The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #2

(Sequences)

Q1. Prove that limit of a sequence is always unique.

Q2. Prove or disprove: the sequence (x_n) is convergent using the definition of convergent if x_n is as follows:

1. $x_n = 2$
2. $x_n = \frac{1}{n^2+2n-4}$
3. $x_n = \frac{n^2+3n}{3n^2+5}$

(Hint: First guess what is limit "l".)

Q3. Let $a_n \to a$ and $a \neq 0$. Then using the definition of limit of a sequence, show that there is $m \in \mathbb{N}$ such that $\frac{|a_n|}{2} > 0$ for all $n \geq m$.
(Hint: $a \neq 0$ implies $|a| > 0$. Also use inequality $||b| - |a|| \leq |b - a|$.)

Q4. Investigate the convergence/divergence of the following sequences:
(a) $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}$
(b) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$
(c) $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0,1)$
(d) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$
(e) $x_n = \frac{n!}{(2n+1)!}$
(f) $x_n = \frac{n}{4^n}$
(g) $x_n = (n!)^{\frac{1}{n^2}}$.

Q5. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.

Q6. Prove that the sequence $\left(n^{\frac{1}{n}}\right)$ converges to 1 as $n \to \infty$.
(Hint: Consider $d_n := n^{\frac{1}{n}} - 1$. Then prove $d_n \to 0$ using binomial expansion and Squeeze theorem.)

Q7. Prove or disprove: "Every bounded sequence of real number is convergent."

Q8. Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for all $n \in \mathbb{N}$. Using Mathematical Induction prove that $\frac{x_{n+1}}{x_n} < 1$ for every $n \in \mathbb{N}$. However, prove that (x_n) does not converge to 0.