Hyperbolic functions (CheatSheet)

1 Intro

For historical reasons hyperbolic functions have little or no room at all in the syllabus of a calculus course, but as a matter of fact they have the same dignity as trigonometric functions. Unfortunately this can be completely understood only if you have some knowledge of the complex numbers. Roughly speaking ordinary trigonometric functions are trigonometric functions of purely real numbers, and hyperbolic functions are trigonometric functions of purely imaginary numbers. For the moment we have to postpone this discussion to the end of Calc3 or Calc4, but still we should be aware of the fact that the impressive similarity between trig formulas and hyperbolic formulas is not a pure coincidence.

Most of the formulas that follow correspond precisely to a trig formula or they differ by at most a change of sign. For each formula I will explicitly state if some change of sign occurs or not (the different sign is marked in green).

The main purpose of this paper is not to give you a bunch of formulas to memorize, but to make you aware of the fact that hyperbolic formulas are just like trig formulas up to signs; and correct signs can always be checked with some very quick calculation.

2 Definitions

Definition of hyperbolic sine and cosine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

There are two equivalent formulas for sine and cosine (Euler's formulas) but they require some knowledge of the complex numbers:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

where $i = \sqrt{-1}$ or if you prefer $i^2 = -1$. Substituting x with ix in these two formulas and keeping in mind that $i^2 = -1$ it's immediate to deduce that $\cosh x = \cos(ix)$ and $\sinh x = -i\sin(ix)$ (I mention this just for the sake of completeness and because it's fun!). From the definition of hyperbolic sine and cosine we define hyperbolic tangent, cotangent, secant, cosecant in the same

way we did for trig functions:

$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\sinh x}$$

3 Basic properties

First of all we notice that hyperbolic functions have the same parity as the corresponding trig functions:

$$\sinh(-x) = -\sinh x \tag{1}$$
$$\cosh(-x) = \cosh x \tag{2}$$
$$\tanh(-x) = -\tanh x \tag{3}$$
$$\coth(-x) = -\coth x \tag{4}$$
$$\operatorname{sech}(-x) = \operatorname{sech} x \tag{5}$$
$$\operatorname{csch}(-x) = -\operatorname{csch} x \tag{6}$$
$$\tag{7}$$

All these formulas follow immediately from the definitions we gave in the previous section. Unlike trig functions, **hyperbolic functions are not periodic!**

Using the definition of hyperbolic sine and cosine it's possible to derive identities similar to $\cos^2 x + \sin^2 x = 1$ and $\tan^2 x + 1 = \sec^2 x$:

$$\cosh^2 x - \sinh^2 x = 1 \tag{8}$$
$$\tanh^2 x + \operatorname{sech}^2 x = +1 \tag{9}$$

These identities do not require Pythagoras' theorem, they can be derived from the definition with a direct calculation and using properties of the exponential.

4 Addition formulas

Unlike the case of ordinary addition trig formulas, the two basic addition formulas for hyperbolic functions can be retrieved immediately from the definition.

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \tag{10}$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \tag{11}$$

Combining these formulas with (1) we easily derive the following:

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y \tag{12}$$
$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y \tag{13}$$

Using the definition of hyperbolic tangent and equations (10,11) we can derive the addition formula:

$$\tanh\left(x \pm y\right) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{14}$$

5 Formulas for the double and half angle

Using equations (10,11) with $x = y$ we immediately have:

$$\sinh 2x = 2 \sinh x \cosh x \tag{15}$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{16}$$

By plugging (8) into (16) we have the following two formulas for the squares of sine and cosine:

$$\cosh^2 x = \frac{1 + \cosh 2x}{2} \tag{17}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2} \tag{18}$$

Notice that both (16) and (8) differ from the corresponding trig formulas by a sign, but the resulting formula for $\cosh^2$ is the same as in the trigonometric case, and the formula for $\sinh^2$ has a global change of sign. By substituting x with $\frac{x}{2}$ and taking the square root we have formulas for the half angle:

$$\cosh \frac{x}{2} = \sqrt{\frac{1 + \cosh x}{2}} \tag{19}$$
$$\sinh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \tag{20}$$

The first formula is the same[1] as the trigonometric one, and in the second one we have a global change of sign in the radicand. In the same way, but using (14) we have:

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \tag{21}$$

6 Inverse Hyperbolic functions

It's easy to check that hyperbolic sine is a monotonic increasing function on the real numbers, and for this reason it's invertible on all the real axis . Hyperbolic cosine has a global minimum at $x = 0$ whose value is $y = 1$ and it's decreasing from $-\infty$ to 0 and increasing from 0 to $+\infty$, for this reason we can invert it on the positive half-axis or the negative one. By convention we choose the positive one $[0, \infty)$. Hyperbolic tangent is defined for all real numbers, it's monotonic increasing and it has horizontal asymptotes:

$$\lim_{x \to \mp\infty} \tanh x = \mp 1 \tag{22}$$

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[1] You don't need to choose the sign in front of the radical since cosh is always positive

The inverse of an hyperbolic function can always be written as the logarithm of an algebraic[2] function:

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad \text{Domain} = (-\infty, +\infty), \text{ Range} = (-\infty, +\infty) \tag{23}$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad \text{Domain} = [1, +\infty), \text{ Range} = [0, +\infty) \tag{24}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad \text{Domain} = (-1, 1), \text{ Range} = (-\infty, +\infty) \tag{25}$$

Remember that the domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function. To retrieve these formulas we rewrite the definition of the hyperbolic function as a degree two polynomial in e^x; then we solve for e^x and invert the exponential. For example:

$$y = \sinh x = \frac{e^x - e^{-x}}{2} \iff e^{2x} - 2ye^x - 1 = 0 \iff e^x = y \pm \sqrt{y^2 + 1}$$

and since the exponential must be positive we select the positive sign.

7   Derivatives

The calculation of the derivative of an hyperbolic function is completely straightforward, so I will just report a list of formulas with no additional comments:

$$\frac{d}{dx}\sinh x = \cosh x \tag{26}$$

$$\frac{d}{dx}\cosh x = +\sinh x \tag{27}$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{28}$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \tag{29}$$

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x \tag{30}$$

$$\frac{d}{dx}\operatorname{csch} x = -\coth x \operatorname{csch} x \tag{31}$$

Formulas for the derivative of an inverse hyperbolic function can be quickly calculated from (23) using basic properties of derivatives. They can also be calculated using the formula for the derivative of the inverse:

$$\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{+x^2 + 1}} \tag{32}$$

$$\frac{d}{dx}\operatorname{arcosh} x = \frac{+1}{\sqrt{x^2 - 1}} \tag{33}$$

$$\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1 - x^2} \tag{34}$$

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[2]An algebraic function is a function containing the four operations and radicals only.

8   Obvious Primitives

The list of primitives of hyperbolic functions that you should actually remember is incredibly short:

$$\int \sinh x \; dx = + \cosh x \tag{35}$$

$$\int \cosh x \; dx = \sinh x \tag{36}$$

Every other primitive can be derived very quickly using some technique of integration and some of the formulas that we have seen so far; and if you really don't know how to integrate an hyperbolic function, just remember that an hyperbolic function can be written using the exponential, and usually functions containing the exponential are easy to integrate.