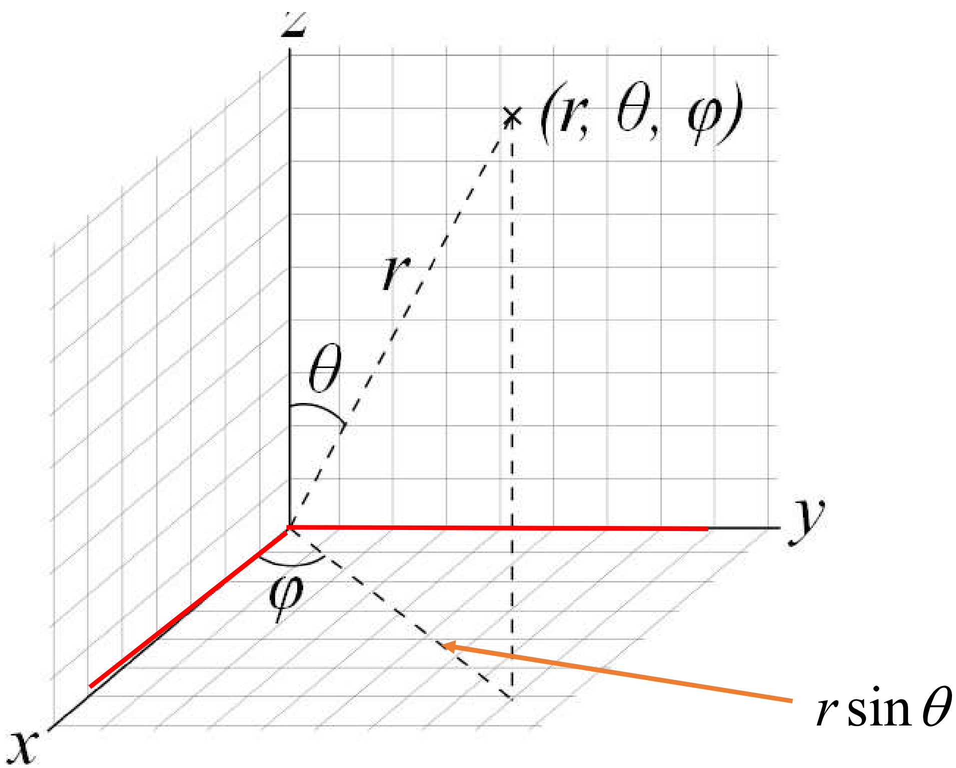
Spherical Co-ordinate



$$0 \le r \le \infty$$

$$0 \le \theta \le \pi$$

$$0 \le \varphi \le 2\pi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

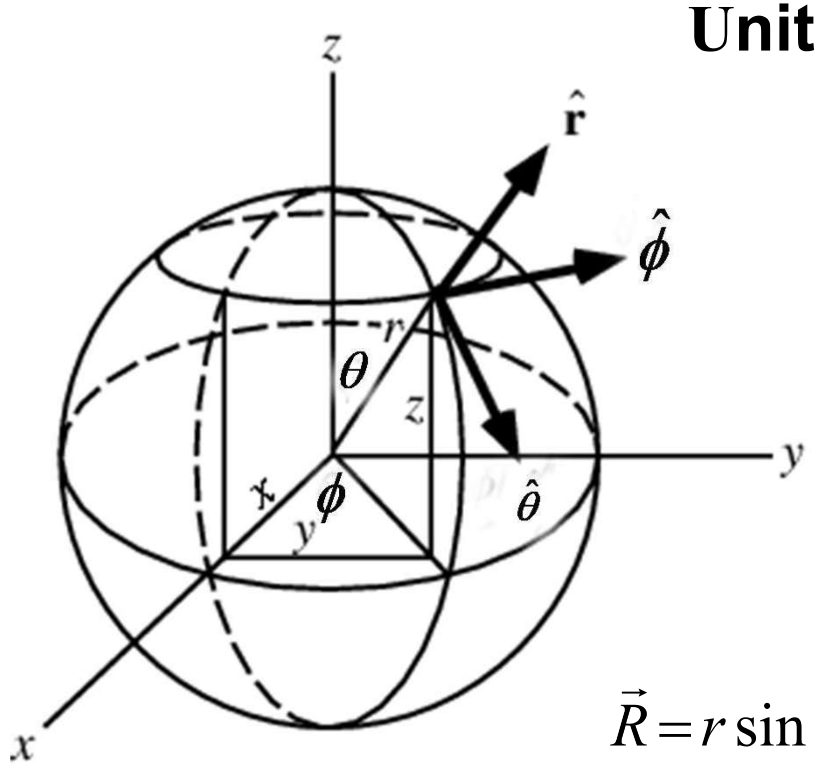
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Unit Vectors



$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{r} = \frac{\dfrac{\partial \vec{R}}{\partial r}}{\left|\dfrac{\partial \vec{R}}{\partial r}\right|} \qquad \hat{\theta} = \frac{\dfrac{\partial \vec{R}}{\partial \theta}}{\left|\dfrac{\partial \vec{R}}{\partial \theta}\right|} \qquad \hat{\varphi} = \frac{\dfrac{\partial \vec{R}}{\partial \varphi}}{\left|\dfrac{\partial \vec{R}}{\partial \varphi}\right|}$$

$$\vec{R} = r\sin\theta\cos\varphi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\hat{k}$$

$$\hat{r} = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

http://mathworld.wolfram.com/SphericalCoordinates.html

$$\vec{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$$

$$\vec{A} = a_r \hat{r} + a_\theta \hat{\theta} + a_\phi \hat{\phi}$$

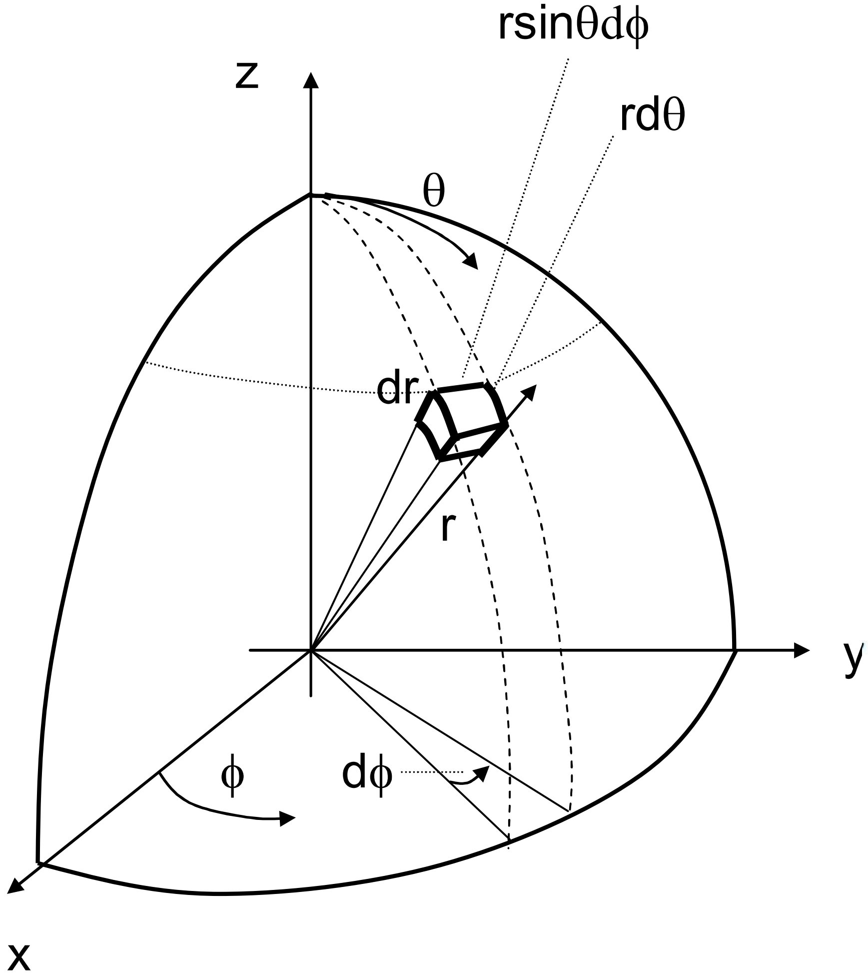
$$a_x = a_r \sin\theta\cos\phi + a_\theta \cos\theta\cos\phi - a_\phi \sin\phi$$

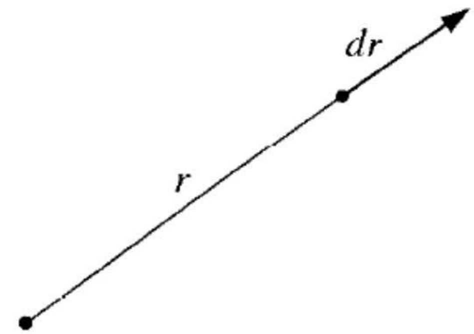
$$a_y = a_r \sin\theta\sin\phi + a_\theta \cos\theta\sin\phi + a_\phi \cos\phi$$

$$a_z = a_r \cos\theta - a_\theta \sin\theta$$

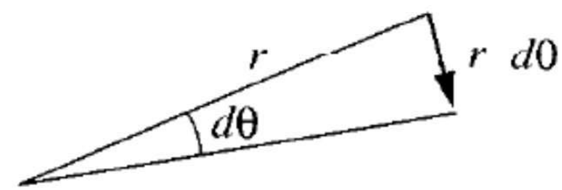
$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}$$

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

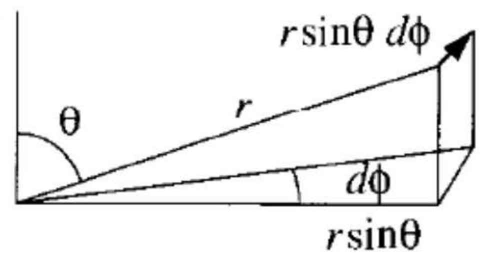




$$dl_r = dr.$$



$$dl_\theta = r\,d\theta.$$



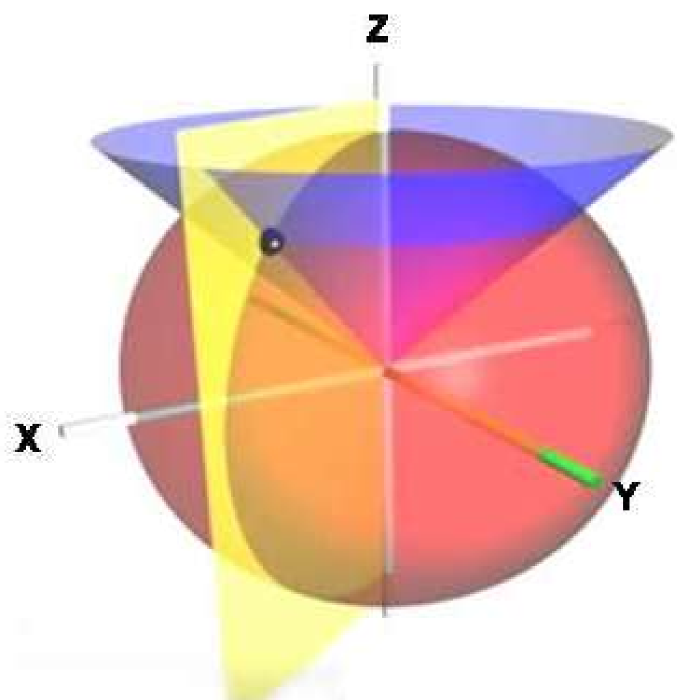
$$dl_\phi = r\sin\theta\,d\phi$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

The infinitesimal volume element $d\tau$

$$d\tau = dl_r\, dl_\theta\, dl_\phi = \boxed{r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

The differential surface elements



$$d\vec{a}_r = dI_\theta \hat{\theta} \times dI_\phi \hat{\phi} = r d\theta \hat{\theta} \times r \sin\theta d\phi \hat{\phi}$$
$$= r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$d\vec{a}_\theta = dI_\phi \hat{\phi} \times dI_r \hat{r} = r \sin\theta d\phi \hat{\phi} \times dr \hat{r}$$
$$= r \sin\theta\, d\theta\, dr\, \hat{\theta}$$

$$d\,\vec{a}_\phi = dI_\theta \hat{\theta} \times dI_r \hat{r} = r d\theta \hat{\theta} \times dr \hat{r}$$
$$= r dr d\theta \hat{\phi}$$

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\nabla \times \mathbf{v} \;&=\; \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ &+\; \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Question Check the divergence theorem.

$$\vec{V} = r\hat{r} + r\sin\theta\hat{\theta} + \hat{\phi}$$

Problem 1.53 Check the divergence theorem for the function

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}},$$

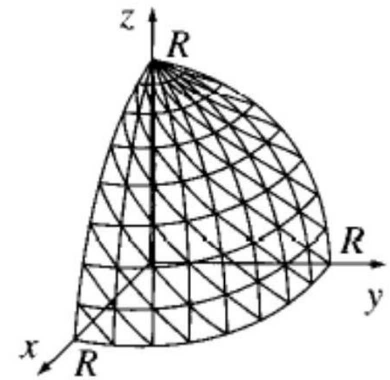


Figure 1.48

$$d\vec{a}_r = dI_\theta\hat{\theta}\times dI_\phi\hat{\phi} = rd\theta\hat{\theta}\times r\sin\theta d\phi\hat{\phi}$$
$$= r^2\sin\theta\, d\theta\, d\phi\,\hat{r}$$

$$d\vec{a}_\theta = dI_\phi\hat{\phi}\times dI_r\hat{r} = r\sin\theta d\phi\hat{\phi}\times dr\hat{r}$$
$$= r\sin\theta\, d\theta\, dr\,\hat{\theta}$$

$$d\,\vec{a}_\phi = dI_\theta\hat{\theta}\times dI_r\hat{r} = rd\theta\hat{\theta}\times dr\hat{r}$$
$$= rdrd\theta\hat{\phi}$$

Problem 1.58 Check the divergence theorem for the function

$$\mathbf{v} = r^2\sin\theta\,\hat{\mathbf{r}} + 4r^2\cos\theta\,\hat{\boldsymbol{\theta}} + r^2\tan\theta\,\hat{\boldsymbol{\phi}},$$

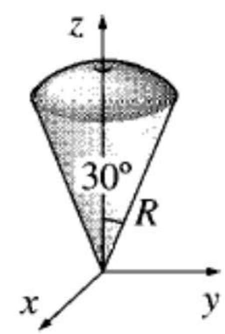


Figure 1.52