# Lecture 12: Continuity

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**Definition 12.1** *Let $D \subseteq R$. Consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that $f$ is continuous at $c$ if for every sequence $(x_n)$* **in** *$D$ such that $x_n \to c$, we have $f(x_n) \to f(c)$. If $f$ is not continuous at $c$, we say that $f$ is discontinuous at $c$. In case $f$ is continuous at every $c \in D$, we say that $f$ is continuous on $D$.*

**Remark 12.2** *The crucial point of the definition is that (i) the sequences are in the domain of $f$ converging to a point of the domain and (ii) we need to verify the condition of the definition for each such sequence in the domain converging to $c$. The second crucial point is that even if $x_n \to c$, it may happen that $(f(x_n))$ may converge to a limit other than $f(a)$ or worse, $(f(x_n))$ may not converge at all.*

**Example 12.3**

1. *Let $a$ and $b$ be real numbers and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax+b$ for $x \in \mathbb{R}$. Then $f$ is continuous on $\mathbb{R}$. To see this, let $c \in \mathbb{R}$ and $(x_n)$ be any sequence in $\mathbb{R}$ such that $x_n \to c$. By parts (i) and (ii) of Limit Theorems for sequences, $ax_n + b \to ac + b$, that is, $f(x_n) \to f(c)$. Thus $f$ is continuous on $\mathbb{R}$.*

2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$$

*Then $f$ is discontinuous at every $c \in \mathbb{R}$. To see this, we note that if $c$ is rational then by density of rationals in $\mathbb{R}$ and Sandwich Theorem we can find a sequence $(x_n)$ of irrationals such that $x_n \to c$. Then $f(x_n) = 0$ for all $n \in \mathbb{N}$, while $f(c) = 1$. On the other hand, if $c$ is irrational and then by density of irrationals in $\mathbb{R}$ and Sandwich Theorem we can find a sequence $(x_n)$ of rationals such that $x_n \to c$. Then $f(x_n) = 1$ for all $n \in \mathbb{N}$, while $f(c) = 0$. Thus in both cases, $x_n \to c$, but $f(x_n) \nrightarrow f(c)$. This function is known as the Dirichlet function.*

**Example 12.4** *Let $f(x) = 0$ when $x$ is rational and $f(x) = x$ when $x$ is irrational. We will see that this function is continuous only at $x = 0$. Let $(x_n)$ be any sequence such that $x_n \to 0$. Because, $|f(x_n)| \leq |x_n|$, $f(x_n) \to f(0)$. Therefore $f$ is continuous at $0$. Suppose $x_0 \neq 0$ and it is rational. We will show that $f$ is not continuous at $x_0$. Choose $(x_n)$ such that*

*$x_n \to x_0$ and all $x_n$'s are irrational numbers. Then $f(x_n) = x_n \to x_0 \neq f(x_0)$. This proves that $f$ is not continuous at $x_0$.*

*When $x_0$ is irrational, Choose $(x_n)$ such that $x_n \to x_0$ and all $x_n$'s are rational numbers. Then $f(x_n) = 0 \to 0 \neq = x_0 = f(x_0)$. This proves that $f$ is not continuous at $x_0$.*

**Example 12.5** *Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}, |f(x) - f(y)| \leq |x - y|$. Show that $f$ is continuous.*

**Solution:** Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Since $|f(x_n) - f(x_0)| \leq |x_n - x_0|$, $f(x_n) \to f(x_0)$. Therefore $f$ is continuous at $x_0$. Since $x_0$ is arbitrary, $f$ is continuous everywhere. ■

**Example 12.6** *Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where $f$ takes the value 0. Show that $f(0) = 0$.*

**Solution:** For every $n$, there exists $x_n \in (-1/n, 1/n)$ such that $f(x_n) = 0$. Since $f$ is continuous at 0 and $x_n \to 0$, we have $f(x_n) \to f(0)$. Therefore, $f(0) = 0$. ■