The LNM Institute of Information Technology
Jaipur, Rajasthan

MATH-I ■ Solutions Assignment #6

(Riemann Integration & Improper Integrals)

Q1. Let $f : [0,1] \to \mathbb{R}$ be defined as $f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 10^9 & \text{if} \quad x = 1 \end{cases}$. Show that f is integrable on $[0,1]$ and that $\int_0^1 f = 1$.

Ans. Let $P = \{x_0, \cdots, x_n\}$ be a partition of $[0,1]$. Note that $m_i(f) = M_i(f) = 1$ for $i = 1, 2, \cdots, n-1$ and $M_n(f) = 10^9$ and $m_n(f) = 1$. Hence, we have

$$L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = 1,$$
$$U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) = \sum_{i=1}^{n-1}(x_i - x_{i-1}) + M_n(f)(x_n - x_{n-1}) = x_{n-1} + 10^9(1 - x_{n-1})$$

Claim. $\inf\{U(P,f) : P$ *is a partition of* $[0,1]\} = \inf\{t + 10^9(1-t) : 0 < t < 1\} = 1.$ *First we show that* 1 *is a lower bound. Note that* $10^9 > 1$. *Hence* $10^9(1-t) > 1 - t$ *if* $t < 1$. *This implies* $t + 10^9(1-t) > 1$. *In order to show that* 1 *is infimum, let* $\epsilon > 0$ *be given. Then we can find* $t \in (0,1)$ *such that* $t > \dfrac{10^9 - 1 - \epsilon}{10^9 - 1}$. *This would imply* $(10^9 - 1)t > 10^9 - 1 - \epsilon \implies 1 + \epsilon > t + 10^9(1-t)$.

Q2. Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 < x \le 1. \end{cases}$$

Is the function continuous on $[-1,1]$? Is the function Riemann integrable?

Ans. Clearly f is not continuous at $x = 0$. Since f is monotonically increasing on $[-1,1]$ hence it is integrable.

Q3. Does there exist a continuous function f on $[0,1]$ such that

$$\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}} \quad \text{for all} \quad n \in \mathbb{N}.$$

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Ans. Suppose there is such a function f and $\sup f = M$. Then

$$\frac{1}{\sqrt{n}} = \left|\int_0^1 f(x)x^n dx\right| \leq M\left|\int_0^1 x^n dx\right| = \frac{M}{n+1}.$$

This implies that $1 \leq \frac{M\sqrt{n}}{n+1} \to 0$ which is a contradiction.

Q4. For each $n \in \mathbb{N}$, let $g_n : [0,1] \to \mathbb{R}$ be defined as $g_n(x) := \begin{cases} \frac{(n+1)x^n}{1+x}, & \text{if } 0 \leq x < 1 \\ 0, & x = 1. \end{cases}$.

Then prove that $\lim_{n\to\infty}\int_0^1 g_n(x)dx = \frac{1}{2}$ whereas $\int_0^1 \lim_{n\to\infty} g_n(x)dx = 0$.

Ans.

Claim. $\lim_{n\to\infty} g_n(x) = 0$ *for all* $x \in [0,1]$.

Proof. 1. If $x = 1, 0$, then $g_n(x) = 0$ for all $n \in \mathbb{N}$.

2. If $x \in (0,1)$, then $g_n(x) > 0$ for all n. Also

$$\lim_{n\to\infty}\frac{g_{n+1}(x)}{g_n(x)} = \lim_{n\to\infty}\frac{(n+2)x^{n+1}}{1+x} \times \frac{1+x}{(n+1)x^n} = \lim_{n\to\infty}\frac{(n+2)x}{n+1} = x < 1$$

By ratio test, it follows that $g_n(x) \to 0$.

□

Therefore $\int_0^1 \lim_{n\to\infty} g_n(x)dx = 0$.

For the other part, use integration by parts to see that $\int_0^1 \frac{(n+1)y^n}{1+y}dy = \frac{1}{2} + \int_0^1 \frac{y^{n+1}}{(1+y)^2}dy$. Note that $\int_0^1 \frac{y^{n+1}}{(1+y)^2}dy \leq \int_0^1 y^{n+1}dy = \frac{1}{n+2} \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_0^1 g_n(y)dy = \frac{1}{2}.$$

Q5. Let $f : [-1,1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} a, & \text{if } -1 \leq x < 0 \\ 0, & \text{if } x = 0 \\ b, & 0 < x \leq 1 \end{cases}$$

Assume that $a, b > 0$.

Ans. Without loss of generality assume that $a < b$. For each $\epsilon > 0$, find a partition $P := \{-1, \frac{-1}{N}, \frac{1}{N}, 1\}$ of $[-1, 1]$ such that $\frac{2b}{N} < \epsilon$.This follows from Archimedian property. Note that $U(P, f) = a(1-\frac{1}{N})+b(1+\frac{1}{N})$ and $L(P, f) = a(1-\frac{1}{N})+b(1-\frac{1}{N})$. Therefore $U(P, f) - L(P, f) = \frac{2b}{N} < \epsilon$. (For more details see Page 183, Ajit kumar-Kumaresan book of Real Analysis.)

Q6. Consider $a_n := \displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{n^2 + in}}$ for $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$.

Ans. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1 + \frac{i}{n}}}$. Therefore using Riemann sum

$$\lim_{n\to\infty} a_n = \int_0^1 \frac{1}{\sqrt{1+x}} = 2(\sqrt{2} - 1).$$

Q7. Find the intervals in which the function $f(x) = 2x^3 + 2x^2 - 2x - 1$ is convex, concave, increasing, decreasing. Also find local maxima, local minima and point of inflection.

Ans. Let $f(x) = x^3 - 6x^2 + 9x + 1$. Note that $f'(x) = 3(x - 1)(x - 3)$. Therefore, f is increasing on $(-\infty, 1) \cup (3, \infty)$ and f is decreasing on $(1, 3)$. Moreover, f has a local maximum at $x = 1$ and local minimum at $x = 3$. Since $f''(x) = 6(x - 2)$, f is convex on $(2, \infty)$ and concave on $(-\infty, 2)$. Moreover, f has a point of inflection at $x = 2$.