Lecture 17: Rolle's Theorem and Mean Value Theorem

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Example 17.1

1. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := -x^2$ has a local maximum at the origin and f is differentiable at origin, hence by Theorem 16.9, $f'(0) = 0$.*

2. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$. $f'(0) = 0$ but f does not have local extrema at the origin. Hence condition that for a differentiable function derivative becoming zero is necessary for local extremum but it is not sufficient.*

3. *Let $D := [-1, 1]$, and $f(x) = |x|$ for $x \in D$. Then f has a local minimum at the interior point 0 of D. But f is not differentiable at 0.*

Theorem 17.2 (Rolle's Theorem) *Let $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.*

Rolle's Theorem can be used together with the IVP of continuous functions to check the uniqueness and the existence of roots in certain intervals, especially for polynomials with real coefficients.

Example 17.3 *If $f(x) = x^3 + px + q$ for $x \in \mathbb{R}$, where $p, q \in \mathbb{R}$ and $p > 0$, then show that f has a unique real root.*

Solution: Note that if f had more than one real root, then there would be $a, b \in \mathbb{R}$ with $a < b$ and $f(a) = f(b) = 0$. Also, f being polynomial is differentiable everywhere, hence by Rolle's Theorem, there would be $c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + p$ is not zero for any $x \in \mathbb{R}$ since $p > 0$. On the other hand, since f is a polynomail of odd degree hence at least one real root exist. $f(c) = 0$ for some $c \in \mathbb{R}$. Thus f has a unique real root.

Example 17.4 *Show that $f(x) := x^4 + 2x^3 - 2$ has a unique root in $(0, \infty)$.*

Solution: Since f is continuous on $[0, 1]$, it has the IVP on $[0, 1]$. Also, $f(0) = -2 < 0$, and $f(1) = 1 > 0$, and so there is $c \in (0, 1)$ such that $f(c) = 0$. Thus f has a unique real root. Suppose there are two roots in $(0, \infty)$. Then there are $a, b \in \mathbb{R}$ with $0 < a < b$ such that $f(a) = 0 = f(b)$. By Rolle's theorem, there is $c \in (a, b) \subset (0, \infty)$ such that $f'(c) = 0$. But $f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) \neq 0$ for $x \in (0, \infty)$. Hence f has unique root in $(0, \infty)$. ■

Example 17.5 *Consider $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = [x]$. Then f is not continuous at $x = 0$, hence not differentiable at $x = 0$. Also $f(1) \neq f(-1)$. But $f'(c) = 0$ for every $c \in (-1, 0) \cup (0, 1)$. This shows that conditions in Rolle's Theorem are sufficient but not necessary.*

Theorem 17.6 (Mean Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Example 17.7 *Using mean value theorem, prove that*

$$\frac{\sin x}{x} \leq 1 \textit{ for } x \neq 0.$$

Solution: Let us first consider the case $x > 0$. Consider $f(t) = \sin t$ on $[0, x]$. Clearly, f satisfies the hypothesis of MVT, therefore $\exists\, c \in (0, x)$ such that

$$\sin x - \sin 0 = \cos c\ (x - 0).$$

Since $x > 0, \cos c \leq 1$, therefore $x \cos c \leq x$. So we get $\sin x \leq x$ for $x > 0$. If $x < 0$, then $-x > 0$ and we have

$$\frac{\sin(-x)}{(-x)} \leq 1.$$

But $\sin(-x) = -\sin x$, hence we get the desired thing. ∎

Example 17.8 *Show that $e^x > 1 + x$ for all $x \neq 0$.*

Solution: Suppose $x < 0$. Then apply MVT for the function $f(t) = e^t$ for $t \in [x, 0]$. There exist $c \in (x, 0)$ such that

$$e^0 - e^x = e^c(0 - x) \implies 1 - e^x = -xe^c \tag{17.1}$$

Since $c < 0$, $e^c < 1$. Therefore $1 - e^x < -x$. We get the desired result. Repeat the same argument for $x > 0$. ∎

The corollary below is perhaps the most important consequence of the MVT in calculus.

Corollary 17.9 *Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be any function. Then f is a constant function on I if and only if f' exists and is identically zero on I.*

Proof: If f is a constant function on I, then it is obvious that f' exists on I and $f'(x) = 0$ for all $x \in I$. Conversely, if f' exists and vanishes identically on I, then for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $[x_1, x_2] \subseteq I$ and applying the MVT to the restriction of f to $[x_1, x_2]$, we obtain $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. Since $f'(c) = 0$, we have $f(x_1) = f(x_2)$. This proves that f is a constant function on I. ∎

Example 17.10 *Let* $D := \mathbb{R} \setminus \{0\}$*, and* $f(x) := \frac{x}{|x|}$ *for* $x \in D$*. Then* $f' = 0$ *on* D*, but* f *is not constant on* D*; this is because* D *is not an interval.*