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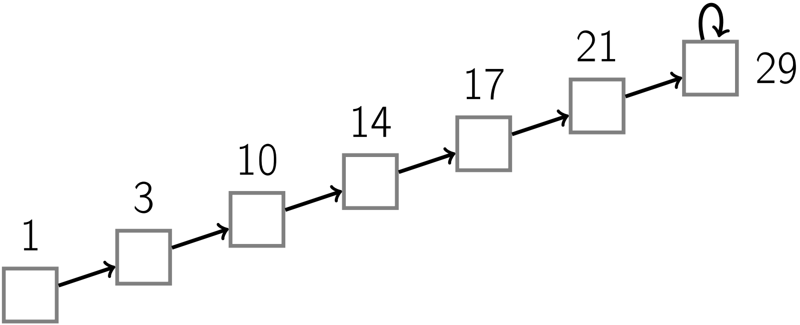

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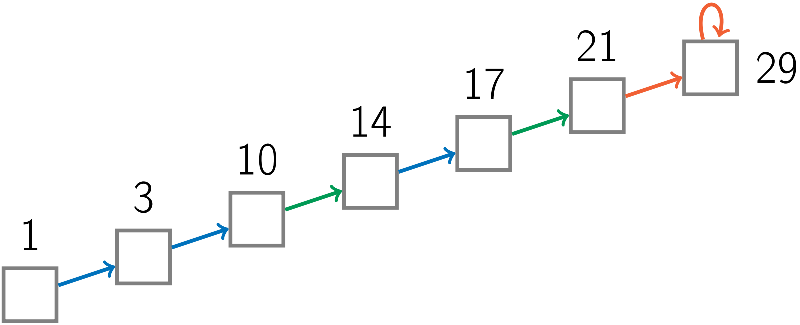

Claim

$\#(i \to j : j \text{ is a root}) \leq O(m)$

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Proof

There are at most m calls to `Find`. $\square$

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Proof

There are at most $\log^* n$ different values for $\log^*(\mathtt{rank})$. $\square$

Claim

$$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) = \log^*(\texttt{rank}[j])) \leq O(n \log^* n)$$

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- after at most 2^k calls to $\texttt{Find}(i)$, the parent of i will have rank from a different interval

Proof (Continued)

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- the number of different intervals is $\log^* n$
- thus, the contribution of all nodes is $O(n \log^* n)$ $\square$