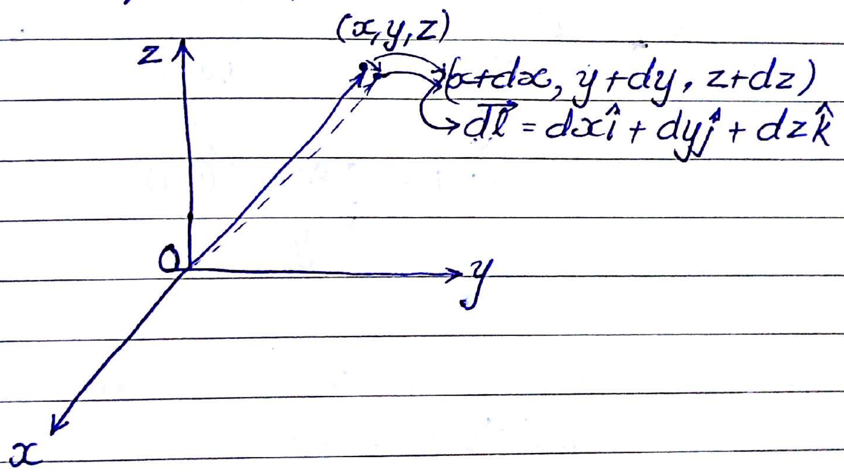
Mathematical Tools
Electrodynamics

* Reference Books:
(1) Introdⁿ to Electrodynamics by David J. Griffith;

* Criteria:
(1) Suprise quiz, attendance — 10%.
(2) End Term — 40%.

* Vector Calculus:

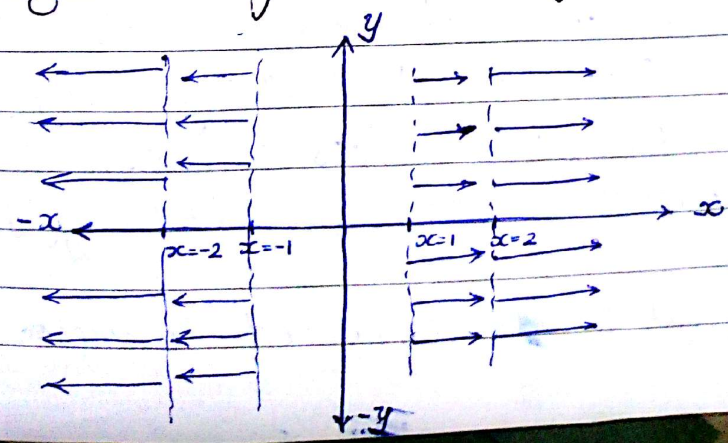
$\vec{r}$ = position vector
$\vec{\mathscr{r}}$ = separation vector
$\vec{dl}$ = infinitesimal displacement vector



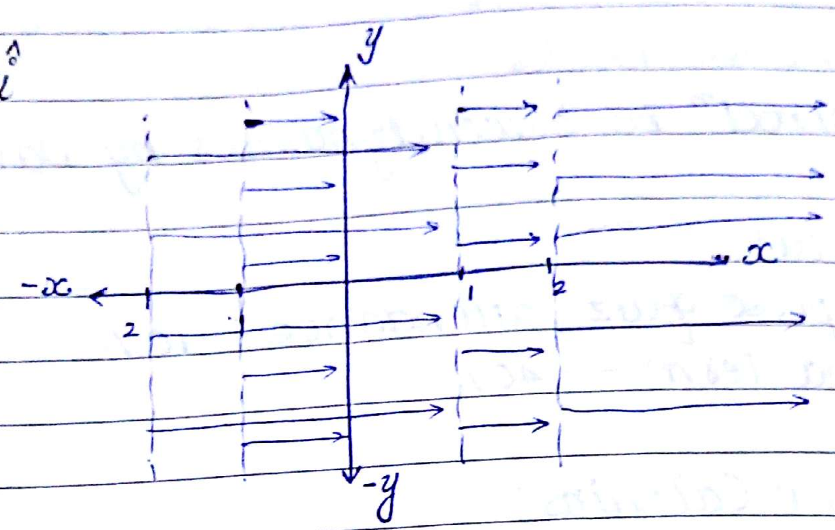
Temperature = scalar f^n (depends upon x,y,z)

→ Sketching Vector field / Vector f^n:

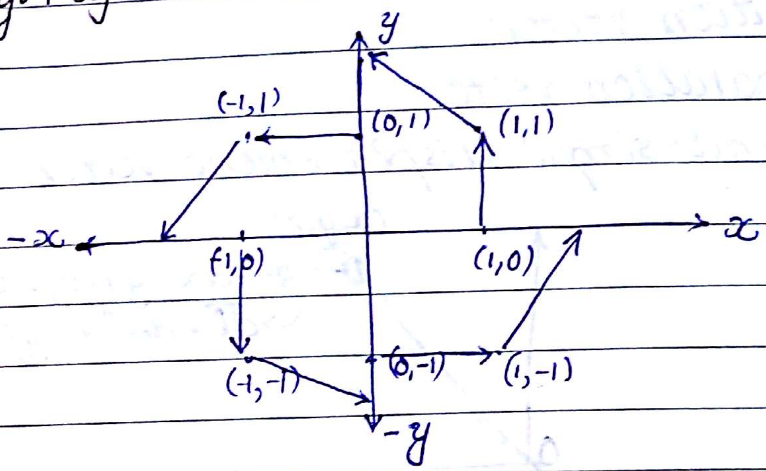
(i) $V = x\hat{i}$



(ii) $V = x^2\hat{i}$



(iii) $V = -y\hat{i} + x\hat{j}$



pt.	mag.	Dir.
1,0	1	$\hat{j}$
1,1	$\sqrt{2}$	$-\hat{i}+\hat{j}$
0,1	1	$-\hat{i}$
-1,1	$\sqrt{2}$	$-\hat{i}-\hat{j}$
-1,0	1	$-\hat{j}$
-1,-1	$\sqrt{2}$	$\hat{i}-\hat{j}$
0,-1	1	$\hat{i}$
1,-1	$\sqrt{2}$	$\hat{i}+\hat{j}$

Vector f^n: A f^n of vectors that exists in entire field or region of space.

- Gradient Operator (∇): $\vec{\nabla} = \left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}$

Ex: $T(x,y) = -(\cos^2 x + \cos^2 y)^2$

$\nabla T(x,y) = [-2(\cos^2 x + \cos^2 y) \times 2\cos x \times -\sin x]\hat{i}$
$+ [-2(\cos^2 x + \cos^2 y) \times 2\cos y \times -\sin y]\hat{j} + 0\hat{k}$
$= 4(\cos^2 x + \cos^2 y)[\cos x \sin x\, \hat{i} + \cos y \sin y\, \hat{j}]$
$= 2(\cos^2 x + \cos^2 y)[\sin 2x\, \hat{i} + \sin 2y\, \hat{j}]$

Scalar field can be converted to vector field.

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Electric field $\leftarrow E = -\nabla V \rightarrow$ potent. field

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If $T = T(x,y,z)$

$$dT = \frac{\partial T}{\partial x} \cdot dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$= \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \vec{\nabla} T \cdot \vec{dl}$$

$$= |\vec{\nabla} T||\vec{dl}|\cos\theta$$

greater magnitude shows that value of f^n changes fast.

& in direc^n of vector field we get max. value of f^n.

$$\underbrace{\nabla T}_{\text{Scalar field}} = \underbrace{\left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}}_{\text{vector field}}$$

- Divergence: Change of flux per unit vol.

Dot prod. of $\vec{\nabla}$ vector field with another vector field

$$\vec{\nabla} \cdot \vec{v} = \left[\left(\frac{\partial}{\partial x}\hat{i}\right) + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}\right] \cdot (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$\nabla \cdot \vec{v} \neq \vec{v} \cdot \nabla$

Ex: $v_c = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

$\therefore \nabla \cdot \vec{v_c} = 0\hat{i} \; 2x\hat{j} + 2y\hat{k}$

$= 2x\hat{j} + 2y\hat{k} = 2(x+y)$

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Meaning of divergence:
We consider a closed surface surrounding P & analyzing the flow over boundary. Thus, its change in flux per unit volume.

$\nabla \cdot \vec{F}$ = outflow − Inflow

Divergence = +ve if outflow > inflow.
⇒ there is a source (or there is ⊕ charge)
= −ve ⇒ there is sink

- The curl:

Cross-product of ∇ and vector $\vec{V}$. Its degree of rot^n of field

If $\nabla \times \vec{V}$ is non-zero ⇒ rotating field

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\quad \right) + \hat{k}\left(\quad \right)$$

Ex: $V_a = -y\hat{i} + x\hat{j}$ (drawn earlier)

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

If curl = 0, ⇒ Irrotational vector field

Ex: Electric field is conservative & irrotational field
Non-conservative field is always rotational field

$\nabla(fg) = f\nabla g + g\nabla f$

$\vec{\nabla} \times \vec{\nabla}(T) \equiv 0$ (Always) = Curl of gradient

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) \equiv 0$ (Always)

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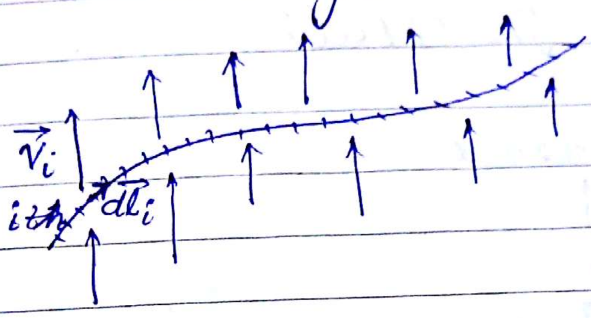
* Vector Line Integration:

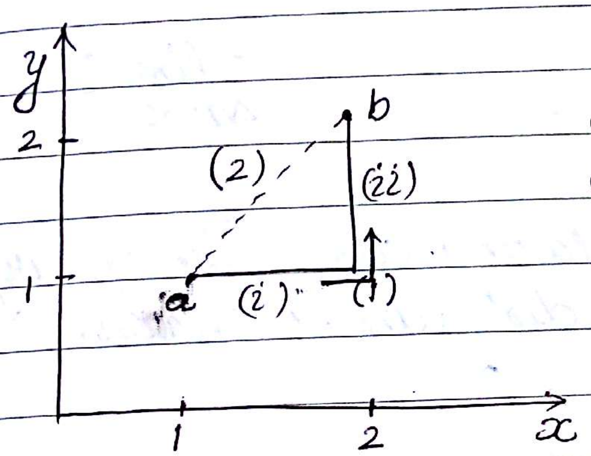
$\int_C \vec{V}\cdot\vec{dl}$ $\vec{V}$ = vector field

$\vec{dl}$ = infinitesimal displacement along path C.



$$\lim_{\Delta l_i \to 0} \sum \vec{V_i}\cdot\vec{\Delta l_i}$$

Ex:



(i)' $\vec{V} = -y\hat{i} + x\hat{j}$

(ii)' $\vec{V} = x\hat{i} + y\hat{j}$

(i) Solⁿ: $\int_{(i)} (-y\hat{i}+x\hat{j})\cdot(dx\hat{i}+dy\hat{j}) = \int -y\,dx$

$= -1\int_1^2 dx = -1$

$\int_{(ii)} (-y\hat{i}+x\hat{j})\cdot dy\hat{j} = 2\int_1^2 dy = 2$

$\therefore \int_{(1)} \vec{V}\cdot dl = (i) + (ii) = 2 + (-1) = 1$

$\int_{(2)} (-y\hat{i}+x\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \int -y\,dx + x\,dy \quad = 0$

(ii)' $\int_{(i)} (x\hat{i}+y\hat{j})\,dx\hat{i} = \int_1^2 x\,dx = \frac{3}{2}$

$\int_{(ii)} (x\hat{i}+y\hat{j})\,dy\hat{j} = \int_1^2 y\,dy = \frac{3}{2}$ $\quad \therefore \int_{(1)} = \frac{3}{2}+\frac{3}{2} = 3$

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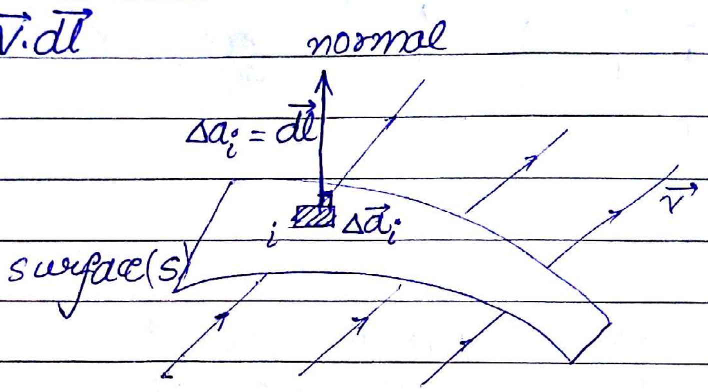
Vol. current density = current per unit area ⊥ to it
$(\vec{J})$

$$\int_{(2)} (x\hat{i} + y\hat{j})(dx\hat{i} + dy\hat{j}) = \int_1^2 x\,dx + \int_1^2 y\,dy = 3$$

$\to y = x \Rightarrow dy = dx$

* Vector Surface Integral:

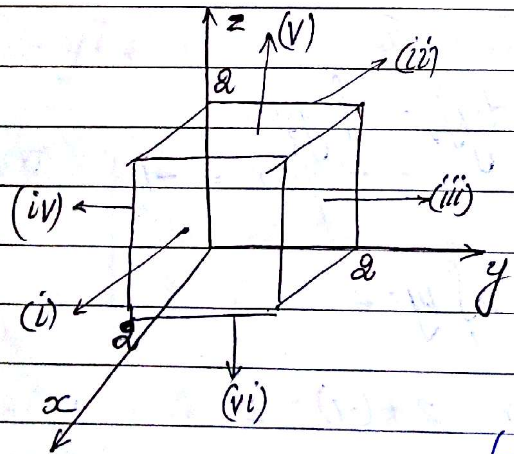
$$\iint_S \vec{V}\cdot d\vec{a}$$



$$= \lim_{\Delta a_i \to 0} \sum \vec{V}_i \cdot \Delta\vec{a}_i$$

In closed surface ⇒ Outward dirⁿ = dirⁿ of surface
In Open surface ⇒ Any dirⁿ can be chosen.

Ex:



(i) $\vec{V}_1 = -y\hat{i} + x\hat{j}$
(ii) $\vec{V}_2 = -y\hat{i} + xy\hat{j}$
(iii) $\vec{V}_3 = xy\hat{i} + yz\hat{j} + zx\hat{k}$

Solⁿ: $$\iint_{(ii)} \vec{V}\cdot d\vec{s}_{yz} = \iint(-y\hat{i} + x\hat{j})\cdot dz\,dy\,\underbrace{\hat{i}(-1)}_{(-\hat{i})}$$

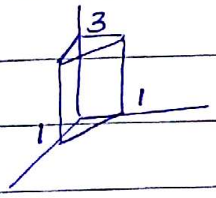
$$= \iint y\,dy\,dz = \int dz \int y\,dy = 2\int_0^2 y\,dy = 4$$

* Vector Volume Integral:

$$\int_{V} T\,d\tau \qquad d\tau = dx\,dy\,dz$$

Ex: If: $T = xyz^2$

$$V = \iiint (xyz^2)\,dx\,dy\,dz = \int_0^1 x\left[\int_0^1 y\left\{\int_0^3 z^2 dz\right\}dy\right]dx$$



$x = \big|_0^1 \qquad y = \big|_0^{1-y} \qquad z = \big|_0^3$

* Fundamental Theorem for Gradients:

$$\int_a^b \underbrace{\overline{\nabla} T \cdot \vec{dl}} = T(b) - T(a)$$

$$dT = \overline{\nabla} T \cdot \vec{dl}$$

If a vector field can be written as gradient (of scalar), its conservative field.

$\therefore,\ \vec{F} = \overline{\nabla}\phi$

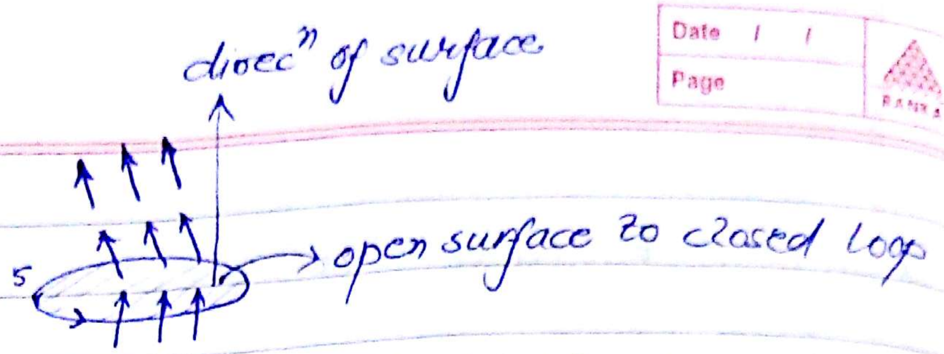
and $\overline{\nabla} \times \overline{\nabla}\phi = 0$

* Fundamental theorem for divergences:

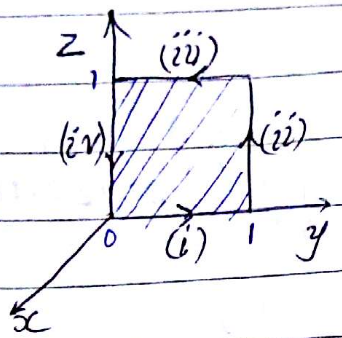
$$\underbrace{\iiint_V (\vec{\nabla} \vec{F})\,d\tau}_{\text{Vol. integrat}^n} = \underbrace{\oiint \vec{F} \cdot \vec{da}}_{\text{Surface integr}^n}$$

* Fundamental theorem for curls: (Stoke's theorem):

$$\int_S (\nabla \times v)\cdot da = \oint_P v \cdot dl \quad \text{OR} \quad \iint_S (\vec{\nabla} \times \vec{F}) \cdot ds = \oint \vec{F} \cdot \vec{dl}$$



Q $V = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check stokes theorem.



Sol^n $\iint_S (\vec{\nabla} \times \vec{F}) ds = \oint \vec{F} \cdot \vec{dl}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} = \hat{x}(4z^2 - 2x) - \hat{y}(0) + \hat{z}(2z)$$

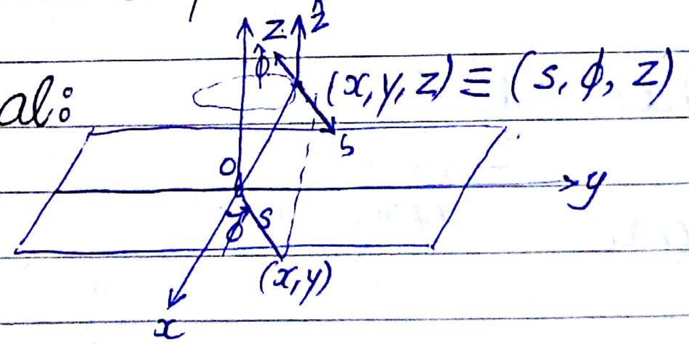
$$= (4z^2 - 2x)\hat{x} + (2z)\hat{z}$$

$$\iint_S \left((4z^2 - 2x)\hat{x} + (2z)\hat{z}\right)\cancel{(dy\,\hat{y} + dz\,\hat{z})}(dy\,dz\,\hat{x})$$

$$= \cancel{\int_0^1\int_0^1 2z\,dz} = \int_0^1\int_0^1 4z^2\,dy\,dz = \frac{4}{3} \quad (\because x = 0)$$

* Cylindrical & Spherical Coordinate System:

* Cylindrical:



$\hat{s}, \hat{\phi}, \hat{z}$ are orthogonal or mutually $\perp$.

$\hat{s} = \hat{s}(\phi)$

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→ if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
we can find unit vector as: $\hat{i} = \dfrac{\frac{\partial \vec{r}}{\partial x}}{\left|\frac{\partial \vec{r}}{\partial x}\right|}$

similiarly,

$$\hat{s} = \frac{\frac{\partial \vec{r}}{\partial s}}{\left|\frac{\partial \vec{r}}{\partial s}\right|}, \quad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

→ Rd^n bet^n $\hat{i}, \hat{j}, \hat{k}$ and $\hat{s}, \hat{\phi}$:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{r} = s\cos\phi\,\hat{i} + s\sin\phi\,\hat{j} + z\hat{k}$$

$$\hat{s} = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi} = \frac{-s\sin\phi\,\hat{i} + s\cos\phi\,\hat{j}}{\sqrt{s^2(\cos^2\phi + \sin^2\phi)}} = \frac{s(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})}{s}$$
$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$\therefore\ \hat{z} = \hat{k}$

$$\vec{A} = a_x\hat{x} + a_y\hat{y} + a_z\hat{k}$$
$$\vec{A} = a_s\hat{s} + a_\phi\hat{\phi} + a_z\hat{z}$$
$$= a_s(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + a_\phi(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) + a_z\hat{z}$$

$a_x = (a_s\cos\phi - a_\phi\sin\phi)\hat{i}$
$= x$-compoment

→ For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $a_x = x,\ a_y = y,\ a_z = z$

$$\begin{bmatrix} a_s \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$a_s = x\cos\phi + y\sin\phi$

$a_\phi = -x\sin\phi + y\cos\phi$

$a_z = z$

$\therefore \vec{r} = a_s\hat{s} + a_\phi\hat{\phi} + a_z\hat{k}$

$= (x\cos\phi + y\sin\phi)\hat{s} + (-x\sin\phi + y\cos\phi)\hat{\phi} + z\hat{k}$

Now, $x = s\cos\phi$, $y = s\sin\phi$

$\therefore \vec{r} = s(\cos^2\phi + \sin^2\phi)\hat{s} + (-s\cos\phi\sin\phi + s\sin\phi\cos\phi)\hat{\phi} + z\hat{k}$

$= s\hat{s} - \frac{s}{\cancel{2}}(\cancel{2}\sin 2\phi)\hat{\phi} + z\hat{k}$
$\quad -\sin 2\phi$

→ has info of ϕ

$\therefore \vec{r} = s\hat{s} - s\cancel{\sin 2\phi}\,\hat{\phi} + z\hat{k} = s\hat{s} + z\hat{k}$

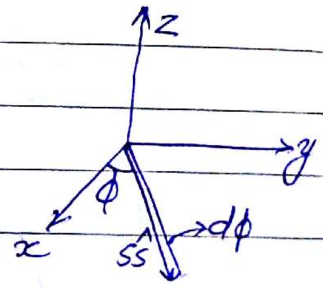
Hence, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = s\hat{s} + z\hat{k}$

→ Infinitesimal displacement vector:

Cartesian: $\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Cylindrical: $\vec{dl} = dl_s\hat{s} + dl_\phi\hat{\phi} + dl_z\hat{z}$

$= ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$



$d\vec{a}_s$ = diff. surface element keeping s constant $= dl_z\hat{z} \times dl_\phi\hat{\phi}$

$= s\,d\phi\,dz\,\hat{s}$

$d\vec{a}_\phi = dl_z\hat{z} \times dl_s\hat{s} = \cancel{dz\,d\phi}\ dz\,ds\,\hat{\phi}$

$d\vec{a}_z = dl_s\hat{s} \times dl_\phi\hat{\phi} = s\,d\phi\,ds\,\hat{z}$

$d\tau$ = vol. element $= s\,ds\,d\phi\,dz$

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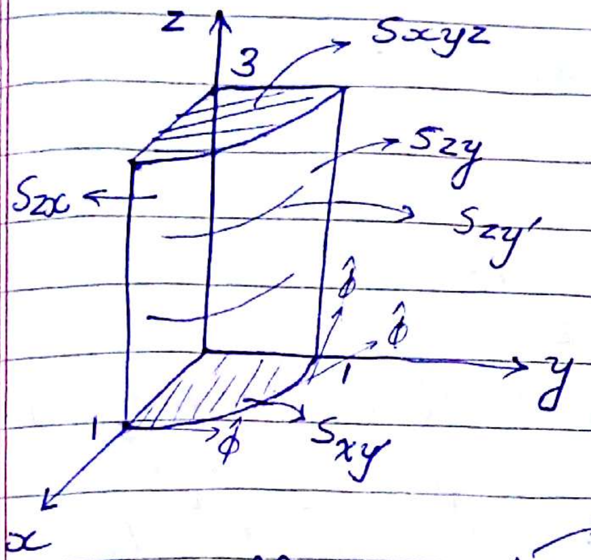
Ex:



$\vec{V} = \hat{s} + \sin\phi\,\hat{\phi} + z\hat{z}$

for zx its $-\hat{\phi}$

(i) $\phi_{S_{zx}} = \iint \vec{V}\cdot dz\,ds(-\hat{\phi})$ = Flux through zx.

dir^n of zx = outward.

$= \iint \sin\phi\, dz\, ds$

$\sin\phi \to 0$ (since $\phi = 0$)

(ii) $\phi_{S_{xy'}} = \iint \vec{V}\cdot s\,ds\,d\phi\,(-\hat{z}) = \iint z\,s\,ds\,d\phi$, $z \to 0$

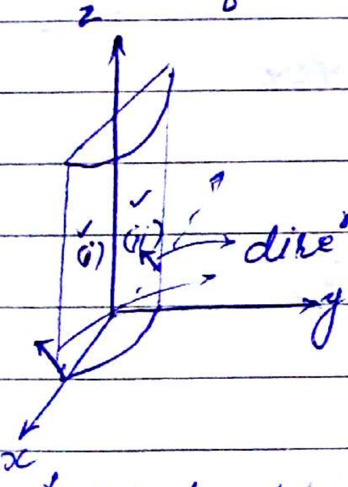
(iii) $\phi_{S_{zy}} = \int_0^3\int_0^1 \vec{V}\cdot dz\,ds\,\hat{\phi}$

(iv) $\phi_{zy'} = \int_0^3\int_0^{\pi/2} \vec{V}\cdot s\,d\phi\,dz\,\hat{s} = \int_0^3\int_0^{\pi/2} s\,d\phi\,dz = \int_0^3\int_0^{\pi/2} d\phi\,dz$

$s = 1$

$= \int_0^3 \frac{\pi}{2}\,dz = \frac{3\pi}{2}$

Ex:



dir^n of surface.

Total Flux through [(i) + (ii)] ? $V = \hat{\phi}$

$= 0$

ϕ = constant.

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→ $\overline{\nabla}$ in cylindrical system:

$T = T(s, \phi, z)$

$dT = \frac{\partial T}{\partial s} ds + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz$

$dT = \overline{\nabla} T . \overrightarrow{dl}$ (where, $\overrightarrow{dl} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$)

$= [(Grad\, T)_s \hat{s} + (Grad\, T)_\phi \hat{\phi} + (Grad\, T)_z \hat{z}] \cdot [ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}]$

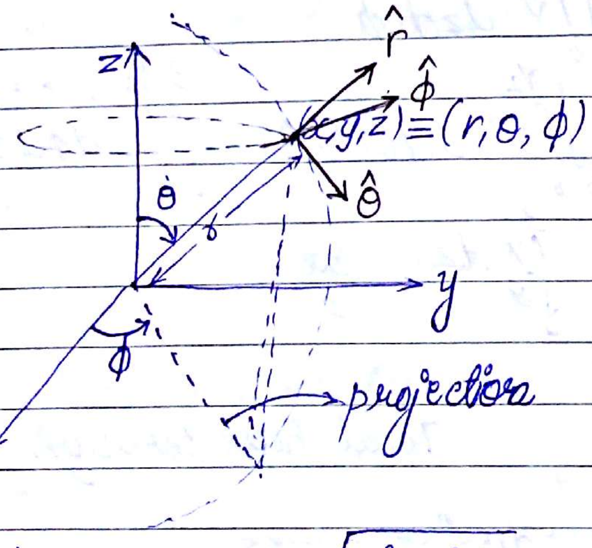
$= (Grad\, T)_s ds + s(Grad\, T)_\phi d\phi + (Grad.T)_z dz$

$(Grad\, T)_s = \frac{\partial T}{\partial s}$, $(Grad\, T)_\phi = \frac{1}{s}\frac{\partial T}{\partial \phi}$, $(Grad.T)_z = \frac{\partial T}{\partial z}$

$$\boxed{\overline{\nabla} = \frac{\partial}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}}$$

$\nabla . v = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$ (will be given in q's)

* Spherical:



$\hat{r}, \hat{\theta}, \hat{\phi}$ = orthogonal

$\hat{r} \times \hat{\theta} = \hat{\phi}$

$r \in [0, \infty)$

$\theta \in [0, \pi]$

$\phi = [0, 2\pi]$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

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$\hat{r} = \hat{r}(\theta, \phi)$ $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$\hat{\theta} = \hat{\theta}(\theta, \phi)$

$\hat{\phi} = \hat{\phi}(\phi)$ $d\vec{l} = dl_r\hat{r} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$

$= dr\hat{r} + rd\theta\,\hat{\theta} + \cancel{rsind}\; r\sin\theta d\phi\,\hat{\phi}$

- Differential Elements:

$d\vec{a}_r = r^2\sin\theta d\theta d\phi\,\hat{r}$ = surface of sphere

$d\vec{a}_\theta = r\sin\theta dr d\phi\,\hat{\theta}$ = surface of cone

$d\vec{a}_\phi = r dr d\theta\,\hat{\phi}$ = a semicircle (on zx plane for $\phi = 0$)
(on zy plane for $\phi = \frac{\pi}{2}$)

Vol. Element: $d\tau = dl_r\, dl_\theta\, dl_\phi = r^2\sin\theta dr d\theta d\phi$

$\therefore$, Vol. of sphere $= \tau$

$\therefore\ \tau = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$

- $\vec{\nabla}$ in spherical system:

$T = T(r, \theta, \phi)$

$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$

$dT = \vec{\nabla}T \cdot d\vec{l}$ (where, $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$)

$= [(\text{Grad}T)_r\, \cancel{d}\hat{r} + (\text{Grad}T)_\theta\, \cancel{d}\hat{\theta} + (\text{Grad}\cdot T)_\phi\, \cancel{d}\hat{\phi}]\, d\vec{l}$

$= (\text{Grad}T)_r\, dr + r(\text{Grad}T)_\theta d\theta + r\sin\theta(\text{Grad}T)\, d\phi$

$$\therefore\ \boxed{\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}}$$

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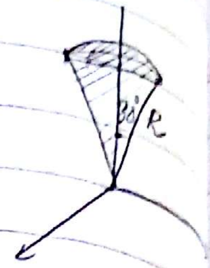
Prob. 1.58: Check divergence theorem for f^n:

$$v = r^2 \sin\theta \,\hat{r} + 4r^2 \cos\theta \,\hat{\theta} + r^2 \tan\theta \,\hat{\phi}$$



Solⁿ: $$\iiint (\vec{\nabla}\cdot\vec{V})\, d\tau = \oint \vec{V}\cdot d\vec{a}$$

$$\phi_{top} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \vec{V}\cdot r^2 \sin\theta \, d\theta \, d\phi \,\hat{r}$$

$$\phi_{cone} = \int_{r=0}^{R}\int_{\phi=0}^{2\pi} \cancel{r}\vec{V}\cdot r\sin\theta \, dr \, d\phi \,\hat{\theta}$$

$$\oint \text{(i.e RHS = total flux)} = \phi_{top} + \phi_{cone}$$

$$\phi_{top} = \int_0^{2\pi}\int_0^{\pi/6} r^4 \sin^2\theta \, d\theta \, d\phi$$

$$= r^4 \int_0^{\pi/6} \sin^2\theta \, d\theta \int_0^{2\pi} d\phi = 2\pi r^4 \int_0^{\pi/6} \sin^2\theta \, d\theta$$

$$\int_0^{\pi/6} \frac{1}{2} - \frac{\cos 2\theta}{2} \, d\theta$$

$$= 2\pi r^4 \times \left(\frac{\pi}{12} - \frac{\sin 2\theta}{4}\Big|_0^{\pi/6} \right) = 2\pi r^4 \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right)$$

*Q Verify divergence theorem: $F = \dfrac{\hat{r}}{r^2}$

Solⁿ: $$\iiint (\vec{\nabla}\cdot F)\cdot d\tau = \oint \vec{F}\cdot d\vec{a}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \cdot v_\theta)}_{0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{0}$$

$$= \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(\cancel{r^2}\times\frac{1}{\cancel{r^2}}\right)\right) = \frac{1}{r^2}\frac{\partial(1)}{\partial r}^{0} \quad \& \quad v_r = \frac{1}{r^2}$$

$$= 0 = LHS$$

$$RHS = \iint \frac{\hat{r}}{\cancel{r^2}} \cancel{r^2} \sin\theta \, d\theta \, d\phi \, \hat{r} = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \underbrace{(-\cos\theta)\Big|_0^{\pi}}_{} d\phi = \int_0^{2\pi} 2 \, d\phi = 4\pi$$

$$-(-1) - (-1) = 1 + 1 = 2$$

These aren't equal since $\frac{\hat{r}}{r^2}$ isn't defined at 0 & RHS is non-zero only for +ve-charge at origin but restwhere in sphere its zero.

* Dirac Delta function:

→ $\delta(x) = 0 \; , \; x \neq 0$
$\quad = \infty \; , \; x = 0$

→ $\delta(x-a) = 0 \; ; \; x \neq a$
$\quad = \infty \; ; \; x = a$

(i) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

(ii) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

(i) $\int_{-\infty}^{\infty} \delta(x-a) dx = 1$

(ii) $\int_{-\infty}^{\infty} \delta(\cancel{x}\, x-a) f(x) dx = f(a)$

→ $\delta^3(r) = 0 \; ; \; r \neq 0$
$\quad = \infty \; ; \; r = 0$

(i) $\iiint_{\text{all space}} \delta(r) d\tau = 01$

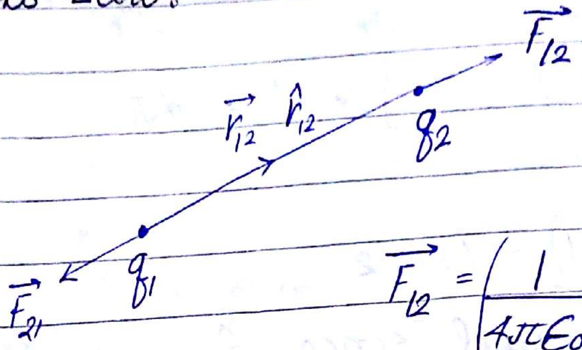
(ii) $\iiint_{\text{all space}} \delta(r) f(r) d\tau = f(0)$

∴ In $\overset{*}{Q}$, $LHS = 4\pi \overbrace{\iiint \delta^3(r) d\tau}^{1} = 4\pi$

$$\bar{\nabla} \cdot F = 4\pi \delta^3(r)$$

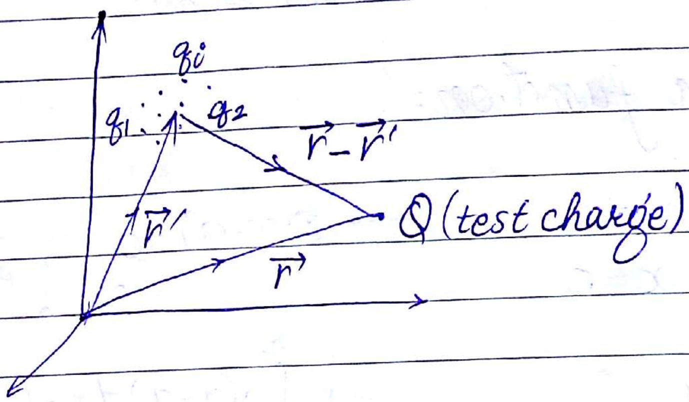
Electrostatics

* Columb's Law:



$$\vec{F_{12}} = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{q_1 q_2}{r_{12}^2}\right)\hat{r}_{12}$$

ϵ_0 ↓ permitivity of free space



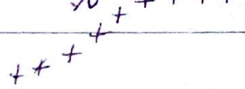
$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q(\vec{r}-\vec{r}_1')}{|\vec{r}-\vec{r}_1'|^3} + \frac{1}{4\pi\epsilon_0}\frac{q_2 Q(\vec{r}-\vec{r}_2')}{|\vec{r}-\vec{r}_2|^3} + \cdots$$

$$= Q\sum \frac{1}{4\pi\epsilon_0}\left(\frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}\right)$$

$\vec{F} = Q\vec{E}(r)$; $\vec{E}(r)$ = Electric Field.

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\int \frac{dq\,\hat{r}}{r^2}$$

Ex:

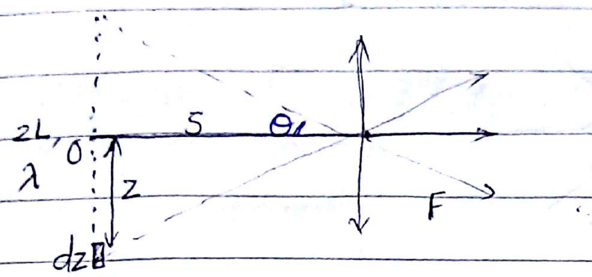


$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda dl'(\vec{r}-\vec{r}')}{|(\vec{r}-\vec{r}')|^3}$$ — ~~str~~ Line charge density

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma\, da'(\vec{r}-\vec{r}')}{|(\vec{r}-\vec{r}')|^3}$$ = surface charge density

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \iint \frac{\rho\, d\tau'(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ = volume charge density

Ex:



Method 1:

$$dE = \frac{1}{4\pi\epsilon_0} \frac{(\lambda dz)\cos\theta}{\sqrt{z^2+s^2}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda s dz}{\sqrt{(s^2+z^2)}^{3/2}}$$

$$\vec{E} = \frac{2\lambda L}{4\pi\epsilon_0 \cdot s\sqrt{s^2+L^2}}$$

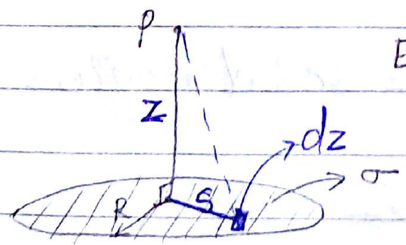
Method 2:

$$\vec{r}-\vec{r}' = s\hat{s} - z\hat{z}$$
$$|\vec{r}-\vec{r}'| = \sqrt{s^2+z^2}$$

$$\vec{E}(s) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dz(s\hat{s}-z\hat{z})}{(s^2+z^2)^{3/2}}$$

$$\vec{E}(s) = \frac{1}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{s\lambda dz\,\hat{s}}{(s^2+z^2)^{3/2}} - \frac{1}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{z\hat{z}\,\lambda dz}{(s^2+z^2)^{3/2}}$$

Ex:



E_p?

$$\vec{r}-\vec{r}' = z\hat{z} - s\hat{s}$$

$$|\vec{r}-\vec{r}'| = (s^2+z^2)^{3/2}$$

$$\vec{E}_p(z) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma(s\,d\phi\,ds)(z\hat{z}-s\hat{s})}{(s^2+z^2)^{3/2}}$$

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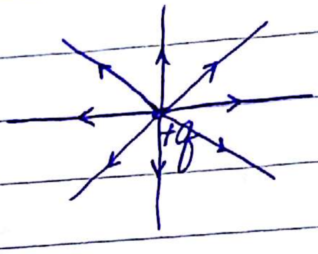
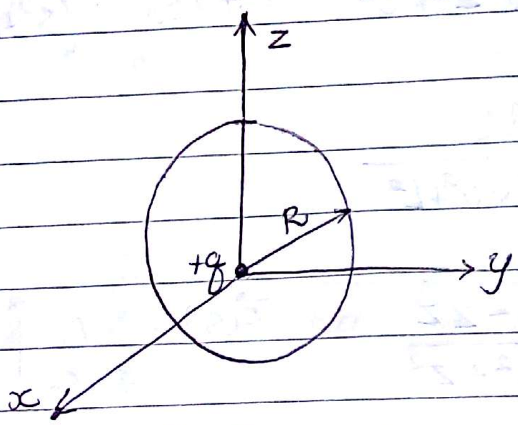
$$\vec{E}_p(z) = \frac{\sigma}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi} \frac{z s\,ds\,d\phi\,\hat{z}}{(s^2+z^2)^{3/2}} - \underbrace{\left(\frac{1}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi} \frac{\sigma s^2\,d\phi\,ds\,\hat{s}}{(s^2+z^2)^{1/2}}\right)}_{=0}$$

$$\therefore \vec{E}_p(z) = \frac{z a}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]$$

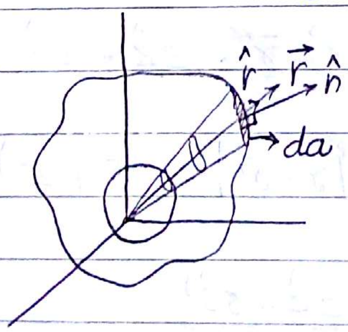


Q calculate Flux through spherical surface:



Sol^n: $\phi = \oint \vec{E}\cdot d\vec{a} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{q\hat{r}}{4\pi\epsilon_0 r^2}\right)(r^2\sin\theta\,d\theta\,d\phi\,\hat{r})$

$$\phi = \frac{q}{\epsilon_0}$$



$d\Omega$ = solid angle = $\frac{\hat{r}\cdot\hat{n}\,da}{r^2}$

$\oint d\Omega = 4\pi$

$\nabla \cdot E = 0 \Rightarrow$ Change of flux = zero
$\not\Rightarrow E = 0$

Solⁿ 2: $\phi = \oiint E \cdot d\vec{a} = \oiint \frac{q}{4\pi\epsilon_0} \underbrace{\frac{\hat{r}\cdot\hat{n}\,da}{r^2}}_{d\Omega} = \frac{q}{4\pi\epsilon_0}\oiint d\Omega = \frac{q}{\epsilon_0}$

$$\boxed{\oiint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}} \Rightarrow \text{Gauss's Law}$$

$$\oiint \vec{E}\cdot d\vec{a} = \iiint (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{q}{\epsilon_0} = \frac{\iiint \rho\, d\tau}{\epsilon_0}$$

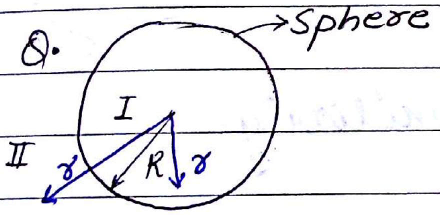
$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \text{Differential form of Gauss's Law}$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(r')\, d\tau'\, \hat{r}}{r^2}$$

$$\vec{\nabla}\cdot\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\iiint \rho(r')\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau'$$

$$= \frac{4\pi}{4\pi\epsilon_0}\iiint \rho(r')\,\delta(r-r')\, d\tau' = \frac{\rho(r)}{\epsilon_0}$$

Ex:



I: $\oiint \vec{E}\cdot d\vec{a} = E_1 \oiint d\vec{a} = E_1(4\pi \cancel{R^2}\,r^2) = \frac{0}{\epsilon_0} \Rightarrow E_1 = 0$

II: $E_2(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$

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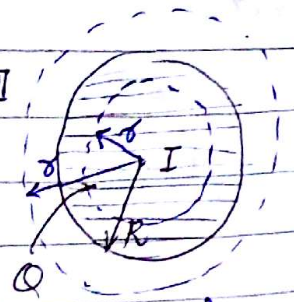
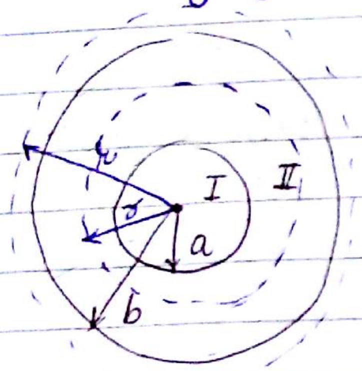
Ex:



$$\text{I}: \; E_1(4\pi r^2) = \frac{\frac{Q}{\left(\frac{4}{3}\pi R^3\right)} \times \left(\frac{4}{3}\pi r^3\right)}{\epsilon_0} = \frac{Q r^3}{\epsilon_0 R^3}$$

Similiarly for region II.

Ex:



charge distribution:

$\rho(r) = \frac{K}{r^2}$; $a \le r \le b$

Find $\vec{E}$ for each region.

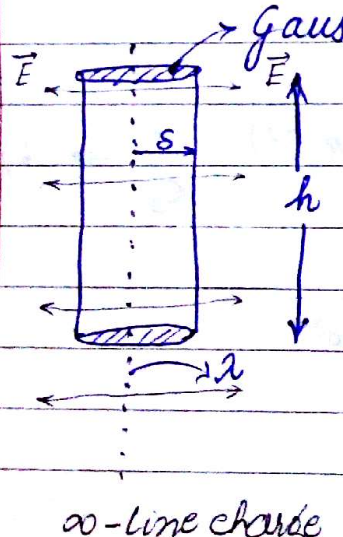
Sol^n $\text{I}: \; E_1(4\pi r^2) = \frac{0}{\epsilon_0} \Rightarrow E_1 = 0$

$$\text{II}: \; E_{II}(4\pi r^2) = \frac{\int_a^r \rho(r) \times 4\pi r^2 dr}{\epsilon_0} \quad \text{OR} = \int_a^r \int_0^\pi \int_0^{2\pi} \frac{K}{r^2} \times r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{4\pi K(r-a)}{\epsilon_0}$$

III: E_{III} can be found similiarly.

Ex:



∞-line charge distrib^n

$$E(2\pi s)h = \frac{\lambda h}{\epsilon_0}$$

$$\therefore \boxed{\vec{E} = \frac{\lambda}{2\pi s \epsilon_0} \hat{s}}$$

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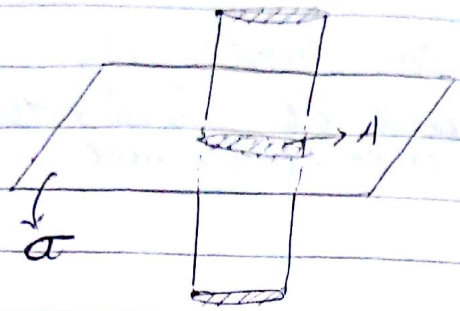
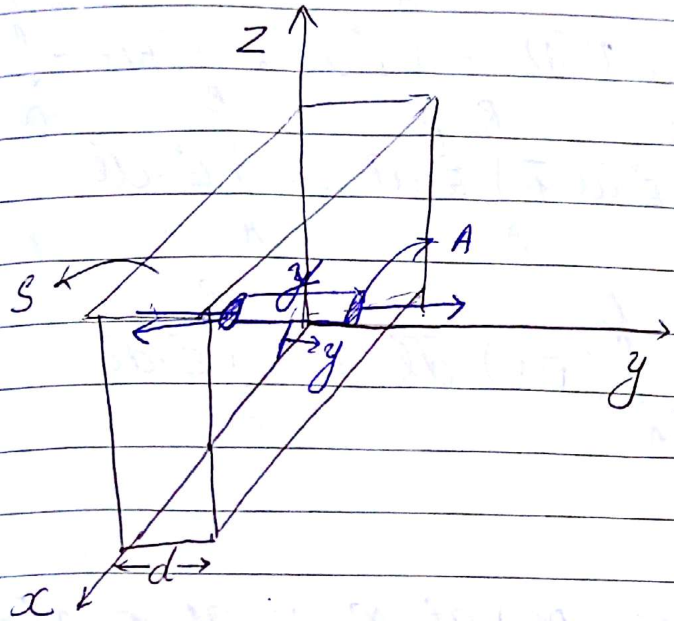
Ex:



$$E(2A) = \frac{\sigma A}{\epsilon_0}$$

$$\therefore \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}}$$

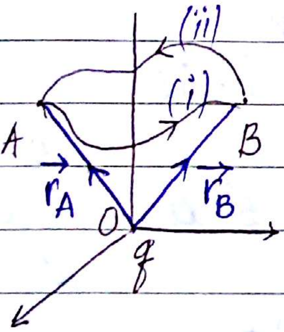
Ex:



Find $\vec{E}$ within & outside the slab?

$$EA = \frac{(Ay)\rho}{\epsilon_0} \; ; \; y < \frac{d}{2}$$

Ex:



Prove: $\oint \vec{E}\cdot\vec{dl} = 0$

$$= \int_A^B \frac{q\hat{r}}{4\pi\epsilon_0 r^2}\cdot(dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi})$$

$$= \frac{q}{4\pi\epsilon_0}\int_A^B \frac{1}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Similiarly,

$$\int_B^A \vec{E}\cdot dl = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\therefore \oint E\cdot dl = \int_A^B \vec{E}\cdot\vec{dl} + \int_B^A \vec{E}\cdot\vec{dl} = 0$$

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* Electrostatic Potential Function :-

$$\boxed{V(r) = -\int_R^r \vec{E}\cdot\vec{dl}}$$

$r \rightarrow$ point at which potential is to be defined.

$R \rightarrow$ Refrence point

In above example : $V(A) = -\int_R^A \vec{E}\cdot\vec{dl}$, $V(B) = -\int_R^B \vec{E}\cdot\vec{dl}$

$$\therefore V(B) - V(A) = -\int_R^B \vec{E}\cdot\vec{dl} + \int_A^R \vec{E}\cdot\vec{dl} = -\int_A^B \vec{E}\cdot\vec{dl}$$

$$\therefore V(B) - V(A) = \int_A^B (\vec{\nabla} V)\cdot\vec{dl} = -\int_A^B \vec{E}\cdot\vec{dl}$$

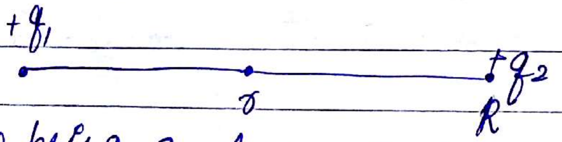
$$\therefore \boxed{\vec{E} = -\vec{\nabla} V}$$

We use R (reference pt.) at ∞. $\because$ at ∞, $V(\infty) = 0$

$$V(r) = -\int_R^r E\cdot\vec{dl} \qquad V'(r) = -\int_{R'}^r \vec{E}\cdot\vec{dl}$$

$$= -\int_R^r \vec{E}\cdot\vec{dl} - \int_{R'}^R \vec{E}\cdot\vec{dl}$$

$$V'(r) = V(r) + C$$



To bring q_2 from R to r ; $W_{done} = \int_R^r \vec{F}_{me}\cdot\vec{dl}$

$$= -\int_R^r \vec{F}_{ele}\cdot\vec{dl}$$

$$= -q_2 \int_R^r \vec{E}\cdot\vec{dl} = q_2 V(r)$$

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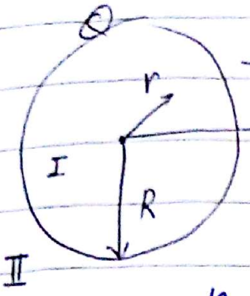


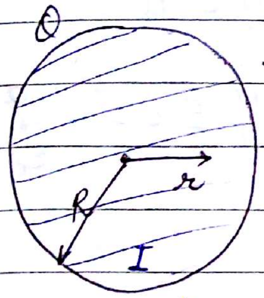
Spherical shell

$$V_{II}(r) = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0 r}$$

At surface, $V(R) = \dfrac{Q}{4\pi\epsilon_0 R}$

$$V_I(r) = -\int_{\infty}^{r} \vec{E}\cdot\vec{dl} = -\int_{+\infty}^{R} E\cdot\vec{dl} + \Big(\underbrace{-\int_{R}^{r} \vec{E}\cdot\vec{dl}}_{0}\Big) \Rightarrow = \frac{Q}{4\pi\epsilon_0 R}$$

Ex:



solid sphere

$$V_I(r) = -\int_{+\infty}^{r} \vec{E}\cdot\vec{dl} = -\int_{+\infty}^{R} \vec{E}\cdot dl - \int_{R}^{r} \vec{E}\cdot\vec{dl}$$

$$= -\int_{+\infty}^{R} \frac{Q\,dr}{4\pi\epsilon_0 r^2} - \int_{R}^{r} \frac{Q r^3\,dr}{\epsilon_0 R^3} = \left(\frac{Q}{4\pi\epsilon_0 r}\right)\Bigg|_{+\infty}^{R} - \left(\frac{Q r^4}{4\epsilon_0 R^3}\right)\Bigg|_{R}^{r}$$

$$= \left(\frac{Q}{4\pi\epsilon_0} - 0\right) -$$

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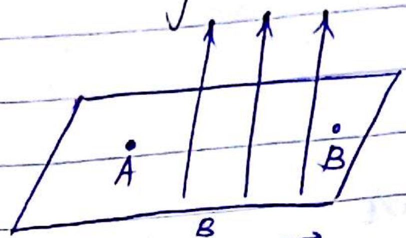
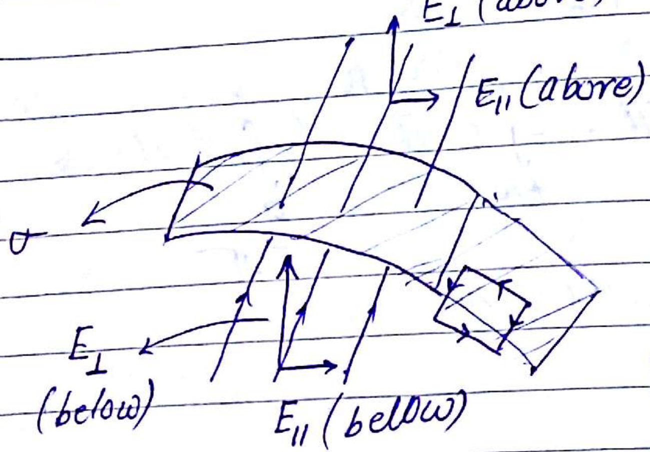
→ Equipotential Surface:



$$0 = V(B) - V(A) = -\int_A^B \vec{E}\cdot\vec{dl}$$

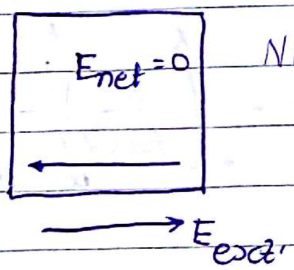


Electrostatic field

$$\oint \vec{E}\cdot\vec{dl} = 0$$

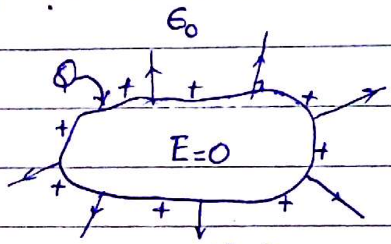
$$\oiint \vec{E}\cdot\vec{da} = \frac{q}{\epsilon_0}$$

* Properties of Conductors:-

(i) 

Neutral conductor

(ii) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

(iii) 

(iv) Equipotential, $E_{\parallel} = 0$

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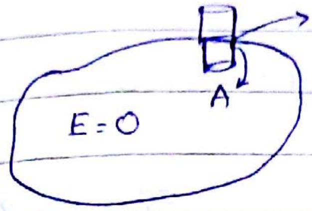
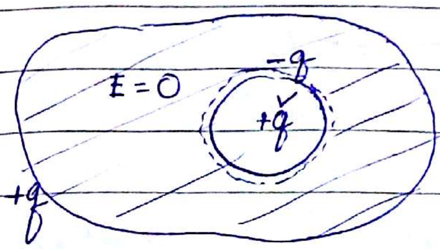
(V)



$$EA = \frac{\sigma' A}{\epsilon_0} \Rightarrow E = \frac{\sigma'}{\epsilon_0}\hat{n}$$

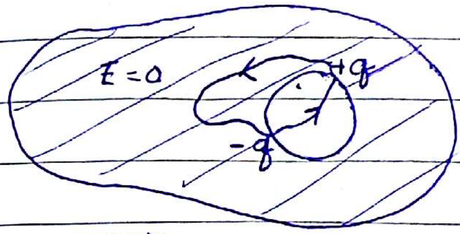
Ex:



$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{encl.}}{\epsilon_0}$$

$$0 = \frac{+q+(-q)}{\epsilon_0}$$

Ex:

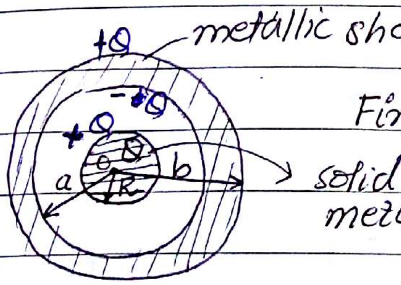


$$\oint \vec{E}\cdot d\vec{l} = 0 \;\times\; (0 + \neq 0)$$

$\vec{E}_{ext.}$ ✓

Faraday Cage

Ex:



Find σ_R, σ_a, σ_b and $V(0)$.

Sol^n: $\sigma_R = \dfrac{+Q}{4\pi R^2}$ $\quad \sigma_a = \dfrac{\mp Q}{4\pi a^2}$ $\quad \sigma_b = \dfrac{+Q}{4\pi b^2}$

$$V(0) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r}$$

$$= -\int_{\infty}^{b} E\cdot dr - \int_{b}^{a} E\cdot dr \mp \int_{a}^{R} E\cdot dr - \int_{R}^{0} E\cdot dr$$

$$\downarrow \qquad\quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow$$

$$\frac{kq}{r^2} \qquad 0 \qquad \frac{kq}{R^2} \qquad 0$$

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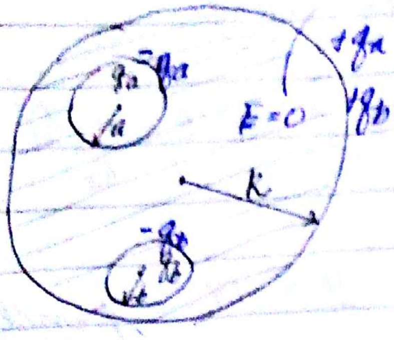
Ex:



Find $\sigma_a, \sigma_b, \sigma_R$.

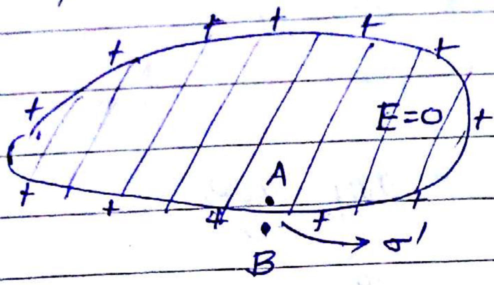
$$\sigma_a = \frac{-q_a}{4\pi a^2} \qquad \sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

F_{betn} q_a & q_b = 0

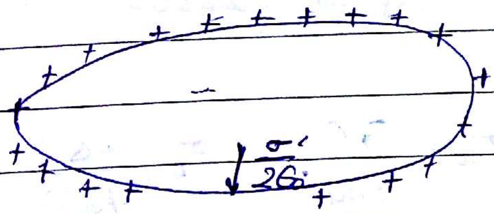
If Q is brought outside, $\vec{E}$ = non-uniform.

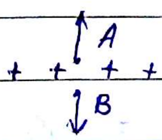
* Electrostatic pressure in a conductor:



$E_A = 0$

E_B depends on σ' i.e $E_B = \frac{\sigma'}{\epsilon_0}$





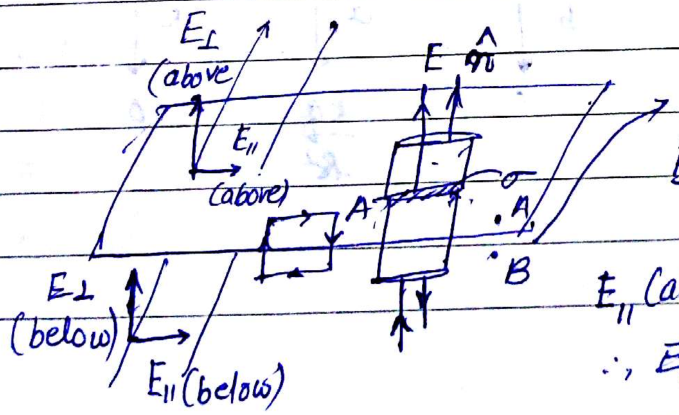
$E_A = \frac{\sigma'}{2\epsilon_0}$

$E_B = \frac{\sigma'}{2\epsilon_0}$

$|\vec{E}|\sigma'$

$= \left(\frac{\sigma'}{2\epsilon_0}\right)\sigma' = \frac{\sigma'^2}{2\epsilon_0}$ ✓

Pr Q.



V is continuous everywhere

$E_\perp(\text{above}) - E_\perp(\text{below}) = \frac{\sigma}{\epsilon_0}$ (discontinuity)

$E_\parallel(\text{above}) - E_\parallel(\text{below}) = 0$

$\therefore E_\parallel(\text{below}) = E_\parallel(\text{above})$ continuous

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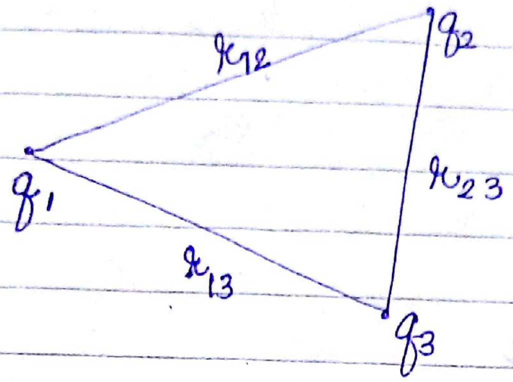
* Electrostatic Potential Energy (U):



$W_1 = 0$

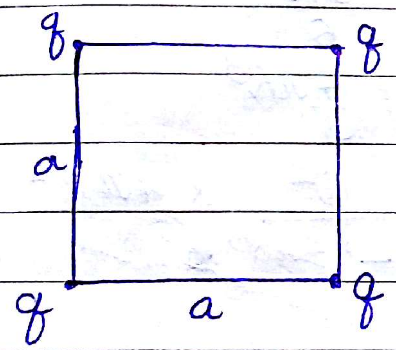
$W_2 = \left(\frac{q_1}{4\pi\epsilon_0 r_{12}}\right) q_2$

$W_3 = \left(\frac{q_1}{4\pi\epsilon_0 r_{13}}\right) q_3 + \left(\frac{q_2}{4\pi\epsilon_0 r_{23}}\right) q_3$

$$U = \sum_{i=1}^{n} q_i \sum_{j>i} \frac{1}{4\pi\epsilon_0}\left(\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{N} q_i \sum_{j\neq i}\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{q_j}{r_{ij}}\right)$$

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i)}$$

Ex:



$$U = \frac{1}{2}\left[4q\left\{\frac{2q}{4\pi\epsilon_0 a} + \frac{q}{4\pi\epsilon_0\sqrt{2}a}\right\}\right]$$

$$U = \underbrace{\frac{1}{2}\iiint \rho V d\tau}_{\text{vol. charge distribut}^{n}} = \underbrace{\frac{1}{2}\iint \sigma V da}_{\text{surface charge distribution}}$$

$$= \frac{\epsilon_0}{2}\iiint (\vec{\nabla}\cdot\vec{E}) V d\tau \qquad \left(\text{using } \nabla\cdot E = \frac{\rho}{\epsilon_0}\right)$$

$$\cancel{\nabla\cdot(V\vec{E})} = \nabla(f\vec{V}) = f(\nabla\cdot\vec{V}) + \vec{V}(\vec{\nabla} f)$$

$$(-\nabla V) = \vec{E}$$

$$= \frac{\epsilon_0}{2}\iiint \vec{\nabla}\cdot(\vec{E}V) d\tau + \frac{\epsilon_0}{2}\iiint E\cdot(\overline{\nabla V}) d\tau$$

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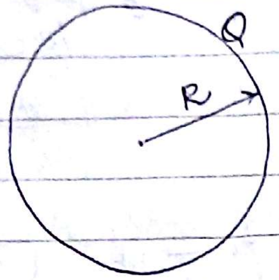
$= 0$ (if obj. is of ∞ size)

$$U = \frac{\epsilon_0}{2}\oiint V\vec{E}\cdot d\vec{a} + \frac{\epsilon_0}{2}\iiint E^2 d\tau$$

$$\boxed{U = \frac{\epsilon_0}{2}\iiint_{\text{All space}} E^2 d\tau}$$

Ex:



Sol^n: $U = \frac{\epsilon_0}{2}\iiint \left(\frac{KQ}{R}\right) r^2 \sin\theta\, d\theta\, d\phi\, dr$

$= \frac{\epsilon_0}{2} \times \frac{1}{4\pi\epsilon_0} \times \frac{Q}{R} \underbrace{\iiint d\tau}_{\text{vol. of sphere} = \frac{4}{3}\pi R^3} = \frac{QR^2}{6}$

M:1 $U = \frac{1}{2}\iint \frac{Q}{4\pi R^2} \times \frac{KQ}{R} \times da \Rightarrow \frac{1}{4\pi\epsilon_0} \times \frac{Q^2}{2\times \cancel{4\pi R^3}} \times \cancel{4\pi R^2} = \frac{Q^2}{8\pi\epsilon_0 R}$

M2: $U = \frac{\epsilon_0}{2}\iiint_R^\infty E^2 d\tau = \frac{\epsilon_0}{2}\int_R^\infty\int_0^\pi\int_0^{2\pi} \frac{K^2Q^2}{r^4} r^2 \sin\theta\, d\theta\, d\phi\, dr$

$= \frac{\epsilon_0}{16\pi^2\epsilon_0^2} \times \frac{1}{2} \times Q^2 \left\{ \frac{-1}{r}\Big|_R^\infty \times -\cos\theta\Big|_0^\pi \times \phi\Big|_0^{2\pi} \right\}$

$= \frac{\epsilon_0 Q^2}{4\ 32\pi^2\epsilon_0^2} \times \frac{1}{R} \times 2 \times 2\pi = \frac{Q^2}{2\times 4\pi\epsilon_0 R} = \frac{Q^2}{8\pi\epsilon_0 R}$

For $\int_R^a$ rather than $\int_R^\infty$

$U = \left(\frac{1}{R} - \frac{1}{a}\right) \times \frac{Q^2}{8\pi\epsilon_0}$

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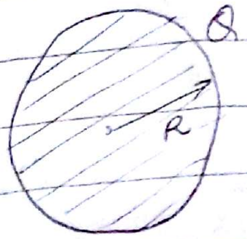
$\rightarrow$ For $\int_R^a$ $\quad U = \frac{\epsilon_0}{2}\oint V\cdot\vec{E}\,d\vec{a} + \frac{\epsilon_0}{2}\iiint E^2 d\tau$

$$U = \frac{\epsilon_0}{2}\oint \frac{Q}{4\pi\epsilon_0 a}\,\frac{Q}{4\pi\epsilon_0 a^2}\times a^2\sin\theta\, d\theta\, d\phi$$

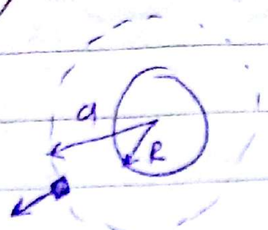
$$U = \frac{\cancel{Q^2}}{8\pi\epsilon_0 a} + \frac{Q^2}{8\pi\epsilon_0 R} - \frac{\cancel{Q^2}}{8\pi\epsilon_0 a} = \frac{Q^2}{8\pi\epsilon_0 R}$$

Ex:



$$U = \frac{\epsilon_0}{2}\oint V\vec{E}\cdot d\vec{a} + \frac{\epsilon_0}{2}\iiint E^2 d\tau = \frac{\epsilon_0}{2}\underbrace{\iiint}_{\text{All space}} E^2 d\tau$$

$$\frac{\epsilon_0}{2}\iint \frac{Q}{4\pi\epsilon_0 a}\cdot\frac{Q}{4\pi\epsilon_0 a^2}\cdot a^2\sin\theta\, d\theta\, d\phi$$



$$\boxed{U = \frac{\epsilon_0}{2}E^2}$$

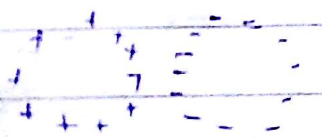
$\downarrow$ electrostatic potential energy density

* Capacitance:

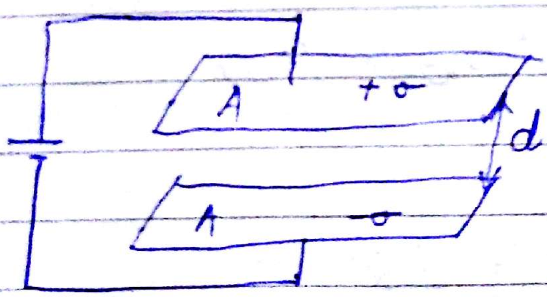
$$V = \frac{Q}{4\pi\epsilon_0 R} = \left(\frac{1}{4\pi\epsilon_0 R}\right)Q \Rightarrow C = \frac{Q}{V}$$

$$C = 4\pi\epsilon_0 R$$

$$C = \frac{Q}{\Delta V}$$



Parallel-Plate capacitor:



$$C = \frac{Q}{\Delta V} = \frac{\sigma d}{\epsilon_0}$$

$$C = \frac{A\sigma\epsilon_0}{\sigma d} = \frac{A\epsilon_0}{d}$$

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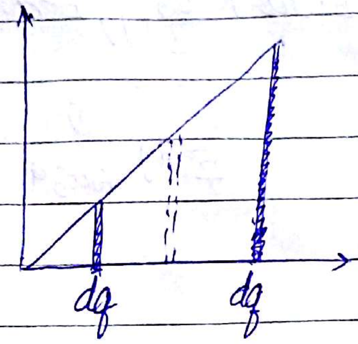
$$U = \frac{\epsilon_0 E^2}{2} = \frac{\epsilon_0}{2} \times \frac{\sigma^2}{\epsilon_0^2} Ad = \frac{\sigma^2 Ad}{2\epsilon_0}$$

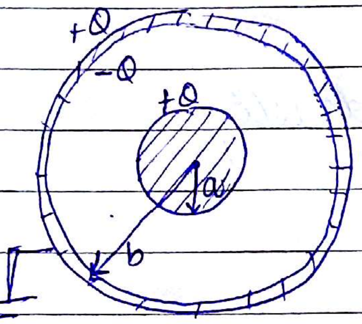
$$= \frac{Q^2 d}{2\epsilon_0 A} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

OR

$$U = \int_0^Q V(q)\,dq = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$



Ex:



(i) $C = \frac{Q}{\Delta V}$

(ii) $V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{r}$

(iii) $U = \frac{\epsilon_0}{2}\iiint_{\text{All space}} E^2 d\tau$

Sol^n: (i) $C = \frac{Q}{\Delta V} = \frac{4\pi\epsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$

(ii) $V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{r} = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$

(iii) $U = \frac{\epsilon_0}{2}\iiint_{r=a}^{b} \frac{Q^2}{(4\pi\epsilon_0 r^2)^2}\, r^2 \sin\theta\, d\theta\, d\phi\, dr$

$$= \frac{1}{2} \times \frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{2}\frac{Q^2}{C}$$

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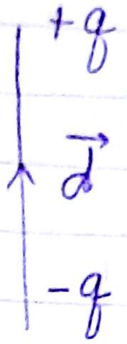
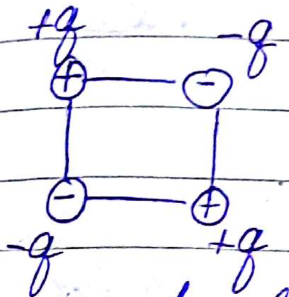
* Electric dipole:
(or Physical dipole)



$$\vec{p} = q\vec{d}$$

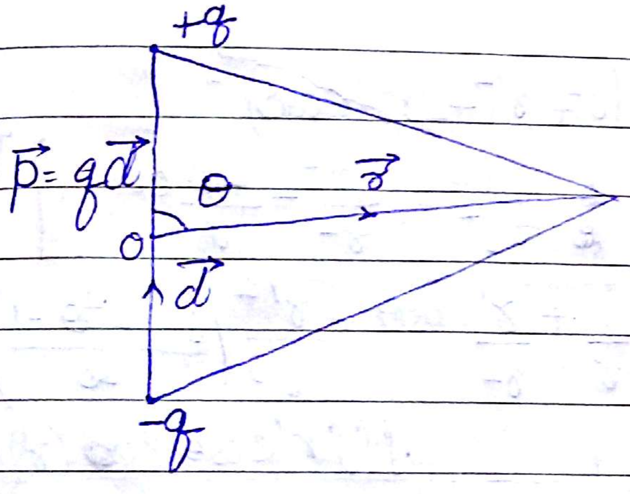
$$V(r) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \Rightarrow V(r) \propto \frac{1}{r^2} \text{ and } E(r) \propto \frac{1}{r^3}$$



(quadruple)

$$V(r) \propto \frac{1}{r^3}$$

$$E(r) \propto \frac{1}{r^4}$$



$V(r, \theta)$

$$\cancel{V(r)} = \frac{1}{4\pi\epsilon_0} \cdot \frac{p\cos\theta}{r^2} \; ; \; |\vec{r}| \gg |\vec{d}|$$

$\theta < 90°$; $V(r) = +ve$

$\theta > 90°$; $V(r) = -ve$

$\theta = 90°$; $V(r) = 0$

$$\vec{E}(r) = -\nabla V = \left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right] \cdot \frac{Kp\cos\theta}{r^2}$$

$$= \frac{P}{4\pi\epsilon_0}\left[\frac{2p\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta} + 0\right]$$

$$E(\vec{r}, \theta) = \frac{P}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

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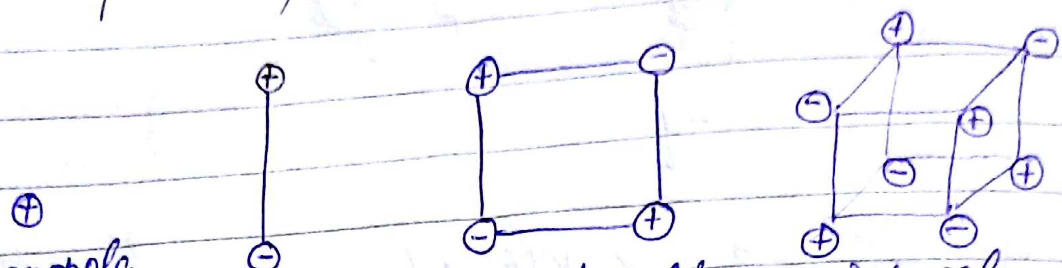
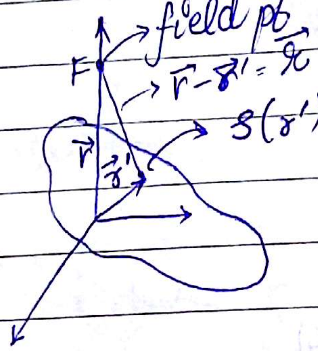
$V(r,\theta) = \cancel{1} \quad V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{p}\cdot\hat{r}}{r^2}$

* Multipole Expansion:



monopole — Dipole — quadruple — Octapole

$V \propto \frac{1}{r}$ $\quad V \propto \frac{1}{r^2}$ $\quad V \propto \frac{1}{r^3}$ $\quad V \propto \frac{1}{r^4}$



$$\cancel{V} \quad V_F = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r')\,d\tau'}{|\vec{\mathscr{r}}|}$$

$$|\vec{\mathscr{r}}| = \left(r^2 + r'^2 - 2rr'\cos\theta\right)^{1/2}$$

$$\frac{1}{\mathscr{r}} = \cancel{\frac{1}{a}}\, \frac{1}{r}\left[1 + \left(\frac{r'^2}{r^2} - \frac{2r'\cos\theta}{r}\right)\right]^{-1/2}$$

$$= \frac{1}{r} + \frac{r'\cos\theta}{r^2} + \frac{r'^2}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots$$

$$A = V_F = \frac{1}{4\pi\epsilon_0} \cdot \underbrace{\frac{\iiint \rho(r')\,d\tau'}{r}}_{\text{monopole term}} \leftarrow K_0 + \frac{1}{4\pi\epsilon_0} \cdot \underbrace{\frac{\iiint r'\cos\theta\, \rho(r')\,d\tau'}{r^2}}_{\text{dipole contribut}^n} \leftarrow K_1 + \ldots$$

$$K_1 = \frac{1}{4\pi\epsilon_0} \iiint r'\cos\theta\, \rho(r')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{\iiint \vec{r}'\cdot\hat{r}\, \rho(r')\,d\tau'}{r^2} = \frac{1}{4\pi\epsilon_0}\, \frac{\hat{r}\cdot\iiint \vec{r}'\, \rho(r')\,d\tau'}{r^2}$$

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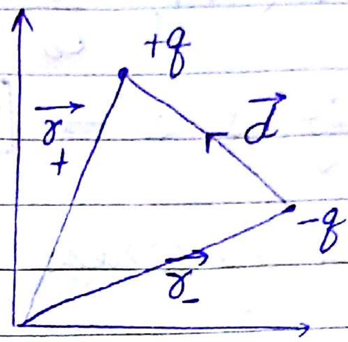
If total charge is non-zero, specify where dipole moment is being found/calc.
For total charge = 0 ⇒ write dipole contribute, no monopole.
≠ 0 ⇒ max. contrib^n of monopole.

general Eq^n of dipole moment,

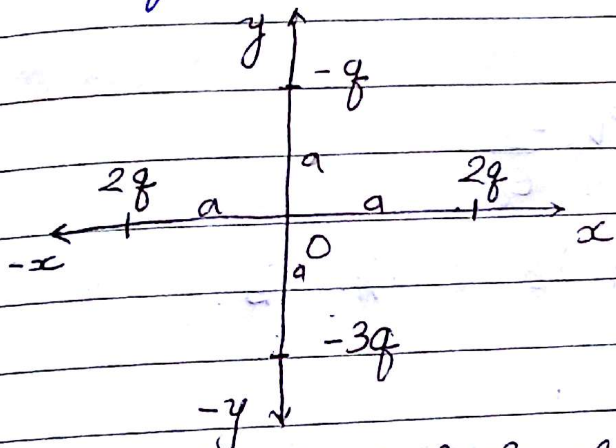
$$A = V_F = \iiint \vec{r}' \rho(\vec{r}')\, d\tau'$$



$$\vec{p} = \sum_{i=1}^{N} \vec{r}_i' Q_i$$

$$= \vec{r}_+(+q) + \vec{r}_-(-q)$$
$$= q(\vec{r}_+ - \vec{r}_-)$$
$$= q\vec{d}$$

Ex:



$$p = \sum_{i=1}^{4} \vec{r}_i Q_i$$
$$= (-a\hat{i})(2q) + (a\hat{i})2q + a\hat{j}(-q) + (a\hat{j})(-3q)$$

$$\therefore \boxed{\vec{p} = 2aq\hat{j}}$$

Find dipole of System?
Find $V(30a, 40a, 0)$? $\vec{r} = 30a\hat{i} + 40a\hat{j}$. Now, use
$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}.$$

→ $\vec{p} = \iiint_V \vec{r}'' \rho(r'')\, d\tau''$; $\vec{r}'' - \vec{r}' = \vec{r}$.

$$= \iiint (\vec{r}' + \vec{r}) \rho(\vec{r}' + \vec{r})\, d\tau'' = \iiint (\vec{r}' + \vec{r}) \rho(r')\, d\tau'$$
$$= \iiint (\vec{r}') \rho(r')\, d\tau' + \vec{r} \iiint \rho(r')\, d\tau'$$

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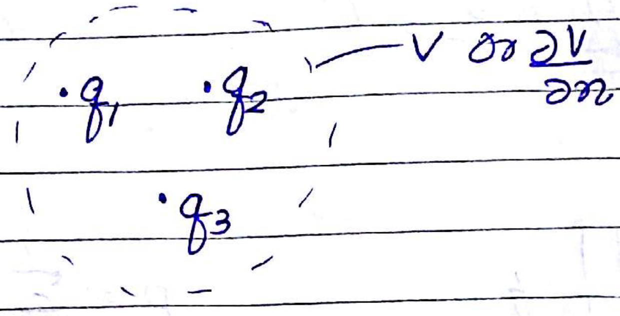
$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0} \Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \Rightarrow \text{Poison's Eq}^n$$

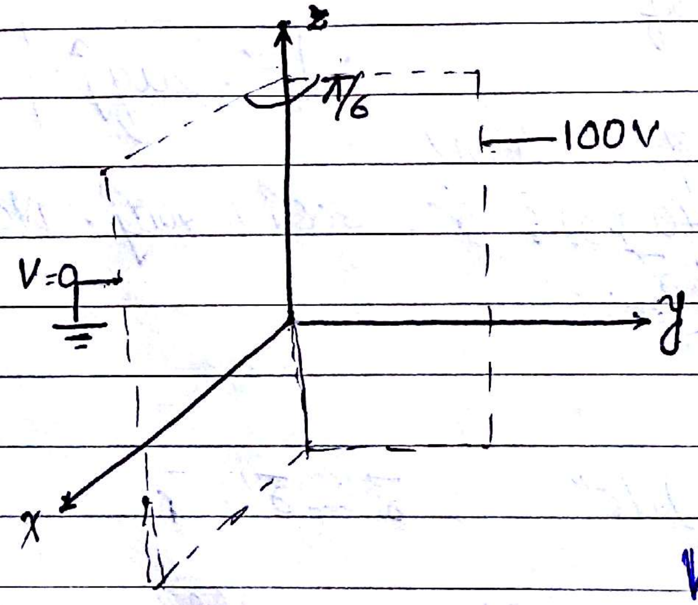
If $\rho = 0 \Rightarrow \boxed{\nabla^2 V = 0}$

→ Uniqueness Theorem:



Known charge : $\nabla^2 V = -\frac{\rho}{\epsilon_0}$
magnitude
location V_A, V_B ; $V_A = V_B$

Ex:



$V = (s, \phi, z)$

Sol^n $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$= \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad ; \Rightarrow V = A\phi + B$

$\phi = 0,\ V = 0,\ B = 0$

$\phi = \pi/6,\ V = 100,\ A = \frac{600}{\pi}$ $\quad \therefore \boxed{V = \frac{600\phi}{\pi}}$

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Ex: Find $\vec{E}$:

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right)$$

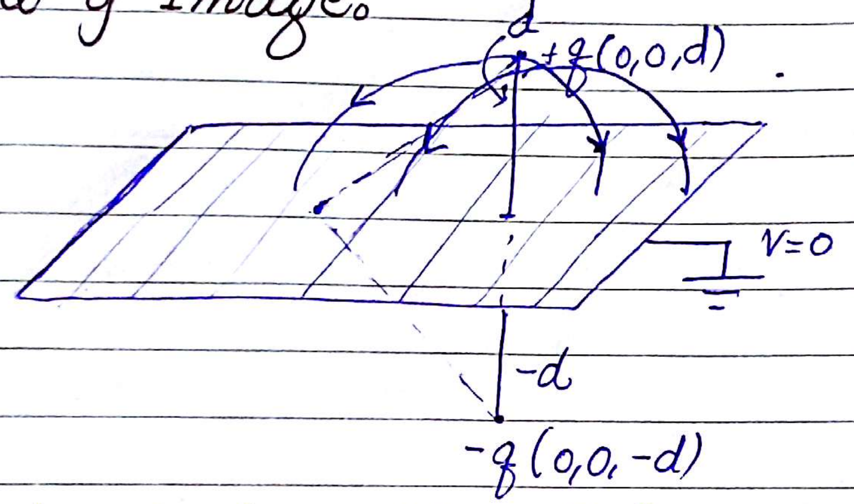
Sol^n: $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

$$E = -\nabla V \Rightarrow -\left(\frac{\partial}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right) \times \left(\frac{600\phi}{\pi}\right)$$

$$= -\frac{1}{s} \times \frac{600}{\pi}\hat{\phi}$$

Hence, $E = \frac{600s}{\pi}\hat{\phi}$

* method of Image:



$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{AT} - \frac{1}{BT}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\{x^2+y^2+(z-d)^2\}^{1/2}} - \frac{1}{\{x^2+y^2+(z+d)^2\}^{1/2}}\right]$$

$E = -\nabla E$

$E = \frac{\sigma}{\epsilon_0}$, $\sigma = \epsilon_0 \vec{E}$

$$= \epsilon_0(-\nabla V) = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$\therefore \sigma = -\frac{q}{2\pi}\frac{d}{\{x^2+y^2+d^2\}^{1/2}} = -\frac{q}{2\pi}\frac{d}{\{s^2+d^2\}^{3/2}}$$

$$Q_{\text{induced charge}} = \iint_{s=0}^{\infty} \sigma s \, d\phi \, ds$$

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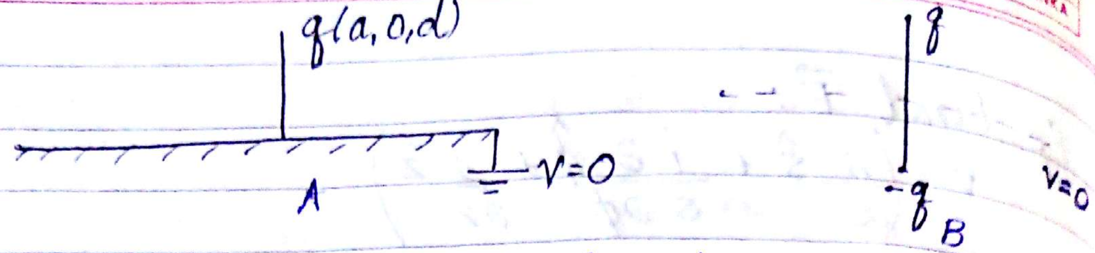
Planar symmetry



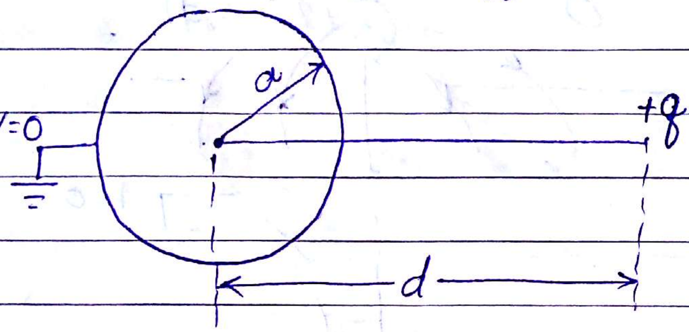
$$\sigma = \frac{-q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-q}{2\pi}\frac{d}{(s^2+d^2)^{3/2}}$$

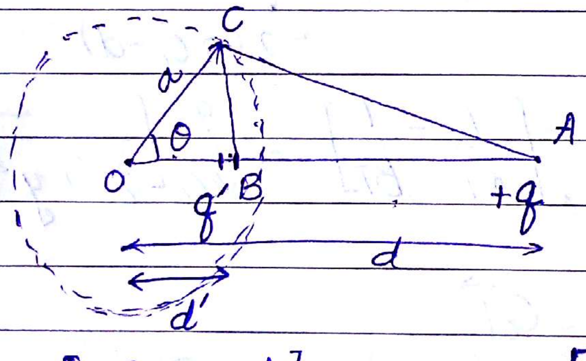
$$Q_{total} = \iint_0^{\infty} \sigma\, s\, d\phi\, ds = -q$$

$$F = \frac{-q^2}{4\pi\epsilon_0(2d)^2} \quad ; \quad U_B = \frac{-q^2}{4\pi\epsilon_0(2d)} = \frac{U_A}{2}$$

$$U_A = \int_{\infty}^{d} F\cdot dl = -\int_{\infty}^{d} \frac{q^2}{4\pi\epsilon_0(2z)^2}\,dz = -\frac{q^2}{16\pi\epsilon_0 d}$$

Spherical symmetry





$$V_C = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{AC} + \frac{q'}{BC}\right] = 0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\{a^2+d^2-2ad\cos\theta\}^{1/2}} + \frac{q'}{\{a^2+d'^2+2ad'\cos\theta\}^{1/2}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q/a}{\{1+\frac{d^2}{a^2}-\frac{2d}{a}\cos\theta\}^{1/2}} + \frac{q'/d'}{\{1+\frac{a^2}{d'^2}-\frac{2a\cos\theta}{d'}\}^{1/2}}\right] = 0$$

$$\left(\because \frac{q}{a} = \frac{-q'}{d'}\right), \quad \frac{d}{a} = \frac{a}{d'}$$

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σ_{max} at $\theta = 0$

Date σ_{min} at $\theta = \pi$

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$\therefore$ $\boxed{d' = \frac{a^2}{d}}$ $\boxed{q' = -\frac{qa}{d}}$

$\rightarrow$ $V(r) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\{r^2 + d^2 - 2rd\cos\theta\}^{1/2}} - \frac{a}{\{r^2d^2 + a^4 - 2rda^2\cos\theta\}^{1/2}}\right]$

$\downarrow$
any arbitrary pt

$$E_r = -\frac{\partial V(r,\theta)}{\partial r}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V(r,\theta)}{\partial \theta}$$

$$\sigma = \epsilon_0 E_\perp = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=a} = \frac{-q}{4\pi(a^2 + d^2 - 2ad\cos\theta)^{3/2}} \times \frac{(d^2 - a^2)}{a}$$

= induced charge density.

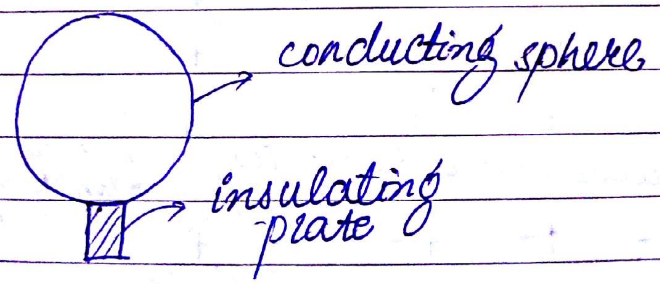
$$Q_{total} = \iint \sigma\, a^2 \sin\theta \, d\theta \, d\phi = -\frac{qa}{d}$$

$\rightarrow$ $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\{r^2 + d^2 - 2rd\cos\theta\}^{1/2}} - \frac{q'}{\{r^2 + d'^2 - 2rd'\cos\theta\}^{1/2}}\right] + \frac{V_0(4\pi\epsilon_0 a)}{4\pi\epsilon_0 r}$

(q'' at O $\therefore$ $V_0 = \frac{q''}{4\pi\epsilon_0 a}$)

$$\sigma = -\frac{q \cdot (d^2 - a^2)}{4\pi a (d^2 + a^2 - 2ad\cos\theta)^{3/2}} + \frac{q''}{4\pi a^2}$$

$\rightarrow$



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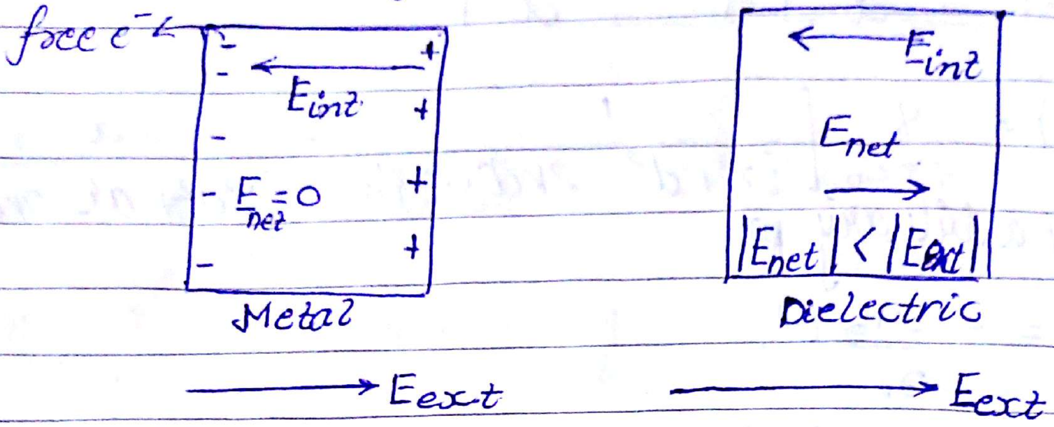
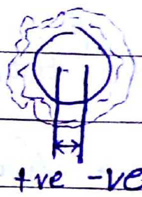
→ Electrostatic field in Dielectric Materials:



In insulator,

→ +ve & -ve charge centre coincide, ∴, net dipole moment = 0

→ When ext. field is applied, +ve & -ve charge do not coincide ∴ net dipole moment is generated. This is called polarisation.

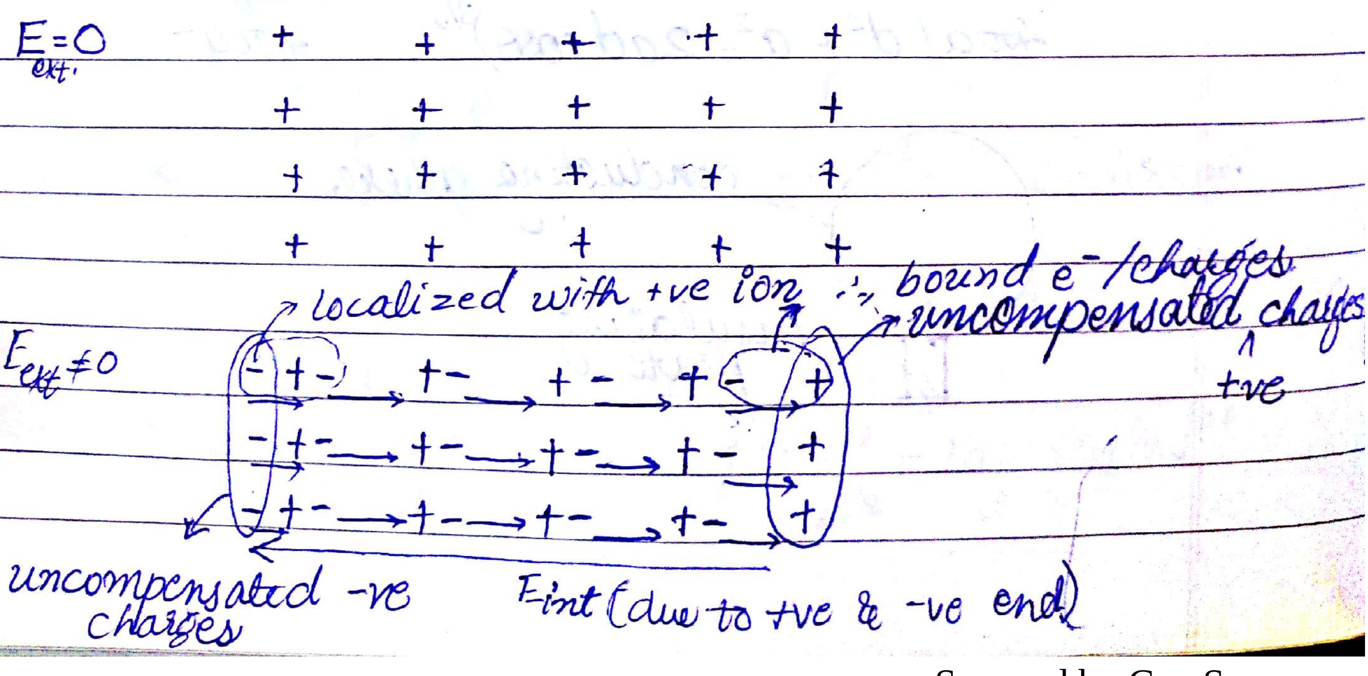


$\vec{P}$ = polarization vector.

$\vec{P} = \dfrac{\Sigma \vec{p}_i}{\Delta V}$

= dipole moment per unit vol. = charge / unit area

⇒ Polarization:



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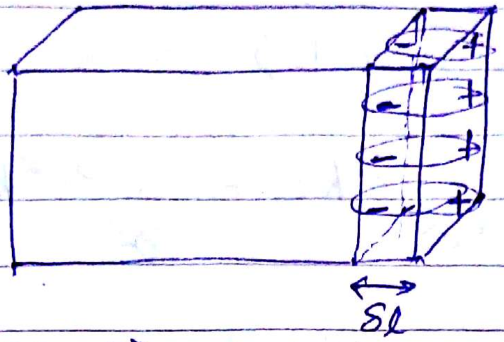


$$\vec{P} = \frac{\sum \vec{p_i}}{\Delta V}$$

$$\vec{P} = \frac{\delta \vec{l} \sum Q_i}{A \cdot \delta l}$$

$$\vec{P}\cdot\hat{n} = \vec{P}\cdot\frac{\vec{A}}{|\vec{A}|}$$

$$= \frac{\sum Q_i}{|\vec{A}|}$$

$= \sigma_b =$ Bound surface charge density.

Q_b = Bound charge ~~over~~ on all surfaces

$= \oiint \sigma_b\, d\vec{a} = \oiint \vec{P}\cdot\hat{n}\, da = \oiint \vec{P}\cdot d\vec{a}$

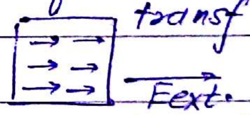
= Total amount of bound surface charge.

$Q_{b\,(vol.)}$ = Total amount of bound vol. charge $= -\oiint \vec{P}\cdot d\vec{a}$

$$= \iiint(-\vec{\nabla}\cdot\vec{P})\,d\tau$$

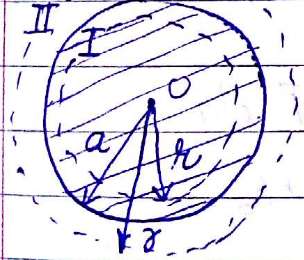
$\rho_b = -\vec{\nabla}\cdot\vec{P}$ = Bound vol. charge density.

liq → solid transf^n



* Electret:

material with permanent dipole moment (i.e. already polarized). It doesn't require E_{ext} to get polarised.



Ex: Find $\vec{E_I}$ & $\vec{E_{II}}$ if its an electret.

$\vec{P} = k\vec{r}$

Sol^n: $\oiint \vec{E}\cdot d\vec{a} = \frac{q_{encl.}}{\epsilon_0} \Rightarrow E(4\pi r^2) = ?$

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$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r)$$

$$= -\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 k r) = -\frac{1}{\cancel{r^2}}\, 3\cancel{r^2} k \Rightarrow -3k$$

$\therefore \rho_b = -3k$.

$$\therefore\ E_I(4\pi r^2) = \frac{-3k \times \frac{4}{3}\pi r^3}{\epsilon_0} \Rightarrow \vec{E}_I = \frac{-kr}{\epsilon_0}\hat{r}$$

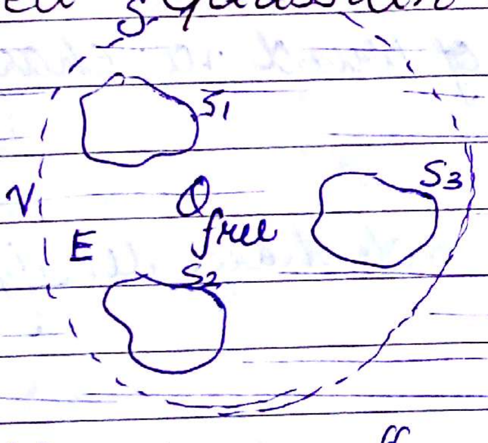
$$\underset{(\text{vol})}{Q_b} = -3k \times \frac{4\pi a^3}{3} = -4\pi k a^3$$

$$\sigma_b = \vec{P}\cdot\hat{n} = k\vec{r}\cdot\hat{n} = ka$$

$$Q_b = ka \times 4\pi a^2 = 4\pi k a^3$$

Total Charge = 0.

→ Modified Gaussian Law:-



Huge dielectric with 3 cavities.

$$\epsilon_0 \oint_S \vec{E}\cdot d\vec{a} = Q_{free} + \oint_{S_1+S_2+S_3} \sigma_b\, da + \iiint \rho_b\, d\tau$$

$$= Q_{free} + \oint_{S_1+S_2+S_3} \vec{P}\cdot d\vec{a} + \iiint(-\vec{\nabla}\cdot\vec{P})\, d\tau$$

$$\epsilon_0 \oint_S \vec{E}\cdot d\vec{a} = Q_{free} + \oint_{S_1+S_2+S_3} \vec{P}\cdot d\vec{a} - \oint_{S_1+S_2+S_3+S} \vec{P}\cdot d\vec{a}$$

$$\therefore\ \oint_S (\underbrace{\epsilon_0\vec{E} + \vec{P}}_{\vec{D}})\, da = Q_{free}$$

$\vec{D}$ = electric displacement vector

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$$\boxed{\oiint \vec{D}\cdot d\vec{a} = Q_{free}}$$ modified Gaussian Law.

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ Dimension of $\vec{D}$ is same as $\vec{P}$ = charge/Area.

$$\vec{E} = \left(\frac{\vec{D}}{\epsilon_0}\right) - \left(\frac{\vec{P}}{\epsilon_0}\right)$$

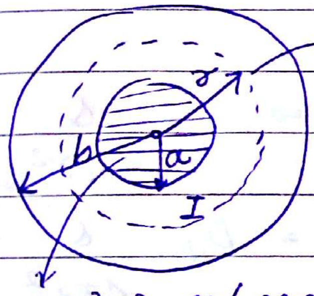
$\frac{\vec{P}}{\epsilon_0}$: E produced due to bound charges.

$\frac{\vec{D}}{\epsilon_0}$: E produced due to free charges

$\vec{E}$: E_{int}

$\left.\begin{array}{l}\vec{\nabla}\times\vec{D} \\ \oint \vec{D}\cdot d\vec{l}\end{array}\right\}$ may be or may not be 0

Q



$\vec{P} = \frac{k}{r}\hat{r}$

Sol^n: (MI) $\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = -\frac{k}{r^2}$

$$\sigma_b = \vec{P}\cdot\hat{n} = \frac{k}{r}\hat{r}\cdot(-\hat{r})\Big|_{r=a} = -\frac{k}{a}$$

$$\oiint \vec{E}\cdot d\vec{a} = \frac{Q_{encl.}}{\epsilon_0} = -\frac{k}{a}\times 4\pi a^2 + \iiint_{r=a}^{r=r} -\frac{k}{r^2}\times r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$\therefore E_I(4\pi r^2) = -4\pi k a - 4\pi k(r-a)$

$\Rightarrow E_I = \frac{-k}{\epsilon_0 r}\hat{r}$

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(M2) $\vec{D}(4\pi r^2) = 0 \Rightarrow \vec{D} = 0 = \epsilon_0 \vec{E_I} + \vec{P}$

$\therefore \vec{E}_I = \frac{-\vec{P}}{\epsilon_0} = \frac{-k\hat{r}}{\epsilon_0 r}$

$\rightarrow \vec{P} = \epsilon_0 \chi \vec{E}$

└→ Electric susceptibility

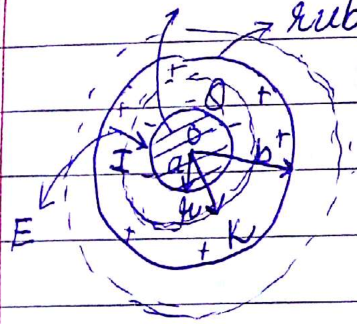
$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi \vec{E} = \epsilon_0(1+\chi)\vec{E}$ → Dielectric const.

$\vec{D} = \epsilon_0 \epsilon_r \vec{E}$; ϵ_r = Relative permittivity

$\vec{D} = \epsilon \vec{E}$; ϵ = permittivity

$\epsilon_r = \frac{\epsilon}{\epsilon_0}$

Ex:



$D_I(4\pi r^2) = Q \Rightarrow D_I = \frac{Q}{4\pi r^2}$

$D_{II}(4\pi r^2) = Q \Rightarrow D_{II} = \frac{Q}{4\pi r^2}\hat{r}$

$E_I = \frac{D}{\epsilon_0 K} = \frac{Q}{4\pi\epsilon_0 r^2 k}\hat{r}$; $E_{II} = \frac{D}{\epsilon_0} = \frac{Q}{4\pi r^2 \epsilon_0}\hat{r}$

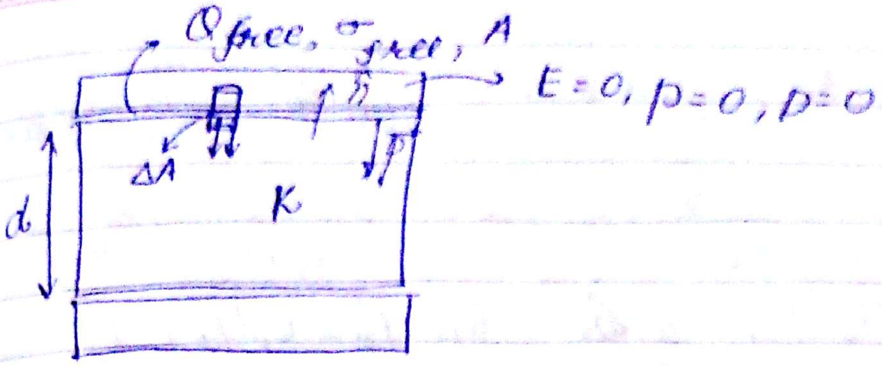
$\vec{P}$ can be found

$\vec{P} = \epsilon_0(k-1)E_I = \frac{Q(k-1)}{4\pi r^2 k}\hat{r}$; $\rho_b = 0$

$Q_b \Big|_{\text{surface of } r=a} = 4\pi a^2 \times \frac{-Q(k-1)}{4\pi a^2 k} = \frac{-Q(k-1)}{k}$

$V(0) = \int_\infty^r -\vec{E}\cdot d\vec{r} = -\int_\infty^b E_{II}\cdot d\vec{r} - \int_b^a E_I \cdot d\vec{r} - \underbrace{\int_a^0 E\cdot d\vec{r}}_{0}$

Ex:



$$\oiint \vec{D}\cdot d\vec{a} = Q_{free}$$

$$D\,\Delta A = \sigma_{free}\,\Delta A \Rightarrow \boxed{D = \sigma_{free}}$$

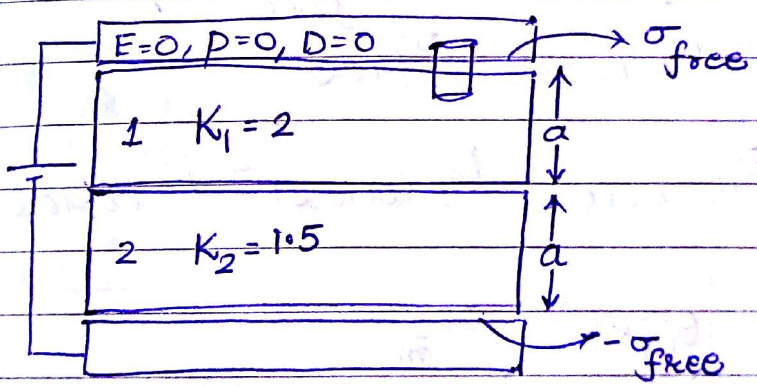
$$E = \frac{D}{\epsilon_0 k} = \frac{\sigma}{\epsilon_0 k} \quad \text{and} \quad Ed = \Delta V = \frac{\sigma d}{\epsilon_0 k}$$

$$\therefore\; C = \frac{Q}{\Delta V} = \left(\frac{A\epsilon_0}{d}\right)k$$

$$\vec{p} = \vec{D} - \epsilon_0\vec{E} = \sigma - \frac{\cancel{\epsilon_0}\sigma}{\cancel{\epsilon_0}k} \Rightarrow \frac{\sigma(k-1)}{k}$$

$$(\sigma_b)_{top} = \frac{-(k-1)\sigma}{k}$$

Ex:



Find: (i) E_1, E_2, $\vec{P_1}$, $\vec{P_2}$, $\sigma_{b_1}(top)$, $\sigma_{b_1}(bottom)$, $\sigma_{b_2}(top)$, $(\sigma_{b_2})_{bottom}$, ΔV.

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Sol^n: $\oint \vec{D}\cdot d\vec{a} = Q_{free} \Rightarrow D\cdot \Delta A = \sigma_{free}\,\Delta A \Rightarrow D = \sigma_{free}$

(i) $\vec{E}_1 = \frac{D}{\epsilon_0 K_1} = \frac{\sigma}{2\epsilon_0}\hat{x}$ (ii) $E_2 = \frac{D}{\epsilon_0 K_2} = \frac{2\sigma}{3\epsilon_0}\hat{x}$

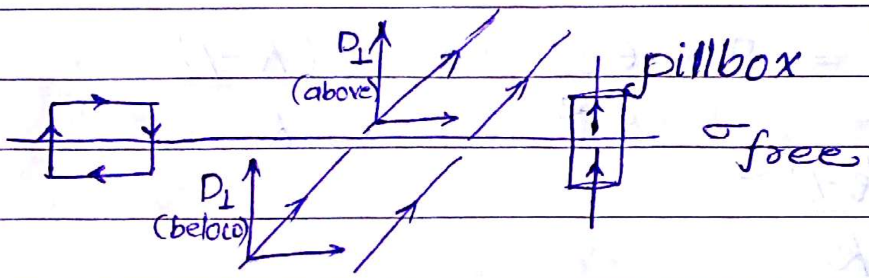
(iii) $\vec{P}_1 = \epsilon_0(K_1-1)E_1 = \frac{\sigma}{2}\hat{x}$ (iv) $\vec{P}_2 = \epsilon_0(K_2-1)E_2 = \frac{\sigma}{3}\hat{x}$

(v) $\sigma_{b1(top)} = \vec{P}\cdot(-\hat{n}) = -\frac{\sigma}{2}$ (vi) $\sigma_{b1(bottom)} = \vec{P}\cdot\hat{n} = \frac{\sigma}{2}$

(vii) $\sigma_{b2(top)} = \vec{P}\cdot(-\hat{n}) = -\frac{\sigma}{3}$ (viii) $\sigma_{b2(bottom)} = \vec{P}\cdot\hat{n} = \frac{\sigma}{3}$

(ix) $\Delta V = E_1 a + E_2 a = a(E_1+E_2)$

$\Rightarrow$ **Boundary Condition of $\vec{D}$:**

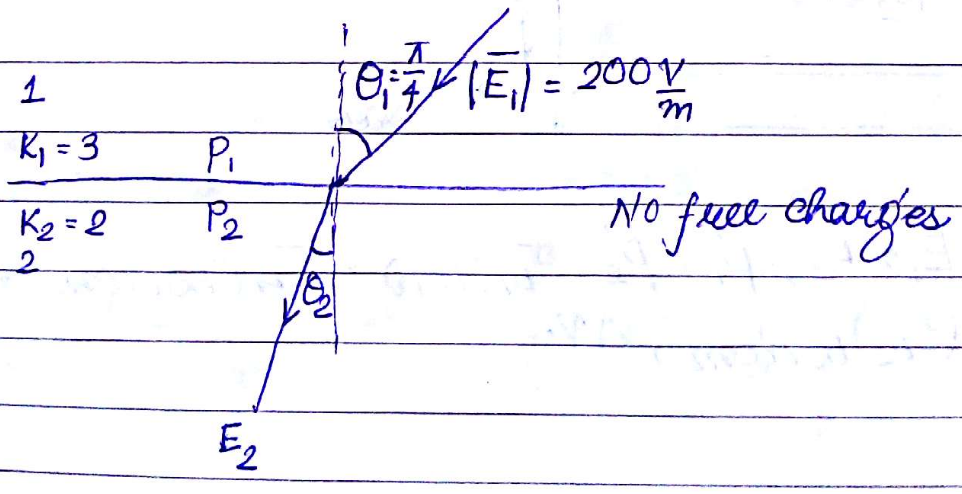


$D_\perp(above)\cancel{A} - D_\perp(below)\cancel{A} = \sigma_{free}\cancel{A}$

$\Rightarrow D_{\perp(above)} - D_{\perp(below)} = \sigma_{free}$

$D_{\parallel(above)} - P_{\parallel(above)} = D_{\parallel(below)} - P_{\parallel(below)}$

Ex:



Sol: $D_{\perp(above)} = D_{\perp(below)} \Rightarrow D_1 \cos 45^\circ = D_2 \cos\theta_2$

$= \epsilon_0 K_1 |E_1| \cos 45^\circ = \epsilon_0 K_2 |E_2| \cos\theta_2$

$= \epsilon_0 \times 3 \times 200 \times \frac{1}{\sqrt{2}} = \epsilon_0 \times 2 \times |E_2| \cos\theta_2$

$\therefore \; |E_2| \cos\theta_2 = 150\sqrt{2}$ —— (1)

$|E_1| \sin 45^\circ = |E_2| \sin \cancel{45}\,\theta_2$ —— (2) $\Rightarrow 100\sqrt{2} = |\vec{E_2}| \sin\theta_2$

$\therefore \; \theta_2 = 33.61^\circ$ and $|\vec{E_2}| = 262.5\, V/m$

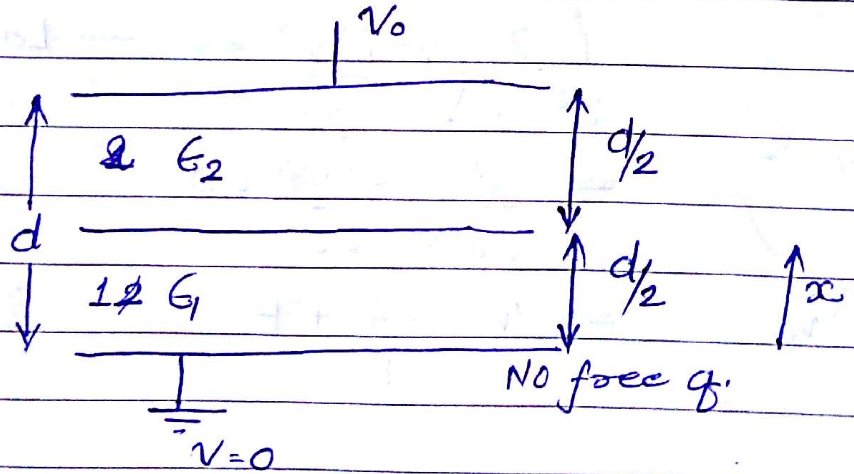
$\vec{p} \cdot \hat{n} = \vec{P_1} \cdot \hat{n} + \vec{P_2} \cdot \hat{n} = \epsilon_0 \times 400 (\cos\theta_1) - \epsilon_0 (E_2 \cos\theta_2)$

$\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho_{free}$

$\vec{D} = \epsilon \vec{E} \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho_{free}}{\cancel{\epsilon_0}\epsilon} \Rightarrow +\vec{\nabla}^2 V = \frac{-\rho_{free}}{\cancel{\epsilon_0}\epsilon}$

$\therefore \; \nabla^2 V = \frac{-\rho_{free}}{\epsilon}$; If $\rho_{free} = 0 \Rightarrow \nabla^2 V = 0$

Ex:



$V(x)$? $E(x)$? $\vec{P_1}$ & $\vec{P_2}$? σ_{free}? σ_b?

Solⁿ: $\nabla^2 V = 0$ and $\frac{\partial^2 V}{\partial x^2} = 0$

$V_I = A_1 x + B_1$ $\qquad$ $E_1 = -\frac{\partial V_1}{\partial x} = -A_1$

$V_{II} = A_2 x + B_2$ $\qquad$ $E_2 = -A_2$

$V_I = 0\ (x=0) = A_1(0) + B_1 \Rightarrow B_1 = 0$ —— ①

$V_{II}(x=d) = V_0 = A_2 d + B_2$ —— ②

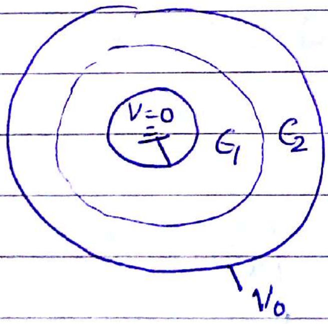
$D_{\perp(above)} = D_{\perp(below)} \Rightarrow D_1 = D_2 \Rightarrow \epsilon_1 E_1 = \epsilon_2 E_2$

$\therefore \epsilon_1 A_1 = \epsilon_2 A_2$ —— ③

$V_I\left(x = \frac{d}{2}\right) = V_{II}\left(x = \frac{d}{2}\right) \Rightarrow A_1 \frac{d}{2} = A_2 \frac{d}{2} + B_2$ —— ④

$E_1 = \dfrac{V_0}{\frac{d}{2}\left(\frac{\epsilon_1}{\epsilon_2} + 1\right)}$ $\qquad$ $E_2 = \dfrac{V_0}{\frac{d}{2}\left(\frac{\epsilon_2}{\epsilon_1} + 1\right)}$

Ex:



$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_r}{\partial r}\right) = 0$ — Laplace Eqⁿ

$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$

$\Rightarrow V = \frac{-A}{r} + B$

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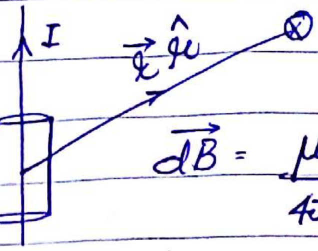
Magnetostatics

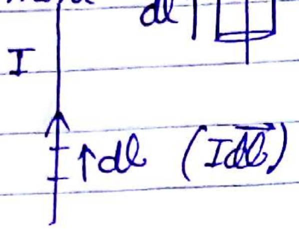
⟹ Biot-Savart's Law:



$\vec{\nabla}\cdot\vec{J}=0$

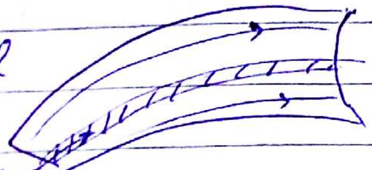
$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

current ~~density~~ element



$$\vec{B}=\frac{\mu_0}{4\pi}\int\frac{\vec{I}\times\hat{r}}{r^2}\,dl'$$

Q per unit l

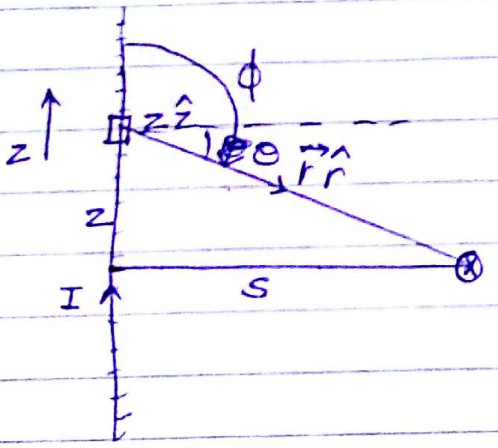


$$\vec{B}=\frac{\mu_0}{4\pi}\iint\frac{\vec{K}\times\hat{r}}{r^2}\,da'$$

Q per unit area.

$$\vec{B}=\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}\times r'}{r^2}\,d\tau'$$

Ex:



$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{I\,dz\,\hat{z}\times\hat{r}}{r^2}$$

$$=\frac{\mu_0}{4\pi}\frac{I\,dz\sin\phi}{r^2}$$

$$=\frac{\mu_0}{4\pi}\frac{I\,dz\sin(90+\theta)}{r^2}$$

$$=\frac{I\mu_0\,dz\cos\theta}{4\pi r^2}$$

$\frac{s}{r}=\cos\theta \rightarrow r=s\sec\theta$

$\frac{z}{s}=\tan\theta \Rightarrow dz=s\sec^2\theta\,d\theta$

$$\therefore\ d\vec{B}=\frac{\mu_0 I}{4\pi}\times\frac{s\sec^2\theta\cos\theta\,d\theta}{s^2\sec^2\theta}$$

$$\vec{B}=\frac{\mu_0 I}{4\pi s}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta=\frac{\mu_0 I}{2\pi s}\hat{\phi}$$

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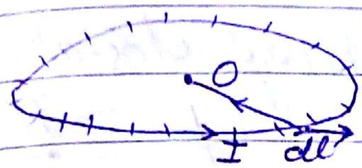
Ex:

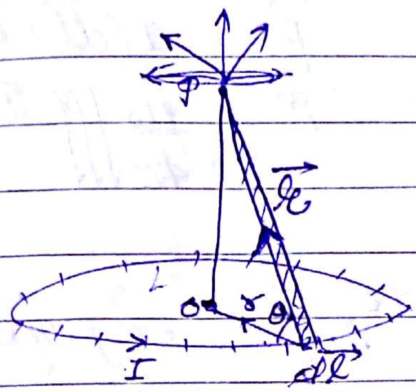


$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{I\vec{dl}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{Idl}{r^2} = \frac{\mu_0 I}{4\pi r^2}\underbrace{\int dl}_{2\pi r}$$

$$\therefore \vec{B} = \frac{\mu_0 I}{2r}\hat{z}$$

Ex:



$$\vec{dB} = \frac{\mu_0 \cancel{I\times\hat{r}}\, I\vec{dl}\times\hat{r}}{4\pi\cdot r^2} = \frac{\mu_0 I dl}{4\pi r^2}$$

$$\vec{dB} = \frac{\mu_0 I dl\cos\theta}{4\pi r^2}$$

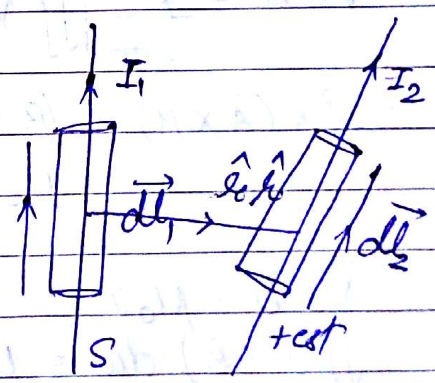
→ Lorentz Force:

$F_L = q(\vec{v}\times\vec{B})$; $\vec{v}$ = velocity of charged particle

$= I(\vec{dl}\times\vec{B}) = (I\vec{dl})\times\vec{B}$

$\vec{F_e} = q\vec{E}$

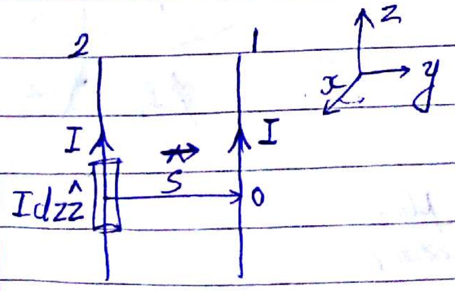
$\vec{F_L}\cdot\vec{dl} = I(\vec{dl}\times\vec{B})\cdot\vec{dl} = 0$



$$F = (I_2\vec{dl_2})\times\frac{\mu_0}{4\pi}\left(\frac{I_1\vec{dl_1}\times\hat{r}}{r^2}\right)$$

$$= \frac{\mu_0}{4\pi}\left(\frac{I_2\vec{dl_2}\times I_1\vec{dl_1}\times\hat{r}}{r^2}\right)$$

Ex:



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\vec{F} = Idz\hat{z}\times\frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$= \frac{\mu_0 dz I^2}{2\pi s}\underbrace{(\hat{z}\times\hat{\phi})}_{-\hat{s}}$$

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Comparison: Electrostatics | magnetostatics

→ $d\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r^2}\hat{r}$ | $d\vec{B} = \frac{\mu_0}{4\pi}\frac{(I\times d\vec{l})\times\hat{r}}{r^2}$

→ $\vec{F_e} = q\vec{E}$ | $\vec{F_L} = I(d\vec{l}\times\vec{B})$

→ $E = \frac{1}{4\pi\epsilon_0}\iiint\frac{\rho\, d\tau'}{r^2}\hat{r}$ | $\vec{B} = \frac{\mu_0}{4\pi}\iiint\frac{\vec{J}\times\hat{r}}{r^2}d\tau'$

→ $\vec{B}(r) = \frac{\mu_0}{4\pi}\iiint\frac{\vec{J}(r')\times\hat{r}}{r^2}d\tau'$ [curl w.r.t prime = 0 $d\tau'$]

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\left[\iiint\frac{\hat{r}}{r^2}\underbrace{(\vec{\nabla}\times\vec{J}(r'))}_{0}d\tau' - \iiint\vec{J}(r')\cdot\underbrace{\left(\vec{\nabla}\times\frac{\hat{r}}{r^2}\right)}_{0}\right]$$

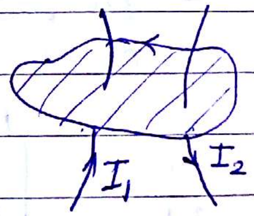
$$= \vec{\nabla}\cdot\vec{B} = 0 = \oint\vec{B}\cdot d\vec{a} = 0$$

→ $\vec{B} = \frac{\mu_0}{4\pi}\iiint\left(\vec{J}\times\frac{\hat{r}}{r^2}\right)d\tau'$

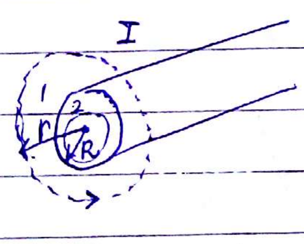
$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\iiint\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{r}}{r^2}\right)d\tau'$$

$$\vec{\nabla}\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\bar{\nabla})\bar{A} - (\bar{A}\cdot\bar{\nabla})\bar{B} + \bar{A}(\bar{\nabla}\cdot\bar{B}) - \bar{B}(\bar{\nabla}\cdot\bar{A})$$
$$= \mu_0 J(r)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$
$$\iint(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\iint\vec{J}\cdot d\vec{a} \implies \boxed{\oint\vec{B}\cdot d\vec{l} = \mu_0 I_{encl}}$$



Ex:



Sol^n:

(1) $\vec{B_1}(2\pi r) = \mu_0 I \Rightarrow \vec{B_1} = \frac{\mu_0 I}{2\pi r}$

(2) $\vec{B_2}(2\pi r) = \mu_0\left(\frac{I}{\pi R^2}\times\pi r^2\right) \Rightarrow \vec{B_2} = \frac{\mu_0 I r}{2\pi R^2}$

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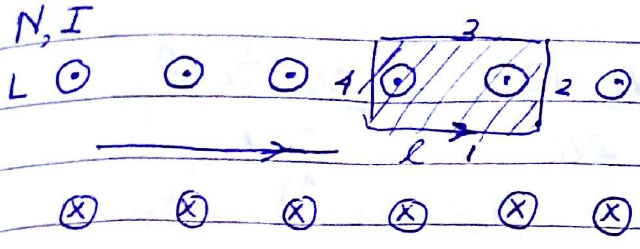
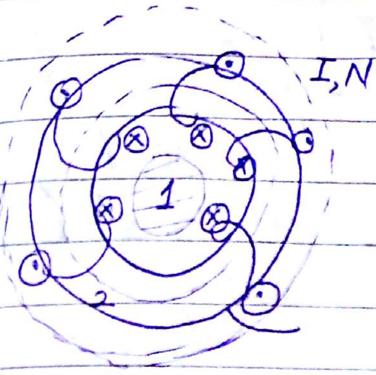
Ex:



Sol^n: $\vec{B}\cdot l = \mu_0\left(\frac{N l I}{L}\right) \Rightarrow \vec{B} = \frac{\mu_0 N I}{L} = \mu_0 n I \;;\; n = \frac{N}{L}$

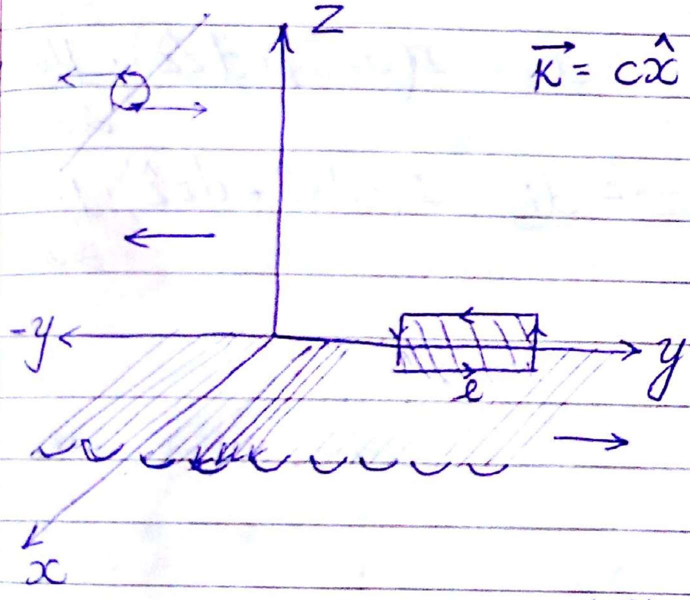
Ex:



$B_2(2\pi r) = \mu_0 N I \Rightarrow B_2 = \frac{\mu_0 N I}{2\pi r}$

$B_3 = B_1 = 0$

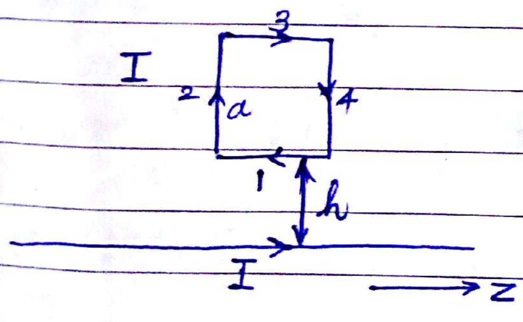
Ex:



$\vec{K} = c\hat{x}$

Sol^n: $Bl + Bl = \mu_0(cl) \Rightarrow |\vec{B}| = \frac{\mu_0 c}{2}$

Ex:



$Ia(-\hat{z})$ $\qquad\qquad\qquad s = h$

$$\vec{F_1} = \cancel{Ia} \times \frac{\mu_0 I}{2\pi s}\hat{\phi} = \frac{\mu_0 a I^2 \hat{s}}{2\pi s} = \frac{\mu_0 I^2 a}{2\pi h}\hat{s}$$

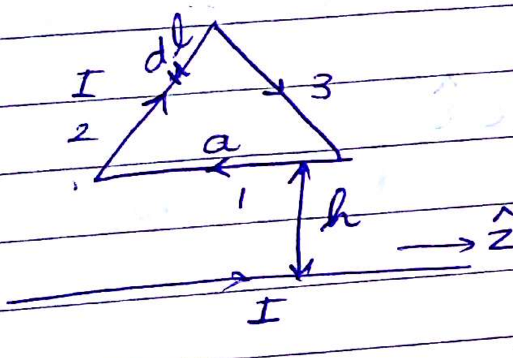
$$\vec{F_3} = Ia(\hat{z}) \times \frac{\mu_0 I \hat{\phi}}{2\pi h} = \frac{\mu_0 I^2 a}{2\pi h}(-\hat{s})$$

$$\vec{F_2} =$$

$$\cancel{\vec{B_2} = \int_h^{a+h} \frac{\mu_0 I \times ds\,\hat{s}}{s^2} = \left[-\frac{\mu_0 I}{s}\right]_h^{a+h} =}$$

$$\vec{dF_2} = I ds\,\hat{s} \times \frac{\mu_0 I}{2\pi s}\hat{\phi} \Rightarrow \vec{F_2} = \frac{\mu_0 I^2}{2\pi}\int_h^{a+h} \frac{ds}{s}\hat{z}$$

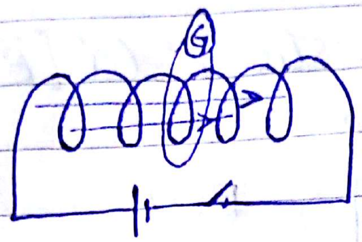
Ex



$$dF_2 = I(ds\,\hat{s} + dz\,\hat{z}) \cdot \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$dF_3 = I(-ds\,\hat{s} + dz\,\hat{z}) \cdot \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Electrodynamics

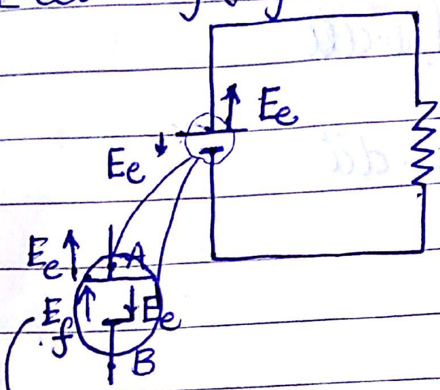


$$\mathcal{E} \propto N \frac{\partial \phi_B}{\partial t}$$

$$\phi_B = \vec{B} \cdot \vec{A}$$

$$\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = \oint (\vec{E_f} + \vec{E_e}) \cdot \vec{dl} \quad \text{——} \ (1)$$

↓
Emf



Chemical rxn

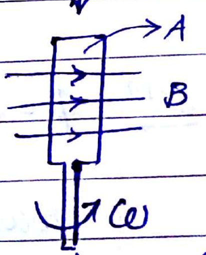
from (1),

$$\therefore \ \mathcal{E} = \oint \vec{E_f} \cdot \vec{dl} + \underbrace{\oint \vec{E_e} \cdot \vec{dl}}_{0} = \int_B^A \vec{E_f} \cdot \vec{dl} = -\int_B^A \vec{E_e} \cdot \vec{dl}$$

$$\mathcal{E} = -\frac{\partial (BA\cos\theta)}{\partial t}$$

change B, or A or relative angle to change Emf.

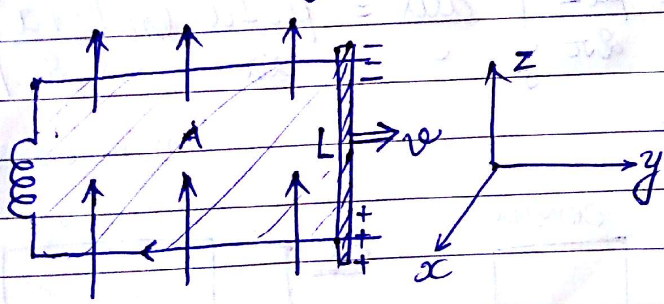
→ Generator



$$\mathcal{E} = \frac{-\partial \phi}{\partial t} = \frac{-\partial (BA\cos\theta)}{\partial t}, \quad \theta = \omega t$$

$$\mathcal{E} = -AB\omega \sin\omega t$$

→ motional Emf:



$$\vec{F_m} = q(v\hat{j} \times B\hat{k}) = qvB\,\hat{i}$$

$$\mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot \vec{dl} = vBL$$

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B, A – vary
→ add.

→ Time varying magnetic field $[\vec{B}(t)]$:

B(t)

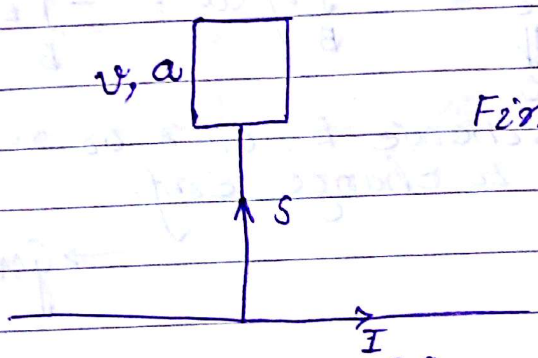
$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{a}$$

$$\iint (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\therefore \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

not electrostatic field.
Its non-conservative

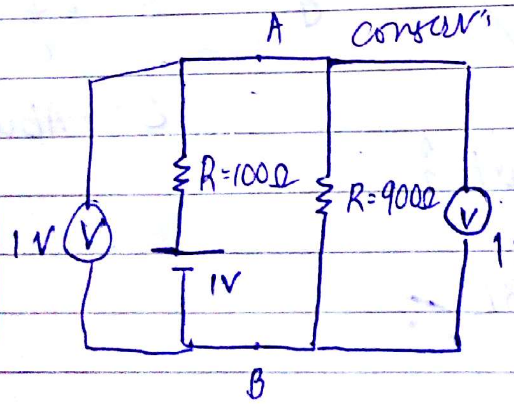
Ex: Find $\mathcal{E}$?



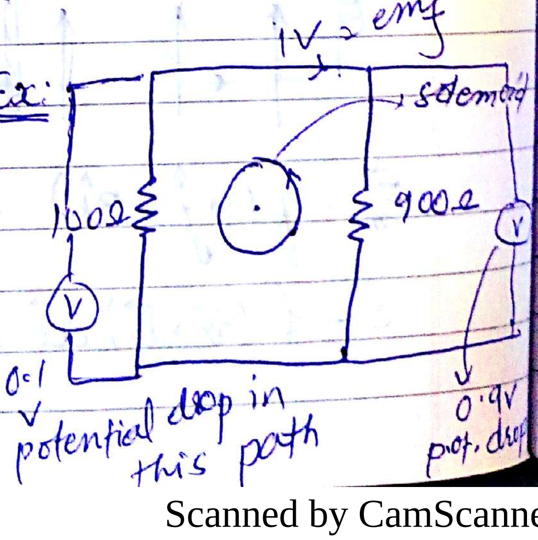
Sol^n → $$\phi = \iint \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi}\int_S^{S+a} \frac{a\,ds}{s} = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{S+a}{S}\right)$$

Now, $\mathcal{E} = -\dfrac{\partial \phi}{\partial t}$

Ex:



Ex:



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