Lecture 10: Absolute Convergence, Comparison Test, Limit Comparison Test

August 25, 2019

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

The next example illustrates that condition in Proposition 9.5 is not sufficient.

Example 10.1 *The Harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges because*

$$\begin{aligned} S_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\overbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}^{2^{(n-1)} \textit{ times}}\right) \\ &= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^{(n-1)} \cdot \frac{1}{2^n} \\ &= 1 + \frac{n}{2}. \end{aligned}$$

So $S_{2^n} \to \infty$ *and also* S_n *is an increasing sequence so* $S_n \to \infty$*. But* $a_k = \frac{1}{k} \to 0$.

Theorem 10.2 (Cauchy Criterion for series) *A series* $\sum_k a_k$ *is convergent if and only if for every* $\epsilon > 0$*, there is* $n_0 \in \mathbb{N}$ *such that*

$$\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon \qquad \textit{for all } m > n \geq n_0$$

Proof: A series $\sum_k a_k$ is convergent if and only if the sequence (S_n) of partial sums is convergent $\iff$ (S_n) is Cauchy $\iff$ every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|S_m - S_n| < \epsilon \qquad \text{for all } m, n \geq n_0$$

Note that $S_m - S_n = \sum_{k=n+1}^{m} a_k$ for all $m > n$. ∎

Definition 10.3 *A series* $\sum_k a_k$ *is said to be absolutely convergent if the series* $\sum_k |a_k|$ *is convergent.*

Theorem 10.4 *An absolutely convergent series is convergent.*

Proof: Convergence of $\sum_k |a_k|$ implies that every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{k=n+1}^{m} |a_k|\right| < \epsilon \qquad \text{for all } m > n \geq n_0$$

Now observe that

$$\left|\sum_{k=n+1}^{m} |a_k|\right| = \sum_{k=n+1}^{m} |a_k| \geq \left|\sum_{k=n+1}^{m} a_k\right|.$$

Hence series $\sum_k a_k$ satisfies Cauchy criteria. So it converges. ■

10.1 Tests for absolute convergence of an Infinite series

We shall now give a variety of tests to determine the absolute convergence (and hence, the convergence) of a series.

Theorem 10.5 (Comparison Test) *Let* $a_k, b_k \in \mathbb{R}$ *be such that* $|a_k| \leq b_k$ *for all* $k \in \mathbb{N}$.

1. *If* $\sum_k b_k$ *is convergent, then* $\sum_k a_k$ *is absolutely convergent and*

$$\left|\sum_k a_k\right| \leq \sum_k b_k.$$

2. *If* $\sum_k |a_k|$ *diverges to* ∞*, then* $\sum_k b_k$ *also diverges to* ∞.

Example 10.6 *Show that* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *diverges for* $p < 1$ *and converges for* $p > 1$.

Solution: Note that for $0 \leq p < 1$

$$0 < \frac{1}{k} \leq \frac{1}{k^p}, \ \forall\, k \in \mathbb{N}$$

Hence by comparison test series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $0 \leq p < 1$.

If $p < 0$ then $-p > 0$ and hence $k^{-p} \geq 1$ for all $k = 1, 2, \cdots$. Therefore by comparison test series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p < 0$.

. Now for $p > 1$

$$\begin{aligned}
&1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots \\
&< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{8^p}\right) + \cdots \\
&= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\
&= 1 + \sum_{k=1}^{\infty} \frac{2^k}{2^{kp}} \\
&= 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k
\end{aligned}$$

Given a series $\sum_k a_k$, it may be difficult to look for a convergent series $\sum_k b_k$ such that $|a_k| \leq b_k$ for each k. It is often easier to find a convergent series $\sum_k b_k$ of nonzero terms such that the ratio $\frac{a_k}{b_k}$ approaches a limit as $k \to \infty$. In these cases, the following result is useful.

Theorem 10.7 (Limit comparison Test) *Let (a_k) and (b_k) be sequences such that a_k and $b_k > 0$ for all k. Assume that $\frac{a_k}{b_k} \to l$ as $k \to \infty$ where $l \in \mathbb{R} \cup \{\infty\}$.*

1. *If $l > 0$ and $l \in \mathbb{R}$ then, $\sum_k a_k$ is convergent $\iff$ $\sum_k b_k$ is convergent.*
2. *If $l = 0$ and $\sum_k b_k$ converges then $\sum_k a_k$ is convergent.*
3. *If $l = \infty$ and $\sum_k a_k$ converges then $\sum_k b_k$ is convergent.*

Example 10.8 *Determine the convergence of the series* $\sum_{k=1}^{\infty} \frac{2^k + k}{3^k - k}$.

Solution: *Let* $a_k = \frac{2^k+k}{3^k-k}$ *and* $b_k = \left(\frac{2}{3}\right)^k$. *Moreover,*

$$\frac{a_k}{b_k} = \frac{2^k + k}{3^k - k} \times \frac{3^k}{2^k} = \frac{1 + \frac{k}{2^k}}{1 - \frac{k}{3^k}} \to 1$$

Hence by limit comparison test series converges.