irrationals (x_n) converging to x. However, $(f(x_n)) \to 1 \neq f(x) = 0$. Similarly, f is not continuous at any irrational point.

Q2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Ans. Since $f(x_0) - c > 0$, choose ϵ such that $0 < \epsilon < f(x_0) - c$. Since, f is continuous at x_0, for this choice of ϵ, there exists a $\delta > 0$, such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Hence, for all $x \in (x_0 - \delta, x_0 + \delta)$ $f(x) > f(x_0) - \epsilon > c$.

Q3. Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ is also one-one then either f is a strictly increasing function or f is a strictly decreasing function.

Ans. Since f is $1-1$, if $x \neq y$, $f(x) \neq f(y)$. Assume $x < y$ and $f(x) < f(y)$. Let $c \in (x, y)$. We claim that $f(c) \in (f(x), f(y))$.
Clearly, $f(c) \neq f(x), f(y)$. If possible, assume that $f(x) > f(c)$. Then, $\frac{f(x)+f(c)}{2}$ lies in both the intervals $(f(c), f(x))$ and $(f(c), f(y))$. By the intermediate value theorem, we can find, $x_1 \in (x, c)$ and $x_2 \in (c, y)$ such that $f(x_1) = \frac{f(x)+f(c)}{2}$ and $f(x_2) = \frac{f(x)+f(c)}{2}$. Here, $x_1 \neq x_2$, but $f(x_1) = f(x_2)$, a contradiction. Thus $f(x) < f(c)$.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which takes only rational values. Show that f is a constant function.

Ans. Suppose $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}$. Let us choose an irrational number α between $f(x)$ and $f(y)$. Since f is continuous by IVP, there exists $z \in (x, y)$ such that $f(z) = \alpha$ which is a contradiction (as f takes only rational values).

Q5. Show that the polynomial $x^4 + 2x^3 - 9$ has at least two real roots.

Ans. Let $p(x) = x^4 + 2x^3 - 9$. Then $p(0) = -9$ and $p(x) \to \infty$ as $x \to \pm\infty$. Therefore, by intermediate value theorem there exist two real roots say $a > 0, b < 0$ of $p(x)$. Note that $p'(x) = 4x^3 + 6x^2 = 2x^2(2x+3)$ and $p\left(-\frac{3}{2}\right) =\neq 0$. a, b are simple roots of p. Since complex roots occur in pair, if p has three real roots, it will have all four as real roots. Also none of them is a repeated root. Therefore, p' must vanish at three distinct points, which is not true. Hence p has exactly two real roots.

Q6. Let $f : [1,3] \to \mathbb{R}$ be a continuous function. Prove that there exist real numbers $x_1, x_2 \in [1,3]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}(f(3) - f(1)).$$

Ans. Define $g(x) = f(x+1) - f(x) - \frac{1}{2}(f(3) - f(1))$. Then $g(1) = -g(2)$. By the intermediate value property, there exists $c \in (1,2)$ such that $g(c) = 0$ i.e. $f(c+1) - f(c) = \frac{1}{2}(f(3) - f(1))$. Here $x_1 = c$ and $x_2 = c+1$.

Q7. Show that the function $f(x) = x|x|$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

Ans. Easy. Use definition of the derivative of a function.

Q8. Check the function $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ for differentiability.

Ans. $f'(0)$ does not exist as $\displaystyle\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \sin \frac{1}{h}$, which doesn't exist.

Q9. Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not bounded on the interval $[-1,1]$. From this deduce that $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

Ans. Note that $f'(0) = \displaystyle\lim_{h\to 0} h \sin \frac{1}{h^2} = 0$. Therefore, f is differentiable at 0. At any other point f is differentiable being the product of two differentiable functions. Hence f is differentiable for all real x.

We have $f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Here $2x \sin \frac{1}{x^2}$ is bounded in $[-1,1]$ but $\frac{2}{x} \cos \frac{1}{x^2}$ is not bounded in any interval containing 0. Hence $f'(x)$ is not bounded on $[-1,1]$ and so it can not be continuous at $[-1,1]$.

Q10. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function $[(f(-x) = f(x)$ for all $x \in \mathbb{R}]$ and has a derivative at every point, then the derivative f' is an odd function $[(f(-x) = -f(x)$ for all $x \in \mathbb{R}]$.

Ans. $f'(-x) = \lim\limits_{h\to 0} \dfrac{f(-x+h) - f(-x)}{h} = \lim\limits_{x\to 0} \dfrac{f(x-h) - f(x)}{h} = -\lim\limits_{k\to 0} \dfrac{f(x+k) - f(x)}{k} = -f'(x)$.

i.e. the derivative of f is an odd function.

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MATH-I ■ Assignment #5

(Rolle's Theorem, Mean Value Theorem, Taylor's Theorem)

Q1. Prove that the ploynomial $f(x) = x^3 - 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.

Q2. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f'(x)f(x) = x$ has at least one root in (a, b).

Q3. Verify that $x^3 + 2x + 1$ satisfies the hypotheses of the Mean Value Theorem on $[0, 1]$. Then find all numbers that satisfy the conclusion of the Mean Value Theorem.

Q4. Using Mean Value Theorems (MVT or CMVT) show that
(a) $\log(1 + x) > \frac{x}{1+x}$, for all $x > 0$
(b) $e^x \geq 1 + x$ for $x \in \mathbb{R}$
(c) $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$
(d) $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x > 0$
(e) $1 - \frac{x^2}{2!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $x \neq 0$.

Q5. Find $\lim_{x \to 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$.

Q6. Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0, f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem prove that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

Q7. For $x > -1$, $x \neq 0$ prove that
(a) $(1 + x)^\alpha > 1 + \alpha x$ whenever $\alpha < 0$, or $\alpha > 1$
(b) $(1 + x)^\alpha < 1 + \alpha x$ whenever $0 < \alpha < 1$.

Q8. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k}x^{2k} < \log(1 + x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$

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MATH-I ■ Solutions Assignment #5

(Series, Power Series, Taylor Series)

Q1. Let $a_n \geq 0$. Then show that both the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

Ans. Suppose $\sum_{n\geq 1} a_n$ converge. Since $0 \leq \frac{a_n}{a_n+1} \leq a_n$ by comparison test $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. Suppose $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. By the necessary condition $\frac{a_n}{a_n+1} \to 0$. Hence $a_n \to 0$ and therefore $1 \leq 1 + a_n < 2$ eventually. Hence $0 \leq \frac{1}{2} a_n \leq \frac{a_n}{a_n+1}$. Apply the comparison test.

Q2. Prove that $\sum_{n\geq 1} (a_n - a_{n+1})$ converges if and only if the sequence a_n converges. Use this to decide the convergence/divergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

Ans. Note that the sequence of partial sums of the series $\sum_{n\geq 1} (a_n - a_{n+1})$ is $(a_1 - a_n)$. The conclusion now follows from the definition of convergence of the sequence a_n.

Q3. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$, where a_n equals:

(a) $1 - n \sin \frac{1}{n}$, (b) $\frac{1}{n} \log(1 + \frac{1}{n})$, (c) $1 - \cos \frac{1}{n}$,
(d) $2^{-n-(-1)^n}$, (e) $\left(1 + \frac{1}{n}\right)^{n(n+1)}$, (f) $\frac{n \log n}{2^n}$.

Ans. (a) Use Limit Comparison Test (LCT) with $\frac{1}{n^2}$. Since $1 - n \sin \frac{1}{n} \leq \frac{1}{3!n^2} < \frac{1}{n^2}$, one can also use comparison test.
(b) Use LCT or comparison test with $\frac{1}{n^2}$.
(c) Use LCT with $\frac{1}{n^2}$ or comparison test because $1 - \cos \frac{1}{n} \leq \frac{1}{2!n^2} < \frac{1}{n^2}$ or $1 - \cos \frac{1}{n} = 2 \sin^2 \frac{1}{2n} < \frac{1}{2n^2}$.
(d) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $\frac{1}{2}$ and therefore the series converges.
(e) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $e > 1$ and hence the series is divergent.
(f) By ratio test, we get $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ and therefore the series converges.

Q4. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Show that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges. Test the series $\sum_{n\geq 1} \frac{n^{n-2}}{e^n n!}$ for convergence.

Ans. Clearly, $c_{n+1} = \frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} = c_n \; \forall n \geq \mathbb{N}$. Thus $0 < c_n = \frac{a_n}{b_n} < \frac{a_N}{b_N} \; \forall n > \mathbb{N}$. Use the comparison test.

For the other part, note that $\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n}\right)^{n-2}}{e} = \frac{\left(1+\frac{1}{n}\right)^{n}}{\left(1+\frac{1}{n}\right)^2 e} < \frac{e}{\left(1+\frac{1}{n}\right)^2 e} = \frac{\frac{1}{(1+n)^2}}{\frac{1}{n^2}}$.

Q5. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\ldots+a_n}{n}$. Show that $\sum_{n\geq 1}(-1)^n b_n$ converges.

Ans. $b_{n+1} - b_n = \frac{a_1+a_2+\ldots+a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n} = \frac{a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n(n+1)}$. Since $\{a_n\}$ is decreasing, $a_1 + a_2 + \ldots + a_n \geq n a_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1}-a_n}{n+1} \leq 0$. Therefore, $\{b_n\}$ is decreasing.

We now need to show that $b_n \to 0$. For a given $\epsilon > 0$, since $a_n \to 0$, there exists n_0 such that $a_n < \epsilon/2$, $\forall n \geq n_0$.

Therefore, $\left|\frac{a_1+a_2+\ldots+a_n}{n}\right| = \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n} + \frac{a_{n_0+1}+a_2+\ldots+a_n}{n}\right| \leq \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n}\right| + \frac{n-n_0}{n}\frac{\epsilon}{2}$.

Choose $N \geq n_0$ large enough so that $\frac{a_1+a_2+\ldots+a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1+a_2+\ldots+a_n}{n} < \epsilon$. Hence, $b_n \to 0$. Use the Leibnitz test for convergence.

Q6. Show that if $\sum_{n=1}^{\infty}|a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges. Give an example to show that the converse need not be true.

Ans. Let $\sum_{n=1}^{\infty}|a_n|$ converges. Then the sequence of partial sums of $\sum_{n=1}^{\infty}|a_n|$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ also satisfies the Cauchy criterion (why?). This shows that the series $\sum_{n=1}^{\infty} a_n$ converges.

For the converse part, consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. This series converges by Leibnitz test, however the series $\sum_{n=1}^{\infty}\frac{1}{n}$ obtained on taking the absolute values of the terms of the original series diverges.

Q7. Determine the values of x for which the following series converges:

(a) $\displaystyle\sum_{n\geq 1}\frac{(x-1)^{2n}}{n^2 3^n}$, (b) $\displaystyle\sum_{n\geq 1}\frac{n^3}{3^n}x^n$, (c) $\displaystyle\sum_{n\geq 1}\frac{(2n)!}{(2^n n!)^2}\frac{x^{2n+1}}{2n+1}$.

Ans. (a) By root test the series converges for $|x-1| < \sqrt{3}$. If $x - 1 = \pm\sqrt{3}$ the series converges. Therefore the series converges for $|x-1| \leq \sqrt{3}$.

(b) Use the ratio test to see that the series converges for $|x| < 3$. For $x = \pm 3$, the series diverges.

(c) Use ratio test to see that the series converges for $|x| < 1$. For $x = \pm 1$, the corresponding series will converge.

Q8. Find the Taylor series at 0 for each of the following functions and also the values of x for which the corresponding series converges:

(a) $f(x) = \frac{1}{x-a}, a \neq 0$, (b) $f(x) = \frac{1}{\sqrt{1-x}}$.

Ans. Easy. Leave it for the students to try at their own.

The LNM Institute of Information Technology
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MATH-I ■ Assignment #6

(Riemann Integration, Improper Integral)

Q1. If f is a bounded function such that $f(x) = 0$ except at a point $c \in [a, b]$. Then show that f is integrable on $[a, b]$ and that $\int_a^b f = 0$.

Q2. Let $f : [0, 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$ Show that f is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 0$.

Q3. Define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 < x \le 1. \end{cases}$$

Is the function continuous on $[-1, 1]$? Is the function Riemann integrable?

Q4. Does there exist a continuous function f on $[0, 1]$ such that

$$\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}} \quad \text{for all} \quad n \in \mathbb{N}.$$

Q5. Let $f : [0, 1] \to \mathbb{R}$ such that $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y}, & \text{if } 0 \le y < 1 \\ 0, & y = 1. \end{cases}$. Then prove that $\lim_{n\to\infty} \int_0^1 g_n(y)dy = \frac{1}{2}$ whereas $\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0$.

Q6. Test the convergence/divergence of the following improper integrals:

(a) $\int_0^1 \frac{dx}{\log(1 + \sqrt{x})}$ (b) $\int_0^1 \frac{dx}{x - \log(1 + x)}$ (c) $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ (d) $\int_0^1 \sin\left(\frac{1}{x}\right) dx$

(e) $\int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{x} dx$ (f) $\int_0^\infty e^{-x^2} dx$ (g) $\int_0^{\pi/2} \frac{dx}{x - \sin x}$ (h) $\int_0^{\pi/2} \text{cosec} x dx$.

Q7. In each case, determine the values of p for which the following improper integrals converge

(a) $\int_0^\infty \frac{1 - e^{-x}}{x^p}$ (b) $\int_0^\infty \frac{t^{p-1}}{1 + t} dt$.

Q8. Show that the integrals $\int_0^\infty \frac{\sin x^2}{x^2} dx$ and $\int_0^\infty \frac{\sin x}{x} dx$ converge. Further, prove that

$$\int_0^\infty \frac{\sin x^2}{x^2} dx = \int_0^\infty \frac{\sin x}{x} dx.$$

Q9. Show that $\int_0^\infty \frac{x \log x}{(1+x^2)^2} dx = 0.$

Q10. Prove that $\int_1^\infty \frac{\sin x}{x^p} dx$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$.

Q11. Show that $\int_0^s \frac{1+x}{1+x^2} dx$ and $\int_{-s}^0 \frac{1+x}{1+x^2} dx$ do not approach a limit as $s \to \infty$. However $\lim_{s\to\infty} \int_{-s}^s \frac{1+x}{1+x^2} dx$ exists.

Q12. Investigate the convergence of the improper integral

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^3}}.$$

The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Solutions Assignment #6

(Riemann Integral, Improper Integrals)

Q1. If f is a bounded function such that $f(x) = 0$ except at a point $c \in [a, b]$. then show that f is integrable on $[a, b]$ and that $\int_a^b f = 0$.

Ans. *Using the definition:* Let $f(c) > 0$ and P be any partition. Suppose $c \in [x_i, x_{i+1}]$. Then $L(P, f) = 0$ and $U(P, f) = f(c)\Delta x_i$. Since P is arbitrary, $\inf_P U(P, f) = 0$ and $\sup_P L(P, f) = 0$. Hence f is integrable and $\int_a^b f(x)dx = 0$.
Using the "$\epsilon - P$ argument (essentially the same)": Let $\epsilon > 0$. Note that if P is a partition such that $\max_i \Delta x_i < \delta$ then $L(P, f) = 0$ and $U(P, f) \leq 2f(c)\delta$. Choose $\delta < \frac{\epsilon}{2f(c)}$. Then $U(P, f) - L(P, f) < \epsilon$ and hence f is integrable by the Riemann criterion. Since the lower integral is 0 and the function is integrable, $\int_a^b f(x)dx = 0$.

Q2. Let $f : [0, 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$ Show that f is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 0$.

Ans. We will use the Riemann criterion to show that f is integrable on $[0, 1]$. Let $\epsilon > 0$ be given. We need to find a partition P such that $U(P, f) - L(P, f) < \epsilon$. Since $\frac{1}{n} \to 0$, there exists N such that $\frac{1}{n} \in [0, \epsilon]$ for all $n > N$.
So only finite number of $\frac{1}{n}'$s lie in the interval $[\epsilon, 1]$. Cover these finite number of $\frac{1}{n}'$s by the intervals $[x_1, x_2], [x_3, x_4], \ldots [x_{m-1}, x_m]$ such that $x_i \in [\epsilon, 1]$ for all $i = 1, 2, \ldots m$ and the sum of the length of these m intervals is less than ϵ. Consider the partition $P = \{0, \epsilon, x_1, x_2, \ldots, x_m\}$. It is clear that $U(P, f) - L(P, f) < 2\epsilon$.
Hence by the Reimann criterion the function is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)dx = 0$.

Q3. Define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1. \end{cases}$$

Is the function continuous on $[-1, 1]$? Is the function Riemann integrable?

Ans. Clearly f is not continuous at $x = 0$. Rest part is similar to exercise (1).

Q4. Does there exist a continuous function f on $[0,1]$ such that

$$\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}} \quad \text{for all} \quad n \in \mathbb{N}.$$

.

Ans. Suppose there is such a function f. Then, by the previous problem, for every n there exist $c_n \in [0,1]$ such that $f(c_n)\displaystyle\int_0^1 x^n dx = \frac{1}{\sqrt{n}}$. This implies that $f(c_n) = \frac{n+1}{n!} \to \infty$. That is, f is not bounded on $[0,1]$ which is a contradiction.

Aliter: This problem can also done without using the previous problem. Suppose there is such a function f and $\sup f = M$. Then

$$\frac{1}{\sqrt{n}} = \left|\int_0^1 f(x)x^n dx\right| \leq M\left|\int_0^1 x^n dx\right| = \frac{M}{n+1}.$$

This implies that $1 \leq \frac{M\sqrt{n}}{n+1} \to 0$ which is a contradiction.

Q5. Let $f : [0,1] \to \mathbb{R}$ such that $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y}, & \text{if } 0 \leq y < 1 \\ 0, & y = 1. \end{cases}$ Then prove that $\displaystyle\lim_{n\to\infty}\int_0^1 g_n(y)dy = \frac{1}{2}$ whereas $\displaystyle\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0.$

Ans. From the ratio test for the sequence we can show that $\displaystyle\lim_{n\to\infty}\frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\displaystyle\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0.$

For the other part, use integration by parts to see that $\displaystyle\int_0^1 \frac{ny^{n-1}}{1+y}dy = \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2}dy.$

Note that $\displaystyle\int_0^1 \frac{y^n}{(1+y)^2}dy \leq \int_0^1 y^n = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_0^1 g_n(y)dy = \frac{1}{2}.$$

Q6. Test the convergence/divergence of the following improper integrals:

$(a)\ \displaystyle\int_0^1 \frac{dx}{\log(1+\sqrt{x})}$ $\quad (b)\ \displaystyle\int_0^1 \frac{dx}{x - \log(1+x)}$ $\quad (c)\ \displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}dx$ $\quad (d)\ \displaystyle\int_0^1 \sin\left(\frac{1}{x}\right)dx$

$(e)\ \displaystyle\int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{x}dx$ $\quad (f)\ \displaystyle\int_0^\infty e^{-x^2}dx$ $\quad (g)\ \displaystyle\int_0^{\pi/2} \frac{dx}{x - \sin x}$ $\quad (h)\ \displaystyle\int_0^{\pi/2} \operatorname{cosec} x dx.$

Ans. (a) Converges by limit comparison test (LCT) with $\frac{1}{\sqrt{x}}$.
(b) Diverges by LCT with $\frac{1}{x^2}$.
(c) The integral $-\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}$ converges by LCT with $\frac{1}{x^p}$, where $\frac{1}{2} < p < 1$.
(d) Since $|\sin\frac{1}{x}| \leq 1$, the integral converges. Note that in this case the integral is a proper integral.
(e) Converges by LCT with $\frac{1}{x^2}$.
(f) Converges by LCT with $\frac{1}{x^p}$, where $p \geq 2$.
(g) Apply LCT with $\frac{1}{x^3}$. The integral diverges.
(h) $\displaystyle\int_0^{\pi/2} \operatorname{cosec} x\, dx = \int_0^{\pi/2} \frac{1}{\sin x}\, dx$. Apply LCT with $\frac{1}{x}$. The integral is divergent.

Q7. In each case, determine the values of p for which the following improper integrals converge
(a) $\displaystyle\int_0^{\infty} \frac{1-e^{-x}}{x^p}$ (b) $\displaystyle\int_0^{\infty} \frac{t^{p-1}}{1+t} dt$.

Ans. (a)

$$\int_0^{\infty} \frac{1-e^{-x}}{x^p} = \int_0^{1} \frac{1-e^{-x}}{x^p} + \int_1^{\infty} \frac{1-e^{-x}}{x^p} = I_1 + I_2.$$

Now one has to see how the function $\frac{1-e^{-x}}{x^p}$ behaves in the respective intervals and apply the LCT.
Since $\displaystyle\lim_{x\to 0} \frac{1-e^{-x}}{x} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that I_1 is convergent iff $p-1 < 1$, i.e. $p < 2$. Similarly, I_2 is convergent (by applying LCT with $\frac{1}{x^p}$) iff $p > 1$. Therefore $\displaystyle\int_0^{\infty} \frac{1-e^{-x}}{x^p}$ converges iff $1 < p < 2$.

(b)

$$\int_0^{\infty} \frac{t^{p-1}}{1+t} dt = \int_0^{1} \frac{t^{p-1}}{1+t} dt + \int_1^{\infty} \frac{t^{p-1}}{1+t} dt = I_1 + I_2.$$

For I_1, use LCT with t^{p-1}. We see that the integral converges iff $p > 0$. Similarly, for I_2, Use LCT with t^{p-2}. The integral converges iff $p < 1$. Therefore, $\displaystyle\int_0^{\infty} \frac{t^{p-1}}{1+t} dt$ converges iff $0 < p < 1$.

Q8. Show that the integrals $\displaystyle\int_0^{\infty} \frac{\sin x^2}{x^2} dx$ and $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx$ converge. Further, prove that

$$\int_0^{\infty} \frac{\sin x^2}{x^2} dx = \int_0^{\infty} \frac{\sin x}{x} dx.$$

Ans.
$$\int_0^\infty \frac{\sin x^2}{x^2}dx = \int_0^1 \frac{\sin x^2}{x^2}dx + \int_1^\infty \frac{\sin x^2}{x^2}dx = I_1 + I_2.$$

I_1 is a proper integral and I_2 converges by a comparison with $\frac{1}{x^2}$.
Similarly $\displaystyle\int_0^\infty \frac{\sin x}{x}dx.$ converges by Dirichlet test.
Using integration by parts we see that

$$\int_0^\infty \frac{\sin x^2}{x^2}dx = -\frac{\sin x^2}{x}\bigg|_0^\infty + \int_0^\infty \frac{2\sin x\cos x}{x}dx = \int_0^\infty \frac{\sin 2x}{2x}d(2x) = \int_0^\infty \frac{\sin x}{x}dx.$$

Q9. Show that $\displaystyle\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = 0.$

Ans.
$$\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx = I_1 + I_2.$$

Since, $\lim\limits_{x\to 0} x\log x = 0$, I_1 is a proper integral.
For large x, $\log x \le x$. Hence $\frac{x\log x}{(1+x^2)^2} \le \frac{x^2}{(1+x^2)^2} \le \frac{1}{1+x^2}$ and I_2 converges. Use the substitution $x = \frac{1}{t}$ in I_1 to get $I_1 = -I_2$.

Q10. Prove that $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \le 1$ and and absolutely for $p > 1$.

Ans. By Dirichlets Test, $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges for all $p > 0$.
$\displaystyle\int_1^\infty \frac{|\sin x|}{x^p}dx \le \int_1^\infty \frac{1}{x^p}dx.$ Therefore, the function converges absolutely for $p > 1$.
Now, let $0 < p \le 1$. Since, $|\sin x| \ge \sin^2 x$, we see that $\left|\frac{\sin x}{x^p}\right| \ge \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p}$. By Dirichlets Test, $\displaystyle\int_1^\infty \frac{\cos 2x}{x^p}$ converges $\forall p > 0$. But $\displaystyle\int_1^\infty \frac{1}{2x^p}$ diverges for $p \le 1$.
Hence, $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \le 1$ and absolutely for $p > 1$.

Q11. Show that $\displaystyle\int_0^s \frac{1+x}{1+x^2}dx$ and $\displaystyle\int_{-s}^0 \frac{1+x}{1+x^2}dx$ do not approach a limit as $s \to \infty$. However $\displaystyle\lim_{s\to\infty}\int_{-s}^s \frac{1+x}{1+x^2}dx$ exists.

Ans. $\displaystyle\int_0^s \frac{1+x}{1+x^2}dx$ diverges by limit comparison with $\frac{1}{x}$.

$$\begin{aligned}\int_{-s}^s \frac{1+x}{1+x^2}dx &= \int_{-s}^0 \frac{1+x}{1+x^2}dx + \int_0^s \frac{1+x}{1+x^2}dx\\ &= \int_0^s \frac{1-u}{1+u^2}du + \int_0^s \frac{1+x}{1+x^2}dx\\ &= \int_0^s \frac{2du}{1+u^2}du,\end{aligned}$$

which converges.

Q12. Investigate the convergence of the improper integral

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^3}}.$$

Ans. Note that $1-x^3 = (1-x)(1+x+x^2)$. Let us compare the given function with $\frac{1}{\sqrt{1-x}}$.

$$\lim_{x\to 1} \frac{1/\sqrt{1-x^3}}{1/\sqrt{1-x}} = \lim_{x\to 1} \frac{\sqrt{1-x}}{\sqrt{1-x^3}} = \lim_{x\to 1} \frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}}.$$

Now

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \left[-\frac{2}{\sqrt{1-x}}\right]_0^1 = 2.$$

and so by LCT the integral I converges.

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MATH-I ■ Assignment #7

(Calculus of functions of several variables, Directional derivatives, Max/Min and Lagrange Multipliers)

Q1. Examine the following functions for continuity at the point $(0,0)$ where $f(0,0) = 0$ and $f(x,y)$ for $(x,y) \neq (0,0)$ is given by
(a) $|x| + |y|$, (b) $\frac{-x}{\sqrt{x^2+y^2}}$, (c) $\frac{2x}{x^2+x+y^2}$, (d) $\frac{x^4-y^2}{x^4+y^2}$, (e) $\frac{x^4}{x^4+y^2}$.

Q2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or if } y = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that the function satisfy the following:
(a) The iterated limits $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right]$ and $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$ exist and equals 0,
(b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist,
(c) $f(x,y)$ is not continuous at $(0,0)$,
(d) the partial derivatives exist at $(0,0)$.

Q3. Let

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that
(a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y,
(b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and
(c) $f(x,y)$ is differentiable at $(0,0)$.

Q4. Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that f is constant.

Q5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Q6. Examine the following functions for local maxima, local minima and saddle points:
(i) $4xy - x^4 - y^4$, (ii) $x^3 - 3xy$, (iii) $(x^2+y^2)\exp^{-(x^2+y^2)}$.

Q7. Let $f(x,y) = 3x^4 - 4x^2y + y^2$. Show that f has a local minimum at $(0,0)$ along every line through $(0,0)$. Does f have a minimum at $(0,0)$? Is $(0,0)$ a saddle point for f?

Q8. Find the absolute maxima of $f(x,y) = xy$ on the unit disc $\{f(x,y) : x^2 + y^2 \leq 1\}$.

Q9. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the $z-$axis.

Q10. Given n positive numbers $a_1, a_2, \ldots, a_n$, find the maximum value of the expression the function $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ where $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$.

Q11. Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

Q12. Minimize the function $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

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MATH-I ■ Solutions Assignment #7

(Functions of several variables: Continuity, Differentiability, Directional derivatives, Maxima, Minima and Lagrange Multipliers)

Q1. Examine the following functions for continuity at the point $(0, 0)$ where $f(0, 0) = 0$ and $f(x, y)$ for $(x, y) \neq (0, 0)$ is given by
(a) $|x| + |y|$, (b) $\frac{-x}{\sqrt{x^2+y^2}}$, (c) $\frac{2x}{x^2+x+y^2}$, (d) $\frac{x^4-y^2}{x^4+y^2}$, (e) $\frac{x^4}{x^4+y^2}$.

Ans. (a) Given that $f(0, 0) = 0$. Let $\epsilon > 0$ be given then $\big|(|x| + |y|) - 0\big| = \big||x| + |y|\big| \leq |x| + |y| < \epsilon$, whenever $|x| < \delta = \epsilon/2$ and $|y| < \delta = \epsilon/2$. Therefore the function is continuous at (0,0).
Alternatively, the given function is continuous being the sum of two continuous functions.
(b) Let $y = mx$. Then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2 + y^2}} = \frac{-1}{\sqrt{1 + m^2}}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0, 0)$.
(c) Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{2x}{x^2 + x + y^2} = \frac{2r\cos\theta}{r^2\cos^2\theta + r\cos\theta + r^2\sin^2\theta} = \frac{2\cos\theta}{r + \cos\theta}.$$

Now, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 + x + y^2} = \lim_{r\to 0} \frac{2\cos\theta}{r + \cos\theta} = 2$. Therefore the function is continuous at (0,0).
(d) Let $y = mx^2$. Then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2} = \frac{1 - m^2}{1 + m^2}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0, 0)$.
(e) Let $y = mx^2$. Then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} = \frac{m}{1 + m^2}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0, 0)$.

Q2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or if } y = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that the function satisfy the following:
(a) The iterated limits $\displaystyle\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right]$ and $\displaystyle\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right]$ exist and equals 0,
(b) $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist,
(c) $f(x, y)$ is not continuous at $(0, 0)$,
(d) the partial derivatives exist at $(0, 0)$.

Ans. (a) Let $x \neq 0$, then $\lim_{y\to 0} f(x,y) = 0$.

Similarly, if $y \neq 0$, then $\lim_{x\to 0} f(x,y) = 0$.

Therefore, $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0$.

(b) Along the line $x = 0$, we have $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Along the line $y = x$, we have $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Hence, the limit does not exist.

(c) From above, the function is not continuous.

(d) Easy. Leave it to the students.

Q3. Let

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Prove that

(a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y,

(b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and

(c) $f(x,y)$ is differentiable at $(0,0)$.

Ans. ***Discussed in the class.***

(a) Note that

$$f_x(0,k) = \lim_{h\to 0}\frac{f(0+h,k) - f(0,k)}{h} = \lim_{h\to 0}\frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} k\left(\frac{h^2-k^2}{h^2+k^2}\right) = -k.$$

Thus $f_x(0,y) = -y$.

Similarly, $f_y(x,0) = x$.

(b) Note that $f_{xy}(0,0) = \lim_{h\to 0}\dfrac{f(h,0) - f(0,0)}{h} = -1$ and $f_{xy}(0,0) = 1$.

(c) We need to show that

$$f(\Delta x, \Delta y) - f(0,0) = f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1(\Delta x, \Delta y)\Delta x + \epsilon_2(\Delta x, \Delta y)\Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. Since $f_x(0,0) = 0$ and $f_y(0,0) = 0$ we will show that

$$f(\Delta x, \Delta y) - f(0,0) = \epsilon_1(\Delta x, \Delta y)\Delta x + \epsilon_2(\Delta x, \Delta y)\Delta y.$$

Now $f(\Delta x, \Delta y) - f(0,0) = f(\Delta x, \Delta y) = \Delta x \Delta y\left(\frac{(\Delta x)^2 - (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right) = \frac{(\Delta x)^2 \Delta y}{(\Delta x)^2 + (\Delta y)^2}\Delta x - \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\Delta y = \epsilon_1 \Delta x + \epsilon_2 \Delta y$.

Here $\epsilon_1(\Delta x, \Delta y) = \frac{(\Delta x)^2 \Delta y}{(\Delta x)^2 + (\Delta y)^2}$, $\epsilon_2(\Delta x, \Delta y) = \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \to 0$ as Δx and $\Delta y \to 0$.

So f is differentiable at $(0,0)$.

Q4. Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that f is constant.

Ans. This follows immediately from the MVT for functions of several variables.

Q5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Ans. Using polar coordinates, we see that

$$\frac{y^3}{x^2+y^2} = \frac{r^3 \sin^3\theta}{r^2} = r\sin^3\theta.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = \lim_{r\to 0} r\sin^3\theta = 0.$$

$\Longrightarrow f$ is continuous at $(0,0)$.
Let $U = (u_1,u_2)$ be a unit vector. Now $D_{(0,0)}f(U) = \lim\limits_{t\to 0}\dfrac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \lim\limits_{t\to 0}\dfrac{f(tu_1,tu_2)}{t} = 0$. Therefore directional derivatives in all directions exist.
Note that $f_x(0,0) = 0$ and $f_y(0,0) = 1$. If f is differentiable at $(0,0)$ then $f'(0,0) = (0,1)$. Now

$$\begin{aligned} f(\Delta x,\Delta y) - f(0,0) = f(\Delta x,\Delta y) &= \Delta x + \frac{(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2} - \Delta x \\ &= \Delta x + \frac{(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\Delta y - \Delta x. \end{aligned}$$

Here $\epsilon_1 = -1$, $\epsilon_2 = \frac{(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} \nrightarrow 0$ as $\Delta x, \Delta y \to 0$.
Therefore the function is not differentiable at $(0,0)$.

Q6. Examine the following functions for local maxima, local minima and saddle points: (i) $4xy - x^4 - y^4$, (ii) $x^3 - 3xy$, (iii) $(x^2+y^2)\exp^{-(x^2+y^2)}$.

Ans. (i) For $f(x,y) = 4xy - x^4 - y^4$, $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0), (1,1)$ or $(-1,-1)$. These are the critical points. By second derivative test, $(0,0)$ ia a saddle point and $(-1,1)$ and $(1,1)$ are local maxima.

(ii) $f(x,y) = x^3 - 3xy^2$, $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ for $(x_0, y_0) = (0,0)$. So $(0,0)$ is the only critical point. Second derivative fails here. Along $y = 0, f(x,y) = x^3$, hence $(0,0)$ is a saddle point.
(iii) Similar, leave it to the students as an exercise.

Q7. Let $f(x,y) = 3x^4 - 4x^2y + y^2$. Show that f has a local minimum at $(0,0)$ along every line through $(0,0)$. Does f have a minimum at $(0,0)$? Is $(0,0)$ a saddle point for f?

Ans. Let $f(x,y) = 3x^4 - 4x^2y + y^2$. Along, the $x-$axis, the local minimum of the function is at $(0,0)$. Let $x = r\cos\theta$ and $y = r\sin\theta$, for a fixed $\theta \neq 0, \pi$ (or let $y = mx$). Then, $f(r\cos\theta, r\sin\theta) = 3r^4\sin^4\theta - 4r^3\cos 2\theta\sin\theta + r^2\sin^2\theta$ which is function of one variable. By the second derivative test (for functions of one variable), we see that $(0,0)$ is a local minima.
Since $f(x,y) = (3x^2 - y)(x^2 - y)$, we see that in the region between the parabolas $3x^2 = y$ and $y = x^2$, the function takes negative values and is positive everywhere else. Thus, $(0,0)$ is a saddle point for f.

Q8. Find the absolute maxima of $f(x,y) = xy$ on the unit disc $\{f(x,y) : x^2 + y^2 \leq 1\}$.

Ans. Given that $f(x,y) = xy$. Clearly, f is differentiable so f can assume extreme values at the points where $f_x = 0$, $f_y = 0$ and boundary points on the disk.
$f_x = 0$, $f_y = 0 \Longrightarrow (x,y) = (0,0)$. The value of f at $(0,0)$ is $f = 0$.
On the boundary of the disk we have $f(x,y) = g(x) = x\sqrt{1-x^2}$, $-1 \leq x \leq 1$. For maxima/minima we have $g'(x) = 0$. This gives $x = \pm\frac{1}{\sqrt{2}}$ and for this value of x, we have $y = \pm\frac{1}{\sqrt{2}}$. Moreover, $g''(x) = -\frac{1}{2} < 0$ at $x = \frac{1}{\sqrt{2}}$ and $g''(x) = \frac{1}{2}$ at $x = -\frac{1}{\sqrt{2}}$. Therefore, the function function $f(x,y)$ takes values $-\frac{1}{2}$ and $\frac{1}{2}$ at the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ respectively.
Thus all maxima/minima for f are $-\frac{1}{2}, 0, \frac{1}{2}$. Hence, the maximum of $f(x,y)$ is $\frac{1}{2}$ which occur at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and the minimum is $-\frac{1}{2}$ which occur at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Q9. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the $z-$axis.

Ans. $f(x,y,z) = x + 2yz + xyz^2 = 0$. Any point P_0 on the $z-$axis is of the form $(0,0,z_0)$. The gradient is

$$\nabla f|_{P_0} = ((1 + yz^2)\overrightarrow{i} + (2z + xz^2)\overrightarrow{j} + 2(y + xyz)\overrightarrow{k})_{(0,0,z_0)} = \overrightarrow{i} + 2z_0\overrightarrow{j}.$$

Equation of the normal lines is given by

$$\frac{x-0}{1} = \frac{y-0}{2z_0} = \frac{z-z_0}{0}$$

Solving, we get

$$y = 2xz_0, z = z_0.$$

Eliminating z_0, we get equation of the surface as

$$2xz - y = 0.$$

Q10. Given n positive numbers $a_1, a_2, \ldots, a_n$, find the maximum value of the expression the function $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ where $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$.

Ans. Note that here $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$.
Using the method of lagrange multipliers let λ be such that $\nabla f = \lambda \nabla g$. This gives,

$$a_1 = \lambda x_1, a_2 = \lambda x_2, \ldots, a_n = \lambda x_n \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 - 1 = 0.$$

Therefore, $a_1^2 + a_2^2 + \ldots + a_n^2 = 4\lambda^2$. This gives $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$. Since the continuous function f achieves its minimum and maximum on the closed and bounded set $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$, $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$ leads to the maximum value $f\left(\frac{a_1}{2\lambda}, \frac{a_2}{2\lambda}, \ldots, \frac{a_n}{2\lambda}\right) = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$ and $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$ leads to the minimum value of f.

Q11. Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

Ans. Let the box have sides of length $x, y, z > 0$. Then $V(x, y, z) = xyz$ and $xy+yz+xz = 10$. Using the method of lagrange multipliers, we see that $yz = \lambda(y+z), xz = \lambda(x+z)$ and $xy = \lambda(x+y)$. It is easy to see that $x, y, z > 0$. Now, we can see that $x = y = z$ and therefore, $x = y = z = \sqrt{\frac{10}{3}}$.

Q12. Minimize the function $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

Ans. Let $F(x, y, z) = x^2 + y^2 + z^2, g(x, y, z) = x + 2y + 3z$ and $h(x, y, z) = x + 3y + 9z$, where $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.
Using the method of lagrange multipliers let λ and μ be such that $\nabla F = \lambda\nabla h+\mu\nabla g$.
We get

$$\lambda + \mu = 2x, 2\lambda + 3\mu = 2y \quad \text{and} \quad 3\lambda + 9\mu = 2z. \tag{1}$$

From here, using $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$, we get $7\lambda + 17\mu = 6$ and $34\lambda + 91\mu = 18$.
Hence, $\mu = -\frac{78}{59}$ and $\lambda = \frac{240}{59}$.
From equation (1), we get $2(x^2 + y^2 + z^2) = 6\lambda + 9\mu$, hence the minimum value of of f is $\frac{369}{59}$.

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MATH-I ■ Assignment #8

(Applications of Integration, Vectors, Curves, Surfaces, Vector Functions)

Q1. Sketch the graphs $r = \cos(2\theta)$ and $r = \sin(2\theta)$. Also, find their points of intersection.

Q2. Find the area of the inner loop of $r = 2 + 4\cos\theta$.

Q3. Find the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Q4. Find the length of the following curves
(a) $C_1 : y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$, $1 \leq x \leq 4$
(b) $C_2 : \{(a\cos^3 t, a\sin^3 t) : t \in [0, 2\pi]\}$ for some $a > 0$,
(c) $C_3 : r = a(1 + \cos\theta)$, where $a > 0, 0 \leq \theta \leq 2\pi$.

Q5. Determine the equation of the cylinder generated by a line through the curve $(x-2)^2+y^2 = 4$, $z = 0$ moving parallel to the vector $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

Q6. Determine the equation of a cone with vertex $(0, -a, 0)$ generated by a line passing through the curve $x^2 = 2y$, $z = h$.

Q7. Reparametrize the following curves in terms of arc-length
(a) $c(t) = \frac{t^2}{2}\overrightarrow{i} + \frac{t^3}{3}\overrightarrow{k}$,
(b) $c(t) = 2\cos t\,\overrightarrow{i} + 2\sin t\,\overrightarrow{j}$.

Q8. If a plane curve has the Cartesian equation $y = f(x)$ where f is a twice differentiable function, then show that the curvature at the point $(x, f(x))$ is

$$\frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

Q9. Show that the parabola $y = ax^2$, $a \neq 0$ has its largest curvature at its vertex and has no minimum curvature.

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MATH-I ■ Assignment #9

(Calculus of functions of several variables, Directional derivatives, Max/Min and Lagrange Multipliers)

Q1. Examine the following functions for continuity at the point $(0,0)$ where $f(0,0)=0$ and $f(x,y)$ for $(x,y)\neq(0,0)$ is given by
(a) $|x|+|y|$, (b) $\frac{-x}{\sqrt{x^2+y^2}}$, (c) $\frac{x^4-y^2}{x^4+y^2}$, (d) $\frac{x^4}{x^4+y^2}$.

Q2. Let $f(x,y)$ be defined in $S=\{f(x,y)\in\mathbb{R}^2 : a<x<b, c<y<d\}$. Suppose that the partial derivatives of f exist and are bounded in S. Then show that f is continuous in S.

Q3. Let

$$f(x,y)=\begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & \text{if} \quad (x,y)\neq(0,0)\\ 0, & \text{if} \quad (x,y)=(0,0).\end{cases}$$

Prove that
(a) $f_x(0,y)=-y$ and $f_y(x,0)=x$ for all x and y,
(b) $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$ and
(c) $f(x,y)$ is differentiable at $(0,0)$.

Q4. Suppose f is a function with $f_x(x,y)=f_y(x,y)=0$ for all (x,y). Then show that f is constant.

Q5. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases} \frac{y^3}{x^2+y^2}, & \text{if } (x,y)\neq 0\\ 0, & \text{otherwise.}\end{cases}$$

Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Q6. Examine the function $f(x,y)=4xy-x^4-y^4$ for local maxima, local minima and saddle points.

Q7. Let $f(x,y)=3x^4-4x^2y+y^2$. Show that f has a local minimum at $(0,0)$ along every line through $(0,0)$. Does f have a minimum at $(0,0)$? Is $(0,0)$ a saddle point for f?

Q8. Find the absolute maxima of $f(x,y)=xy$ on the unit disc $\{f(x,y): x^2+y^2\leq 1\}$.

Q9. Given n positive numbers $a_1,a_2,\ldots,a_n$, find the maximum value of the expression the function $a_1x_1+a_2x_2+\ldots+a_nx_n$ where $x_1^2+x_2^2+\ldots+x_n^2=1$.

Q10. Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

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MATH-I ■ Assignment #10

(Double/Triple Integrals, Green's/Stoke's/Gauss's Theorems)

Q1. Evaluate the integral $\int\int_R (x+y)^2 dxdy$ over the triangle R with vertices $(0,0)$, $(2,2)$ and $(0,1)$.

Q2. Sketch the region R in the $xy-$plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

Q3. Evaluate $\int\int_R x\, dxdy$ where R is the region $1 \le x(1-y) \le 2$ and $1 \le xy \le 2$.

Q4. Change the order of integration to prove that
(a) $\int_0^x \int_0^u exp(m(x-t))f(t)\, dtdu = \int_0^x (x-t)exp(m(x-t))f(t)\, dt,$
(b) $\int_0^x \int_0^v \int_0^u exp(m(x-t))f(t)\, dtdudv = \int_0^x \frac{(x-t)^2}{2} exp(m(x-t))f(t)\, dt.$

Q5. Find the volume of the region B bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy-$plane.

Q6. Find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$.

Q7. Compute $\int\int_S xy d\sigma$, where S is the surface of the cone $x = r\cos t$, $y = r\sin t$, $z = r$ for $0 \le r \le 1$ and $0 \le t \le 2\pi$.

Q8. Find the line integral of the vector field $F(x,y,z) = y\hat{i} - x\hat{j} + \hat{k}$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \le t \le 2\pi$ joining $(1,0,0)$ to $(1,0,1)$.

Q9. Evaluate $\int_C \frac{xdy - ydx}{x^2+y^2}$ along any simple closed curve in the $xy-$plane not passing through the origin. Distinguish the cases where the region R enclosed by C:
(a) includes the origin (b) does not include the origin.

Q10. Show that the integral $\int_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1,0,0)$ and $(2,1,4)$.

Q11. Use Green's Theorem to compute $\int_C (2x^2 - y^2)dx + (x^2+y^2)dy$ where C is the boundary of the region $\{(x,y) : x, y \ge 0 \;\&\; x^2 + y^2 \le 1\}$.

Q12. Use Stoke's Theorem to evaluate the line integral $\int_C (-y^3 dx + x^3 dy - z^3 dz)$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the $xy-$plane.

Q13. Verify the Stoke's Theorem where $\overrightarrow{F} = (y, z, x)$ and S is the part of the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$, oriented with $\overrightarrow{n}$ pointing outward.

Q14. Let $\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|^3}$ where $\overrightarrow{r} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$ and let S be any surface that surrounds the origin. Prove that $\int\int_S F \cdot n d\sigma = 4\pi$.

Q15. Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\overrightarrow{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\int\int_{\partial D} \overrightarrow{F}.\overrightarrow{n}\, d\sigma$.