Lecture 24: Improper Integral

October 10, 2019

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Remark 24.1 *It is useful to keep in mind that the convergence of an improper integral* $\int_a^\infty f(x)dx$ *is not affected by changing the initial point* a *of the interval* $[a,\infty)$*, although the value of improper integral may change by doing so. Indeed, if* $a' \geq a$*, then*

$$\int_a^\infty f(x)dx \text{ is convergent } \iff \int_{a'}^\infty f(x)dx \text{ is convergent }.$$

and if this holds, then the values are related by the equation

$$\int_a^\infty f(x)dx = \int_a^{a} f(x)dx + \int_{a'}^\infty f(x)dx$$

Example 24.2 *The improper integral* $\int_0^\infty \cos x \, dx$ *is divergent, since for all* $x \in \mathbb{R}$, $\int_0^x \cos t \, dt = \sin x$ *and* $\lim_{x\to\infty} \sin x$ *does not exist.*

Definition 24.3 *An improper integral* $\int_a^\infty f(x)dx$ *is said to be absolutely convergent if the improper integral* $\int_a^\infty |f(x)|dx$ *is convergent.*

Proposition 24.4 *An absolutely convergent improper integral is convergent.*

Converse is not true. We shall see an example later.

24.1 Convergence Tests for Improper Integrals

Theorem 24.5 (Comparison Test for Improper Integrals) *Suppose* $a \in \mathbb{R}$ *and* $f, g : [a,\infty) \to \mathbb{R}$ *are such that both* f *and* g *are integrable on* $[a,x]$ *for every* $x \geq a$ *and* $|f| \leq g$*. If* $\int_a^\infty g(x)dx$ *is convergent, then* $\int_a^\infty f(x)dx$ *is absolutely convergent and*

$$\left|\int_a^\infty f(x)dx\right| \leq \int_a^\infty g(x)dx.$$

Example 24.6 *Determine if the improper integral* $\displaystyle\int_1^\infty \frac{\cos x}{x^2}dx$ *converges.*

Solution: Note that $\left|\dfrac{\cos x}{x^2}\right| \leq \dfrac{1}{x^2}$ for all $x \geq 1$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges, hence improper integral $\displaystyle\int_1^\infty \frac{\cos x}{x^2}dx$ converges.

Theorem 24.7 (Limit Comparison Test for Improper Integrals) *Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ be such that both f and g are integrable on $[a, x]$ for every $x \geq a$ with $f(t) > 0$ and $g(t) > 0$ for all large t. Assume that*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = l, \qquad \text{where } l \in \mathbb{R} \text{ or } l = \infty$$

1. *If $l \in \mathbb{R}, l \neq 0$, then $\displaystyle\int_a^\infty f(x)dx$ converges $\iff$ $\displaystyle\int_a^\infty g(x)dx$ converges.*
2. *If $l = 0$ and $\displaystyle\int_a^\infty g(x)dx$ converges, then $\displaystyle\int_a^\infty f(x)dx$ converges.*
3. *If $l = \infty$ and $\displaystyle\int_a^\infty f(x)dx$ converges, then $\displaystyle\int_a^\infty g(x)dx$ converges absolutely.*

Example 24.8 *Let $q \in \mathbb{R}$. Discuss the convergence of improper integral* $\displaystyle\int_1^\infty e^{-t}t^q\, dt$.

Solution Let $f(t) = t^q e^{-t}, g(t) = t^{-2}$. Clearly, $f(t) > 0, g(t) > 0$ for all $t \geq 1$. Now

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{e^{-t}t^q}{t^{-2}} = \lim_{t\to\infty} \frac{t^{q+2}}{e^t} = 0 \quad \text{for any } q \in \mathbb{R} \text{ (By L'Hôpital rule in case } q+2 > 0)$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges, hence by Limit Comparison Test $\displaystyle\int_1^\infty e^{-t}t^q\, dt$ converges.

Theorem 24.9 (Dirichlet's Test for Improper Integrals) *Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ be such that f is monotonic, $f(x) \to 0$ as $x \to \infty$, g is continuous, and the function $G : [a, \infty) \to \mathbb{R}$ defined by $G(x) := \displaystyle\int_a^x g(t)dt$ is bounded. Then the improper integral $\displaystyle\int_a^\infty f(t)g(t)dt$ is convergent.*

Example 24.10 *Discuss the convergence of* $\displaystyle\int_1^\infty \frac{\sin t}{t}dt$.

Solution Let $f(t) = \frac{1}{t}, g(t) = \sin t$. Clearly, f is monotonically decreasing on $[1,\infty)$, $f(x) \to 0$ as $x \to \infty$, g is continuous, and $\int_1^x g(t)dt = \cos 1 - \cos x$. Hence $|\int_1^x g(t)dt| \le |\cos 1| + 1$ for all $x \ge 1$. Hence by Dirichlet's test $\displaystyle\int_1^\infty \frac{\sin t}{t}dt$ converges. But $\displaystyle\int_1^\infty \frac{\sin t}{t}dt$ does not converge absolutely. Note that $a^2 \le |a|$ if $|a| \le 1$. This tells us that $\sin^2 x \le |\sin x|$ for all $x \in \mathbb{R}$. Hence

$$\left|\frac{\sin t}{t}\right| \ge \frac{\sin^2 t}{t} = \frac{1-\cos 2t}{2t}, \ \forall t \ge 1.$$

Hence for each $x \ge 1$ we have

$$\int_1^x \left|\frac{\sin t}{t}\right| dt \ge \int_1^x \frac{1}{2t}dt - \int_1^x \frac{\cos 2t}{2t}dt$$

Now $\displaystyle\int_1^\infty \frac{\cos 2t}{2t}dt$ converges by Dirichlet's test $\left(\left|\displaystyle\int_a^x \cos 2t dt\right| = \left|\frac{\sin 2x - \sin 2}{2}\right| \le \frac{1+|\sin 2|}{2}\right)$. But $\displaystyle\int_1^\infty \frac{1}{2t}dt$ diverges to infinity, hence $\displaystyle\int_1^\infty \left|\frac{\sin t}{t}\right| dt$ diverges to infinity.

24.2 Other improper integrals of the first kind

Suppose $b \in \mathbb{R}$ and $f : (-\infty, b] \to \mathbb{R}$ is integrable on $[x,b]$ for every $x \le b$. Then an integral of the form $\displaystyle\int_{-\infty}^b f(x)dx$ is called an improper integral of the first kind. One can convert improper integral $\displaystyle\int_{-\infty}^b f(x)dx$ to the type $\displaystyle\int_a^\infty g(t)dt$ by using substitution $x = -t$.

Next, let $f : \mathbb{R} \to \mathbb{R}$ be a function that is integrable on $[a,b]$ for all $a, b \in \mathbb{R}$ with $a \le b$. We say that $\displaystyle\int_{-\infty}^\infty f(t)dt$ is convergent if both $\displaystyle\int_0^\infty f(t)dt$ and $\displaystyle\int_{-\infty}^0 f(t)dt$ are convergent.

24.3 Cauchy Principle value

If the limit $\displaystyle\lim_{x\to\infty} \int_{-x}^x f(t)dt$ exists, then this limit is called the Cauchy principal value of the improper integral $\displaystyle\int_{-\infty}^\infty f(t)dt$.

If improper integral $\displaystyle\int_{-\infty}^{\infty} f(t)dt$ converges, then since

$$\int_{-x}^{x} f(t)dt = \int_{-x}^{0} f(t)dt + \int_{0}^{x} f(t)dt, \text{ for all } x \geq 0.$$

Therefore Cauchy principal value of $\displaystyle\int_{-\infty}^{\infty} f(t)dt$ exists and is equal to $\displaystyle\int_{-\infty}^{\infty} f(t)dt$. But Cauchy principal value may exit even when integral $\displaystyle\int_{-\infty}^{\infty} f(t)dt$ is divergent. For example consider the improper integral $\displaystyle\int_{-\infty}^{\infty} \sin t dt$. It diverges because $\int_0^x \sin t\ dt = 1 - \cos x$,and $\displaystyle\lim_{x\to\infty} \cos x$ does not exist. But

$$\int_{-x}^{x} \sin t dt = 0, \text{ for all } x \geq 0$$

So Cauchy principle value exists and it is equal to zero.

24.4 Improper Integral of Second Kind

Definition 24.11 *Let $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : (a, b] \to \mathbb{R}$ be such that f is unbounded on $(a, b]$ but integrable on $[x, b]$ for each $x \in (a, b]$. Then the symbol $\displaystyle\int_a^b f(t)dt$ is called an improper integral of the second kind. We say that $\displaystyle\int_a^b f(t)dt$ is convergent if the right (hand) limit*

$$\lim_{x\to a^+} \int_x^b f(t)dt$$

exists.

Example 24.12 *Let $p \in \mathbb{R}$ and $f : (0, 1] \to \mathbb{R}$ be defined by $f(x) := \dfrac{1}{x^p}$. (Since f is continuous hence integrable on $[x, 1]$ for every $x \in (0, 1]$.) As $x \to 0$, $f(x) \to \infty$, hence f is unbounded on $(0, 1]$. consider the improper integral $\displaystyle\int_0^1 \frac{1}{x^p}dx$. Given any $x \in (0, 1]$, we have*

$$\int_x^1 \frac{1}{t^p}dt = \begin{cases} \dfrac{1 - x^{1-p}}{1 - p} & \text{if} \quad p \neq 1 \\ -\ln x & \text{if} \quad p = 1 \end{cases}$$

It follows that if $p > 1$, then $\displaystyle\lim_{x\to 0^+} x^{1-p} = \infty$ and if $p < 1$, then $\displaystyle\lim_{x\to 0^+} x^{1-p} = 0$. Hence if $p < 1$, $\displaystyle\int_0^1 \frac{1}{x^p}dx$ converges to $\dfrac{1}{1 - p}$, while if $p \geq 1$, then it diverges to ∞.