

ROBOT DYNAMICS & ANALYSIS HOMEWORK - 1:

1. a) \tan : domain = \mathbb{R} , image = $(-\infty, \infty)$, differentiability (C^{-1})
 not one-to-one nor onto. C^∞
- b) \arctan : domain = \mathbb{R} , image = $(-\frac{\pi}{2}, \frac{\pi}{2})$, differentiability (C^∞)
 one-to-one
- c) abs : domain = \mathbb{R} , image = $[0, \infty)$, differentiability (C^{-1})
 not one-to-one, not onto
- d) \ln : domain = $(0, \infty)$, image = $(-\infty, \infty)$, differentiability (C^∞)
 one-to-one and onto
- e) floor : domain = \mathbb{R} , image = \mathbb{Z} , differentiability (C^{-1})
 neither one-to-one nor onto
- f) $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$: domain = \mathbb{R} , image = $(-\infty, \infty)$
 differentiability, one-to-one
 $(f(0)) = 0$ is a bottom $\rightarrow |x-1| = (\text{constant})^2 = (x)^2$ and onto
 $(1, 1) \times [(\text{constant}), \infty) = V$

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g) $g(x, y) = x^2 + xy$ domain = \mathbb{R}^2 , image = $(-\infty, \infty)$

$\left(\begin{array}{l} \frac{\partial g}{\partial x} = 2x + y \\ \frac{\partial g}{\partial y} = x \end{array} \right)$ differentialability \Leftrightarrow onto \mathbb{C}^2
other non onto form

h) $f(x) = x^2$, $A = \text{interval}$: $x \in \mathbb{R}$

h) $f \circ g$ domain = \mathbb{R} , image = $(-\infty, \infty)$ differentialability C^∞

Hilfslinie $y = x$ onto (function) $A = \text{interval}$: $x \in \mathbb{R}$

other non onto at -ve for

(x, ∞)

1.2 Hilfslinie $y = x$ $c(x(t), y(t)) = x^2 + y^2 = x^2 = 1$ \Rightarrow $x = \pm 1$

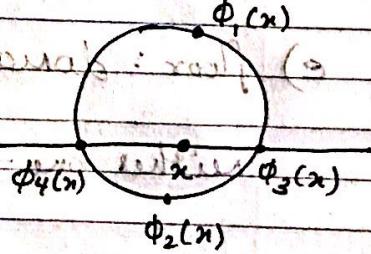
2.1 $c(x(t), y(t)) = x^2 + y^2 - x^2 = 0$ where $x = 1$

$\therefore c(x(t), y(t)) = x^2 + y^2 = 1$

Hilfslinie $y = \sqrt{1-x^2}$ $A = \text{interval}$: $x \in \mathbb{R}$

$(x, -\sqrt{1-x^2}) = \phi_1(x)$

Interval = $[-1, 1]$



2.2 $g_1(x)^2 = x^2 + x^2$ initial: $0 \leq x \leq x_0$ $x_0 = ?$

$g_1(x) = \sqrt{x^2 + x^2} = \sqrt{1-x^2}$ initial point = $(0, 1)$

$g_2(x) = -\sqrt{x^2 + x^2} = -\sqrt{1-x^2}$ initial point = $(0, -1)$

$V_1 = \{[x, g_1(x)] \times \mathcal{E}(-1, 1)\}, V_2 = \{[x, g_2(x)] \times \mathcal{E}(-1, 1)\}$

$$2.3. \quad g_3(y) = \sqrt{x^2 - y^2} = \sqrt{1 - y^2} - 1 \text{ initial point } = (0, 1)$$

$$g_4(y) = -\sqrt{x^2 - y^2} = -\sqrt{1 - y^2} - \text{initial point } = (0, -1)$$

$$V_1 = \{ [g_3(y), y] \mid y \in (-1, 1)\}$$

$$O = 1 - \frac{1}{2}x^2 + \frac{1}{2}y^2 \quad O = 1 - \left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) = 0$$

$$V_2 = \{ [g_4(y), y] \mid y \in (-1, 1)\}$$

$$2.4. \quad C = x^2 + y^2 = r^2 = 1 \quad [x^2 + y^2 = 1]$$

$$\frac{\partial C}{\partial x} = 2x \quad [x^2 + y^2 = 1]$$

$$\frac{\partial C}{\partial y} = 2y \quad [x^2 + y^2 = 1]$$

$$\Rightarrow [2x, 2y] = \text{Jacobian} \quad [x^2 + y^2 = 1]$$

$$\therefore \dot{c} = [2x, 2y] \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 2x\dot{x} + 2y\dot{y} = 0 \quad (K)$$

2.5. For $g(x) \Rightarrow S^1$ manifold: $S^1 \rightarrow S$

$$\text{If } x \rightarrow s, \quad y \rightarrow \sqrt{s^2 - x^2}$$

$$\therefore c(s) = s^2 + (\sqrt{s^2 - x^2})^2 - 1 = s^2 + s^2 - x^2 - 1 \Rightarrow c(s) = 2s^2 - 1$$

$$\text{Jacobian} \rightarrow [2s, \frac{1}{2}(1-s^2)^{-1/2}(-2s)]$$

$$\rightarrow [2s, \frac{-s}{\sqrt{1-s^2}}]$$

$$2.5 \quad x \rightarrow s, y = \sqrt{1-s^2}, z = \sqrt{1-s^2} = (1, 0, 0) \text{ P.S.}$$

$$c = x^2 + y^2 - z^2 = 0 \quad \Rightarrow \quad [x, y, z] = (1, 0, 0) \text{ P.R.}$$

$$c = x^2 + y^2 = 1$$

$$c = x^2 + y^2 - 1 = 0 \quad \text{f}(1, 0, 0) \text{ B.C. } [y, (y)_c] \text{ P.R.}$$

$$c(s) = x^2 + (\sqrt{1-s^2})^2 - 1 = 0 \quad \Rightarrow \quad x^2 + 1-s^2 - 1 = 0$$

$$\text{Jacobian} = \begin{bmatrix} 2x & 2y \end{bmatrix} \left| \begin{bmatrix} \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} \end{bmatrix} \right| = 0 \text{ P.S.}$$

$$= \begin{bmatrix} 2s & 2\sqrt{1-s^2} \end{bmatrix} \left| \begin{bmatrix} \frac{\partial c}{\partial x} \\ \frac{\partial c}{\partial y} \end{bmatrix} \right| = \sqrt{1-s^2} = 0 \text{ P.S.}$$

$$= \begin{bmatrix} 2s & 2\sqrt{1-s^2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{s}{\sqrt{1-s^2}} \end{bmatrix} \quad \dot{s} = \frac{-2s}{2\sqrt{1-s^2}}$$

$$\dot{c} = 2s - 2s = 0$$

$$\text{midpoint} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \text{ P.S.}$$

$$g_2(x) \Rightarrow x \rightarrow s, y = -\sqrt{1-s^2}, z = \sqrt{1-s^2} = \dots$$

$$\dot{c} = \begin{bmatrix} 2s & -2\sqrt{1-s^2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{s}{\sqrt{1-s^2}} \end{bmatrix}$$

$$-s^2 - s\sqrt{1-s^2} = 0 \quad 2s - 2\sqrt{1-s^2} = 0 \quad \text{f}(0, 1, 0) \text{ P.R.} \quad \text{P.S.}$$

$$\dot{c} = 2s - 2s = 0$$

$$1 - s^2 - (2s)^2 = 0 \quad 1 - (s_2 + s_3)(1+s_2 - (2s)) = 0$$

$$[(s_2 + s_3)(1+s_2 - (2s))] = 0 \quad \text{midpoint}$$

$$\begin{bmatrix} 2 & -2s \end{bmatrix} \times$$

$$g_3(y) = \sqrt{y^2 - s^2}, \quad y = s$$

$$c = g_3(y)^2 + y^2 - s^2 = 0 \Rightarrow g_3(y)^2 + y^2 = 1$$

$$\Rightarrow (\sqrt{y^2 - s^2})^2 + s^2 - 1 = 0$$

$$\text{Jacobian} = \begin{bmatrix} 2g_3(y) & 2y \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\sqrt{y^2 - s^2} & 2y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s/\sqrt{1-s^2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\sqrt{1-s^2} & 2s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s \\ \sqrt{1-s^2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{0 = c}$$

$$g_4(y) = -\sqrt{1-s^2}, \quad y = s$$

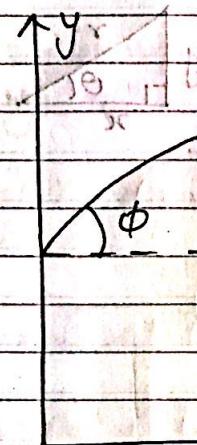
$$c = g_4(y)^2 + s^2 = 1$$

$$c = (-\sqrt{1-s^2})^2 + s^2 = 1$$

$$\text{Jacobian} = \begin{bmatrix} -2\sqrt{1-s^2} & 2s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ \sqrt{1-s^2} \\ 1 \end{bmatrix}$$

$$\boxed{c = 0}$$

3.



$$x(0) = 0$$

$$y(0) = 0$$

$$\dot{x}(0) = v \cos(\phi)$$

$$\dot{y}(0) = v \sin(\phi)$$

$$\ddot{x}(t) = 0$$

$$\ddot{y}(t) = -g$$

$$(1) \quad \ddot{y}(t) + g = 0$$

$$\begin{aligned} & \text{Initial conditions: } \\ & \ddot{y}(0) = -g \quad \ddot{y}(0) = 0 \\ & \text{Solving: } \ddot{y}(t) = -gt + C_1 \\ & \text{At } t=0: -g(0) + C_1 = 0 \Rightarrow C_1 = 0 \end{aligned}$$

3.1

$$y(t) = v_0 t - \frac{gt^2}{2}$$

$$\ddot{y}(t) = -g$$

$$\Rightarrow y(t) = v_0 t - \frac{gt^2}{2}$$

$$\ddot{y}(t) = -gt + C_2$$

$$(1) \quad \ddot{y}(0) = -g(0) + C_2 = v \sin(\phi)$$

$$\ddot{x}(t) = 0$$

$$\therefore \ddot{y}(t) = -gt + v \sin(\phi)$$

$$\dot{x}(t) = C_3$$

$$\dot{x}(0) = C_3 = v \cos(\phi) \quad \Rightarrow \quad \ddot{x}(t) = -gt^2 + v t \sin(\phi) + D$$

$$\therefore \dot{x}(t) = v \cos(\phi)$$

$$y(0) = 1$$

$$x(t) = v t \cos(\phi) + D$$

$$\Rightarrow -\frac{C_3}{2} \Rightarrow D = 1$$

$$y(0) = 0$$

$$\therefore y(t) = -\frac{gt^2}{2} + v t \sin(\phi) + 1$$

$$\therefore x(t) = v t \cos(\phi)$$

3.2

$$x = vt \cos \theta$$

$$y = vt \sin \theta$$

$$x(t) = vt \cos(\phi), \quad y(t) = -\frac{gt^2}{2} + vt \sin(\phi) + 1$$

$$r = \sqrt{x^2 + y^2}$$

$$\Rightarrow r(t) = \sqrt{(vt \cos(\phi))^2 + \left(1 - \frac{gt^2}{2} + vt \sin(\phi)\right)^2}$$

$$\theta(t) = \arctan \left(\frac{\frac{-gt^2}{2} + vt \sin(\phi) + 1}{vt \cos(\phi)} \right)$$

3.3

Assuming $\phi = 0$,

$$r(t) = \sqrt{(vt)^2 + \left(1 - \frac{gt^2}{2}\right)^2}$$

$$= \sqrt{v^2 t^2 + 1 + \frac{g^2 t^4}{4} - gt^2} = \left(v^2 t^2 + 1 + \frac{g^2 t^4}{4} - gt^2\right)^{1/2}$$

$$\theta(t) = \arctan \left(\frac{\frac{-gt^2}{2} + 1}{vt} \right)$$

$$\dot{\theta}(t) = \frac{1}{2} \left(v^2 t^2 + 1 + \frac{g^2 t^4}{4} - gt^2\right)^{-1/2} \cdot \frac{(2v^2 t + g^2 t^3 - 2gt)}{\left(v^2 t + \frac{g^2 t^3}{2} - gt\right)}$$

$$\dot{\theta}(t) = \frac{2v(2+gt^2)}{(g^2 t^4 - 4gt^2 + 4t^2 v^2 + 4)}$$

$$\begin{array}{ccc} TM & \xrightarrow{f} & TN \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

Basis
Basis

Q.E.D.

Where $T_f = D_f$ - Jacobian

$$\begin{aligned} \dot{x} &= v \cos(\phi) \left(1 + \frac{s_{tp}}{s} - 1 \right) + s((\phi) \omega_0 + y) = (\pm)x \\ \dot{y} &= -gt + v \sin(\phi) \cdot \frac{s}{s} \\ \text{Sub } \phi = 0, \quad \dot{x} &\neq v \left(1 + \frac{(1)(\omega_0^2 + v^2)}{s} + \frac{s_{tp}}{s} \right) \\ \dot{y} &= -gt + 0 \end{aligned}$$

Using mapping,

$$\frac{dr}{dt} = \frac{d}{dt} \sqrt{x^2 + y^2}$$

$$r^2 \left(\frac{s_{tp}}{s} + \frac{s_{\omega_0}}{s} (x^2 + y^2) \right)^{-1/2} = (2x\dot{x} + 2y\dot{y}) + \frac{s_{\omega_0}}{s} v^2$$

$$\Rightarrow \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = \dot{r} \left(\left(1 + \frac{s_{tp}}{s} \right) + \frac{s_{\omega_0}}{s} v^2 \right)$$

$$\Rightarrow v \cos(\phi) (v \cos \phi) + \left(-\frac{gt^2}{s} + v t \sin(\phi) + 1 \right) \left(-gt + v \sin \phi \right)$$

$$= \left(\frac{s_{tp}}{s} + \frac{s_{\omega_0}}{s} + \frac{s_{\omega_0}}{s} v^2 \right) + \left(1 + \frac{gt^2}{s} + v t \sin(\phi) \right)^2$$

$$= \frac{(\dot{r} + \frac{s_{tp}}{s} + \frac{s_{\omega_0}}{s} v^2)^2}{s^2} = (\pm) \dot{r}$$

Substituting $\phi = 0$,

$$\dot{\gamma} = vt(v) + \left(1 - \frac{gt^2}{2}\right)(-gt)$$
$$\sqrt{(vt)^2 + \left(1 - \frac{gt^2}{2}\right)^2}$$

$$\dot{\gamma} = \frac{v^2 t - gt + g^2 t^3}{2}$$
$$\sqrt{v^2 t^2 + 1 + \frac{g^2 t^4}{4} - 2gt^2}$$
$$\therefore \dot{\gamma} = \ddot{\gamma}$$

= Using mapping,

$$\frac{d\theta}{dt} = \frac{d}{dt} \left(\arctan \left(\frac{-gt^2}{2} + \frac{vt \sin(\phi)}{v t \cos(\phi)} + 1 \right) \right)$$

Using mathematical

$$\frac{d\theta}{dt} = \sec(\phi) \frac{t(v \sin(\phi) - gt)}{\left(-\frac{gt^2}{2} + vt \sin(\phi) + 1\right)^2 + 1} - \arctan \left(\frac{-gt^2 + vt \sin(\phi)}{2} + 1 \right)$$

Sub $\phi = 0$,

$$\frac{d\theta}{dt} = \frac{t(-gt)}{\left(-\frac{gt^2}{2} + 1\right)^2 + 1} - \arctan \left(\frac{-gt^2 + 1}{2} \right)$$

using mapping:

$$\frac{d\theta}{dt} = \frac{d(\arctan(\frac{y}{x}))}{dt} = \frac{d(\arctan(\frac{-gt^2 + vtsin(\phi) + v}{2}))}{dt}$$
$$\frac{d}{dt} (\arctan(\frac{-\frac{gt^2}{2} + vtsin(\phi) + v}{vt\cos(\phi)}))$$

Using Mathematica,

$$\frac{d\theta}{dt} =$$

$$\frac{d\theta}{dt} = \frac{d}{dt} (\arctan(\frac{y}{x}))$$

Using Mathematica,

$$\frac{d\theta}{dt} = \frac{x\dot{y} - y\dot{x}}{(x^2 + y^2)}$$

Sub x, y, \dot{x} and \dot{y} :

$$[vt\cos(\phi)(-gt + v\sin(\phi))] - \left[\left(\frac{-gt^2 + vtsin(\phi) + v}{2} \right) (v\cos(\phi)) \right]$$
$$(v\cos(\phi))^2 + \left(1 - \frac{gt^2}{2} + v\sin(\phi) \right)^2$$

Sub $\phi = 0$:

$$vt(-gt) + \left[\left(\frac{-gt^2 + v}{2} \right) (v) \right] = \cancel{0} \quad \cancel{+ gt^2 v} \neq 2v$$
$$\frac{(vt)^2 + \left(1 - \frac{gt^2}{2} \right)^2}{2} = \frac{(g^2 t^4 - 2gt^2 + 2t^2 v^2 + 2)}{2}$$

$$\therefore \text{proved} = \sqrt{\frac{2v(2 + gt^2)}{(g^2 t^4 - 4gt^2 + 4t^2 v^2 + 2)}}$$