Lecture 02: Supremum and Infimum

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2.1 Supremum

Once again as far as the definition is concerned, we are not saying that supremum of a set is unique, ('a supremum'). Here is another very important point that in mathematics that, we try to put minimum hypothesis to define anything and rest of the properties are deduced by the logical arguments. So definition does not say itself that supremum of a set is unique but we can deduce that supremum has to be unique.

Proposition 2.1 *If S has a supremum, then it is unique.*

Proof: Let $M, M' \in \mathbb{R}$ be two supremum of S. First using the fact that M is an upper bound and M' is supremum we have $M' \leq M$. Now using the fact that M' is an upper bound and M is supremum we have $M \leq M'$. This implies $M' = M$. ∎

From the Proposition 2.1, it follows that whenever supremum of a set S exists, we can call it the supremum of set S rather than a supremum of set S, we denote it by $\sup S$.

Proposition 2.2 (Necessary & Sufficient condition for the supremum) *Let S be a nonempty subset of $\mathbb{R}$ and α be a real number. Then $\alpha = \sup S$ if and only if*

(a) α is an upper bound of S.

(b) If for each $\epsilon > 0$, there is some $x \in S$ such that $\alpha - \epsilon < x$.

Proof:

Necessity: Suppose $\alpha = \sup S$. Then by definition α is an upper bound of S. Now we prove condition (b) by contradiction. Suppose that condition (b) is false. Then $\exists\ \epsilon > 0$ such that $\alpha - \epsilon < x$, for all $x \in S$. Then $x \leq \alpha - \epsilon < \alpha$ for all $x \in S$. This contradicts that α is supremum.

Sufficiency: We assume condition (a) and (b). It is left to show that if M is an upper bound for S then $\alpha \leq M$. Assume contrary, that $\alpha > M$. Then for $\epsilon = \alpha - M > 0$, there is a $x \in S$ such that $\alpha - (\alpha - M) < x$. That is $M < x$. This contradicts that M is an upper bound for S. ■

Example 2.3 *Let* $S_1 = \{1, 2, 3, 4\}, S_2 = (0, 2)$. *Show that* $\sup S_1 = 4, \sup S_2 = 2$.

Solution: 4 is clearly an upper bound for S_1. Let $\epsilon > 0$ is given. Then $4 \in S_1$ is such that $4 - \epsilon < 4$. Hence by Proposition 2.2, it follows that $\sup S_1 = 4$.

For S_2, note that 2 is an upper bound. Let $\epsilon > 0$ is given. Then for any $x \in (\alpha, 2) \subset S_2$, where $\alpha = \max\{2 - \epsilon, 0\}$, we have $2 - \epsilon < x$

2.2 Infimum

Definition 2.4 *We say that S is bounded below if there exists $\beta \in \mathbb{R}$ such that $x \geq \beta$ for all $x \in S$. Any such β is called a lower bound of S.*

For example, $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \right\}$, $\{x \in \mathbb{Q} | x^2 < 2\}$ are bounded below by 1 and -2 respectively.

Definition 2.5 *An element $m \in \mathbb{R}$ is called a infimum or a greatest lower bound of the set S if*

1. *m is a lower bound of S, that is, $x \geq m$ for all $x \in S$, and*
2. *$m \geq \beta$ for any lower bound β of S.*

It follows from the definition that whenever infimum of a set exists, it is unique.

Proposition 2.6 (Necessary & Sufficient condition for the infimum) *Let S be a nonempty subset of $\mathbb{R}$ and β be a real number. Then $\beta = \inf S$ if and only if*

(a) β is a lower bound of S.

(b) If for each $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$.

2.3 Maximum of a set versus Supremum of a Set

Let S be a non-empty subset of $\mathbb{R}$. If the supremum of set S is an element of S then it is called the maximum of S and denoted by $\max S$.

Example 2.7 *If* $S = (0, 1)$, *then* $\max S$ *does not exists, where as* $\sup S = 1$. *On the other hand if* $S = (0, 1]$ *then* $\max S = 1$.

Likewise, If the infimum of set S is an element of S then it is called the minimum of S and denoted by $\min S$.