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ROBOT DYNAMICS & ANALYSIS HOMEWORK - 1:

1. a) tan : domain = $\mathbb{R}$, image = $(-\infty, \infty)$, differentiability (C^{-1})

not one-to-one nor onto. C^∞

b) arctan : domain = $\mathbb{R}$, imge = $(-\frac{\pi}{2}, \frac{\pi}{2})$, differentiability ($C^\infty$)

one-to-one

c) abs : domain = $\mathbb{R}$, image = $(0, \infty)$, differentiability (C^{-1})

not one-to-one, not onto

d) ln : domain = $(0, \infty)$, image = $(-\infty, \infty)$, differentiability (C^∞)

one-to-one and onto

e) floor : domain = $\mathbb{R}$, image = $\mathbb{Z}$, differentiability (C^{-1})

neither one-to-one nor onto

f) $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$: domain = $\mathbb{R}$, image $(-\infty, \infty)$

differentiability (C^2), one-to-one and onto

g) $g(x,y) = x^2+xy$ domain $= \mathbb{R}^2$, image $= (-\infty, \infty)$ differentiability (C^2) onto

$$\frac{\partial g}{\partial x} = 2x + y$$

$$\frac{\partial g}{\partial y} = x$$

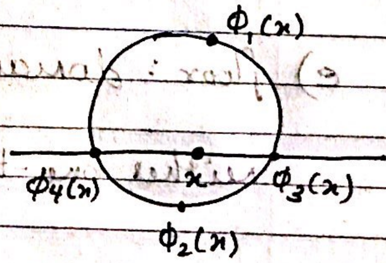
h) $f \circ g$ domain $= \mathbb{R}$, image $= (-\infty, \infty)$ differentiability C^∞

onto function

2.1 $c(x(t), y(t)) = x^2+y^2-r^2 = 0$, where $r = 1$

$\therefore c(x(t), y(t)) = x^2+y^2 = 1$

$(x, \sqrt{1-x^2}) = \Phi_1(x)$



$(x, -\sqrt{1-x^2}) = \Phi_2(x)$

Interval $= [-1, 1]$

2.2 $g_1(x)^2 = r^2 - x^2$

$g_1(x) = \sqrt{r^2-x^2} = \sqrt{1-x^2}$ initial point $= (0, 1)$

$g_2(x) = -\sqrt{r^2-x^2} = -\sqrt{1-x^2}$ initial point $= (0, -1)$

$V_1 = \{[x, g_1(x)]\ x \in (-1, 1)\}$, $V_2 = \{[x, g_2(x)]\ x \in (-1, 1)\}$

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2.3 $g_3(y) = \sqrt{r^2-y^2} = \sqrt{1-y^2}$ initial point $= (0, 1)$

$g_4(y) = -\sqrt{r^2-y^2} = -\sqrt{1-y^2}$ initial point $= (0,-1)$

$V_1 = \{[g_3(y), y] \mid y \in (-1,1)\}$

$V_2 = \{[g_4(y), y] \mid y \in (-1,1)\}$

2.4 $c = x^2+y^2 = r^2$

$\frac{\partial c}{\partial x} = 2x$

$\frac{\partial c}{\partial y} = 2y$

$\Rightarrow [2x \quad 2y]$ = Jacobian

$\therefore \dot{c} = [2x \quad 2y]\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{2x\dot{x}+2y\dot{y}=0}$

2.5 For $g_1(x) \Rightarrow S^1$ Manifold : $S^1 \to S$ $\quad c = x^2+y^2 = r^2$

$\Rightarrow x^2+y^2=1$

If $x \to s$, $y \to \sqrt{r^2-s^2}$

$\therefore c(s) = s^2+(\sqrt{r^2-s^2})^2-1 \quad \Rightarrow c(s) = r^2-1$

$= \cancel{s^2+\sqrt{1-s^2}}$

Jacobian $\Rightarrow [2s \quad \frac{1}{2}(1-s^2)^{-1/2}(-2s)]$

$\Rightarrow \left[2s \quad \frac{-s}{\sqrt{1-s^2}}\right]$

2.5 $x \to s$, $y = \sqrt{1-s^2}$

$c = f = x^2 + y^2 - r^2 = 0$

$c = x^2 + y^2 = 1$

$c = x^2 + y^2 - 1 = 0$

$c(s) = x^2 + (\sqrt{1-s^2})^2 - 1 = 0 \Rightarrow x^2 + 1 - s^2 - 1 = 0$

$x^2 - s^2 = 0$

$\text{Jacobian} = \begin{bmatrix} 2x & 2y \end{bmatrix}$

$= \begin{bmatrix} 2s & 2\sqrt{1-s^2} \end{bmatrix} \begin{bmatrix} \frac{\partial c}{\partial x} \\ \frac{\partial c}{\partial y} \end{bmatrix}$

$= \begin{bmatrix} 2s & 2\sqrt{1-s^2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-s}{\sqrt{1-s^2}} \end{bmatrix}$

$$\boxed{\begin{aligned} \dot{c} &= 2s - 2s \\ &= 0 \end{aligned}}$$

$g_2(x) \Rightarrow x \to s$, $y = -\sqrt{1-s^2}$

$\dot{c} = \begin{bmatrix} 2s & -2\sqrt{1-s^2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{s}{\sqrt{1-s^2}} \end{bmatrix}$

$$\boxed{\begin{aligned} \dot{c} &= 2s - 2s \\ &= 0 \end{aligned}}$$

$$g_3(y) = \sqrt{r^2 - \cancel{s}^2} \quad , y = s$$

$$c = g_3(y)^2 + y^2 - r^2 = 0 \Rightarrow g_3(y)^2 + y^2 = 1$$

$$\Rightarrow \left(\sqrt{\cancel{r}^2 - s^2}\right)^2 + s^2 - 1 = 0$$

$$\underset{w.r.t\ c}{\text{Jacobian}} = \begin{bmatrix} 2g_3(y) & 2y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\sqrt{\cancel{r}^2 - s^2} & 2y \end{bmatrix} \begin{bmatrix} -s/\sqrt{1-s^2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\sqrt{1-s^2} & 2s \end{bmatrix} \begin{bmatrix} \frac{-s}{\sqrt{1-s^2}} \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{0 = \dot{c}}$$

$$g_4(y) = -\sqrt{1-s^2} \quad , y = s$$

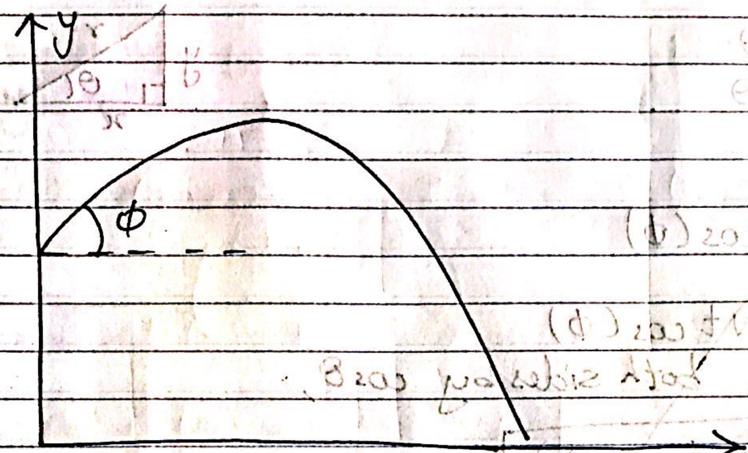
$$c = g_4(y)^2 + \cancel{s}^2 = 1$$

$$c = \left(-\sqrt{1-s^2}\right)^2 + \cancel{s}^2 = 1$$

$$\text{Jacobian} = \begin{bmatrix} -2\sqrt{1-s^2} & 2s \end{bmatrix} \begin{bmatrix} \frac{s}{\sqrt{1-s^2}} \\ 1 \end{bmatrix}$$

$$\boxed{\dot{c} = 0}$$

3.



$x(0) = 0$
$y(0) = 1$
$\dot{x}(0) = v\cos(\phi)$
$\dot{y}(0) = v\sin(\phi)$
$\ddot{x}(t) = 0$
$\ddot{y}(t) = -g$

3.1

$\cancel{y(t) = y_0 t - \frac{1}{2} g t^2}$

$\Rightarrow \cancel{y(t) = t - \frac{1}{2} g t^2}$

$\ddot{x}(t) = 0$
$\dot{x}(t) = C$
$\dot{x}(0) = C = v\cos\phi$

$\boxed{\therefore \dot{x}(t) = v\cos\phi}$

$x(t) = vt\cos\phi + D$
$x(0) = 0$
$\Rightarrow D = 0$

$\boxed{\therefore x(t) = vt\cos\phi}$

$\ddot{y}(t) = -g$
$\dot{y}(t) = -gt + C$

$\dot{y}(0) = -g(0) + C = v\sin(\phi)$

$\boxed{\therefore \dot{y}(t) = -gt + v\sin(\phi)}$

$\Rightarrow y(t) = -\frac{gt^2}{2} + vt\sin(\phi) + D$

$y(0) = 1$

$\Rightarrow D = 1$

$\boxed{\therefore y(t) = -\frac{gt^2}{2} + vt\sin(\phi) + 1}$

3.2

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x(t) = vt\cos(\phi) \quad , \quad y(t) = \frac{-gt^2}{2} + vt\sin(\phi) + 1$$

$$r = \sqrt{x^2 + y^2}$$

$$\Rightarrow r(t) = \sqrt{(vt\cos(\phi))^2 + \left(1 - \frac{gt^2}{2} + vt\sin(\phi)\right)^2}$$

$$\theta(t) = \arctan\left(\frac{\left(\frac{-gt^2}{2} + vt\sin(\phi) + 1\right)}{vt\cos(\phi)}\right)$$

3.3 Assuming $\phi = 0$,

$$r(t) = \sqrt{(vt)^2 + \left(1 - \frac{gt^2}{2}\right)^2}$$

$$= \sqrt{v^2t^2 + 1 + \frac{g^2t^4}{4} - gt^2} = \left(v^2t^2 + 1 + \frac{g^2t^4}{4} - gt^2\right)^{1/2}$$

$$\theta(t) = \arctan\left(\frac{\left(\frac{-gt^2}{2} + 1\right)}{vt}\right)$$

$$\boxed{\begin{array}{l} \dot{r}(t) = \frac{1}{2}\left(v^2t^2 + 1 + \frac{g^2t^4}{4} - gt^2\right)^{-1/2} \cdot \left(2v^2t + g^2t^3 - 2gt\right) \\ \quad \hookrightarrow \dfrac{\left(v^2t + \frac{g^2t^3}{2} - gt\right)}{\sqrt{\left(v^2t^2 + 1 + \frac{g^2t^4}{4}\right)}} \rightarrow \dot{r} \\ \dot{\theta}(t) = \dfrac{2v(2 + gt^2)}{(g^2t^4 - 4gt^2 + 4t^2v^2 + 4)} \end{array}}$$

$$
\begin{tikzcd}
TM \arrow[r, "T_f"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}
$$

Where $T_f = D_f$ - Jacobian

$$\dot{x} = v\cos(\phi)$$
$$\dot{y} = -gt + v\sin(\phi)$$

Sub $\phi = 0$,

$$\dot{x} = v$$
$$\dot{y} = -gt + 0$$

Using Mapping,

$$\frac{dr}{dt} = \frac{d}{dt}\sqrt{x^2 + y^2}$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2}(2x\dot{x} + 2y\dot{y})$$

$$\Rightarrow \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} = \dot{r}$$

$$\Rightarrow \frac{vt\cos(\phi)(v\cos\phi) + \left(-\frac{gt^2}{2} + vt\sin(\phi) + 1\right)\left(-gt + v\sin\phi\right)}{\sqrt{(vt\cos(\phi))^2 + \left(1 - \frac{gt^2}{2} + vt\sin(\phi)\right)^2}}$$

Substituting $\phi = 0$,

$$\dot{r} = \frac{vt(v) + \left(1 - \frac{gt^2}{2}\right)(-gt)}{\sqrt{(vt)^2 + \left(1 - \frac{gt^2}{2}\right)^2}}$$

$$\dot{r} = \frac{v^2 t - gt + \frac{g^2 t^3}{2}}{\sqrt{v^2 t^2 + 1 + \frac{g^2 t^4}{4} - gt^2}} \qquad \therefore \dot{r} = \dot{r},$$

~~Using mapping~~,

$$\frac{d\theta}{dt} = \frac{d}{dt}\left(\arctan\left(\frac{\frac{-gt^2}{2} + vt\sin(\phi) + 1}{vt\cos(\phi)}\right)\right)$$

Using mathematica:

$$\frac{d\theta}{dt} = \frac{\sec(\phi)\left(\frac{t(v\sin(\phi) - gt)}{\left(\frac{-gt^2}{2} + tv\sin(\phi) + 1\right)^2 + 1} - \arctan\left(\frac{-gt^2}{2} + tv\sin\phi + 1\right)\right)}{t^2 v}$$

Sub $\phi = 0$,

$$\frac{d\theta}{dt} = \frac{\frac{t(-gt)}{\left(\frac{-gt^2}{2} + 1\right)^2 + 1} - \arctan\left(\frac{-gt^2}{2} + 1\right)}{t^2 v}$$

using mapping:

$$\cancel{\frac{d\theta}{dt} = \frac{d}{dt}\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{d}{dt}\left(\arctan\left(\frac{-\frac{gt^2}{2}+vt\sin(\phi)+1}{vt\cos(\phi)}\right)\right)}$$

$$\cancel{\frac{d}{dt}\left(\arctan\left(\frac{-\frac{gt^2}{2}+vt\sin(\phi)+1}{vt\cos(\phi)}\right)\right)}$$

~~Using mathematica,~~

$$\cancel{\frac{d\theta}{dt} =}$$

$$\frac{d\theta}{dt} = \frac{d}{dt}\left(\arctan\left(\frac{y}{x}\right)\right)$$

Using mathematica,

$$\frac{d\theta}{dt} = \frac{x\dot{y}-y\dot{x}}{(x^2+y^2)}$$

Sub x, y, $\dot{x}$ and $\dot{y}$:

$$\frac{\left[vt\cos(\phi)(-gt+v\sin(\phi))\right]-\left[\left(\frac{-gt^2}{2}+vt\sin(\phi)+1\right)(v\cos(\phi))\right]}{(vt\cos(\phi))^2+\left(1-\frac{gt^2}{2}+vt\sin(\phi)\right)^2}$$

Sub $\phi=0$:

$$\frac{vt(-gt)-\left[\left(\frac{-gt^2}{2}+1\right)(v)\right]}{(vt)^2+\left(1-\frac{gt^2}{2}\right)^2} = \frac{-gt^2v-2v}{-\left(\frac{g^2t^4}{2}-2gt^2+2t^2v^2+2\right)}$$

$\therefore$ proved $= \boxed{\frac{2v(2+gt^2)}{(g^2t^4-4gt^2+4t^2v^2+4)}}$