Claim

$$\#(i \rightarrow j : \log^*(\mathrm{rank}[i]) = \log^*(\mathrm{rank}[j])) \leq$$
$$O(n \log^* n)$$